Analytical foundations of a class of hybrid models with applications to collective dynamics

Candidate: **Marta Menci**

Supervisor: **Prof. Marco Papi**

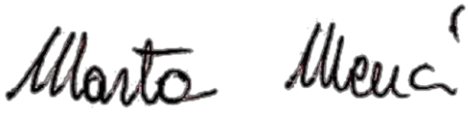
A Thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Bioengineering and Bioscience

Department of Engineering
Università Campus Bio-Medico di Roma

XXXII cycle
Academic Year 2016-2017
12 March 2020

Marta Menci

"A Chiara e Stefano"



Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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General Introduction

This thesis concludes my XXXII PhD course of Bioengineering and Bioscience at Università Campus Bio-Medico di Roma, under the supervision of Professor Marco Papi. The initial research program originates from personal research interests concerning mathematical modeling of collective dynamics, both from numerical and analytical point of views. A first approach to the topic is due to the master's degree thesis on Mathematics, with supervisor Roberto Natalini, Head of Istituto per Le Applicazioni del Calcolo "Mauro Picone", CNR, Rome. Moreover, the interdisciplinary nature of this PhD Course, bringing together mathematicians, engineers with different knowledge and biologists, allowed me to give a contribution in other research areas. In particular, collective motions of agents, regardless of their nature, has been linked to consensus problems and dynamics on network. In this sense, the thesis collects two different yet complementary approaches to address the above research directions. This classification loosely reflects Parts I and II of the thesis, respectively.

Synopsis of the thesis

The thesis as a whole focuses on the study and mathematical modeling of collective dynamics arising in different contexts. The results we present follow a preliminary in-depth study of the literature, carried out during the three-years term of the PhD course. When reporting multiple references, we choose to cite, among the large amount of papers collected, those that have played a major role on our researches.

The work is organized in two parts, that are structured to be read independently. Both parts start with a brief introduction, and discussion and future works conclude the topic of each chapter. Since Part I and Part II use concepts from different fields, the notation we employ is consistent within each part. In particular, the terminology *hybrid* has two different connotations.

In the first part, we study a distinct type of systems of differential equations, arising from mathematical models that combine discrete and continuum approaches, known as *hybrid* or *discrete-continuum models*. In the last

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years, these kind of models has gained particular interest, among others, in biological phenomena involving cell-cell interactions and cell-matrix interactions, specifically chemotactic ones. The terminology *hybrid* thus refers to the presence of different scales within the same model: while cells are treated as discrete units, the chemotactic signal influencing their dynamics is represented as a continuum.
Part I of the thesis is devoted to analytical results concerning existence and uniqueness of the solution of coupled hybrid systems. From a mathematical point of view, the general structure of the investigated models combines second order ordinary differential equations, which model inter-individual interactions, with partial differential equations describing the evolution of the signal further influencing the dynamics. The fact that the great variety of applications does not correspond to a relevant literature concerning theoretical fundaments is shared by experts of these kind of modeling. Moreover, there are at least two main aspects that differentiate our results compared to the few available in the current state of the art concerning this class of coupled systems. First, the solution to the chemotaxis equation is a path-dependent function : for every time t, in order to evaluate the solution at that time, it is necessary to know the entire trajectory of the agents up to time t. Second, the coupling is achieved through the gradient of the solution of the parabolic equation, evaluated, at each time step, in the position of each agent.

With regard to the second order structure of coupled hybrid systems, we propose an overview of the literature concerning the basic inter-individual mechanical interactions arising in collective phenomena. Within this framework, we present the first step of a current joint collaboration with Emiliano Cristiani (Istituto per le Applicazioni del Calcolo-CNR), aiming at developing hybrid models with applications to crowds and flocks [H].
For the rest of Part I, we shall deal with hybrid systems, both with analytical and numerical purposes.
We start proposing well-posedness theorems for the solution in $\mathbb{R}^N$ assuming, in particular, the source term in the parabolic equation describing chemotaxis and the initial data continuous functions, with particular growth conditions [A].
We extend the previous study investigating the case of a source term with less regularity properties [C]. Finally, we further generalize the structure of hybrid models, assuming in particular initial data in the Lebesgue space $L^2(\mathbb{R}^N)$,and source term in $Lip_{loc}(\mathbb{R}^{N\times n}; L^2(\mathbb{R}^N))$ [D].

The analytical approach is linked to a more applied and numerical oriented work [B]. In this study we consider a hybrid model, adapting to those phenomena involving alignment and chemotaxis mechanisms. The evolution in time of the exogenous chemical signal is modeled by a parabolic equation with constant coefficient and initial null concentration. From the analytic point of view, we prove existence and uniqueness of the solution. With re-

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spect to the previous researches, the focus is on the asymptotic behavior of the system. In particular, we present the results obtained on a linearised version of the investigated system. Theoretical findings are compared with numerical simulations, based on finite difference schemes, concerning the behavior of the full nonlinear system.

Part II of the thesis introduces a different approach to collective dynamics of agents, as a result of a joint work with the Research Unit of Automation of Università Campus Bio-Medico di Roma. The collaboration started with a published book chapter focused on robustness and control of distributed systems [F]. In order to provide evidence of the robustness of distributed biological systems, we considered a case study describing chemotaxis processes for a colony of E. Coli bacteria. Afterwards, distributed systems of agents have been investigated in a decision-making perspective. The research in this area has led to two contributions. In a first study, we find effective distributed algorithms to solve the Sparse Analytic Hierarchy Process problem, where a set of networked agents (e.g., wireless sensors, mobile robots or IoT devices) needs to be ranked based on their utility/importance [E], using information on the relative utility of pairs of agents (e.g., agent i is twice as important as agent j).
Specifically, we provide a numerical comparison of the performances of four methods over networks with different characteristics.
In a second contribution, we consider a scenario where a set of agents, interconnected by a network topology, aim at computing an estimate of their own utility, based on pairwise relative information having hybrid nature, i.e., combining information regarding the ratio of the utilities with information regarding the difference in utility of selected pairs of agents. Specifically, we developed a distributed algorithm that lets each agent asymptotically compute a utility value [G]. The novelty with respect to previous work in literature relies in the fact that different types of information are considered within the same scenario. For that reason, in Part II the terminology *hybrid* refers to the framework of the investigated problem, which is characterized by the coexistence of informations with different nature.

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Published or submitted papers related to this Ph.D. thesis

A) Menci, M., Papi, M. (2019). Global solutions for a path-dependent hybrid system of differential equations under parabolic signal. Nonlinear Analysis, 184, 172-192.

B) Di Costanzo, E., Menci, M., Messina, E., Natalini, R. and Vecchio, A. (2020). A hybrid model of collective motion of discrete particles under alignment and continuum chemotaxis. Discrete Cont. Dyn-B, 25(1), 443–472. (doi: 10.3934/dcdsb.2019189)

C) Menci, M., Papi, M., Existence of solutions for hybrid systems of differential equations under chemotactic signal with discontinuous source term, (Submitted, October 2019)

D) Menci, M., Papi, M., Porzio, M., Smarrazzo, F., Coupled hybrid system of nonlinear differential equations with a non-local concentration and L^σ initial data. (Submitted, December 2019).

E) Menci, M., Oliva, G., Papi, M., Setola, R., Scala, A. (2018). A suite of distributed methodologies to solve the sparse analytic hierarchy process problem. In 2018 European Control Conference (ECC) (pp. 1447-1453). IEEE.

F) Menci, M., Oliva, G. (2018). Robustness vs. Control in Distributed Systems. In Biological Robustness (pp. 189-205). Springer, Cham.

G) Menci, M., Oliva, G., Papi, M., Setola, R., Zoppello, M. (2018). Distributed Utility Estimation With Heterogeneous Relative Information. IEEE Control Systems Letters, 2(2), 248-253.

H) Cristiani, E., Menci, M., Papi, M., A microscopic model for collective motions of birds with turning. In preparation.

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Part I

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Introduction

In the first part of this thesis, we study a particular type of systems of differential equations, arising from mathematical models that simultaneously combine discrete and continuum approaches. In recent years, these kinds of models have been mainly adopted to describe phenomena concerning living systems, such as cell aggregates ([1], [2], [3], [4], [5]) or crowds ([6], [7]. For an updated survey of recent results in the mathematical modeling and control of crowd, at the various level of modeling, see also [8]). These are regarded as collections of agents presenting two fundamental aspects: a proper behavior, and the ability to sense and actively interact with other individuals and the surrounding environment. When modeling collective phenomena, the choice of the scale of observation represents one of the first point to address. The majority of mathematical models in the literature treats agents aggregates either as a continuum or as a discrete set of individuals.

Discrete, continuous and hybrid approaches

Discrete models operate at the scale of individuals ([9], [10], [11], [12]). For example, in the biomedical field, each cell is treated as a unit of finite volume, which is able to move, divide and die individually according to biological observations. Agents have been modelled with simple points, spheres and ellipsoids, both with fixed volume and size, or more complex evolving deformable structures ([13],[14]). A discrete approach allows to easily model mechanical interactions with other cells and with the surrounding matrix, and to incorporate details concerning individual cells (e.g. size, metabolic state). The possibility to model agents in fine details leads to the drawback of a large computational cost, which rapidly increases with the number of agents considered. Thus, discrete models are suitable for a microscopic description of phenomena when the number of agents is relatively limited.

Since a same problem can be modelled at different scales depending on the aspects of interest, the choice of the approach to adopt is often not unique. Sometimes the distinction among different approaches is not extremely defined, other times it is quite obvious. In fact, to model regions in which mechanical and rheological properties are of primary interest, it

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is completely unnecessary to focus on the cell scale. Aiming at a global description of the agents interactions, continuum models better fulfil the requirements ([15],[16]). From a macroscopic point of view, the units' aggregate is described through its spatial mass density, and continuum equations are used to model cell-cell and cell-matrix interactions. On the one hand, continuous models are easier to be computationally analyzed, and do not present limitations in the number of agents involved. On the other hand, they suffer from the fact that generally the averaging over space realized in continuum formalisms cannot fully account for the diversity of cellular and sub-cellular dynamical features.
Advantages and disadvantages of the two categories seem to be complementary. The emergent hybrid approaches, in which some of the variables of the model are continuous and other are discrete, gain the advantages offered by both, providing a link between macro and microscale descriptions [17]. In the literature, there is no a commonly agreed definition for the class of hybrid models. In [1], authors investigate the effects of individual-based cell interactions in the different stages of tumor growth, presenting a model defined by a system of coupled non-linear partial differential equations.

As already mentioned, modelling certain cell processes with a pure continuum approach would be challenging. The authors classify their model as *hybrid*, since they discretize the reaction-diffusion model using finite-difference methods in order to focus on the individual cell level. In particular, they consider a random-walk model, assuming that the coefficients arising from the discretization process correspond to the probabilities for cells to move or remaining at their current location. Authors refer to this kind of procedure as *hybrid discrete-continuum*. This technique has been used in other works related to cancer growth [18], and also in the context of angiogenesis [19] and retinal vasculature development [20]. Other hybrid models are the so called *interfacing hybrid models* [21], in which the spatial domain is partitioned in few time-dependent domains, and the cells are described by discrete variable in a portion of the spatial domain, or by continuous variables in the rest of domain (see for instance [22] or [23] about cancer cells invasion).

A further generalization has been developed on the basis of measure-theoretic approaches ([24], [25]). The basic idea is that a same constituent can be described, in the whole domain, in a unique framework in which microscopic and macroscopic points of view coexist ([6], [26]). These models, first applied to crowd dynamics, allow a balance between micro- and macro-contribution through a combination of singular and absolutely continuous measures with respect to the Lebesgue measure of the space. In this way, one can suitably focalize the granularity of individuals (interaction with a single) and the macroscopic description of crowd flow (interaction with the mass). A first attempt to apply this framework in a biological context, about the cell behavior, can be found in [27].

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In our work, we focus on a hybrid differential system which generalizes the mathematical setting of the models presented in [2] and [3]. In those papers, cells are treated as a set of localized agents whereas chemical concentrations are described through the spatial distribution of their concentrations. In the above mentioned works, authors focus on the construction of the models and provide some numerical simulations able to reproduce the related biological phenomena. To this end, only a numerical approximation of the solution of the system occurring in the mathematical model has been proposed.

From a mathematical point of view, these models share some common features. The dynamic of the cells is driven by second-order ODEs, whereas the evolution in time of the chemical signals is described by reaction-diffusion equations. The dynamics of agents are not only influenced by mechanical interactions, but also by *chemotaxis*, which leads the agents from regions with low concentration to the ones at an higher level. The main contribution in our model consists in the coupling through the gradient of the concentration of the chemoattractant. In particular, the solution to the chemotaxis equation (see Equation (2) below) is a *path-dependent* function: for every time t, in order to evaluate the solution at that time, it is necessary to know the entire trajectory of the agents up to time t. In the investigated system, the coupling is achieved through the gradient of the solution of the parabolic equation, evaluated, at each time step, in the position of each agent. We stress the realism of this assumption, since it is known that some living organisms [28] are able to construct local approximations to gradients. This kind of dependence requires further regularity for the chemotaxis solution in order to achieve a uniqueness result.

To the best of our knowledge, these features differentiate our formulation from those available in the current state of the art concerning analytical results for coupled systems. We point out that, while applications of hybrid models are increasingly frequent, the literature concerning well-posedness (existence and uniqueness) of the solution of the resulting differential systems is still lacking. In [29], authors investigate an initial value problem for systems characterized by the coupling of conservation laws and ordinary differential equations through boundary conditions. From an analytic point of view, the particular structure of model in [29] has been investigated in some previous works ([30], [31], [32]).

Hybrid models: the general structure

We consider a group of n interacting agents in $\mathbb{R}^N$. The dynamic of each agent $i = 1, ..., n$ is expressed by the following second-order differential equation:

$$\ddot{\mathbf{x}}_i(t) = F_i\left(t, \mathbf{X}(t), \dot{\mathbf{X}}(t), \nabla f\left(\mathbf{x}_i(t), t; \mathbf{X}\right)\right), \tag{1}$$

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where $\mathbf{X}(t) = [\mathbf{x}_1(t), ..., \mathbf{x}_n(t)], \dot{\mathbf{X}}(t) = [\dot{\mathbf{x}}_1(t), ..., \dot{\mathbf{x}}_n(t)] \in \mathbb{R}^{N\times n}$ are the collections of position and velocity of each agents, at each time t, and f models a signal influencing the dynamics (e.g. the concentration of a chemoattractant as in [3], [2]). The notation $f = f(x, t; \mathbf{X})$ aims to emphasize the dependence on the whole trajectory of all agents till time t, that will be clear in the next section (see equation (3.5)) .
In the proposed model, f is the solution of the Cauchy problem

$$\begin{cases} Lf(x,t;\mathbf{X}) &= g(x,\mathbf{X}(t)), \qquad (x,t) \in \mathbb{R}^N \times (0,T) \\ f(x,0;\mathbf{X}) &= \varphi(x) \qquad\qquad x \in \mathbb{R}^N, \end{cases} \tag{2}$$

where L is the following differential operator:

$$L = \sum_{i,j=1}^{N} a_{i,j}(x,t)\partial^2_{i,j} + \sum_{i=1}^{N} b_i(x,t)\partial_i + c(x,t) - \partial_t. \tag{3}$$

Moreover we investigate a different variation of (1), in which $\nabla f(\mathbf{x}_i(t), t; \mathbf{X})$ is replaced by the average over a ball centered in $\mathbf{x}_i(t)$ and having radius $\delta > 0$. The introduction of an average gradient in [2] and [3] aims at modeling the fact that, in biological system, a cell feels the presence of chemical signals not only in its center, but also in the region surrounding it.
For the case of a non-local concentration, we generalize the above structure of hybrid models (1)-(3), considering systems of the following coupled hybrid form

$$\begin{cases} \dot{\mathbf{x}}_i(t) = F_i\left(t, \mathbf{X}(t), \displaystyle\int_{\mathbb{R}^N} \nabla u\,(x,t)\, h(x, x_i(t))dx\right) \quad t \in (0,T)\,, \\ u_t = \operatorname{div}\left(\mathbf{a}(x,u,\nabla u)\right) + b(x,u) + g(x,\mathbf{X}(t)) \qquad \text{in } \mathbb{R}^N \times (0,T)\,, \\ x_i(0) = x_{i0} \in \mathbb{R}^N, \\ u(\cdot,0) = u_0 \in L^2(\mathbb{R}^N), \end{cases} \tag{4}$$

It is worth pointing out that the set of assumptions we will consider for problem (4) generalizes the previous for (1) -(3), and cover the particular structure of our inspiring models [2], [3].
Another crucial investigated aspect concerns the asymptotic behavior of the solution. In a more applied and numerical oriented paper [33], we presented a simplified model of system (2)-(3). Finally, in a more applied perspective, we present a novel hybrid model, inspired by collective motions of cells occurring in different biological phenomena, such as wound healing and morphogenesis processes. We consider an alignment and chemotaxis mechanism, whose evolution in time is modeled by a parabolic equation, with term of source

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and degradation. The basic structure is given by the following system:

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\beta}{n} \displaystyle\sum_{i=1}^{n} \phi\left(|\mathbf{x}_i - \mathbf{x}_j|\right)(\mathbf{v}_j - \mathbf{v}_i) + \gamma \nabla f(\mathbf{x}_i), \\ \partial_t f = D\Delta f + g_{\mathbf{X}} - \eta f, \\ f(\cdot, 0; \mathbf{X}) = 0. \end{cases} \tag{5}$$

where β, γ, δ, D, η are positive constants, and ϕ is the influence function, modeling alignment interactions among agents and depending on the relative distances.

Plan of Part I

In Chapter 1 we provide an overview of the literature concerning second order agent-based models. In particular, we performe numerical simulations of these models, based on finite difference schemes. Moreover, we present the first step of a current joint collaboration with Emiliano Cristiani (Istituto per le Applicazioni del Calcolo-CNR). The model we propose aims at reproducing flocking behaviors typically displayed by starlings, with a particular focus on turning phenomena. On the basis of comprehensive researches in the literature of the field, we developed a novel agent-based second-order model with delay. Numerical simulations show that the model is able to reproduce the investigated behaviour .
Chapters 2, 3, 4 are devoted to analytical results concerning wellposedeness of hybrid systems. With a step-by-step perspective, in Chapter 2 focus on the solution of (1)-(3) assuming, in particular, g and φ continuous functions. In Chapter 3 we extend the obtained results to the case of g with less regularity properties. This study is motivated by the fact that in our inspiring models ([2], [3]), the source term is modeled by a discontinuous function, able to differentiate regions from which the signal arises from the others.
In Chapter 4 we prove global existence and uniqueness results for the general form of hybrid systems in (4). In particular, we assume $\varphi \in L^2(\mathbb{R}^N)$ and $g \in Lip_{loc}(\mathbb{R}^{N\times n}; L^2(\mathbb{R}^N))$. In this Chapter we shall introduce a different technique, based on a preliminary study concerning well-posedness of pseudo-parabolic approximating problems, in order to prove existence and uniqueness of the solution.
In Chapter 5 we present the result of a joint collaboration with Roberto Natalini (Istituto per le Applicazioni del Calcolo-CNR) and other coauthors. The aim is to combine analytical and numerical approaches in order to study the asymptotic behavior of the solution of (5). From the analytic point of

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view we prove existence and uniqueness of the solution. Then, the asymptotic behavior of a linearised version of the system is investigated. We prove that the migrating aggregate exponentially converges to a state in which all the particles have a same position with zero velocity. Theoretical results are compared to numerical simulations performed on the full nonlinear system.

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Chapter 1

A microscopic model for collective motions of birds with turning

In this chapter [1] we present a microscopic model for collective behaviors specifically conceived for modeling flocks of birds and swarms. In particular, the aim is to model turning scenarios, due to sudden direction changes in the motion of the group as a whole. With respect to flocking, the mechanisms regulating turning events are rarely investigated by the present literature. We first present a brief overview of mathematical models related to collective motions, in order to highlights the basic interactions that need to be taken into account when dealing with flocks. Then, we focus on a novel microscopic model for flocking and turning phenomena. Numerical simulations show that the introduced model is able to reproduce the desired behavior.

1.1 Collective motions: modeling background

Collective motions of agents can be regarded as a multidisciplinary area of study, on the borderline of several scientific disciplines. The main feature of collective behaviour is that the individual unit's action is strongly dominated by the influence of other units, so that its motion turns out to be very different from how it would be if it was alone. Everyone could think at those fascinating sceneries typically displayed by a flock of birds, or a school of fish, but collective motions concern also bacteria colonies, amoebae, cells, insects, mammals and humans ([34], [35], [36], [37], [38]).

Many reasons of aggregations in a given system have been proposed. In the biological field, coordinated motion of cells results in making a biological

[1] This chapter is based on joint work with E. Cristiani, M. Papi

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process more efficient. In particular, a large variety of models of the literature focus on the process of cell invasions and biological processes related to tumor growth ([1], [11], [18], [39], [40]). In a different context, mathematical modeling of crowd dynamics gives advantages to better forecast the behavior of pedestrian flows ([6], [8], [26], [41], [42]). Regardless of the nature of the agents and the field of application, mathematical research on collective motions aims at modeling those phenomena in which individual units interact in simple or more complex ways, which turn out to be combination of simpler interactions.

In this section we present an overview of the literature, in order to highlight the basic interactions which need to be taken into account when modeling collective behaviors. Moreover, we present numerical simulations of the described models, which are performed using finite difference schemes for ordinary differential equations.

1.2 Alignment Models

Standard Vicsek Model

Perhaps the first widely-known flocking model was introduced in [43], in order to establish a quantitative interpretation of the behavior of flocks in the presence of perturbations. What we are going to present here is the so called *"Standard Vicsek Model" (SVM)*, which is discrete with respect to time. According to SVM, the single organisms move with a fixed absolute velocity v_0, and assume the average direction of others within a given distance R. Therefore equations for the position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ of particle i are

$$\begin{cases} \mathbf{v}_i(t+1) = v_0\mathbf{d}(t) + perturbation, \\ \mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1) \end{cases} \tag{1.1}$$

with $\mathbf{d}(t) := \dfrac{\langle \mathbf{v}_j(t)\rangle_R}{|\langle \mathbf{v}_j(t)\rangle_R|}$, having labeled with j the neighbors of i, in the sense of the flocking organisms belonging to a circle of radius R surrounding particle i.
The notation $\langle \mathbf{v}_j(t)\rangle_R$ denotes averaging of the velocities in that region, and by definition $\mathbf{d}(t)$ is a unit vector, pointing in the average direction of motion. We refer to the presence of noise with general *perturbation*, as it can be explained in various way. In SVM it means adding a random angle to the one corresponding to the average direction of motion in the neighborhood of i. In other words, the final direction of particle i is obtained after rotating the average direction of the neighbors with a random angle.
SVM is the starting point for the model we are now going to introduce, pointing out differences and similarities between them.

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Cucker-Smale Model

Originally presented for a flocks of birds, the model [10] can be extended to all those phenomena in which single units reach a sort of consesus: in case of birds, a common direction. For example, opinion formation dynamic can be regarded as a collective behavior problem involving an alignment process. This is the approach, for example, of [44]: the law describing the evolution in time of the opinion of each agent is similar to the law prescribed by Cucker and Smale for the velocities.
Let us consider a flock of n birds, labeled with $i = 1, ..., n$, moving in $\mathbb{R}^3$. At each instant of their motion, every bird adjusts its velocity by adding to it a weighted average of the differences between its velocity and those of the others in the flock.

$$\mathbf{v}_i(t+h) = \mathbf{v}_i(t) + h\sum_{j=1}^{n} a_{ij}(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \tag{1.2}$$

for every $t \in \mathbb{N}$ and $i = 1, ..., n$.

The weights a_{ij} quantify the ability of bird i to communicate with j, that is the way birds influence each other. A reasonable assumption could be that a_{ij} depends on the distance between i and j:

$$a_{ij} = \eta(|\mathbf{x}_i - \mathbf{x}_j|^2) \tag{1.3}$$

Here $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a non increasing function, called *influence function*. The prototype influence function chosen by Cucker-Smale is given by:

$$\begin{aligned} \eta : \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ y \quad &\longmapsto \frac{\alpha_1}{(\alpha_2 + y)^\sigma}, \end{aligned}$$

where σ, α_1, α_2 are fixed positive constants.

Letting h tend to zero, we obtain the equations of flocking: the dynamic of each agent $i = 1, ..., n$ at a certain time instant t in the interval of observation $(0, T]$ is given by

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \\ \\ \dot{\mathbf{v}}_i(t) = \displaystyle\sum_{j=1}^{n} a_{ij}(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \end{cases} \tag{1.4}$$

where $\mathbf{x}_i(t)$,$\mathbf{v}_i(t) \in \mathbb{R}^3$, are the position and velocity of i-th bird. We will refer to this system simply with CS.

There are two main differences between these two models. First of all,

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while in SVM interactions are restricted to a region surrounding a given particle, in CS every flocking organism influences all the others. Furthermore, CS does not consider perturbation, but only an alignment effect.

System (1.4) is the original formulation given by Cucker and Smale, but more often with "Cucker-Smale model" we refer to the following modification, which contain a normalization pre-factor $1/n$ [45]:

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \\ \\ \dot{\mathbf{v}}_i(t) = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \frac{\alpha_1}{\left(\alpha_2 + |\mathbf{x}_i(t) - \mathbf{x}_j(t)|^2\right)^{\sigma}} (\mathbf{v}_j(t) - \mathbf{v}_i(t)). \end{cases} \tag{1.5}$$

The Cucker-Smale paper [10] is widely known to have established an analytical exact result on the convergence to the same velocity in a group of interacting agents through an alignment effect.

Consider system (1.5) with initial condition $\mathbf{x}_i(0) = \mathbf{x}_{i0}$, $\mathbf{v}_i(0) = \mathbf{v}_{i0}$.

The main convergence result proved in [10] has been improved in [46], using an explicit Lyapunov functional approach. First, as in this last reference, we can give the definition of *time-asymptotic flocking* as follows:

Definition 1. *Let* $\mathbf{x}_{CM} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$ *and* $\mathbf{v}_{CM} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{v}_i$ *position and velocity of the centre of mass. System* (1.5) *has a time-asymptotic flocking if and only if* $(\mathbf{x}_i, \mathbf{v}_i)$, $i = 1, \dots, n$, *satisfy the two conditions:*

1. *the velocity fluctuations go to zero time-asymptotically (velocity alignment):*

$$\lim_{t \to +\infty} \sum_{i=1}^{n} |\mathbf{v}_i(t) - \mathbf{v}_{CM}(t)|^2 = 0; \tag{1.6}$$

2. *the position fluctuations are uniformly bounded in time t (forming a group):*

$$\sup_{0 \le t \le +\infty} \sum_{i=1}^{n} |\mathbf{x}_i(t) - \mathbf{x}_{CM}(t)|^2 < +\infty. \tag{1.7}$$

Notice that the square root of the quantities under the limit and supremum operations in (1.6)–(1.7) is proportional to the standard deviations of $\mathbf{v}_i(t)$ and $\mathbf{x}_i(t)$ around the centre of mass system. In [10] and [46] authors proved that for $\sigma \in [0, 1/2]$ a global *unconditional flocking* of system (1.5) occurs, as stated in Definition 1, regardless of initial configurations, whereas for $\sigma \in (1/2, +\infty)$ there is *conditional flocking*, that is only some parameters and initial data lead to a flocking state, but in general the dispersion of the

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flock may occur.

1.2.1 Numerical simulations

In this section we present numerical simulations of the model in (1.5), based on finite differences scheme in a two dimensional domain. In particular, varying the value of $\sigma \in [0, \infty)$, we verify the analytical results concerning unconditional and conditional flocking.

We consider the interval of observation $[0, T]$ divided into $M > 0$ subintervals: if Δt is the time step, we denote by t_k the $k-th$ temporal step, i.e. $t_k = k\Delta t$ for any $k = 0, ..., M$. Finally, with $\mathbf{x}_i^k$ and $\mathbf{v}_i^k$ we denote respectively the approximations of position and velocity of agent i at time t_k. Putting in implicit the term depending on the velocities, the scheme for system (1.5) reads:

$$\begin{cases} \dfrac{\mathbf{x}_i^{k+1} - \mathbf{x}_i^k}{\Delta t} = \mathbf{v}_i^k, \\ \dfrac{\mathbf{v}_i^{k+1} - \mathbf{v}_i^k}{\Delta t} = \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \frac{\alpha_1}{(\alpha_2 + |\mathbf{x}_i^k - \mathbf{x}_j^k|^2)^\sigma} (\mathbf{v}_j^{k+1} - \mathbf{v}_i^{k+1}). \end{cases} \tag{1.8}$$

The second equation in (1.8) can be rewritten as

$$\left(1 + \frac{\Delta t}{n} \sum_{j=1}^{n} \frac{\alpha_1}{(\alpha_2 + |\mathbf{x}_i^k - \mathbf{x}_j^k|^2)^\sigma}\right) \mathbf{v}_i^{k+1} - \frac{\Delta t}{n} \sum_{j=1}^{n} \frac{\alpha_1 \mathbf{v}_j^{k+1}}{(\alpha_2 + |\mathbf{x}_i^k - \mathbf{x}_j^k|^2)^\sigma} = \mathbf{v}_i^k. \tag{1.9}$$

Thus, for a flock of cardinality n, at each time step we solve a system of n equations to find the unknown velocities $\mathbf{V}^{k+1} := (\mathbf{v}_1^{k+1}, ..., \mathbf{v}_n^{k+1})$. We observe the simmetry property of the coefficient matrix, denoted with B, with entries:

$$B(i,j) = \begin{cases} 1 + \dfrac{\Delta t}{n} \displaystyle\sum_{l=1}^{n} \sum_{l \neq i} \frac{\alpha_1}{(\alpha_2 + |\mathbf{x}_i - \mathbf{x}_l|^2)^\sigma} & \text{if } i{=}j \ , \\ -\dfrac{\Delta t}{n} \dfrac{\alpha_1}{(\alpha_2 + |\mathbf{x}_i - \mathbf{x}_j|^2)^\sigma} & \text{if } i \neq j. \end{cases} \tag{1.10}$$

The proposed implicit method requires to solve a linear system at each time step, and gains from the fact there there is no need to impose restriction on the time step Δt in order to guarantee stability. This represent an advantage, since the time interval $[0, T]$ could be even very wide, virtually infinite. In that case, the possibility to choose the time step relatively large

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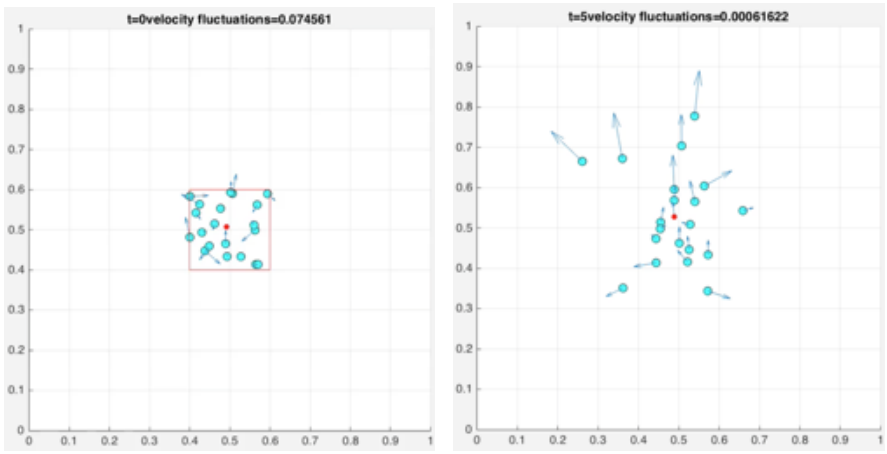
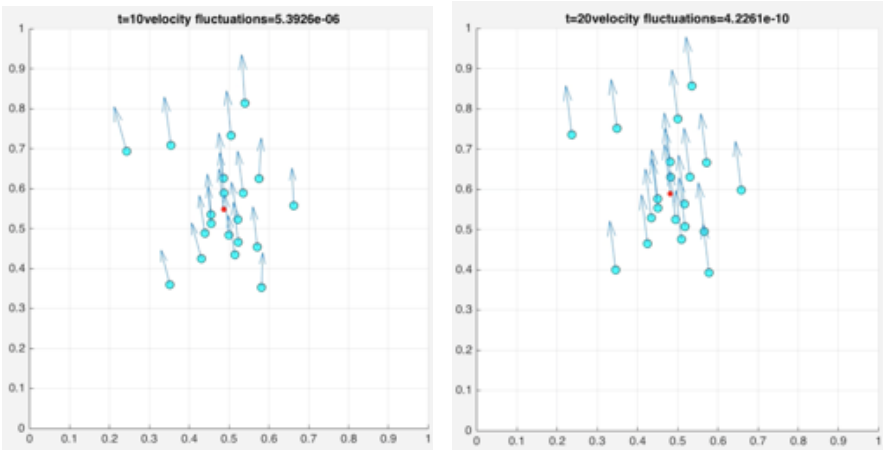
plays an essential role in order to reduce the number of iterations, the cost, and the accumulated errors.

Test 1

In this test we consider a group of $n = 20$ agents, initially located in $\mathbf{x}_0$, as in Figure 1.1(a).We set the parameters $\alpha_1 = 0.5$, $\alpha_2 = 1$, $\Delta t = 0.1$, $T = 20$, $\sigma = 0.4$. As $\sigma \in [0, \frac{1}{2}]$, we observe the time-asymptotic behavior, regardless of the initial configurations of positions and velocities. In particular, at each time step, we evaluate the velocity fluctuation around the centre of mass, and observe that it reduces as time increases.



(a) (b)



(c) (d)

Figure 1.1: Numerical simulation of Cucker-Smale model with σ in the range of unconditional flocking.

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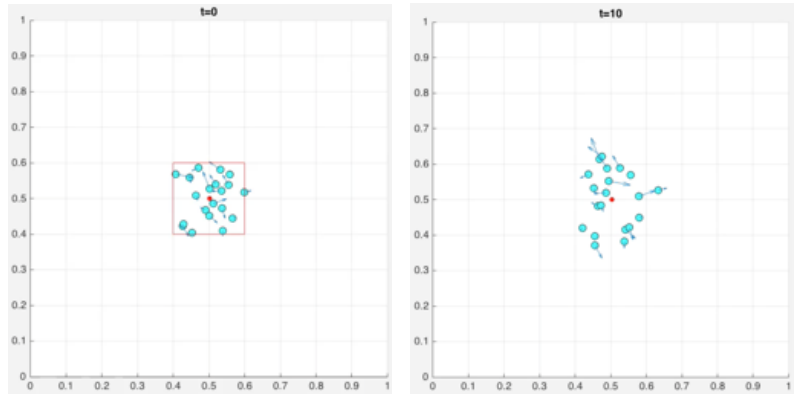
Test 2

In this test we consider a group of $n = 20$ agents, initially located in $\mathbf{x}_0$, as in Figure 1.2(a). We set the parameters as in Test 1, apart from σ and T, which are now $\sigma = 1$, $T = 50$. As $\sigma > \frac{1}{2}$, we remember that time-asymptotic behavior of the group is ensured only under certain conditions for the initial velocities and positions. In this simulation, parameters are chosen to simulate the dispersion of the group, as initial velocities do not satisfy the necessary condition for flocking.



(a) (b)

(c) (d)

Figure 1.2: Numerical simulation of Cucker-Smale model with σ in the range of conditional flocking.

1.3 Attraction-repulsion models

The model proposed by Cucker and Smale in [10] takes into account only an alignment mechanism of the individuals by adjusting their relative

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velocities with all the others, in order to move in the same direction, and the strength of this process depends on the mutual distance. Other important models in the literature investigate systems in which units do not follow any kind of explicit alignment rule. *Attraction-repulsion models* are useful to describe the social tendency of units to produce grouping (attraction/cohesion), avoiding collision with other members and crowding (repulsion/separation). Among the great variety of attraction-repulsion models, the model in [47] is well-known to reproduce the so called *milling* formation:

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \\ \dot{\mathbf{v}}_i(t) = (\alpha - \beta|\mathbf{v}_i(t)|^2)\mathbf{v}_i(t) - \dfrac{1}{n}\displaystyle\sum_{j\neq i} \nabla_{\mathbf{x}_i(t)} U(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|) \end{cases} \tag{1.11}$$

for $i = 1, ..., n$, where $\alpha, \beta \geq 0$, and $U : \mathbb{R}^d \to \mathbb{R}$ is a given potential modelling short-range repulsion and long-range attraction. In [47] the form of function U is given by

$$U(x) = k(|x|), \qquad k(r) = -C_A e^{-\frac{r}{l_A}} + C_R e^{-\frac{r}{l_R}},$$

where $l_A, l_R > 0$ represent the attractive and repulsive potential ranges, and C_A, C_R their respective amplitudes. We can distinguish two components:

- the self-propelled component, due to the presence of a self-propulsive force, $\alpha\mathbf{v}_i$, and a friction force, which also depend on the Euclidean norm of $\mathbf{v}_i$.
- the component relating to the potential. For agent i, this term is sum of an attractive component and a repulsive one, characterized by the intensities C_A, C_R, which dacay exponetially as a function of the distance between i and every other j in the group. The positive constant l_A, l_R denote the range within which attraction and repulsion forces are perceived by the agents .

Consider the quantities $C = C_r/C_a$, and $l = l_r/l_a$. If

$$C > 1 \quad \text{and} \quad l < 1, \tag{1.12}$$

the repulsion prevails on the attractive component, and its range l_r is shorter than l_a. Thus we expect that particles relativity far attract each others, keeping the cohesion of the group, but also avoid collision in case they tend to become too close. In other words, particles tend to form mills, vortices with velocity of center of mass equal to zero, similar to concentric rings. In that configuration, all agents rotate in the same way, and their velocities tend to the asymptotic value $|\mathbf{v}_i|^2 = \frac{\alpha}{\beta}$.

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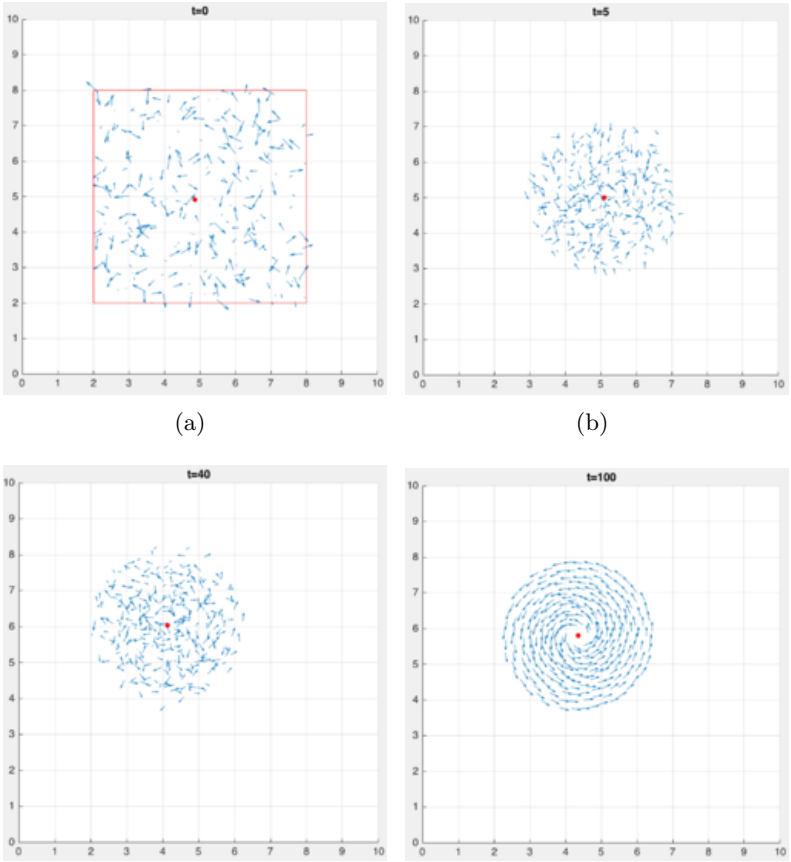
In our simulation we set $C_r = 1$, $C_a = 0.5$, $l_r = 0.5$, $l_a = 3$, thus condition (1.12) is satisfied, and we follow the dynamic of (1.11) in the time interval $[0, 100]$, using $\Delta t = 0.05$. Figure (1.3) shows numerical results obtained for $n = 300$ agents. Numerical simulations performed for a smaller group of agents (data not shown here), highlight the fact that the vortex size decreases as a function of particle number.



(a) (b)

(c) (d)

Figure 1.3: Numerical simulation of D'Orsogna model for a group of $n = 300$ agents.

1.4 Zone models

A model combining alignment, attraction and repulsion effects is called *three zone model*. One of the first simulations was carried in [48]. Based on the observational and empirical investigation of fish behavior in schooling phenomenon, it distinguishes three fundamental regions of influence.

The *repulsion zone* is characterized by the tendency of moving apart from another individual in near proximity, in order to avoid physical collision or being of mutual obstacle. An agent moving out from this region finds itself in the *alignment or orientation zone*, where it tries to identify the possible direction of the group and to align with it. Finally, when the individual is too far apart from the group, it will try to reach the others which are located in the *attraction zone*.
Many behavioral models stand on Aoki's model, which is consider to be a proper base for further considerations. In [49], authors distinguish between predators and prey, and focus on predators' behavior. Using a three-zone model, the region surrounding predator is divided in *prey*, *chase* and *searching* area.

In [34] authors propose a model based on the three interaction zones, modelled as spheres, centered on the individual, with radius R_r,R_o,R_a, $0 \leq R_r \leq R_o \leq R_a$. This model also takes into account the fact that the perception zone of an individual is limited, introducing the concept of *cone of vision*. For example, the visual field of a bird or fish, is defined by an angle α: $\alpha = 360°$ correspond to an individual which can respond to others in any direction within the behavioral zones. For all the others values of α, the cone with interior angle $(360 - \alpha)$ defines a blind area within which neighbors are undetectable. Varying the width of the perception zones and some parameters of the model, the simulations exhibit various basic type of collective motion: swarm, with little or no parallel orientation; milling, with individuals that rotate around an empty core; dynamic parallel group, where the individual are aligned but can move throughout the group, causing fluctuation in density and group form, and highly parallel group (Figure 1.4). In [50], the concept of field of vision is enhanced by the introduction of sensitivity zones. Clearly, the behavioral rules prescribed in the model do not concern the blind region. Within the cone of vision, each agent has its own zones of interaction, different in size and shape. Authors show that simply changing, for example, the sensitivity angles for attraction and repulsion forces, we obtain cluster, line and V-like formations.

1.4.1 Metric and topological distance

All models seen in our brief review adopt a metric concept of distance, assuming that the motion of every unit is influenced by those mates within a fixed range of interaction. This hypothesis seems to be natural, but it

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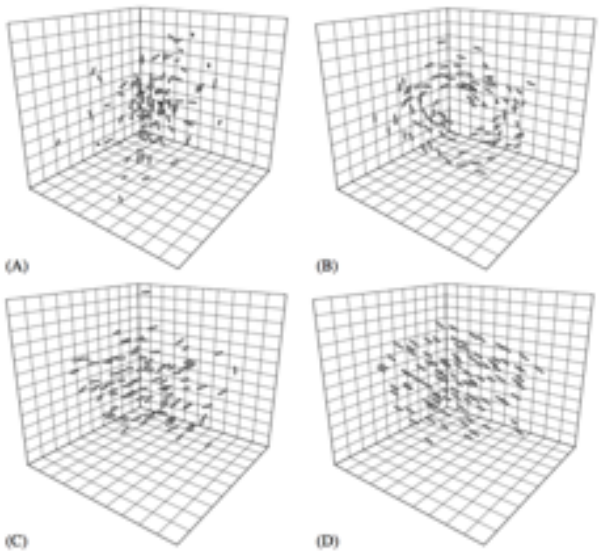

Figure 1.4: The collective behaviour exhibited by the model in [34] : (A) swarm with little orientation; (B) milling formations, in which individuals rotate around an empty core; (C) dynamic parallel group; (D) highly parallel group (source [34]).

presents several drawbacks. For example, how to explain those typical scenarios in which flocks abruptly contract, expand and even split, before reforming as a whole? In a metric context, one would expect cohesion to be lost when the mutual distances become larger than the range of interaction, thus the density changes, typical of animal aggregations, is not well reproduced. To cope with this problem, an alternative definition of distance is been introduced, referred as "topological distance": the relevant quantity is how many agents separate two members of the flock, and not how close or how far they are. It follows that each individual interacts with a fixed number of mates, and that the aggregation can be dense or sparse, change shape and fluctuate, but interactions do not vanish and cohesion under perturbation is kept.
Recent studies conducted for flock of birds [51] support the topological approach. Analyzing several independent flocking events recorded between December 2005 and February 2006 in Rome, they observed that each bird changes its position influenced by a range of six, seven neighbors. Moreover, they compared numerical results of Vicsek Model simulations, with the ones obtained replacing the metric interaction with the topological seen above. Using the strictly metric interaction, a flock exposed to a perturbation reproducing the attack of a predator tends to split into many small components, which are of course more vulnerable, while the topological modification better preserve the cohesion of the group.

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In [50] authors show that, choosing a suitably restricted angle of vision, together with a topological interaction, it is possible to reproduce the main structure of animal group (e.g. clusters, lines and V-like structures). The model is an agent-based zone model, including only the superposition of attraction and repulsion effect, without alignment. In particular, the choice of a linear attraction and a repulsion term proportional to the inverse of the distance between agents, are similar to those described in the previous subsection. A development of the three zone model, combining multiscale and both topological and metric approaches, can be found in [52]. It describes the dynamics of a large group of agents influenced by a small number of external agents. This scenario is typically displayed in nature, when the presence of external agents such as predators or leaders modify the internal dynamics of a group (flock of birds, school of fishes, herd of sheep). The proposed model, endowed of both metric and topological interactions, considers different level of descriptions, from the microscopic to a kinetic one, obtained through a mean-field limit, and finally to a macroscopic system through a hydrodynamic limit.

In the following section we present a novel model, aiming at reproducing turning phenomena in flock of birds, which includes the three basic interactions above seen, and adopts a topological concept of distance. Our approach is neither exhaustive nor definitive, since it represents the starting point of an ongoing work. However, numerical results obtained in the first step of our researches have been able to reproduce the investigated phenomena.

1.5 A novel mathematical model for turning

Before presenting our model, we clarify the reasons of the assumptions that will be made, mainly justified by the nature of starlings. We consider a flock of n units. Since males and females are similar in size and weight, we assume that the birds have all the same mass. Each member of the flock is represented by a dimensionless point having unit mass, corresponding to its position $\mathbf{x}_i(t) \in \mathbb{R}^N$. Hence, in our second-order model, the terminology force and acceleration have to be considered as synonyms.

The dynamic of a generic agent i is described by a second-order particle model with time delay $\delta > 0$, in the form

$$\ddot{\mathbf{x}}_i(t) = A_i\left(\mathbf{X}(t-\delta), \mathbf{V}(t-\delta)\right), \tag{1.13}$$

where $\mathbf{X}(\cdot), \mathbf{V}(\cdot)$ denote the collection of positions and velocities of all the agents. At each time $t \geq \delta$, the algorithm updates the values of the acceleration of the agents, which is a function of positions and velocities. Integrating the above equation we get the general expressions for velocity and position,

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namely

$$\dot{\mathbf{x}}_i(t+dt) = \dot{\mathbf{x}}_i(t) + \ddot{\mathbf{x}}_i(t)dt, \tag{1.14}$$

$$\mathbf{x}_i(t+dt) = \mathbf{x}_i(t) + \dot{\mathbf{x}}_i(t+dt)dt. \tag{1.15}$$

In the general form, the acceleration is thus a function of the velocities and positions of all birds at a previous instant of the motion. The introduction of a delay parameter δ, modeling the natural delay of living systems in the response to external stimuli, allow to control the speed at which information spreads through the flock.
In order to model turning scenarios, we define *leader* an agent that during the motion randomly decides to change its direction, and *follower* any other agent. Our main assumption, which particularly fits reality, is that each member of the group is potentially a leader. In accordance with the biological and mathematical literature, we reject the idea of a hierarchical structure within the flock. Moreover, the reasons why a bird suddenly decides to move in a different direction with respect to the rest of its group are not clear. On the contrary, a large number of studies agree that a leader-bird trigger a new phase of consensus within the group, moving in a different direction, for a short period of time [34],[51],[53].
In order to reproduce this feature, we introduce the term *persistency* to denote the period of time, or number of time steps in numerical simulations, in which a bird is a leader and moves in a fixed direction. Behavioral studies of the literature [54] state that, when a bird stops being a leader, it cannot be immediately one again. We will refer to *negative persistency* in order to denote the period of time in which a bird cannot become a leader. Clearly, assuming a null negative persistency, we are also able to consider the case in which a leader bird stop moving in a direction and immediately become again a leader, eventually choosing another direction.

The function $A_i = A_i(\mathbf{X}(t), \mathbf{V}(t))$ includes different types of interactions: some are common to every agent of the flock, others are characteristic of leaders or followers.

According to the empirical value in [55], a bird flies at a characteristic speed between 10 and 12 km/h. We model this aspect introducing a force that limits the velocity of each agent to be close to the characteristic one. Namely, we define

$$A_i^{cha}(t) = C^{cha}\left(S^{cha} - |\mathbf{v}_i(t)|\right)\mathbf{v}_i(t), \tag{1.16}$$

where S^{cha} is the characteristic speed, C^{cha} a positive constant.

We consider only topological interactions, meaning that the dynamics of each agent i are influenced by a fixed number M of mates, which are the nearest ones [55]. In the following $N(i, M)$ denotes the set of the $M-nearest$ neighbors of agent i. In greater details, we model the attitude of each agent

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to be close to the rest of the group avoiding collisions with attraction and repulsion term, respectively. We model these two kinds of interactions with the following expressions:

$$A_i^{att}(t) = C^{att} \sum_{j \in N(i,M)} e^{|\mathbf{x}_i(t) - \mathbf{x}_j(t)|} \frac{\mathbf{x}_j(t) - \mathbf{x}_i(t)}{|\mathbf{x}_i(t) - \mathbf{x}_j(t)|}, \tag{1.17}$$

$$A_i^{rep}(t) = -C^{rep} \sum_{j \in N(i,M)} e^{-|\mathbf{x}_i(t) - \mathbf{x}_j(t)|} \frac{\mathbf{x}_j(t) - \mathbf{x}_i(t)}{|\mathbf{x}_i(t) - \mathbf{x}_j(t)|}, \tag{1.18}$$

with C^{att}, C^{rep} positive fixed constants. We observe that attraction-repulsion contributions are function of the distances among the agents. In particular, attraction force grows with the distance, whereas the repulsion one decreases. In fact, when two agents are close, the attraction is low and collisions have to be avoided. On the contrary, agents do not need to repel themselves when they are at a long distance, and cohesion is kept due to a high level of attraction. In order to follow the flock direction, the model includes a topological alignment term: each agent tends to align to the M nearest agents, always avoiding collisions with them and keeping close enough. We choose the Cucker-Smale model form, defining

$$A_i^{ali}(t) = \frac{C^{ali}}{n} \sum_{j \in N(i,M)} (\mathbf{v}_j(t) - \mathbf{v}_i(t)). \tag{1.19}$$

In order to model the turning phase of a flock, we define the turning contribution. In our model, a bird i is picked randomly within the flock, and a random turning velocity $\mathbf{v}_i^{tur}$ is assigned to it. The associated contribution is expressed as

$$A_i^{tur}(t) = C^{tur} \left(\mathbf{v}_i^{tur} - \mathbf{v}_i(t) \right), \tag{1.20}$$

with $C^{tur} > 0$.

The consensus is reached if the leader is followed by its neighbors, heading them in the new chosen direction. Birds finding in the proximity of the border of the flock are more likely to success [56], [57]. In fact, leaders in denser areas of the flock would have to avoid many other members of the group to change direction. On the contrary, birds not too far from the border can easily find their way out.

We can summarize the model with the following system. Firstly, we introduce the variable

$$s_i(t) = \begin{cases} 1 & \text{if } i \text{ is a leader at time } t, \\ 0 & \text{if } i \text{ is not a leader at time } t. \end{cases} \tag{1.21}$$

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For any time t, the dynamic of each agent $i = 1, ..., n$ is modeled by the following equation

$$\begin{aligned} \ddot{\mathbf{x}}_i(t) &= A_i^{cha}(t-\delta) + A_i^{att}(t-\delta) + A_i^{rep}(t-\delta) \\ &+ s_i(t-\delta)A_i^{tur}(t) + (1 - s_i(t-\delta))A_i^{ali}(t-\delta), \end{aligned} \tag{1.22}$$

where $\delta > 0$ denotes the delay parameter.

1.6 Numerical simulation

In this section we present some numerical simulations of the model in (1.22), based on finite difference schemes. The aim is to validate, from a qualitative perspective, the model introduced. To this end, the values used for the parameters have to been consider dimensionless. In an ongoing work, concerning the three-dimensional domain, the availability of data arising from empirical observations, will allow us to estimate their values.

Figure 1.5 shows four different time steps of a preliminary test, which neglects the characteristic velocity ($S^{cha} = 0$, $C^{cha} = 0$) and the delay ($\delta = 0$) . We consider a flock of 200 agent, randomly displaced in a circle of radius 0.1. We assume that each bird finding on the border at the initial time, becomes a leader with infinity persistence, and moves with a prescribed horizontal direction. We observe that the inner part of the flock decide to move on the left or on the right depending on the position of each agent with respect to the nearest leaders.

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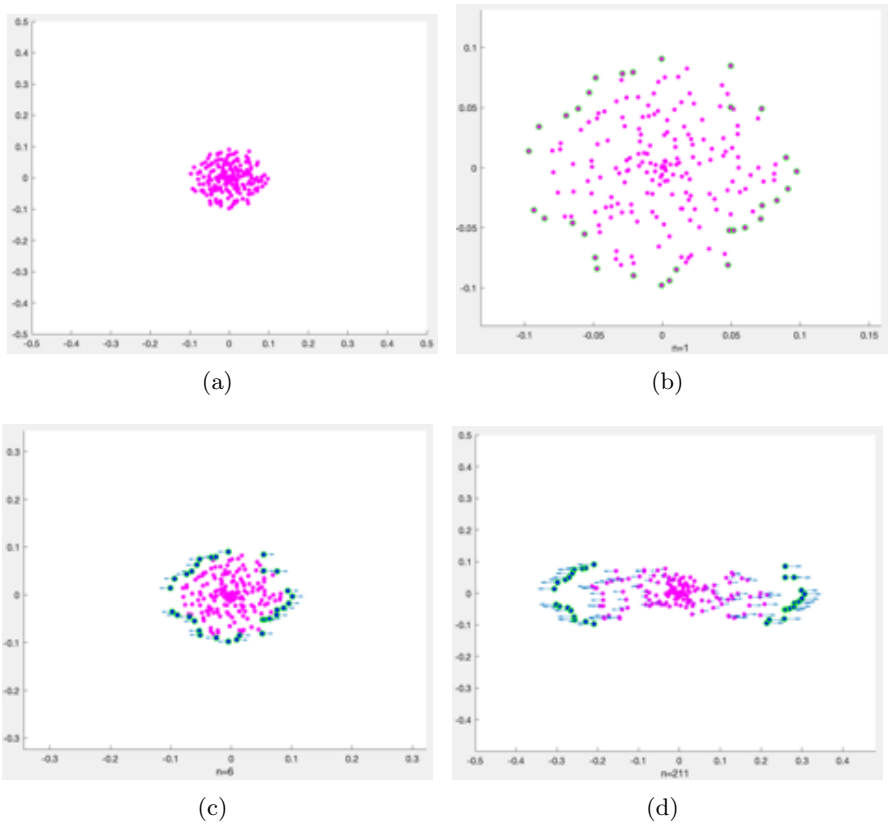

(a) (b)

(c) (d)

Figure 1.5: Numerical simulation of (1.22) for a flock of 200 agents (pink dots). We impose leaders (blu dots) only at the border, with infinite persistence and fixed direction.

We now focus on the model (1.22), without imposing additional conditions on leaders birds or on the directions. Clearly, the videos obtained merging the frames of the different time steps have a greater visual impression. For the next numerical tests we present some significant frames, which supports our conclusions.

We discretize (1.22) with classic Explicit Euler finite difference scheme. We consider the interval of observation $[0, T]$ divided into $M > 0$ subintervals: if Δt is the time step, we denote by t_k the $k - th$ temporal step, i.e. $t_k = k\Delta t$ for any $k = 0, ..., M$. Finally, $\mathbf{x}_i^k$ and $\mathbf{v}_i^k$ denote respectively the approximations of position and velocity of agent i at time t_k, and $A_i^{k-\delta}$ the sum of the different contributions to the acceleration above mentioned,

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evaluated at time $t_k - \delta$. The scheme reads:

$$\begin{cases} \dfrac{\mathbf{x}_i^{k+1} - \mathbf{x}_i^k}{\Delta t} = \mathbf{v}_i^k, \\ \dfrac{\mathbf{v}_i^{k+1} - \mathbf{v}_i^k}{\Delta t} = A_i^{k-\delta}. \end{cases}$$

In order to implement the leadership status, we provide each agent with a counter, initialized with null value. At each time step, the counter is decremented. When its value reaches the negative persistence one, the agent can become a leader. A random variable in $(0, 1]$ is extracted: if its value is greater than a leadership threshold parameter (in our simulations equal to 0.002), the agent remains in the follower status, otherwise it becomes a leader. The introduction of the leadership threshold allows to increase or decrease the number of leader within the flock.

In Figure 1.6(a) we consider a flock of 500 agents, randomly displaced in the square domain $[0, 6] \times [0, 6]$. Leader birds are marked as blue dots, whereas followers are red. According to the model, each bird i can randomly become leader. In this case, our numerical implementation provides for the assignment of a random direction $\mathbf{v}_i^{tur}$.

The simulation shows the deformation of the flock in the time interval $[0, 1000]$. We observe that cohesion within the flock is kept: the leaders model the shape of the flock, but their influence is not strong enough to split the group.

On the contrary, the scenario envisaged in Figure 1.7 shows an original flock of 200 agents, randomly displaced in the square domain $[0, 10] \times [0, 10]$, dividing into two/three other groups. This comes as a result of using a greater value for the parameter regulating turning forces, characteristic of leaders. The simulation also outlines a typical behavior displayed by starlings: during the motion, new leader birds in a group, moving in the direction of another group, can merge again the two part. This is what happens in the couple of Figure 1.7(c)-(d), and Figure 1.7(e)-(f).

In particular, during the motion, new leader birds in a group moving in the direction of another group, can merge again the two part.

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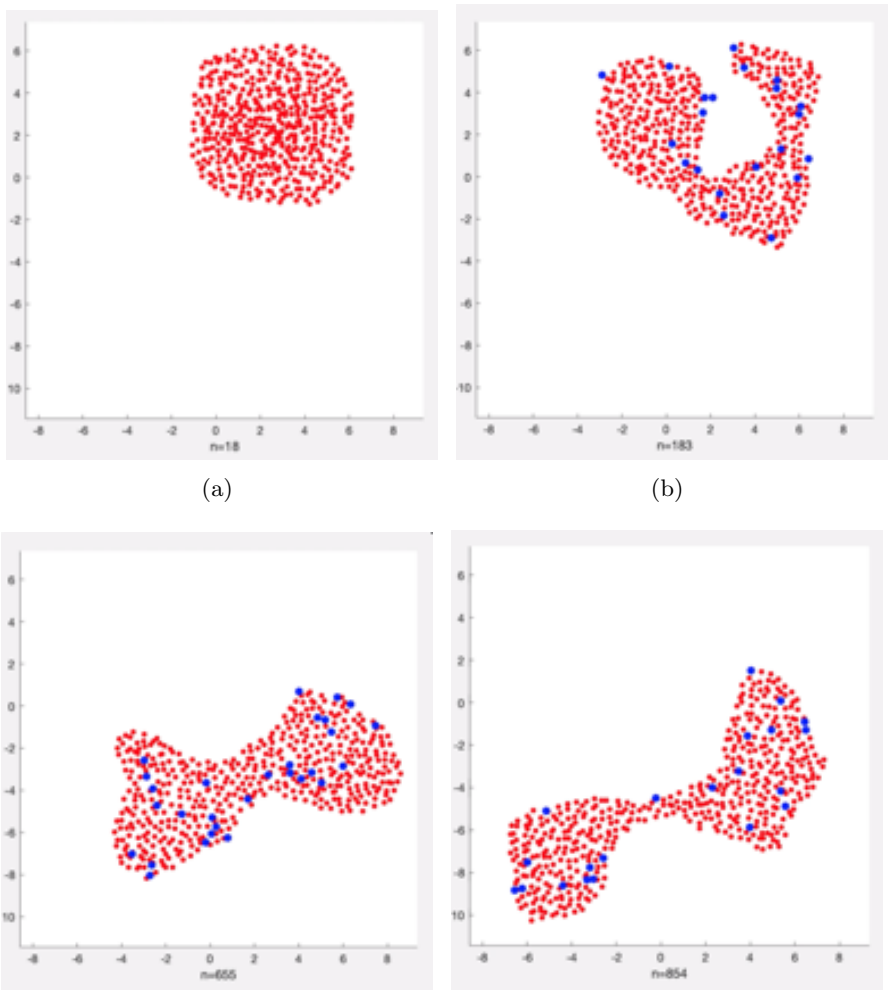

(a) (b)

(c) (d)

Figure 1.6: Numerical simulation of (1.22) for a flock of $n = 500$ agents. We fix the following dimensionless values for the parameters: $T = 900$, $\Delta t = 10^{-3}$, $C^{att} = 0.5$, $C^{rep} = 2$, $C^{ali} = 3$, $C^{turn} = 1$.

1.7 Discussion and future perspective

In this chapter we propose an overview of the literature concerning collective motions of agents, in order to highlights the basic interactions occurring in collective dynamics. Moreover, we present a first step of our studies, with application to flocking of birds. The model aims at reproducing those particular scenarios displayed by flock of birds, named as turning phenomena, in which a flock moving as a whole abruptly changes its flight direction, following the one chosen by one or few agents. Taking into account the literature concerning behavioral aspects of flocks, the model relies on three

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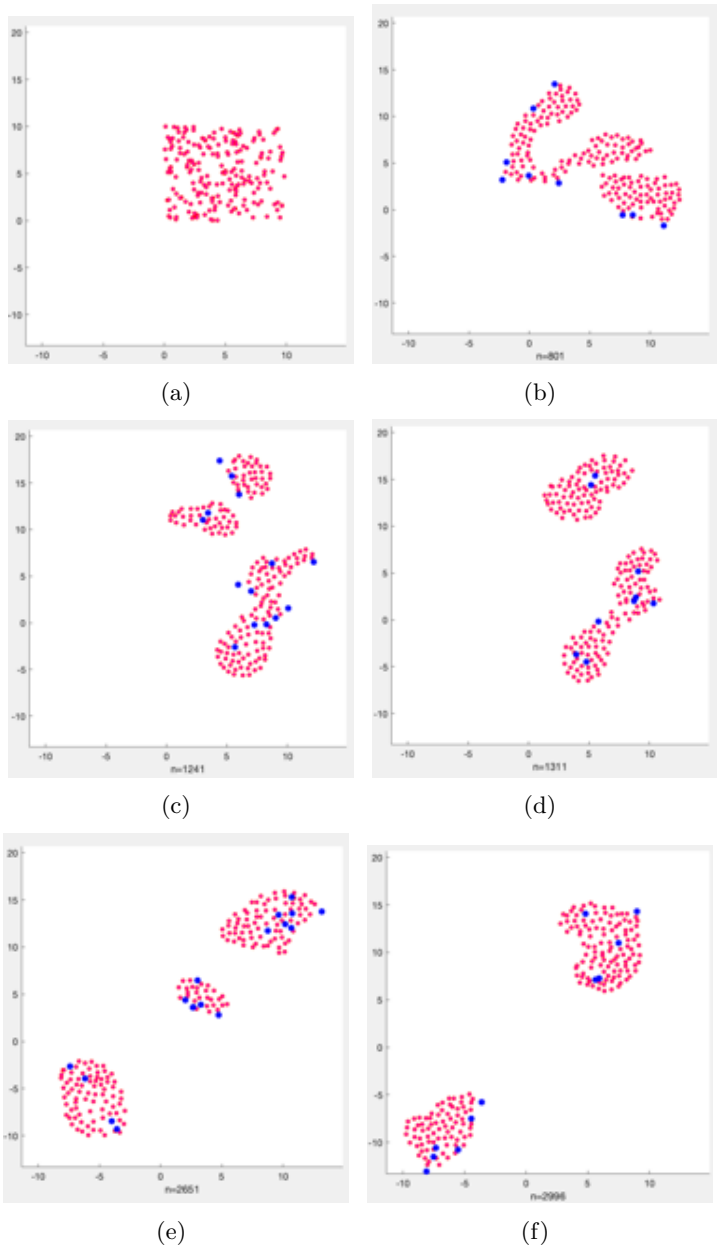

Figure 1.7: Numerical simulation of (1.22) for a flock of $n = 200$ agents. We fix the following dimensionless values for the parameters: $T = 3000$, $\Delta t = 10^{-3}$, $C^{att} = 0.5$, $C^{rep} = 2$, $C^{ali} = 1.5$, $C^{turn} = 3.5$.

main assumptions. Firstly, we replace the classic concept of metric distance with the topological one [51]: the relevant quantity is how many agents separate two members of the flock, and not how close or how far they are. It follows that each individual interacts with a fixed number of mates. The

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second assumption concerns the fact that there is not a hierarchical structure in a flock, and each member is potentially a *leader*, meaning that it can decide to suddenly change direction and leave the consensus state. As in the real phenomena, in our numerical simulation we observe that if a leader bird find itself in crowded region of the flock, it is not able to trigger the turning phase. On the contrary, birds finding in the proximity of the border when become leaders, have better chances to success. Finally, the model accounts for the natural delay in the response to external stimuli, thanks to the presence of a delay parameter.
The first stage of our study is thus an agent-based model. We are working in order to create a multiscale model, as stated in the Introduction. Moreover, the different parameters of the model will be estimated in the three-dimensional version, using data coming from empirical observations. Future perspective also involve the possibility to model the fact that the turning phase is not triggered by agents within the flock, but by an exogenous signal, that spreads through the entire group and influences the dynamics. The class of coupled *hybrid* models, representing the core of Part I of the thesis, particularly fits to this last scenario.

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Chapter 2

Existence and uniqueness of solutions for coupled hybrid systems

In this chapter we propose well-posedness results for the solution of hybrid systems (1)-(3) under specific assumptions. In particular, we investigate the case in which the source term and initial data of the parabolic equation are modeled by continuous functions. Using a fixed-point argumentation, we first present an existence and uniqueness result of the solution, locally in time. By a principle of continuation of the solution, the result is then extended to a global one.

2.1 Problem statement

We consider the following system of differential equations

$$\begin{cases} \dot{\mathbf{x}}_i(t) &= \mathbf{v}_i(t), & 0 < t < T, \\ \dot{\mathbf{v}}_i(t) &= F_i\left(t, \mathbf{X}(t), \mathbf{V}(t), \nabla f\left(\mathbf{x}_i(t), t; \mathbf{X}\right)\right), \end{cases} \tag{2.1}$$

with initial data $\mathbf{x}_i(0) = \mathbf{x_0}_i, \mathbf{v}_i(0) = \mathbf{v_0}_i \ \in \mathbb{R}^N$, $i = 1, ..., n$. Here

$$F_i : [0,T] \times \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N,$$

and $f = f(x, t; \mathbf{X})$ is the solution to the Cauchy Problem in $\Omega = \mathbb{R}^N \times [0, T]$

$$\begin{cases} Lf(x,t;\mathbf{X}) &= g(x, \mathbf{X}(t)), & (x,t) \in \mathbb{R}^N \times (0,T), \\ f(x,0;\mathbf{X}) &= \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.2}$$

with

$$\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad g : \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$$

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continuous functions, and L the following differential parabolic-type operator:

$$L = \sum_{i,j=1}^{N} a_{i,j}(x,t)\partial^2_{i,j} + \sum_{i=1}^{N} b_i(x,t)\partial_i + c(x,t) - \partial_t. \tag{2.3}$$

We denote $\mathbf{X}(t) := [\mathbf{x}_1(t), ..., \mathbf{x}_n(t)]$, $\mathbf{V}(t) := [\mathbf{v}_1(t), ..., \mathbf{v}_n(t)] \in \mathbb{R}^{N\times n}$, and $F := [F_1, ..., F_n]$. In particular, we denote $\mathbf{X}(0) = \mathbf{X}_0$ and $\mathbf{V}(0) = \mathbf{V}_0$.
When referring to vectors in $\mathbb{R}^N$, we use the bold notation to distinguish component i of $\mathbf{X}$ and $\mathbf{V}$, namely $\mathbf{x}_i$ and $\mathbf{v}_i$, from a general vector of the $N-$dimensional space. The complete list of notation introduced can be found in the Glossary chapter. We use the notation $f = f(x,t;\mathbf{X})$ in order to highlight the fact that, for every $t \in [0,T]$, f depends on the previous states $\mathbf{X}(s)$, with $s \in [0,t]$.
Moreover we denote $g(x,\mathbf{X}(t))$ with $g_{\mathbf{X}}(x,t)$.

We shall consider the following standing assumptions:

H1) For every $i = 1, ..., n$, F_i is a continuous function, satisfying the following condition: $\exists L_F \geq 0$ such that $\forall K \subset \mathbb{R}^{N\times n}$ compact, $\exists L_F^K \geq 0$ such that

$$\begin{aligned}\left|F_i(t,\mathbf{X},\mathbf{V},\mathbf{w}) - F_i(t,\widehat{\mathbf{X}},\widehat{\mathbf{V}},\widehat{\mathbf{w}})\right| \leq \; & L_F^K\left(\left|\mathbf{X}-\widehat{\mathbf{X}}\right| + \left|\mathbf{V}-\widehat{\mathbf{V}}\right|\right) \\ & + L_F\left|\mathbf{w}-\widehat{\mathbf{w}}\right|,\end{aligned} \tag{2.4}$$

for any $\mathbf{X}, \widehat{\mathbf{X}}, \mathbf{V}, \widehat{\mathbf{V}} \in K$, $\mathbf{w}, \widehat{\mathbf{w}} \in \mathbb{R}^N$, $t \in [0,T]$.

H2) The coefficients $a_{i,j}$, b_i, c, are bounded Hölder continuous function in Ω, with coefficient $\alpha \in (0,1)$ with respect to x and $\alpha/2$ with respect to t.

H3) L is uniformly parabolic in Ω, meaning that there exist $\mu_0, \mu_1 > 0$ such that, for every $\xi \in \mathbb{R}^N$ it holds

$$\mu_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \mu_1|\xi|^2 \qquad \forall (x,t) \in \Omega. \tag{2.5}$$

Moreover, there exists a constant $0 \leq C < \lambda_0/4T$, with $\lambda_0 \leq \mu_0/\mu_1^2$ (See [58] and also Appendix for more details) such that:

H4)

$$|\varphi(x) - \varphi(\widehat{x})| \leq H\exp\left[C\max\left(|x|^2, |\widehat{x}|^2\right)\right]|x-\widehat{x}|^\alpha, \tag{2.6}$$

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for any x, $\widehat{x} \in \mathbb{R}^N$, for some constant $H \geq 0$.

H5) For every $R > 0$, there exists a constant $H_R \geq 0$ such that

$$\left|g(x,\mathbf{X}) - g(\widehat{x},\widehat{\mathbf{X}})\right| \leq H_R \exp\left[C \max\left(|x|^2, |\widehat{x}|^2\right)\right] \left\{|x-\widehat{x}|^\alpha + \left|\mathbf{X}-\widehat{\mathbf{X}}\right|\right\}$$

for any x, $\widehat{x} \in \mathbb{R}^N$, $\mathbf{X}$, $\widehat{\mathbf{X}} \in B_R$.

In the following we shall make use of the following amount $\ell(\theta,\nu) := e^{-\theta}(\theta/\nu)^\theta$ for every $\theta > 0$, $\nu > 0$ that is easy to check that corresponds to the global maximum of the function $y \in [0,\infty) \mapsto y^\theta e^{-\nu y}$.

Remark 1. If Assumption H5) holds true, for every $\mathbf{X} = \mathbf{X}(t)$ $\mathbb{R}^{N\times n}$-valued continuous function on $[0,T]$, we define $g_{\mathbf{X}}(x,t) := g(x,\mathbf{X}(t))$, which has the following properties: let $\bar{R} = \sup_{t\in[0,T]} |\mathbf{X}(t)|$, then, for every $M > 0$,

$$|g_{\mathbf{X}}(x,t) - g_{\mathbf{X}}(\widehat{x},t)| \leq H_{\bar{R},M}|x-\widehat{x}|^\alpha, \tag{2.7}$$

for any x, $\widehat{x} \in \mathbb{R}^N$ such that $|x|$, $|\widehat{x}| \leq M$, $t \in [0,T]$, where $H_{\bar{R},M} = H_{\bar{R}} \exp[CM^2]$, that is $g_{\mathbf{X}}$ is locally Hölder continuous in x with exponent α, uniformly with respect to t.
Moreover, let $C < C' < \lambda_0/4T$, then, from H5), we obtain

$$\begin{aligned}
&|g_{\mathbf{X}}(x,t)| \leq |g(0,0)| + H_{\bar{R}} e^{C|x|^2}\left(|x|^\alpha + \bar{R}\right)\\
&\leq |g_{\mathbf{X}}(0,0)| e^{C'|x|^2} + H_{\bar{R}} e^{C'|x|^2}\left(|x|^\alpha e^{-(C'-C)|x|^2} + \bar{R} e^{C'|x|^2}\right)\\
&\leq \left[|g_{\mathbf{X}}(0,0)| + H_{\bar{R}}(\ell(\alpha/2, C'-C) + \bar{R})\right] e^{C'|x|^2}.
\end{aligned} \tag{2.8}$$

Similarly, such an upper bound is satisfied by the function φ, thanks to the Assumption H4) . These properties, satisfied by φ and $g_{\mathbf{X}}$, in light of Theorem 12 (page 25) in [58], allow to establish that $f(\cdot,\cdot;\mathbf{X}) : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ defined by

$$f(x,t;\mathbf{X}) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\varphi(\xi)d\xi - \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\, g_{\mathbf{X}}(\xi,\tau)d\xi d\tau \tag{2.9}$$

is the unique solution to the Cauchy problem (2.2) associated to $\mathbf{X}$, meaning in particular that f is continuous on $\mathbb{R}^N \times [0,T]$, $\partial_t f$,$\partial_i f$,$\partial_{i,j} f$ are continuous on $\mathbb{R}^N \times (0,T)$. Here the function Γ is a fundamental solution of $Lu = 0$ (See [58] and Appendix for the details).
Later on, we will use estimates concerning Γ and its derivatives:

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for every $\lambda_0^* < \lambda_0$ (see assumption H3)) there exists a constant C_Γ such that

$$|\Gamma(x,t;\xi,\tau)| \le C_\Gamma \frac{1}{(t-\tau)^{\frac{N}{2}}} e^{-\frac{\lambda_0^*|x-\xi|^2}{4(t-\tau)}}, \tag{2.10}$$

$$|\nabla\Gamma(x,t;\xi,\tau)| \le C_\Gamma \frac{1}{(t-\tau)^{\frac{N+1}{2}}} e^{-\frac{\lambda_0^*|x-\xi|^2}{4(t-\tau)}}, \tag{2.11}$$

$$\left|\nabla^2\Gamma(x,t;\xi,\tau)\right| \le C_\Gamma \frac{1}{(t-\tau)^{\frac{N+2}{2}}} e^{-\frac{\lambda_0^*|x-\xi|^2}{4(t-\tau)}}, \tag{2.12}$$

$$\left|\nabla\Gamma(x,t;\xi,\tau) - \nabla\Gamma(x',t;\xi,\tau)\right| \le C_\Gamma \frac{|x-x'|^{\alpha/2}}{(t-\tau)^{1/2+\alpha/4}} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|x-\xi|^2}{t-\tau}}}{(t-\tau)^{N/2}} \tag{2.13}$$

for any $x, x', \xi \in \mathbb{R}^N$, $0 \le \tau < t \le T$. for any $x, \xi \in \mathbb{R}^N$, $0 \le \tau < t \le T$. For (2.10)-(2.11) the reader is referred to [59], Theorem 4.5, whereas for (2.12) and (2.13) to [60]. We observe that in [60], inequality (2.12) is proved for a hypoelliptic differential operator. However, such estimate is clearly satisfied by the uniformly parabolic operator L defined in (2.3). An essential ingredient for subsequent results is the estimation of the first and second-order derivatives of $f(x,t;\mathbf{X})$ with respect to x, in terms of the supremum of $\mathbf{X}$ over $[0,T]$. The inequalities in Proposition 2 improve the usual estimates available for the solution of the Cauchy problem (2.2), thanks to the local Hölder continuity of the data. Here we get a precise estimate of the constants for such bounds.

Proposition 2. *Let the assumptions H2)-H5) be satisfied. Then, for every* $0 < \nu_0 < \lambda_0/4 - CT$, $x \in \mathbb{R}^N$, $0 < t \le T$, $\mathbf{X} \in C([0,T];\mathbb{R}^{N\times n})$, $i, j = 1,\dots,N$, *the following inequalities hold true:*

$$|\partial_i f(x,t;\mathbf{X})| \le K e^{\kappa|x|^2} \left(\frac{H}{t^{\frac{1-\alpha}{2}}} + \frac{2}{\alpha+1} t^{\frac{\alpha+1}{2}} H_{\mathbf{X}} \right), \tag{2.14}$$

$$|\partial_{ij}^2 f(x,t;\mathbf{X})| \le K e^{\kappa|x|^2} \left(\frac{H}{t^{1-\frac{\alpha}{2}}} + \frac{2}{\alpha} t^{\frac{\alpha}{2}} H_{\mathbf{X}} \right), \tag{2.15}$$

where C_Γ *is obtained from inequalities (2.10)-(2.12) for* $\lambda_0^* = \lambda_0 - 4\nu_0$, $H_{\mathbf{X}}$ *stands for the constant in H5), for* $R = \|\mathbf{X}\|_{\infty,T}$, H *the one in H4), and*

$$K := \frac{\pi^{N/2} C_\Gamma \ell(\alpha/2, \nu_0)}{[\lambda_0/2 - 2CT]^{N/2}}, \qquad \kappa := \frac{C^2 T}{\lambda_0/4 - CT} + 2C. \tag{2.16}$$

Proof. Proposition 2

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We focus on (2.14). First we recall that the Fundamental solution Γ satisfies

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)d\xi = 1, \tag{2.17}$$

for any $x \in \mathbb{R}^N$, $0 \le \tau < t \le T$. Therefore, we have

$$0 = \partial_i \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)d\xi = \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;\xi,\tau)d\xi, \tag{2.18}$$

$$0 = \partial^2_{ij} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)d\xi = \int_{\mathbb{R}^N} \partial^2_{ij} \Gamma(x,t;\xi,\tau)d\xi. \tag{2.19}$$

Let $\mathbf{X} \in C([0,T];\mathbb{R}^{N\times n})$ and set $\overline{R} = \|\mathbf{X}\|_{\infty,T}$. Then, the representation formula (3.5) implies, for every $x \in \mathbb{R}^N$, $t \in (0,T]$, the inequality $|\partial_i f(x,t;\mathbf{X})| \le G_1 + G_2$, where

$$G_1 := \Big| \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;\xi,0)\varphi(\xi)d\xi - \varphi(x) \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;\xi,0)d\xi \Big|, \tag{2.20}$$

$$G_2 := \Big| \int_0^t \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;\xi,\tau) g_{\mathbf{X}}(\xi,\tau)d\xi - \int_0^t g_{\mathbf{X}}(x,\tau) \int_{\mathbb{R}^N} \partial_i \Gamma(x,t;\xi,\tau)d\xi d\tau \Big|. \tag{2.21}$$

By the inequalities in $H4$)-$H5$), since $C < \lambda_0/4T$, we choose $\lambda_0^* \in (4TC, \lambda_0)$ and we set $\nu_0 = (\lambda_0 - \lambda_0^*)/4$. Therefore from (2.11), for $0 < t \le T$, we get

$$\begin{aligned}
G_1 &\le \frac{C_\Gamma H}{t^{\frac{N+1}{2}}} \int_{\mathbb{R}^N} e^{-\frac{\lambda_0^*}{4}\frac{|x-\xi|^2}{t} + C|x|^2 + C|\xi|^2} |x-\xi|^\alpha d\xi \\
&\le C_\Gamma H\, t^{\frac{\alpha-1}{2}} \int_{\mathbb{R}^N} e^{-\left(\lambda_0^*/4 - Ct\right)|u|^2 + 2C\sqrt{t}\langle x,u\rangle + 2C|x|^2} |u|^\alpha du \\
&\le C_\Gamma H \ell(\alpha/2,\nu_0)\, t^{\frac{\alpha-1}{2}} \int_{\mathbb{R}^N} e^{-\left(\lambda_0^*/4 - Ct + \nu_0\right)|u|^2 + 2C\sqrt{t}\langle x,u\rangle + 2C|x|^2} du \\
&\le C_\Gamma H \ell(\alpha/2,\nu_0) t^{\frac{\alpha-1}{2}} e^{\frac{C^2 t|x|^2}{\lambda_0/4 - Ct}} \int_{\mathbb{R}^N} e^{-\left|u\sqrt{\lambda_0/4 - Ct} - \frac{C\sqrt{t}x}{\sqrt{\lambda_0/4-Ct}}\right|^2 + 2C|x|^2} du \\
&= \frac{C_\Gamma H \ell(\alpha/2,\nu_0)}{t^{\frac{1-\alpha}{2}} \left[\lambda_0/4 - Ct\right]^{N/2}} e^{\left(\frac{C^2 t}{\lambda_0/4 - Ct} + 2C\right)|x|^2} \left(\frac{\pi}{2}\right)^{N/2}.
\end{aligned} \tag{2.22}$$

Here we have applied the changes of variable $\xi = x + \sqrt{t}u$ and $v/2 = u\sqrt{\lambda_0/4 - Ct} - \frac{C\sqrt{t}x}{\sqrt{\lambda_0/4-Ct}}$ and we have used the well known relation $\int_{\mathbb{R}^N} e^{-\frac{1}{2}|v^2|} dv = [2\pi]^{N/2}$. Since $t \in\mapsto [\lambda_0/4 - Ct]^{-1}$ is an increasing function,

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we finally obtain the estimate

$$G_1 \leq \frac{C_\Gamma H \ell(\alpha/2,\nu_0)}{t^{\frac{1-\alpha}{2}}\left[\lambda_0/4 - CT\right]^{N/2}} e^{\left(\frac{C^2T}{\lambda_0/4-CT}+2C\right)|x|^2}\left(\frac{\pi}{2}\right)^{N/2}. \tag{2.23}$$

By similar arguments applied to G_2, using $H5$) (with $\mathbf{X} \equiv \widehat{\mathbf{X}}$) and replacing t with $t-\tau$ in the integral over $\mathbb{R}^N$, we can write

$$\begin{aligned} G_2 &\leq \frac{C_\Gamma H_{\overline{R}} \ell(\alpha/2,\nu_0)}{\left[\lambda_0/4 - CT\right]^{N/2}} e^{\left(\frac{C^2T}{\lambda_0/4-CT}+2C\right)|x|^2}\left(\frac{\pi}{2}\right)^{N/2} \int_0^t \frac{1}{(t-\tau)^{\frac{1-\alpha}{2}}} d\tau \\ &= \frac{2C_\Gamma H_{\overline{R}} \ell(\alpha/2,\nu_0)}{(\alpha+1)\left[\lambda_0/4 - CT\right]^{N/2}} e^{\left(\frac{C^2T}{\lambda_0/4-CT}+2C\right)|x|^2}\left(\frac{\pi}{2}\right)^{N/2} t^{\frac{\alpha+1}{2}}, \end{aligned} \tag{2.24}$$

Inequalities (2.23) and (2.24) yield (2.14). We observe that the proof of inequality (2.15) follows through similar passages using the estimate for the second-order derivatives of the fundamental solution in (2.12) together with (2.19), hence we omit the details.

□

2.2 Existence and Uniqueness of a local solution

Using a fixed point argument, we prove the local existence and uniqueness of the solution of (2.1)-(2.2). We state more clearly that, by solution of (2.1)-(2.2), we refer to the couple $\mathbf{Y} = (\mathbf{X}, \mathbf{V})$, where $\mathbf{X}, \mathbf{V} \in C\left([0,T];\mathbb{R}^{N\times n}\right) \cap C^1\left((0,T);\mathbb{R}^{N\times n}\right)$, and $f \in C^{2,1}\left(\mathbb{R}^N \times (0,T)\right)$, satisfying (2.2), is expressed as in (3.5).

Theorem 3. ***Local Exiscence and Uniqueness.***
Under hypotheses H1)-H5), system (2.1)-(2.2) has a unique solution on $[0,\overline{T}]$, *where* $\overline{T} \in (0,T]$ *depends on* $\mathbf{X_0}$, $\mathbf{V_0}$, α, n, N, R.

Proof. Theorem 3
Let $R > 0$ and $0 < \overline{T} \leq T$.
In the following we denote $C_0 := \max_{\tau\in[0,\overline{T}]} |F(\tau, \mathbf{X}_0, \mathbf{V}_0, 0)|$. We consider the mapping Ψ defined as follows

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$$\Psi\left(\mathbf{X},\mathbf{V}\right)(t)=\begin{pmatrix} x_{01}+\int_0^t \mathbf{v}_1(\tau)\,d\tau \\ \vdots \\ x_{0n}+\int_0^t \mathbf{v}_n(\tau)\,d\tau \\ \mathbf{v}_{01}+\int_0^t F_1\left(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\nabla f\left(\mathbf{x}_1(\tau),\tau;\mathbf{X}\right)\right)\,d\tau \\ \vdots \\ \mathbf{v}_{0n}+\int_0^t F_n\left(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\nabla f\left(\mathbf{x}_n(\tau),\tau;\mathbf{X}\right)\right)\,d\tau \end{pmatrix} \tag{2.25}$$

for any $(\mathbf{X},\mathbf{V})\in E_R$, $t\in[0,\overline{T}]$, where $E_R:=C\left([0,\overline{T}];B_R(\mathbf{X}_0)\times B_R(\mathbf{V}_0)\right)$, and f is the function in (3.5).
Due to (2.4) and (2.14), we have that $\Psi\left(\mathbf{X},\mathbf{V}\right)$ is continuous at $t=0$, thus

$$\Psi\left(\mathbf{X},\mathbf{V}\right)\in C\left([0,\overline{T}];\mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\right).$$

We observe that a fixed point of Ψ, $\left(\overline{\mathbf{X}},\overline{\mathbf{V}}\right)\in E_R$, is a solution to (2.1)-(2.2). We shall define suitable conditions on $\overline{T}$, in order to guarantee $\Psi: E_R\longrightarrow E_R$, and the fact that Ψ is a contraction operator. Since E_R, endowed with the uniform norm, is clearly a Banach space, the result follows from fixed-point theorem ([61], Theorem 1, p.534).

$$\left|\Psi_j\left(\mathbf{X},\mathbf{V}\right)(t)-\mathbf{x}_{0j}\right|\leq\overline{T}\left(R+|\mathbf{V}_0|\right)\qquad\forall t\in[0,\overline{T}],j=1,...,n.$$

Thus

$$\left|\begin{pmatrix}\Psi_1\left(\mathbf{X},\mathbf{V}\right)(t)-\mathbf{x}_{01}\\ \vdots\\ \Psi_n\left(\mathbf{X},\mathbf{V}\right)(t)-\mathbf{x}_{0n}\end{pmatrix}\right|\leq n\overline{T}\left(R+|\mathbf{V}_0|\right). \tag{2.26}$$

Hence $\left[\Psi_1\left(\mathbf{X},\mathbf{V}\right)(t),...,\Psi_n\left(\mathbf{X},\mathbf{V}\right)(t)\right]\in B_R\left(\mathbf{X}_0\right)$ for any $t\in[0,\overline{T}]$ if we impose the condition:

$$\overline{T}\leq\frac{R}{n(R+|\mathbf{V}_0|)}. \tag{2.27}$$

Let us consider the component $l\in\{n+1,...,2n\}$.

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Recalling (2.4) and (2.14), we obtain in particular that

$$\begin{aligned}
&\left|\Psi_{j+n}\left(\mathbf{X},\mathbf{V}\right)(t)-\mathbf{v}_{0j}\right| \leq \int_0^t \left|F_j(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X}))\right| d\tau \\
&\leq \int_0^t |F_j(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X})) - F_j(\tau,\mathbf{X}_0,\mathbf{V}_0,0)| \\
&+|F_j(\tau,\mathbf{X}_0,\mathbf{V}_0,0)|d\tau \\
&\leq 2L_F^R R\overline{T} + \int_0^t L_F\left|\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X})\right| + |F_j(\tau,\mathbf{X}_0,\mathbf{V}_0,0)|d\tau \\
&\leq 2L_F^R R\overline{T} + \int_0^t L_F\sqrt{N}Ke^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{H}{\tau^{\frac{1-\alpha}{2}}}+\frac{2}{\alpha+1}\tau^{\frac{\alpha+1}{2}}H_{\mathbf{X}}\right) \\
&+|F_j(\tau,\mathbf{X}_0,\mathbf{V}_0,0)|d\tau \leq \overline{T}\left(2L_F^R R + C_0\right) \\
&+L_F\sqrt{N}Ke^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{2H}{\alpha+1}\overline{T}^{\frac{\alpha+1}{2}}+\frac{2H_{\mathbf{X}}}{\alpha+1}\frac{2}{\alpha+3}\overline{T}^{\frac{\alpha+3}{2}}\right).
\end{aligned} \tag{2.28}$$

Thus

$$\left|\begin{pmatrix} \Psi_{n+1}\left(\mathbf{X},\mathbf{V}\right)-\mathbf{v}_{01} \\ \vdots \\ \Psi_{2n}\left(\mathbf{X},\mathbf{V}\right)-\mathbf{v}_{0n} \end{pmatrix}\right| \leq n\overline{T}\left(2L_F^R R + C_0\right)$$

$$+ nL_F\sqrt{N}Ke^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{2H}{\alpha+1}\overline{T}^{\frac{\alpha+1}{2}}+\frac{2H_{\mathbf{X}}}{\alpha+1}\frac{2}{\alpha+3}\overline{T}\overline{T}^{\frac{\alpha+1}{2}}\right). \tag{2.29}$$

From (2.27) follows, in particular, that $\overline{T} < 1$, thus $\overline{T} < \overline{T}^{\frac{\alpha+1}{2}}$ since $\alpha \in (0,1)$. Hence it sufficies to impose $\overline{T} \leq T_1$, where $T_1 = T_1\left(\mathbf{X}_0,\mathbf{V}_0,\alpha,n,N,R\right)$ is defined as

$$T_1 := \min\left(\frac{R}{n(R+|\mathbf{V}_0|)}, \left(\frac{R}{\frac{2n\sqrt{N}L_F K e^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}}{\alpha+1}\left(1+\frac{2H_{\mathbf{X}}}{\alpha+3}\right)}\right)^{\frac{2}{\alpha+1}}\right), \tag{2.30}$$

to ensure that the range of Ψ is a subset of E.
We now show that it is possible to determine a condition on $\overline{T}$ such that Ψ is a contraction operator. Let $(\mathbf{X},\mathbf{V})$ and $\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right) \in E$.

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Clearly it holds

$$\begin{aligned}&\left|\Psi_j\left(\mathbf{X},\mathbf{V}\right)(t)-\Psi_j\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)(t)\right|\leq\int_0^t\left|\mathbf{v}_j(\tau)-\hat{\mathbf{v}}_j(\tau)\right|d\tau\\&\leq\overline{T}||\mathbf{V}-\widehat{\mathbf{V}}||_{\infty,\overline{T}}\leq\overline{T}\left|\left|(\mathbf{X},\mathbf{V})-\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\right|\right|_{\infty,\overline{T}}.\end{aligned}\tag{2.31}$$

We now focus on the remaining components. From the Lipschitz condition (2.4), it follows that, for $j=1,...,n$:

$$\begin{aligned}&\left|\Psi_{n+j}\left(\mathbf{X},\mathbf{V}\right)(t)-\Psi_{n+j}\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)(t)\right|\leq\\&\leq L_F^R\int_0^t\left|\mathbf{X}(\tau)-\widehat{\mathbf{X}}(\tau)\right|+\left|\mathbf{V}(\tau)-\widehat{\mathbf{V}}(\tau)\right|d\tau\\&+L_F\int_0^t\left|\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X}))-\nabla f(\widehat{\mathbf{x}}_j(\tau),\tau;\widehat{\mathbf{X}}))\right|d\tau\leq\\&\leq L_F^R\overline{T}\left(||\mathbf{X}-\widehat{\mathbf{X}}||_{\infty,\overline{T}}+||\mathbf{V}-\widehat{\mathbf{V}}||_{\infty,\overline{T}}\right)+L_F\left(\tilde{I}_1+\tilde{I}_2\right),\end{aligned}\tag{2.32}$$

where $\tilde{I}_1,\tilde{I}_2$ are the integrals

$$\tilde{I}_1:=\int_0^t\left|\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X}))-\nabla f(\widehat{\mathbf{x}}_j(\tau),\tau;\mathbf{X}))\right|d\tau,\tag{2.33}$$

$$\tilde{I}_2:=\int_0^t\left|\nabla f(\widehat{\mathbf{x}}_j(\tau),\tau;\mathbf{X}))-\nabla f(\widehat{\mathbf{x}}_j(\tau),\tau;\widehat{\mathbf{X}}))\right|d\tau.\tag{2.34}$$

We observe that, by the mean value theorem and (2.15) , it holds

$$\begin{aligned}&\left|\nabla f(\mathbf{x}_j(\tau),\tau;\mathbf{X}))-\nabla f(\widehat{\mathbf{x}}_j(\tau),\tau;\mathbf{X}))\right|\leq\\&\leq\sum_{i=1}^N\left|\partial_i f(\mathbf{x}_j(\tau),\tau;\mathbf{X}))-\partial_i f(\widehat{\mathbf{x}}_j(\tau),\tau;\mathbf{X}))\right|\leq\\&\leq\sum_{i=1}^N\sum_{m=1}^N\left|\partial_{im}^2 f(\mathbf{x}_{ij}^*,\tau;\mathbf{X})\right|\left(|\mathbf{x}_j(\tau)-\widehat{\mathbf{x}}_j(\tau)|\right)\leq\\&\leq N^2Ke^{\kappa|\mathbf{x}_{ij}^*|^2}\left(\frac{H}{\tau^{1-\alpha/2}}+\frac{2H_{\mathbf{X}}}{\alpha}\tau^{\alpha/2}\right)\left(|\mathbf{x}_j(\tau)-\widehat{\mathbf{x}}_j(\tau)|\right),\end{aligned}\tag{2.35}$$

where $\mathbf{x}^*{}_{ij}$ belongs to the segment connecting $\mathbf{x}_j(\tau)$ to $\widehat{\mathbf{x}}_j(\tau)$ thus, in par-

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ticular, $\mathbf{x}^*{}_{ij} \in B_R(\mathbf{X}_0)$. From the parallelogram inequality we get $|\mathbf{x}^*{}_{ij}|^2 \leq 2\left(|\mathbf{X}_0|^2 + R^2\right)$. We finally achieve the following estimate for $\tilde{I}_1$:

$$\tilde{I}_1 \leq N^2 K e^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)} \left(\frac{2H}{\alpha}\overline{T}^{\alpha/2} + \frac{2H_{\mathbf{X}}}{\alpha(\alpha/2+1)}\overline{T}^{\alpha/2+1}\right) \left|\left|\mathbf{X} - \widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}}. \tag{2.36}$$

We now focus on $\tilde{I}_2$. Since f is a solution of (2.2), we observe that $\eta(\xi,\tau) := f(\xi,\tau;\mathbf{X}) - f(\xi,\tau;\widehat{\mathbf{X}})$ satisfies the following Cauchy problem:

$$\begin{cases} L\eta(\xi,\tau) &= g(\xi,\mathbf{X}(\tau)) - g(\xi,\widehat{\mathbf{X}}(\tau)) \qquad (\xi,\tau) \in \mathbb{R}^N \times (0,T), \\ \eta(\xi,0) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \xi \in \mathbb{R}^N. \end{cases} \tag{2.37}$$

$\tilde{I}_2$ can thus be rewritten as $\tilde{I}_2 = \int_0^t |\nabla\eta(\widehat{x}_j(\tau),\tau)|\, d\tau$.
From (2.11) and H5) we have

$$\begin{aligned} |\nabla\eta(\xi,\tau)| &= \left|\int_0^\tau \int_{\mathbb{R}^N} \nabla\Gamma(\xi,\tau;\overline{\xi},\overline{\tau})\{g(\overline{\xi},\mathbf{X}(\tau)) - g(\overline{\xi},\widehat{\mathbf{X}}(\tau))\} d\overline{\xi}d\overline{\tau}\right| \\ &\leq C_\Gamma \int_0^\tau \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|\xi-\overline{\xi}|^2}{\tau-\overline{\tau}}}}{(\tau-\overline{\tau})^{\frac{N+1}{2}}} \left|g(\overline{\xi},\mathbf{X}(\tau)) - g(\overline{\xi},\widehat{\mathbf{X}}(\tau))\right| d\overline{\xi}d\overline{\tau} \\ &\leq C_\Gamma H_R \int_0^\tau \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|\xi-\overline{\xi}|^2}{\tau-\overline{\tau}}}}{(\tau-\overline{\tau})^{\frac{N+1}{2}}} e^{C|\overline{\xi}|^2} d\overline{\xi}d\overline{\tau} \left|\left|\mathbf{X} - \widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}} \\ &= C_\Gamma H_R \int_0^\tau \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}|y|^2 + C\left|\xi+\sqrt{\tau-\overline{\tau}}y\right|^2}}{(\tau-\overline{\tau})^{\frac{N+1}{2}}} (\tau-\overline{\tau})^{\frac{N}{2}}\, dy d\overline{\tau} \left|\left|\mathbf{X} - \widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}} \end{aligned} \tag{2.38}$$

where, in the last equality, we have performed the change of variable

$$y = \frac{-\xi+\overline{\xi}}{\sqrt{\tau-\overline{\tau}}}. \tag{2.39}$$

Since $\left|\xi+\sqrt{\tau-\overline{\tau}}y\right|^2 \leq 2|\xi|^2 + 2|y|^2$, we rewrite the exponential term in (2.38). We require $\overline{T} < \frac{\lambda_0^*}{8C}$, implying that $\overline{\gamma} := \left(\frac{\lambda_0^*}{4} - 2C\overline{T}\right) > 0$. Recalling the expression of $I_0(\overline{\gamma})$, and its exact value (see Appendix for details), we

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get:

$$\begin{aligned}|\nabla\eta(\xi,\tau)| &\leq C_\Gamma H_R e^{2C|\xi|^2}\int_0^\tau\int_{\mathbb{R}^N}\frac{1}{(\tau-\overline{\tau})^{\frac{1}{2}}}e^{-\overline{\gamma}|y|^2}dyd\overline{\tau}\left|\left|\mathbf{X}-\widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}}\\ &= C_\Gamma H_R e^{2C|\xi|^2}\sqrt{\tau}\left(\frac{\pi}{\overline{\gamma}}\right)^{N/2}\left|\left|\mathbf{X}-\widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}},\end{aligned} \tag{2.40}$$

By (2.40), with $|\xi|^2=|x_j(\tau)|^2\leq 2\left|\mathbf{X}_0\right|^2+2R^2$, the estimate of $\tilde{I}_2$ immediately follows:

$$\tilde{I}_2\leq C_\Gamma H_R e^{2C\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{\pi}{\overline{\gamma}}\right)^{N/2}\overline{T}^{\frac{3}{2}}\left|\left|\mathbf{X}-\widehat{\mathbf{X}}\right|\right|_{\infty,\overline{T}}. \tag{2.41}$$

From (2.32), (2.36) and (2.41) we get the following inequality

$$\begin{aligned}&\left|\Psi_{n+j}\left(\mathbf{X},\mathbf{V}\right)(t)-\Psi_{n+j}\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)(t)\right|\leq\\ &\leq\left(2L_F^R\overline{T}+L_F N^2 K e^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}\left(\tfrac{2H}{\alpha}\overline{T}^{\alpha/2}+\tfrac{2H_{\mathbf{X}}}{\alpha(\alpha/2+1)}\overline{T}^{\alpha/2+1}\right)\right.\\ &\left.+L_F C_\Gamma H_R e^{2C\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{\pi}{\overline{\gamma}}\right)^{N/2}\overline{T}^{\frac{3}{2}}\right)\left|\left|(\mathbf{X},\mathbf{V})-\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\right|\right|_{\infty,\overline{T}},\end{aligned} \tag{2.42}$$

for any $(\mathbf{X},\mathbf{V}),\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\in E$, $j=1,...,n$, $t\in[0,T]$.
From (2.31) and (2.42) we conclude that

$$\begin{aligned}&\left|\left|\Psi\left(\mathbf{X},\mathbf{V}\right)-\Psi\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\right|\right|_{\infty,\overline{T}}\\ &\leq\sqrt{2n}\left[\max\left(2L_F^R,1\right)\overline{T}+S\right]\left|\left|(\mathbf{X},\mathbf{V})-\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\right|\right|_{\infty,\overline{T}},\end{aligned} \tag{2.43}$$

where

$$\begin{aligned}S=S\left(\mathbf{X}_0,\mathbf{V}_0,\alpha,n,N,R,\overline{T}\right):=\left(2L_F^R\overline{T}+L_F N^2 K e^{\kappa\left(|\mathbf{X}_0|^2+R^2\right)}\overline{T}^{\alpha/2}\right.\\ \left.\tfrac{2}{\alpha}\left(H+H_{\mathbf{X}}\overline{T}\right)+L_F C_\Gamma H_R e^{2C\left(|\mathbf{X}_0|^2+R^2\right)}\left(\frac{\pi}{\overline{\gamma}}\right)^{N/2}\overline{T}^{\frac{3}{2}}\right).\end{aligned} \tag{2.44}$$

Finally, a sufficient condition on $\overline{T}$ such that Ψ is a contraction operator is given by

$$\overline{T}<\min\{T_1,T_2\}, \tag{2.45}$$

where T_1 is defined in (2.30), and $T_2=T_2\left(\mathbf{X}_0,\mathbf{V}_0,\alpha,n,\ \ N,R\right)$ is such that

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$T_2 < \dfrac{\lambda_0^*}{8C}$ and $S(\mathbf{X}_0, \mathbf{V}_0, \alpha, n, N, R, T_2) < 1$. We observe that such a choice of T_2 can be obtained since S is an increasing continuous function of $\overline{T}$, and $S(\mathbf{X}_0, \mathbf{V}_0, \alpha, n, N, R, 0) = 0$.
Hence, thesis follows from fixed-point theorem ([61], Theorem 1, p. 534), which ensures the existence and uniqueness of a fixed point for Ψ over the interval $\left[0, \overline{T}\right]$.

□

Remark 2. Let us consider the case of $\varphi \equiv 0$. From Proposition 2 we deduce that

$$|\nabla f(x,t;\mathbf{X})| \leq \frac{2\sqrt{N}}{\alpha+1} K e^{\kappa|x|^2} T^{\frac{\alpha+1}{2}} H_{\mathbf{X}}. \tag{2.46}$$

Therefore Assumption H1) can be weakened, requiring only the local Lipschitz continuity of F_i with respect to all the arguments. Actually $|\nabla f(x,t;\mathbf{X})|$ is bounded on $B_R \times \left[0, \overline{T}\right] \times E_R$, for any $R > 0$.

2.3 The case of a non-local concentration

In the previous sections we established an existence result for a local solution to systems in the form (2.1)-(2.2). Under the same assumptions, we consider a variant of system (2.1), given by

$$\begin{cases} \dot{\mathbf{x}}_i(t) &= \mathbf{v}_i(t), \qquad\qquad 0 < t < T \\ \dot{\mathbf{v}}_i(t) &= F_i\left(t, \mathbf{X}(t), \mathbf{V}(t), \fint_{B_\delta(\mathbf{x}_i(t))} \nabla f(\xi, t; \mathbf{X})\, d\xi\right), \end{cases} \tag{2.47}$$

where f satisfies (2.2). The main difference between (2.1) and (2.47) lies in the fact that $\nabla f(\mathbf{x}_i(t), t; \mathbf{X})$ is replaced by the average over a ball centered in $\mathbf{x}_i(t)$ and having radius $\delta > 0$. A local existence result is stated in the following theorem.

Theorem 4. *Under hypotheses H1)-H5), the system (2.47) has a unique solution on $[0, \widetilde{T}]$, where $\widetilde{T} \in (0, T)$ depends on $\mathbf{X_0}$, $\mathbf{V_0}$, α, n, N, R.*

Proof. Theorem 4
We consider the operator defined in (2.25), replacing $\nabla f(\mathbf{x}_i(\tau), \tau; \mathbf{X})$ with $\fint_{B_\delta(\mathbf{x}_i(\tau))} \nabla f(\xi, \tau; \mathbf{X})\, d\xi$.
We shall sketch the proof of the result looking back to the proof of Theorem 3, analyzing only the main points affected by such a change.

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The mapping defined in (2.25) for (2.1) can be rewritten likewise for (2.47),

$$\Psi\left(\mathbf{X},\mathbf{V}\right)(t)=\begin{pmatrix}\mathbf{x}_{01}+\int_0^t \mathbf{v}_1(\tau)\,d\tau\\ \vdots\\ \mathbf{x}_{0n}+\int_0^t \mathbf{v}_n(\tau)\,d\tau\\ \mathbf{v}_{01}+\int_0^t F_1\left(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\fint_{B_\delta(x_1(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi\right)d\tau\\ \vdots\\ \mathbf{v}_{0n}+\int_0^t F_n\left(\tau,\mathbf{X}(\tau),\mathbf{V}(\tau),\fint_{B_\delta(x_n(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi\right)d\tau\end{pmatrix} \tag{2.48}$$

for any $(\mathbf{X},\mathbf{V})\in E_{\widetilde{R}}$, $t\in[0,\widetilde{T}]$, where $E_{\widetilde{R}}:=C\left([0,\widetilde{T}];B_{\widetilde{R}}(\mathbf{X}_0)\times B_{\widetilde{R}}(\mathbf{V}_0)\right)$, for a fixed $\widetilde{R}>0$, $0<\widetilde{T}\leq T$.
Let $(\mathbf{X},\mathbf{V})$ and $\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)\in E_{\widetilde{R}}$. Clearly (2.31) still remains true. We now focus on the component $l\in\{n+1,...,2n\}$. From the Lipschitz condition (2.4), it follows that

$$\begin{aligned}&\left|\Psi_{n+j}\left(\mathbf{X},\mathbf{V}\right)(t)-\Psi_{n+j}\left(\widehat{\mathbf{X}},\widehat{\mathbf{V}}\right)(t)\right|\\ &\leq L_F^R\int_0^t\left|\mathbf{X}(\tau)-\widehat{\mathbf{X}}(\tau)\right|+\left|\mathbf{V}(\tau)-\widehat{\mathbf{V}}(\tau)\right|d\tau\\ &+L_F\int_0^t\left|\fint_{B_\delta(\mathbf{x}_j(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi-\fint_{B_\delta(\hat{\mathbf{x}}_j(\tau))}\nabla f\left(\xi,\tau;\widehat{\mathbf{X}}\right)d\xi\right|d\tau\\ &\leq L_F^R\overline{T}\left(||\mathbf{X}-\widehat{\mathbf{X}}||_{\infty,\widetilde{T}}+||\mathbf{V}-\widehat{\mathbf{V}}||_{\infty,\widetilde{T}}\right)+L_F\left(J_1+J_2\right),\end{aligned} \tag{2.49}$$

where J_1, J_2 are the integrals

$$J_1:=\int_0^t\left|\fint_{B_\delta(\mathbf{x}_j(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi-\fint_{B_\delta(\hat{\mathbf{x}}_j(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi\right|d\tau, \tag{2.50}$$

$$J_2:=\int_0^t\left|\fint_{B_\delta(\hat{\mathbf{x}}_j(\tau))}\nabla f\left(\xi,\tau;\mathbf{X}\right)d\xi-\fint_{B_\delta(\hat{\mathbf{x}}_j(\tau))}\nabla f\left(\xi,\tau;\widehat{\mathbf{X}}\right)d\xi\right|d\tau. \tag{2.51}$$

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Since $|B_\delta(\mathbf{x}_j(\tau))| = |B_\delta(\hat{\mathbf{x}}_j(\tau))|$, we obtain:

$$\begin{aligned}
&\left| \fint_{B_\delta(\mathbf{x}_j(t))} \nabla f\left(\xi,\tau;\mathbf{X}\right) d\xi - \fint_{B_\delta(\hat{\mathbf{x}}_j(t))} \nabla f\left(\xi,\tau;\mathbf{X}\right) d\xi \right| \\
&\leq \frac{1}{|B_\delta(\mathbf{x}_j(\tau))|} \left| \int_{B_\delta(\mathbf{x}_j(\tau))} \nabla f\left(\xi,\tau;\mathbf{X}\right) d\xi - \int_{B_\delta(\hat{\mathbf{x}}_j(\tau))} \nabla f\left(\xi,\tau;\mathbf{X}\right) d\xi \right| \\
&\leq \frac{1}{\delta^N C_N} \int_{B_\delta} \left| \nabla f\left(\xi - \mathbf{x}_j(\tau),\tau;\mathbf{X}\right) - \nabla f\left(\xi - \hat{\mathbf{x}}_j(\tau),\tau;\mathbf{X}\right) \right| d\xi,
\end{aligned} \tag{2.52}$$

where C_N denotes the volume of $B_\delta \subset \mathbb{R}^N$.
The estimate follows as in (2.36), since from (2.35) we get:

$$\begin{aligned}
J_1 \quad &\leq N^2 K e^{\left(\delta^2 + |\mathbf{X}_0|^2 + \widetilde{R}^2\right)} \int_0^t \left(\frac{H}{\tau^{1-\alpha/2}} + \frac{2H_{\mathbf{X}}}{\alpha} \tau^{\alpha/2} \right) |\mathbf{x}_j(\tau) - \hat{\mathbf{x}}_j(\tau)| \, d\tau \\
&\leq N^2 K e^{\left(\delta^2 + |\mathbf{X}_0|^2 + \widetilde{R}^2\right)} \left(\frac{2H}{\alpha} \widetilde{T}^{\alpha/2} + \frac{2H_{\mathbf{X}}}{\alpha+2} \widetilde{T}^{\alpha/2+1} \right) ||\mathbf{X} - \widehat{\mathbf{X}}||_{\infty,\widetilde{T}}.
\end{aligned} \tag{2.53}$$

Recalling the estimates of I_2 in the previous section, we observe that

$$\begin{aligned}
J_2 \quad &= \int_0^t \left| \fint_{B_\delta(\hat{\mathbf{x}}_j(\tau))} \nabla f\left(\xi,\tau;\mathbf{X}\right) - \nabla f\left(\xi,\tau;\widehat{\mathbf{X}}\right) d\xi \right| d\tau \\
&\leq \int_0^t \frac{1}{|B_\delta(\hat{\mathbf{x}}_j(\tau))|} \int_{B_\delta(\hat{\mathbf{x}}_j(\tau))} \left| \nabla f\left(\xi,\tau;\mathbf{X}\right) - \nabla f\left(\xi,\tau;\widehat{\mathbf{X}}\right) \right| d\xi d\tau \\
&\leq \int_0^t \frac{1}{|B_\delta(\hat{\mathbf{x}}_j(\tau))|} \left(\int_{B_\delta(\hat{\mathbf{x}}_j(\tau))} |\nabla \eta\left(\xi,\tau\right)| \, d\xi \right) d\tau,
\end{aligned} \tag{2.54}$$

where η has been introduced in (2.37). From (2.41) we get

$$\begin{aligned}
J_2 \quad &\leq C_\Gamma H_{\widetilde{R}+\delta} e^{2C\left(|\mathbf{X}_0|^2 + \widetilde{R}^2 + \delta^2\right)} \left(\frac{\pi}{\overline{\gamma}} \right)^{N/2} \int_0^t \sqrt{\tau} d\tau \left|\left| \mathbf{X} - \widehat{\mathbf{X}} \right|\right|_{\infty,\widetilde{T}} \\
&\leq C_\Gamma H_{\widetilde{R}+\delta} e^{2C\left(|\mathbf{X}_0|^2 + \widetilde{R}^2 + \delta^2\right)} \left(\frac{\pi}{\overline{\gamma}} \right)^{N/2} \widetilde{T}^{\frac{3}{2}} \left|\left| \mathbf{X} - \widehat{\mathbf{X}} \right|\right|_{\infty,\widetilde{T}}.
\end{aligned} \tag{2.55}$$

From (2.53) and (2.55) we deduce a sufficient condition, of the same form as (2.45), that $\widetilde{T}$ has to satisfy in order to ensure that Ψ is a contraction operator. As done in (2.29), we can easily show an analogous relation for $\widetilde{T}$ in order to guarantee that the range of Ψ is a subset of $E_{\widetilde{R}}$. In fact, we

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observe that the replacement of the gradient term with the average results in a slight modification of the exponential factor of (2.28), that is $e^{\kappa\left(|\mathbf{x}_0|^2+R^2\right)}$ is replaced by $e^{\kappa\left(|\mathbf{x}_0|^2+\widetilde{R}^2+\delta^2\right)}$. □

2.4 Existence of global solution result

In this section we present a global existence and uniqueness result for (2.1)-(2.2) using a principle of continuation of solutions. Strengthening the growth conditions on functions g and φ, we shall prove that bounded solutions can be continued. For the sake of completeness, an analogous result will be also shown for the variant (2.47)-(2.2).
In light of the previous section, let assume that there exists a local solution $\mathbf{Y} = \mathbf{Y}(t)$ to (2.1)-(2.2) in $[0, \overline{T}]$, with $\overline{T} \leq T$. We replace the previous hypotheses H1) and H4) with the following assumptions:

H6) For every $i = 1, ..., n$

$$F_i : [0,T] \times \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \times \mathbb{R}^N \to \mathbb{R}^N \tag{2.56}$$

is globally Lipschitz continuous, with Lipschitz constant L_F.

H7) $g : \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$, $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ are assumed to be a continuous functions, satisfying:

$$|g(x,\mathbf{X})| \leq M(1+|x|+|\mathbf{X}|), \qquad |\varphi(x)| \leq M(1+|x|) \tag{2.57}$$

for any $x \in \mathbb{R}^N$, $\mathbf{X} \in C\left(\left[0,\overline{T}\right];\mathbb{R}^{N\times n}\right)$, with $\overline{T} \leq T$.

Theorem 5. ***Global existence and uniqueness of the solution***
Under assumptions H6)-H7), the local solution $\mathbf{Y}$ *is a global solution.*

In order to prove Theorem 5, we shall use the following result:

Lemma 6. *Let* $\mathbf{X} \in C\left(\left[0,\overline{T}\right];\mathbb{R}^{N\times n}\right)$, *with* $\overline{T} \leq T$.
Under assumption H7), the following estimate holds true:

$$|\nabla f\,(x,t;\mathbf{X})| \leq K_1\left(\frac{1+|x|}{\sqrt{t}} + K_2 + \int_0^t \left(\frac{1+|x|+|\mathbf{X}(\tau)|}{\sqrt{t-\tau}}\right) d\tau\right), \tag{2.58}$$

for any $x \in \mathbb{R}^N$, $t \in \left(0,\overline{T}\right]$, *where* $K_1 = K_1(C_\Gamma, N, M, \lambda_0^*)$,
$K_2 = K_2(N, \lambda_0^*, T)$, $\lambda_0^* < \lambda_0$, C_Γ *as in* (2.11).

Moreover we adopt the following Gronwall-type inequality:

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Lemma 7. *Let $h \in C([0,T])$, $w \in L^1([0,T])$, $v \in L^1([0,T]\times[0,T])$ be non-negative functions, such that*

$$h(t) \leq \alpha + \int_0^t w(\tau)h(\tau)d\tau + \int_0^t \int_0^\tau v(s,\tau)h(s)dsd\tau \quad \forall t \in [0,T], \quad (2.59)$$

Then the following inequality holds true:

$$h(t) \leq \alpha \ \ exp\left[\int_0^t \left(w(\tau) + \int_0^\tau v(s,\tau)ds\right) d\tau\right] \quad \forall t \in [0,T]. \quad (2.60)$$

We posticipate the proof of the technical Lemmas, focusing first on the main result of this section:

Proof of Theorem 5.
To establish the result, we structure the proof in two steps. In the first we shall prove that a solution $\mathbf{Y}(t)$ of (2.1)-(2.2) on an interval $[0,\overline{T})$ remains bounded. In the second, we will prove that $\mathbf{Y}(t)$ can be extended to $[0,T]$, showing that there exists the $\lim_{t\to\overline{T}^-} \mathbf{Y}(t)$. Thesis follows applying Theorem 3 starting at $t = \overline{T}$.

Step 1:
Recalling $(2.1)_2$ we get that for every $i = 1, ..., n$

$$\mathbf{v}_i(t) = \mathbf{v}_{0i} + \int_0^t F_i\left(\tau, \mathbf{X}(\tau), \mathbf{V}(\tau), \nabla f\left(\mathbf{x}_i(\tau), \tau; \mathbf{X}\right)\right) d\tau, \quad (2.61)$$

thus

$$\begin{aligned} |\mathbf{v}_i(t) - \mathbf{v}_{0i}| &\leq \int_0^t \left|F_i\left(\tau, \mathbf{X}(\tau), \mathbf{V}(\tau), \nabla f\left(\mathbf{x}_i(\tau), \tau; \mathbf{X}\right)\right)\right| d\tau \\ &\leq tC_0 + L_F \int_0^t |\mathbf{X}(\tau) - \mathbf{X}_0| \, d\tau \quad (2.62) \\ &+ L_F \int_0^t |\mathbf{V}(\tau) - \mathbf{V}_0| \, d\tau + L_F \int_0^t \left|\nabla f\left(\mathbf{x}_i(\tau), \tau; \mathbf{X}\right)\right| d\tau. \end{aligned}$$

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From (2.58), we get

$$|\mathbf{V}(t)-\mathbf{V}_0| \leq nTC_0 + nL_F\int_0^t |\mathbf{X}(\tau)-\mathbf{X}_0|\,d\tau$$
$$+nL_F\int_0^t |\mathbf{V}(\tau)-\mathbf{V}_0|\,d\tau + nL_FK_1\left(\int_0^t \frac{1+|\mathbf{X}(\tau)|}{\sqrt{\tau}}d\tau + K_2T\right)$$
$$+nL_FK_1\left(\int_0^t\int_0^\tau \frac{1+|\mathbf{X}(\tau)|+|\mathbf{X}(s)|}{\sqrt{\tau-s}}dsd\tau\right). \tag{2.63}$$

Integrating $(2.1)_1$, we get

$$\mathbf{X}(t) = \mathbf{X}_0 + \mathbf{V}_0t + \int_0^t |\mathbf{V}(\tau)-\mathbf{V}_0|\,d\tau. \tag{2.64}$$

By equations (2.64) and (2.63) we obtain that

$$|\mathbf{Y}(t)-\mathbf{Y}_0| \leq |\mathbf{X}(t)-\mathbf{X}_0| + |\mathbf{V}(t)-\mathbf{V}_0| \leq |\mathbf{V}_0|\,T+$$
$$\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau + nTC_0 + nL_F\sqrt{2}\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau$$
$$+nL_FK_1\left((1+|\mathbf{X}_0|)\,2\sqrt{T} + \int_0^t \frac{|\mathbf{Y}(\tau)-\mathbf{Y}_0|}{\sqrt{\tau}}d\tau + K_2T\right)$$
$$+nL_FK_1\left((1+2\,|\mathbf{X}_0|)\,\tfrac{4}{3}\sqrt{T^3} + \int_0^t\int_0^\tau \frac{|\mathbf{X}(\tau)-\mathbf{X}_0|}{\sqrt{\tau-s}}dsd\tau\right.$$
$$\left. + \int_0^t\int_0^\tau \frac{|\mathbf{X}(s)-\mathbf{X}_0|}{\sqrt{\tau-s}}dsd\tau\right)$$
$$\leq |\mathbf{V}_0|\,T + nTC_0 + nL_FK_1\,(1+|\mathbf{X}_0|)\,2\sqrt{T} + nL_FK_1\,(1+2\,|\mathbf{X}_0|)\,\tfrac{4\sqrt{T^3}}{3}$$
$$+nL_FK_1K_2T + \left(1+nL_F\sqrt{2}\right)\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau + nL_FK_1\cdot$$
$$\left(\int_0^t \frac{|\mathbf{Y}(\tau)-\mathbf{Y}_0|}{\sqrt{\tau}}d\tau + \int_0^t\int_0^\tau \frac{|\mathbf{Y}(\tau)-\mathbf{Y}_0|}{\sqrt{\tau-s}}dsd\tau + \int_0^t\int_0^\tau \frac{|\mathbf{Y}(s)-\mathbf{Y}_0|}{\sqrt{\tau-s}}dsd\tau\right). \tag{2.65}$$

Denoting with $\alpha = \alpha\,(\mathbf{X}_0, \mathbf{V}_0, L_F, K_1, K_2, N, n, T)$ the quantity $|\mathbf{V}_0|\,T + nTC_0 + nL_FK_1\,(1+|\mathbf{X}_0|)\,2\sqrt{T} + nL_FK_1\,(1+2\,|\mathbf{X}_0|)\,\frac{4}{3}\sqrt{T^3} + nL_FK_1K_2T$,

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we rewrite

$$
\begin{aligned}
&|\mathbf{Y}(t)-\mathbf{Y}_0| \le \alpha + \left(1+nL_F\sqrt{2}\right)\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau \\
&+nL_FK_1\int_0^t \frac{|\mathbf{Y}(\tau)-\mathbf{Y}_0|}{\sqrt{\tau}}d\tau + nL_FK_1\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,2\sqrt{\tau}d\tau \\
&+nL_FK_1\int_0^t\int_0^\tau \frac{|\mathbf{Y}(s)-\mathbf{Y}_0|}{\sqrt{\tau-s}}dsd\tau \\
&= \alpha + \left(1+nL_F\sqrt{2}\right)\int_0^t |\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau + nL_FK_1\int_0^t \frac{|\mathbf{Y}(\tau)-\mathbf{Y}_0|}{\sqrt{\tau}}d\tau \\
&+2nL_FK_1\int_0^t \sqrt{\tau}\,|\mathbf{Y}(\tau)-\mathbf{Y}_0|\,d\tau + nL_FK_1\int_0^t\int_0^\tau \frac{|\mathbf{Y}(s)-\mathbf{Y}_0|}{\sqrt{\tau-s}}dsd\tau.
\end{aligned}
\tag{2.66}
$$

Finally, we observe that (2.66) can be rewritten as

$$
h(t) \le \alpha + \int_0^t w(\tau)h(\tau)d\tau + \int_0^t\int_0^\tau v(s,\tau)h(s)dsd\tau, \tag{2.67}
$$

with $h(t) := |\mathbf{Y}(t)-\mathbf{Y}_0| \in C(0,T)$, and $w, v \in L^1([0,T])$ defined by $w(t) := 1+nL_F\sqrt{2}+\dfrac{nL_FK_1}{\sqrt{t}}+2nL_FK_1\sqrt{t}$,

$$
v(t,\tau) := \begin{cases} \dfrac{nL_FK_1}{\sqrt{t-\tau}} & \tau < t \\ 0 & \tau \ge t. \end{cases} \tag{2.68}
$$

Thus, by Lemma 7, we reach the conclusion

$$
|\mathbf{Y}(t)-\mathbf{Y}_0| \le \alpha \ \exp\left[\int_0^t \left(w(\tau)+\int_0^\tau v(s,\tau)ds\right)d\tau\right] := B \tag{2.69}
$$

where we denote with B the following constant:

$$
\begin{aligned}
B &= B\left(n,\alpha,L_F,M,C_\Gamma,\lambda_0^*,N,T\right) := \\
&= \alpha \ \exp\left[\left(1+nL_F\sqrt{2}\right)T + 2\sqrt{T}nL_FK_1 + \frac{2}{3}nL_FK_1T^{3/2}\right] \\
&= \alpha \ \exp\left[\left(1+nL_F\sqrt{2}\right)T + 2\sqrt{T}nL_FC_\Gamma M\frac{2^N\pi^{\frac{N}{2}}}{(\lambda_0^*)^{\frac{N}{2}}}\left(1+\frac{T}{3}\right)\right].
\end{aligned}
\tag{2.70}
$$

Step 2:

In order to prove the existence of a limit for $|\mathbf{Y}(t)|$, as t goes to T^-, we consider $\{t_n\}$ a monotonic increasing sequence, with $\lim_{t\to T}\{t_n\}$, and show

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that $\mathbf{Y}(t_n)$ is a Cauchy sequence.
Let $0 < t_m < t < t_n < T$. From (2.69) follows, in particular, that $|\mathbf{X}(t)| \leq B + |\mathbf{X}_0|$, $|\mathbf{V}(t)| \leq B + |\mathbf{V}_0|$, for any $t \in [0, \overline{T}]$. This allow us to retrace the computation of Step 1, estimating $|\mathbf{X}_0|$, $|\mathbf{V}_0|$ with $2B$, and obtaining that

$$\begin{aligned} |\mathbf{Y}(t) - \mathbf{Y}(t_m)| \leq A(t - t_m) + \int_{t_m}^{t} w(\tau)\, |\mathbf{Y}(\tau) - \mathbf{Y}(t_m)|\, d\tau \\ + \int_{t_m}^{t} \int_{t_m}^{\tau} v(s,\tau)\, |\mathbf{Y}(s) - \mathbf{Y}(t_m)|\, ds d\tau, \end{aligned} \tag{2.71}$$

where

$$\begin{aligned} A(t) \quad &= A\left(B, L_F, N, n, t\right) := 2Bt + n\left(C_0 + L_F K_1 K_2\right) t \\ &+ nL_F K_1 \left(1 + 2B\right) 2\sqrt{t} + nL_F K_1 \left(1 + 4B\right) \tfrac{4}{3}\sqrt{t^3}. \end{aligned} \tag{2.72}$$

By Lemma 7 and (2.69) we get that

$$\begin{aligned} |\mathbf{Y}(t_n) - \mathbf{Y}(t_m)| \quad &\leq A(t_n - t_m) \exp\left[\int_{t_m}^{t_n} \left(w(\tau) + \int_{t_m}^{\tau} v(s,\tau) ds\right) d\tau\right] \\ &= A(t_n - t_m)\, \frac{B}{\alpha}. \end{aligned} \tag{2.73}$$

From (2.72) we observe that $A(t_n - t_m) \to 0$ as $n, m \to \infty$.
Thus $\{\mathbf{Y}(t_n)\}$ is a Cauchy sequence, and admits a limit valute as $t \to T^-$.
□

In the following we present the proof of the previous Lemma 6 and 7.

Proof of Lemma 6.

From (3.5) we get the following expression

$$\begin{aligned} \nabla f\left(x, t; \mathbf{X}\right) \quad &= \int_{\mathbb{R}^N} \nabla\Gamma\left(x, t; \xi, 0\right) \varphi(\xi) d\xi - \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma\left(x, t; \xi, \tau\right) g_{\mathbf{X}}(\xi, \tau) d\xi d\tau \\ &= \widetilde{G_1} - \widetilde{G_2}, \end{aligned} \tag{2.74}$$

where

$$\widetilde{G_1} = \int_{\mathbb{R}^N} \nabla\Gamma\left(x, t; \xi, 0\right) \varphi(\xi) d\xi, \qquad \widetilde{G_2} = \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma\left(x, t; \xi, \tau\right) g_{\mathbf{X}}(\xi, \tau) d\xi d\tau. \tag{2.75}$$

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We now focus on $\widetilde{G_1}$. In particular, from (2.11) and (2.57) we obtain

$$\begin{aligned}
&\left|\widetilde{G_1}\right| \leq \int_{\mathbb{R}^N} C_\Gamma \frac{1}{t^{\frac{N+1}{2}}} e^{-\frac{\lambda_0^*}{4}\frac{|\xi-x|^2}{t}} M(1+|\xi|)d\xi \leq \\
&\leq C_\Gamma M \frac{1}{\sqrt{t}} \int_{\mathbb{R}^N} e^{-\frac{\lambda_0^*}{4}|u|^2} du\,(1+|x|) + C_\Gamma M \int_{\mathbb{R}^N} e^{-\frac{\lambda_0^*}{4}|u|^2}\,|u|\,du \qquad (2.76)\\
&= C_\Gamma M \frac{1}{\sqrt{t}} I_0\left(\frac{\lambda_0^*}{4}\right)(1+|x|) + C_\Gamma M I_1\left(\frac{\lambda_0^*}{4}\right),
\end{aligned}$$

where, in the last inequality, $I_0(.)$, $I_1(.)$ are defined in (6.12), (6.13). Replacing $I_0\left(\frac{\lambda_0^*}{4}\right)$ and $I_1\left(\frac{\lambda_0^*}{4}\right)$ with their exact value (see Appendix for computation), we obtain the following estimate for $\widetilde{G_1}$:

$$\begin{aligned}
&\left|\widetilde{G_1}\right| \leq C_\Gamma M \left(\frac{2^N \pi^{\frac{N}{2}}}{(\lambda_0^*)^{\frac{N}{2}}}\frac{(1+|x|)}{\sqrt{t}} + \left(\frac{4}{\lambda_0^*}\right)^{\frac{N+1}{2}} \frac{\omega_N}{2}\frac{2\pi^{\frac{N+1}{2}}}{\omega_{N+1}}\right)\\
&= C_\Gamma M \left(\frac{2^N \pi^{\frac{N}{2}}}{(\lambda_0^*)^{\frac{N}{2}}}\frac{(1+|x|)}{\sqrt{t}} + \frac{2^N 2}{(\lambda_0^*)^{\frac{N}{2}}\sqrt{\lambda_0^*}}\frac{\omega_N \pi^{\frac{N}{2}}\pi}{\omega_{N+1}}\right)\\
&= C_\Gamma M \frac{2^N \pi^{\frac{N}{2}}}{(\lambda_0^*)^{\frac{N}{2}}}\left(\frac{(1+|x|)}{\sqrt{t}} + \frac{2}{\sqrt{\lambda_0^*}}\frac{\omega_N \pi}{\omega_{N+1}}\right) = K_1\left(\frac{(1+|x|)}{\sqrt{t}} + \widetilde{K}_2\right),
\end{aligned} \qquad (2.77)$$

where $K_1 := C_\Gamma M \frac{2^N \pi^{\frac{N}{2}}}{(\lambda_0^*)^{\frac{N}{2}}}$, and $\widetilde{K}_2 := \frac{2}{\sqrt{\lambda_0^*}}\frac{\omega_N \pi}{\omega_{N+1}}$. As already noted in the proof of Proposition 1, the estimate of $\widetilde{G_2}$ follows by similar computation. For the sake of simplicity, we omit details and state that

$$\left|\widetilde{G_2}\right| \leq K_1\left(\int_0^t \frac{1+|x|+|\mathbf{X}(\tau)|}{\sqrt{t-\tau}}d\tau + \widetilde{K}_2 T\right). \qquad (2.78)$$

Thesis follows from (2.77) and (2.78), denoting with K_2 the constant $\widetilde{K}_2 + \widetilde{K}_2 T$.

□

Proof of Lemma 7.
Let us define $\widetilde{h}(t) := \sup_{0\leq s\leq t} h(s)$. Clearly, $\widetilde{h}(t) \geq h(t)$, for any $t \in [0,T]$. For any fixed $\widehat{t} \leq T$, inequality (2.67) implies that

$$h(t) \leq \alpha + \int_0^{\widehat{t}} w(\tau)\widetilde{h}(\tau)d\tau + \int_0^t \widetilde{h}(\tau)\int_0^\tau v(s,\tau)dsd\tau \qquad (2.79)$$

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for any $t \leq \widehat{t}$. Hence, in particular,

$$\widetilde{h}\left(\widehat{t}\right) \leq \alpha + \int_0^{\widehat{t}} w(\tau)\widetilde{h}(\tau)d\tau + \int_0^{\widehat{t}} \widetilde{h}(\tau) \int_0^{\tau} v(s,\tau)dsd\tau. \tag{2.80}$$

We conclude that

$$\widetilde{h}\left(t\right) \leq \alpha + \int_0^{t} \left(w(\tau) + \int_0^{\tau} v(s,\tau)ds\right) \widetilde{h}(\tau)d\tau \quad \forall t \leq T. \tag{2.81}$$

Applying Gronwall inequality to (2.81), we obtain

$$h(t) \leq \widetilde{h}(t) \leq \alpha \ \exp\left[\int_0^{t} \left(w(\tau) + \int_0^{\tau} v(s,\tau)ds\right) d\tau\right] \quad \forall t \leq T. \tag{2.82}$$

This concludes the proof.

□

Under the same assumptions for the global existence of solutions to system (2.1)-(2.2), we state an analogous result for system (2.47)-(2.2), previously introduced as a modification of (2.1) and already investigated for a local result in Theorem 4.

Theorem 8. *Let* $\mathbf{Y} = \mathbf{Y}(t)$ *local solution to (2.47)-(2.2) in* $[0, \widetilde{T}]$, *with* $\widetilde{T} \leq T$.
Under assumptions H6)-H7), $\mathbf{Y}$ *is a global solution.*

Proof Theorem 8
We present only a sketch of the proof, which follows the same line of reasoning of proof of Theorem 5. In particular, we point out the steps in which the replacement of the gradient term in (2.1) with the average over a ball involves modifications. Integrating $(2.47)_2$, we obtain that

$$\begin{aligned}
|\mathbf{v}_i(t) - \mathbf{v}_{0i}| &\leq \int_0^{t} \left| F_i\left(\tau, \mathbf{X}(\tau), \mathbf{V}(\tau), \fint_{B_\delta(\mathbf{x}_i(t))} \nabla f\left(\xi, t; \mathbf{X}\right) d\xi\right)\right| d\tau \\
&\leq tC_0 + L_F \int_0^{t} |\mathbf{X}(\tau) - \mathbf{X}_0|\, d\tau \\
&+ L_F \int_0^{t} |\mathbf{V}(\tau) - \mathbf{V}_0|\, d\tau + L_F \int_0^{t} \left| \fint_{B_\delta(\mathbf{x}_i(t))} \nabla f\left(\xi, t; \mathbf{X}\right) d\xi \right| d\tau,
\end{aligned} \tag{2.83}$$

The above Lemma 6, recalling that $\xi \in B_\delta(\mathbf{x}_i(t))$, leads to

$$|\nabla f\left(\xi, \tau; \mathbf{X}\right)| \leq K_1 \left(\frac{1 + \delta + |\mathbf{X}(\tau)|}{\sqrt{\tau}} + K_2 + \int_0^{\tau} \left(\frac{1 + \delta + |\mathbf{X}(s)| + |\mathbf{X}(\tau)|}{\sqrt{\tau - s}}\right) ds\right). \tag{2.84}$$

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Hence we get

$$
\begin{aligned}
|\mathbf{V}(t)-\mathbf{V}_0| &\leq nTC_0 + nL_F \int_0^t |\mathbf{X}(\tau)-\mathbf{X}_0|\, d\tau \\
&+nL_F \int_0^t |\mathbf{V}(\tau)-\mathbf{V}_0|\, d\tau + nL_F K_1 \left(\int_0^t \frac{1+\delta+|\mathbf{X}(\tau)|}{\sqrt{\tau}} d\tau + K_2 T \right) \\
&+nL_F K_1 \left(\int_0^t \int_0^\tau \frac{1+\delta+|\mathbf{X}(\tau)|+|\mathbf{X}(s)|}{\sqrt{\tau-s}} ds d\tau \right).
\end{aligned}
\tag{2.85}
$$

We note that the previous upper-bound for $|\mathbf{V}(t)-\mathbf{V}_0|$ can be obtained from (2.63), simply replacing $|\mathbf{X}(\tau)|$ with $|\mathbf{X}(\tau)|+\delta$. Hence the proof of Theorem 5 can be repeated for system (2.47), observing that $1+|\mathbf{X_0}|$ will be replaced by $1+\delta+|\mathbf{X_0}|$, and $1+2\,|\mathbf{X_0}|$ by $1+\delta+2\,|\mathbf{X_0}|$, in (2.65) and thus in the definition of constant α. We observe that modifications occur only in Step 1 of the proof, whereas Step 2 follows as for Theorem 5. □

2.5 Discussion

In this chapter we presented a first step toward a theoretic characterization of emergent coupled systems of differential equations. These kind of systems are widely involved in modelling of collective motions of agents interacting with each other and with the surrounding. The coexistence of distinct mathematical formalisms represents the main feature of *hybrid* approaches, in which the dynamics of the agents are modeled by second-order ODEs, while reaction-diffusion equations are used to model the evolution in time of a signal influencing them. We proved existence and uniqueness results for the solution, assuming continuous source term and initial data in the parabolic equation. In the next chapter, we extend the obtained result to the case of a discontinuous source term, in line with the modeling choice in [3] and [2].

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Chapter 3

Existence results for hybrid systems under exogenous information with discontinuous source term

In Chapter 2 we presented analytical results concerning the existence and uniqueness of solution for systems of differential equations in the form of (2.1)-(2.2). In particular, the source term in the parabolic diffusion equation is assumed to be modeled by a continuous function g. The present literature exhibits models with discontinuous source term, in order to differentiate regions from which a signal arises from the others. For that reason, in the following chapter we investigate (2.1)-(2.2) focusing on the case of a source term with less regularity properties, covering the structure of models in [2] and [3].

3.1 Problem statement

For the sake of simplicity, we here recall the structure of the investigated system and the working assumptions we will consider in this chapter. The dynamics of n agents is modeled by system (2.1), which reads

$$\begin{cases} \dot{\mathbf{x}}_i(t) &= \mathbf{v}_i(t), & 0 < t < T, \\ \dot{\mathbf{v}}_i(t) &= F_i\left(t, \mathbf{X}(t), \mathbf{V}(t), \nabla f\left(\mathbf{x}_i(t), t; \mathbf{X}\right)\right), & i = 1, ..., n, \end{cases}$$

with initial data $\mathbf{x}_i(0) = \mathbf{x}_{0i}, \mathbf{v}_i(0) = \mathbf{v}_{0i} \in \mathbb{R}^N$, for any $i = 1, ..., n$. Here

$$F_i : [0, T] \times \mathbb{R}^{N \times n} \times \mathbb{R}^{N \times n} \times \mathbb{R}^N \to \mathbb{R}^N,$$

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satisfies H6), and $f = f(x,t;\mathbf{X})$ solves the Cauchy problem (3.1)

$$\begin{cases} Lf(x,t;\mathbf{X}) &= g(x,\mathbf{X}(t)), & (x,t) \in \mathbb{R}^N \times (0,T), \\ f(x,0;\mathbf{X}) &= 0, & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where

H8) L is the uniformly parabolic-type operator in (2.3) satisfying H2) and H3). Moreover, we assume that the coefficients $a_{i,j}$ assume weak derivative with respect to x in $L^1_{loc}(\Omega)$.

The functions F_i, $i = 1,\dots,n$, specify inter-individual interactions between the agents, whereas the function f describes the information which affects the behavior of agents. In particular, the source term in equation (2.2), given by the (possibly discontinuous) function g, accounts for both exogenous features coming from the surrounding environment (thanks to the dependence on x) and self-production of information related to the agents, through the state $\mathbf{X}(t)$.

As in the previous chapter, we shall use the notation $f(x,t;\mathbf{X})$ in order to highlight the fact that, for every $t \in [0,T]$, f depends on the previous position of agents $\mathbf{X}(s)$, for $s \in [0,t]$, and we assume $f(x,0;\mathbf{X}) \equiv 0$ in order to fit the scenario of [2] and [3]. Possibile extension to more general initial data φ follows simply replacing (3.5) with (2.9).

Let us focus on the structure of the source term $g : \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$. We consider g a measurable real-valued locally bounded function on $\mathbb{R}^n \times \mathbb{R}^{N\times n}$, satisfying the following assumptions:

G1) $|g(x,\mathbf{X})| \le M\left(1 + |x| + |\mathbf{X}|\right)$,
for any $x \in \mathbb{R}^N$, $\mathbf{X} \in \mathbb{R}^{N\times n}$, for some constant $M \ge 0$.

G2) For every $K \subset \mathbb{R}^N$ compact, and $\psi \in C\left(\mathbb{R}^N\right)$ satisfying $\int_{\mathbb{R}^N} |\psi(x) g(x,\mathbf{X})|\, dx < \infty$, for any $\mathbf{X} \in K^n$, there exists $K' \supset K$, compact, such that

$$\left| \int_{\mathbb{R}^N} \psi(x) g(x,\mathbf{X}_1) - \psi(x) g(x,\mathbf{X}_2)\, dx \right| \le |\mathbf{X}_1 - \mathbf{X}_2| \vee \nu_\psi^{K'}\left(|\mathbf{X}_1 - \mathbf{X}_2|\right), \tag{3.2}$$

for any $\mathbf{X}_1, \mathbf{X}_2 \in K^n$, where $\nu_\psi^{K'}$ denotes the modulus of continuity of the function ψ on K'.

As already stated in the introduction, the case of g discontinuous is of interest for this kind of model. In fact, both in [2] and [3], we find that g is defined as the sum of characteristic functions over circular domains modeling the cells, that is $g(x,\mathbf{X}(t)) = \sum_{i=1}^{n} \chi_{B_\delta(\mathbf{x}_i(t))}(x)$, where $\delta > 0$ denotes

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their radius and $\mathbf{x}_i(t)$ the center coordinates at time t . It is immediate to prove that G1) is satisfied by that choice of g.
We show that G2) holds true for such a case. It suffices to verify the condition for $g(x,\mathbf{X}) = \chi_{B_\delta(\mathbf{x}_j)}(x)$, for some fixed $j \in \{1,...,n\}$. Namely, for every $K \subset \mathbb{R}^N$ compact, let ψ as in G2), and $K' = K + B_\delta$. Denoting with $|B_\delta|$ the measure of $B_\delta \subset \mathbb{R}^N$, it holds:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \psi(x)\chi_{B_\delta(\mathbf{x}_j(t))}(x) - \psi(x)\chi_{B_\delta(\overline{\mathbf{x}}_j(t))}(x)\,dx\right| \\
&\leq \left|\int_{\mathbb{R}^N} \psi(\mathbf{x}_j+\mathbf{z})\chi_{B_\delta}(\mathbf{z}) - \psi(\overline{\mathbf{x}}_j+\mathbf{z})\chi_{B_\delta}(\mathbf{z})\,dz\right| \qquad (3.3)\\
&\leq |B_\delta|\,\nu_\psi^{K'}(|\mathbf{x}_j-\overline{\mathbf{x}}_j|) \leq |B_\delta|\,\nu_\psi^{K'}\left(\left|\mathbf{X}-\overline{\mathbf{X}}\right|\right) \vee \left|\mathbf{X}-\overline{\mathbf{X}}\right|,
\end{aligned}$$

for any $\mathbf{X},\overline{\mathbf{X}} \in K^n$. Finally we also observe that, if ψ is a Lipschitz continuous function with Lipschitz constant L_ψ, (3.2) clearly reduces to

$$\left|\int_{\mathbb{R}^N} \psi(x)g(x,\mathbf{X}_1) - \psi(x)g(x,\mathbf{X}_2)\,dx\right| \leq L_\psi\,|\mathbf{X}_1-\mathbf{X}_2|. \qquad (3.4)$$

Hence we stress that our working assumptions cover the particular modeling choice of function g presented in [2] and [3].

3.2 Strong solutions for approximating problems

In Chapter 2 we have obtained existence and uniqueness results for the solution of the system in the form (2.1)-(2.2) where g is a continuous function, satisfying H5).

Under assumptions H5),H6),H8),we have proved that system (2.1)-(2.2) admits a unique solution $\mathbf{Y} = (\mathbf{X},\mathbf{V})$, locally in time. In particular, $\mathbf{X},\mathbf{V} \in C\left([0,\overline{T}];\ \mathbb{R}^{N\times n}\right) \cap C^1\left((0,\overline{T});\mathbb{R}^{N\times n}\right)$, with $\overline{T} \leq T$, and $f(\cdot,\cdot;\mathbf{X})$ $\in C^{2,1}\left(\mathbb{R}^N \times (0,\overline{T})\right)$ is the unique classical solution of (2.2) .

As stated in [58], Theorem 12, $f(\cdot,\cdot;\mathbf{X})$ can be expressed in terms of the fundamental solution Γ associated to $Lu = 0$, precisely:

$$f(x,t;\mathbf{X}) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,g_{\mathbf{X}}(\xi,\tau)d\xi d\tau. \qquad (3.5)$$

Moreover, if g satisfies the growth G1) the solution $\mathbf{Y}$ is proved to be global in time. In particular, we refer the reader to Theorem 7 (p. 188) in [62] for the details. Therefore, the existence and uniqueness results obtained for the continuous case, ensure the well-posedness of the following approximating problems:

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Let us consider $\{\varepsilon_k\}_{k\in\mathbb{N}}$, with $\varepsilon_k \in (0,1)$, an infinitesimal sequence as $k \to \infty$, let $\{g^k\}_{k\in\mathbb{N}}$ denotes a x-mollifier sequence approximating the discontinuous function g, namely

$$g^k(x,\mathbf{X}) := \varphi_{\varepsilon_k} * g(\cdot,\mathbf{X}) = \int_{\mathbb{R}^N} \varphi_{\varepsilon_k}(x-\xi)\, g(\xi,\mathbf{X})\, d\xi,$$

for any $x \in \mathbb{R}^N$, $\mathbf{X} \in \mathbb{R}^{N\times n}$, where $\{\varphi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is the usual mollifier sequence: $\varphi_{\varepsilon_k}(x) = \frac{1}{\varepsilon_k^N}\varphi\left(\frac{x}{\varepsilon_k}\right)$, with

$$\varphi(x) := \begin{cases} c\exp\left(-\dfrac{1}{1-|x|^2}\right) & |x|<1, \\ \\ 0 & |x|\geq 1, \end{cases} \tag{3.6}$$

and c chosen in order to satisfy $\int_{\mathbb{R}^N}\varphi(x)dx = 1$. In order to simplify the notation, in the following we omit the subscript k when writing ε_k and φ_{ε_k}. We observe that g^k satisfies A4) and G1) for any $k \in \mathbb{N}$. In fact, applying G1) to the function g, we obtain the inequality

$$\begin{aligned} \left|g^k(x,\mathbf{X})\right| \leq M\Bigg[&\int_{\mathbb{R}^N}\varphi_{\varepsilon_k}(x-\xi)\,d\xi + |\mathbf{X}|\int_{\mathbb{R}^N}\varphi_{\varepsilon_k}(x-\xi)\,d\xi + (1+|x|) \\ &\int_{\mathbb{R}^N}\varphi_{\varepsilon_k}(x-\xi)\,d\xi\Bigg] \leq 2M(1+|x|+|\mathbf{X}|), \end{aligned} \tag{3.7}$$

for any $x \in \mathbb{R}^N$, $\mathbf{X} \in \mathbb{R}^{N\times n}$. Moreover, since $g^k(\cdot,\mathbf{X})$ belongs to $C^\infty(\mathbb{R}^N)$ for any $\mathbf{X} \in \mathbb{R}^{N\times n}$, by assumption G2) and (3.4) follows that g^k satisfies A4). Therefore from Theorem 4 (p. 184) in [62], for each k, there exists a unique global solution $\mathbf{Y}^k = (\mathbf{X}^k, \mathbf{V}^k)$ of

$$\begin{cases} \dot{\mathbf{x}}_i^k(t) &= \mathbf{v}_i^k(t), \quad 0<t<T, \\ \\ \dot{\mathbf{v}}_i^k(t) &= \mathbf{F}_i\left(t,\mathbf{X}^k(t),\mathbf{V}^k(t),\nabla f^k\left(\mathbf{x}_i^k(t),t;\mathbf{X}^k\right)\right), \quad \forall\, i=1,...,n, \end{cases} \tag{3.8}$$

with $\mathbf{x}_i^k(0) = \mathbf{x}_{0i}$, $\mathbf{v}_i^k(0) = \mathbf{v}_{0i}$, and f^k is the unique classical solution to $Lf^k = g^k_{\mathbf{X}^k}$, on $\mathbb{R}^N \times (0,T]$, with $f^k(x,0;\mathbf{X}^k) \equiv 0$. Moreover f^k can be also expressed as in (3.5). For the sake of convenience, we recall here the following estimate, proved in [62] (Lemma 5, p. 184), that will be used in

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the following sections:

$$\left|\nabla f^k\left(x,t;\mathbf{X}^k\right)\right| \leq K_1\left(\frac{1+|x|}{\sqrt{t}}+K_2+\int_0^t\left(\frac{1+|x|+|\mathbf{X}^k(\tau)|}{\sqrt{t-\tau}}\right)d\tau\right), \tag{3.9}$$

for all $x\in\mathbb{R}^N$, $t\in(0,T]$, where K_1, $K_2>0$ are constants independent of k.

3.3 Global existence of weak solutions

Classical results of the literature , see for instance [58], [63], [64], do not cover the considered setting for the parabolic equation. In this section, using the results concerning the approximating problems in (3.8), we prove our main contribution, namely the existence of a weak solution for system (2.1)-(2.2). We preliminary clarify the notion of weak solution to (2.2), which is suitable for the particular coupling of the investigated system, in presence of a possibly discontinuous source term.

Let $\mathbf{X}:[0,T]\to\mathbb{R}^{N\times n}$, we recall that $f(\cdot,\cdot;\mathbf{X})$ denotes the solution of (2.2) associated to $\mathbf{X}$, meaning that the source term corresponds to $g_{\mathbf{X}}$.

Definition 9. *For every $\mathbf{X}\in C\left([0,T];\mathbb{R}^{N\times n}\right)$, $f(\cdot,\cdot;\mathbf{X})$ solves the Cauchy problem (2.2) if and only if*

i) $f(\cdot,\cdot;\mathbf{X})$ is continuous;

ii) for every $t\in[0,T]$, $f(\cdot,t;\mathbf{X})\in C^1\left(\mathbb{R}^N\right)$;

iii) $\nabla f(\cdot,\cdot;\mathbf{X})\in C\left(\mathbb{R}^N\times[0,T]\right)$;

iv) for every $\rho\in C_0^1(\Omega)$, satisfies the equation

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} f(\xi,\tau;\mathbf{X})\partial_t\rho(\xi,\tau)d\xi d\tau+\int_0^T\int_{\mathbb{R}^N} c(\xi,\tau)\rho(\xi,\tau)d\xi d\tau\\ &+\int_0^T\int_{\mathbb{R}^N}\sum_i b_i(\xi,\tau)\partial_i f(\xi,\tau;\mathbf{X})\rho(\xi,\tau)d\xi d\tau\\ &-\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}\partial_i\left(a_{ij}\rho\right)(\xi,\tau)\partial_j f(\xi,\tau;\mathbf{X})d\xi d\tau=\int_0^T\int_{\mathbb{R}^N} g_{\mathbf{X}}(\xi,\tau)\rho(\xi,\tau)d\xi d\tau.\end{aligned} \tag{3.10}$$

In the following Definition 10, we specify the concept of solution to system (2.1)-(2.2), taking into account the fact that the source term $g_{\mathbf{X}}$ may be discontinuous as in the reference model considered in [2] and [3].

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Definition 10. *For every* $\mathbf{Y}_0 = (\mathbf{X}_0, \mathbf{V}_0) \in \mathbb{R}^{2(N\times n)}$, *by a solution to problem (2.1)-(2.2) we mean a couple* $\mathbf{Y} = (\mathbf{X}, \mathbf{V}) \in C^1\left([0,T]; \mathbb{R}^{2(N\times n)}\right)$ *such that* $\mathbf{Y}(0) = \mathbf{Y}_0$, $\mathbf{Y}$ *is a solution of (2.1) and* $f(\cdot,\cdot;\mathbf{X})$ *solves (2.2) in the sense of Definition 9.*

We can now formulate our main result as follow:

Theorem 11. *Under Assumptions H6),H8),G1),G2), system (2.1)-(2.2) admits a solution* $\mathbf{Y} = (\mathbf{X}, \mathbf{V})$, *where the function* $f(\cdot,\cdot;\mathbf{X})$ *is given by*

$$f(x,t;\mathbf{X}) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\, g_{\mathbf{X}}(\xi,\tau) d\xi d\tau, \tag{3.11}$$

satisfying $|f(x,t;\mathbf{X})| \leq L_T\left(1 + |x| + ||\mathbf{X}||_{\infty,T}\right)$, *and* $|\nabla f(x,t;\mathbf{X})| \leq L_T(1 + |x| + ||\mathbf{X}||_{\infty,T})$, *for any* $x \in \mathbb{R}^N$, $t \in [0,T]$, *for some constant* $L_T \geq 0$.

In order to prove our result, we introduce the following set: for every $R > 0$, $\delta > 0$, we define

$$\begin{aligned}\Omega_{\delta,R} := \quad & \{(x_1, x_2, t, \mathbf{X}_1, \mathbf{X}_2) : |x_1|, |x_2| < R, \mathbf{X}_1, \mathbf{X}_2 \in C([0,T]; B_R), \\ & t \in (0,T], |x_1 - x_2| < \delta, ||\mathbf{X}_1 - \mathbf{X}_2||_{\infty,T} < \delta\}.\end{aligned} \tag{3.12}$$

Moreover, we shall use the following technical Lemma 12, concerning the behavior of

$$\Sigma^\varepsilon := \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma(x_1,t;\xi,\tau)\, \varphi_\varepsilon * g(\xi, \mathbf{X}_1(\tau)) - \nabla\Gamma(x_2,t;\xi,\tau)\, g(\xi,\mathbf{X}_2(\tau))\, d\xi d\tau, \tag{3.13}$$

with $\varepsilon \in (0,1)$, $(x_1, x_2, t, \mathbf{X}_1, \mathbf{X}_2) \in \Omega_{\delta,R}$. We rewrite Σ^ε as $\Sigma^\varepsilon = \Sigma_1^\varepsilon + \Sigma_2^\varepsilon + \Sigma_3$, where:

$$\begin{aligned}\Sigma_1^\varepsilon := \Sigma_1^\varepsilon(x_1,t,\mathbf{X}_1,\mathbf{X}_2) := \quad & \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma(x_1,t;\xi,\tau)\, \varphi_\varepsilon * g(\xi,\mathbf{X}_1(\tau)) \quad d\xi d\tau + \\ & - \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma(x_1,t;\xi,\tau)\, \varphi_\varepsilon * g(\xi,\mathbf{X}_2(\tau)) \quad d\xi d\tau,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Sigma_2^\varepsilon := \Sigma_2^\varepsilon(x_1,t,\mathbf{X}_2) := \quad & \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma(x_1,t;\xi,\tau)\, \varphi_\varepsilon * g(\xi,\mathbf{X}_2(\tau)) \quad d\xi d\tau + \\ & - \int_0^t \int_{\mathbb{R}^N} \nabla\Gamma(x_1,t;\xi,\tau)\, g(\xi,\mathbf{X}_2(\tau))\, d\xi d\tau,\end{aligned} \tag{3.15}$$

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$$\Sigma_3 := \Sigma_3(x_1, x_2, t, \mathbf{X}_2) := \int_0^t \int_{\mathbb{R}^N} [\nabla\Gamma(x_1, t; \xi, \tau) - \nabla\Gamma(x_2, t; \xi, \tau)] \, g(\xi, \mathbf{X}_2(\tau)) \, d\xi d\tau. \tag{3.16}$$

Lemma 12. *Under assumptions H8) and G1)-G2), for every $R > 0$, it holds:*

i) $\sup_{\substack{\Omega_{\delta,R} \\ \varepsilon\in(0,1)}} |\Sigma_1^\varepsilon(x_1, t, \mathbf{X_1}, \mathbf{X_2})| \to 0$ *as* $\delta \to 0$.

ii) $\sup |\Sigma_2^\varepsilon(x_1, t, \mathbf{X_2})| < \infty$, *where the supremum is taken for* $\varepsilon \in (0,1)$, $t \in [0,T]$, $|x_1| \le R$, $||\mathbf{X}_2||_{\infty,T} \le R$ *and, for every* $\mathbf{X_2} \in C\left([0,T]; \mathbf{R}^{N\times n}\right)$, $t \in (0,T]$, $\lim_{\varepsilon\to 0} \sup_{|x_1|\le R} \Sigma_2^\varepsilon(x_1, t, \mathbf{X_2}) = 0$.

iii)

$$\sup_{\Omega_{\delta,R}} |\Sigma_3(x_1, x_2, t, \mathbf{X_2})| \le C_3 \delta^{\frac{\alpha}{2}}, \tag{3.17}$$

where $C_3 > 0$ depends on $M, C_\Gamma, \alpha, T, \lambda_0^*, R$.

Proof. Lemma 12

Let $\delta, R > 0$, $(x_1, x_2, t, \mathbf{X}_1, \mathbf{X}_2) \in \Omega_{\delta,R}$.
i) We observe that Σ_1^ε can be rewritten as

$$\int_0^t \int_{\mathbb{R}^N} [g(y, \mathbf{X}_1(\tau)) - g(y, \mathbf{X}_2(\tau))] \int_{\mathbb{R}^N} \nabla\Gamma(x_1, t; \xi, \tau) \, \varphi_{\varepsilon_k}(\xi - y) \, d\xi dy d\tau$$

$$= \int_0^t \int_{\mathbb{R}^N} [g(y, \mathbf{X}_1(\tau)) - g(y, \mathbf{X}_2(\tau))] \, dy \, (\nabla\Gamma)_\varepsilon(x_1, t; y, \tau) \, d\tau,$$

where $(\nabla\Gamma)_\varepsilon$ denotes the convolution of $\nabla\Gamma(x_1, t; \cdot, \tau)$ with φ_ε as a function of ξ. Let $c \in (0,T)$. For the properties of Γ, $\nabla\Gamma$ is uniformly continuous in the compact set

$$K_{c,R} := \{(x, t; \xi, \tau) \,|\, |x| \le R, |\xi| \le R+1, t - \tau \ge c, t, \tau \in [0,T]\}.$$

In the following we shall denote with $\nu_\Gamma^{R,c}$ the modulus of continuity of $\nabla\Gamma$ over $K_{c,R}$.

In order to study the behavior of the mollifier $(\nabla\Gamma)_\varepsilon$ with respect to y variable, we observe that, for any $\varepsilon < 1$, since $\int_{\mathbb{R}^N} \varphi_\varepsilon(\xi) \, d\xi = 1$, it holds:

$$|(\nabla\Gamma)_\varepsilon(x, t; y_1, \tau) - (\nabla\Gamma)_\varepsilon(x, t; y_2, \tau)| \le \nu_\Gamma^{R,c}(|y_1 - y_2|), \tag{3.18}$$

for any $x, y_{1,2} \in B_R$, $t - \tau \ge c$. Then we decompose Σ_1^ε as $\Sigma_{1a}^\varepsilon + \Sigma_{1b}^\varepsilon$, where

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$$\Sigma_{1_a}^{\varepsilon} := \int_0^{(t-c)\vee 0} \int_{\mathbb{R}^N} \left[g\left(y, \mathbf{X}_1\left(\tau\right)\right) - g\left(y, \mathbf{X}_2\left(\tau\right)\right)\right] \left(\nabla\Gamma\right)_{\varepsilon}\left(x_1, t; y, \tau\right) dy \quad d\tau,$$

$$\Sigma_{1_b}^{\varepsilon} := \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} \left[g\left(y, \mathbf{X}_1\left(\tau\right)\right) - g\left(y, \mathbf{X}_2\left(\tau\right)\right)\right] \left(\nabla\Gamma\right)_{\varepsilon}\left(x_1, t; y, \tau\right) dy \quad d\tau.$$

Firstly,

$$|\Sigma_{1_a}^{\varepsilon}| \leq \int_0^{(t-c)\vee 0} \sum_{j=1}^{n} \left| \int_{\mathbb{R}^N} \left[g\left(y, \mathbf{X}_1\left(\tau\right)\right) - g\left(y, \mathbf{X}_2\left(\tau\right)\right)\right] \left(\partial_j\Gamma\right)_{\varepsilon}\left(x_1, t; y, \tau\right) dy \right| d\tau.$$

From (3.18) and G2) with $\psi(\xi) = \left(\partial_j\Gamma\right)_{\varepsilon}\left(x_1, t; \xi, \tau\right)$, $\xi \in \mathbb{R}^N$ there exists $R' > R$, dependent only on R and c, such that

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \left[g\left(y, \mathbf{X}_1\left(\tau\right)\right) - g\left(y, \mathbf{X}_2\left(\tau\right)\right)\right] \left(\partial_j\Gamma\right)_{\varepsilon}\left(x_1, t; y, \tau\right) dy \right| \\ &\leq \nu_{\Gamma}^{R',c}\left(|\mathbf{X}_1(\tau) - \mathbf{X}_2(\tau)|\right) \vee |\mathbf{X}_1(\tau) - \mathbf{X}_2(\tau)|. \end{aligned} \tag{3.19}$$

Therefore

$$|\Sigma_{1_a}^{\varepsilon}| \leq \left(\nu_{\Gamma}^{R',c}\left(||\mathbf{X}_1 - \mathbf{X}_2||_{\infty,T}\right) \vee ||\mathbf{X}_1 - \mathbf{X}_2||_{\infty,T}\right)(t-c) \vee 0, \tag{3.20}$$

Let us consider $\Sigma_{1_b}^{\varepsilon}$. From (3.7), we obtain

$$\begin{aligned} |\Sigma_{1_b}^{\varepsilon}| \leq & \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} \left|\nabla\Gamma\left(x_1, t; \xi, \tau\right)\left(\varphi_{\varepsilon} * g\left(\xi, \mathbf{X}_1\left(\tau\right)\right) - \varphi_{\varepsilon} * g\left(\xi, \mathbf{X}_2\left(\tau\right)\right)\right)\right| d\xi d\tau \\ &\leq 4M \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} \left|\nabla\Gamma\left(x_1, t; \xi, \tau\right)\right| \left(1 + |\xi| + R\right) d\xi d\tau. \end{aligned} \tag{3.21}$$

Using (2.11), we compute an upper bound for the integral in (3.21):

$$\begin{aligned} &|\Sigma_{1_b}^{\varepsilon}| \leq 4M \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} C_{\Gamma} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N+1}{2}}} \left(1 + |\xi| + R\right) d\xi d\tau \\ &= 4MC_{\Gamma}\left(1+R\right) \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}|u|^2}}{\sqrt{t-\tau}} du d\tau \\ &+4MC_{\Gamma} \int_{(t-c)\vee 0}^{t} \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}|u|^2}}{\sqrt{t-\tau}} \left|u\sqrt{t-\tau} - \mathbf{x}_1\right| du d\tau \\ &\leq 4MC_{\Gamma}\left(1+2R\right) I_0\left(\frac{\lambda_0^*}{4}\right) \int_{(t-c)\vee 0}^{t} \frac{d\tau}{\sqrt{t-\tau}} + 4MC_{\Gamma} I_1\left(\frac{\lambda_0^*}{4}\right)\left(t - (t-c)\vee 0\right), \end{aligned} \tag{3.22}$$

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From (3.20), (3.22) we finally have

$$|\Sigma_1^\varepsilon| \leq N\left(\nu_\Gamma^{R',c}(\delta) \vee \delta\right) T\mathbb{I}_{t\geq c} + 8MC_\Gamma(1+2R) I_0\left(\tfrac{\lambda_0^*}{4}\right)\sqrt{c} + 4MC_\Gamma I_1\left(\tfrac{\lambda_0^*}{4}\right) c. \tag{3.23}$$

where $\mathbb{I}_{t\geq c}$ denotes the indicator function of $[c,T] \subset [0,T]$. Since c is arbitrary, we get the assertion $\lim\limits_{\delta\to 0} \sup\limits_{\substack{\Omega_{\delta,R}\\ \varepsilon\in(0,1)}} |\Sigma_1^\varepsilon(x_1,t,\mathbf{X}_1,\mathbf{X}_2)| = 0$.

ii) By (2.11), (3.7) and G1), with usual arguments we have

$$\begin{aligned}
|\Sigma_2^\varepsilon| \leq\ & C_\Gamma \int_0^t \int_{\mathbb{R}^N} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N+1}{2}}} |\varphi_\varepsilon * g(\xi,\mathbf{X}_2(\tau)) - g(\xi,\mathbf{X}_2(\tau))|\, d\xi d\tau \\
&\leq 3MC_\Gamma \int_0^t \int_{\mathbb{R}^N} C_\Gamma \frac{e^{-\frac{\lambda_0^*}{4}|u|^2}}{\sqrt{t-\tau}} \left(1+2R+|u|\sqrt{T}\right) du d\tau \\
&\leq 2\sqrt{T}C_\Gamma \left[3M(1+2R) I_0\left(\tfrac{\lambda_0^*}{4}\right) + 3M\sqrt{T} I_1\left(\tfrac{\lambda_0^*}{4}\right)\right].
\end{aligned} \tag{3.24}$$

Therefore, we deduce that Σ_2^ε is uniformly bounded. To prove the second part of the assertion, we estimate Σ_2^ε with the sum of two contributions, $|\Sigma_2^\varepsilon| \leq \Sigma_{2\nu} + \Sigma_{2\nu}^\varepsilon$, where, for a fixed $\nu > 2R$,

$$\begin{aligned}
\Sigma_{2\nu} &:= 3MC_\Gamma \int_0^t \int_{|\xi|>\nu} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N+1}{2}}} (1+|\xi|+|\mathbf{X}_2(\tau)|)\, d\xi d\tau, \\
\Sigma_{2\nu}^\varepsilon &:= C_\Gamma \int_0^t \int_{|\xi|<\nu} \frac{e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N+1}{2}}} |\varphi_\varepsilon * g(\xi,\mathbf{X}_2(\tau)) - g(\xi,\mathbf{X}_2(\tau))|\, d\xi d\tau.
\end{aligned} \tag{3.25}$$

Since $|x_1| \leq R < \dfrac{\nu}{2}$, and $||\mathbf{X}_2||_{\infty,T} \leq R$, we obtain

$$\begin{aligned}
|\Sigma_{2\nu}| \leq\ & \int_0^t \frac{3M du d\tau}{\sqrt{t-\tau}} \int_{|x_1+\sqrt{t-\tau}u|>\nu} e^{-\frac{\lambda_0^*}{4}|u|^2} \left(1+\left|x_1+\sqrt{t-\tau}u\right|+||\mathbf{X}_2||_{\infty,T}\right) \\
&\leq \int_0^t \frac{3M}{\sqrt{t-\tau}} \int_{|u|>\frac{\nu}{\sqrt{t-\tau}}-\frac{|x_1|}{\sqrt{t-\tau}}} e^{-\frac{\lambda_0^*}{4}|u|^2} \left(1+2R+\sqrt{T}|u|\right) du d\tau \\
&\leq \int_0^t \frac{3M}{\sqrt{t-\tau}} \int_{|u|>\frac{\nu}{2\sqrt{t-\tau}}} e^{-\frac{\lambda_0^*}{4}|u|^2} \left(1+2R+\sqrt{T}|u|\right) du d\tau.
\end{aligned} \tag{3.26}$$

We observe that, for any fixed $\tau < t$, the inner integral tends to zero as

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$\nu \to \infty$. Moreover

$$\begin{aligned} &\frac{3M}{\sqrt{t-\tau}} \int_{|u|>\frac{\nu}{2\sqrt{t-\tau}}} e^{-\frac{\lambda_0^*}{4}|u|^2}\left(1+2R+\sqrt{T}\,|u|\right) du d\tau \\ &\leq \frac{3M}{\sqrt{t-\tau}}\left((1+2R)\, I_0\left(\frac{\lambda_0^*}{4}\right)+\sqrt{T} I_1\left(\frac{\lambda_0^*}{4}\right)\right), \end{aligned} \tag{3.27}$$

which is an integrable function with respect to $\tau \in (0,t)$. From Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\nu\to\infty} \sup_{|x_1|\leq R} \Sigma_{2,\nu}\left(x_1,t,\mathbf{X}_2\right)=0. \tag{3.28}$$

Let now focus on $\Sigma_{2\nu}^{\varepsilon}$.
We recall that, for every $h \in L^{\infty}_{loc}\left(\mathbb{R}^N\right)$, $||\varphi_\varepsilon * h - h||_{L^q(B_R)}$ tends to zero for $\varepsilon \to 0$, for any $q \in [1,\infty)$, $R>0$ (see [61]). Thus, let $1<p<\dfrac{N}{N-1}$ be fixed, we define $\theta := \dfrac{N+1}{2}-\dfrac{N}{2p}<1$. Applying Hölder inequality, we get

$$\Sigma_{2,\nu}^{\varepsilon}=\int_0^t \frac{C_\Gamma}{(t-\tau)^{\frac{N+1}{2}}}\left[\int_{|\xi|<\nu} e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2 p}{t-\tau}} d\xi\right]^{\frac{1}{p}} ||(\varphi_\varepsilon * g - g)\left(\mathbf{X}_2(\tau)\right)||_{L^q(B_\nu)}\, d\tau. \tag{3.29}$$

With a change of variable, we rewrite the integral

$$\begin{aligned} &\left[\int_{|\xi|<\nu} e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2 p}{t-\tau}} d\xi\right]^{\frac{1}{p}} = \left[\int_{\left|x_1+\sqrt{t-\tau}u\right|<\nu} e^{-\frac{\lambda_0^*}{4}|u|^2 p} du\right]^{\frac{1}{p}} (t-\tau)^{\frac{N}{2p}} \\ &\leq \left[\int_{\mathbb{R}^N} e^{-\frac{\lambda_0^*}{4}|u|^2 p} du\right]^{\frac{1}{p}} (t-\tau)^{\frac{N}{2p}} = \left(\sqrt{\frac{4\pi}{\lambda_0^* p}}\right)^N (t-\tau)^{\frac{N}{2p}}. \end{aligned} \tag{3.30}$$

Hence we obtain the inequality

$$\sup_{|x_1|\leq R} \Sigma_{2,\nu}^{\varepsilon} \leq \int_0^t \frac{C_\Gamma}{(t-\tau)^{\theta}}\left(\sqrt{\frac{4\pi}{\lambda_0^* p}}\right)^{\frac{N}{p}} ||(\varphi_\varepsilon * g - g)\left(\mathbf{X}_2(\tau)\right)||_{L^q(B_\nu)}\, d\tau. \tag{3.31}$$

where p and q are conjugate exponents.
Furthermore, $C_\Gamma (t-\tau)^{-\theta} ||(\varphi_\varepsilon * g - g)\left(\mathbf{X}_2(\tau)\right)||_{L^q(B_\nu)}$ converges to 0 as $\varepsilon \to 0$, for any $\tau < t$, and, by G1) and (3.7), it is bounded from above by
$C_\Gamma \dfrac{3M\left(1+\nu+|\mathbf{X}_2(\tau)|\right)}{(t-\tau)^{\theta}}$, which is integrable over $(0,t)$. From Lebesgue's

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dominated convergence theorem, we conclude that $\sup_{|x_1|\leq R}\Sigma_{2\nu}^{\varepsilon}(x_1,t,\mathbf{X}_2)\to 0$ as $\varepsilon\to 0$, for any $\nu>2R$. Thus we can write

$$\lim_{\varepsilon\to 0}\sup_{|x_1|\leq R}\left|\Sigma_2^{\varepsilon}\left(x_1,t,\mathbf{X}_2\right)\right|\leq \sup_{|x_1|\leq R}\Sigma_{2\nu}\left(x_1,t,\mathbf{X}_2\right), \tag{3.32}$$

which converges to zero as $\nu\to\infty$, for any $||\mathbf{X}_2||_{\infty,T}\leq R$, $t\in(0,T)$.

iii) From (2.13) and G1), recalling the expressions in (6.12), (6.13) we get

$$|\Sigma_3|\leq 2MC_\Gamma\int_0^t\frac{|x_1-x_2|^{\frac{\alpha}{2}}}{(t-\tau)^{\frac{1}{2}+\frac{\alpha}{4}}}\int_{\mathbb{R}^N}\frac{e^{-\frac{\lambda_0^*}{4}\frac{|x_1-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N}{2}}}\left(1+|\xi|+||\mathbf{X}_2||_{\infty,T}\right)d\xi d\tau$$

$$\leq 2MC_\Gamma|x_1-x_2|^{\frac{\alpha}{2}}\int_0^t\frac{I_0\left(\lambda_0^*\right)\left(1+|x_1|+||\mathbf{X}_2||_{\infty,T}\right)+\sqrt{T}I_1\left(\lambda_0^*\right)}{(t-\tau)^{\frac{1}{2}+\frac{\alpha}{4}}}d\tau\leq C_3\delta^{\frac{\alpha}{2}},$$

with $C_3=C_3\left(M,C_\Gamma,\alpha,T,\lambda_0^*,R\right)=\dfrac{8MC_\Gamma T^{\frac{2-\alpha}{4}}}{2-\alpha}\left[I_0\left(\lambda_0^*\right)(1+2R)+\sqrt{T}I_1\left(\lambda_0^*\right)\right]$.
Therefore $\sup_{\Omega_{\delta,R}}|\Sigma_3\left(x_1,x_2,t,\mathbf{X}_2\right)|\to 0$ as $\delta\to 0$. This complete the proof.

□

The following Lemma 13 provides some regularity properties of the function in (3.11), that will be used to prove Theorem 11.

Lemma 13. *Under assumptions H8),G1),G2), the following properties concerning the function expressed in (3.11) hold true:*

1. *For every $\mathbf{X}\in C\left([0,T];\mathbb{R}^{N\times n}\right)$, $x\in\mathbb{R}^N$, $t_0\in[0,T]$ we get*

$$\lim_{t\to t_0}|f\left(x,t;\mathbf{X}\right)-f\left(x,t_0;\mathbf{X}\right)|=0. \tag{3.33}$$

For every $R>0$,

$$\sup_{||\mathbf{X}||_{\infty,T}\leq R}|f\left(x,t;\mathbf{X}\right)-f\left(x_0,t;\mathbf{X}\right)|\leq C_{T,R}\sqrt{t}\,|x-x_0|, \tag{3.34}$$

with $C_{T,R}$ a suitable positive constant, for any $t\in[0,T]$, $|x|,|x_0|\leq R$, and

$$|f\left(x,t;\mathbf{X}\right)|\leq L_T\left(1+|x|+||\mathbf{X}||\right)t. \tag{3.35}$$

2. *For every $t\in[0,T]$, $\mathbf{X}\in C\left([0,T];\mathbb{R}^{N\times n}\right)$, $f\left(\cdot,t;\mathbf{X}\right)\in C^1\left(\mathbb{R}^N\right)$ and*

$$\nabla f\left(x,t;\mathbf{X}\right)=-\int_0^t\int_{\mathbb{R}^N}\nabla\Gamma\left(x,t;\xi,\tau\right)g_{\mathbf{X}}(\xi,\tau)d\xi d\tau. \tag{3.36}$$

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In particular $\nabla f(\cdot,t;\mathbf{X})$ *is a locally Hölder continuous function with exponent* $\alpha/2$ *with respect to* x*, uniformly with respect to* $t\in[0,T]$*. Moreover* $\nabla f(\cdot,\cdot;\mathbf{X})\in C\left(\mathbb{R}^N\times[0,T]\right)$ *and*

$$|\nabla f(x,t;\mathbf{X})|\leq L_T\left(1+|x|+||\mathbf{X}||\right)\sqrt{t}. \tag{3.37}$$

3. *For every* $\mathbf{X}\in C\left([0,T];\mathbb{R}^{N\times n}\right)$, $R>0$

$$\lim_{k\to\infty}\sup_{\substack{|x|\leq R\\ t\in(0,T]}}\left(\left|f^k(x,t;\mathbf{X})-f(x,t;\mathbf{X})\right|+\left|\nabla f^k(x,t;\mathbf{X})-\nabla f(x,t;\mathbf{X})\right|\right)=0$$

where f^k *is the unique classical solution to* $Lf^k=g^k_{\mathbf{X}}$, $f^k(\cdot,0;\mathbf{X})\equiv 0$.

4. *For every* $R>0$, *it holds:*

$$\lim_{\delta\to 0}\sup_{\Omega_{\delta,R}}|f(x,t;\mathbf{X_1})-f(x,t;\mathbf{X_2})|+|\nabla f(x,t;\mathbf{X_1})-\nabla f(x,t;\mathbf{X_2})|=0. \tag{3.38}$$

Proof. Lemma 13:

1. We preliminary extimate the following integrals:

$$I_{t_1,t_2}(x,t):=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\Gamma(x,t;\xi,\tau)\,|g_{\mathbf{X}}(\xi,\tau)|\,d\xi d\tau, \tag{3.39}$$

$$J_{t_1,t_2}(x,t):=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\nabla\Gamma(x,t;\xi,\tau)\,|g_{\mathbf{X}}(\xi,\tau)|\,d\xi d\tau, \tag{3.40}$$

for $t_2\geq t_1$, $t\in[0,T]$.

Using (2.10), G1) and the change of variable $y=\dfrac{\xi-x}{\sqrt{t-\tau}}$, we have

$$\begin{aligned}
I_{t_1,t_2}(x,t)&\leq\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\frac{C_\Gamma e^{-\frac{\lambda_0^*}{4}\frac{|x-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N}{2}}}M\left(1+|\xi|+||\mathbf{X}||_{\infty,T}\right)d\xi d\tau\\
&\leq C_\Gamma M\int_{t_1}^{t_2}\int_{\mathbb{R}^N}e^{-\frac{\lambda_0^*}{4}y^2}\left(1+|x|+|y|\sqrt{t}+||\mathbf{X}||_{\infty,T}\right)dyd\tau\\
&\leq C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right)I_0\left(\frac{\lambda_0^*}{4}\right)+I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t}\right](t_2-t_1).
\end{aligned} \tag{3.41}$$

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In a similar way we extimate $J_{t_1,t_2}(x,t)$, using (2.11) instead of (2.10)

$$J_{t_1,t_2}(x,t) \leq \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \frac{C_\Gamma e^{-\frac{\lambda_0^*}{4}\frac{|x-\xi|^2}{t-\tau}}}{(t-\tau)^{\frac{N+1}{2}}} M\left(1+|\xi|+||\mathbf{X}||_{\infty,T}\right) d\xi d\tau$$

$$\leq 2C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t}\right]\sqrt{t_2-t_1}. \tag{3.42}$$

Let $x \in \mathbb{R}^N$, $\mathbf{X} \in C\left([0,T];\mathbb{R}^N\right)$. Without loss of generality, we prove the assertion for $t_0 \in (0,T]$, and we consider the case of $t \in (0,t_0)$. Let $c \in (0,t)$, we estimate $|f(x,t;\mathbf{X}) - f(x,t_0;\mathbf{X})|$

$$\leq \int_0^t \int_{\mathbb{R}^N} |\Gamma(x,t;\xi,\tau) - \Gamma(x,t_0;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau$$

$$+ \int_t^{t_0} \int_{\mathbb{R}^N} \Gamma(x,t_0;\xi,\tau)\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau$$

$$= \int_0^{t-c} \int_{\mathbb{R}^N} |\Gamma(x,t;\xi,\tau) - \Gamma(x,t_0;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau + I_{t-c,t}(x,t)$$

$$+ I_{t-c,t_0}(x,t_0) \leq \int_0^{t-c} \int_{\mathbb{R}^N} |\Gamma(x,t;\xi,\tau) - \Gamma(x,t_0;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau +$$

$$+ \left(C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t_0}\right]\right)[2c+t_0-t]. \tag{3.43}$$

We conclude that

$$\limsup_{t\to t_0^-} |f(x,t;\mathbf{X}) - f(x,t_0;\mathbf{X})|$$

$$\leq \limsup_{t\to t_0^-} \int_0^{t-c} \int_{\mathbb{R}^N} |\Gamma(x,t;\xi,\tau) - \Gamma(x,t_0;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau$$

$$+ \left(C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t_0}\right]\right) 2c. \tag{3.44}$$

Since $t-\tau \geq c$, by the above mentioned uniform continuity property of Γ and (2.10), we can take the limit as t tends to t_0^-, through the

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integral, to get

$$\begin{aligned}&\limsup_{t\to t_0^-} |f(x,t;\mathbf{X}) - f(x,t_0;\mathbf{X})| \\ &\leq \left(C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t_0}\right]\right) 2c.\end{aligned} \tag{3.45}$$

The relation (3.33) follows since c is arbitrary.
Let $t \in [0,T]$, $\mathbf{X} \in C\left([0,T];\mathbb{R}^N\right)$, $x, x_0 \in \mathbb{R}^N$ then we write:

$$\begin{aligned}&|f(x,t;\mathbf{X}) - f(x_0,t;\mathbf{X})| \\ &\leq \int_0^t \int_{\mathbb{R}^N} |\Gamma(x,t;\xi,\tau) - \Gamma(x_0,t;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau \\ &\leq \int_0^t \int_{\mathbb{R}^N} \int_0^t |\nabla\Gamma(x_0+\lambda(x-x_0),t;\xi,\tau)|\, d\lambda\, |x-x_0|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau \\ &\leq C_{T,R}\, |x-x_0|\, \sqrt{t},\end{aligned} \tag{3.46}$$

where $C_{T,R} := C_\Gamma M\left[(1+2R)\, I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{T}\right]$. In order to prove (3.35), we observe that from (3.41) it follows

$$|f(x,t;\mathbf{X})| \leq L_T\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) t, \tag{3.47}$$

where L_T denotes the quantity $C_\Gamma M\left(I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{T} + I_0\left(\frac{\lambda_0^*}{4}\right)\right)$.

2. For every $i = 1,...,n$, let us consider $x \in \mathbb{R}^N$, $t \in (0,T]$, $\mathbf{X} \in C\left([0,T];\mathbb{R}^N\right)$, $h \neq 0$. Denoting with e_i the vectors of the standard basis of $\mathbb{R}^N$ for any $i = 1,...,n$, we get

$$\begin{aligned}&\left|\frac{f(x+he_i,t;\mathbf{X}) - f(x,t;\mathbf{X})}{h} + \int_0^t \int_{\mathbb{R}^N} \partial_i\Gamma(x,t;\xi,\tau)\, g_{\mathbf{X}}(\xi,\tau)\, d\xi d\tau\right| \\ &= \left|-\int_0^t \int_{\mathbb{R}^N} \int_0^1 \partial_i\Gamma(x+h\lambda e_i,t;\xi,\tau)\, g_{\mathbf{X}}(\xi,\tau)\, d\lambda d\xi d\tau\right. \\ &\left. + \int_0^t \int_{\mathbb{R}^N} \partial_i\Gamma(x,t;\xi,\tau)\, g_{\mathbf{X}}(\xi,\tau)\, d\xi d\tau\right| \\ &\leq \int_0^1 \int_0^t \int_{\mathbb{R}^N} |\partial_i\Gamma(x+h\lambda e_i,t;\xi,\tau) - \partial_i\Gamma(x,t;\xi,\tau)|\, |g_{\mathbf{X}}(\xi,\tau)|\, d\xi d\tau d\lambda.\end{aligned} \tag{3.48}$$

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By using Lemma 12 *iii)* in the last inequality, taking the limit for $h \to 0$, we argue that the right-hand term in (3.48) goes to zero. Hence

$$\nabla f\left(x,t;\mathbf{X}\right) = -\int_0^t \int_{\mathbb{R}^N} \nabla\Gamma\left(x,t;\xi,\tau\right) g_{\mathbf{X}}(\xi,\tau) d\xi d\tau. \tag{3.49}$$

In order to prove that $\nabla f\left(\cdot,\cdot;\mathbf{X}\right) \in C\left(\mathbb{R}^N \times [0,T]\right)$ we retrace the steps of the proof of point 1). Let $x \in \mathbb{R}^N$, $\mathbf{X} \in C\left([0,T];\mathbb{R}^N\right)$, replacing f with ∇f in (3.43), by the uniform continuity property of $\nabla\Gamma$ for $t-\tau \geq c > 0$, we get the inequality

$$\begin{aligned}&\limsup_{t\to t_0^-} \left|\nabla f\left(x,t;\mathbf{X}\right) - \nabla f\left(x,t_0;\mathbf{X}\right)\right| \\ &\leq \left(2C_\Gamma M\left[\left(1+|x|+||\mathbf{X}||_{\infty,T}\right) I_0\left(\frac{\lambda_0^*}{4}\right) + I_1\left(\frac{\lambda_0^*}{4}\right)\sqrt{t_0}\right]\right) 4\sqrt{c}.\end{aligned} \tag{3.50}$$

By Lemma 12 *iii)* we get the Hölder continuity property of $\nabla f(\cdot,t;\mathbf{X})$ with respect to x, uniformly with respect to $t \in [0,T]$. Inequality (3.37) immediately follows using (3.40) for $J_{0,t}(x,t)$, with computations similar to the ones performed to prove (3.35).

3. By (3.15), we obtain

$$\left|\nabla f^k\left(x,t;\mathbf{X}\right) - \nabla f\left(x,t;\mathbf{X}\right)\right| = \left|\Sigma_2^{\varepsilon_k}\left(x,t,\mathbf{X}\right)\right|, \tag{3.51}$$

and the result follows from Lemma 12 *ii)*. Analogous estimate can be obtained for $\left|f^k\left(x,t;\mathbf{X}\right) - f\left(x,t;\mathbf{X}\right)\right|$, replacing $\nabla\Gamma$ with Γ, and using (2.10) instead of (2.11), in the proof of Lemma 12 *ii)*.

4. Since $\Sigma_1^{\varepsilon_k}\left(x,x,t,\mathbf{X}\right) = \nabla f^k\left(x,t;\mathbf{X_1}\right) - \nabla f^k\left(x,t;\mathbf{X_2}\right)$, Lemma 12 *i)* yields

$$\lim_{\delta\to 0} \sup_{\substack{\Omega_{\delta,R} \\ k}} \left|\nabla f^k\left(x,t;\mathbf{X_1}\right) - \nabla f^k\left(x,t;\mathbf{X_2}\right)\right| = 0. \tag{3.52}$$

With similar arguments, using (2.10) instead of (2.11) one can prove an analogous result for f^k:

$$\lim_{\delta\to 0} \sup_{\substack{\Omega_{\delta,R} \\ k}} \left|f^k\left(x,t;\mathbf{X_1}\right) - f^k\left(x,t;\mathbf{X_2}\right)\right| = 0. \tag{3.53}$$

Combining (3.52), (3.53) and point 3), we deduce (3.38).

□

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Combining the previous results, we finally prove the existence theorem:

Proof. Theorem 11:

Let $\{\mathbf{Y}^k\}_{k\in\mathbb{N}} = (\mathbf{X}^k, \mathbf{V}^k)$ be the solution of (3.8). First, we recall that $|\mathbf{Y}^k| \leq B + |\mathbf{Y}_0|$, for any $k \in \mathbb{N}$, where B is a positive constant independent of k (see Theorem 4 in [62]). We now prove the equicontinuity property of the sequence $\{\mathbf{Y}^k\}_{k\in\mathbb{N}}$. From (3.8) immediately follows the equicontinuity of the first n components of the sequence $\{\mathbf{Y}^k\}_{k\in\mathbb{N}} = (\mathbf{X}^k, \mathbf{V}^k)$, since for every $t, t' \in [0,T]$

$$\left|\mathbf{x}_i^k(t) - \mathbf{x}_i^k(t')\right| \leq (B + |\mathbf{V}_0|)\,|t - t'|, \qquad \forall i = 1, ..., n. \tag{3.54}$$

For the remaining components, we observe that, from $(3.8)_2$ and (3.9), it holds:

$$\begin{aligned}
&\left|\mathbf{v}_i^k(t) - \mathbf{v}_i^k(t')\right| = \left|\int_{t'}^{t} \mathbf{F}_i\left(\tau, \mathbf{X}^k(\tau), \mathbf{V}^k(\tau), \nabla f^k\left(\mathbf{x}_i^k(\tau), \tau, \mathbf{X}^k\right)\right)\, d\tau\right| \\
&\leq \int_{t'}^{t} |\mathbf{F}_i(\tau, \mathbf{X}_0, \mathbf{V}_0, 0)| + \sqrt{2}L_F\left|\mathbf{Y}^k(\tau) - \mathbf{Y}_0\right| + L_F\left|\nabla f^k\left(\mathbf{x}_i^k(\tau), \tau, \mathbf{X}^k\right)\right| d\tau \\
&\leq C_0\,|t - t'| + L_F\sqrt{2}\int_{t'}^{t}\left|\mathbf{Y}^k(\tau) - \mathbf{Y}_0\right| d\tau \\
&+ \int_{t'}^{t} L_F K_1\left(\frac{1 + \left|\mathbf{X}^k(\tau)\right|}{\sqrt{\tau}} + K_2 + \int_0^{\tau}\frac{1 + \left|\mathbf{X}^k(\tau)\right| + \left|\mathbf{X}^k(s)\right|}{\sqrt{\tau - s}} ds\right) d\tau,
\end{aligned} \tag{3.55}$$

where $C_0 = \sup_{\tau\in(0,T]} |\mathbf{F}_i(\tau, \mathbf{X}_0, \mathbf{V}_0, 0)|$. Since K_1 and K_2 in (3.9) are independent of the index k, we get

$$\begin{aligned}
\left|\mathbf{v}_i^k(t) - \mathbf{v}_i^k(t')\right| &\leq \left(C_0 + L_F\sqrt{2}B + L_F K_1 K_2\right)|t - t'| \\
&+ 2L_F K_1\left[1 + (|\mathbf{X}_0| + B)\right]\sqrt{|t - t'|} \\
&+ L_F K_1\left(1 + 2B + 2\,|\mathbf{X}_0|\right) 2\sqrt{T}\,|t - t'|.
\end{aligned} \tag{3.56}$$

By (3.54) and (3.56), the equicontinuity of the sequence $\{\mathbf{Y}^k\}_{k\in\mathbb{N}}$ is proved. Hence, Ascoli-Arzelà theorem implies the existence of a subsequence, still denoted as $\{\mathbf{Y}^k\}_{k\in\mathbb{N}}$, which converges to $\overline{\mathbf{Y}} = (\overline{\mathbf{X}}, \overline{\mathbf{V}}) \in C\left([0,T]; \mathbb{R}^{2(N\times n)}\right)$, such that $||\overline{\mathbf{X}}||_{\infty,T}, ||\overline{\mathbf{V}}||_{\infty,T} \leq B + |\mathbf{Y}_0|$.
Clearly, $f^k\left(\cdot,\cdot;\mathbf{X}^k\right)$ satisfies equation (3.10), being the unique classical solution of $Lf\left(\cdot,\cdot;\mathbf{X}^k\right) = g^k_{\mathbf{X}^k}$. We now prove that the function in (3.11) is a solution of (2.2), associated to $\overline{\mathbf{X}}$, in the sense of Definition 9. From Lemma

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12, $f\left(\cdot,\cdot;\overline{\mathbf{X}}\right)$, as in (3.11), satisfies *i)*, *ii)* and *iii)* of Definition 9. Therefore, it suffices to prove that it satisfies also equation (3.10). We observe that Lemma 13, points 2. and 3., imply

$$\lim_{k\to\infty}\sup_{\substack{|x|\leq R\\ t\in[0,T]}}\left|\nabla f^k\left(x,t;\mathbf{X}^k\right)-\nabla f\left(x,t;\overline{\mathbf{X}}\right)\right|+\left|f^k\left(x,t;\mathbf{X}^k\right)-f\left(x,t;\overline{\mathbf{X}}\right)\right|=0, \tag{3.57}$$

hence $f\left(\cdot,\cdot;\overline{\mathbf{X}}\right)$ satisfies (3.10), since the test functions have compact support. From $(3.8)_1$ immediately follows that, as $k\to\infty$,

$$\mathbf{X}^k(t)\to\overline{\mathbf{X}}(t)=\mathbf{X}_0+\int_0^t\overline{\mathbf{V}}(\tau)\,d\tau,\qquad\forall t\in[0,T]. \tag{3.58}$$

By $(3.8)_2$, for every $i=1,...,n$, it holds:

$$\begin{aligned}
&\left|\overline{\mathbf{v}}_i(t)-\mathbf{v}_{i0}-\int_0^t\mathbf{F}_i\left(\tau,\overline{\mathbf{X}}(\tau),\overline{\mathbf{V}}(\tau),\nabla f\left(\overline{x}(\tau),\tau;\overline{\mathbf{X}}(\tau)\right)\right)\,d\tau\right|\\
&\leq\left|\overline{\mathbf{v}}_i(t)-\mathbf{v}_i^k(t)\right|+\left|\int_0^t\mathbf{F}_i\left(\tau,\mathbf{X}^k(\tau),\mathbf{V}^k(\tau),\nabla f^k\left(\mathbf{x}_i^k(\tau),\tau;\mathbf{X}^k\right)\right)\right.\\
&\left.-\mathbf{F}_i\left(\tau,\overline{\mathbf{X}}(\tau),\overline{\mathbf{V}}(\tau),\nabla f\left(\overline{x}(\tau),\tau;\overline{\mathbf{X}}(\tau)\right)\right)\,d\tau\right|\leq\sqrt{2}L_FT\left|\left|\mathbf{Y}^k-\overline{\mathbf{Y}}\right|\right|_{\infty,T}\\
&+L_F\int_0^t\left|\nabla f^k\left(\mathbf{x}_i^k(\tau),\tau;\mathbf{X}^k\right)-\int_0^\tau\int_{\mathbb{R}^N}\nabla\Gamma\left(\overline{x}(s),\tau;\xi,s\right)g_{\overline{\mathbf{X}}}(\xi,s)\,d\xi d\tau\right|d\tau\\
&\leq\sqrt{2}L_FT\left|\left|\mathbf{Y}^k-\overline{\mathbf{Y}}\right|\right|_{\infty,T}+L_F\int_0^t\left|\Sigma_1^{\varepsilon_k}\left(\mathbf{x}_i^k(\tau),\tau,\mathbf{X}^k,\overline{\mathbf{X}}\right)\right|d\tau+\\
&L_F\int_0^t\left|\Sigma_2^{\varepsilon_k}\left(\mathbf{x}_i^k(\tau),\tau,\overline{\mathbf{X}}\right)\right|d\tau+L_F\int_0^t\left|\Sigma_3\left(\mathbf{x}_i^k(\tau),\overline{\mathbf{x}}_i(\tau)\tau,\overline{\mathbf{X}}\right)\right|d\tau,
\end{aligned} \tag{3.59}$$

where $\Sigma_1^{\varepsilon_k}$, $\Sigma_2^{\varepsilon_k}$, Σ_3 are defined in (3.14)-(3.16), with $\varepsilon_k\in(0,1)$ an infinitesimal sequence as $k\to\infty$.
Let us consider $R=|\mathbf{Y}_0|+B$, and $\delta_k=\left|\left|\mathbf{X}^k-\overline{\mathbf{X}}\right|\right|_{\infty,T}$. Since $\delta_k\to0$ as $k\to\infty$, from Lemma 12 *i)*, we get

$$\int_0^t\left|\Sigma_1^{\varepsilon_k}\left(\mathbf{x}_i^k(\tau),\tau,\mathbf{X}^k,\overline{\mathbf{X}}\right)\right|d\tau\leq T\sup_{\substack{\Omega_{\delta_k,R}\\ \varepsilon\in(0,1)}}\left|\Sigma_1^{\varepsilon}\left(\mathbf{x}_i^k(\tau),\tau,\mathbf{X}^k,\overline{\mathbf{X}}\right)\right|\to0$$

as $k\to\infty$. Moreover, $\int_0^t\left|\Sigma_2^{\varepsilon_k}\left(\mathbf{x}_i^k(\tau),\tau,\overline{\mathbf{X}}\right)\right|d\tau\to0$ as $k\to\infty$.
In fact, $\left|\Sigma_2^{\varepsilon_k}\left(\mathbf{x}_i^k(\tau),\tau,\overline{\mathbf{X}}\right)\right|\leq\sup_{|x|\leq R}\left|\Sigma_2^{\varepsilon_k}\left(x,\tau,\overline{\mathbf{X}}\right)\right|$, and Lemma 12 *ii)* en-

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sures that the function

$$\tau \mapsto \sup_{|x|\leq R} \left|\Sigma_2^{\varepsilon_k}\left(x,\tau,\overline{\mathbf{X}}\right)\right|$$

is uniformly bounded with respect to k and it converges to zero as $k \to \infty$, pointwise in $(0,t)$, for $t \in (0,T]$. Finally, from Lemma 12 *iii)*, we obtain

$$\int_0^t \left|\Sigma_3\left(\mathbf{x}_i^k(\tau), \overline{\mathbf{v}}_i(\tau)\tau, \overline{\mathbf{X}}\right)\right| d\tau \leq T \sup_{\Omega_{\delta_k,R}} \left|\Sigma_3\left(\mathbf{x}_i^k(\tau), \overline{\mathbf{x}}_i(\tau)\tau, \overline{\mathbf{X}}\right)\right| \leq TC_3\delta_k^{\frac{\alpha}{2}},$$

which converges to zero as $k \to \infty$.

Thus, by (3.59), taking the limit for $k \to \infty$, we argue that

$$\overline{\mathbf{V}}(t) = \mathbf{V}_0 + \int_0^t \mathbf{F}_i\left(\tau, \overline{\mathbf{X}}(\tau), \overline{\mathbf{V}}(\tau), \nabla f\left(\overline{x}(\tau), \tau; \overline{\mathbf{X}}(\tau)\right)\right) d\tau \tag{3.60}$$

for any $t \in (0,T]$. We conclude that $\overline{\mathbf{Y}} := \left(\overline{\mathbf{X}}, \overline{\mathbf{Y}}\right)$, solves the coupled system (2.1)-(2.2) in the sense of Definition 10 .

□

3.3.1 Some regularity properties

In the following Theorem 14, we investigate further regularity properties of the function defined in (3.11), which we have already proved to solve (2.2) in the sense of Definition 9-10. Here, we shall make use of an additional hypothesis concerning the structure of the source term g, together with the previous assumptions G1) and G2).
Precisely, under a suitable assumption on the support of the function $g(\cdot, \mathbf{X}(\cdot))$, it is possible to identify a region, in $\mathbb{R}^N$, where the function $f(\cdot,\cdot;\mathbf{X})$ is smooth enough and it is also possibile to deduce some estimates on its derivatives, despite the possible presence of discontinuities of the source term.

Let $\mathbf{X} \in C\left([0,T];\mathbb{R}^{N\times n}\right)$, for every fixed $r > 0$, $h > 0$, we define the following sets:

$$\Gamma_r^{\mathbf{X}} := \bigcup_{i=1}^n \left\{ x \in \mathbb{R}^N \ : \ \inf_{\tau\in[0,T]} |x - \mathbf{x}_i(\tau)| \leq r \right\}, \tag{3.61}$$

$$\Omega_{h,r}^{\mathbf{X}} := \left\{ x \in \mathbb{R}^N \ : \ \mathrm{dist}\left(x, \Gamma_{\mathbf{X}}^r\right) > h \right\}. \tag{3.62}$$

Clearly $\Omega_{h,r}^{\mathbf{X}}$ is an open set in $\mathbb{R}^N$. Roughly speaking, $\Gamma_r^{\mathbf{X}}$ represents the region of points whose distance from at least one of the trajectories, enclosed in $\mathbf{X}$, is less or equal to r. On the other hand, the set $\Omega_{h,r}^{\mathbf{X}}$ includes all the

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points that are *quite far* from the trajectories, the distance being greater than a. Thus, we can establish the following result.

Theorem 14. *Assume H8) and let $f = f(x,t;\mathbf{X})$ be the solution of (3.11) with g satisfying G1)-G2). Suppose that, for some $r > 0$, the support of $(x,t) \mapsto g_{\mathbf{X}}(x,t)$ is a subset of*

$$\Omega_r := \bigcup_{i=1}^{n} \left\{ (\xi,\tau) \in \mathbb{R}^N \times (0,T) \middle| \ |\xi - \mathbf{x}_i(\tau)| \leq r \right\}. \tag{3.63}$$

Then, for every $a > 0$, hold:

i)

$$f(\cdot,\cdot;\mathbf{X}) \in L^{\infty}\left([0,T]; W^{2,\infty}\left(\Omega_{h,r}^{\mathbf{X}}\right)\right). \tag{3.64}$$

with the inequalities:

$$\begin{aligned} ||\partial_j f(\cdot,t;\mathbf{X})||_{\infty,T} \leq \ & M\left|\Gamma_r^{\mathbf{X}}\right| (1 + 2||\mathbf{X}||_{\infty,T} + r) \times \\ & \times \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N-1}{2}} \gamma_e\left(\frac{N-1}{2}, \frac{\lambda_0^* h^2}{4t}\right), \end{aligned} \tag{3.65}$$

$$\begin{aligned} ||\partial_{i,j}^2 f(\cdot,t;\mathbf{X})||_{\infty,T} \leq \ & M\left|\Gamma_r^{\mathbf{X}}\right| (1 + 2||\mathbf{X}||_{\infty,T} + r) \times \\ & \times \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N}{2}} \gamma_e\left(\frac{N}{2}, \frac{\lambda_0^* h^2}{4t}\right). \end{aligned} \tag{3.66}$$

for any $i,j = 1,\ldots,N$.

ii) $f(\cdot,\cdot;\mathbf{X}) \in W^{2,1,\infty}\left(\Omega_{h,r}^{\mathbf{X}} \times (0,T)\right)$ *and*

$$\begin{aligned} & ||\partial_t f||_{L^{\infty}\left(\Omega_{h,r}^{\mathbf{X}} \times (0,T)\right)} \leq ||c||_{\infty} + (1 + 2||\mathbf{X}||_{\infty,T} + r)\, M|\Gamma_r^{\mathbf{X}}| \times \\ & \times \left[\sqrt{n}||b||_{\infty}\left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N-1}{2}} \Gamma\left(\frac{N-1}{2}\right) + n||a||_{\infty}\left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)\right] \\ & + M\,(1 + 2||\mathbf{X}||_{\infty,T} + r) \end{aligned} \tag{3.67}$$

Here $\Gamma(\cdot)$ is the Gamma function, $\gamma_e(\alpha,x) = \int_x^{\infty} u^{\alpha-1} e^{-u} du$ being the upper incomplete Gamma, $\left|\Gamma_r^{\mathbf{X}}\right|$ stands for the N-dimensional Lebesgue measure of $\Gamma_r^{\mathbf{X}}$ and $\|c\|_{\infty}$, $\|b\|_{\infty}$, $\|a\|_{\infty}$ denote the supremum norm of the functions c, $\left(\sum_i b_i^2\right)^{1/2}$ and $\left(\sum_{i,j} a_{ij}^2\right)^{1/2}$, respectively.

Remark 15. We emphasize that the modeling form of g, presented in [2] and [3], falls within the assumptions of Theorem 14.

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Remark 16. The bounds for the x-partial derivatives of $f(\cdot,\cdot;\mathbf{X})$ yield a precise information on the decay rate as $t \to 0^+$, for any fixed $h > 0$. In fact it is well known that $\gamma_e(\alpha, \lambda_0^\star h^2/4t) \approx \left(\lambda_0^\star h^2/4t\right)^{\alpha-1} e^{-\lambda_0^\star h^2/4t}$, as $t \to 0^+$, for $\alpha = N/2$ or $\alpha = (N-1)/2$.

Proof. Theorem 14:
i) In the following ∂_j^p denotes the p-order partial derivative with respect to x_j, for $p = 0,\ 1$. Let $h > 0$ and $\varphi \in C_0^\infty\left(\Omega_{h,r}^{\mathbf{X}}\right)$, with compact support $\Lambda_\varphi \subset \Omega_{h,r}^{\mathbf{X}}$. Thus, by (2.11), (2.12), $G1$) and the assumption that the support of $g_{\mathbf{X}}$ is a subset of Ω_r, for every $x \in \mathbb{R}^N$, $t \in (0,T)$, it holds:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \partial_j^p f\,(x,t;\mathbf{X})\,\partial_i\varphi(x)dx\right| \\
&\leq \int_{\mathbb{R}^N}\int_0^t\int_{\mathbb{R}^N} \left|\partial_j^p \Gamma\,(x,t;\xi,\tau)\,g_{\mathbf{X}}\,(\xi,\tau)\,\partial_i\varphi(x)\right| d\xi d\tau dx \\
&\leq M\,(1+2||\mathbf{X}||_{\infty,T}+r)\int_{\Lambda_{g,t}} d\xi d\tau \int_{\Lambda_\varphi}\left|\partial_j^{p+1}\Gamma\,(x,t;\xi,\tau)\right| |\varphi(x)|\,dx \\
&\leq M\,(1+2||\mathbf{X}||_{\infty,T}+r)\int_{\Lambda_{g,t}} d\xi d\tau \int_{\Lambda_\varphi-\xi} \frac{e^{-\frac{\lambda_0^*|\eta|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{N+p+1}{2}}}\,|\varphi(\eta+\xi)|\,d\eta
\end{aligned} \tag{3.68}$$

where $\Lambda_{g,t}$ denotes the intersection between the support of $g_{\mathbf{X}}$ and $\mathbb{R}^N \times (0,t)$. Here, in order to get the second inequality, we have applied the integration by parts and, in the last passage, we have changed the variable setting $\eta = x - \xi$. We observe that, if $\eta \in \Lambda_\varphi - \xi$, for some $\xi \in \Lambda_g$, since $\Lambda_g \subset \Omega_r$, then $|\eta| = |x-\xi| \geq \mathrm{dist}(x, \Gamma_r^{\mathbf{X}}) > h$. Therefore, we obtain the inequality

$$\begin{aligned}
&\int_{\Lambda_{g,t}}\int_{\Lambda_\varphi-\xi} \frac{e^{-\frac{\lambda_0^*|\eta|^2}{4(t-\tau)}}}{(t-\tau)^{\frac{N+p+1}{2}}}\,|\varphi(\eta+\xi)|\,d\eta d\xi d\tau \\
&\leq \left|\Gamma_r^{\mathbf{X}}\right| \int_0^t \frac{e^{-\frac{\lambda_0^* h^2}{4(t-\tau)}} d\tau}{(t-\tau)^{\frac{N+p+1}{2}}}\,||\varphi||_{L^1} \\
&= \left|\Gamma_r^{\mathbf{X}}\right| \left(\tfrac{4}{\lambda_0^* h^2}\right)^{\frac{N+p-1}{2}} \int_{\frac{\lambda_0^* h^2}{4t}}^\infty e^{-v} v^{\frac{N+p-1}{2}-1} dv\ ||\varphi||_{L^1} \\
&= \left|\Gamma_r^{\mathbf{X}}\right| \left(\tfrac{4}{\lambda_0^* h^2}\right)^{\frac{N+p-1}{2}} \gamma_e\left(\frac{N+p-1}{2}, \frac{\lambda_0^* h^2}{4t}\right) ||\varphi||_{L^1},
\end{aligned} \tag{3.69}$$

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where, in the first equality, we have applied the change of variable $4v = \lambda_0^* h^2 (t-\tau)^{-1}$. Thus, using (3.69) in (3.68) we get

$$\left| \int_{\mathbb{R}^N} \partial_j^p f\,(x,t;\mathbf{X})\,\partial_i \varphi(x) dx \right| \leq \quad M\,(1+2||\mathbf{X}||_{\infty,T}+r)\,\left|\Gamma_r^{\mathbf{X}}\right| \left(\tfrac{4}{\lambda_0^* h^2}\right)^{\frac{N+p-1}{2}} \gamma_e\left(\frac{N+p-1}{2}, \frac{\lambda_0^* h^2}{4t}\right) ||\varphi||_{L^1}\,. \tag{3.70}$$

Hence $f(\cdot,t;\mathbf{X}) \in W^{2,\infty}\left(\Omega_{h,r}^{\mathbf{X}}\right)$ (see [65]), and (3.65), (3.66) follow from (3.70) for $p=0$, $p=1$, respectively.

ii) In light of inequalities (3.65)-(3.66), it suffices to prove the existence of the weak derivative with respect to t of $f(\cdot,\cdot;\mathbf{X})$ and the inequality (3.67). Let ρ be a smooth test function with compact support in $\Omega_{h,r}^{\mathbf{X}} \times (0,T)$, since $f(\cdot,\cdot;\mathbf{X})$ satisfies condition *iv)*-Definition 9, we have:

$$\begin{aligned}
\Big| \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} f(\xi,\tau;\mathbf{X}) \partial_t \rho(\xi,\tau) d\xi d\tau \Big| &\leq \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} |c(\xi,\tau)|\,|\rho(\xi,\tau)| d\xi d\tau \\
&+ \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} |b(\xi,\tau)|\,|\nabla f(\xi,\tau;\mathbf{X})|\,|\rho(\xi,\tau)| d\xi d\tau \\
&+ \Big| \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} \sum_{i,j} a_{ij}(\xi,\tau) \partial_{ij}^2 f(\xi,\tau;\mathbf{X}) \rho(\xi,\tau) d\xi d\tau \Big| \\
&+ M\,(1+2||\mathbf{X}||_{\infty,T}+r)\,||\rho||_{L^1}
\end{aligned} \tag{3.71}$$

where in order to get the last term, we have applied $G1$) and the fact that $\Lambda_g \subset \Omega_r$. Here the L^1-norm refers to the domain $\Omega_{h,r}^{\mathbf{X}} \times (0,T)$. Thus, by inequalities (3.65)-(3.66), we can write:

$$\begin{aligned}
&\Big| \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} f(\xi,\tau;\mathbf{X}) \partial_t \rho(\xi,\tau) d\xi d\tau \Big| \leq ||c||_\infty ||\rho||_{L^1} + (1+2||\mathbf{X}||_{\infty,T}+r) \times \\
&\times \left[\sqrt{n}||b||_\infty M |\Gamma_r^{\mathbf{X}}| \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N-1}{2}} \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} \gamma_e\left(\frac{N-1}{2}, \frac{\lambda_0^* h^2}{4\tau}\right) |\rho(\xi,\tau)| d\xi d\tau \right. \\
&\qquad \left. + n||a||_\infty M |\Gamma_r^{\mathbf{X}}| \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N}{2}} \int_0^T \int_{\Omega_{h,r}^{\mathbf{X}}} \gamma_e\left(\frac{N}{2}, \frac{\lambda_0^* a^2}{4\tau}\right) |\rho(\xi,\tau)| d\xi d\tau \right] \\
&+ M\,(1+2||\mathbf{X}||+r)\,||\rho||_{L^1} \leq ||c||_\infty ||\rho||_{L^1} + (1+2||\mathbf{X}||_{\infty,T}+r)\,M|\Gamma_r^{\mathbf{X}}| \times \\
&\times \left[\sqrt{n}||b||_\infty \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N-1}{2}} \Gamma\left(\frac{N-1}{2}\right) + n||a||_\infty \left(\frac{4}{\lambda_0^* h^2}\right)^{\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) \right] ||\rho||_{L^1} \\
&+ (1+2||\mathbf{X}||_{\infty,T}+r)\,M||\rho||_{L^1}.
\end{aligned} \tag{3.72}$$

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This proves the point *ii*) together with inequality (3.67).

□

3.4 Discussion

In this chapter we presented an existence theorem for the solutions of hybrid systems of differential equations, which generalizes the results obtained in Chapter 2. In particular, our working assumptions allow to deal with the form of the functions considered in [2] and [3] to model the source term of the diffusion equation. We observe that the particular structure, together with the set of chosen assumption, do not allow to establish uniqueness property of the solution. We emphasize that the particular choice of a discontinuous source term is not covered by the classical results of the literature. From these premises, we cannot expect to have classical solutions for the specific problem. Therefore, the main contribution of this work relies on the existence of weak solutions for coupled hybrid systems, in the sense of our given definitions. With a step by step approach, the setting we introduce in Chapter 4 is the higher level of generalization presented in this thesis, and it represents the starting point of ongoing and future works.

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Chapter 4

On a coupled system of nonlinear differential equations with a non-local concentration and L^2 initial data

We present [1] existence and uniqueness results for the solution of hybrid systems which, in particular, represent a further generalization of the models previously investigated. With a step by step approach, we here consider the case of $g \in Lip_{loc}(\mathbb{R}^{N\times n}; L^2(\mathbb{R}^N))$. The equation modeling chemotaxis is given in a divergence form, and initial data belonging to L^2 are considered. The results are obtained using a different technique, based on a preliminary study concerning well-posedness of pseudo-parabolic approximation problems, and a passage to the limit.

4.1 Problem statement and results

In this chapter we investigate, from an analytical view point, a particular coupled system of nonlinear differential equations. The considered structure takes inspiration from the hybrid form of models in [2], [3], and generalizes the ones previously investigated in this thesis. In particular, as in the inspiring papers of the literature, the coupling between ordinary and partial differential equations is realized in a non-local form.

[1] This chapter is based on joint work with F. Smarrazzo, M. Porzio, M.Papi

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Let consider the following hybrid system:

$$\begin{cases} \dot{\mathbf{x}}_i(t) = F_i\left(t, \mathbf{X}(t), \displaystyle\int_{\mathbb{R}^N} \nabla u\,(x,t)\, h(x, \mathbf{x}_i(t))dx\right) & t \in (0,T)\,, \\ u_t = \operatorname{div}\left(\mathbf{a}(x,u,\nabla u)\right) + b(x,u) + g(x, \mathbf{X}(t)) & \text{in } \mathbb{R}^N \times (0,T)\,, \\ \mathbf{x}_i(0) = \mathbf{x}_{i0} \in \mathbb{R}^N, \\ u(\cdot,0) = u_0 \in L^2(\mathbb{R}^N), \end{cases} \tag{4.1}$$

for any $i = 1, ..., n$.
In the following we denote $\mathbf{X}(t) = [\mathbf{x}_1(t), ..., \mathbf{x}_n(t)] \in \mathbb{R}^{N\times n}$, $\mathbf{x}_i(t) \in \mathbb{R}^N$. With respect to the previous chapters, we use a different notation u, for the function modeling the signal, in order to highlight the fact the the model is investigated only on a theoretical basis, and u is not necessarily a chemical concentration. In the same way, $\mathbf{x}_i$ denotes the variable modeling the dynamic of agent i, e.g. position, velocity. In our setting, we assume:

A1) For every $i = 1, ..., n$, the function $F_i : [0,T] \times \mathbb{R}^{N\times n} \times \mathbb{R}^N \to \mathbb{R}^N$ belongs to the space $C\left([0,T] \times \mathbb{R}^{N\times n} \times \mathbb{R}^N; \mathbb{R}^N\right)$, and there exists $L_F > 0$ such that for every $i = 1, ..., n$ it holds

$$\left|F_i\left(t, \mathbf{X}, \mathbf{z}\right) - F_i\left(t, \widehat{\mathbf{X}}, \hat{\mathbf{z}}\right)\right| \leq L_F\left(\left|\mathbf{X} - \widehat{\mathbf{X}}\right| + |\mathbf{z} - \hat{\mathbf{z}}|\right), \tag{4.2}$$

for any $\mathbf{X}, \hat{\mathbf{X}} \in \mathbb{R}^{N\times n}$, $\mathbf{z}, \hat{\mathbf{z}} \in \mathbb{R}^N$, $t \in [0,T]$.

A2) i) The function $\mathbf{a} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ belongs to the space $C\left(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N; \mathbb{R}^N\right)$, and there exists $L_a > 0$ such that

$$\left|\mathbf{a}\left(x, u, \xi\right) - \mathbf{a}\left(x, \hat{u}, \hat{\xi}\right)\right| \leq L_a\left(|u - \hat{u}| + \left|\xi - \hat{\xi}\right|\right), \tag{4.3}$$

for any $u, \hat{u} \in \mathbb{R}$, ξ, $\hat{\xi} \in \mathbb{R}^N$, and for all $x \in \mathbb{R}^N$. Moreover, without loss of generality, in the following we assume $\mathbf{a}(x, 0, \mathbf{0}) = \mathbf{0}$ for any $x \in \mathbb{R}^N$.

ii) For every $\xi_1, \xi_2 \in \mathbb{R}^N$, there holds

$$\left[\mathbf{a}\left(x, u, \xi_1\right) - \mathbf{a}\left(x, u, \xi_2\right)\right] \cdot \left[\xi_1 - \xi_2\right] \geq 0\,, \tag{4.4}$$

for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

iii) There exists $\alpha_0 > 0$ such that

$$\mathbf{a}\left(x, u, \xi\right) \cdot \xi \geq \alpha_0 \left|\xi\right|^2 \tag{4.5}$$

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for all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

A3) The function $b(x,u) \in C\left(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}\right)$, and there exists $L_b > 0$ such that

$$|b(x,u) - b(x,\hat{u})| \leq L_b(|u-\hat{u}|), \tag{4.6}$$

for all $x \in \mathbb{R}^N$ and u, $\hat{u} \in \mathbb{R}$. Moreover, without loss of generality, in the following we assume $b(x,0) = 0$ for any $x \in \mathbb{R}^N$.

A4) $g \in \mathrm{Lip}_{\mathrm{loc}}\left(\mathbb{R}^{N\times n}; L^2\left(\mathbb{R}^N\right)\right)$.
Hence, for every $R > 0$ there exists $L_g^{(R)} > 0$ such that

$$||g(\cdot, \mathbf{X}_1) - g(\cdot, \mathbf{X}_2)||_{L^2(\mathbb{R}^N)} \leq L_g^{(R)} |\mathbf{X}_1 - \mathbf{X}_2| \tag{4.7}$$

for any $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^{N\times n}$, $|\mathbf{X}_1| \leq R$, $|\mathbf{X}_2| \leq R$.

A5) $h \in \mathrm{Lip}_{\mathrm{loc}}\left(\mathbb{R}^N; L^2\left(\mathbb{R}^N\right)\right)$.
Hence for every $R > 0$ there exists $L_h^{(R)} > 0$ such that

$$||h(\cdot, \mathbf{x}_1) - h(\cdot, \mathbf{x}_2)||_{L^2(\mathbb{R}^N)} \leq L_h^{(R)} |\mathbf{x}_1 - \mathbf{x}_2| \tag{4.8}$$

for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$, $|\mathbf{x}_1| \leq R$, $|\mathbf{x}_2| \leq R$.

A6) There exist $C_h, C_g > 0$ such that

$$||h(\cdot, \mathbf{x})||_{L^2(\mathbb{R}^N)} \leq C_h(1 + |\mathbf{x}|)^{\theta_1} \quad \text{for all } \mathbf{x} \in \mathbb{R}^N, \tag{4.9}$$

$$||g(\cdot, \mathbf{X})||_{L^2(\mathbb{R}^N)} \leq C_g(1 + |\mathbf{X}|)^{\theta_2} \quad \text{for all } \mathbf{X} \in \mathbb{R}^{N\times n}, \tag{4.10}$$

where $\theta_1, \theta_2 \in [0,1]$ satisfy $\theta_1 + \theta_2 \leq 1$.

In the next sections, we prove global existence and uniqueness of the solution to (4.1) under the above assumptions. Firstly, we state clearly the definition of solution to problem (4.1) .

Definition 17. *For every $u_0 \in L^2\left(\mathbb{R}^N\right)$, $\mathbf{X}_0 \in \mathbb{R}^{N\times n}$, by a solution to problem (4.1) in $(0,T_0)$, with $0 < T_0 \leq T$, we mean any pair $(\mathbf{X}, u)$, with $u \in C\left([0,T_0]; L^2\left(\mathbb{R}^N\right)\right) \cap L^2\left((0,T_0); H^1(\mathbb{R}^N)\right)$, $\mathbf{X} = (\mathbf{x}_1, ..., \mathbf{x}_n) \in W^{1,2}\left((0,T_0); \mathbb{R}^{N\times n}\right)$ such that*

i) $\mathbf{X}(0) = \mathbf{X}_0$, $u(\cdot, 0) = u_0$ a.e. in $\mathbb{R}^N$;

ii) for any $i = 1, \cdots, n$, equality $(4.1)_1$ is satisfied a.e. in $(0,T)$;

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iii) for any $\rho \in C\left([0,T_0];L^2(\mathbb{R}^N)\right)\cap L^2\left((0,T_0);H^1(\mathbb{R}^N)\right)$, *with* $\rho_t \in L^2(Q_{T_0})$, *there holds*

$$\iint_{Q_{T_0}} \{u\rho_t - \mathbf{a}(x,u,\nabla u)\nabla\rho + b(x,u)\rho + g(x,\boldsymbol{X}(t))\rho\}\, dxdt = -\int_{\mathbb{R}^N} \rho(x,0)u_0(x)\, dx \tag{4.11}$$

We denote $Q_{T_0} := \mathbb{R}^N \times (0,T_0)$.

The main results of our study are the following theorems concerning the existence and uniqueness of the strong solution to problem (4.1).

Theorem 18. *Let assumptions A1)-A6) be satisfied. Then, for every* $(\mathbf{X}_0, u_0) \in \mathbb{R}^{N\times n} \times H^1(\mathbb{R}^N)$, *problem (4.1) admits a global solution in* $[0,T]$.

In order to prove uniqueness, we replace assumption $A2) - ii)$ with the stronger condition

A2) $-iv)$ There exists $\beta_0 > 0$ such that, for every $\xi_1, \xi_2 \in \mathbb{R}^N$, there holds

$$[\mathbf{a}(x,u,\xi_1) - \mathbf{a}(x,u,\xi_2)] \cdot [\xi_1 - \xi_2] \geq \beta_0 |\xi_1 - \xi_2| \,. \tag{4.12}$$

Observe that (4.12) and the assumption $\mathbf{a}(x,0,\mathbf{0}) = \mathbf{0}$ (see $A2)$-$i)$) plainly give $A2)$-$iii)$ (with $\alpha_0 = \beta_0$).

In Theorem 19 below we shall denote by $A2)'$ the set of assumptions $A2)$-$i)$ and $A2)$-$iv)$.

Theorem 19. *Let assumptions A1), A2)', A3)-A6) be satisfied. Then, for every* $(\mathbf{X}_0, u_0) \in \mathbb{R}^{N\times n} \times H^1(\mathbb{R}^N)$, *problem (4.1) has at most one solution.*

In this chapter, the proof of existence will rely on a pseudoparabolic approximation of the parabolic equation in (4.1) (see [66], [67], [68], [69]).

In Section 4.2, we prove well-posedness of the approximating problems. In particular, we present some a priori estimates on the approximating solutions . The results previously obtained are combined in Section 4.3, devoted to the proof of global existence and uniqueness of the solution to (4.1).

4.2 Approximating Problems

For every $\varepsilon > 0$, let us consider the problem

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$$\begin{cases} \dot{\mathbf{x}}_i(t) = F_i\left(t, \mathbf{X}(t), \displaystyle\int_{\mathbb{R}^N} \nabla u\,(x,t)\, h(x, \mathbf{x}_i(t))dx\right) & t \in (0,T)\,, \\ u_t - \varepsilon \Delta u_t = \operatorname{div}\left(\mathbf{a}(x,u,\nabla u)\right) + b(x,u) + g(x, \mathbf{X}(t)) & \text{in } \mathbb{R}^N \times (0,T)\,, \\ \mathbf{x}_i(0) = \mathbf{x}_{i0} \in \mathbb{R}^N, \\ u(\cdot,0) = u_0 \in H^1(\mathbb{R}^N), \end{cases} \tag{4.13}$$

for $i = 1, ..., n$.

In the sequel, besides hypothesis $A1$), $A2$)-i) and $A3$), we shall always assume that

$A4)'$ $g \in \operatorname{Lip}\left(\mathbb{R}^{N\times n}; L^2\left(\mathbb{R}^N\right)\right) \cap L^\infty\left(\mathbb{R}^{N\times n}; L^2\left(\mathbb{R}^N\right)\right)$.
Hence there exists $L_g > 0$ such that

$$||g\,(\cdot, \mathbf{X}_1) - g\,(\cdot, \mathbf{X}_2)||_{L^2(\mathbb{R}^N)} \leq L_g\,|\mathbf{X}_1 - \mathbf{X}_2| \tag{4.14}$$

for any $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^{N\times n}$.

$A5)'$ $h \in \operatorname{Lip}\left(\mathbb{R}^N; L^2\left(\mathbb{R}^N\right)\right) \cap L^\infty\left(\mathbb{R}^N; L^2\left(\mathbb{R}^N\right)\right)$.
Hence there exists $L_h > 0$ such that

$$||h\,(\cdot, \mathbf{x}_1) - h\,(\cdot, \mathbf{x}_2)||_{L^2(\mathbb{R}^N)} \leq L_h\,|\mathbf{x}_1 - \mathbf{x}_2| \tag{4.15}$$

for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$.

Definition 20. *For every $u_0 \in H^1\left(\mathbb{R}^N\right)$, $\mathbf{X}_0 \in \mathbb{R}^{N\times n}$, by a solution to problem (4.13) in $(0,T)$ we mean any pair $(u, \mathbf{X})$, with $u \in C^1\left([0,T]; H^1\left(\mathbb{R}^N\right)\right)$, $(\mathbf{x}_1, ..., \mathbf{x}_n) = \mathbf{X} \in C^1\left([0,T]; \mathbb{R}^{Nn}\right)$ such that*

i) $\mathbf{X}\,(0) = \mathbf{X}_0$, $u\,(\cdot, 0) = u_0$ *a.e. in* $\mathbb{R}^N$;

ii) $\mathbf{x}_i$ *is a classical solution of* $(4.13)_1$, *for any* $i = 1, ..., n$;

iii) for every $t \in [0,T]$ *and* $\rho \in H^1(\mathbb{R}^N)$ *there holds*

$$\int_{\mathbb{R}^N} u_t(t)\rho + \varepsilon \nabla u_t(t) \nabla \rho + \mathbf{a}\,(x, u(t), \nabla u(t))\,\nabla \rho - b\,(x, u(t))\,\rho - g\,(x, \mathbf{X}(t))\,\rho\,dx = 0. \tag{4.16}$$

Remark 21. For any solution $(u, \mathbf{X})$ of problem (4.13), assumptions $A4)'$ and $A5)'$ ensure that the mappings $t \mapsto g\,(\cdot, \mathbf{X}(t))$ and $t \mapsto h\,(\cdot, \mathbf{x}_i(t))$ belong to the space $C\left([0,T]; L^2(\mathbb{R}^N)\right)$ (here $i = 1, \cdots, n$).

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The following theorem states well-posedness results for (4.13):

Theorem 22. *Let assumptions A1), A2)-i), A3), and A4)′–A5)′ be satisfied. Then, for every* $(u_0, \mathbf{X}_0) \in H^1(\mathbb{R}^N) \times \mathbb{R}^{N\times n}$ *problem (4.13) admits a unique solution in* $[0,T]$.

Before proving the theorem, we consider an equivalent abstract formulation of system (4.13). Let introduce the mapping

$$\begin{aligned} \mathcal{L} : [0,T] \times \mathbb{R}^{N\times n} \times H^1(\mathbb{R}^N) &\longrightarrow \mathbb{R}^{N\times n} \times H^1(\mathbb{R}^N) \\ (t, \mathbf{X}, u) &\longrightarrow \mathcal{L}(t, \mathbf{X}, u) := (\overline{\mathbf{X}}, \overline{u}) \end{aligned} \tag{4.17}$$

($\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$, with $\mathbf{x}_i \in \mathbb{R}^N$ for every $i = 1, \cdots, n$), defined as follows:

$\mathcal{L}1)$ $\overline{\mathbf{X}} = (\overline{\mathbf{x}}_1, ..., \overline{\mathbf{x}}_n)$, where

$$\overline{\mathbf{x}}_i = F_i\left(t, \mathbf{X}, \int_{\mathbb{R}^N} \nabla u\, h(x, \mathbf{x}_i)\, dx\right) \qquad \forall i = 1, ..., n\,; \tag{4.18}$$

$\mathcal{L}2)$ $\overline{u} \in H^1(\mathbb{R}^N)$ is the unique weak solution of problem

$$-\varepsilon \Delta \overline{u} + \overline{u} = \operatorname{div}(\mathbf{a}(x, u, \nabla u)) + b(x, u) + g(x, \mathbf{X}) \quad \text{in } \mathbb{R}^N\,. \tag{4.19}$$

We observe that the existence and uniqueness of $\overline{u}$ follows from $A2)$–$i)$, $A3)$, and $A4)'$ which guarantee, respectively, $|\mathbf{a}(\cdot, u, \nabla u)| \in L^2(\mathbb{R}^N)$, $b(\cdot, u) \in L^2(\mathbb{R}^N)$, and $g(\cdot, \mathbf{X}) \in L^2(\mathbb{R}^N)$ for any $u \in H^1(\mathbb{R}^N)$ and $\mathbf{X} \in \mathbb{R}^{N\times n}$.

Thus the approximating problem (4.13) can be rewritten in the equivalent form

$$\begin{cases} \dot{y}(t) = \mathcal{L}(t, y(t)) \qquad y \in Y, \\ \\ y(0) = (\mathbf{X}_0, u_0) \in Y \end{cases} \tag{4.20}$$

where we have set

$$Y := \mathbb{R}^{N\times n} \times H^1(\mathbb{R}^N), \qquad y = (\mathbf{X}, u) \in Y\,. \tag{4.21}$$

The following Lemmas 23 and 25 ensure the existence and uniqueness of a local solution $(\mathbf{X}, u)$ of (4.20).

Lemma 23. *Let* Y *be the Banach space in (4.21), and let* $\mathcal{L} : [0,T] \times Y \to Y$ *be the mapping defined in (4.17). Then* $\mathcal{L} \in C([0,T] \times Y; Y)$.

Proof. Lemma 23
Let $t_k \to t \in [0,T]$, $\mathbf{X}_k = (\mathbf{x}_{1,k}, ..., \mathbf{x}_{n,k}) \to \mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ in $\mathbb{R}^{N\times n}$, and

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$u_k \to u$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$. Set

$$\left(\overline{\mathbf{X}}_k, \overline{u}_k\right) = \mathcal{L}\left(t_k, \mathbf{X}_k, u_k\right), \qquad \left(\overline{\mathbf{X}}, \overline{u}\right) = \mathcal{L}\left(t, \mathbf{X}, u\right). \tag{4.22}$$

We shall prove that $\overline{\mathbf{X}}_k \to \overline{\mathbf{X}}$ in $\mathbb{R}^{N\times n}$, and $\overline{u}_k \to \overline{u}$ in $H^1(\mathbb{R}^N)$.

Since $h \in C\left(\mathbb{R}^N; L^2(\mathbb{R}^N)\right)$, whence $h(\cdot, \mathbf{x}_{i,k}) \to h(\cdot, \mathbf{x}_i)$ in $L^2(\mathbb{R}^N)$ as $k \to \infty$, for every $i = 1, ..., n$ we have

$$\int_{\mathbb{R}^N} \nabla u_k \, h(x, \mathbf{x}_{i,k})\, dx \to \int_{\mathbb{R}^N} \nabla u \, h(x, \mathbf{x}_i)\, dx\,,$$

and

$$\overline{\mathbf{x}}_{i,k} := F_i\left(t_k, \mathbf{X}_k, \int_{\mathbb{R}^N} \nabla u_k \, h(x, \mathbf{x}_{i,k})\, dx\right) \to \overline{\mathbf{x}}_i := F_i\left(t, \mathbf{X}, \int_{\mathbb{R}^N} \nabla u \, h(x, \mathbf{x}_i)\, dx\right),$$

as F_i is continuous.

In order to prove that $\overline{u}_k \to \overline{u}$ in $H^1(\mathbb{R}^N)$, we observe that $\overline{u}_k, \overline{u} \in H^1(\mathbb{R}^N)$ are the unique weak solutions in $\mathbb{R}^N$ of

$$-\varepsilon\Delta\overline{u}_k + \overline{u}_k = \operatorname{div}\left(\mathbf{a}\left(\cdot, u_k, \nabla u_k\right)\right) + b\left(\cdot, u_k\right) + g\left(\cdot, \mathbf{X}_k\right), \tag{4.23}$$

and

$$-\varepsilon\Delta\overline{u} + \overline{u} = \operatorname{div}\left(\mathbf{a}\left(\cdot, u, \nabla u\right)\right) + b\left(\cdot, u\right) + g\left(\cdot, \mathbf{X}\right). \tag{4.24}$$

By assumptions $A2$)-i), $A3$), and $A4)'$ we get, respectively,

$$\mathbf{a}\left(\cdot, u_k, \nabla u_k\right) \to \mathbf{a}\left(\cdot, u, \nabla u\right) \quad \text{in } \left[L^2(\mathbb{R}^N)\right]^N \tag{4.25}$$

$$b\left(\cdot, u_k\right) \to b\left(\cdot, u\right) \quad \text{in} \quad L^2(\mathbb{R}^N) \tag{4.26}$$

$$g\left(\cdot, \mathbf{X}_k\right) \to g\left(\cdot, \mathbf{X}\right) \quad \text{in} \quad L^2(\mathbb{R}^N) \tag{4.27}$$

Then, by standard results on elliptic equations, it follows that $\overline{u}_k \to \overline{u}$ in $H^1(\mathbb{R}^N)$.

□

Lemma 24. *There exists $L_1 > 0$ such that for any $(\mathbf{X}_1, u_1)$, $(\mathbf{X}_2, u_2) \in Y$,*

$$||\overline{u}_1 - \overline{u}_2||_{H^1(\mathbb{R}^N)} \le L_1\left(|\mathbf{X}_1 - \mathbf{X}_2| + ||u_1 - u_2||_{H^1(\mathbb{R}^N)}\right), \tag{4.28}$$

where $\overline{u}_1 \in H^1(\mathbb{R}^N)$ is the solution of (4.19) with $u = u_1$ and $\mathbf{X} = \mathbf{X}_1$, and $\overline{u}_2 \in H^1(\mathbb{R}^N)$ is the solution of (4.19) with $u = u_2$ and $\mathbf{X} = \mathbf{X}_2$.

Proof. Lemma 24
First, we observe that $(\overline{u}_1 - \overline{u}_2) \in H^1\left(\mathbb{R}^N\right)$ is the weak solution in $\mathbb{R}^N$ of

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the equation

$$\begin{aligned} -\varepsilon\Delta\left(\overline{u}_1-\overline{u}_2\right)+\left(\overline{u}_1-\overline{u}_2\right) &= \operatorname{div}\left(\mathbf{a}\left(\cdot,u_1,\nabla u_1\right)-\mathbf{a}\left(\cdot,u_2,\nabla u_2\right)\right) \\ &\quad +b\left(\cdot,u_1\right)-b\left(\cdot,u_2\right)+g\left(\cdot,\mathbf{X}_1\right)-g\left(\cdot,\mathbf{X}_2\right). \end{aligned} \tag{4.29}$$

Choosing as test function $\rho=\overline{u}_1-\overline{u}_2$, it follows that

$$\begin{aligned} &\varepsilon\int_{\mathbb{R}^N}\left|\nabla\left(\overline{u}_1-\overline{u}_2\right)\right|^2dx+\int_{\mathbb{R}^N}\left(\overline{u}_1-\overline{u}_2\right)^2dx \\ &\leq L_a\int_{\mathbb{R}^N}\left(\left|u_1-u_2\right|+\left|\nabla(u_1-u_2)\right|\right)\left|\nabla\left(\overline{u}_1-\overline{u}_2\right)\right|\,dx \\ &+L_b\int_{\mathbb{R}^N}\left|u_1-u_2\right|\left|\overline{u}_1-\overline{u}_2\right|\,dx+\|g(\cdot,\mathbf{X}_1)-g(\cdot,\mathbf{X}_2)\|_{L^2(\mathbb{R}^N)}\|\overline{u}_1-\overline{u}_2\|_{L^2(\mathbb{R}^N)} \\ &\leq\left(L_a+L_b+L_g\right)\left\{\|u_1-u_2\|_{H^1(\mathbb{R}^N)}+|\mathbf{X}_1-\mathbf{X}_2|\right\}\|\overline{u}_1-\overline{u}_2\|_{H^1(\mathbb{R}^N)}, \end{aligned} \tag{4.30}$$

with L_a, L_b, L_g as in $A2)$-$i)$, $A3)$ and $A4)'$. Then the conclusion immediately follows from (4.30), with a suitable choice of a constant L_1. □

Lemma 25. *For every $R>0$, there exists $L_R>0$ such that, for any $(\mathbf{X}_1,u_1),(\mathbf{X}_2,u_2)\in Y$, with $||u_1||_{H^1(\mathbb{R}^N)},||u_2||_{H^1(\mathbb{R}^N)}\leq R$, and for every $t\in[0,T]$ it holds*

$$\begin{aligned} &\left|F_i\left(t,\mathbf{X}_1,\int_{\mathbb{R}^N}\nabla u_1\,h(x,\mathbf{x_{1}}_i)\,dx\right)-F_i\left(t,\mathbf{X}_2,\int_{\mathbb{R}^N}\nabla u_2\,h(x,\mathbf{x_{2}}_i)\,dx\right)\right| \\ &\leq L_R\left(|\mathbf{X}_1-\mathbf{X}_2|+\|u_1-u_2\|_{H^1(\mathbb{R}^N)}\right) \end{aligned} \tag{4.31}$$

for any $i=1,...,n$.

Proof. Lemma 25
Let $\mathbf{X}_j=\left(\mathbf{x}_{\mathbf{j}_1},...,\mathbf{x}_{\mathbf{j}_n}\right)\in\mathbb{R}^{N\times n}$ $(j=1,2)$. Then, by $A1)$ it holds

$$\begin{aligned} &\left|F_i\left(t,\mathbf{X}_1,\int_{\mathbb{R}^N}\nabla u_1\,h(x,\mathbf{x}_{1i})\,dx\right)-F_i\left(t,\mathbf{X}_2,\int_{\mathbb{R}^N}\nabla u_2\,h(x,\mathbf{x}_{2i})\,dx\right)\right| \\ &\leq\tilde{L}_{F_i}\left(|\mathbf{X}_1-\mathbf{X}_2|+\int_{\mathbb{R}^N}\left|\nabla u_1\,h(x,\mathbf{x}_{1i})-\nabla u_2\,h(x,\mathbf{x}_{2i})\right|dx\right). \end{aligned} \tag{4.32}$$

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Since $\|u_1\|_{H^1(\mathbb{R}^N)} \leq R$, the thesis follows observing that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_1\, h(x, \mathbf{x}_{1i}) - \nabla u_2\, h(x, \mathbf{x}_{2i})|\, dx \\
&\leq R\|h(\cdot, \mathbf{x}_{1i}) - h(\cdot, \mathbf{x}_{2i})\|_{L^2(\mathbb{R}^N)} + ||u_1 - u_2||_{H^1(\mathbb{R}^N)} \|h\|_{L^\infty(\mathbb{R}^N; L^2(\mathbb{R}^N))} \\
&\leq RL_h |\mathbf{x}_{1i} - \mathbf{x}_{2i}| + \|h\|_{L^\infty(\mathbb{R}^N; L^2(\mathbb{R}^N))} \|u_1 - u_2\|_{H^1(\mathbb{R}^N)}
\end{aligned} \tag{4.33}$$

Here, use of assumption $A5)'$ has been made. □

Proof. Theorem 22
By Lemmas 23-25 we get that for every $(\mathbf{X}_0, u_0) \in Y$, there exists a unique local solution $(\mathbf{X}, u)$ of (4.20), thus of the equivalent problem (4.13),

$$\mathbf{X} \in C^1\left([0, T_\delta]; \mathbb{R}^{N\times n}\right), \qquad u \in C^1\left([0, T_\delta]; H^1\left(\mathbb{R}^N\right)\right) \tag{4.34}$$

for some $T_\delta \in (0, T)$. We now show that $(\mathbf{X}, u)$ can be continued in $[0, T]$, being the unique solution of (4.13). To this end, we prove the existence of a positive constant independent of T_δ, which is an upper-bound for $|\mathbf{X}(\tau)|$ and $||u(\cdot, \tau)||_{H^1(\mathbb{R}^N)}$, for any $0 < \tau < T_\delta$.

From $(4.13)_1$ and $A1)$ we get, for all $t \in (0, T_\delta)$,

$$\begin{aligned}
&|\dot{\mathbf{x}}_i(t)| \leq C_1\left(1 + |\mathbf{X}(t)| + \textstyle\int_{\mathbb{R}^N} |\nabla u(x,t)|\, |h(x, \mathbf{x}_i(t))|\, dx\right) \\
&\leq C_2\left(1 + |\mathbf{X}(t)| + \left(\textstyle\int_{\mathbb{R}^N} |\nabla u(x,t)|^2\, dx\right)^{1/2} \left(\textstyle\int_{\mathbb{R}^N} |h(x, \mathbf{x}_i(t))|^2\, dx\right)^{1/2}\right) \\
&\leq C_3\left(1 + |\mathbf{X}(t)| + ||u(\cdot, t)||_{H^1(\mathbb{R}^N)}\right)
\end{aligned} \tag{4.35}$$

with $C_3 > 0$ independent of T_δ.
Here we have also used that $h \in L^\infty(\mathbb{R}^N; L^2(\mathbb{R}^N))$; see $A5)'$. Integrating the above estimate over $[0, \tau]$, with $0 < \tau < T_\delta$ we obtain

$$|\mathbf{X}(\tau)| \leq C_4\left(1 + \int_0^\tau \|u(\cdot, t)\|_{H^1(\mathbb{R}^N)} dt + \int_0^\tau |\mathbf{X}(t)|\, dt\right). \tag{4.36}$$

By Gronwall inequality it follows

$$|\mathbf{X}(\tau)| \leq C_5\left(1 + \int_0^\tau ||u(\cdot, t)||_{H^1(\mathbb{R}^N)} dt\right) \qquad \forall \tau \in (0, T_\delta) \tag{4.37}$$

where $C_5 = C_{5,T} > 0$ can be made independent of T_δ. We now focus on an estimate of $\|u(\cdot, \tau)\|_{H^1(\mathbb{R}^N)}$. For every $t \in (0, T_\delta)$, choosing $\rho = u_t(t)$ as test function in (4.16), by assumptions $A2)$-$i)$, $A3)$ and $A4)'$ it holds

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$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(\varepsilon \left|\nabla u_t(x,t)\right|^2 + \left|u_t(x,t)\right|^2 \, dx\right) \\
&\leq \int_{\mathbb{R}^N} \left|\mathbf{a}(x,u(x.t),\nabla u(x,t))\right| \left|\nabla u_t(x,t)\right| dx \\
&+ \int_{\mathbb{R}^N} \left(\left|b(x,u(t))\right| + \left|g(x,\mathbf{X}(t))\right|\right) \left|u_t(x,t)\right| dx \\
&\leq L_a \int_{\mathbb{R}^N} \left(\left|u(x,t)\right| + \left|\nabla u(x,t)\right|\right) \left|\nabla u_t(x,t)\right| dx + L_b \int_{\mathbb{R}^N} \left|u(x,t)\right| \left|u_t(x,t)\right| dx \\
&+ \|g\|_{L^\infty(\mathbb{R}^{N\times n};L^2(\mathbb{R}^N))} \left(\int_{\mathbb{R}^N} \left(u_t(x,t)\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon \left|\nabla u_t(x,t)\right|^2 + \left|u_t(x,t)\right|^2\right) dx \\
&+ C_{1,\varepsilon} \left[1 + \int_{\mathbb{R}^N} \left(u^2(x,t) + \left|\nabla u(x,t)\right|^2\right) dx\right],
\end{aligned}
\tag{4.38}
$$

with $C_{1,\epsilon} := \frac{L_a^2}{\epsilon} + L_b^2 + \|g\|^2_{L^\infty(\mathbb{R}^{N\times n};L^2(\mathbb{R}^N))}$. Then we conclude that

$$
||u_t(\cdot,t)||_{H^1(\mathbb{R}^N)} \leq C_{2,\varepsilon} \left(||u(\cdot,t)||_{H^1(\mathbb{R}^N)} + 1\right). \tag{4.39}
$$

Integrating the above inequality in $[0,\tau]$ and using the Gronwall's inequality, we get that there exists $C_{3,\varepsilon} > 0$ independent of T_δ, such that for all $\tau \in [0,T_\delta)$ there holds

$$
||u(\cdot,\tau)||_{H^1(\mathbb{R}^N)} \leq C_{3,\varepsilon}\,. \tag{4.40}
$$

Combining (4.40), (4.37) and (4.35) gives, for some $C_{4,\varepsilon} > 0$ indepenedent of T_δ,

$$
|\mathbf{X}(\tau)| \leq C_{4,\varepsilon}, \qquad \left|\dot{\mathbf{X}}(\tau)\right| \leq C_{4,\varepsilon}\,,
$$

for all $\tau \in [0,T_\delta)$. Therefore the conclusion immediately follows. □

4.2.1 A priori estimates

In the following we suppose assumptions A1)-A6) to be satisfied. In particular, from A1) and A3) it follows that there exists $C_F > 0$ $C_a > 0$ and $C_b > 0$ such that

$$
\left|F_i\left(t,\mathbf{X},\mathbf{z}\right)\right| \leq C_F\left(1 + |\mathbf{X}| + |\mathbf{z}|\right) \qquad \forall i = 1,...,n, \tag{4.41}
$$

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$$|\mathbf{a}(x,u,\mathbf{z})| \le C_a(|x|+|u|) \qquad \forall i=1,...,n, \tag{4.42}$$

$$|b(x,u)| \le C_b|u| \tag{4.43}$$

for any $\mathbf{X}\in\mathbb{R}^{N\times n}$, $\mathbf{z},x\in\mathbb{R}^N$, $u\in\mathbb{R}$ and $t\in[0,T]$.

For every $j\in\mathbb{N}$, we consider the function $l_j\in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(l_j)\subseteq[-j-1,j+1]$, such that

$$0\le l_j(s)\le 1, \qquad l_j(s)=1 \qquad \forall s\in\mathbb{R}:|s|\le j. \tag{4.44}$$

Let $\mathbf{X}\in\mathbb{R}^{N\times n}$, $\hat{\mathbf{x}}\in\mathbb{R}^N$. We define the sequences $\{g^j(\cdot,\mathbf{X})\}_{j\in\mathbb{N}}$ and $\{h^j(\cdot,\hat{\mathbf{x}})\}_{j\in\mathbb{N}}$ by setting

$$g^j(\cdot,\mathbf{X}) := l_j(|\mathbf{X}|)\,g(\cdot,\mathbf{X}) \qquad a.e.\ \text{in}\ \mathbb{R}^N, \tag{4.45}$$

$$h^j(\cdot,\hat{\mathbf{x}}) := l_j(|\hat{\mathbf{x}}|)\,h(\cdot,\hat{\mathbf{x}}) \qquad a.e.\ \text{in}\ \mathbb{R}^N. \tag{4.46}$$

Clearly, there holds

$$g^j\to g \quad \text{in}\ C_{\mathrm{loc}}(\mathbb{R}^{N\times n};L^2(\mathbb{R}^N)), \tag{4.47}$$

$$h^j\to h \quad \text{in}\ C_{\mathrm{loc}}(\mathbb{R}^{N};L^2(\mathbb{R}^N)). \tag{4.48}$$

By $A4$)–$A5$) and the very definition of $\{g^j\}$ and $\{h^j\}$, for every $j\in\mathbb{N}$ there exist C_g^j, $C_h^j>0$ such that

$$\|g^j\|_{L^\infty(\mathbb{R}^{N\times n};L^2(\mathbb{R}^N))}\le C_g^j, \tag{4.49}$$

$$\|h^j\|_{L^\infty(\mathbb{R}^{N};L^2(\mathbb{R}^N))}\le C_h^j, \tag{4.50}$$

whereas by assumption $A6$) and (4.44) we get

$$\left\|g^j(\cdot,\mathbf{X})\right\|_{L^2(\mathbb{R}^N)}\le C_g(1+|\mathbf{X}|)^{\theta_2}, \tag{4.51}$$

$$\left\|h^j(\cdot,\hat{\mathbf{x}})\right\|_{L^2(\mathbb{R}^N)}\le C_h(1+|\hat{\mathbf{x}}|)^{\theta_1}, \tag{4.52}$$

for any $\mathbf{X}\in\mathbb{R}^{N\times n}$, $\hat{\mathbf{x}}\in\mathbb{R}^N$.

Let $u_0\in L^2(\mathbb{R}^N)$ and $\mathbf{X}_0\in\mathbb{R}^{N\times n}$. In Lemmas 26–29 below, we present some a priori estimates on the solution $(\mathbf{X}^j,u^j)$ of the approximating prob-

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lem

$$\begin{cases} \dot{\mathbf{x}}_i^j(t) = F_i\left(t, \mathbf{X}^j(t), \int_{\mathbb{R}^N} \nabla u^j(x,t)\, h^j(x, \mathbf{x}_i^j(t))dx\right) & t \in (0,T), \\ u_t^j - \varepsilon_j \Delta u_t^j = \operatorname{div}\left(\mathbf{a}(x, u^j, \nabla u^j)\right) + b(x, u^j) + g^j(x, \mathbf{X}^j(t)) \quad \text{in } Q, \\ \mathbf{x}_i^j(0) = \mathbf{x}_{i0} \in \mathbb{R}^N, \\ u^j(\cdot, 0) = u_0^j \in H^1(\mathbb{R}^N), \end{cases} \tag{4.53}$$

for $i = 1, ..., n$. Here

$$\varepsilon_j \to 0^+, \tag{4.54}$$

and the sequence $\{u_0^j\} \subseteq H^1(\mathbb{R}^N)$ is chosen so that

$$u_0^j \to u_0 \quad \text{in } L^2(\mathbb{R}^N), \tag{4.55}$$

$$\sqrt{\varepsilon_j}\left|\nabla u_0^j\right| \to 0 \quad \text{in } L^2(\mathbb{R}^N) \tag{4.56}$$

as $j \to \infty$ (such a sequence can be constructed by a convolution of u_0 with a sequence of standard mollifiers).

Notice that the existence of a unique solution to problem (4.53) follows from Theorem 22, since for every j the functions g^j and h^j satisfy assumptions $A4)'$ and $A5)'$, respectively.

Lemma 26. *There exist $C_1, C_2 > 0$ independent of j, such that for every $\tau \in (0,T]$ there holds*

i) if $0 \le \theta_1 < 1$

$$\sup_j \left|\mathbf{X}^j(\tau)\right| \le C_1 \left[1 + \left(\int_0^\tau ||\nabla u^j(\cdot,t)||^2_{L^2(\mathbb{R}^N)} dt\right)^{\frac{1}{2(1-\theta_1)}}\right], \tag{4.57}$$

$$\sup_j \left|\left|g^j\left(\cdot, \mathbf{X}^j(\tau)\right)\right|\right|_{L^2(\mathbb{R}^N)} \le C_2 \left[1 + \left(\int_0^\tau ||\nabla u^j(\cdot,t)||^2_{L^2(\mathbb{R}^N)} dt\right)^{\frac{\theta_2}{2(1-\theta_1)}}\right]; \tag{4.58}$$

ii) if $\theta_1 = 1$

$$\sup_j \left|\mathbf{X}^j(\tau)\right| \le C_1 \left[1 + e^{C_1\left(\int_0^\tau ||\nabla u^j(\cdot,t)||^2_{L^2(\mathbb{R}^N)} dt\right)^{\frac{1}{2}}}\right], \tag{4.59}$$

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$$\sup_j \left|\left|g^j\left(\cdot, \mathbf{X}^j(\tau)\right)\right|\right|_{L^2(\mathbb{R}^N)} \leq C_g\,, \tag{4.60}$$

with $C_g > 0$ *defined in* (4.51).

Proof. Lemma 26
For every $\tau \in (0,T]$, integrating equation $(4.53)_1$ in $(0,\tau)$ and using (4.41) gives

$$\begin{aligned}\left|\mathbf{x}_i^j(\tau)\right| &\leq |\mathbf{x}_i(0)| + C_F T + C_F \int_0^\tau \left|\mathbf{X}^j(t)\right| dt \\ &+C_F \int_0^\tau ||\nabla u^j(\cdot,t)||_{L^2(\mathbb{R}^N)} ||h^j\left(\cdot, \mathbf{x}_i^j(t)\right)||_{L^2(\mathbb{R}^N)} dt\end{aligned} \tag{4.61}$$

for all $i = 1, \cdots, n$. By (4.52), considering the sum over $i = 1, ..., n$, it follows

$$\begin{aligned}1 + \left|\mathbf{X}^j(\tau)\right| &\leq \overline{C}\left[1 + \int_0^\tau \left(1 + \left|\mathbf{X}^j(t)\right|\right) dt + \right. \\ &\left. \int_0^\tau \|\nabla u^j(\cdot,t)\|_{L^2(\mathbb{R}^N)} \left(1 + \left|\mathbf{X}^j(t)\right|\right)^{\theta_1} dt\right],\end{aligned} \tag{4.62}$$

for some $\overline{C} > 0$ independent of j.

i) Let $0 \leq \theta_1 < 1$, $\tau \in (0,T]$ (whence $0 < \theta_2 \leq 1$; see assumption $A6$)). Using a non linear generalized form of Gronwall inequality ([70], Theorem 21, p.11), we obtain

$$\begin{aligned}1 + \left|\mathbf{X}^j(\tau)\right| &\leq \left\{\overline{C}^{(1-\theta_1)} e^{(1-\theta_1)\overline{C}\tau} \right. \\ &\left. +(1-\theta_1)\int_0^\tau \overline{C} ||\nabla u^j(\cdot,t)||_{L^2(\mathbb{R}^N)} e^{(1-\theta_1)\overline{C}(\tau-t)} dt\right\}^{\frac{1}{1-\theta_1}}.\end{aligned} \tag{4.63}$$

Since $\tau \in (0,T]$, we finally get

$$\begin{aligned}\left|\mathbf{X}^j(\tau)\right| &\leq \tilde{C}\left\{1 + \int_0^\tau ||\nabla u^j(\cdot,t)||_{L^2(\mathbb{R}^N)} dt\right\}^{\frac{1}{(1-\theta_1)}} \\ &\leq C_1\left\{1 + \left(\int_0^\tau ||\nabla u^j(\cdot,t)||^2_{L^2(\mathbb{R}^N)} dt\right)^{\frac{1}{2(1-\theta_1)}}\right\},\end{aligned} \tag{4.64}$$

with $C_1 > 0$ independent of j.

In order to prove (4.58), it suffices to combine (4.57) and (4.51).

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ii) Let $\theta_1 = 1$. In this case (4.62) reduces to

$$\begin{aligned} 1 + \left|\mathbf{X}^j(\tau)\right| \quad &\leq \overline{C} \left[1 + \int_0^\tau \left(1 + \left|\mathbf{X}^j(t)\right|\right) dt + \right. \\ &\left. \int_0^\tau \left|\left|\nabla u^j\left(\cdot, t\right)\right|\right|_{L^2(\mathbb{R}^N)} \left(1 + \left|\mathbf{X}^j(t)\right|\right) dt \right], \end{aligned} \tag{4.65}$$

and (4.60) follows from the Gronwall's inequality (*e.g.*, see [70] Theorem 21). Moreover, in this case assumption A6) implies $\theta_2 = 0$, hence estimate (4.60) immediately follows from (4.51). This concludes the proof.

□

Lemma 27. *There exists $C_3 > 0$ such that, for every $j \in \mathbb{N}$,*

$$\|u^j\|_{L^\infty((0,T);L^2(\mathbb{R}^N))} + \| \left|\nabla u^j\right| \|_{L^2(Q)} \leq C_3, \tag{4.66}$$

$$\|\sqrt{\varepsilon_j}\, u_t^j\|_{L^2(Q)} + \|\varepsilon_j \left|\nabla u_t^j\right| \|_{L^2(Q)} \leq C_3\,. \tag{4.67}$$

Remark 28. The estimates obtained in (4.57)-(4.60) and (4.66) ensure the existence of $C_4 > 0$, independent of $j \in \mathbb{N}$ and $t \in (0,T)$, such that

$$\sup_j \|\mathbf{X}^j\|_{L^\infty(0,T;\mathbb{R}^{N\times n})} \leq C_4, \tag{4.68}$$

$$\sup_j \left\|g^j\left(\cdot, \mathbf{X}^j(t)\right)\right\|_{L^2(\mathbb{R}^N)} \leq C_4 \quad \text{for all } \ t \in [0,T]. \tag{4.69}$$

In particular, combining (4.66) and (4.69) with assumptions $A2)$-$i)$ and $A3)$, it follows that $\left\{\text{div}\left(\mathbf{a}(\cdot, u^j, \nabla u^j)\right)\right\}$ is bounded in $L^2\left((0,T); H^{-1}(\mathbb{R}^N)\right)$ and $\{b(\cdot, u^j)\}$ is bounded in $L^\infty\left(0,T; L^2(\mathbb{R}^N)\right)$, whence (see $(4.53)_2$, (4.67) and (4.69))

$$\{u_t^j\} \text{ is bounded in } \ L^2\left(0,T; H^{-1}(\mathbb{R}^N)\right). \tag{4.70}$$

Proof. Lemma 27
We rewrite the pseudoparabolic equation in (4.53), choosing u^j as test function in its weak fromulation. For every $\tau \in (0,T]$, we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} \left\{\left(u^j(\tau)\right)^2 + \varepsilon_j \left|\nabla u^j(\tau)\right|^2\right\} dx - \frac{1}{2}\int_{\mathbb{R}^N} \left\{\left(u_0^j\right)^2 + \varepsilon_j \left|\nabla u_0^j\right|^2\right\} dx \\ &+ \iint_{Q_\tau} \mathbf{a}(x, u^j, \nabla u^j)\cdot \nabla u^j dx dt = \iint_{Q_\tau} b(x, u^j) u^j dx dt + \iint_{Q_\tau} g^j\left(x, \mathbf{X}^j(t)\right) u^j dx dt \\ &\leq C_b \iint_{Q_\tau} \left(u^j\right)^2 dx dt + \frac{\overline{C}}{2}\iint_{Q_\tau} \left(g^j\left(x, \mathbf{X}^j(t)\right)\right)^2 dx dt + \frac{1}{2\overline{C}}\iint_{Q_\tau} \left(u^j\right)^2 dx dt\,, \end{aligned} \tag{4.71}$$

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for a suitable constant $\overline{C} > 0$ which will be chosen below; observe that in the last inequality, we have used (4.43) and the Hölder inequality. Let focus on the case of $0 \leq \theta_1 < 1$. Estimating the integral

$$\int_{\mathbb{R}^N} \left\{ (u_0^j)^2 + \varepsilon_j |\nabla u_0^j|^2 \right\} dx$$

with the constant $\overline{C}_0$ (see (4.55)–(4.56)), by $A2$)-iii), $A6$) (in particular, recall that $\theta_1 + \theta_2 \leq 1$) and (4.58) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} \left\{ \left(u^j(\tau)\right)^2 + \varepsilon_j \left|\nabla u^j(\tau)\right|^2 \right\} dx + \alpha_0 \iint_{Q_\tau} \left|\nabla u^j\right|^2 dxdt - \overline{C}_0 \\
&\leq \left(C_b + \frac{1}{2\overline{C}}\right) \iint_{Q_\tau} \left(u^j\right)^2 dxdt + \frac{\overline{C}}{2} \int_0^\tau C_2^2 \left[1 + \left(\int_0^t ||\nabla u^j(s)||^2_{L^2(\mathbb{R}^N)} ds\right)^{\frac{\theta_2}{2(1-\theta_1)}}\right]^2 dt \\
&\leq \left(C_b + \frac{1}{2\overline{C}}\right) \iint_{Q_\tau} \left(u^j\right)^2 dxdt + \overline{C} C_2^2 \int_0^\tau \left[1 + \left(1 + \int_0^t ||\nabla u^j(s)||^2_{L^2(\mathbb{R}^N)} ds\right)^{\frac{\theta_2}{(1-\theta_1)}}\right] dt \\
&\leq \left(C_b + \frac{1}{2\overline{C}}\right) \iint_{Q_\tau} \left(u^j\right)^2 dxdt + \overline{C} C_2^2 \left[2T + \int_0^\tau \int_0^\tau ||\nabla u^j(s)||^2_{L^2(\mathbb{R}^N)} ds dt\right] \\
&\leq \left(C_b + \frac{1}{2\overline{C}}\right) \iint_{Q_\tau} \left(u^j\right)^2 dxdt + 2T\overline{C}C_2^2 + \overline{C}C_2^2 T \iint_{Q_\tau} \left|\nabla u^j\right|^2 dxdt.
\end{aligned} \tag{4.72}$$

Choosing $\overline{C} = \frac{\alpha_0}{2C_2^2 T}$ we get that $\overline{C}C_2^2 T = \frac{\alpha_0}{2}$, hence the above inequality gives

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} \left\{ \left(u^j(\tau)\right)^2 + \varepsilon_j \left|\nabla u^j(\tau)\right|^2 \right\} dx + \frac{\alpha_0}{2} \iint_{Q_\tau} \left|\nabla u^j\right|^2 dxdt \\
&\leq \overline{C}_1 \left[1 + \iint_{Q_\tau} \left(u^j\right)^2 dxdt\right],
\end{aligned} \tag{4.73}$$

where $\overline{C}_1$ is a positive constante independent of j. By Gronwall inequality, there exists $\overline{C}_4 > 0$ such that

$$||u^j||_{L^\infty(0,T;L^2(\mathbb{R}^N))} \leq \overline{C}_2. \tag{4.74}$$

Combining (4.73) and (4.74) (the former with $\tau = T$), we conclude that

$$\sup_j \iint_Q \left|\nabla u^j\right|^2 \leq \overline{C}_3. \tag{4.75}$$

Then (4.66) follows with $C_3 = \max\{\overline{C}_2, \overline{C}_3\}$.
We leave the proof of the easier case $\theta_1 = 1$, which follows using (4.60) instead of (4.58) in the above computations.

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The proof of (4.67) relies on similar steps. The key factor is to choose the test function $\rho = \varepsilon_j u_t^j$ in the weak formulation of the pseudoparabolic equation in (4.53), using the a priori estimates (4.66) and (4.69) (we omit the details). □

Lemma 29. *There exists $C_5 > 0$ such that, for every $j \in \mathbb{N}$,*

$$\int_0^T \left|\dot{\mathbf{X}}^j(t)\right|^2 dt \leq C_5. \tag{4.76}$$

Proof. Lemma 29
From the first equation in (4.53), (4.41), (4.68) and assumption $A6$)), we get, for every $i = 1, ..., n$

$$\begin{aligned}
\left|\dot{\mathbf{x}}_i^j(t)\right|^2 &\leq C_F^2 \left(1 + \left|\mathbf{X}^j(t)\right| + \int_{\mathbb{R}^N} \left|\nabla u^j(x,t)\right| \left|h^j\left(x, \mathbf{x}_i^j(t)\right)\right| dx\right)^2 \\
&\leq C_F^2 \left(1 + C_4 + \left|\left|\nabla u^j(\cdot,t)\right|\right|_{L^2(\mathbb{R}^N)} \left|\left|h^j\left(\cdot, \mathbf{x}_i^j(t)\right)\right|\right|_{L^2(\mathbb{R}^N)}\right)^2 \\
&\leq C_F^2 \left(1 + C_4 + C_h\left(1 + \left|\mathbf{X}^j(t)\right|\right)^{\theta_1} \left|\left|\nabla u^j(\cdot,t)\right|\right|_{L^2(\mathbb{R}^N)}\right)^2 \\
&\leq C\left(1 + \left|\left|\nabla u^j(\cdot,t)\right|\right|^2_{L^2(\mathbb{R}^N)}\right),
\end{aligned} \tag{4.77}$$

for a suitable $C > 0$ independent of j and $t \in (0,T)$. Thus,

$$\int_0^T \left|\dot{\mathbf{x}}_i^j(t)\right|^2 dt \leq CT + C \iint_Q \left|\nabla u^j(x,t)\right|^2 dxdt\,,$$

and the conclusion follows combining the above inequality and (4.66). □

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4.3 Global Existence and Uniqueness

4.3.1 Letting $\varepsilon \to 0$

Throughout this subsection we shall always assume that assumptions $A1$)–$A6$) are satisfied.

For every $j \in \mathbb{N}$, let $\left(u^j, \mathbf{X}^j\right)$ be the solution to problem (4.53), in the sense of Definition 20.
By the a priori estimates (4.66)-(4.70) and (4.76) we deduce the following convergence results.

Proposition 30. *(i) There exists* $u \in C\left([0,T]; L^2\left(\mathbb{R}^N\right)\right) \cap L^2\left(0,T; H^1\left(\mathbb{R}^N\right)\right)$, *with* $u_t \in L^2\left(0,T; H^{-1}\left(\mathbb{R}^N\right)\right)$, *such that possibly up to a subsequence (not relabeled) it holds*

1) $u^j \to u$ *in* $L^2(Q)$, *and* $u^j \to u$ *a.e. in* Q;

2) $\nabla u^j \rightharpoonup \nabla u$ *in* $\left[L^2(Q)\right]^N$;

3) $\varepsilon_j u_t^j \to 0$ *in* $L^2(Q)$;

4) $\varepsilon_j \nabla u_t^j \rightharpoonup 0$ *in* $\left[L^2(Q)\right]^N$.

(ii) There exists $\mathbf{X} \in W^{1,2}\left(0,T; \mathbb{R}^{N\times n}\right)$ *such that possibly up to a subsequence (not relabeled) it holds*

5) $\mathbf{X}^j \to \mathbf{X}$ *in* $C\left([0,T]; \mathbb{R}^{N\times n}\right)$,

6) $\dot{\mathbf{X}}^j \rightharpoonup \dot{\mathbf{X}}$ *in* $L^2\left(0,T; \mathbb{R}^{N\times n}\right)$.

Remark 31. Recalling the definition of the approximating sequences $\{g^j\}$ and $\{h^j\}$ in (4.45)-(4.46), together with the continuity properties of g and h in assumptions $A4$) and $A5$), from 5) it follows that for all $t \in (0,T)$ there holds

7) $g^j\left(\cdot, \mathbf{X}^j(t)\right) \longrightarrow g\left(\cdot, \mathbf{X}(t)\right)$ in $L^2\left(\mathbb{R}^N\right)$,

8) $h^j\left(\cdot, \mathbf{x}_i^j(t)\right) \longrightarrow h\left(\cdot, \mathbf{x}_i(t)\right)$ in $L^2\left(\mathbb{R}^N\right)$, for any $i = 1, ..., n$,

as $j \to \infty$. Since by (4.68)–(4.69) and assumption $A6$) we have

$$\sup_{(0,T)} \left\| g^j\left(\cdot, \mathbf{X}^j(t)\right)\right\|_{L^2(\mathbb{R}^N)} \le C_4 \tag{4.78}$$

$$\sup_{(0,T)} \left\| h^j\left(\cdot, \mathbf{x}_i^j(t)\right)\right\|_{L^2(\mathbb{R}^N)} \le C_h(1 + \|\mathbf{X}^j\|_{L^\infty(\mathbb{R}^{N\times n})})^{\theta_1} \le C_h(1+C_4)^{\theta_1}, \tag{4.79}$$

by the dominated convergence theorem we get

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9) $\lim_{j\to\infty}\int_0^T \left\|g^j\left(\cdot,\mathbf{X}^j(t)\right)-g\left(\cdot,\mathbf{X}(t)\right)\right\|^p_{L^2(\mathbb{R}^N)}dt=0,$

10) $\lim_{j\to\infty}\int_0^T \left\|h^j\left(\cdot,\mathbf{x}_i^j(t)\right)-h\left(\cdot,\mathbf{x}_i(t)\right)\right\|^p_{L^2(\mathbb{R}^N)}dt=0$

for any $p\in[1,\infty)$ and $i=1,\cdots,n$.

Proof of Proposition 30.
Claims 2)–6) immediately follow from the a priori estimates (4.66)-(4.70) and (4.76). By (4.66), (4.70) and classical compactness arguments [71] possibly up to a subsequence (not relabeled) we have

$$u^j \rightharpoonup u \quad \text{in } L^2(Q), \tag{4.80}$$

$$u^j \to u \quad \text{in } L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^N)), \tag{4.81}$$

for some $u\in C\left([0,T];L^2\left(\mathbb{R}^N\right)\right)\cap L^2\left(0,T;H^1\left(\mathbb{R}^N\right)\right)$, with distributional derivative $u_t\in L^2\left(0,T;H^{-1}\left(\mathbb{R}^N\right)\right)$. Observe that without loss of generality, by standard diagonal arguments, we may assume that $u^j\to u$ *a.e.* in Q. Then, it only remains to prove that

$$u^j \to u \quad \text{in } L^2(Q). \tag{4.82}$$

For every $p\in\mathbb{N}$, let $\rho_p\in C^\infty(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ be chosen so that $\rho_p(x)=0$ if $|x|\le p$, $\rho_p(x)=1$ if $|x|\ge 2p$, $0\le\rho_p(x)\le 1$ for all $x\in\mathbb{R}^N$, and $\||\nabla\rho_p|\|_{L^\infty(\mathbb{R}^N)}\le\tilde{C}$, for some $\tilde{C}>0$ independent of p. Choosing $\psi(x,t)=\rho_p^2(x)u^j(x,t)$ as test function in the weak formulation of the pseudoparabolic equation in (4.53) gives, for every $\tau\in(0,T]$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\left(u^j(x,\tau)\right)^2\rho_p^2(x)\,dx-\frac{1}{2}\int_{\mathbb{R}^N}(u_0^j)^2\rho_p^2dx\\
&+\frac{\varepsilon_j}{2}\int_{\mathbb{R}^N}|\nabla u^j(x,\tau)|^2\rho_p^2(x)\,dx-\frac{\varepsilon_j}{2}\int_{\mathbb{R}^N}|\nabla u_0^j|^2\rho_p^2\,dx\\
&+2\varepsilon_j\iint_{Q_\tau}\nabla u_t^j\cdot\nabla\rho_p\,u^j\rho_p\,dxdt+\iint_{Q_\tau}\mathbf{a}(x,u^j,\nabla u^j)\cdot\nabla u^j\,\rho_p^2\,dxdt\\
&\le 2\iint_{Q_\tau}|\mathbf{a}(x,u^j,\nabla u^j)|\,|\nabla\rho_p|\,\rho_p\,|u^j|\,dxdt+\iint_{Q_\tau}|b(x,u^j)|\,|u^j|\,\rho_p^2dxdt\\
&+\iint_{Q_\tau}|g^j(x,\mathbf{X}^j(t))|\,|u^j|\,\rho_p^2\,dxdt\le 2\iint_{Q}|\mathbf{a}(x,u^j,\nabla u^j)|\,|\nabla\rho_p|\,\rho_p\,|u^j|\,dxdt\\
&+\left(C_b+\frac{1}{2}\right)\iint_{Q_\tau}(u^j)^2\,\rho_p^2\,dxdt+\iint_{Q}(g^j(x,\mathbf{X}^j(t)))^2\,\rho_p^2\,dxdt
\end{aligned}$$

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(here we have used (4.43)). Then, from the Gronwall's inequality, for all $\tau \in (0,T]$ we have

$$\int_{\mathbb{R}^N} \left(u^j(x,\tau)\right)^2 \rho_p^2(x)\,dx \le \tilde{C}_1 \left\{ \int_{\mathbb{R}^N} (u_0^j)^2 \rho_p^2\,dx \right.$$

$$+\varepsilon_j \int_{\mathbb{R}^N} |\nabla u_0^j|^2 \rho_p^2\,dx - 2\varepsilon_j \iint_Q \nabla u_t^j \cdot \nabla \rho_p\, u^j \rho_p\,dxdt$$

$$\left. + \iint_Q \left[|\mathbf{a}(x,u^j,\nabla u^j)|\,|\nabla \rho_p|\,\rho_p\,|u^j| + \left(g^j(x,\mathbf{X}^j(t))\right)^2 \rho_p^2 \right]\,dxdt \right\}$$

for some $\tilde{C}_1 > 0$ independent of both j and p.

In view of (4.80)–(4.81), (4.55)–(4.56), and point 4) in Proposition 30, Remark 31, and since we may assume $|\mathbf{a}(x,u^j,\nabla u^j)| \rightharpoonup F$ in $L^2(Q)$ for some nonnegative $F \in L^2(Q)$, integrating the above inequality in $(0,T)$ and letting $j \to \infty$, we get

$$\limsup_{j\to\infty} \iint_Q \left(u^j\right)^2 \rho_p^2\,dxdt \le \tilde{C}_2 \left\{ \int_{\mathbb{R}^N} u_0^2 \rho_p^2\,dx + \iint_Q \left(g(x,\mathbf{X}(t))\right)^2 \rho_p^2\,dxdt \right.$$

$$\left. + \iint_{\{(x,t)\in Q:\, p\le|x|\le 2p\}} F\,|\nabla \rho_p|\,\rho_p\,|u|\,dxdt \right\}$$

(here we have used that $\nabla \rho_p$ has compact support in $\mathbb{R}^N$). Since the functions in the right-hand side of the previous inequality belong to $L^2(\mathbb{R}^N)$ and $L^2(Q)$, by the very definition of the function ρ_p we have

$$\limsup_{p\to\infty} \left(\limsup_{j\to\infty} \iint_Q \left(u^j\right)^2 \rho_p^2\,dxdt \right) = 0\,. \tag{4.83}$$

Then, it is enough to observe that

$$\limsup_{j\to\infty} \|u^j - u\|_{L^2(Q)} \le \lim_{j\to\infty} \iint_Q (u^j-u)^2(1-\rho_p^2)\,dxdt$$

$$+ \limsup_{j\to\infty} \iint_Q (u^j-u)^2 \rho_p^2\,dxdt = \limsup_{j\to\infty} \iint_Q (u^j-u)^2 \rho_p^2\,dxdt$$

(notice that $\iint_Q (u^j-u)^2(1-\rho_p^2)\,dxdt \to 0$ by (4.81), as the function $\phi(x) = (\rho_p^2(x)-1)$ has compact support in $\mathbb{R}^N$), and the convergence in (4.82) easily follows from the above inequality, (4.83) and the condition $u \in L^2(Q)$. □

Remark 32. (*i*) Observe that (4.42) and (4.66) ensure the existence of

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$\mathbf{z} =\in \left[L^2(Q)\right]^N$ such that

$$\mathbf{a}\left(\cdot, u^j, \nabla u^j\right) \rightharpoonup \mathbf{z} \quad in \left[L^2\left(Q\right)\right]^N. \tag{4.84}$$

Moreover, by (4.43) and the convergences in Proposition 30- 1), it holds

$$b\left(\cdot, u^j\right) \to b\left(\cdot, u\right) \quad \text{in} \quad L^2(Q). \tag{4.85}$$

(ii) In view of (4.54)–(4.56), and the convergences in Proposition 30 and in Remark 31, letting $j \to \infty$ in the weak formulation of the pseudoparabolic equations in (4.53) we get

$$\begin{aligned} &\iint_Q u\phi_t \, dxdt - \iint_Q \left\{\mathbf{z} \cdot \nabla\phi - b(x,u)\phi - g(x, \mathbf{X}(t))\phi\right\} \, dxdt \\ &= \int_{\mathbb{R}^N} u(x,T)\phi(x,T)\, dx - \int_{\mathbb{R}^N} u_0(x)\phi(x,0)\, dx \end{aligned} \tag{4.86}$$

for all $\phi \in C([0,T]; L^2(\mathbb{R}^N)) \cap L^2(0,T; H^1(\mathbb{R}^N))$, such that $\phi_t \in L^2(Q)$.

The following proposition states the relationship between u and $\mathbf{z}$.

Proposition 33. *Let u and $\mathbf{z}$ be the limiting functions in Proposition 30 and (4.84), respectively. Then, for a.e. $(x,t) \in Q_T$, it holds*

$$\mathbf{z}\left(x,t\right) = \mathbf{a}\left(x, u(x,t), \nabla u(x,t)\right). \tag{4.87}$$

Proof. Proposition 33
Let $\rho \in C_c^1\left(\mathbb{R}^N\right)$ and $\psi \in C_c^1\left(Q\right)$, $\psi \geq 0$. Choosing $\left(u^j - \rho\right)\psi$ as test function in the weak formulation of the pseudoparabolic equation in (4.53), we get

$$\begin{aligned} &\iint_Q \mathbf{a}\left(x, u^j, \nabla u^j\right) \cdot \nabla\left(u^j - \rho\right)\psi \, dxdt \\ &= -\varepsilon_j \iint_Q \nabla u_t^j \cdot \nabla\left[\left(u^j - \rho\right)\psi\right] \, dxdt - \iint_Q \mathbf{a}\left(x, u^j, \nabla u^j\right) \cdot \nabla\psi\left(u^j - \rho\right) \, dxdt \\ &+ \iint_Q \psi_t \frac{\left(u^j - \rho\right)^2}{2} \, dxdt + \iint_Q b\left(x, u^j\right)\left(u^j - \rho\right)\psi \, dxdt \\ &+ \iint_Q g\left(x, \mathbf{X}^j\left(t\right)\right)\left(u^j - \rho\right)\psi \, dxdt. \end{aligned} \tag{4.88}$$

Focusing on the first integral in the right-hand side of the above equality,

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we observe that

$$\begin{aligned}\varepsilon_j \iint_Q \nabla u_t^j \cdot \nabla \left[\left(u^j-\rho\right)\psi\right] dxdt &= \varepsilon_j \iint_Q \nabla u_t^j \cdot \nabla\psi \left(u^j-\rho\right) dxdt \\ -\frac{\varepsilon_j}{2}\iint_Q |\nabla u_j|^2 \psi_t \, dxdt &- \varepsilon_j \iint_Q \nabla u_t^j \cdot \nabla\rho\,\psi \, dxdt\,.\end{aligned} \tag{4.89}$$

We now pass to the limit as $j\to\infty$ in (4.88)-(4.89).

$$\begin{aligned}&\lim_{j\to\infty}\iint_Q \mathbf{a}\left(x,u^j,\nabla u^j\right)\cdot\nabla\left(u^j-\rho\right)\psi dxdt \\ &= -\iint_Q \mathbf{z}\cdot\nabla\psi\left(u-\rho\right)dxdt + \iint_Q \psi_t\frac{(u-\rho)^2}{2}dxdt \\ &+\iint_Q b\left(x,u\right)\left(u-\rho\right)\psi\, dxdt + \iint_Q g\left(x,\mathbf{X}\left(t\right)\right)\left(u-\rho\right)\psi\, dxdt \\ &= \iint_Q \mathbf{z}\cdot\nabla\left(u-\rho\right)\psi\, dxdt\,;\end{aligned} \tag{4.90}$$

here we have formally used the test function $\phi=(u-\rho)\psi$ in the equation $u_t=\mathrm{div}(\mathbf{z})+b(\cdot,u)+g(\cdot,\mathbf{X})$ in $L^2(0,T;H^{-1}(\mathbb{R}^N))$ (see (4.86)).

On the other hand,

$$\begin{aligned}&\iint_Q \mathbf{a}\left(x,u^j,\nabla u^j\right)\cdot\nabla\left(u^j-\rho\right)\psi\, dxdt \\ &= \iint_Q \left[\mathbf{a}\left(x,u^j,\nabla u^j\right)-\mathbf{a}\left(x,u^j,\nabla\rho\right)\right]\cdot\nabla\left(u^j-\rho\right)\psi\, dxdt \\ &+\iint_Q \mathbf{a}\left(x,u^j,\nabla\rho\right)\cdot\nabla\left(u^j-\rho\right)\psi\, dxdt \\ &\geq \iint_Q \mathbf{a}\left(x,u^j,\nabla\rho\right)\cdot\nabla\left(u^j-\rho\right)\psi\, dxdt,\end{aligned} \tag{4.91}$$

since $A2)-ii)$ holds true. Hence, by the convergences in Proposition 30 we have

$$\lim_{j\to\infty}\iint_Q \mathbf{a}\left(x,u^j,\nabla u^j\right)\cdot\nabla\left(u^j-\rho\right)\psi\, dxdt \geq \iint_Q \mathbf{a}\left(x,u,\nabla\rho\right)\cdot\nabla\left(u-\rho\right)\psi\, dxdt\,. \tag{4.92}$$

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Since ψ is an arbitrary function, by (4.90) and (4.92) it follows that

$$[\mathbf{a}(x,u(x,t),\nabla\rho(x))-\mathbf{z}(x,t)]\cdot[\nabla u(x,t)-\nabla\rho(x)]\leq 0 \tag{4.93}$$

for any $(x,t)\in Q\backslash N_0$, for some null set $N_0\subset Q$ (observe that the choice of N_0 can be made independent of ρ by standard separability arguments).

Let $(\overline{x},\overline{t})\in Q\backslash N_0$ be fixed. Then by the arbitrariness of ρ in (4.93) it can be easily seen that

$$\left[\mathbf{a}\left(\overline{x},u\left(\overline{x},\overline{t}\right),l\right)-\mathbf{z}\left(\overline{x},\overline{t}\right)\right]\cdot\left[\nabla u\left(\overline{x},\overline{t}\right)-l\right]\leq 0 \tag{4.94}$$

for all $l\in\mathbb{R}^N$. Choosing $l=\nabla u\left(\overline{x},\overline{t}\right)-\tau\nu$ (with $\nu\in\mathbb{R}^N$, $\nu\neq 0$) and dividing (4.94) by $\tau\neq 0$, in the limit as $\tau\to 0^+$ and $\tau\to 0^-$ we get

$$\left[\mathbf{a}\left(\overline{x},u\left(\overline{x},\overline{t}\right),\nabla u\left(\overline{x},\overline{t}\right)\right)-\mathbf{z}\left(\overline{x},\overline{t}\right)\right]\cdot\nu=0 \tag{4.95}$$

(here we have also used the continuity of the mapping $\xi\mapsto\mathbf{a}(x,u,\xi)$). Then (4.87) follows from the arbitrariness of ν. □

4.3.2 Proof of Theorem 18 and Theorem 19

We can finally prove our main results, stated in Theorem 18 and Theorem 19. In the former, we shall use the following Proposition 34, which refers to convergence properties of the terms in the ordinary differential equations in 4.1.

Proposition 34. *Let $\{(\mathbf{X}^j,u^j)\}$ be the subsequence given in Proposition 30. Then, possibly up to a subsequence (not relabelled for simplicity), for a.e. $t\in(0,T)$ it holds*

i) $\nabla u^j(\cdot,t)\rightharpoonup\nabla u(\cdot,t)$ *in* $\left[L^2\left(\mathbb{R}^N\right)\right]^N$,

ii) $\displaystyle\int_{\mathbb{R}^N}\nabla u^j(x,t)\,h^j(x,\mathbf{x}_i^j(t))dx\to\int_{\mathbb{R}^N}\nabla u(x,t)\,h(x,\mathbf{x}_i(t))dx,$

for all $i=1,\cdots,n$.

Proof. Proposition 34 The convergence in *ii)* is a direct consequence of *i)* and claim 8) in Remark 31.

In order to prove *i)*, we follow the same line of reasoning of Proposition 33. Let us consider the function $u^j\psi(t)$, with $\psi\in C_c^1(0,T)$, $\psi\geq 0$ as test function in the pseudoparabolic equation of (4.53). Arguing as in (4.90), and using (4.87), we get

$$\lim_{j\to\infty}\iint_Q\mathbf{a}\left(x,u^j,\nabla u^j\right)\cdot\nabla u^j\psi\,dxdt=\iint_Q\mathbf{a}\left(x,u,\nabla u\right)\cdot\nabla u\,\psi\,dxdt\,. \tag{4.96}$$

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For simplicity of notations, we define

$$\beta_j(x,t) = \left[\mathbf{a}\left(x,u^j,\nabla u^j\right) - \mathbf{a}\left(x,u^j,\nabla u\right)\right]\cdot\nabla\left(u^j-u\right). \tag{4.97}$$

Clearly $\beta_j(x,t)\geq 0$ *a.e.* in Q (see assumption $A2$)-ii)). From Proposition 30, (4.84), (4.87) and (4.96) we get

$$\lim_{j\to\infty}\iint_Q \beta_j(x,t)\psi(t)\,dxdt = 0. \tag{4.98}$$

Since ψ is an arbitrary nonnegative function, there exists a subset $\mathcal{N}\subset(0,T)$, of zero Lebesgue measure, such that

$$\beta_j(\cdot,t)\to 0 \quad \text{in} \quad L^1\left(\mathbb{R}^N\right) \tag{4.99}$$

for any $t\in(0,T)\backslash\mathcal{N}$.

Let $t\in(0,T)\backslash\mathcal{N}$ be fixed. Observe that without loss of generality we may assume that $u(\cdot,t)\in H^1(\mathbb{R}^N)$, and (possibly up to a subsequence, again not relabeled) $u^j(\cdot,t)\to u(\cdot,t)$ in $L^2(\mathbb{R}^N)$ for any such t (see Proposition 30-1)) . From the definition (4.97) of β_j, and using (4.42), for *a.e.* $x\in\mathbb{R}^N$ it holds

$$\begin{aligned}
&\mathbf{a}\left(x,u^j(x,t),\nabla u^j(x,t)\right)\cdot\nabla u^j(x,t)\\
&\leq \beta_j(x,t) + C_a\left(\left|u^j(x,t)\right| + \left|\nabla u^j(x,t)\right|\right)|\nabla u(x,t)|\\
&+C_a\left(\left|u^j(x,t)\right| + |\nabla u(x,t)|\right)\left|\nabla u^j(x,t)\right|\\
&+C_a\left(\left|u^j(x,t)\right| + |\nabla u(x,t)|\right)|\nabla u(x,t)|\\
&\leq \beta_j(x,t) + C_1\left(\left|u^j(x,t)\right|^2 + |\nabla u(x,t)|^2\right) + \frac{\alpha_0}{2}\left|\nabla u^j(x,t)\right|^2,
\end{aligned} \tag{4.100}$$

where in the last inequality we applied Young inequality with suitable constants. Moreover, by $A2)-iii)$ it follows that

$$\mathbf{a}\left(x,u^j(x,t),\nabla u^j(x,t)\right)\cdot\nabla u^j(x,t)\geq\alpha_0\left|\nabla u^j(x,t)\right|^2. \tag{4.101}$$

Combining (4.100)–(4.101) plainly gives

$$\begin{aligned}
&\frac{\alpha_0}{2}\int_{\mathbb{R}^N}|\nabla u^j(x,t)|^2\,dx\\
&\leq C_2\left[\int_{\mathbb{R}^N}\beta_j(x,t)\,dx + \int_{\mathbb{R}^N}|\nabla u(x,t)|^2\,dx + \int_{\mathbb{R}^N}(u^j(x,t))^2\,dx\right],
\end{aligned}$$

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whence (see (4.66) and (4.99))

$$\frac{\alpha_0}{2}\int_{\mathbb{R}^N}\left|\nabla u^j(x,t)\right|^2 dx \le C_t\,, \tag{4.102}$$

for some $C_t > 0$ depending on the choice of $t \in (0,T)\setminus\mathcal{N}$. Then the weak convergence

$$\nabla u^j(\cdot,t) \rightharpoonup \nabla u(\cdot,t) \quad \text{in } [L^2(\mathbb{R}^N)]^N$$

will easily follow from (4.102), since $u^j(\cdot,t) \to u(\cdot,t)$ in $L^2(\mathbb{R}^N)$. □

Proof of Theorem 18.
Let us check that the pair $(\mathbf{X},u)$ given by Proposition 30 is a solution of problem (4.1) in $(0,T)$. Clearly, the regularity requirements of Definition 17 and the equality $\mathbf{X}(0)=\mathbf{X}_0$ are satisfied (concerning the latter, recall that $\mathbf{X}^j(0)=\mathbf{X}_0$ for all j and $\mathbf{X}^j \to \mathbf{X}$ in $C([0,T];\mathbb{R}^{N\times n})$). The weak formulation (4.11) – whence the equality $u(\cdot,0)=u_0$ *a.e.* in $\mathbb{R}^N$ – immediately follows combining (4.86) and (4.87). Finally, in view of $(4.53)_1$, claims 5) and 6) in Proposition 30 and Proposition 34, by standard arguments we get, for every $i=1,\cdots,n$,

$$F_i\left(\cdot,\mathbf{X}^j,\int_{\mathbb{R}^N}\nabla u^j(x,\cdot)\,h^j(x,\mathbf{x}_i^j(\cdot))\,dx\right)=\dot{\mathbf{X}}_i^j \rightharpoonup \dot{\mathbf{X}}_i \quad \text{in } [L^2(0,T)]^N\,,$$

and

$$F_i\left(\cdot,\mathbf{X}^j,\int_{\mathbb{R}^N}\nabla u^j(x,\cdot)\,h^j(x,\mathbf{x}_i^j(\cdot))\,dx\right)\rightharpoonup \mathcal{F}_i \quad \text{in } [L^2(0,T)]^N\,,$$

where

$$\mathcal{F}_i(t)=F_i\left(t,\mathbf{X}(t),\int_{\mathbb{R}^N}\nabla u(x,t)\,h(x,\mathbf{x}_i(t))\,dx\right) \quad (a.e.\ t\in(0,T))$$

(here we have also used assumption $A1$). From the above convergences, it follows that equality $(4.1)_1$ holds for *a.e.* $t\in(0,T)$. This concludes the proof. □

Proof of Theorem 19.
Let $(\mathbf{X}_1,u_1)$ and $(\mathbf{X}_2,u_2)$ be two solutions to (4.1).

For every $i=1,...,n$, and for *a.e.* $t\in(0,T)$, by A1) we get

$$\begin{aligned}\left|\dot{\mathbf{X}}_{1i}(t)-\dot{\mathbf{X}}_{2i}(t)\right| &\le L_{F_i}\left(|\mathbf{X}_1(t)-\mathbf{X}_2(t)|\right.\\ &\left.+\left|\int_{\mathbb{R}^N}\nabla u_1(x,t)h\left(x,\mathbf{x_1}_i(t)\right)\,dx-\int_{\mathbb{R}^N}\nabla u_2(x,t)h\left(x,\mathbf{x_2}_i(t)\right)\,dx\right|\right).\end{aligned} \tag{4.103}$$

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Since $A5$) holds true, and $\mathbf{X}_1$, $\mathbf{X}_2 \in C([0,T];\mathbb{R}^{N\times n})$, the above integral can be estimated as follows:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \nabla u_1(x,t)h\left(x,\mathbf{x_1}_i(t)\right)\,dx - \nabla u_2(x,t)h\left(x,\mathbf{x_2}_i(t)\right)\,dx\right| \\
&\leq ||\nabla u_1(t)-\nabla u_2(t)||_{L^2(\mathbb{R}^N)}\,||h\left(\cdot,\mathbf{x_1}_i(t)\right)||_{L^2(\mathbb{R}^N)} \\
&+||\nabla u_2(t)||_{L^2(\mathbb{R}^N)}\,||h\left(\cdot,\mathbf{x_1}_i(t)\right)-h\left(\cdot,\mathbf{x_2}_i(t)\right)||_{L^2(\mathbb{R}^N)} \\
&\leq C_1\,||\nabla u_1(t)-\nabla u_2(t)||_{L^2(\mathbb{R}^N)} + L_h\,||\nabla u_2(t)||_{L^2(\mathbb{R}^N)}\,|\mathbf{X}_1(t)-\mathbf{X}_2(t)|\,,
\end{aligned} \tag{4.104}$$

for a suitable choice of $C_1>0$ and $L_h>0$ depending on the norms $\|\mathbf{X}_1\|_\infty$, $\|\mathbf{X}_2\|_\infty$. Hence, using (4.104) in (4.103) and integrating over $(0,\tau)$, with $\tau\in(0,T]$, we obtain

$$\begin{aligned}
&|\mathbf{X}_1(\tau)-\mathbf{X}_2(\tau)| \leq C_2\left[\int_0^\tau |\mathbf{X}_1(t)-\mathbf{X}_2(t)|\,dt\right. \\
&\left.+\int_0^\tau ||\nabla u_1(t)-\nabla u_2(t)||_{L^2(\mathbb{R}^N)} + ||\nabla u_2(t)||_{L^2(\mathbb{R}^N)}\,|\mathbf{X}_1(t)-\mathbf{X}_2(t)|\,dt\right] \\
&\leq C_2\left[\sqrt{\tau}\left(\int_0^\tau |\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt\right)^{1/2} + \sqrt{\tau}\left(\iint_{Q_\tau}|\nabla u_1-\nabla u_2|^2\,dxdt\right)^{1/2}\right. \\
&\left.+\left(\iint_{Q_\tau}|\nabla u_2|^2\,dxdt\right)^{1/2}\left(\int_0^\tau |\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt\right)^{1/2}\right]
\end{aligned} \tag{4.105}$$

whence,

$$\begin{aligned}
&|\mathbf{X}_1(\tau)-\mathbf{X}_2(\tau)|^2 \\
&\leq C_3\left[\int_0^\tau |\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt + \iint_{Q_\tau}|\nabla u_1-\nabla u_2|^2\,dxdt\right],
\end{aligned} \tag{4.106}$$

for some $C_3>0$ also depending on $\|\|\nabla u_2\|\|_{L^2(Q)}$.

Since u_1-u_2 satisfies in the weak sense the parabolic equation

$$\begin{aligned}
&\partial_t\left(u_1-u_2\right) = \operatorname{div}\left(\mathbf{a}(x,u_1,\nabla u_1)-\mathbf{a}(x,u_2,\nabla u_2)\right) \\
&+b(x,u_1)-b(x,u_2)+g\left(x,\mathbf{X}_1(t)\right)-g\left(x,\mathbf{X}_2(t)\right)\,,
\end{aligned} \tag{4.107}$$

choosing $\rho(x,t)=u_1(x,t)-u_2(x,t)$ as test function and integrating in $Q_\tau=$

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$\mathbb{R}^N \times (0,\tau)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}(u_1(\tau)-u_2(\tau))^2\,dx+\iint_{Q_\tau}[\mathbf{a}(x,u_1,\nabla u_1)-\mathbf{a}(x,u_1,\nabla u_2)]\cdot\nabla(u_1-u_2)\,dxdt\\
&=-\iint_{Q_\tau}[\mathbf{a}(x,u_1,\nabla u_2)-\mathbf{a}(x,u_2,\nabla u_2)]\cdot\nabla(u_1-u_2)\,dxdt\\
&+\iint_{Q_\tau}[b(x,u_1)-b(x,u_2)]\,[u_1-u_2]\,dxdt\\
&+\iint_{Q_\tau}[g(x,\mathbf{X}_1(t))-g(x,\mathbf{X}_2(t))]\,[u_1-u_2]\,dxdt\\
&\le L_a\iint_{Q_\tau}|u_1-u_2|\,|\nabla u_1-\nabla u_2|\,dxdt+L_b\iint_{Q_\tau}|u_1-u_2|^2\,dxdt\\
&+\frac{1}{2}\int_0^\tau L_g^2\,|\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt+\frac{1}{2}\int_0^\tau\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{R}^N)}dt\,.
\end{aligned}\tag{4.108}$$

In the last inequality we have used assumptions $A2)-i)$, $A3)$ and $A4)$,the latter with a suitable choice of $L_g>0$ depending on the norms $\|\mathbf{X}_1\|_\infty$, $\|\mathbf{X}_2\|_\infty$. From (4.108) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}(u_1(\tau)-u_2(\tau))^2\,dx+\iint_{Q_\tau}[\mathbf{a}(x,u_1,\nabla u_1)-\mathbf{a}(x,u_1,\nabla u_2)]\cdot\nabla(u_1-u_2)\,dxdt\\
&\le\left(L_b+\frac{1}{2}\right)\iint_{Q_\tau}|u_1-u_2|^2\,dxdt+\frac{L_g^2}{2}\int_0^\tau|\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt\\
&+\frac{L_a^2}{2\beta_0}\iint_{Q_\tau}|u_1-u_2|^2\,dxdt+\frac{\beta_0}{2}\iint_{Q_\tau}|\nabla u_1-\nabla u_2|^2\,dxdt\,,
\end{aligned}\tag{4.109}$$

where β_0 is the constant in (4.12).

Finally, recalling $A2)-iv)$ we get that, for every $\tau\in(0,T]$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}(u_1(\tau)-u_2(\tau))^2\,dx+\frac{\beta_0}{2}\iint_{Q_\tau}|\nabla u_1-\nabla u_2|^2\,dxdt\\
&\le C_4\left[\int_0^\tau|\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt+\iint_{Q_\tau}|u_1-u_2|^2\,dxdt\right].
\end{aligned}\tag{4.110}$$

By (4.110) and Gronwall inequality, we conclude that

$$\|u_1(t)-u_2(t)\|^2_{L^2(\mathbb{R}^N)}\le C_5\int_0^t|\mathbf{X}_1(s)-\mathbf{X}_2(s)|^2\,ds\,,\tag{4.111}$$

for any $t\in(0,T]$. By (4.111) and (4.110) we obtain

$$\iint_{Q_\tau}|\nabla u_1-\nabla u_2|^2\,dxdt\le C_6\int_0^\tau|\mathbf{X}_1(t)-\mathbf{X}_2(t)|^2\,dt\tag{4.112}$$

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for any $\tau \in (0, T]$. Using (4.112) in (4.106) it follows that $\mathbf{X}_1(t) = \mathbf{X}_2(t)$ for all $t \in [0, T]$. Then, inequality (4.111) implies that $u_1 = u_1$ a.e. in Q. This concludes the proof.

□

4.4 Discussion

In this chapter we presented well-posedeness results for a coupled system of nonlinear differential equations. The coupling is realized in a non-local form, and L^2 initial data are considered for the parabolic equation. The strategy adopted in order to prove our results relies on a preliminary study of the pseudo-parabolic approximating problem, and several a priori estimates of the different quantities in object. The study here presented has to be regarded as the starting point of a more general on-going work, which takes in consideration L^σ initial data, with $\sigma \in [1, 2]$. Further perspectives include the case of a bounded domain $\Omega \subset \mathbb{R}^N$ as the spatial domain of the investigated system.

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Chapter 5

A hybrid model for collective motions under alignment and chemotaxis

In this chapter we propose and study a hybrid mathematical model of collective motion under alignment and chemotaxis effect. In particular, the alignment term has the form of Cucker-Smale [10], seen in Chapter 1, whereas the chemotaxis equation is based on classic Keller-Segel model [72]. The proposed model is then studied both from an analytical and a numerical point of view. From the analytic point of view we prove, globally in time, existence and uniqueness of the solution. With respect to the previous Chapters, we also investigate the asymptotic behaviour of the solution. A comparison between the analytical findings and numerical results concludes our study.

5.1 Chemotaxis: the Keller-Segel model

It is well known that the movement of living species is influenced by the environment they live in. In general, the reaction of an organism to an external stimulus is called *taxis*. Different types of taxis can be mentioned, for instance *phototaxis* refers to the response to variation in light intensity, *aerotaxis* if the variation concerns oxygen concentration. In this chapter we are interested in *chemotaxis*: the influence of chemical substances on the movement of mobile species. In particular, if the organism is driven away from the source of chemical signal, we have negative chemotaxis and the chemical substance is called chemorepellent. On the contrary, we have positive chemotaxis when the chemical factor, which is now a chemoattractant, tends to attract the organism. In multicellular organisms, chemotaxis of cell populations plays a crucial role throughout the life cycle. Starting from embryogenesis, it takes part in organizing cell positioning, for example dur-

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ing gastrulation [11]and patterning of the nervous system. In the adult, it directs the migration of immune cells to sites of inflammation [73], and the same mechanisms occur in cancer growth, allowing cancerous cells to move and develop faster than healthy ones [74].

Before presenting our model, we introduce the mathematical modelling of chemotaxis, exploring the most classical model to describe collective motion of cell, proposed by Keller and Segel [72], [75].

Keller-Segel Model

The most famous model of partial differential equations was proposed in 1970 by the mathematicians Evelyn F. Keller and Lee A. Segel. They followed a macroscopic approach, where the behavior of a population in considered as a whole, despite of a microscopic one, which focuses on the irregular movements of a single member. Their model was elaborated to describe the aggregation behavior of living organism like Amebae Dictyostelium Discoideum. The hallmark of these eukariotes is their tendence to aggregate under food restrictions. Once food resources are over, they emit a chemical signal, called cyclic Adenosine Monophosphate (cAMP), which attracts the other amoebae. After the aggregation process, they differentiate to form a multicellular organism, where individuals maintain their integrity, and move on slime toward light. The original techniques used to understand the behavior of Dictyostelium, are currently being used to understand phenomena in other domains of science, such as in the study of angiogenesis and atherogenesis.

Consider a cell population $c(\mathbf{x}, t)$ and a chemoattractant $u(\mathbf{x}, t)$. In the original form of Keller-Segel model, authors distinguish between different types of chemical substances, and the model was given by a system of four equations. Further semplifications allowed the problem to be reduced to the two equations we present.

In the following we derive the chemotaxis equation following the approach in [76]. Many further approaches, like stochastic and discrete methods, have been, as it is reviewed in [77].

We consider a cell population $c(x, t)$ and a chemoattractant $u(x, t)$ which give Let V be an arbitrary fixed region in $\mathbb{R}^d$. The principle of mass conservation states that the change of mass in V is equal to the rate of flow $\mathbf{J}$ of material across ∂V, plus the material created in V:

$$\frac{\partial}{\partial t}\int_V c(x,t)\,dv = -\int_{\partial V} \mathbf{J}^c \cdot \mathbf{n}\,ds + \int_V Q^c\,dv,$$

$$\frac{\partial}{\partial t}\int_V u(x,t)\,dv = -\int_{\partial V} \mathbf{J}^u \cdot \mathbf{n}\,ds + \int_V Q^u\,dv$$

where $\mathbf{n}$ is the outward unit normal to V and $Q^{c/u}$ represents the source of material, depending possibly on c/u, $\mathbf{x}$ and t. Using the divergence theorem,

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we have

$$\int_V \frac{\partial c}{\partial t} + \nabla \cdot \mathbf{J}^c - Q^c \, dv = 0, \tag{5.1}$$

$$\int_V \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{J}^u - Q^u \, dv = 0 \tag{5.2}$$

Since the equations have to hold for arbitrary V, then the integrand must vanish, giving the continuity equation

$$\frac{\partial a}{\partial t} + \nabla \cdot \mathbf{J}^a = Q^a \qquad a = c, u$$

To characterize the flux and growth terms, authors made the following assumptions for the concentration of the chemical substance u and the density of cell population c. For equation (5.1), they assumed the total number of amoebae to remain relatively fixed, thus $Q^c = 0$. A reasonable form for the flux term $\mathbf{J}^c$ is

$$\mathbf{J}^c = -D_c \nabla c + D_u \nabla u$$

which captures two important aspects of the movement of the species. The term $-D_c \nabla c$ with $D_c > 0$ says that the organisms avoid increasing concentrations of their own kind of species. The second term $D_u \nabla u$ with $D_u > 0$, illustrates the positive chemotaxis phenomenon, and can be interpreted as the movement of individuals from low to higher concentrations of u. For equation (5.2), $\mathbf{J}^u$ is supposed to follow the classic Fick's law $\mathbf{J}^u = -D\nabla u$, with D constant diffusion coefficient.
Finally, Keller and Segel assume Q^u in the linear form $Q^u(c, u) = k_1 c - k_2 u$, where $k_1, k_2 > 0$. With these two terms, they take into account a spontaneous production of chemoattractant proportional to the number of cells, and a molecular exponential degradation if the chemotactic signal is not produced by the cells. Then the Keller Segel model can be written as:

$$\begin{cases} \dfrac{\partial c}{\partial t} = \nabla \cdot (D_c \nabla c - D_u \nabla u), \\ \dfrac{\partial u}{\partial t} = D\Delta u + k_1 c - k_2 u. \end{cases} \tag{5.3}$$

In literature there exist several variations of (5.3), which differ in assumptions based on additional biological realism. Among these, we remember the so called *Minimal Keller-Segel Model*

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$$\begin{cases} \dfrac{\partial c}{\partial t} = \nabla \cdot (\mu \nabla c - \chi(c,u)\nabla u), \\ \dfrac{\partial u}{\partial t} = \Delta u + c - u \end{cases} \tag{5.4}$$

where χ is called *chemotactic sensitivity function* and μ *mobility coefficient*. Typically, χ is assumed to be linear in the species c, in keeping with the notion that the flux of a species should be proportional to its density. We write

$$\chi(c,u) = c\chi_0(u).$$

As already said, positive or negative chemotaxis can occur, and here the coefficient χ_0 distinguishes between the two situations: if $\chi_0 > 0$, we are in presence of a chemoattractor, whereas if $\chi_0 < 0$ the chemical signal behaves as a chemoinhibitor. We observe that, in case of $\chi_0 \equiv 0$, $(5.4)_1$ reduces to a pure diffusion equation, and chemotaxis is not taken into account.

Parameter μ is the analog of the diffusion coefficient for nonliving species and, in general, it can depend on space, density of the living specie considered c, concentration of the chemical signal u, or a combination of these variables. Whether it is constant or not, typically it is $\mu \geq 0$.

5.2 The basic mathematical model

Let consider a group of n particles. We assume that the force acting on each particle is given by an alignment term, proportional to the differences of velocity with the other particles and weighed on the distances, and by a chemotactic attraction towards higher concentration of a chemical signal $f(x,t)$, produced by the particles themselves. Typically, this last force is proportional to the gradient of the concentration ∇f (see [78] for biological backgrounds, and [76, 79] for some mathematical references). In our hybrid description, while particles are considered discrete entities, endowed of a radius R describing their circular shape, the signal f is supposed to be continuous and its rate of change in time is equal to a diffusion term, a source term depending on the position of each particle, and a degradation term.

To summarize our hypotheses, we write the following system:

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\beta}{n} \displaystyle\sum_{j=1}^{n} \frac{1}{\left(1 + \frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{R^2}\right)^{\sigma}} (\mathbf{v}_j - \mathbf{v}_i) + \gamma \nabla f(x_i), \\ \partial_t f = D\Delta f + g_{\mathbf{X}} - \eta f, \end{cases} \tag{5.5}$$

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where β, σ, γ, D and η are positive constants.

Initial data are given by initial position and velocity for each particle:

$$\mathbf{X}(0) = \mathbf{X}_0, \quad \mathbf{V}(0) = \mathbf{V}_0,$$

with $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$, $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_N)$, and by the initial concentration of signal, that we assume

$$f(x, 0) := f_0 = 0. \tag{5.6}$$

We note that equation $(5.5)_3$ can be analytically solved making the classical exponential transformation:

$$f = e^{-\eta t} u,$$

with $u(x, t)$ solution of

$$\partial_t u = D\Delta u + e^{\eta t} g_{\mathbf{X}},$$

and $u(x, 0) = f_0$. Denoting with

$$\Gamma(x, t) := \frac{1}{(4\pi D t)^{N/2}} e^{-\frac{|x|^2}{4Dt}}$$

the fundamental solution of the heat equation on $\mathbb{R}^N$, the unique solution of $(5.5)_3$ (see [58] and Appendix for details) can be written as

$$\begin{aligned} f(x, t) &= (\Gamma(x, t) * f_0) e^{-\eta t} + e^{-\eta t} \int_0^t \Gamma(x, t - \tau) * \left(e^{\eta \tau} g_{\mathbf{X}}(x, \tau)\right) d\tau \\ &= \int_0^t e^{-\eta(t-\tau)} \int_{\mathbb{R}^n} \Gamma(x - \bar{x}, t - \tau) g_{\mathbf{X}}(\bar{x}, \tau) \, d\bar{x} d\tau, \end{aligned} \tag{5.7}$$

where $*$ is the convolution operation in the variable x, and $f_0 = 0$ for our initial condition (5.6).

In the following, for analytical and numerical simplicity, we will discuss the case of n particles in $\mathbb{R}^2$. Moreover, analytical results in $\mathbb{R}^N$ have already been discussed in the previous chapters.

For the source term $g_{\mathbf{X}}$ in the chemical signal equation, we assume

$$g_{\mathbf{X}}(x, t) := \xi \sum_{j=1}^{n} \varphi\left(|x_1 - x_{j1}|\right) \varphi\left(|x_2 - x_{j2}|\right), \tag{5.8}$$

where $\xi > 0$, $x = (x_1, x_2)$, $\mathbf{x}_j = (x_{j1}, x_{j2})$ is the position of $j-$th cell, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded function, $\varphi \in C_c^1([-R, R])$. This function is intended to take into account for the production of chemical signal due to each cell.

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First, if $x = (x_1, x_2)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$, from (5.7) and (5.8) we can thus write

$$f(x_1,x_2,t) = \xi \sum_{j=1}^{n} \int_0^t \frac{e^{-\eta(t-\tau)}}{4\pi(t-\tau)D} \int_{\mathbb{R}} \phi\left(x_1 - \bar{x}_1, t-\tau\right) \varphi\left(\bar{x}_1 - x_{j1}(\tau)\right)\, d\bar{x}_1$$
$$\int_{\mathbb{R}} \phi\left(x_2 - \bar{x}_2, t-\tau\right) \varphi\left(\bar{x}_2 - x_{j2}(\tau)\right)\, d\bar{x}_2\, d\tau, \tag{5.9}$$

where, for the sake of notational simplicity, we denote with $\phi : \mathbb{R} \times (0,T] \longrightarrow \mathbb{R}$ the function given by

$$\phi(x,t) := e^{-\frac{x^2}{4Dt}}.$$

Equation (5.9) can be rewritten in a more compact way, introducing

$$W : E \longrightarrow C\left(\mathbb{R} \times (0,T]\right),$$

with $E := \left\{\psi \in C(\mathbb{R}) : |\psi(x)| \leq Ce^{hx^2} \text{ with } C > 0, h < \frac{1}{4DT}\right\}$, defined by

$$W(\psi)(x,t) := \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi Dt}} \phi(x-y,t)\, \psi(y)\, dy. \tag{5.10}$$

We observe that (5.10) is the unique classical solution to the parabolic Cauchy problem, given by

$$\begin{cases} \partial_t W(\psi)(x,t) = D\partial^2_{xx} W(\psi)(x,t), \quad (x,t) \in \mathbb{R} \times (0,T], \\ W(\psi)(x,0) = \psi(x). \end{cases}$$

Hence, (5.9) can be rewritten as

$$f(x_1,x_2,t) = \xi \sum_{j=1}^{n} \int_0^t e^{-\eta(t-\tau)} \left(W(\varphi)\left(x_1 - x_{j1}(\tau), t-\tau\right)\right.$$
$$\left. W(\varphi)\left(x_2 - x_{j2}(\tau), t-\tau\right)\right) d\tau. \tag{5.11}$$

Before computing the chemotactic gradient $\nabla f = (\partial_1 f, \partial_2 f)$, we observe that for any $\psi \in E$, $(x,t) \in \mathbb{R} \times (0,T]$, it follows

$$\begin{aligned} \partial_x W(\psi)(x,t) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi Dt}} \left(-\frac{x-y}{2Dt} \phi(x-y,t)\right) \psi(y)\, dy \\ &= \int_{\mathbb{R}} \frac{\phi(x-y,t)\, \psi'(y)}{\sqrt{4\pi Dt}} dy \\ &= W(\psi')(x,t). \end{aligned}$$

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Thus

$$\begin{aligned}\partial_1 f(x_1,x_2,t) = \xi \sum_{j=1}^{n} \int_0^t e^{-\eta(t-\tau)} \big(W(\varphi')\left(x_1 - x_{j1}(\tau), t-\tau\right) \\ W(\varphi)\left(x_2 - x_{j2}(\tau), t-\tau\right)\big)\, d\tau, \\ \partial_2 f(x_1,x_2,t) = \xi \sum_{j=1}^{n} \int_0^t e^{-\eta(t-\tau)} \big(W(\varphi)\left(x_1 - x_{j1}(\tau), t-\tau\right) \\ W(\varphi')\left(x_2 - x_{j2}(\tau), t-\tau\right)\big)\, d\tau. \end{aligned} \tag{5.12}$$

Finally, substituting (5.12) into (5.5) we can summarize, for $i = 1, \dots, n$, the following system:

$$\begin{cases} \dot{v}_{i1} &= \dfrac{\beta}{n} \displaystyle\sum_{j=1}^{n} \frac{1}{\left(1 + \frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{R^2}\right)^{\sigma}} (v_{j1} - v_{i1}) \\ &\quad + \displaystyle\sum_{j=1}^{n} \int_0^t K(t-\tau) \big(W(\varphi')\left(x_{i1}(t) - x_{j1}(\tau), t-\tau\right) \\ &\qquad W(\varphi)\left(x_{i2}(t) - x_{j2}(\tau), t-\tau\right)\big)\, d\tau, \\ \dot{v}_{i2} &= \dfrac{\beta}{n} \displaystyle\sum_{j=1}^{n} \frac{1}{\left(1 + \frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{R^2}\right)^{\sigma}} (v_{j2} - v_{i2}) \\ &\quad + \displaystyle\sum_{j=1}^{n} \int_0^t K(t-\tau) \big(W(\varphi)\left(x_{i1}(t) - x_{j1}(\tau), t-\tau\right) \\ &\qquad W(\varphi')\left(x_{i2}(t) - x_{j2}(\tau), t-\tau\right)\big)\, d\tau, \\ \dot{x}_{i1} &= v_{i1}, \\ \dot{x}_{i2} &= v_{i2}, \end{cases} \tag{5.13}$$

with

$$K(t-\tau) := \gamma \xi e^{-\eta(t-\tau)}. \tag{5.14}$$

5.3 Local existence and uniqueness of the solution

5.3.1 Preliminary results

In this section, using a fixed point argumentation, we prove for (5.13) the local existence and uniqueness of solutions. In the next section, the result will be extended to a global result in time. For analogous results in the general case of $\mathbb{R}^N$, the reader is referred to Chapters 2.

First, let $\mathbf{y} = (\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{x}_1, \dots, \mathbf{x}_n)$ the solution vector. We rewrite (5.13) in a compact form. To this end, let introduce the following vectors

$$\mathbf{q} := (\mathbf{q}_1, \mathbf{q}_2),$$

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with

$$q_{1,i1} := \frac{\beta}{n}\sum_{j=1}^{n}\frac{1}{\left(1+\frac{|\mathbf{x}_j-\mathbf{x}_i|^2}{R^2}\right)^{\sigma}}(v_{j1}-v_{i1}), \tag{5.15}$$

$$q_{1,i2} := \frac{\beta}{n}\sum_{j=1}^{n}\frac{1}{\left(1+\frac{|\mathbf{x}_j-\mathbf{x}_i|^2}{R^2}\right)^{\sigma}}(v_{j2}-v_{i2}), \tag{5.16}$$

$$q_{2,i1} := v_{i1}, \tag{5.17}$$

$$q_{2,i2} := v_{i2}, \tag{5.18}$$

and

$$\mathbf{p} := (\mathbf{p}_1, \mathbf{p}_2),$$

with

$$p_{1,i1} := \sum_{j=1}^{n} W(\varphi^{'})\left(x_{i1}(t)-x_{j1}(\tau), t-\tau\right) W(\varphi)\left(x_{i2}(t)-x_{j2}(\tau), t-\tau\right), \tag{5.19}$$

$$p_{1,i2} := \sum_{j=1}^{n} W(\varphi)\left(x_{i1}(t)-x_{j1}(\tau), t-\tau\right) W(\varphi^{'})\left(x_{i2}(t)-x_{j2}(\tau), t-\tau\right), \tag{5.20}$$

$p_{2,i1} = p_{2,i2} = 0.$

With the introduced notations, system (5.13) can be written as

$$\dot{\mathbf{y}} = \mathbf{q}(\mathbf{y}) + \int_0^t K(t-\tau)\mathbf{p}(t-\tau, \mathbf{y}(t), \mathbf{y}(\tau))\, d\tau,$$

with $K(t-\tau)$ given in (5.14). Integrating from 0 to t we have

$$\mathbf{y} = \mathbf{y}_0 + \int_0^t \mathbf{q}(\mathbf{y}(\tau))\, d\tau + \int_0^t \int_0^s K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, d\tau\, ds,$$

with $\mathbf{y}_0 = \mathbf{y}(0)$. Then, interchanging the order of integration in the second integral, we have

$$\mathbf{y} = \mathbf{y}_0 + \int_0^t \left[\mathbf{q}(\mathbf{y}(\tau)) + \int_\tau^t K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds\right] d\tau, \tag{5.21}$$

or

$$\mathbf{y} = \mathbf{y}_0 + \int_0^t \left[\mathbf{q}(\mathbf{y}(\tau)) + \mathbf{h}(t, \tau, \mathbf{y}(\tau))\right] d\tau, \tag{5.22}$$

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with

$$\mathbf{h}(t,\tau,\mathbf{y}(\tau)) := \int_\tau^t K(s-\tau)\mathbf{p}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))\,ds.$$

For a discussion of such type of equations see, for example, [80, 81], and also [82, 83].

Now, let $a, b > 0$. We consider the set

$$S = \{(t,\tau,s,\mathbf{y}) : 0 \le \tau \le s < t \le a,\ |\mathbf{y}(t) - \mathbf{y}_0| \le b\}\,.$$

Since $\mathbf{q}(\mathbf{y})$ is continuous on S, we can define

$$M_1 = \max_S |\mathbf{q}(\mathbf{y})|\,. \tag{5.23}$$

Then we prove that $K(s-\tau)\mathbf{p}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))$ is $L^1(\tau,t)$ with respect to the variable s, in order to prove that $\mathbf{h}(t,\tau,\mathbf{y}(\tau))$ is continuous in S. It is enough to demonstrate the integrability around $s=\tau$. From (5.19) and (5.10) it follows

$$\begin{aligned} p_{1,i1}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau)) &= \sum_{j=1}^{n} W(\varphi')\left(x_{i1}(s) - x_{j1}(\tau), s-\tau\right) \\ &\qquad W(\varphi)\left(x_{i2}(s) - x_{j2}(\tau), s-\tau\right) \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}} \frac{e^{-\frac{(x_{i1}(s)-x_{j1}(\tau)-\tilde{y}_1)^2}{4D(s-\tau)}}}{\sqrt{4\pi D(s-\tau)}} \varphi'(\tilde{y}_1)d\tilde{y}_1 \\ &\qquad \int_{\mathbb{R}} \frac{e^{-\frac{(x_{i2}(s)-x_{j2}(\tau)-\tilde{y}_2)^2}{4D(s-\tau)}}}{\sqrt{4\pi D(s-\tau)}} \varphi(\tilde{y}_2)d\tilde{y}_2. \end{aligned}$$

Considering the change of variables

$$\frac{x_{i1}(s) - x_{j1}(\tau) - \tilde{y}_1}{\sqrt{4(s-\tau)D}} = z_1, \qquad \frac{x_{i2}(s) - x_{j2}(\tau) - \tilde{y}_2}{\sqrt{4(s-\tau)D}} = z_2, \tag{5.24}$$

recalling that $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$, it follows

$$\begin{aligned} |p_{1,i1}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))| &\le \frac{1}{\pi}\sum_{j=1}^{n} \int_{\mathbb{R}} e^{-z_1^2} dz_1 \int_{\mathbb{R}} e^{-z_2^2} dz_2\, ||\varphi||_{L^1(\mathbb{R})} \left|\left|\varphi'\right|\right|_{L^1(\mathbb{R})} \\ &= n\, ||\varphi||_{L^1(\mathbb{R})} \left|\left|\varphi'\right|\right|_{L^1(\mathbb{R})}. \end{aligned}$$

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The same holds for $p_{1,i2}$ in (5.20), so we can write

$$|\mathbf{p}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))| \leq \sqrt{n\left(2\left(n\,||\varphi||_{L^1(\mathbb{R})}\,||\varphi'||_{L^1(\mathbb{R})}\right)^2\right)} \tag{5.25}$$

$$\tag{5.26}$$

$$= \sqrt{2n^3}\,||\varphi||_{L^1(\mathbb{R})}\,\left|\left|\varphi'\right|\right|_{L^1(\mathbb{R})}, \tag{5.27}$$

and

$$|K(s-\tau)|\,|\mathbf{p}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))| \leq \gamma\xi e^{-\eta(s-\tau)}\sqrt{2n^3}\,||\varphi||_{L^1(\mathbb{R})}\,\left|\left|\varphi'\right|\right|_{L^1(\mathbb{R})}. \tag{5.28}$$

Now $K(s-\tau)\mathbf{p}(s-\tau,\mathbf{y}(s),\mathbf{y}(\tau))$ is continuous in $\mathbf{y}$ and, from (5.28), it is $L^1(\tau,t)$ with respect to the variable s, so $\mathbf{h}(t,\tau,\mathbf{y}(\tau))$ is continuous in S, and we can define

$$M_2 = \max_S |\mathbf{h}(t,\tau,\mathbf{y}(\tau))|. \tag{5.29}$$

5.3.2 Local existence and uniqueness

To prove local existence and uniqueness, we want to obtain a Lipschitz condition in S for the functions $\mathbf{q}$ and $\mathbf{p}$ with respect to the variable $\mathbf{y}$. First, because of $\mathbf{q}$ is C^1 on S, the Jacobian matrix $[\partial\mathbf{q}/\partial\mathbf{y}]$ is bounded on S uniformly in τ, so $\mathbf{q}$ satisfies the Lipschitz condition

$$|\mathbf{q}(\mathbf{y}_1)-\mathbf{q}(\mathbf{y}_2)| \leq L_1\,|\mathbf{y}_1-\mathbf{y}_2|, \tag{5.30}$$

with L_1 positive constant and $(t,\tau,s,\mathbf{y}_1)$, $(t,\tau,s,\mathbf{y}_2) \in S$.

To obtain a Lipschitz condition in S for $\mathbf{p}$ with respect to $\mathbf{y}$, we preliminary observe that, for any $\psi \in E$, $(x,t),(x',t) \in \mathbb{R}\times(0,T]$, by definition (5.10) immediately follows

$$|W(\psi)(x,t)| \leq \frac{||\psi||_{L^1(\mathbb{R})}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-z^2}dz = ||\psi||_{L^1(\mathbb{R})}. \tag{5.31}$$

Moreover, using the mean value theorem, and the estimate of the derivative

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of function Γ [58], we obtain

$$\begin{aligned}\left|W(\psi)(x,t)-W(\psi)(x',t)\right| &\leq \int_{\mathbb{R}} \left|\Gamma(x-y,t)-\Gamma\left(x'-y,t\right)\right| |\psi(y)|\, dy \\ &\leq ||\psi||_{L^1(\mathbb{R})} \frac{|x-x'|}{t} \int_{\mathbb{R}} e^{-|u|^2}\sqrt{4Dt}\, du \\ &\leq 2\sqrt{D\pi}\, ||\psi||_{L^1(\mathbb{R})} \frac{|x-x'|}{\sqrt{t}}.\end{aligned} \tag{5.32}$$

Let now focus on the component $p_{1,i1}$ of $\mathbf{p}$. We observe that, for any $(t,\tau,s,\mathbf{y_1})$, $(t,\tau,s,\mathbf{y_2}) \in S$, denoting with $X^{(1)}$, $X^{(2)}$ the variables belonging respectively to $\mathbf{y_1}$ and $\mathbf{y_2}$, we obtain

$$\begin{aligned}&|p_{1,i1}(s-\tau,\mathbf{y}_1(s),\mathbf{y}_1(\tau)) - p_{1,i1}(s-\tau,\mathbf{y}_2(s),\mathbf{y}_2(\tau))| \\ &\leq \sum_{j=1}^{n} \Big| W(\varphi')\left(x_{i1}^{(1)}(s)-x_{j1}^{(1)}(\tau),s-\tau\right) W(\varphi)\left(x_{i2}^{(1)}(s)-x_{j2}^{(1)}(\tau),s-\tau\right) \\ &\quad -W(\varphi')\left(x_{i1}^{(2)}(s)-x_{j1}^{(2)}(\tau),s-\tau\right) W(\varphi)\left(x_{i2}^{(2)}(s)-x_{j2}^{(2)}(\tau),s-\tau\right)\Big| \\ &\leq \sum_{j=1}^{n} \Big(\Big|W(\varphi')\left(x_{i1}^{(1)}(s)-x_{j1}^{(1)}(\tau),s-\tau\right)\Big| \Big| W(\varphi)\left(x_{i2}^{(1)}(s)-x_{j2}^{(1)}(\tau),s-\tau\right) \\ &\quad -W(\varphi)\left(x_{i2}^{(2)}(s)-x_{j2}^{(2)}(\tau),s-\tau\right)\Big| + \Big|W(\varphi)\left(x_{i2}^{(2)}(s)-x_{j2}^{(2)}(\tau),s-\tau\right)\Big| \\ &\quad \Big|-W(\varphi')\left(x_{i1}^{(2)}(s)-x_{j1}^{(2)}(\tau),s-\tau\right) + W(\varphi')\left(x_{i1}^{(1)}(s)-x_{j1}^{(1)}(\tau),s-\tau\right)\Big|\Big) \\ &\leq \sum_{j=1}^{n} \frac{2\sqrt{D\pi}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})}}{\sqrt{s-\tau}} \left(\left|x_{i2}^{(1)}(s)-x_{i2}^{(2)}(s)\right| + \left|x_{j2}^{(1)}(\tau)-x_{j2}^{(2)}(\tau)\right| \right. \\ &\quad \left. + \left|x_{i1}^{(2)}(s)-x_{i1}^{(1)}(s)\right| + \left|x_{j1}^{(2)}(\tau)-x_{j1}^{(1)}(\tau)\right|\right).\end{aligned} \tag{5.33}$$

The same can be done for $p_{1,i2}$, so $\mathbf{p}$ satisfies the following condition in $\mathbf{y}$ on S:

$$\begin{aligned}&|\mathbf{p}(s-\tau,\mathbf{y}_1(s),\mathbf{y}_1(\tau)) - \mathbf{p}(s-\tau,\mathbf{y}_2(s),\mathbf{y}_2(\tau))| \\ &\leq \frac{L_2}{\sqrt{s-\tau}} \left(|\mathbf{y}_1(s)-\mathbf{y}_2(s)| + |\mathbf{y}_1(\tau)-\mathbf{y}_2(\tau)|\right),\end{aligned} \tag{5.34}$$

with L_2 a suitable positive constant that incorporates previous constants, and $(\tau,s,\mathbf{y}_1)$, $(\tau,s,\mathbf{y}_2) \in S$.

Now, we fix

$$T = \min\left[a, \frac{b}{M_1+M_2}, \frac{1}{L_1+2L_2M}\right], \tag{5.35}$$

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with L_1, L_2 Lipschitz constants in (5.30), (5.34),M_1, M_2 given by (5.23), (5.29), and

$$M := \int_0^{+\infty} \frac{|K(z)|}{\sqrt{z}}\, dz. \tag{5.36}$$

Then we prove the following

Theorem 35. *Equation* (5.22) *has a unique solution on* $[0, T]$, *where* T *is defined in* (5.35).

Proof. Theorem 35:
We consider the functional space

$$\mathcal{B} = \left\{ \mathbf{y} \in C^0\left([0,T]\right) : ||\mathbf{y} - \mathbf{y}_0||_{C^0} \leq b \right\},$$

where

$$||\mathbf{y} - \mathbf{z}||_{C^0} := \sup_{0 \leq t \leq T} |\mathbf{y}(t) - \mathbf{z}(t)|,$$

and we define the functional $\mathbf{A} : \mathcal{B} \to \mathcal{B}$ as

$$\mathbf{A}(\mathbf{y})(t) := \mathbf{y}_0 + \int_0^t \left[\mathbf{q}(\mathbf{y}(\tau)) + \mathbf{h}(t, \tau, \mathbf{y}(\tau))\right] d\tau,$$

To see that $\mathbf{A} : \mathcal{B} \to \mathcal{B}$ notice that $\mathbf{y}$ continuous implies $\mathbf{A}(\mathbf{y})$ continuous, because $\mathbf{q}$ and $\mathbf{h}$ are continuous, and that

$$\begin{aligned} \|\mathbf{A}(\mathbf{y}) - \mathbf{y}_0\|_{C^0} &= \sup_{0 \leq t \leq T} |\mathbf{A}(\mathbf{y})(t) - \mathbf{y}_0| \\ &\leq \sup_{0 \leq t \leq T} \int_0^t \left(|\mathbf{q}(\mathbf{y}(\tau))| + |\mathbf{h}(t, \tau, \mathbf{y}(\tau))|\right) d\tau \\ &\leq (M_1 + M_2)T \leq b, \end{aligned}$$

where we have used (5.23), (5.29) and, in the last inequality, (5.35). To see

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that $\mathbf{A}$ is a contraction mapping, notice that if $\mathbf{y}_1$ and $\mathbf{y}_2 \in \mathcal{B}$ then

$$
\begin{aligned}
&\|\mathbf{A}(\mathbf{y}_1) - \mathbf{A}(\mathbf{y}_2)\|_{C^0} = \sup_{0\le t\le T} |\mathbf{A}(\mathbf{y}_1)(t) - \mathbf{A}(\mathbf{y}_2)(t)| \\
&\le \sup_{0\le t\le T} \int_0^t (|\mathbf{q}(\mathbf{y}_1(\tau)) - \mathbf{q}(\mathbf{y}_2(\tau))| \\
&+ \int_\tau^t |K(s-\tau)|\,|\mathbf{p}(s-\tau, \mathbf{y}_1(s), \mathbf{y}_1(\tau)) - \mathbf{p}(s-\tau, \mathbf{y}_2(s), \mathbf{y}_2(\tau))|)\,ds\,d\tau \\
&\le_{(5.30),(5.34)} \sup_{0\le t\le T} \int_0^t [L_1\,|\mathbf{y}_1(\tau) - \mathbf{y}_2(\tau)| + L_2 \int_\tau^t \frac{1}{\sqrt{s-\tau}}\,|K(s-\tau)| \\
&\qquad (|\mathbf{y}_1(s) - \mathbf{y}_2(s)| + |\mathbf{y}_1(\tau) - \mathbf{y}_2(\tau)|)]\,ds\,d\tau \\
&\le L_1 T\,\|\mathbf{y}_1 - \mathbf{y}_2\|_{C^0} + 2L_2\,\|\mathbf{y}_1 - \mathbf{y}_2\|_{C^0} \sup_{0\le t\le T} \int_0^t \int_\tau^t |K(s-\tau)|\,\frac{1}{\sqrt{s-\tau}}\,ds\,d\tau \\
&= \left(L_1 T + 2L_2 \sup_{0\le t\le T} \int_0^t \int_0^{t-\tau} |K(z)|\,\frac{1}{\sqrt{z}}\,dz\,d\tau\right) \|\mathbf{y}_1 - \mathbf{y}_2\|_{C^0} \\
&\le \left(L_1 T + 2L_2 \int_0^T \int_0^{+\infty} |K(z)|\,\frac{1}{\sqrt{z}}\,dz\,d\tau\right) \|\mathbf{y}_1 - \mathbf{y}_2\|_{C^0} \\
&= (L_1 + 2L_2 M)T\,\|\mathbf{y}_1 - \mathbf{y}_2\|_{C^0}\,.
\end{aligned}
$$

From (5.35) the constant $(L_1 + 2L_2 M)T \in (0,1)$. The Banach-Caccioppoli fixed-point theorem completes the proof. □

5.4 Global existence of the solution

To obtain global existence for (5.21) we will use a principle of continuation of solutions. We will prove that bounded solutions can be continued to $t = +\infty$. The following general result, adapted to equation (5.21), provides a condition for the continuation of solutions.

Proposition 36. *Let $\mathbf{y}(t)$ be a solution of (5.21) on an interval $[0,T)$, if there is a constant P with $|\mathbf{y} - \mathbf{y}_0| \le P$ on $[0,T)$, then there is a $\bar{T} > T$ such that $\mathbf{y}(t)$ can be continued to $[0,\bar{T}]$.*

Proof. Proposition 36:
We show that $\lim_{t\to T^-} \mathbf{y}(t)$ exists, so we can apply Theorem 35 starting at $t = T$, and this completes the proof.

Let t_n be a monotonic increasing sequence with limit T, and let

$$\bar{U} = \{(t,\tau,s,\mathbf{y}) : 0 \le \tau \le s \le t \le T,\ |\mathbf{y} - \mathbf{y}_0| \le P\}\,.$$

We prove that $\{\mathbf{y}(t_n)\}$ is a Cauchy sequence. If $t_m > t_n$, from (5.21) we

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have

$$
\begin{aligned}
|\mathbf{y}(t_m) - \mathbf{y}(t_n)| &= \left| \int_0^{t_m} \left[\mathbf{q}(\mathbf{y}(\tau)) + \int_\tau^{t_m} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right] d\tau \right. \\
&\left. - \int_0^{t_n} \left[\mathbf{q}(\mathbf{y}(\tau)) + \int_\tau^{t_n} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right] d\tau \right| \\
&\leq \int_0^{t_n} \left| \int_\tau^{t_m} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right. \\
&\left. - \int_\tau^{t_n} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right| d\tau \\
&+ \left| \int_{t_n}^{t_m} \left[\mathbf{q}(\mathbf{y}(\tau)) + \int_\tau^{t_m} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right] d\tau \right| \\
&\leq \int_0^{t_n} \left| \int_{t_n}^{t_m} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds \right| d\tau \\
&+ \left| \int_{t_n}^{t_m} \int_\tau^{t_m} K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))\, ds\, d\tau \right| + \left| \int_{t_n}^{t_m} \mathbf{q}(\mathbf{y}(\tau))\, d\tau \right| \\
&\leq \int_0^{t_n} \int_{t_n}^{t_m} |K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))|\, ds\, d\tau \\
&+ \int_{t_n}^{t_m} \int_\tau^{t_m} |K(s-\tau)\mathbf{p}(s-\tau, \mathbf{y}(s), \mathbf{y}(\tau))|\, ds\, d\tau + \int_{t_n}^{t_m} |\mathbf{q}(\mathbf{y}(\tau))|\, d\tau.
\end{aligned}
$$

In the last inequality the third integral tends to zero as $n, m \to +\infty$, because $\mathbf{q}$ is bounded on $\bar{U}$ and $t_m, t_n \to T$. Also the first two integrals converge to zero as $n, m \to +\infty$, because of (5.28). The proof is completed. □

Now, from Proposition 36, we obtain the following

Theorem 37. *Equation (5.22) has a unique global solution for all $t \geq 0$.*

Proof. Theorem 37:
First, equations (5.15)–(5.18) imply

$$
\begin{aligned}
|q_{1,i1}| \leq 2\beta|\mathbf{y}|, \quad |q_{1,i2}| \leq 2\beta|\mathbf{y}|, \\
|q_{2,i1}| \leq |\mathbf{y}|, \quad |q_{2,i2}| \leq |\mathbf{y}|,
\end{aligned}
$$

so that

$$
|\mathbf{q}| \leq \sqrt{2N(4\beta^2+1)}|\mathbf{y}|.
$$

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Then (5.21), (5.25) yield

$$\begin{aligned}
|\mathbf{y}| &\leq |\mathbf{y}_0| + \sqrt{2n(4\beta^2+1)} \int_0^t |\mathbf{y}(\tau)|\, d\tau \\
&\qquad + \sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})} \int_0^t \int_\tau^t \frac{|K(s-\tau)|}{\sqrt{s-\tau}}\, ds\, d\tau \\
&= |\mathbf{y}_0| + \sqrt{2n(4\beta^2+1)} \int_0^t |\mathbf{y}(\tau)|\, d\tau \\
&\qquad + \sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})} \int_0^t \int_0^{t-\tau} \frac{|K(z)|}{\sqrt{z}}\, dz\, d\tau \\
&\leq |\mathbf{y}_0| + \sqrt{2n(4\beta^2+1)} \int_0^t |\mathbf{y}(\tau)|\, d\tau \\
&\qquad + \sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})} \int_0^t \int_0^{+\infty} \frac{|K(z)|}{\sqrt{z}}\, dz\, d\tau \\
&= |\mathbf{y}_0| + \sqrt{2n(4\beta^2+1)} \int_0^t |\mathbf{y}(\tau)|\, d\tau + M\sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})}\, t,
\end{aligned}$$

where we have set $z = s - \tau$, and M is given by (5.36).

Now, for each $0 \leq t < T$,

$$|\mathbf{y}| \leq \left(|\mathbf{y}_0| + M\sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})}\, T\right) + \sqrt{2n(4\beta^2+1)} \int_0^t |\mathbf{y}(\tau)|\, d\tau,$$

so that

$$|\mathbf{y}| \leq \left(|\mathbf{y}_0| + M\sqrt{2n^3}\, ||\varphi||_{L^1(\mathbb{R})}\, ||\varphi'||_{L^1(\mathbb{R})}\, T\right) e^{t\sqrt{2n(4\beta^2+1)}}$$

by the Gronwall's inequality. Since the solution remains bounded, for Proposition 36, it can be continued to all $[0, +\infty)$. □

5.5 Asymptotic properties on the linearised system

In this section we prove some asymptotic properties on the linearised form of system (5.5). To simplify some computations and the following numerical simulations, here we consider the case in which the source term in the chemical signal equation is given by a characteristic function on a ball of radius R centered on each particle, as in [2], [3], where R represents the cell radius. Namely, we assume

$$g_{\mathbf{X}} = \xi \sum_{j=1}^{n} \chi_{\mathbf{B}(\mathbf{x}_j, R)}, \quad \text{with} \quad \xi > 0,\ R > 0, \tag{5.37}$$

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and

$$\chi_{\mathbf{B}(\mathbf{x}_j,R)} := \begin{cases} 1, & \text{if} \quad x \in \mathbf{B}(\mathbf{x}_j, R) := \{x : ||x - \mathbf{x}_j|| \leq R\}; \\ 0, & \text{otherwise.} \end{cases} \tag{5.38}$$

With similar computations to those of the previous section, from (5.7), (5.37), we get

$$f(x_1, x_2, t) = \xi \sum_{j=1}^{n} \int_0^t \iint_{\mathbf{B}(\mathbf{x}_j(\tau),R)} \frac{e^{-\eta(t-\tau)} e^{-\frac{(x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2}{4(t-\tau)D}}}{4\pi(t-\tau)D} \, d\bar{x}_1 \, d\bar{x}_2 \, d\tau. \tag{5.39}$$

Performing the change of variables $\tilde{x}_2 = \bar{x}_2 - x_{j2}(\tau)$, we express the chemotactic gradient $\nabla f = (\partial_1 f, \partial_2 f)$, obtaining

$$\partial_1 f(x_1, x_2, t) = -\xi \sum_{j=1}^{n} \int_0^t \frac{e^{-\eta(t-\tau)}}{4\pi(t-\tau)D} \int_{-R}^{+R} e^{-\frac{(x_2-x_{j2}(\tau)-\tilde{x}_2)^2}{4(t-\tau)D}} \left(e^{-\frac{\left(x_1-x_{j1}(\tau)-\sqrt{R^2-\tilde{x}_2^2}\right)^2}{4(t-\tau)D}} - e^{-\frac{\left(x_1-x_{j1}(\tau)+\sqrt{R^2-\tilde{x}_2^2}\right)^2}{4(t-\tau)D}} \right) d\tilde{x}_2 \, d\tau, \tag{5.40}$$

and similarly we can proceed for $\partial_2 f(x_1, x_2, t)$.

Hence, substituting (5.40) into $(5.5)_1$ we can summarize, for $i = 1, \ldots, n$, the following system:

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$$
\begin{cases}
\dot{v}_{i1} &= \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n}\frac{1}{\left(1+\frac{|\mathbf{x}_i-\mathbf{x}_j|^2}{R^2}\right)^{\sigma}}(v_{j1}-v_{i1}) \\
&\quad -\displaystyle\int_0^t C(t-\tau)\sum_{j=1}^{n}\int_{-R}^{+R} e^{-\frac{(x_{i2}(t)-x_{j2}(\tau)-\tilde{x}_2)^2}{4(t-\tau)D}}\left(e^{-\frac{\left(x_{i1}(t)-x_{j1}(\tau)-\sqrt{R^2-\tilde{x}_2^2}\right)^2}{4(t-\tau)D}}\right. \\
&\quad \left.-e^{-\frac{\left(x_{i1}(t)-x_{j1}(\tau)+\sqrt{R^2-\tilde{x}_2^2}\right)^2}{4(t-\tau)D}}\right) d\tilde{x}_2\, d\tau, \\
\dot{v}_{i2} &= \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n}\frac{1}{\left(1+\frac{|\mathbf{x}_i-\mathbf{x}_j|^2}{R^2}\right)^{\sigma}}(v_{j2}-v_{i2}) \\
&\quad -\displaystyle\int_0^t C(t-\tau)\sum_{j=1}^{n}\int_{-R}^{+R} e^{-\frac{(x_{i1}(t)-x_{j1}(\tau)-\tilde{x}_1)^2}{4(t-\tau)D}}\left(e^{-\frac{\left(x_{i2}(t)-x_{j2}(\tau)-\sqrt{R^2-\tilde{x}_1^2}\right)^2}{4(t-\tau)D}}\right. \\
&\quad \left.-e^{-\frac{\left(x_{i2}(t)-x_{j2}(\tau)+\sqrt{R^2-\tilde{x}_1^2}\right)^2}{4(t-\tau)D}}\right) d\tilde{x}_1\, d\tau, \\
\dot{x}_{i1} &= v_{i1}, \\
\dot{x}_{i2} &= v_{i2},
\end{cases}
\tag{5.41}
$$

with

$$
C(t-\tau) := \frac{\gamma\xi e^{-\eta(t-\tau)}}{4\pi(t-\tau)D}. \tag{5.42}
$$

We are interested in the equilibrium points that satisfy the condition:

$$
\begin{cases} \mathbf{x}_i(t) = \mathbf{x}_{\text{eq}}(t), & \forall i, \quad \forall t; \\ \mathbf{v}_i(t) = \mathbf{0}, & \forall i; \end{cases} \quad \Leftrightarrow \quad \mathbf{X}_i(t) = \mathbf{X}_{\text{eq}} = \text{constant}, \quad \forall i. \tag{5.43}
$$

Equation (5.43) means that all particles are in a same position for all times.

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Now, to make a first-order approximation of (5.41), we consider the following Taylor expansions around points (5.43):

$$F_1\left(\mathbf{x}_j-\mathbf{x}_i, v_{j1}-v_{i1}\right) := \frac{1}{\left(1+\frac{|\mathbf{x}_j-\mathbf{x}_i|^2}{R^2}\right)^{\sigma}}(v_{j1}-v_{i1}) \tag{5.44}$$

$$=v_{j1}-v_{1i}+\rho_1(\mathbf{x}_j-\mathbf{x}_i, v_{j1}-v_{i1}), \tag{5.45}$$

$$F_2\left(t-\tau, x_{i2}(t)-x_{j2}(\tau), \tilde{x}_2\right) := e^{-\frac{\left(x_{i2}(t)-x_{j2}(\tau)-\tilde{x}_2\right)^2}{4(t-\tau)D}} \tag{5.46}$$

$$=e^{-\frac{\tilde{x}_2^2}{4(t-\tau)D}}+\rho_2(t-\tau, x_{i2}(t)-x_{j2}(\tau), \tilde{x}_2), \tag{5.47}$$

$$F_3\left(t-\tau, x_{i1}(t)-x_{j1}(\tau), \tilde{x}_2\right) := e^{-\frac{\left(x_{i1}(t)-x_{j1}(\tau)\pm\sqrt{R^2-\tilde{x}_2^2}\right)^2}{4(t-\tau)D}} \tag{5.48}$$

$$\begin{aligned}=&e^{-\frac{R^2-\tilde{x}_2^2}{4(t-\tau)D}}\mp e^{-\frac{R^2-\tilde{x}_2^2}{4(t-\tau)D}}\frac{\sqrt{R^2-\tilde{x}_2^2}}{2(t-\tau)D}\left(x_{i1}(t)-x_{j1}(\tau)\right)\\ &+\rho_3(t-\tau, x_{i1}(t)-x_{j1}(\tau), \tilde{x}_2),\end{aligned} \tag{5.49}$$

where the functions ρ_1 and ρ_3 contain the nonlinear terms, while ρ_2 contains the linear and the nonlinear terms. Similarly we can treat equation $(5.41)_2$.

From (5.44)–(5.49), we linearise equation $(5.41)_1$ in the form

$$\begin{aligned}\dot{v_{i1}} =& \frac{\beta}{n}\sum_{j=1}^{n}(v_{j1}(t)-v_{i1}(t))-\int_0^t\frac{C(t-\tau)}{2(t-\tau)D}\int_{-R}^{+R}e^{-\frac{\tilde{x}_2^2}{4(t-\tau)D}}e^{-\frac{R^2-\tilde{x}_2^2}{4(t-\tau)D}}\sqrt{R^2-\tilde{x}_2}\\ &\sum_{j=1}^{n}\left(x_{i1}(t)-x_{j1}(\tau)+x_{i1}(t)-x_{j1}(\tau)\right)\,d\tilde{x}_2\,d\tau\\ =&\frac{\beta}{n}\sum_{j=1}^{n}(v_{j1}(t)-v_{i1}(t))-\int_0^t\frac{C(t-\tau)e^{-\frac{R^2}{4(t-\tau)D}}}{(t-\tau)D}\int_{-R}^{+R}\sqrt{R^2-\tilde{x}_2^2}\,d\tilde{x}_2\\ &\sum_{j=1}^{n}\left(x_{i1}(t)-x_{j1}(\tau)\right)\,d\tau\\ =&\frac{\beta}{n}\sum_{j=1}^{n}(v_{j1}(t)-v_{i1}(t))-\int_0^t\bar{C}(t-\tau)\sum_{j=1}^{n}\left(x_{i1}(t)-x_{j1}(\tau)\right)\,d\tau,\end{aligned}$$

with

$$\bar{C}(t-\tau) := \frac{\pi R^2 C(t-\tau)e^{-\frac{R^2}{4(t-\tau)D}}}{2(t-\tau)D}. \tag{5.50}$$

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Similarly it can be done for $\dot{V}_{i2}$. Finally, we obtain

$$\begin{cases} \dot{v_{i1}} &= \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n}(v_{j1}(t)-v_{i1}(t)) - \int_0^t \bar{C}(t-\tau)\sum_{j=1}^{n}(x_{i1}(t)-x_{j1}(\tau))\,d\tau, \\ \dot{v_{i2}} &= \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n}(v_{j2}(t)-v_{i2}(t)) - \int_0^t \bar{C}(t-\tau)\sum_{j=1}^{n}(x_{i2}(t)-x_{j2}(\tau))\,d\tau, \\ \dot{x_{i1}} &= v_{i1}, \\ \dot{x_{i2}} &= v_{i2}. \end{cases} \tag{5.51}$$

We are interested to establish the following time-asymptotic convergence property of (5.51):

Theorem 38. *Let*

$$\mathbf{x}_{CM}(t) := \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(t), \tag{5.52}$$

$$\mathbf{v}_{CM}(t) := \frac{1}{n}\sum_{i=1}^{n}\mathbf{v}_i(t), \tag{5.53}$$

position and velocity of the centre of mass of the system of n particles, of the same mass, satisfying (5.51). For $t \to +\infty$, the following properties hold:

i) the velocity and position of all particles converge to the same values. This position is their centre of mass (5.52);

ii) the velocity of the centre of mass (5.53), tends to zero.

Remark 39. Theorem 38 ensures a condition of time-asymptotic flocking such as stated in Definition 1. Moreover, we have also the stronger condition that all particles converge asymptotically to their centre of mass and the velocity of the centre of mass decays to zero.

Next propositions and lemmas will lead to the proof of Theorem 38 at the end of the section. First, it is convenient to introduce the centre of mass system, in which equations (5.51) become a nonautonomous system of ordinary differential equation, decoupled with respect to the i-th particle and with respect to the two components of each position and velocity vector. Then the equation of the centre of mass can be studied apart.

Starting from (5.52)–(5.53) we define the new variables

$$\bar{\mathbf{x}}_i := \mathbf{x}_i - \mathbf{x}_{\mathrm{CM}}, \tag{5.54}$$

$$\bar{\mathbf{v}}_i := \mathbf{v}_i - \mathbf{v}_{\mathrm{CM}}. \tag{5.55}$$

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In variables (5.54)–(5.55) the equilibrium condition (5.43) becomes

$$(\bar{\mathbf{x}}_i, \bar{\mathbf{v}}_i) = (\mathbf{0}, \mathbf{0}), \quad \forall i = 1, \dots, n, \tag{5.56}$$

moreover the following identities hold:

$$\sum_{i=1}^{n} \bar{\mathbf{x}}_i = \mathbf{0}, \tag{5.57}$$

$$\sum_{i=1}^{n} \bar{\mathbf{v}}_i = \mathbf{0}. \tag{5.58}$$

If $\mathbf{v}_{\mathrm{CM}} = (v_{\mathrm{CM1}}, v_{\mathrm{CM2}})$ and $\mathbf{x}_{\mathrm{CM}} = (x_{\mathrm{CM1}}, x_{\mathrm{CM2}})$, from (5.51) we have

$$\begin{aligned}
\dot{v}_{\mathrm{CM1}} &= \frac{1}{n}\sum_{i=1}^{n} \dot{v}_{i1} = \frac{1}{n}\frac{\beta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(v_{j1}(t) - v_{i1}(t)) \\
&- \frac{1}{n}\int_0^t \bar{C}(t-\tau)\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{i1}(t) - x_{j1}(\tau))\, d\tau \\
&= \frac{1}{n}\frac{\beta}{n}\sum_{i=1}^{n}(n v_{\mathrm{CM1}}(t) - n v_{i1}(t)) \\
&- \frac{1}{n}\int_0^t \bar{C}(t-\tau)\sum_{i=1}^{n}(n x_{i1}(t) - n x_{\mathrm{CM1}}(\tau))\, d\tau \\
&= \frac{1}{n}\frac{\beta}{n}(n^2 v_{\mathrm{CM1}}(t) - n^2 v_{\mathrm{CM1}}(t)) \\
&- \frac{1}{n}\int_0^t \bar{C}(t-\tau)\left(n^2 x_{\mathrm{CM1}}(t) - n^2 x_{\mathrm{CM1}}(\tau)\right) d\tau \\
&= -\int_0^t \bar{C}(t-\tau)\,(n x_{\mathrm{CM1}}(t) - n x_{\mathrm{CM1}}(\tau))\, d\tau,
\end{aligned} \tag{5.59}$$

where we have used definitions (5.52)–(5.53). The same holds for $\dot{v}_{\mathrm{CM2}}$:

$$\dot{v}_{\mathrm{CM2}} = -\int_0^t \bar{C}(t-\tau)\,(n x_{\mathrm{CM2}}(t) - n x_{\mathrm{CM2}}(\tau))\, d\tau. \tag{5.60}$$

In the variables $(\bar{\mathbf{x}}_i, \bar{\mathbf{v}}_i)$, taking into account (5.59)–(5.60), equations $(5.51)_{1,3}$ become

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$$
\begin{aligned}
\dot{\bar{v}}_{i1} &= -\dot{v}_{\mathrm{CM1}} + \frac{\beta}{n}\sum_{j=1}^{n}(\bar{v}_{j1}(t) - \bar{v}_{i1}(t)) \\
&\quad - \int_0^t \bar{C}(t-\tau)\sum_{j=1}^{n}\left(\bar{x}_{i1}(t) - \bar{x}_{j1}(\tau) + x_{\mathrm{CM1}}(t) - x_{\mathrm{CM1}}(\tau)\right) d\tau \\
&= \frac{\beta}{n}\left(\sum_{j=1}^{n}\bar{v}_{j1}(t) - \sum_{j=1}^{n}\bar{v}_{i1}(t)\right) \\
&\quad - \int_0^t \bar{C}(t-\tau)(n\bar{x}_{i1}(t) - \sum_{j=1}^{n}\bar{x}_{j1}(\tau) + nx_{\mathrm{CM1}}(t) - nx_{\mathrm{CM1}}(\tau) \\
&\quad - nx_{\mathrm{CM1}}(t) + nx_{\mathrm{CM1}}(\tau))\, d\tau \\
&= -\beta\bar{v}_{i1}(t) - n\left(\int_0^t \bar{C}(t-\tau)\, d\tau\right)\bar{x}_{i1}(t),
\end{aligned}
$$

where we have used (5.57)–(5.58), and

$$\dot{\bar{x}}_{i1} = -\dot{x}_{\mathrm{CM1}} + \bar{v}_{i1} + v_{\mathrm{CM1}} = \bar{v}_{i1}.$$

Similarly for $\dot{\bar{v}}_{i2}$ and $\dot{\bar{x}}_{i2}$.

Finally, we can write the following system:

$$
\begin{cases}
\dot{\bar{v}}_{i1} = -\beta\bar{v}_{i1}(t) - g(t)\bar{x}_{i1}(t), \\
\dot{\bar{v}}_{i2} = -\beta\bar{v}_{i2}(t) - g(t)\bar{x}_{i2}(t), \\
\dot{\bar{x}}_{i1} = \bar{v}_{i1}, \\
\dot{\bar{x}}_{i2} = \bar{v}_{i2},
\end{cases}
\tag{5.61}
$$

where we denote for simplicity

$$g(t) := n\int_0^t \bar{C}(t-\tau)\, d\tau. \tag{5.62}$$

Now we will prove the uniform asymptotic stability of equilibrium (5.56) providing a suitable Lyapunov function for system (5.61). For simplicity, system (5.61) can be written, for each particle and for each component, as a planar system in the variable $\mathbf{y} = (V, X)$:

$$
\begin{cases}
\dot{V} = -\beta V - g(t)X, \\
\dot{X} = V,
\end{cases}
\tag{5.63}
$$

with $g(t)$ given in (5.62).

In relation to (5.63) we prove the following two propositions.

Proposition 40. *Fixed a $\bar{t} > 0$, the system (5.63), admits a Lyapunov function $U(t, \mathbf{y})$ with the properties:*

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1.

$$k_2 \left|\mathbf{y}\right|^2 \leq U(t,\mathbf{y}) \leq k_1 \left|\mathbf{y}\right|^2; \tag{5.64}$$

2.

$$\dot{U}(t,\mathbf{y}) = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial V}\dot{V} + \frac{\partial U}{\partial X}\dot{X} \leq -k_3 \left|\mathbf{y}\right|^2; \tag{5.65}$$

for all $t \geq \bar{t}$*, where* k_1*,* k_2*, and* k_3 *are positive constants.*

Proof. Proposition 40: Let $\bar{t} > 0$, we define the Lyapunov function

$$U(t,V,X) := (V^2 + kXV + g(t)X^2)\psi(t),$$

where

$$\psi(t) := e^{-\frac{g(t)}{\underline{g}}}, \tag{5.66}$$

$$\underline{g} := \inf_{t\geq\bar{t}} g(t) = n\int_0^{\bar{t}} \bar{C}(t-\tau)\,d\tau, \quad \overline{g} := \sup_{t\geq\bar{t}} g(t) = n\int_0^{+\infty} \bar{C}(t-\tau)\,d\tau, \tag{5.67}$$

$$\underline{\psi} := \inf_{t\geq\bar{t}} \psi = e^{-\frac{\bar{g}}{\underline{g}}}, \quad \overline{\psi} := \sup_{t\geq\bar{t}} \psi = e^{-1}, \tag{5.68}$$

$$\overline{\overline{\psi}} := \sup_{t\geq\bar{t}} |\dot{\psi}| = \sup_{t\geq\bar{t}} \frac{e^{-\frac{g(t)}{\underline{g}}}\dot{g}}{\underline{g}}, \tag{5.69}$$

$$k := \min\left[\frac{\underline{\psi}}{\overline{\overline{\psi}}}, \frac{\underline{g}\,\underline{\psi}}{\overline{\overline{\psi}}}, \frac{2\beta\underline{g}\,\underline{\psi}^2}{2\underline{g}\,\underline{\psi}\overline{\psi} + (\overline{\overline{\psi}} + \beta\overline{\psi})^2}\right]. \tag{5.70}$$

Similar functional can be found in [84]. In the following equations we consider the inequalities:

$$-\frac{X^2+V^2}{2} \leq XV \leq \frac{X^2+V^2}{2}.$$

Then, because of $g(t)$ is an increasing function, $\psi(t)$ is nonincreasing, so

$$\dot{\psi} \leq 0, \tag{5.71}$$

and finally

$$\begin{aligned}\dot{\psi}g + \psi\dot{g} &= -e^{-\frac{g(t)}{\underline{g}}}\frac{\dot{g}}{\underline{g}}g + e^{-\frac{g(t)}{\underline{g}}}\dot{g} \\ &= \dot{g}e^{-\frac{g(t)}{\underline{g}}}\left(1 - \frac{g(t)}{\underline{g}}\right) \leq 0, \quad \forall\, t \geq \bar{t}.\end{aligned} \tag{5.72}$$

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To prove the second inequality in a) we consider

$$\begin{aligned} U(t,V,X) &\leq \left(V^2 + k\frac{X^2+V^2}{2} + \overline{g}X^2\right)\overline{\psi} \\ &= \left[\left(1+\frac{k}{2}\right)V^2 + \left(\overline{g}+\frac{k}{2}\right)X^2\right]\overline{\psi} \leq k_1 \left|\mathbf{y}\right|^2, \end{aligned}$$

where

$$k_1 := \max\left[\left(1+\frac{k}{2}\right)\overline{\psi}, \left(\overline{g}+\frac{k}{2}\right)\overline{\psi}\right].$$

To prove the remain inequality in 1. we consider

$$\begin{aligned} U(t,V,X) &\geq \underline{\psi}V^2 + \underline{g}\,\underline{\psi}X^2 - k\overline{\psi}\frac{X^2+V^2}{2} \\ &= V^2\left(\underline{\psi} - \frac{k\overline{\psi}}{2}\right) + X^2\left(\underline{g}\,\underline{\psi} - \frac{k\overline{\psi}}{2}\right) \\ &\geq_{(5.70)} V^2\frac{\underline{\psi}}{2} + X^2\frac{\underline{g}\,\underline{\psi}}{2} \geq k_2\left|\mathbf{y}\right|^2, \end{aligned}$$

where

$$k_2 := \min\left[\frac{\underline{\psi}}{2}, \frac{\underline{g}\,\underline{\psi}}{2}\right].$$

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To prove point 2. we consider the following inequalities:

$$
\begin{aligned}
\dot{U}(t,V,X) &= \dot{\psi}(V^2 + kXV + gX^2) \\
&\quad + \psi\left[2V(-\beta V - gX) + kV^2 + kX(-\beta V - gX) + \dot{g}X^2 + 2gXV\right] \\
&= \left(\dot{\psi} - 2\beta\psi + k\psi\right)V^2 + \left(k\dot{\psi} - \beta k\psi\right)XV + \left(\dot{\psi}g + \psi\dot{g} - kg\psi\right)X^2 \\
&\leq_{(5.71),(5.72)} \left(-2\beta\psi + k\psi\right)V^2 + \left(k\dot{\psi} - \beta k\psi\right)XV - kg\psi X^2 \\
&\leq \left(-2\beta\psi + k\psi\right)V^2 + \left(k|\dot{\psi}| + \beta k\psi\right)|X||V| - kg\psi X^2 \\
&\leq \left(-2\beta\underline{\psi} + k\overline{\psi}\right)V^2 + \left(k\overline{\overline{\psi}} + \beta k\overline{\psi}\right)|X||V| - k\underline{g}\,\underline{\psi}X^2 \\
&= \left(-2\beta\underline{\psi} + k\overline{\psi}\right)V^2 - \frac{k\underline{g}\,\underline{\psi}}{2}\left[|X| - \frac{|V|\left(\overline{\overline{\psi}} + \beta\overline{\psi}\right)}{\underline{g}\,\underline{\psi}}\right]^2 \\
&\quad - \frac{k\underline{g}\,\underline{\psi}}{2}X^2 + \frac{k\left(\overline{\overline{\psi}} + \beta\overline{\psi}\right)^2}{2\underline{g}\,\underline{\psi}}V^2 \\
&\leq \left(-2\beta\underline{\psi} + k\overline{\psi} + \frac{k\left(\overline{\overline{\psi}} + \beta\overline{\psi}\right)^2}{2\underline{g}\,\underline{\psi}}\right)V^2 - \frac{k\underline{g}\,\underline{\psi}}{2}X^2 \\
&\leq_{(5.70)} -\left(\beta\underline{\psi}V^2 + \frac{k\underline{g}\,\underline{\psi}}{2}X^2\right) \leq -k_3\left|\mathbf{y}\right|^2,
\end{aligned}
$$

where

$$
k_3 := \min\left[\beta\underline{\psi}, \frac{k\underline{g}\,\underline{\psi}}{2}\right].
$$

This completes the proof. □

Starting from Proposition 40 we can state the following

Proposition 41. *The equilibrium point* $(V,X) = (0,0)$ *of the linearised system* (5.63) *is globally uniformly asymptotically stable with exponential rate of convergence.*

Proof. Proposition 41: Inequalities (5.65) and (5.64) imply that U satisfies the differential inequality

$$
\dot{U} \leq -\frac{k_3}{k_1}U, \quad \forall\, t \geq \bar{t}.
$$

By the Gronwall's inequality,

$$
U(t,\mathbf{y}(t)) \leq U\left(\bar{t},\mathbf{y}\left(\bar{t}\right)\right)e^{-(k_3/k_1)(t-\bar{t})}.
$$

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Then, using again (5.64), we have

$$|\mathbf{y}(t)| \leq \left(\frac{U(t,\mathbf{y}(t))}{k_2}\right)^{1/2} \leq \left(\frac{U\left(\bar{t},\mathbf{y}\left(\bar{t}\right)\right)e^{-(k_3/k_1)(t-\bar{t})}}{k_2}\right)^{1/2}$$
$$\leq \left(\frac{k_1\left|\mathbf{y}\left(\bar{t}\right)\right|^2 e^{-(k_3/k_1)(t-\bar{t})}}{k_2}\right)^{1/2} = \left(\frac{k_1}{k_2}\right)^{1/2}\left|\mathbf{y}\left(\bar{t}\right)\right| e^{-(k_3/(2k_1))(t-\bar{t})}.$$

Hence the conditions for the uniform asymptotic stability are satisfied with exponential convergence. The proof is completed. □

Remark 42. Returning to system (5.61), Proposition 41 can be applied for each particle and for each component of the position and velocity vectors. Recalling transformations (5.54)–(5.55), this proves the first part of Theorem 38.

Now, to prove the second part of Theorem 38, we investigate equation (5.59) for the motion of the centre of mass. Taking into account that $\dot{X}_{\mathrm{CM1}} = V_{\mathrm{CM1}}$, it assumes the form of a Volterra Integro-Differential Equations (VIDEs) of the type

$$\dot{v}(t) = -\int_0^t c(t-\tau)\int_\tau^t v(s)dsd\tau, \tag{5.73}$$

with

$$c(u) = c_1\frac{e^{-c_2u}e^{-\frac{c_3}{u}}}{u^2}, \tag{5.74}$$

c_1, c_2 and c_3 being positive constants, that arise from (5.50) and (5.42). The same holds for equation (5.60).

While existence and uniqueness of the solution of (5.73) follow from classic techniques, here we want to study its asymptotic properties, which give the asymptotic behaviour of the centre of mass. By Dirichlet formula, equation (5.73) becomes

$$\dot{v}(t) = -\int_0^t\int_0^s c(t-\tau)d\tau v(s)ds,$$

or equivalently

$$\dot{v}(t) = -\int_0^t K(t,s)v(s)ds, \quad K(t,s) = \int_{t-s}^t c(u)du. \tag{5.75}$$

The analytic form of the kernel K is not known. Observing that $\lim_{u\to\infty} u^2c(u) =$

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0, we can write $K(t,s) = F(t) - \alpha + \alpha - F(t-s)$, where

$$F(t) = \int_0^t c(u)du, \tag{5.76}$$

and

$$\alpha = \alpha(c_1, c_2, c_3) = \int_0^\infty c(u)du = \lim_{t\to\infty} F(t). \tag{5.77}$$

With this notation equation (5.75) can be written in the form (VIDEs) of the type [85, eq. (9.9)], i.e.

$$\dot{v}(t) = \bar{f}(t) + \int_0^t B(t-s)v(s)ds, \tag{5.78}$$

with

$$\bar{f}(t) = -(F(t) - \alpha)\int_0^t v(s)ds, \tag{5.79}$$

and

$$B(t) = F(t) - \alpha. \tag{5.80}$$

Our aim now is to apply the theorem by Miller and Grossman in [85, Th. 9.2], that we report here adapted to a scalar equation of the type (5.78).

Theorem 43 (Miller and Grossman). *Assume that, in equation (5.78), the kernel B is in $L^1(0,\infty)$. Then $v(t) \to 0$ whenever $\bar{f}(t) \to 0$ if and only if*

$$w(z) = z - \int_0^{+\infty} B(u)e^{-zu}du \neq 0, \quad Re(z) \geq 0. \tag{5.81}$$

The following results are then the fundamental premise for applying Theorem 43.

Lemma 44. *The function $B(t)$, defined in (5.80) with $F(t)$ given in (5.76), satisfies (5.81).*

Proof. Lemma 44:
For $z = 0$, we have $w(0) = \int_0^{+\infty} B(u)du < 0$, since $F(t) < \alpha$. When $z \neq 0$, according to the elementary properties of the Laplace transform, we get

$$w(z) = z - \frac{1}{z}\int_0^{+\infty} c(u)e^{-zu}du + \frac{\alpha}{z} = \frac{1}{z}\varphi(z),$$

with $\varphi(z) = z^2 - \int_0^{+\infty} c(u)e^{-zu}du + \alpha$. Then, with $z = x + iy$, we have

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$\varphi(z) = \varphi(x,y) = a(x,y) + ib(x,y)$, with

$$a(x,y) = x^2 - y^2 + \alpha - \int_0^{+\infty} c(u)e^{-xu}cos(yu)du$$
$$b(x,y) = 2xy + \int_0^{+\infty} c(u)e^{-xu}sin(yu)du.$$

Consider the case $x > 0$, $y > 0$. Let $p > 0$ be an arbitrary constant, denote by $\tilde{x} = \frac{x}{\sqrt{p}}$ and $\tilde{y} = \frac{y}{\sqrt{p}}$, and define

$$\begin{aligned}\tilde{a}(\tilde{x},\tilde{y}) &= \tilde{x}^2 - \tilde{y}^2 + \frac{\alpha}{p} - \frac{1}{p}\int_0^{+\infty} c(u)e^{-\sqrt{p}\tilde{x}u}\cos(\sqrt{p}\tilde{y}u)du \\ \tilde{b}(\tilde{x},\tilde{y}) &= 2\tilde{x}\tilde{y} + \frac{1}{p}\int_0^{+\infty} c(u)e^{-\sqrt{p}\tilde{x}u}\sin(\sqrt{p}\tilde{y}u)du\end{aligned} \tag{5.82}$$

Then $a(x,y) = 0 \Leftrightarrow \tilde{a}(\tilde{x},\tilde{y}) = 0$ and $b(x,y) = 0 \Leftrightarrow \tilde{b}(\tilde{x},\tilde{y}) = 0$. Taking into account that

$$\left|\int_0^{+\infty} c(u)e^{-xu}\cos(yu)du\right| \leq \alpha, \quad \forall x > 0,\ y \in \mathbb{R},$$

and

$$\left|\int_0^{+\infty} c(u)e^{-xu}\sin(yu)du\right| \leq \alpha, \quad \forall x > 0,\ y \in \mathbb{R},$$

we deduce that

$$\tilde{x}^2 - \tilde{y}^2 < \tilde{a}(\tilde{x},\tilde{y}) < \tilde{x}^2 - \tilde{y}^2 + 2\frac{\alpha}{p},$$
$$\tilde{b}(\tilde{x},\tilde{y}) > 2\tilde{x}\tilde{y} - \frac{\alpha}{p},$$

and hence $\tilde{a}(\tilde{x},\tilde{y}) > 0$ for $0 < \tilde{y} < \tilde{x}$, and $\tilde{a}(\tilde{x},\tilde{y}) < 0$ for $\tilde{y} > \sqrt{\tilde{x}^2 + 2\frac{\alpha}{p}}$, whereas $\tilde{b}(\tilde{x},\tilde{y}) > 0$ for $\tilde{y} > \frac{\alpha}{2p\tilde{x}}$. This assures that $\tilde{a}$ and $\tilde{b}$ may vanish simultaneously only in a region of the $\tilde{x}\tilde{y}$ plane which, by choosing p properly, can be made small and included in a square having side of length 1. An example of this region R_A has been drawn in Figure 7.1, for specific values of α and p. Finally, by plotting $\tilde{a}(\tilde{x},\tilde{y})$ and $\tilde{b}(\tilde{x},\tilde{y})$ for $0 < \tilde{x} < 1$, $0 < \tilde{y} < 1$, we can verify that it definitely results that they never vanish at the same time. Hence, also $a(x,y)$ and $b(x,y)$ are never both zero at a point (x,y) and (5.81) is true.

The case $x \geq 0$, $y < 0$ can be treated analogously, since $a(x,y) = a(x,-y)$ and $b(x,y) = -b(x,-y)$. □

Lemma 45. *Assume that, in equation (5.78),*

i) there exists $M > 0$ and $c_4 \leq c_2$ such that $|v(u)| < Me^{c_4 t}$,

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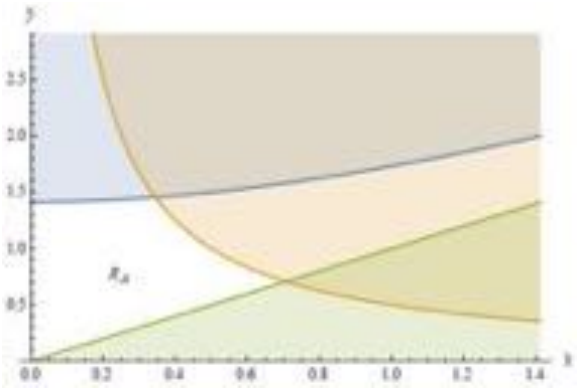

Figure 5.1: R_A : vanishing region for $\tilde{a}$ and $\tilde{b}$ defined in (5.82), bounded by the curves $\tilde{y} = \sqrt{\tilde{x}^2 + 2\frac{\alpha}{p}}$, $\tilde{y} = \frac{\alpha}{2p\tilde{x}}$ and $\tilde{y} = \tilde{x}$.

where c_2 is one of the positive constants appearing in the definition of kernel B by (5.74), (5.76) and (5.80), then $\lim_{u\to+\infty} |v(u)| = 0$.

Proof. Lemma 45:
In the assumptions i), the function $\bar{f}$ in (5.79) satisfies

$$|\bar{f}(t)| \leq (\alpha - F(t))\frac{M}{c_4}(e^{c_4 t} - 1).$$

By the de l'Hospital rule,

$$\lim_{t\to+\infty} (\alpha - F(t))e^{c_4 t} = \frac{1}{c_4} \lim_{t\to+\infty} c(t)e^{c_4 t}.$$

Then by recalling the expression (5.74) of $c(t)$, the hypothesis $c_4 \leq c_2$ and (5.77), we get

$$\lim_{t\to+\infty} \bar{f}(t) = 0. \tag{5.83}$$

By analogous considerations it can be shown that $\lim_{t\to+\infty}(F(t) - \alpha)t^2 = 0$, so that

$$B(t) \in L^1. \tag{5.84}$$

Lemma 44, together with (5.83) and (5.84) assure that all the assumptions of [85, Th. 9.2] are accomplished, so that $\lim_{t\to+\infty} v(t) = 0$. □

Remark 46. Lemma 45 requires that the solution of (5.75) is bounded by an increasing exponential function. So, the class of functions involved in this result is quite large and assures that any bounded solution of equation (5.78), and thus of (5.73), vanishes at infinity. Nevertheless we want to prove that this is true for any solution of (5.73).

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The following result, which generalizes Lemma 45, considerably relaxes the hypotheses on the function $v(t)$.

Lemma 47. *Assume that, in equation* (5.78),

$$\exists M_0 > 0 \text{ and } n \in \mathbb{N} : |v(t)| \leq M_0 e^{nc_2 t}. \tag{5.85}$$

Then $\lim_{t\to+\infty} v(t) = 0$.

Proof. Lemma 47:
If we multiply both sides of equation (5.78) by $e^{-(n-i)c_2 t}$ and add and subtract the quantity $(n-i)c_2 e^{-(n-i)c_2 t} v(t)$, we obtain

$$\begin{aligned} &e^{-(n-i)c_2 t}\dot{v}(t) - (n-i)c_2 e^{-(n-i)c_2 t} v(t) = e^{-(n-i)c_2 t} f(t) \\ &- (n-i)c_2 e^{-(n-i)c_2 t} v(t) + \int_0^t B(t-s) e^{-(n-i)c_2(t-s)} e^{-(n-i)c_2 s} v(s) ds. \end{aligned}$$

Denoting by $v_i(t) = e^{-(n-i)c_2 t} v(t)$, $B_i(t) = e^{-(n-i)c_2 t} B(t)$ and

$$\bar{f}_i(t) = e^{-(n-i)c_2 t} \bar{f}(t), \tag{5.86}$$

this reads

$$\dot{v}_i(t) = \bar{f}_i(t) - (n-i)c_2 v_i(t) + \int_0^t B_i(t-s) v_i(s) ds. \tag{5.87}$$

Now we want to prove that, for $i = 0, \ldots, n-1$, equation (5.87) satisfies the hypotheses of [85, Th. 9.2]. First of all we note that $B_i \in L^1[0, +\infty)$, for all $i = 0, \ldots, n-1$. Consider

$$z + (n-i)c_2 - \int_0^{+\infty} B_i(u) e^{-zu} du, \tag{5.88}$$

which, setting $\zeta = z + (n-i)c_2$, becomes

$$\zeta - \int_0^{+\infty} B(u) e^{-\zeta u} du. \tag{5.89}$$

In Lemma 44 ve have proved that the expression in (5.89) is nonzero for all $\mathrm{Re}(\zeta) \geq 0$, hence also (5.88) is nonzero for all $\mathrm{Re}(z) \geq 0$.
Now consider $f_i(t)$. For $i = 1$, from (5.85) and (5.86),

$$|\bar{f}_1(t)| \leq (\alpha - F(t)) \left(\frac{M_0}{nc_2} e^{nc_2 t} - 1 \right) e^{-(n-1)c_2 t}.$$

According to the proof of Lemma 45, it is easy to see that

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$$\lim_{t\to+\infty}(\alpha - F(t))e^{nc_2t}e^{-(n-1)c_2t} = \lim_{t\to+\infty}(\alpha - F(t))e^{c_2t} = 0.$$

Then, $\lim_{t\to+\infty}\bar{f}_1(t) = 0$ and, in view of [85, Th. 9.2], we have $\lim_{t\to+\infty} v_1(t) = 0$. This in turn implies that

$$\exists M_1 > 0 : |v(t)| \leq M_1 e^{(n-1)c_2t}. \tag{5.90}$$

Now consider $i = 2$, and assume that (5.90) holds. By analogous considerations on $\bar{f}_2(t)$ we obtain that $\exists M_2 > 0 : |v(t)| \leq M_2 e^{(n-2)c_2t}$. Proceeding further for $i = 3,\ 4, \ldots, n-1$, we obtain that $\exists M_{n-1} > 0 : |v(t)| \leq M_{n-1}e^{c_2t}$, thus satisfying the hypotheses of Lemma 45 with $M = M_{n-1}$ and $c_4 = c_2$. □

Proposition 48. *For any triple of positive constants c_1, c_2 and c_3, the solution $v(t)$ of equation (5.73) satisfies $\lim_{t\to+\infty} v(t) = 0$.*

Proof. Proposition 48
By integrating both sides of the VIDE in (5.75) we obtain the Volterra integral equation

$$v(t) = v(0) - \int_0^t A(t,s)v(s)ds,$$

with $A(t,s) = \int_s^t \int_{\tau-s}^{\tau} c(u)du d\tau$. Observe that, since $\frac{e^{-\frac{c_3}{u}}}{u^2} \leq \frac{4}{c_3^2e^2}$, it holds that $c(u) \leq \frac{4c_1}{c_3^2e^2}e^{-c_2u}$. Therefore, for $\tilde{v}(t) = e^{-c_2t}v(t)$, the following inequality holds.

$$|\tilde{v}(t)| \leq e^{-c_2t}|v(0)| + \int_0^t e^{-c_2(t-s)} \int_s^t \int_{\tau-s}^{\tau} \frac{4c_1}{c_3^2e^2}e^{-c_2u}du d\tau |\tilde{v}(s)|ds,$$

hence,

$$|\tilde{v}(t)| \leq |v(0)| + \frac{4c_1}{c_2^2c_3^2e^2}\int_0^t \Lambda(t,s)|\tilde{v}(s)|ds,$$

where $\Lambda(t,s) = e^{-2c_2t}(e^{c_2s}-1)(e^{c_2t}-e^{c_2s}) \leq 1$. So, by Gronwall inequality (see e.g. [86, pg. 79])

$$|v(t)| \leq |v(0)|e^{\left(\frac{4c_1}{c_2^2c_3^2e^2}+c_2\right)t},$$

and the result follows from Lemma 47. □

Remark 49. Proposition 48 holds for both components V_{CM1}, V_{CM2} with equations (5.59), (5.60)

We can now prove our main result of this section:

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Proof. Theorem 38:
The results immediatelly follows combining the previous results. In greater details, First, Proposition 41 and Remark 42 prove point i), the proof of ii) rely on Proposition 48 and Remark 49 . □

5.6 Numerical simulations

In this section we present some numerical simulations, performed in order to show the dynamical behaviour of the model introduced in Section 5.2. Finally, we compare the obtained results with the analytical ones presented in Section 5.3. Firstly, we consider the following dimensionless quantities:

$$t^* := t\eta, \quad \mathbf{x}^* := \frac{\mathbf{x}}{R}, \quad f^* := \frac{f}{f_{\max}}, \quad \beta^* := \frac{\beta}{\eta},$$
$$\gamma^* := \frac{\gamma f_{\max}}{R^2\eta^2}, \quad D^* := \frac{D}{R^2\eta}, \quad \xi^* := \frac{\xi}{f_{\max}\eta},$$

where $f_{\max}$ is the maximum concentration of signal f. With these definitions, system (5.5) can be written as

$$\begin{cases} \dot{\mathbf{v}}_i = \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n} \frac{1}{\left(1+|\mathbf{x}_i-\mathbf{x}_j|^2\right)^{\sigma}}(\mathbf{v}_j-\mathbf{v}_i) + \gamma\nabla f(\mathbf{x}_i), \\ \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \partial_t f = D\Delta f + \xi\displaystyle\sum_{j=1}^{n}\chi_{\mathbf{B}(\mathbf{x}_j,1)} - f, \end{cases} \tag{5.91}$$

where we have dropped, for simplicity, the asterisks for the nondimensional quantities. Notice that, due to the choice of R as characteristic length, the dimensionless particle ray turns out to be a unit value.

5.6.1 Numerical method

The numerical approximation scheme used here employs a 2D finite difference method on a spatial domain $\Omega := [a,b]\times[c,d]$.

The computation grid for our problem consists of points (x_m, y_l) where $x_m = m\Delta x$, $m = 0,...,N_x$, and $y_l = l\Delta y$, $l = 0,...,N_y$. The same is done for the period of observation $[0,T]$: choosing the time step Δt, it follows that the generic $k-th$ temporal step is given by $t_k = k\Delta t$, $k = 0,...,N_T$.

For the parabolic equation $(5.91)_3$, we perform the classical exponential transformation in order to eliminate the stiff term $-f$. In greater details, we write $f = e^{-t}u$, where u solves

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$$\partial_t u = D\Delta u + \xi e^t \sum_{j=1}^{n} \chi_{\mathbf{B}(\mathbf{x}_j,1)}. \tag{5.92}$$

At each time step, the solution of (5.91)$_3$ will be approximated at nodes of the obtained grid: we denote the approximation of $u((x,y),t)$ at point $((x_m,y_l),t_k)$ with $u^k_{m,l}$.

We now discuss the discretization for the Laplace operator, using a five-point scheme. The second order partial derivates can be approximated with a second order accuracy. Defining

$$\begin{aligned} u_{xx}((x_m,y_l),t_k) &\approx D_x^2 u^k := \frac{u^k_{m-1,l} - 2u^k_{m,l} + u^k_{m+1,l}}{\Delta_x^2}, \\ u_{yy}((x_m,y_l),t_k) &\approx D_y^2 u^k := \frac{u^k_{m,l-1} - 2u^k_{m,l} + u^k_{m,l+1}}{\Delta_y^2}, \end{aligned} \tag{5.93}$$

we approximate $\Delta u((x_m,y_l),t_k)$ as

$$\begin{aligned} \Delta u((x_m,y_l),t_k) &\approx (D_x^2 u^k + D_y^2 u^k) \\ &= \frac{u^k_{m-1,l} - 2u^k_{m,l} + u^k_{m+1,l}}{h_x^2} + \frac{u^k_{m,l-1} - 2u^k_{m,l} + u^k_{m,l+1}}{h_y^2}. \end{aligned} \tag{5.94}$$

Since this model at this stage does not require to reproduce a particular experimental setting, from a numerical point of view it was natural for us to choose periodic boundary conditions to avoid to introduce artificial boundary conditions that would affect the solution.

With the notation used in (5.93), we write

$$\begin{aligned} \frac{u^{k+1}_{m,l} - u^k_{m,l}}{\Delta t} &= \frac{D}{2}\left(D_x^2 u^{k+1} + D_y^2 u^{k+1}\right) + \frac{D}{2}\left(D_x^2 u^k + D_y^2 u^k\right) \\ &+ \frac{1}{2} e^{(k+1)\Delta t}\xi \sum_{j=1}^{n} \chi_{\mathbf{B}(\mathbf{x}_j^k,1)} + \frac{1}{2} e^{k\Delta t}\xi \sum_{j=1}^{n} \chi_{\mathbf{B}(\mathbf{x}_j^k,1)} \end{aligned} \tag{5.95}$$

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With simple calculation, it can be rewritten

$$\left(1+\frac{D\Delta t}{\Delta_x^2}+\frac{D\Delta t}{\Delta_y^2}\right)u_{m,l}^{k+1}-\frac{D\Delta t}{2\Delta_x^2}u_{m-1,l}^{k+1}-\frac{D\Delta t}{2\Delta_x^2}u_{m+1,l}^{k+1}-\frac{D\Delta t}{2\Delta_y^2}u_{m,l-1}^{k+1}-\frac{D\Delta t}{2\Delta_y^2}u_{m,l+1}^{k+1}=$$
$$\left(1-\frac{D\Delta t}{\Delta_x^2}-\frac{D\Delta t}{\Delta_y^2}\right)u_{m,l}^{k}+\frac{D\Delta t}{2\Delta_x^2}u_{m-1,l}^{k}+\frac{D\Delta t}{2\Delta_x^2}u_{m+1,l}^{k}+\frac{D\Delta t}{2\Delta_y^2}u_{m,l-1}^{k}+\frac{D\Delta t}{2\Delta_y^2}u_{m,l+1}^{k}+$$
$$+\frac{1}{2}\Delta t e^{(k+1)\Delta t}\xi\sum_{j=1}^{n}\chi_{\mathbf{B}(\mathbf{X}_j^k,1)}+\frac{1}{2}\Delta t e^{k\Delta t}\xi\sum_{j=1}^{n}\chi_{\mathbf{B}(\mathbf{X}_j^k,1)}. \tag{5.96}$$

The numerical scheme for (5.91)$_{1,2}$ reads:

$$\begin{cases}\dfrac{\mathbf{x}_i^{k+1}-\mathbf{x}_i^k}{\Delta t} &= \mathbf{v}_i^k,\\[2ex] \dfrac{\mathbf{v}_i^{k+1}-\mathbf{v}_i^k}{\Delta t} &= \dfrac{\beta}{n}\displaystyle\sum_{j=1}^{n}\frac{1}{\left(1+|\mathbf{x}_i^k-\mathbf{x}_j^k|^2\right)^{\sigma}}\left(\mathbf{v}_j^{k+1}-\mathbf{v}_i^{k+1}\right)\\[2ex] &\quad +\gamma F_{\mathbf{x}_i^k}\left(\nabla_{m,l}f^k\right).\end{cases} \tag{5.97}$$

We now focus on (5.97)$_2$. It can be rewritten as

$$\left(1+\Delta t\frac{\beta}{n}\sum_{j=1}^{n}\frac{1}{(1+|\mathbf{x}_i^k-\mathbf{x}_j^k|^2)^{\sigma}}\right)\mathbf{v}_i^{k+1}-\Delta t\frac{\beta}{n}\sum_{j=1}^{n}\frac{\mathbf{v}_j^{k+1}}{(1+|\mathbf{x}_i^k-\mathbf{x}_j^k|^2)^{\sigma}}=$$
$$\mathbf{v}_i^k+\gamma F_{\mathbf{x}_i^k}\left(\nabla_{m,l}f^k\right). \tag{5.98}$$

Finally, let us deepen the structure of the term $F_{\mathbf{X}_i^k}\left(\nabla_{m,l}f^k\right)$. Finite difference method (5.96) returns the values of function u and thus of f, reminding the exponential transformation, only at the points of our computational grid. Using a central difference scheme, we discretize the gradient at those points. Namely we write

$$\nabla_{m,l}f^k:=\left(\frac{f_{m+1,l}^k-f_{m-1,l}^k}{2\Delta x},\frac{f_{m,l+1}^k-f_{m,l-1}^k}{2\Delta y}\right). \tag{5.99}$$

Our problem requires the value $\nabla f(\mathbf{X}_i)$, for all $i=1,...,n$. The difficulty is that the position $\mathbf{x}_i$ is identified by a general point in the domain, which could not belong to our computational grid. For that reason we use an interpolation of values $\nabla_{m,l}f^k$ concerning the grid points nearest to $\mathbf{X}_i^k$, denoted

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with $F_{\mathbf{X}_i^k}\left(\nabla_{m,l} f^k\right)$. In particular, we choose the following expression:

$$F_{\mathbf{X}_i^k}\left(\nabla_{m,l} f^k\right) = \frac{\sum_{j=1}^4 \nabla f^k(\mathbf{x}^j)}{\sum_{j=1}^4 d^j}, \tag{5.100}$$

where $\mathbf{x}^j$, $j = 1, ..., 4$ denote the four points of the grid which are nearest to $\mathbf{X}_i^k$, and d^j the value of the distance, namely $d^j = \left|\mathbf{X}_i^k - \mathbf{x}^j\right|$.

We summarize the numerical scheme for (5.91) in the following system

$$\begin{cases} \dfrac{\mathbf{x}_i^{k+1} - \mathbf{x}_i^k}{\Delta t} = \mathbf{v}_i^k, \\ \dfrac{\mathbf{v}_i^{k+1} - \mathbf{v}_i^k}{\Delta t} = \dfrac{\beta}{n} \displaystyle\sum_{j=1}^n \frac{1}{\left(1 + |\mathbf{x}_i^k - \mathbf{x}_j^k|^2\right)^\sigma} \left(\mathbf{v}_j^{k+1} - \mathbf{v}_i^{k+1}\right) \\ \qquad\qquad + \gamma F_{\mathbf{x}_i^k}\left(\nabla_{m,l} f^k\right), \\ f_{m,l}^k = e^{-t_k} u_{m,l}^k, \end{cases} \tag{5.101}$$

where $u_{m,l}^k$ is obtained solving (5.96).

5.6.2 Numerical tests

In the following numerical tests we consider a spatial domain $\Omega = [0, 50] \times [0, 50]$ with periodic boundary conditions, and we choose a suitable time interval of observation $[0, T]$. For the initial data we fix $f(x, 0) = 0$ and, for $i = 1, \ldots, n$, $\mathbf{X}_i(0) = \mathbf{X}_{i0}$, $\mathbf{V}_i(0) = \mathbf{V}_{i0}$. In particular $\mathbf{X}_{i0}$ is chosen as a random vector, such that all the particles at $t = 0$ are contained in a suitable initial region, fixed in the domain. Then $\mathbf{V}_{i0} = (V_{i0} \cos\theta_i, V_{i0} \sin\theta_i)$ are chosen with V_{i0} random numbers in $[0, V_{0,\max}]$, and θ_i random numbers in $[0, 2\pi]$. Let us now describe some meaningful tests.

Test 1

In this test we set the parameters $\sigma = 0.5$, $\beta = 5$, $\gamma = 2 \times 10^2$, $D = 2 \times 10^2$, $\xi = 0.5$, $V_{0,\max} = 3$, and we consider $n = 10$ particles located in $\mathbf{X}_0$ as in Figure 5.2 (a). The time interval of observation is $[0, 500]$. Spatial and temporal discretizations are given respectively by $\Delta x = \Delta y = 0.25$ and $\Delta t = 10^{-4}$.

Figure 5.2 shows four time steps of the numerical simulation. Here and in the next tests for each time step we plot, on the left the chemoattractant concentration $f(x, t)$, while on the right the positions and the velocities of the particles in the spatial domain. The red square at $t = 0$ is the region in

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which the initial positions are taken. The red marker indicates the centre of mass of the system, and the blue arrows are the velocity vectors. We observe an initial stage in which the particles tend to move somewhat aligned about until $t = 5$ (Figure 5.2 (b)), then they begin to converge to their centre of mass about at $t = 30$ (Figure 5.2 (c)), finally all particles stop in a same position (Figure 5.2 (d)).

In Figure 5.3 (a) is shown the spatial fluctuation around the centre of mass system

$$Fl_X(t) := \sum_{i=1}^{n} |\mathbf{x}_i(t) - \mathbf{x}_{\mathrm{CM}}(t)|^2 ,$$

as a function of the time. In Figure 5.3 (b) is shown the velocity fluctuation around the centre of mass

$$Fl_V(t) := \sum_{i=1}^{n} |\mathbf{v}_i(t) - \mathbf{v}_{\mathrm{CM}}(t)|^2 .$$

For $t \geq 61$ $Fl_X(t)$ and $Fl_V(t)$ are less than 10^{-10}. Notice that the square root of $Fl_X(t)$ and $Fl_V(t)$ is proportional to the standard deviations of $\mathbf{x}_i(t)$ and $\mathbf{v}_i(t)$ with respect to the position and velocity of the centre of mass. Figure 5.3 (c) displays the norm of the velocity of the centre of mass $|\mathbf{v}_{\mathrm{CM}}(t)|$ versus time. For $t \geq 52$ this velocity is less than 7.8×10^{-2}. The oscillating pattern, shown here and in the next tests, can be attributed to the numerical error, as it is confirmed using finer meshes for the discretization.

Test 2

In this simulation we fix the parameters as in Test 1, and we double the number of the interacting agents, considering $N = 20$ cells as in Figure 5.4 (a). The time interval of observation is $[0, 500]$.

In Figure 5.4 we can observe four different time steps of the numerical simulation showing the aggregation of the initial group and the convergence to zero of its velocity.

In Figures 5.5 (a)–(b) we plot, the quantities $Fl_X(t)$ and $Fl_V(t)$. Here, for $t \geq 34$ we have values less than 10^{-10}. In Figure 5.5 (c) we show the quantity $\|\mathbf{V}_{\mathrm{CM}}(t)\|$ versus time, with values smaller than 8.39×10^{-2} for $t \geq 28$. Comparing the results of this test with those of Test 1 we can state that, with the same parameters, an increasing number of cells enhances the rate of convergence due to the greater amount of the expressed chemoattractant.

Test 3

In this test we consider a case in which, for the pure Cucker-Smale model (1.5), i.e. $\gamma = 0$ in our model, the flocking behaviour does not occur. We fix

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the other parameters $\sigma = 0.8$, $\beta = 5$, $D = 2 \times 10^2$, $\xi = 0.5$, $V_{0,\max} = 3$, and $\mathbf{X}_0$ as in Figure 5.6 (a). The time interval of observation is $[0, 15]$. Spatial and temporal discretizations are as in Test 1. Clearly in this case equations $(5.6)_{1,2}$ and $(5.6)_3$ are decoupled. Since $\sigma > 1/2$, and taking into account the initial data, according to the results in [87], the flocking of the system is not guaranteed.

Figure 5.6 shows our numerical simulation at three time steps. We can observe, in fact, a dispersion of the initial group of particles.

In the next test we will show that, adding the chemotactic effect, we can recover the time-asymptotic convergence of the migrating group.

Test 4

Inspired by the mathematical model proposed in [3] for the zebrafish lateral line development, we consider a simulation in which two kinds of cells are involved: the *leaders* that produce the chemotactic signal and the *followers* that do not produce any signal, both subjected to the alignment effect and to the attraction of the chemical gradient.

For the numerical simulation set $\sigma = 0.5$, $\beta = 5$, $\gamma = 1.5 \times 10^2$, $D = 2 \times 10^2$, $\xi = 3$, $V_{0,\max} = 0.3$, and $\mathbf{X}_0$ as in Figure 5.7 (a). We consider a time interval of observation $[0, 500]$. Spatial and temporal discretizations are as in Test 1.

Figure 5.7 shows four time steps of our simulation. A single leader cell is marked in green colour, while the other follower cells are in red colour. Here the centre of mass is marked in blue. We observe, about at $t = 15$, that cells begin to be attracted toward the chemoattractant source (Figure 5.7 (b)). In the next time steps the cells tend to converge in the centre of mass and then they stop (Figure 5.7 (c)–(d)).

In Figures 5.8 (a)–(b) are shown the spatial and velocity fluctuations, $Fl_X(t)$ and $Fl_V(t)$, around the centre of mass system. For $t \geq 131$ these quantities are less than 10^{-10}. Figure 5.8 (c) displays the quantity $\|\mathbf{V}_{\mathrm{CM}}(t)\|$. For $t \geq 328$ we have values smaller than 5.3×10^{-2}.

Test 5

In this test we simulate the system only under the chemotactic effect, neglecting the alignment term, that is we set $\beta = 0$. The other parameters are $\gamma = 10^2$, $D = 2 \times 10^2$, $\xi = 1.5$, and the initial data $V_{0,\max} = 0.8$, $\mathbf{X}_0$ as in Figure 5.9 (a). The time interval of observation is $[0, 4000]$. Spatial and temporal discretizations are fixed as in Test 1.

In Figure 5.9 we plot four time steps. Although the initial group aggregates, we do not observe, in our time of observation, a convergence of the particles, rather they show an oscillating behaviour around their centre of mass.

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In Figures 5.10 (a)–(b) we plot respectively $Fl_X(t)$ and $Fl_V(t)$. In this case, the spatial and velocity fluctuations around the centre of mass remain bounded but do not converge to zero. In particular, in the time interval of observation, we have $Fl_X(t) \geq 0.88$ and, if we consider a trend line, its slope seems to decreases monotonically approaching to zero. For example, in relation to Figure 5.10 (a), the slope of the linear fit on the values of $Fl_X(t)$, computed on the time intervals $[0, 800]$ and $[3200, 4000]$, changes from -1.46×10^{-1} to -1.84×10^{-4}. Moreover, we remark that in previous simulations, containing the alignment effect, on time intervals much smaller than this test we have obtained values of $Fl_X(t)$ less than 10^{-10}.

In Figure 5.10 (c) is shown $|\mathbf{v}_{\mathrm{CM}}(t)|$ as a function of time. For $t \geq 254$ we have values smaller than 3.79×10^{-2}. Performing the same numerical test with the finer discretization $\Delta x = \Delta y = 1.25$ and $\Delta t = 10^{-5}$, we find that, for $t \geq 120$, $|\mathbf{v}_{\mathrm{CM}}(t)|$ is smaller than 1.41×10^{-2}, see Figure 5.10 (d). From this we can deduce that the velocity of the centre of mass goes to zero.

Our numerical results suggest that our model, in absence of alignment and with the only chemotactic effect, is unable to reproduce stationary patterns.

5.7 Discussion

In this chapter we have proposed simplified version of the models in [2] and [3]. The model can be regarded as an extension of the Cucker-Smale model, coupling alignment and chemotaxis effects, and introducing a hybrid description. Differently from the previous chapters, we have studied our model by both an analytical and a numerical point of view. From the analytical point of view, we have proved local and global existence and uniqueness of the solution to the nonlinear system (see [62] and Chapter 2 for an extension to a more general framework). The novelty rely on the study of the asymptotic behaviour of the linearised system. We have proved the asymptotic convergence of the particles in their centre of mass with same velocity. Then, the velocity of the centre of mass is proved to go time-asymptotically to zero. From a numerical point of view this property has been tested on the full nonlinear system, finding a complete concordance with the analytical results.

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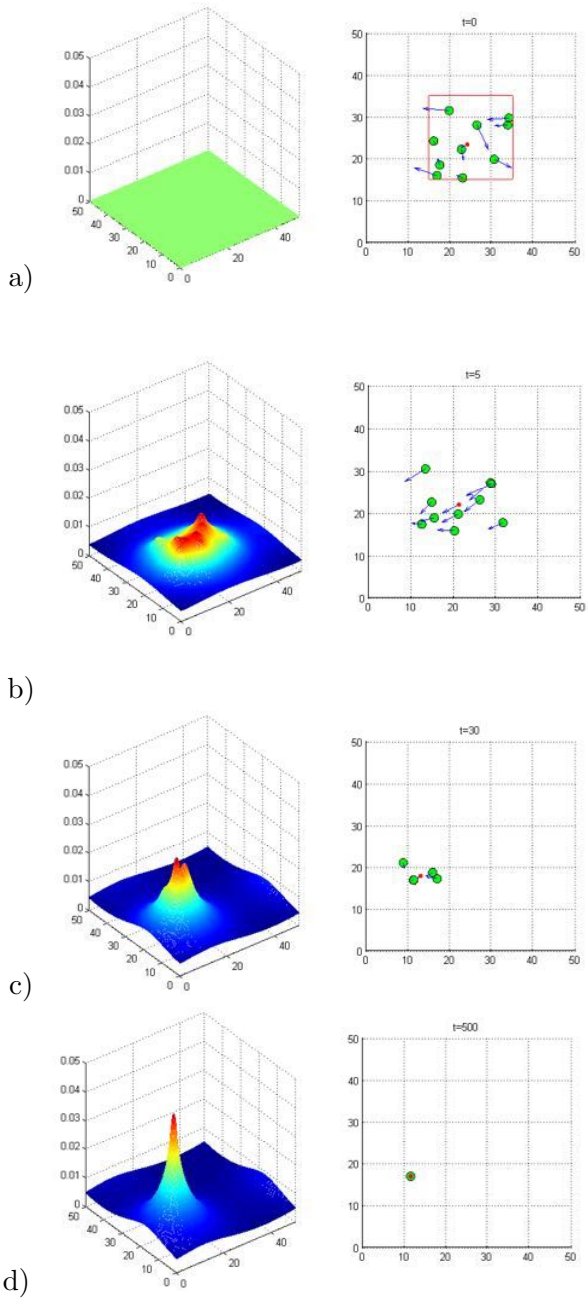

Figure 5.2: **Test 1**. Simulation with parameters $\sigma = 0.5$, $\beta = 5$, $\gamma = 2 \times 10^2$, $D = 2 \times 10^2$, $\xi = 0.5$, $V_{0,\max} = 3$, and $\mathbf{X}_0$ randomly taken in the red square shown in the top panel. The chemoattractant concentration $f(x,t)$ is on the left, while on the right there is the positions and the velocities of the particles. The red marker marks the centre of mass of the system, and the blue arrows are the velocity vectors.

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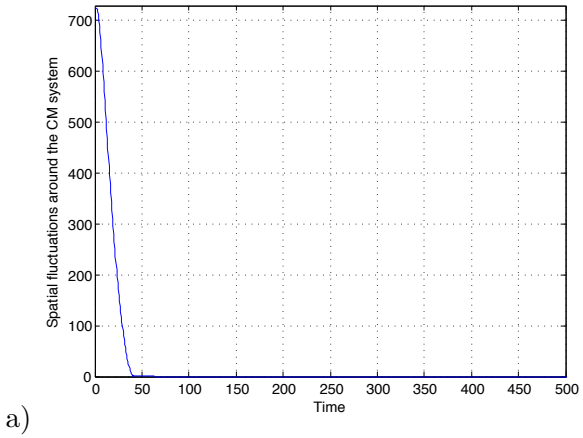
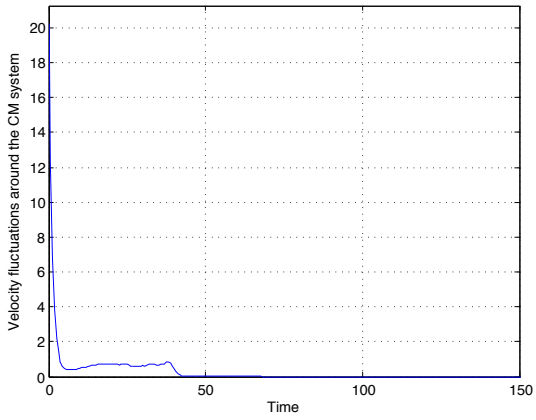


a)



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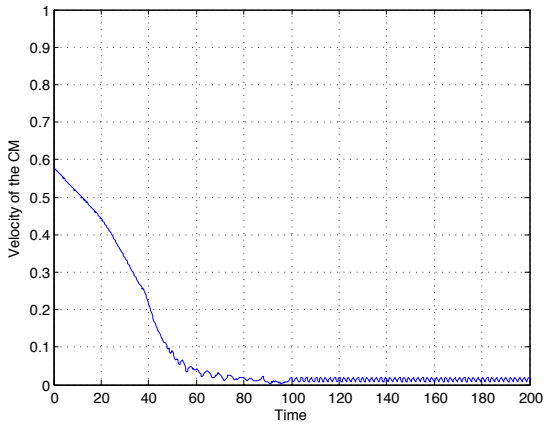

c)

Figure 5.3: **Test 1**. Functions $Fl_X(t)$, $Fl_V(t)$ and $|\mathbf{v}_{\mathrm{CM}}(t)|$ versus time (x-axis shows only a part of the time domain), as defined in Section 5.6.2.

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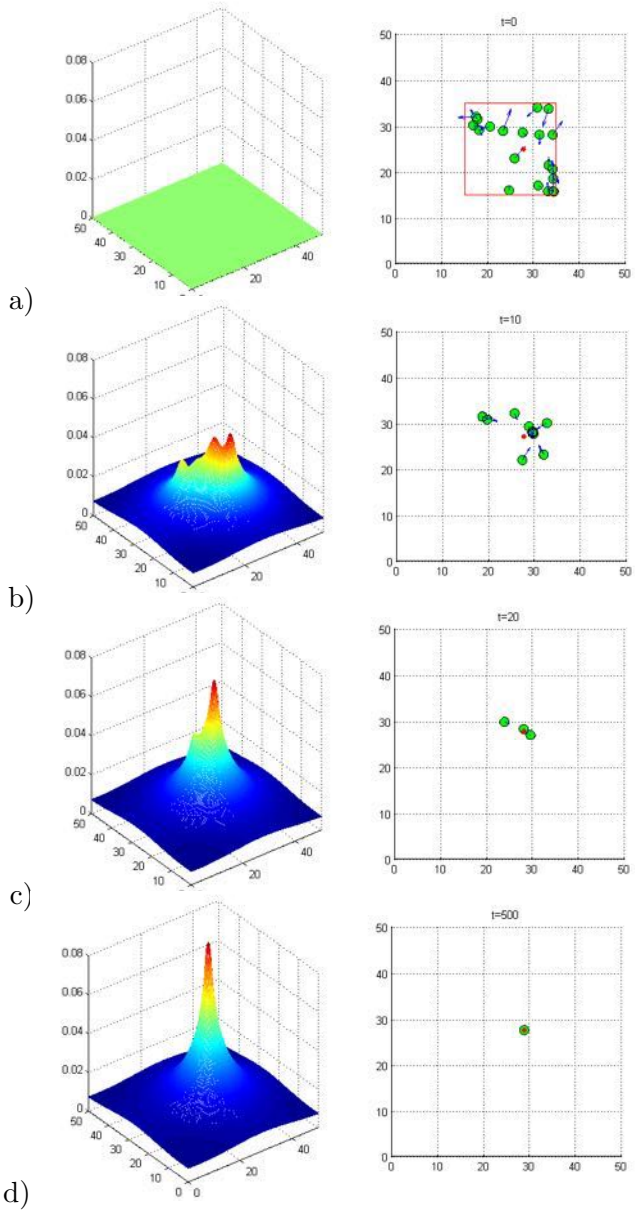

Figure 5.4: **Test 2**. Numerical simulation in a spatial domain $\Omega = [0, 50] \times [0, 50]$ with periodic boundary conditions, and in the time interval $[0, 500]$. The parameters values are $N = 20$, $\sigma = 0.5$, $\beta = 5$, $\gamma = 2 \times 10^2$, $D = 2 \times 10^2$, $\xi = 0.5$, $V_{0,\max} = 3$, and $\mathbf{X}_0$ randomly taken in the red square shown in (a). Spatial and temporal discretizations are respectively $\Delta x = \Delta y = 0.25$ and $\Delta t = 10^{-4}$. The four plots are respectively at time steps $t = 0, 10, 20, 500$.

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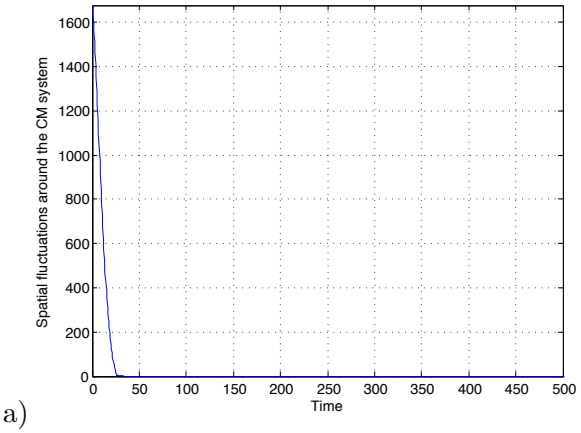
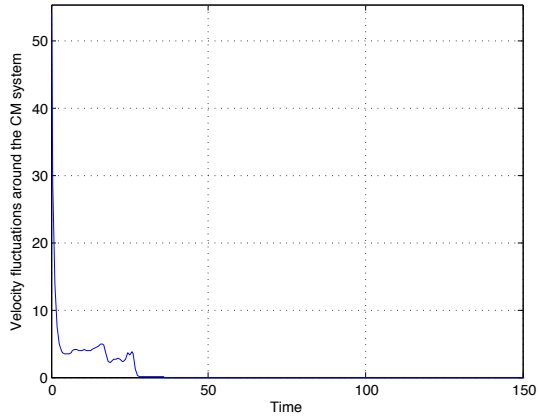


a)



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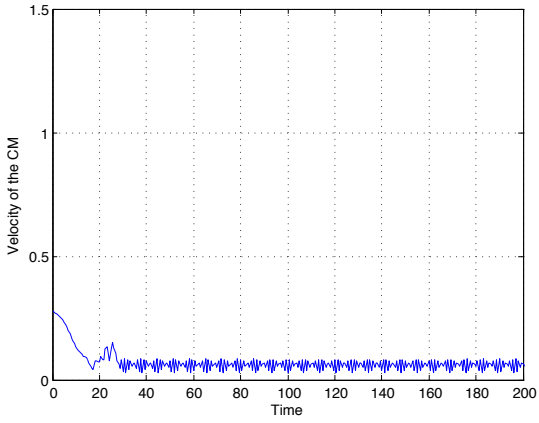

c)

Figure 5.5: **Test 2**. (a)–(b) Spatial and velocity fluctuations around the centre of mass system $Fl_X(t)$ and $Fl_V(t)$ (x-axis shows only a part of the time domain). For $t \geq 34$ we have values less than 10^{-10}. (c) Norm of the velocity of the centre of mass $|\mathbf{v}_{\mathrm{CM}}(t)|$ versus time (only a part of the time interval is shown on the x-axis). For $t \geq 28$ this velocity is less than 8.39×10^{-2}.

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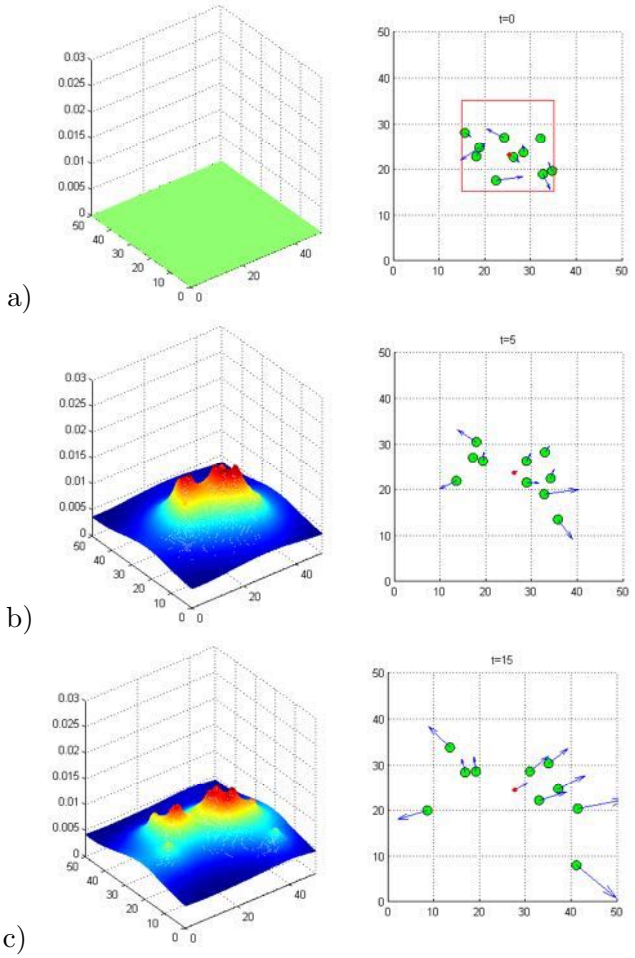

Figure 5.6: **Test 3**. Numerical simulation in a spatial domain $\Omega = [0, 50] \times [0, 50]$ with periodic boundary conditions, and in the time interval $[0, 15]$. The parameters values are $\sigma = 0.8$, $\beta = 5$, $\gamma = 0$, $D = 2 \times 10^2$, $\xi = 0.5$, $V_{0,\max} = 3$, and $\mathbf{X}_0$ randomly taken in the red square shown in (a). Spatial and temporal discretizations are respectively $\Delta x = \Delta y = 0.25$ and $\Delta t = 10^{-4}$. In this case the motion equations and the chemoattractant equation are decoupled and for the particles we simulate the pure Cucker-Smale model (1.5). Taken into account the parameters values and the initial data the flocking behaviour is not ensured by results in [87]. In fact, from the three plots, taken respectively at time steps $t = 0, 5, 15$, we can observe a dispersion of the initial group.

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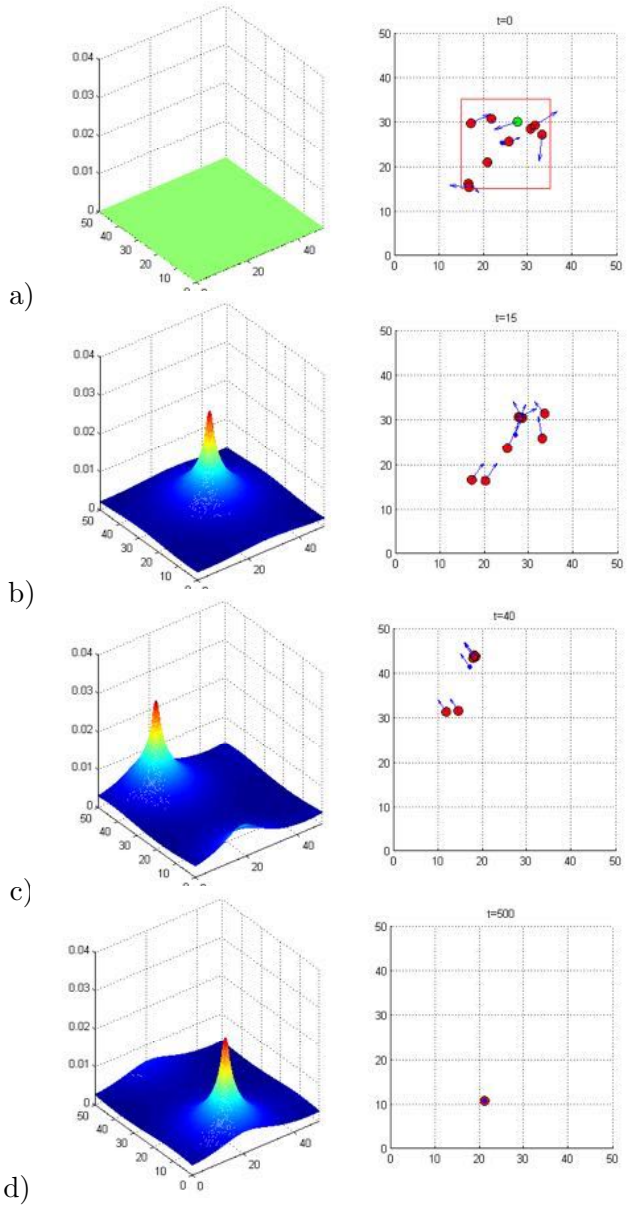

Figure 5.7: **Test 4**. Numerical simulation in a spatial domain $\Omega = [0, 50] \times [0, 50]$ with periodic boundary conditions, and in the time interval $[0, 500]$. The parameters values are $\Delta x = \Delta y = 0.25$, $\Delta t = 10^{-4}$, $\sigma = 0.5$, $\beta = 5$, $\gamma = 1.5 \times 10^2$, $D = 2 \times 10^2$, $\xi = 3$, $V_{0,\max} = 0.3$, and $\mathbf{X}_0$ randomly taken in the red square shown in (a). The green cell marks a leader cell (●), that produce the chemical signal, while the other red cells are the followers (●), that do not produce any signal and follow the chemoattractant gradient. The four plots are respectively at time steps $t = 0, 15, 40, 500$. On the left there is the chemoattractant concentration, while on the right the positions and the velocities of the particles.

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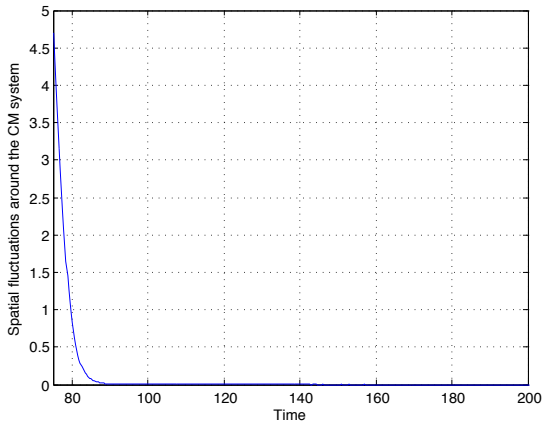
a)



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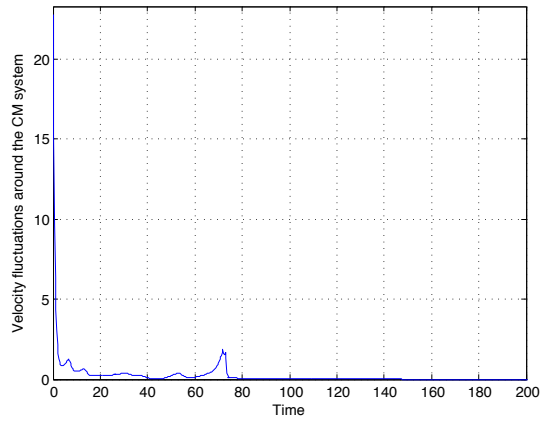

c)

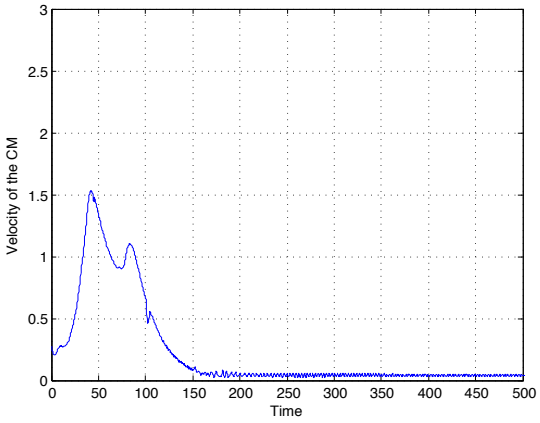

Figure 5.8: **Test 4**. (a)-(b) Spatial and velocity fluctuations, $Fl_X(t)$ and $Fl_V(t)$, around the centre of mass system (only a part of the time domain is shown on the x-axis). For $t \geq 131$ $Fl_X(t)$ and $Fl_V(t)$ are less than 10^{-10}. (c) Velocity of the centre of mass $|\mathbf{v}_{\mathrm{CM}}(t)|$ as a function of the time. For $t \geq 328$ we have values smaller than 5.3×10^{-2}.

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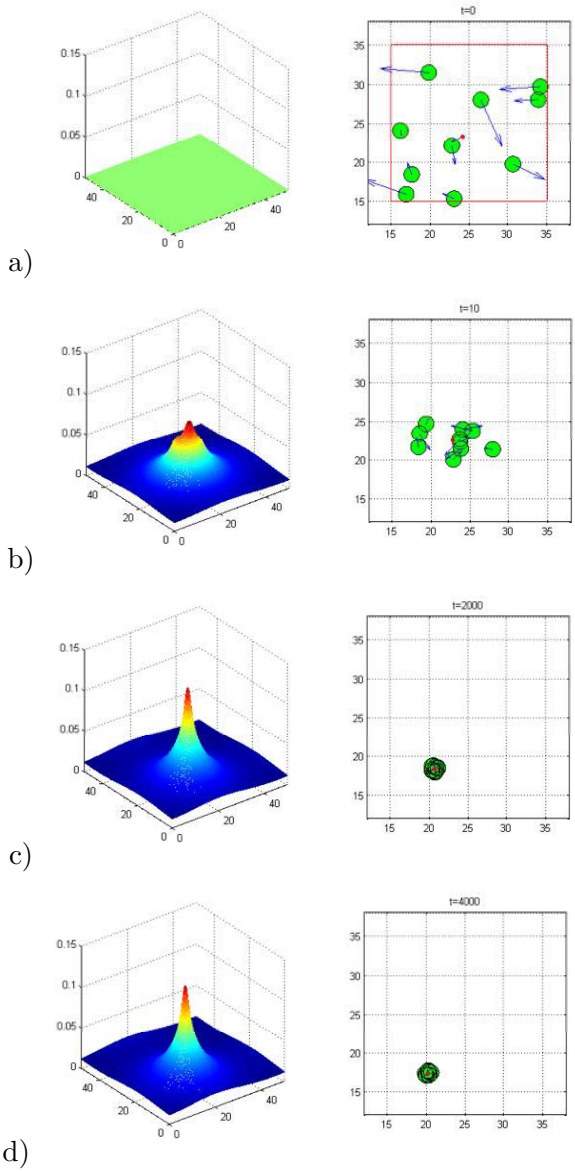

Figure 5.9: **Test 5**. Numerical simulation in a spatial domain $\Omega = [0, 50] \times [0, 50]$ with periodic boundary conditions, and in the time interval $[0, 4000]$ (plots on the right shown only a part of the spatial domain). Spatial and temporal discretizations are respectively $\Delta x = \Delta y = 0.25$ and $\Delta t = 10^{-4}$. In this test only the chemotactic force is considered, neglecting the alignment effect ($\beta = 0$). For the other values we fix $\gamma = 10^2$, $D = 2 \times 10^2$, $\xi = 1.5$, $V_{0,\max} = 0.8$, and $\mathbf{X}_0$ randomly taken in the red square shown in (a). The plots taken at time steps $t = 0, 10, 2000, 4000$ display the aggregation of the initial group of particles. In this case the convergence is not observed, but an oscillating behaviour around their centre of mass is shown.

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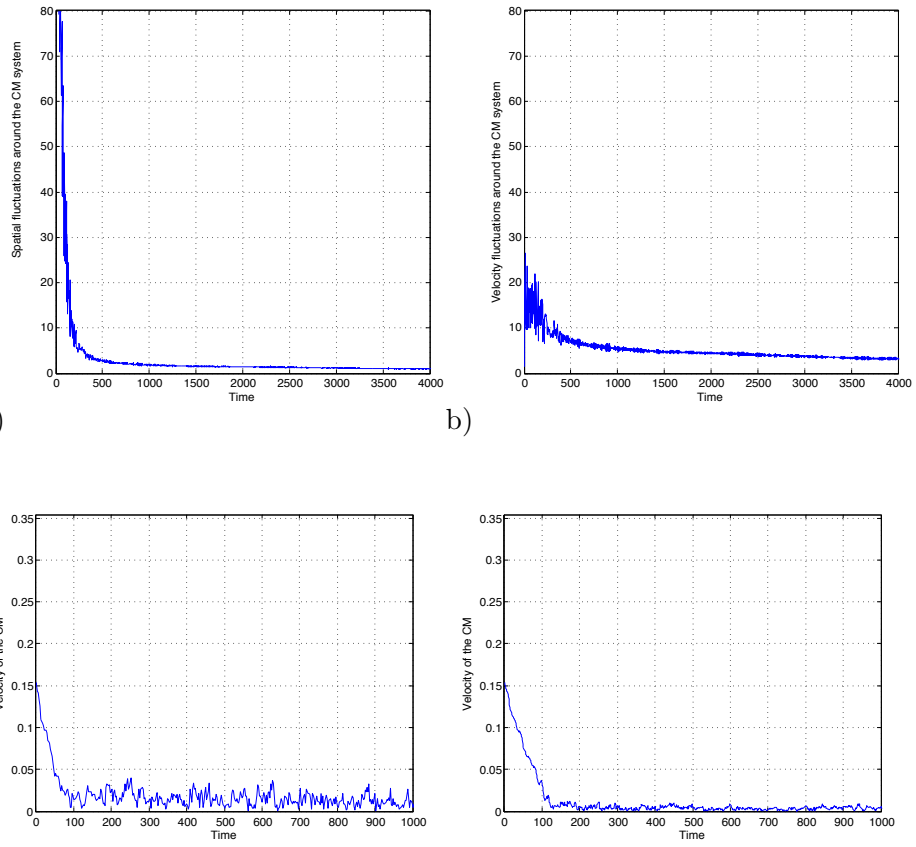

a) b)

c) d)

Figure 5.10: **Test 5**. (a)-(b) Spatial and velocity fluctuations, $Fl_X(t)$ and $Fl_V(t)$, around the centre of mass system versus time (only a part of the time domain is shown on the x-axis). In this case $Fl_X(t)$ and $Fl_V(t)$ remain bounded but do not converge to zero. In particular we have $Fl_X(t) \geq 0.88$ and, if we consider a trend line, its slope seems to decrease monotonically. In particular in the time intervals $[0, 800]$ and $[3200, 4000]$, the slope of the fit line changes from -1.46×10^{-1} to -1.84×10^{-4}. (c) Velocity of the centre of mass $\|\mathbf{V}_{\mathrm{CM}}(t)\|$ versus time, using a spatial and a temporal discretization given by $\Delta x = \Delta y = 2.5$ and $\Delta t = 10^{-4}$ (x-axis shows a part of the time interval). For $t \geq 254$ this quantity is smaller than 3.79×10^{-2}. (d) With a finer mesh, $\Delta x = \Delta y = 0.125$ and $\Delta t = 10^{-5}$, we obtain smaller values: for $t \geq 120$ we have values less than 1.41×10^{-2}.

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Chapter 6

Conclusions and Future Perspective

The purpose of Part I is to introduce the reader to the emerging class of what we call coupled hybrid models. With a step by step approach, the previous chapters investigate well-posedeness properties and the asymptotic behavior of the solutions to systems generally expressed as in (1)-(3). After a preliminary chapter, in which we give a brief overview of seminal second-order models of the literature, we present part of an ongoing work, consisting in a novel second-order model with delay. The aim is to couple the actual discrete-particle model with a continuum one, describing the evolution in time of an exogenous information, which influence the dynamics. For the rest of this Part, we focus on the coupled structure in (1)-(3). From a modeling point of view, the idea to treat the agents as discrete entities, and the exogenous signal, regardless of its nature, as a continuum, combines the advantages of individual-based models with continuous ones. The particular structure has been already adopted by other work of the literature, which only have modeling and numerical purposes. For that reasons, our work has to be considered as a first step towards a more detailed analytical characterization. Since the literature concerning analytical findings is still lacking, there are several future perspective in this field we are going to address. In particular, one of the future work directions concerns another critical issue, that is the shortcoming of a detailed technique to estimate the parameters occurring in a model.

Suppose to introduce a set of real parameters in a suitable domain $D \in \mathbb{R}^p$, denoted with $\theta = (\theta_1, ..., \theta_p)$, in the reaction-diffusion equation. It means to assume $a_{ij} = a_{ij}(.;\theta)$, $b_i = b_i(.;\theta)$, $c = c(.;\theta)$ in equation (3), and $f = f(.;\theta)$ as the solution to (3), once assigned initial conditions. Recalling that f influences the dynamics expressed in (1), the introduction of the parameters affects the solution $\mathbf{X} = \mathbf{X}(\cdot;\theta)$. The introduction of parameters

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leads to the following modifications of equations (1)-(3):

$$\ddot{\mathbf{x}}_i(t;\theta) = F_i\left(t, \mathbf{X}(t;\theta), \dot{\mathbf{X}}(t;\theta), \nabla f\left(\mathbf{x}_i(t;\theta), t; \mathbf{X}(t;\theta), \theta\right), \theta\right), \tag{6.1}$$

$$\begin{cases} L^\theta f(x,t;\mathbf{X}(t;\theta)) &= g(x,\mathbf{X}(t;\theta)) \qquad (x,t) \in \mathbb{R}^N \times (0,T), \\ f(x,0;\mathbf{X}) &= \varphi(x) \qquad\qquad x \in \mathbb{R}^N, \end{cases} \tag{6.2}$$

where

$$L^\theta = \sum_{i,j=1}^{N} a_{i,j}(x,t;\theta)\partial^2_{i,j} + \sum_{i=1}^{N} b_i(x,t;\theta)\partial_i + c(x,t;\theta) - \partial_t. \tag{6.3}$$

Given a set of observable data, $\{x_i^*(t)\colon\ i = 1,...,n,\ t \in I\ \}$ where I is a discrete set of times, $I \subset (0,T)$, (e.g. the set of positions of n cells recorded at different time steps of an experiment), the issue to address is to find the optimal values of the parameters for which a considered model fits reality, meaning that the model is able to reproduce the observed behavior.
Mathematically, this can be regarded as a least squares problem, aiming at finding $\theta_{opt} \in D$ arising from $\inf_{\theta \in D} \sum_{i=1}^{n} \sum_{t \in I} |x_i(\cdot;\theta) - x_i^*(t)|^2 + \epsilon(\theta)$, where $\epsilon(\theta)$ is a suitable penalization term, and $x_i(\cdot;\theta)$ is the solution to model (6.1)-(6.2). Standard numerical procedures used to solve the above optimization problem require the computation of the derivative $\mathbf{u}_{ik} := \partial_{\theta_k} x_i(\cdot;\theta)$ for $k = 1,...,p$. By differentiating (6.1) and (6.2) with respect to θ_k, $k = 1,...,p$, we obtain that $\mathbf{u}_{ik}$ is solution, at least formally, to a problem of the same form of (6.1)-(6.2). For that reason, we argue that the obtained analytical results can be extended to investigate the dependence of the solution on the parameters.

Another extension of our researches concern the possibility to give analytical foundations to the case in which the $N-$dimensional domain is replaced by a bounded domain, and to consider the coupling with different advection-diffusion equations could be considered in the coupling with the dynamics equations.

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Appendix Part I

Fundamental solution and Cauchy problem

Let consider the Cauchy problem

$$\begin{cases} Lf(x,t) &= g(x,t) \qquad (x,t) \in \Omega := \mathbb{R}^N \times (0,T], \\ f(x,0) &= \varphi(x), \qquad x \in \mathbb{R}^N, \end{cases} \tag{6.4}$$

where L is the operator

$$L = \sum_{i,j=1}^{N} a_{i,j}(x,t)\partial^2_{i,j} + \sum_{i=1}^{N} b_i(x,t)\partial_i + c(x,t) - \partial_t. \tag{6.5}$$

Let L uniformly parabolic in Ω, i.e. the matrix $(a_{ij}(x,t))$ is positive definite and there exist positive constants $\overline{\lambda}_0$, $\overline{\lambda}_1$ such that for any $\xi \in \mathbb{R}^n$

$$\overline{\lambda}_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \leq \overline{\lambda}_1 |\xi|^2 \qquad \forall (x,t) \in \Omega.$$

Moreover let $a_{i,j}$, b_i, c bounded Hölder continuous function in Ω, with coefficient $\alpha \in (0,1)$ with respect to x and $\alpha/2$ with respect to t.
Let $g(x,t)$, $\varphi(x)$ be continuous functions respectively in Ω and $\mathbb{R}^N$, satisfying

$$|g(x,t)| \leq Ce^{h|x|^2}, \qquad |\varphi(x)| \leq Ce^{h|x|^2}, \tag{6.6}$$

with h positive constant satisfying $h < \dfrac{\lambda_0}{4T}$.
Finally, let assume g locally Hölder continuous with exponent α in $x \in \mathbb{R}^N$, uniformly with respect to t. Then the function

$$f(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\,\varphi(\xi)\,d\xi - \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,g(\xi,\tau)\,d\xi d\tau \tag{6.7}$$

is a solution of (6.4)-(6.5).
$\Gamma(x,t;\xi,\tau)$ is a fundamental solution of $Lf = 0$,and it is a continuous function of (x,t), uniformly with respect to (ξ,τ) if $t - \tau \geq constant > 0$,

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and it is a continuous function of (ξ,τ), uniformly with respect to (x,t) if $t-\tau \geq constant$. Hence $\Gamma(x,t;\xi,\tau)$ is a continuous function of $(x,t;\xi,\tau)$. Moreover also the first and second derivative with respect to space, and the derivative with respect to time of Γ are continuous functions of $(x,t;\xi,\tau)$, where $x,\xi \in \mathbb{R}^N$ and $0 \leq \tau < t \leq T$.

Computation of a possible upper bound for λ_0 in Chapter 2.

Let A be a symmetric matrix satisfying

$$\mu_0 |\xi|^2 \leq \langle A\xi,\xi\rangle \leq \mu_1 |\xi|^2 \qquad \forall \xi \in \mathbb{R}^N \tag{6.8}$$

where μ_0,μ_1 are positive constants.
Defining $\eta = A\xi \in \mathbb{R}^N$, (2.5) can be rewritten as

$$\mu_0 \left|A^{-1}\eta\right|^2 \leq \left\langle \eta, A^{-1}\eta\right\rangle \leq \mu_1 \left|A^{-1}\eta\right|^2 . \tag{6.9}$$

We choose $W=\{w_1,...,w_N\}$ an orthonormal basis of eigenvectors of A, and rewrite η as a linear combination, with coefficients $c_1,...,c_N \in \mathbb{R}$,namely $\eta = \sum_{i=1}^{N} c_i w_i$. Denoting with θ_i, $i=1,...,N$, the eigenvalues of A, we thus obtain:

$$A^{-1}\eta = \sum_{i=1}^{N} \frac{c_i}{\theta_i} w_i .$$

Since $|\eta|^2 = \sum_{i=1}^{N} c_i^2$, we have

$$\left|A^{-1}\eta\right|^2 = \left\langle A^{-1}\eta, A^{-1}\eta\right\rangle = \sum_{i=1}^{N} \frac{c_i^2}{\theta_i^2} > \frac{1}{\max\limits_{i=1,...,N}(\theta_i^2)} \sum_{i=1}^{N} c_i^2 > \frac{1}{\mu_1^2} |\eta|^2 . \tag{6.10}$$

By (6.9), we get the following lower bound:

$$\left\langle \eta, A^{-1}\eta\right\rangle \geq \mu_0 \left|A^{-1}\eta\right|^2 \geq \frac{\mu_0}{\mu_1^2} |\eta|^2 . \tag{6.11}$$

We performed the algebraic computation above in order to show the relation between λ_0 and μ_0,μ_1 in H3), following the approach in [58]. Let A be the matrix having entries $A_{ij} = a_{ij}(x,t)$, where a_{ij} are the coefficients introduced in (2.3): we note that, since L is of parabolic-type, A is symmetric, and (2.5) corresponds to condition (6.8). Hence, from equation (6.11), we get an upper bound for the parameter λ_0.

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Computation of $I_0(\gamma)$, $I_1(\gamma)$.

Let γ be a positive constant.
We define $I_0(\gamma)$, $I_1(\gamma)$ the integrals

$$I_0(\gamma) := \int_{\mathbb{R}^N} e^{-\gamma|y|^2} dy, \tag{6.12}$$

$$I_1(\gamma) := \int_{\mathbb{R}^N} e^{-\gamma|y|^2} |y|\, dy. \tag{6.13}$$

With the changes of variable given by $v := \sqrt{2\gamma}y$ and then $u := \frac{v}{\sqrt{2}}$, we rewrite:

$$I_0(\gamma) := \int_{\mathbb{R}^N} e^{-\gamma|y|^2} dy = \int_{\mathbb{R}^N} e^{-\frac{v^2}{2}} \frac{1}{(2\gamma)^{N/2}} dv = \frac{1}{\gamma^{N/2}} \int_{\mathbb{R}^N} e^{-u^2} du = \left(\frac{\pi}{\gamma}\right)^{N/2}. \tag{6.14}$$

For the second, we convert it in polar coordinates and perform the change of variable given by $t = \gamma h^2$, obtaining

$$\begin{aligned} I_1(\gamma) &:= \int_{\mathbb{R}^N} e^{-\gamma|y|^2} |y|\, dy = \int_0^\infty e^{-\gamma h^2} h^N \omega_N dh \\ &= \int_0^\infty e^{-t} \left(\frac{1}{\gamma}\right)^{\frac{N-1}{2}} t^{\frac{N-1}{2}} \frac{\omega_N}{2} \frac{1}{\gamma} dt = \Gamma_e\left(\frac{N+1}{2}\right) \left(\frac{1}{\gamma}\right)^{\frac{N+1}{2}} \frac{\omega_N}{2} = \\ &= \left(\frac{1}{\gamma}\right)^{\frac{N+1}{2}} \frac{\omega_N}{2} \frac{2\pi^{\frac{N+1}{2}}}{\omega_{N+1}}, \end{aligned} \tag{6.15}$$

recalling that $\Gamma_e(N/2)\omega_N = 2\pi^{\frac{N}{2}}$.

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Glossary Part I

Throughout Part I, we will identify each element of $\mathbb{R}^{N\times n}$ with the column vector of $\mathbb{R}^{Nn}$ obtained putting in column the n columns one after another. Moreover, we will refer to $B_R(P)$ and B_R as subsets of $\mathbb{R}^{N\times n}$ or $\mathbb{R}^N$, depending on the context. In the following we denote $\Omega \subset \mathbb{R}^N$ open.

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$B_R(P)$	Closed ball of $\mathbb{R}^p$, with center $P \in \mathbb{R}^p$,radius R
B_R	Closed ball of $\mathbb{R}^p$, with center $O \in \mathbb{R}^p$, radius R
$\|\| \cdot \|\|_{\infty,T}$	$\sup_{[0,T]} \| \cdot \|$
ω_N	Surface area of the (N-1)-dimensional sphere, in $\mathbb{R}^N$
$\|\mathbf{V}\|$	Euclidean norm of $\mathbf{V} \in \mathbb{R}^p$
A^n	$A \times ... \times A$, for any $A \subset \mathbb{R}^N$
$A = [a_1 ... a_d]$	$A \in \mathbb{R}^{n \times d}$, with $a_i \in \mathbb{R}^n$ $i = 1, ..., n$ column vector
$\nabla f\,(x_0, t)$	Gradient of $f = f(x,t)$, with respect to x variable for any f such that $f(\cdot, t)$ is differentiable at x_0
$\dot{\mathbf{X}}$	Derivative with respect to t of function $\mathbf{X}$ for any $\mathbf{X} : [0,T] \to \mathbb{R}^p$
∂_i^α	Partial derivative operator with respect to x_i variable of order $\alpha > 1$. We denote $\partial_i^1 = \partial_i$
∂_t	Partial derivative operator with respect to t variable
$(f \vee g)(x)$	Maximum value between $f(x)$ and $g(x)$
$C^k\,(\Omega; \mathbb{R}^p)$	Class of C^k functions $f : \Omega \to \mathbb{R}^p$, $k = 0$ is the class of continuos functions. We denote $C^0 = C$
$C^{k,1}\,(\Omega \times [0,T]; \mathbb{R}^p)$	Class of functions $f : \Omega \times [0,T] \to \mathbb{R}^p$, $p \geq 2$, $\Omega \subset \mathbb{R}^N$ open, C^k in Ω and C^1 in [0,T]
$\|\|f\|\|_{L^p(\Omega)}$ $p \neq \infty$	$\left(\int_\Omega \|f\|^p d\mu \right)^{1/p}$
$W^{k,p}(\Omega)$ $p \geq 1$	Sobolev space of functions $f \in L^p(\Omega)$ with $D^\alpha f \in L^p(\Omega)$, $\|\alpha\| \leq k$
$W^{p,q,\infty}(\Omega = \Omega_1 \times (0,T))$	Sobolev space of functions $f \in L^\infty(\Omega)$ with $D^\alpha f \in L^\infty(\Omega)$, $\|\alpha\| \leq p$, $\partial_t^\beta f \in L^\infty(\Omega)$, $\|\beta\| \leq q$
$Lip_{loc}(\Omega)$	Locally Lipschitz functions
$H^1(\Omega)$	$W^{1,2}(\Omega)$
$H^{-1}(\Omega)$	The dual space to $H_0^1(\Omega) = W_0^{1,2}(\Omega)$
$C_c^\infty\,(U; \mathbb{R}^p)$	Space of infinitely differentiable functions $\phi : U \to \mathbb{R}^p$, with compact support in U
$L^p\,(0,T;\Omega)$ $p \geq 1$	Strongly measurable functions $f : [0,T] \to \Omega$ with $\|\|f\|\|_{L^p(0,T;\Omega)} := \left(\int_0^T \|f(t)\|^p dt \right)^{1/p} < \infty$

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Part II

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Introduction

In Part II of this thesis, the research area of collective dynamics is approached with a different point of view. This comes as a result of an ongoing collaboration with the Research Unit of Automation of Università Campus Bio-Medico di Roma. The aim of our studies is to investigate distributed systems of interacting agents over networks, in a decision making perspective.

In the following we present a well-established technique in this field, that represents the starting point of our researches in the next two chapters.

Sparse Analytic Hierarchy Process

The *Analytic Hierarchy Process* (AHP) [88] is an effective decision-making technique aimed at ranking a set of alternatives based on their utility or importance. This task is typically done by resorting to relative preference information, i.e., by considering the ratio of the utilities. Such ratios are typically perturbed or affected by subjective biases or errors, and do not represent a perfectly transitive ordering, thus calling for approximated solutions. In its standard formulation, the AHP problem requires knowledge on the ratio of the utilities of each pair of alternatives; however, in the literature several results have been provided in order to handle incomplete information [89, 90, 91, 92].

The AHP problem appears of particular interest in a distributed decision-making context; in particular we are interested to the scenario envisaged in [92], where a set of networked agents need to rank themselves in a distributed manned, based on the knowledge of perturbed ratios of utilities with respect to their neighbors. An example, in this sense, is a network of mobile agents, each provided with different equipment or tools in order to perform their tasks; agent i, by comparing its equipment with the one of a neighboring agent j, is able to assess its relative priority; by composing such a local information, the agents are able to rank themselves and compute an absolute priority.

Notice that, in the literature, typical distributed ranking approaches aim at developing a ranking based on the topological structure of the network

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(see, among other methodologies, the *pagerank* problem [93]). Conversely, the ranking achieved within the AHP approach has no relation with the topological structure, but is the result of the composition of the relative importance information involving the nodes and their neighbors.

We point out that the standard approach to solve the AHP problem, the so-called *eigenvector method*, has been often regarded as an arbitrary or questionable choice for approximating the unknown utilities (see for instance the debate in [94, 95]); the main reason for such a criticism is that the ranking obtained via the eigenvector method might be reversed when additional alternatives are considered. Moreover, the procedure developed in [92], although effective, has nontrivial computational requirements. To the best of our knowledge, no other approach has been provided in the literature in the distributed case.

The research in this area has led, for the time being, to two contributions, that will be presented in the two chapters of Part II.

In Chapter 7 we investigate the possibility of solving the distributed AHP problem in three alternative ways which are based on different approaches with respect to the eigenvector method, and require less computational resources with respect to the approach in [92]. Specifically, the proposed methods are inspired to well known algorithms in the literature, i.e., Metropolis-Hastings Markov chains [96], Heat-Bath Markov chains [97] and formation control [98]. In the end, we identify strengths and weak points of such methodologies, evaluating their performance on an experimental basis.

In Chapter 8, we consider a scenario where a set of agents, interconnected by a network topology, aim at computing an estimate of their own utility, based on pairwise relative information having hybrid nature. In greater detail, the agents are able to measure the difference between their value and the value of some of their neighbors, or have an estimate of the ratio between their value and the value of the remaining neighbors. This setting finds application in problems where estimations provided by humans have to be merged with sensor measurements: the human is able to give qualitative informations, whereas the sensors represent those agents giving quantitative informations. We observe that this coupled scenario, in which different kind of information are considered, represents the novelty with respect to previous work in literature, where the two types of information are treated separately. In this sense, also in Part II we will use the terminology *hybrid*: here, it highlights the possibility to deal with different kind of information within the same model, whereas in Part I it concerns the presence of different scales at which a phenomena is simultaneously observed.

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Chapter 7

Distributed Methodologies to Solve Sparse Analytic Hierarchy Process Problem

In this chapter we present distributed algorithms to solve the *Sparse Analytic Hierarchy Process* (SAHP) problem, where a set of networked agents (e.g., wireless sensors, mobile robots or IoT devices) need to be ranked based on their utility/importance. Recently, the *Sparse Eigenvector Method* proved its effectiveness in tackling this problem. However, such a method has several drawbacks, such as demanding computation/communication requirements and lack of control on the magnitude of the computed estimate. With the aim to mitigate such issues, we inspect the possibility to resort to a suite of different methodologies, each inspired to well known algorithms in the literature. Strength points and drawbacks of the proposed methods are highlighted by numerical comparisons.

7.1 Problem Statement

In the Introduction to Part II, we gave an overview of SAHP. We now formalize the graph structure that underlines this kind of problem. For a detailed description of the basic concepts and notations here adopted, the reader is referred to the Appendix of Part II.

Let us consider a set of n agents, interconnected by an undirected and connected graph $G = \{V, E\}$, and suppose that each agent is characterized by an unknown utility or relevance $w_i > 0$. Suppose further that each agent, identified with the a vertex of the graph v_i, is provided with just relative information regarding the ratios between its utility w_i and the utility w_j of each of its neighbors $v_j \in \mathcal{N}_i$ over G. In particular, let us assume that each ratio is perturbed by a multiplicative error or bias $\epsilon_{ij} > 0$; in other words, for each neighbor $v_j \in \mathcal{N}_i$, the i-th agent v_i knows just $S_{ij} = \frac{w_i}{w_j}\epsilon_{ij}$.

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In particular, as implicitly done in [92], we assume that there is no vector $\boldsymbol{f} \in \mathbb{R}^n$ such that $\epsilon_{ij} = f_i/f_j$ for all $(v_i, v_j) \in E$; in other terms, the perturbations ϵ_{ij} represent a distortion of the utilities and there is no obvious way to compute a vector $\widetilde{\boldsymbol{w}}$ that satisfies $\widetilde{w}_i/\widetilde{w}_j = \epsilon_{ij} w_i/w_j$ for all links $(v_i, v_j) \in E$.

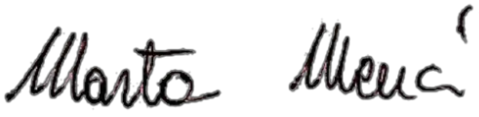
Let S be the $n \times n$ matrix collecting the overall information available to the agents; we assume that, for all $(v_i, v_j) \in E$, it holds $S_{ji} = S_{ij}^{-1}$, i.e., we assume that $\epsilon_{ji} = \epsilon_{ij}^{-1}$. Such an assumption is common practice in the AHP literature [88, 99, 100]. The aim of each agent v_i is to compute an estimate $\widetilde{w}_i$ for its utility w_i. Note that, since the agents are provided with ratios of utilities, we assume that such utilities are defined up to a multiplicative scaling factor that is the same for all the agents. In the following, we denote by $\boldsymbol{w} \in \mathbb{R}^n$ and $\widetilde{\boldsymbol{w}} \in \mathbb{R}^n$ the stack of all w_i and $\widetilde{w}_i$, respectively.

7.2 Sparse Eigenvector Method

The Sparse Eigenvector Method (SEM), developed in [92], is an effective distributed methodology to solve the AHP problem in the case of a matrix S having the structure of a connected and undirected graph $G = \{V, E\}$. The main idea behind the approach is that, in the nominal case where all perturbations $\epsilon_{ij} = 1$, the dominant eigenvector of matrix $D^{-1}S$ (we recall that D is a diagonal matrix having the degree d_i of the i-th node at its i-th diagonal entry) is indeed the desired utility vector $\boldsymbol{w}$, while the dominant eigenvalue associated to it is equal to one, i.e., it holds $D^{-1}S\boldsymbol{w} = \boldsymbol{w}$.

As demonstrated in [92], the dominant eigenvalue of $D^{-1}S$ is equal to one if and only if the perturbations are not present, while it is in general different from one. Following the path of the traditional eigenvector method for the complete information case [88], the algorithm in [92] aims at letting the agents compute the dominant eigenvector of $D^{-1}S$, and specifically each agent aims at computing the corresponding component of the eigenvector. This is done by implementing a power iteration [101], which corresponds to a distributed algorithm given the sparse nature of matrix $D^{-1}S$. However, since in general the dominant eigenvalue of $D^{-1}S$ is not equal to one, a naive power iteration (i.e., without normalization) would either converge to zero or diverge; in order to address this challenge, the algorithm in [92] aims at letting the agents compute also an estimate of the dominant eigenvalue of $D^{-1}S$, which is used as a normalizing factor in the power iteration. In more detail, the agents have knowledge on an upper bound $\widetilde{\delta}$ of the graph diameter δ and execute a max-consensus procedure which is reinitialized at time steps that are multiple of $\widetilde{\delta}$, in order to reach an agreement on the estimate of the dominant eigenvalue as of $\widetilde{\delta}$ steps earlier.

The pseudocode of the algorithm developed in [92] is given in Algorithm 1. Specifically, Algorithm 1 is initialized with an estimate $h_i(0) = 1$ for



Algorithm 1 Sparse Eigenvector Method. The pseudocode represents the point of view of the i-th agent.

procedure $SEM(\widetilde{\delta})$
▷ Initial Condition
$\widetilde{w}_i(0) =$ random positive real number
$y_i(0) = \ell_i(0) = h_i(0) = 1$
▷ Synchronous Iteration

$$\widetilde{w}_i(k+1) = \frac{1}{h_i(k) d_i} \sum_{j \in \mathcal{N}_i} S_{ij} \widetilde{w}_j(k)$$

$$y_i(k+1) = \frac{1}{\widetilde{w}_i(k) d_i} \sum_{j \in \mathcal{N}_i} S_{ij} \widetilde{w}_j(k)$$

$$\ell_i(k+1) = \begin{cases} y_i(k), & \text{if } \mathrm{mod}(k, \widetilde{\delta}) = 0, \\ \max\limits_{v_j \in \mathcal{N}_i \cup \{v_i\}} \{\ell_j(k)\}, & \text{otherwise.} \end{cases}$$

$$h_i(k+1) = \begin{cases} \ell_i(k), & \text{if } \mathrm{mod}(k, \widetilde{\delta}) = 0, \\ h_i(k), & \text{otherwise.} \end{cases}$$

end procedure

the dominant eigenvalue that is the same for all agents; moreover, each agent relies on $\widetilde{w}_i(k)$ in order to compute a quantity $y_i(k)$ that tends to the dominant eigenvalue of $D^{-1}S$ as k approaches infinity. While doing so, the agents execute several max-consensus procedures, which are re-initialized every $\widetilde{\delta}$ steps, so that every time a max-consensus procedure terminates at a given time instant k, they compute an updated h_i that is the same for all agents and corresponds to the greatest among the values $y_i(k - \widetilde{\delta})$ over all agents i. By repeatedly doing so, the agent i is able to compute the i-th component of a finite and nonzero vector in the span of the dominant eigenvector of $D^{-1}S$, as well as the dominant eigenvalue of $D^{-1}S$.

7.2.1 Discussion

We point out that Algorithm 1, although effective, has several drawbacks: (1) the agents need to know a global parameter such as an upper bound $\widetilde{\delta}$ of the network diameter δ (e.g., computing it via the approach in [102]); (2) each agent needs to let four state variables evolve in parallel, relying at each step on the state variables of their neighbors; (3) the agents need to run a max-consensus iteration in parallel with the main iteration, and such a max-consensus procedure has to be reinitialized at prescribed time instants; (4) no guarantee on the magnitude of $\widetilde{\boldsymbol{w}}$ is given, and the agents might need to further interact to calculate a normalization of the result. However, the main advantage of Algorithm 1 is that, by computing the dominant eigenvalue of $D^{-1}S$, the agents become aware of valuable meta-information regarding the degree of inconsistency of the available data.

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7.3 Metropolis-Hastings Method

In this section we inspect the possibility to resort to a *Metropolis-Hastings* (MH) Markov chains [96, 103] to solve the AHP problem in a distributed way. We point out that Metropolis-Hastings Markov chains are quite a popular tool in the literature to sample from a known discrete distribution (or, at least, a distribution for which probability ratios are known). However, to the best of our knowledge, the adoption of such a tool in order to rank a set of alternatives based on perturbed ratios can be regarded as an original contribution of this paper.

Let an $n \times n$ matrix M having the same structure as the graph G, such that, for all $i, j \in \{1, \dots, n\}$ it holds

$$M_{ij} = \begin{cases} \frac{1}{d_i} \min\left\{1, \frac{d_i S_{ji}}{d_j}\right\}, & \text{if } (v_i, v_j) \in E \\ 1 - \frac{1}{d_i} \sum_{h \in \mathcal{N}_i} \min\left\{1, \frac{d_i S_{hi}}{d_h}\right\}, & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

It can be easily noted that, by construction, it holds $M_{ij} \in [0, 1]$ and $M\mathbf{1}_n = \mathbf{1}_n$; hence, matrix M is the transition probability matrix of a Markov chain in the form

$$\boldsymbol{p}(k+1) = M^T \boldsymbol{p}(k). \tag{7.2}$$

The standard Metropolis-Hastings algorithm corresponds to the above Markov chain, in the special case when all terms $S_{ij} = w_i/w_j$, i.e., when no perturbation is considered. In such a special case, assuming that G is undirected and connected and that all $w_i > 0$, as shown in [96, 103], the Markov chain in Eq. (7.2) is aperiodic and has $\boldsymbol{w}$ as its limiting distribution.

Note that if the initial condition $\boldsymbol{p}(0)$ of the Markov chain is a probability distribution vector, then at each time k the state $\boldsymbol{p}(k)$ is a probability distribution vector. Conversely, as noted in Remark 64, when $\boldsymbol{p}(0)$ is not a probability distribution vector but it has nonnegative entries and nonzero sum, the state converges to a vector in the span of the limiting distribution vector having sum of the entries equal to the sum of the entries of $\boldsymbol{p}(0)$. This means that, differently from the SEM approach, if the agents select random positive initial conditions, there is a guarantee that the sum of the entries of the estimated utilities will remain constant during the execution of the algorithm. Moreover, since the algorithm exhibits asymptotic convergence, if a normalized vector with sum equal to one is required, it is more convenient to let the agents cooperate in order to select an initial distribution beforehand, rather than performing normalization after the execution of the asymptotic algorithm is truncated. In order to do this, the agents need to execute an initialization phase where $\widetilde{\boldsymbol{w}}(0)$ is constructed. Among other possible choices, a feasible approach is to elect a *leader* via max-consensus or other techniques (see [104] and references therein for recent works on this

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topic) and then select $\widetilde{w}_i(0) = 1$ if node v_i is the leader and $\widetilde{w}_i(0) = 0$ otherwise.

Algorithm 2 Distributed Metropolis-Hastings Algorithm. The pseudocode represents the point of view of the i-th agent.

procedure $MH(\widetilde{\delta}, \alpha, normalize)$
▷ Initialization
send d_i to all neighbors;
receive d_j from each neighbor $v_j \in \mathcal{N}_i$;
▷ Choose initial condition $\widetilde{\boldsymbol{w}}(0)$
if *normalize* **then**
elect a leader, e.g., via max-consensus($\widetilde{\delta}$)
select $\widetilde{w}_i(0) = 1$ if i leader $w_i(0) = 0$, otherwise
else
$\widetilde{w}_i(0)$ = random positive real number
end
▷ Synchronous Iteration
$\widetilde{w}_i(k+1) = (\alpha M_{ii} + 1 - \alpha)\widetilde{w}_i(k) + \alpha \sum_{j \in \mathcal{N}_i} M_{ji}\widetilde{w}_j(k)$
end procedure

Let us now discuss the proposed algorithm when perturbations are present; the pseudocode is reported in Algorithm 2. As said above, the algorithm either requires the agents to choose an initial condition $\widetilde{w}_i(0)$ that corresponds to a distribution vector $\widetilde{\boldsymbol{w}}(0)$ or a positive random number.

Moreover, in order to compute the entries M_{ij} according to Eq. (7.1), the agents need to know the degree d_j of their neighbors $v_j \in \mathcal{N}_j$. Let $\alpha \in (0, 1)$ be a parameter known to all agents and let us consider the stacked dynamics of all the agents, i.e., $\widetilde{\boldsymbol{w}}(k+1) = (\alpha M^T + (1-\alpha)I)\widetilde{\boldsymbol{w}}(k)$. As discussed in Remark 63, the dynamic matrix of the above discrete-time system is a convex combination of M^T and the identity matrix, hence $(\alpha M^T + (1-\alpha)I)$ is aperiodic. Moreover, since the graph G is connected for hypothesis, it follows that $(\alpha M^T + (1-\alpha)I)$ is irreducible. Therefore, the proposed dynamics corresponds to a Markov chain having a limiting distribution, which is given by the dominant left eigenvector of M.

7.3.1 Discussion

The approach discussed in this section has several advantages with respect to Algorithm 1: (1) except for an initialization in order to select a starting distribution, the algorithm requires less memory and communication among the agents and a less complex interaction, since only a standard discrete-time linear iteration is executed; (2) the vector computed is intrinsically normalized (provided that the initial state is a distribution) or, in any case, the sum of the entries of the estimated utilities is constant over time. However, the main drawback of this approach is that the parameter α, which is used to guarantee the existence of a limiting distribution, has to be known to all agents; moreover, if a distribution is required, the agents

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need to know a global parameter (e.g., the network diameter, in order to execute a leader election procedure).

7.4 Heat-Bath Method

The *Heat-Bath* (HB) method, similarly to the Metropolis-Hastings approach, is yet another popular way to design a Markov chain that has a desired limiting distribution (see among others [97, 105]). Note that the adoption of a Heat-Bath Markov chain for the distributed ranking can be regarded as an original contribution of this paper.

Let an $n \times n$ matrix Q having the same structure as the graph G, such that, for all $i, j \in \{1, \dots, n\}$ it holds

$$Q_{ij} = \begin{cases} \frac{\gamma}{1+S_{ij}}, & \text{if } (v_i, v_j) \in E \\ 1 - \gamma \sum_{h \in \mathcal{N}_i} \frac{1}{1+S_{ih}}, & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{7.3}$$

where γ is a global parameter known to all agents and such that $\gamma < 1/\max_{i=1,\dots,n} d_i$. It can be easily noted that, by construction, it holds $Q_{ij} \in [0, \gamma] \subseteq [0, 1]$ and $Q\mathbf{1}_n = \mathbf{1}_n$; hence, matrix Q is the transition probability matrix of a Markov chain. Moreover, we observe that it holds

$$\sum_{j \in \mathcal{N}_i} Q_{ij} = \gamma \sum_{j \in \mathcal{N}_i} \frac{1}{1+S_{ij}} < \frac{1}{\max\{d_i\}} \sum_{j \in \mathcal{N}_i} 1 = \frac{d_i}{\max\{d_i\}} \leq 1;$$

hence, we have that, for all $i \in \{1, \dots, n\}$ it holds $Q_{ii} > 1 - d_i/\max\{d_i\} \geq 0$. Therefore, as discussed in Remark 62, the Heat-Bath Markov chain, in the general case, is intrinsically aperiodic and since G is connected we conclude that such a dynamics always reaches a limiting distribution. As a consequence, differently from the Metropolis-Hastings case, there is no need to implement convex combinations with the identity matrix. We point out that, in the nominal case where all terms $\epsilon_{ij} = 1$, the Heat-Bath Markov chain is known[1] to have $\boldsymbol{w}$ as its dominant left eigenvector [97, 105]; therefore, when no perturbation is present, the Heat-Bath Markov chain has a limiting distribution that corresponds to $\boldsymbol{w}$.

The pseudocode of the proposed Heat-Bath algorithm is reported in Algorithm 3. Similarly to the MH algorithm, if an estimated utility vector having sum of the entries equal to one is required, a feasible path is to resort to leader election; otherwise, if the agents select random positive initial condition then the sum of the entries is preserved as discussed in

[1] This is a consequence of the fact that the Heat-Bath Markov chain satisfies the *detailed balance* $w_i Q_{ij} = w_j Q_{ji}$ for all $(v_i, v_j) \in E$; the interested reader is referred to [97, 105] for further information.

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Algorithm 3 Distributed Heat-Bath Algorithm. The pseudocode represents the point of view of the i-th agent.

procedure $HB(\gamma, \widetilde{\delta}, normalize)$
▷ Initialization
▷ Choose initial condition $\widetilde{\boldsymbol{w}}(0)$
if *normalize* **then**
elect a leader, e.g., via max-consensus($\widetilde{\delta}$)
select $\widetilde{w}_i(0) = 1$ if i leader $w_i(0) = 0$, otherwise
else
$\widetilde{w}_i(0) =$ random positive real number
end
▷ Synchronous Iteration
$\widetilde{w}_i(k+1) = \sum_{j \in \mathcal{N}_i \cup \{i\}} Q_{ji} \widetilde{w}_j(k)$
end procedure

Remark 64.

7.4.1 Discussion

Similarly to the MH algorithm, the Heat-Bath approach has several benefits with respect to the eigenvector method, i.e., lower memory and communication requirements. Moreover, differently from MH, there is no need to force aperiodicity and the agents need not to know the degrees of their neighbors. However, just like MH, in case there is a need to select an initial condition, the agents must be aware of a global parameter such as the diameter.

7.5 Sparse Logarithmic Least Squares Method

In this section we develop an algorithm, namely *Sparse Logarithmic Least Squares* (SLLS) Method that extends the *Logarithmic Least Squares* (LLS) Method developed in [99, 100] for the complete information case; such an algorithm is based on Fax and Murray's formation control algorithm [98, 106]. Specifically, within the SLLSM algorithm, the agents aim at finding a logarithmic least squares approximation $\widetilde{\boldsymbol{w}}$ of the vector $\boldsymbol{w}$ that is the stack of the utilities w_i. In other words, the problem corresponds to finding the vector $\widetilde{\boldsymbol{w}}$ such that

$$\widetilde{\boldsymbol{w}} = \arg\min_{\boldsymbol{q} \in \mathbb{R}^n_+} \left\{ \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i}^{n} \left(\ln(S_{ij}) - \ln\left(\frac{q_i}{q_j}\right) \right)^2 \right\}; \tag{7.4}$$

notice that, since the above function is convex, such a vector always exists. We now provide a theorem that is the basis for the extension of the LLSM

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method to a sparse information context[2].

Theorem 50. *Let us consider a set of n agents, interconnected by an undirected and connected graph $G = \{V, E\}$, and let S be the $n \times n$ matrix collecting the overall information available to the agents. Let us define $s_i = \sum_{j \in \mathcal{N}_i}^{n} \ln(S_{ij})$ and let $\boldsymbol{s}$ be the stack of all s_i. Moreover, let $\boldsymbol{r}^* \in \mathbb{R}^n$ be the vector that satisfies $L\boldsymbol{r}^* = \boldsymbol{s}$, where L is the laplacian matrix corresponding to the graph G. It holds $\widetilde{\boldsymbol{w}} = \exp(\boldsymbol{r}^*)$, where by $\exp(\boldsymbol{r}^*)$ we mean the component-wise exponentiation with base equal to e of the vector $\boldsymbol{r}^*$.*

Proof. Theorem 50 Let $r_i = \ln(q_i)$; Eq. (7.4) can be rearranged as

$$\begin{aligned} \widetilde{\boldsymbol{w}} &= \exp\left(\underset{\boldsymbol{r} \in \mathbb{R}^n}{\arg\min} \left\{ \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i}^{n} \left(\ln(S_{ij}) - r_i + r_j \right)^2 \right\} \right) = \\ &= \exp\left(\underset{\boldsymbol{r} \in \mathbb{R}^n}{\arg\min} \left\{ f(\boldsymbol{r}) \right\} \right). \end{aligned} \tag{7.5}$$

According to Eq. (7.5), $\boldsymbol{y}^*$ is the solution of a convex and unconstrained minimization problem. Therefore the vector $\boldsymbol{r}^*$ that minimizes $f(\boldsymbol{r})$ is such that

$$\frac{\partial f(\boldsymbol{r})}{\partial r_i}\Big|_{\boldsymbol{r}=\boldsymbol{r}^*} = 0, \quad \forall i = 1, \ldots, n,$$

which corresponds to $\sum_{j \in \mathcal{N}_i}^{n} (r_i^* - r_j^*) = s_i$, for all $i = 1, \ldots, n$. Stacking the above equation for all agents i we get $L\boldsymbol{r}^* = \boldsymbol{s}$. This completes our proof. □

We point out that, since the Laplacian matrix L is singular by definition, the vector $\boldsymbol{r}^*$ can not be computed by matrix inversion. A possible way to compute $\boldsymbol{r}^*$ is thus to resort to an asymptotic algorithm, whose pseudocode is reported in Algorithm 4.

Algorithm 4 Sparse Logarithmic Least Squares Method. The pseudocode represents the point of view of the i-th agent.

procedure $SLLSM(\theta)$
▷ Initialization
$r_i(0) =$ random positive real number
$\widetilde{w}_i(0) = \exp(r_i(0))$
$s_i = \sum_{j \in \mathcal{N}_i}^{n} \ln(S_{ij})$
▷ Synchronous Iteration
$r_i(k+1) = r_i(k) + \theta \sum_{j \in \mathcal{N}_i} (r_j(k) - r_i(k)) + \theta s_i$
$\widetilde{w}_i(k+1) = \exp(r_i(k+1))$
end procedure

[2] We point out that some of the ideas of this theorem are similar to the results in [98, 106]; the theorem is given with a proof for the sake of self-containedness and clarity.

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Remark 51. We point out that Algorithm 4 and, in particular, the dynamics chosen for $r_i(k)$, amounts to a discrete-time version of the continuous time *formation control* algorithm developed by Fax and Murray [98, 106], which has the form $\dot{\boldsymbol{r}}(t) = -L\boldsymbol{r}(t) + \boldsymbol{s}$; over connected undirected graphs, such a continuous time equation is known to converge to a vector $\boldsymbol{r}^*$ that satisfies $L\boldsymbol{r}^* = \boldsymbol{s}$. Let us consider a discrete-time setting and let us write down the stack of the dynamics for all the agents within Algorithm 4, which is $\boldsymbol{r}(k+1) = (I - \theta L)\boldsymbol{r}(k) + \theta\boldsymbol{s}$. The above dynamics is indeed a discrete-time average consensus dynamics (plus a constant exogenous input that does not affect stability); such a dynamics is known to converge asymptotically [106] over connected undirected graphs if the parameter θ, which can be regarded as a sampling time, satisfies $\theta \leq 1/\max_{i=1,\dots,n} d_i$.

7.5.1 Discussion

We notice that Algorithm 4 has several advantages with respect to Algorithm 1: (1) instead of approximating the unknown utilities with the dominant eigenvector of $D^{-1}S$, a procedure that has raised some criticism in the literature, the solution computed by Algorithm 4 represents a clear log-quadratic minimization of the error between the perturbed ratios S_{ij} and the ratios $\widetilde{w}_i/\widetilde{w}_j$; (2) the algorithm requires less memory and communication among the agents and a less complex interaction, since only a standard discrete-time average consensus iteration (with an exogenous constant input) is executed. We point out that Algorithm 4, being essentially an asymptotic average-consensus algorithm, can be the base for several extensions, such as distributed stopping criteria [107, 108], finite-time [109, 110] or asynchronous implementations [111]. However, Algorithm 4 has some drawbacks: (1) there is no control on the norm of the estimate, which might need to undergo normalization; (2) each agent needs to known the same global parameter θ.

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7.6 Experimental Comparison

In this section we analyze in an experimental way the performances of the four methodologies discussed in this paper, considering random networks of different typologies with $n = 50$ nodes. In more detail, we consider a Watts-Strogatz small-world network with 3 links per node and rewiring probability 30%, a Barabási-Albert scale free network with 3 preferential attachments per node, a random geometric network (i.e., a network where the nodes are generated at random in the unit square and are connected if their Euclidean distance is less than a radius ρ) with $\rho = 0.3$ and an Erdős-Renyí random network model with connection probability 30%. In order to evaluate the performance of the four methodologies, we consider log-normal perturbation terms $\epsilon_{ij} = exp(\phi_{ij})$, where $\phi_{ij} = N(0, \sigma^2)$ is sampled from a Gaussian distribution with zero mean and standard deviation σ; we consider different values of $\sigma \in [0, 1]$ and for each choice of σ we show the results over $m = 50$ random networks. Note that we select random terms ϕ_{ij} for $i < j$, while we set $\phi_{ji} = -\phi_{ij}$, in order to obtain $\epsilon_{ji} = \epsilon_{ij}^{-1}$.

In Figures 1–4 we show, plotted against the standard deviation of the terms ϕ_{ij}, the Kendall correlation coefficient τ between the ranking obtained via the nominal utility vector and the one obtained based on the approximation $\widetilde{\boldsymbol{w}}$ computed via the different algorithms. According to the figures, as the perturbation magnitude grows, the Sparse AHP methodology exhibits a comparatively large degree of shuffling with respect to the nominal ranking; conversely, the other methodologies have remarkably better results, especially for large perturbation. We point out that the MH, HB and SLLS methodologies have comparable results, although the SLLS approach has slightly better correlation with the nominal ranking than the other approaches, especially for large perturbations.

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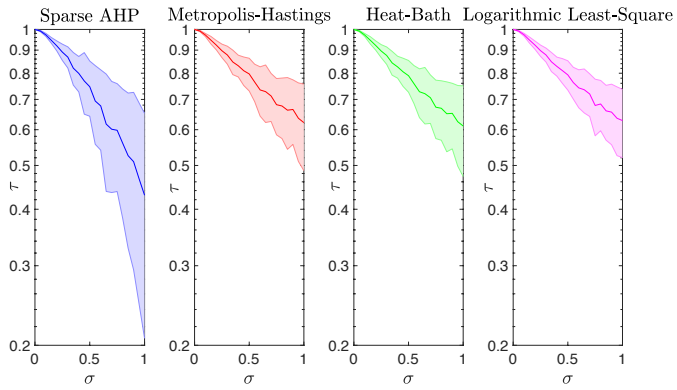

Figure 7.1: Kendall correlation coefficient between the nominal and perturbed ranking, plotted against the standard deviation of the log-normal perturbations in the case of Small-World networks with $n = 50$ nodes, 3 links per node and rewiring probability equal to 30%. Results are the average over $m = 50$ runs.

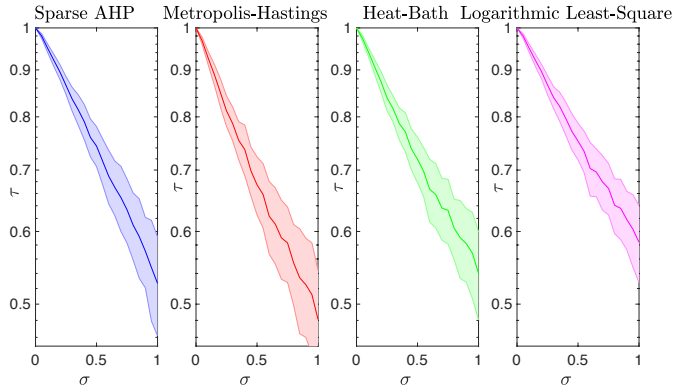

Figure 7.2: Kendall correlation coefficient between the nominal and perturbed ranking, plotted against the standard deviation of the log-normal perturbations in the case of Scale-Free networks with $n = 50$ nodes and 3 preferential attachments per node. Results are the average over $m = 50$ runs.

7.7 Conclusions and Future Work

In this chapter we provided three different methodologies to solve the Sparse Analytic Hierarchy Process problem in a distributed way, based on popular algorithms in the literature such has Metropolis-Hastings Markov chains, Heat-Bath Markov chains and formation control. Future work will be aimed at providing a formal characterization of the performance of such

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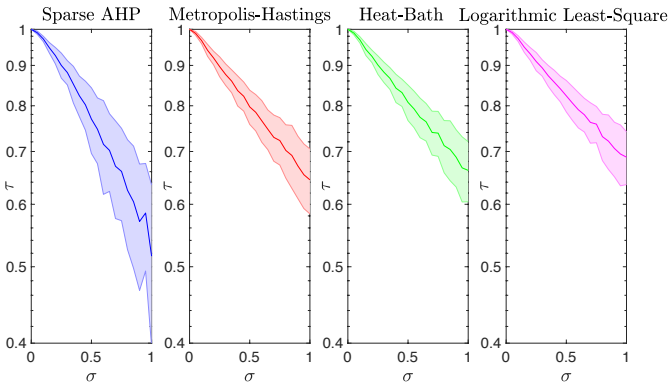

Figure 7.3: Kendall correlation coefficient between the nominal and perturbed ranking, plotted against the standard deviation of the log-normal perturbations in the case of Random Geometric networks with $n = 50$ nodes and $\rho = 0.3$. Results are the average over $m = 50$ runs.

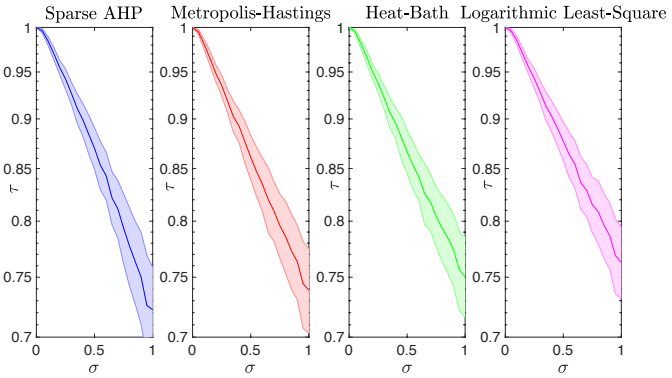

Figure 7.4: Kendall correlation coefficient between the nominal and perturbed ranking, plotted against the standard deviation of the log-normal perturbations in the case of Erdős-Renyí networks with $n = 50$ nodes and connection probability equal to 30%. Results are the average over $m = 50$ runs.

algorithms as a function of the perturbations, as well as to identify topological conditions that can be used to select the most appropriate methodology depending on the structure of the network. We highlight the fact that the methods here presented focuses on information expressed only as ratio. In the following chapter we present a novel methodology, which is able to deal with different kind of relative informations, e.g. ratios and differences of the utility.

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Chapter 8

Distributed Utility Estimation with Heterogeneous Relative Information

In this chapter we consider a scenario where a set of agents, interconnected by a network topology, aim at computing an estimate of their own utility, importance or value, based on pairwise relative information, having heterogeneous nature. In greater detail, the agents are able to measure the difference between their value and the value of some their neighbors, or have an estimate of the ratio between their value and the value the remaining neighbors. This setting may find application in problems involving information provided by heterogeneous sensors (e.g., differences and ratios), as well as in scenarios where estimations provided by humans have to be merged with sensor measurements. Specifically, we develop a distributed algorithm that lets each agent asymptotically compute a utility value. To this end, we first characterize the task at hand in terms of a least-squares minimum problem, providing a necessary and sufficient condition for the existence of a unique global minimum, and then we show that the proposed algorithm asymptotically converges to a global minimum. Numerical analyses corroborate the theoretical findings.

8.1 Introduction

In Chapter 7, we presented three different methods to solve the Sparse Analytic Hierarchy Process. In particular, the agents aim at ranking their utility knowing their relative, eventually perturbed, importance with respect to their neighbors. In this chapter we enhance the previous scenario, dealing

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with information of heterogeneous nature.

In recent years, a large body of scientific literature has been aimed at endowing networked agents with the ability to distributedly compute absolute information based on relative measurements. A relevant example in this sense is *Sensor Network Localization*, where networked sensors aim at computing their location based on relative information such as bearings [112, 113], distances [114], presence within the sensing range [115], or combinations of distance and presence information [116, 117]. Other examples include *Formation Control* and *Distributed Analytic Hierarchy Process*. Within Formation Control problems [98, 106, 118, 119], networked mobile agents aim at occupying locations that satisfy prescribed relative positions (e.g., in a least-squares sense as done in [98, 106] or exactly, under the assumption that the network is rigid, as done in [118]). Conversely, within distributed Analytic Hierarchy Process algorithms, the nodes in the network aim at computing their own utility or importance value based on the knowledge of perturbed utility ratios [89, 90, 92, 120].

To the best of our knowledge, current approaches in the literature operate based on homogeneous information. However, there are situations where one can improve the quality of the estimate by mixing heterogeneous pieces of information. For instance, consider a scenario where humans and machines cooperate; in this case, while sensors might be able to provide measurements of the difference between two quantities, while humans might be able to provide ratio information, e.g., assessing how many times one light or sound source is brighter or louder than another (see for instance [121]). Another example is the fusion of the information provided by sensors of heterogeneous nature, e.g., some able to measure distances [114], some able to measure ratios, such as signal strength ratios [122] or hop-count ratios [123]. To overcome some of the limitations of previous works, in this paper we consider a hybrid scenario where networked agents aim at computing their own utility, position or importance, based on heterogeneous pairwise relative information. Specifically, each agent knows: (i) an estimate of the differences between its utility and the utility of some of its neighbors; (ii) an estimate of the utility ratio with respect to the remaining neighbors. Based on such heterogeneous relative information, the agents cooperate in order to compute the absolute utility of each agent. To this end, we first define a least-squares minimization problem, by characterizing its global minima and providing a necessary and sufficient condition for the existence of a unique global minimum. Then, we develop a synchronous continuous-time distributed algorithm and we show that its dynamics converges to a global minimum, discussing the conditions guaranteeing that such a problem admits a unique or several global minima. We point out that the proposed problem setting is a mixture of the formation control approach, where agents are equipped with sensors able to measure relative positions, and the case where just ratios are considered. However, due to the presence of informa-

tion of heterogeneous nature, there is no straightforward way to apply either of the above methodologies, thus calling for a different approach. Indeed, we point out that a simple replacement of the ratios by their logarithm, with the aim to resort to an approach able to handle just distances, would not be an effective choice. In fact, there would be the need to introduce additional variables and constraints (i.e., constraints in the form $z_i = \log(x_i)$), which would need to be carefully handled. However, this represents an interesting direction that we leave for future research.

8.2 Problem Statement

In the following section we shall make use of basic notions and definitions concerning graph theory. For a detailed description see the Appendix section of Part II.
Let us consider a bidirectional strongly connected graph $G = \{V, E\}$ with n nodes, where each node $v_i \in V$ represents an agent and each link (v_i, v_j) captures the existence of a communication channel from agent v_i to agent v_j. Each agent $v_i \in V$ has the task to compute a value $f_i > 0$ (e.g., its utility, position or importance), based on relative information with respect to its neighboring agents; we assume that such an information has an heterogeneous nature, as discussed next. In the following, we refer to the i-th value f_i simply as utility. In more detail, for each agent v_i the in-neighborhood $\mathcal{N}_i^{in}$ over G is partitioned into two mutually exclusive[1] sets $\mathcal{D}_i^{in}$ and $\mathcal{R}_i^{in}$, i.e., $\mathcal{D}_i^{in} \bigcap \mathcal{R}_i^{in} = \emptyset$ and $\mathcal{N}_i^{in} = \mathcal{D}_i^{in} \bigcup \mathcal{R}_i^{in}$. The set $\mathcal{D}_i^{in}$ contains the in-neighbors of i for which relative information on the *difference* of the values is available; in other words, for all $v_j \in \mathcal{D}_i^{in}$ the agent v_i knows the value d_{ij} for the difference $f_i - f_j$. The set $\mathcal{R}_i^{in}$ contains the in-neighbors of i for which relative *ratios* are available; in other words, for all $v_j \in \mathcal{R}_i^{in}$ the agent v_i knows the value $r_{ij} > 0$ for the ratio f_i/f_j. Note that, for simplicity, we assume that $v_j \in \mathcal{D}_i^{in}$ whenever $v_i \in \mathcal{D}_j^{in}$ and $v_j \in \mathcal{R}_i^{in}$ whenever $v_i \in \mathcal{R}_j^{in}$. Moreover, for each available difference d_{ij} it holds $d_{ji} = -d_{ij}$, while for each available ratio r_{ij} it holds $r_{ji} = 1/r_{ij}$. Note that we can express E as $E = E^d \bigcup E^r$, where $E^d = \{(v_i, v_j) \in E \,|v_i \in \mathcal{D}_j^{in}\}$ and $E^r = \{(v_i, v_j) \in E \,|v_i \in \mathcal{R}_j^{in}\}$; clearly, it holds $E^d \bigcap E^r = \emptyset$.

In this paper we provide a distributed algorithm to let each agent asymptotically estimate its utility. To this end, we first formulate a least-squares optimization problem; then, we prove that our distributed algorithm asymptotically converges to a global optimal solution of the least-squares optimization problem.

[1] Note that the proposed approach can be easily extended to the case where the graph is a *multigraph* with at most two links connecting any pair of nodes, i.e., a node v_j may belong to both sets. In this way it would be possible to handle situations where both difference and ratio information is provided for the same link.

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8.3 Optimization Problem

In this section we consider the problem of finding an $\boldsymbol{x}^* \in \mathbb{R}^n$ that satisfies all the distance and ratio constraints in an optimal least-squares sense. To this end, we consider a function $g : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\begin{aligned} g(\boldsymbol{x}) = &\frac{1}{2} \sum_{(v_i,v_j)\in E^d} (x_i - x_j - d_{ij})^2 \\ &+ \frac{1}{2} \sum_{(v_i,v_j)\in E^r} \left(\frac{r_{ji}}{1+r_{ji}} x_i - \frac{r_{ij}}{1+r_{ij}} x_j \right)^2 . \end{aligned} \tag{8.1}$$

In order to solve the problem at hand in this paper, we look for a global minimum $\boldsymbol{x}^*$ of $g(\cdot)$, i.e., we aim at finding $\boldsymbol{x}^*$ that satisfies $g(\boldsymbol{x}^*) = \min_{\boldsymbol{x}\in\mathbb{R}^n} \{g(\boldsymbol{x})\}$. It is immediate to recognize that $g(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and that $g(\boldsymbol{x}) = 0$ if and only if for all $(v_i, v_j) \in E^d$ it holds $x_i - x_j = d_{ij}$ and for all $(v_i, v_j) \in E^r$ it holds $x_i/x_j = r_{ij}$; hence, in order to solve the problem at hand in this paper, we seek a global minimum $\boldsymbol{x}^*$ for $g(\cdot)$.

Let us now characterize the structure of the optimal solutions of the above problem. By straightforward computations, and since by assumption $j \in \mathcal{D}_i^{in}$ whenever $i \in \mathcal{D}_j^{in}$ and $j \in \mathcal{R}_i^{in}$ whenever $i \in \mathcal{R}_j^{in}$, it follows that the first order partial derivative of $g(\cdot)$ with respect to x_i is given by

$$\begin{aligned} \frac{\partial g(\boldsymbol{x})}{\partial x_i} = &\sum_{j\in\mathcal{D}_i^{in}} (x_i - x_j) - \sum_{j\in\mathcal{D}_i^{in}} d_{ij} \\ &+ \sum_{j\in\mathcal{R}_i^{in}} \frac{r_{ji}}{1+r_{ji}} \left(\frac{r_{ji}}{1+r_{ji}} x_i - \frac{r_{ij}}{1+r_{ij}} x_j \right) . \end{aligned} \tag{8.2}$$

Again, by simple computations, it can be shown that the $n \times n$ Hessian matrix $H(\cdot)$ associated to $g(\cdot)$ is such that

$$H_{ij}(\boldsymbol{x}) = \frac{\partial^2 g(\boldsymbol{x})}{\partial x_i \partial x_j} = \begin{cases} |\mathcal{D}_i^{in}| + \sum_{j\in\mathcal{R}_i^{in}} \frac{r_{ji}^2}{(1+r_{ji})^2}, & \text{if } i = j \\ -1 & \text{if } v_j \in \mathcal{D}_i^{in} \\ -\frac{1}{(1+r_{ij})(1+r_{ji})} & \text{if } v_j \in \mathcal{R}_i^{in} \\ 0, & \text{otherwise.} \end{cases} \tag{8.3}$$

Let us now collect some observations about $g(\cdot)$ and $H(\cdot)$.

Remark 52. We point out that, since H is positive semidefinite, the function $g(\cdot)$ is convex [2] (see, for instance [124], Chapter 2). Finally, we notice that for $i \neq j$ it holds $H_{ij} \neq 0$ if and only if $(v_j, v_i) \in E$; hence, $H \in \mathbb{A}_G$.

[2] The positive semidefiniteness of H implies convexity but not strict convexity, i.e., $g(\cdot)$ might have multiple global minima.

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Remark 53. Since $g(\cdot)$ is convex, any of its global minima $\boldsymbol{x}^*$ satisfies

$$\frac{\partial g(\boldsymbol{x})}{\partial x_i}\Big|_{\boldsymbol{x}=\boldsymbol{x}^*} = 0, \quad \forall i \in \{1,\dots,n\}.$$

Stacking Eq. (8.2) for all $i \in \{1,\dots,n\}$, setting $\delta_i = \sum_{j\in\mathcal{D}_i^{in}} d_{ij}$, $\boldsymbol{\delta} = [\delta_1,\dots,\delta_n]^T$ and evaluating at $\boldsymbol{x} = \boldsymbol{x}^*$, we conclude that the global minima $\boldsymbol{x}^*$ of $g(\cdot)$ satisfy $H\boldsymbol{x}^* = \boldsymbol{\delta}$. Therefore, we observe that $g(\cdot)$ has a unique global minimum $\boldsymbol{x}^* = H^{-1}\boldsymbol{\delta}$ if and only if $rank(H) = n$. Otherwise, it holds $rank(H) = m < n$ and the set of global minima of $g(\cdot)$ is a subspace of $\mathbb{R}^n$ with dimension equal to $n - m$.

8.4 Existence of a Unique Global Minimum

In this section, we provide a necessary and sufficient condition that guarantees the existence of the unique global optimal solution to the minimization problem. To this end, let us now provide a necessary and sufficient condition for a vector $\boldsymbol{x} \in \mathbb{R}^n$ to belong to the kernel of the Hessian matrix H of $g(\cdot)$.

Proposition 54. *Let $G = \{V, E^d \bigcup E^r\}$ be a connected bidirectional graph with n nodes, where E^d and E^r reflect, respectively, the difference and ratio information available; moreover, let $g(\cdot)$ be defined as in Eq (8.1). A vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $H\boldsymbol{x} = \mathbf{0}_n$, where H is the Hessian matrix associated to $g(\cdot)$, if and only if it holds*

$$\sum_{(v_i,v_j)\in E^d} (x_i - x_j)^2 + \sum_{(v_i,v_j)\in E^r} \frac{(x_i - r_{ij}x_j)^2}{(1+r_{ij})^2} = 0. \tag{8.4}$$

Proof. Proposition 54 To establish the result we notice that, being H symmetric, it holds $H\boldsymbol{x} = \mathbf{0}_n$ if and only if $\boldsymbol{x}^T H\boldsymbol{x} = 0$. Moreover, we have that $H = H' + H''$, where H' and H'' are symmetric matrices having entries given by

$$H'_{ij} = \begin{cases} |\mathcal{D}_i^{in}|, & \text{if } i = j \\ -1 & \text{if } v_j \in \mathcal{D}_i^{in} \\ 0, & \text{otherwise.} \end{cases} \tag{8.5}$$

and

$$H''_{ij} = \begin{cases} \sum_{j\in\mathcal{R}_i^{in}} \frac{r_{ji}^2}{(1+r_{ji})^2}, & \text{if } i = j \\ -\frac{1}{(1+r_{ij})(1+r_{ji})} & \text{if } v_j \in \mathcal{R}_i^{in} \\ 0, & \text{otherwise.} \end{cases} \tag{8.6}$$

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In other words, we have that $H\boldsymbol{x} = \mathbf{0}_n$ if and only if it holds

$$\boldsymbol{x}^T H\boldsymbol{x} = \boldsymbol{x}^T H'\boldsymbol{x} + \boldsymbol{x}^T H''\boldsymbol{x} = 0. \tag{8.7}$$

We observe that

$$\boldsymbol{x}^T H'\boldsymbol{x} = \sum_{i=1}^{n} x_i \sum_{j\in\mathcal{D}_i^{in}} (x_i - x_j) = \sum_{(v_i,v_j)\in E^d} (x_i - x_j)^2, \tag{8.8}$$

and that

$$\boldsymbol{x}^T H''\boldsymbol{x} = \sum_{i=1}^{n} x_i \sum_{j\in\mathcal{R}_i^{in}} \left(\frac{r_{ji}^2}{(1+r_{ji})^2} x_i - \frac{1}{(1+r_{ij})(1+r_{ji})} x_j \right).$$

At this point, we notice that, by some algebra, it holds

$$\boldsymbol{x}^T H''\boldsymbol{x} = \sum_{(v_i,v_j)\in E^r} \frac{(x_i - r_{ij}x_j)^2}{(1+r_{ij})^2}. \tag{8.9}$$

The proof follows. □

We now show that $rank(H) \geq n-1$.

Lemma 55. *Let $G = \{V, E^d \bigcup E^r\}$ be a connected bidirectional graph with n nodes, where E^d and E^r reflect, respectively, the difference and ratio information available; moreover, let $g(\cdot)$ be defined as in Eq (8.1). The Hessian matrix H associated to $g(\cdot)$ is such that $rank(H) \geq n-1$.*

Proof. Lemma 55 Let $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ be such that $H\boldsymbol{x} = \mathbf{0}_n$. By Proposition 54, $\boldsymbol{x}$ satisfies Eq. (8.4). Therefore, it must hold $x_i - x_j = 0$, for all $(v_i, v_j) \in E^d$ and $x_i - r_{ij}x_j = 0$, for all $(v_i, v_j) \in E^r$. The above conditions can be rearranged as $x_i/x_j = 1$, for all $(v_i, v_j) \in E^d$ and $x_i/x_j = r_{ij}$, for all $(v_i, v_j) \in E^r$. In other words, for each link $(v_i, v_j) \in E$, the ratio of the entries x_i and x_j of the vector $\boldsymbol{x}$ must be equal to a given w_{ij}, where $w_{ij} = r_{ij}$ for links corresponding to ratio information and $w_{ij} = 1$ for links corresponding to difference information. Let $W \in \mathbb{A}_G$ be the $n \times n$ matrix collecting such ratios, i.e., $W_{ij} = w_{ij}$ if $(v_j, v_i) \in E^d \bigcup E^r$ and $W_{ij} = 0$, otherwise. In [92], the authors demonstrate that, when a matrix W collecting sparse ratio information has the same structure as a connected bidirectional graph, a necessary and sufficient condition for the existence of a vector $\boldsymbol{x}$ such that $W_{ij} = x_i/x_j$ for all $W_{ij} \neq 0$ is that the product of the entries W_{ij} along any cycle of the graph is equal to one; otherwise, no solution exists. When such a condition is satisfied, we observe that the ratios x_i/x_j are defined up to a scaling factor. Hence, the kernel of H has dimension one and $rank(H) = n-1$. In the latter case, no solution exists (other than the triv-

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ial one) and therefore the kernel of H coincides with $\{\mathbf{0}_n\}$ and $rank(H) = n$. The proof is complete. □

As a consequence of Lemma 55, we can state a necessary and sufficient condition for $rank(H)$ to be equal to n.

Proposition 56. *Let H be the Hessian matrix associated to $g(\cdot)$ and let us assign a weight $w_{ij} = 1$ to all $(v_i, v_j) \in E^d$ and a weight $w_{ij} = r_{ij}$ to all $(v_i, v_j) \in E^r$. It holds $rank(H) = n$ if and only if there is a cycle $c = \{(v_1, v_2), \ldots, (v_m, v_1)\}$ over $G = \{V, E^d \bigcup E^r\}$ such that $\prod_{(v_i,v_j)\in c} w_{ij} \neq 1$.*

Proof. Proposition 56 By Lemma 55, it holds $rank(H) = n - 1$ if and only if all cycles over G satisfy $\prod_{(v_i,v_j)\in c} w_{ij} = 1$, otherwise $rank(H) = n$. The proof follows. □

8.5 Proposed Algorithm

If all the information can be collected and processed in a centralized way then, as noted in Remark 52, a solution to the problem at hand in this paper is to find $\boldsymbol{x}^*$ that satisfies $H\boldsymbol{x}^* = \boldsymbol{\delta}$. In several situations it might be impossible to solve the problem by means of a centralized supervisory entity; in those cases, each agent aims at computing its own utility in a distributed way. Specifically, based on the information regarding its neighbors, each agent i aims at computing a value x_i^* such that, overall, the vector $\boldsymbol{x}^*$ satisfies all difference and ratio constraints in a least-squares sense; in other words, the agents aim at computing a vector $\boldsymbol{x}^*$ that is a global minimum for $g(\cdot)$. Within the proposed algorithm, each agent i executes the following continuous-time and synchronous update algorithm

$$\begin{aligned}\dot{x}_i(t) = \alpha &\sum_{j\in\mathcal{D}_i^{in}} (x_j - x_i) \\ &+ \alpha \sum_{j\in\mathcal{R}_i^{in}} \frac{r_{ij}}{1+r_{ij}} \left(\frac{r_{ij}}{1+r_{ij}} x_j - \frac{r_{ji}}{1+r_{ji}} x_i \right) + \alpha\delta_i,\end{aligned} \tag{8.10}$$

where $\alpha > 0$ and $\delta_i = \sum_{j\in\mathcal{D}_i^{in}} d_{ij}$.

Let us now show that the proposed distributed algorithm asymptotically converges to a global minimum[3] $\boldsymbol{x}^*$ for $g(\cdot)$.

Theorem 57. *Let us consider a connected bidirectional graph $G = \{V, E^d \bigcup E^r\}$ with n nodes, where E^d and E^r reflect, respectively, the difference and ratio information available to the agents. Let the agents execute the synchronous update rule in Eq. (8.10), with initial condition*

[3] As discussed in the previous section, the solution is unique if and only if the condition in Proposition 56 is satisfied.

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$x_i(0) > 0$ *and* $\alpha > 0$. *It holds* $\lim_{t\to\infty} x_i(t) = x_i^*$, *where* $\boldsymbol{x}^* = [x_1^*, \ldots, x_n^*]^T$ *is a global minimum for* $g(\cdot)$.

Proof. Theorem 57 Stacking Eq. (8.10) for all the agents and setting $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ and $\boldsymbol{\delta} = [\delta_1, \ldots, \delta_n]^T$, we get

$$\dot{\boldsymbol{x}}(t) = -\alpha H \boldsymbol{x}(t) + \alpha \boldsymbol{\delta}. \tag{8.11}$$

As noted in Remark 52, we have that H is positive semidefinite and by Lemma 55 it holds $rank(-\alpha H) \geq n-1$. Hence, matrix $-\alpha H$ is stable in the continuous-time sense and the system $\dot{\boldsymbol{x}}(t) = -\alpha H \boldsymbol{x}(t)$ converges to an equilibrium point. We point out that the presence of the constant input $\alpha\boldsymbol{\delta}$ does not affect stability; hence, also the dynamics in Eq. (8.11) converges to an equilibrium point $\boldsymbol{x}_{eq}$, which satisfies $\mathbf{0}_n = -\alpha H \boldsymbol{x}_{eq} + \alpha\boldsymbol{\delta}$, that is, $H\boldsymbol{x}_{eq} = \boldsymbol{\delta}$. Therefore, using the same reasoning as in Remark 53, we conclude that the equilibrium reached corresponds to a global minimum of $g(\cdot)$. The proof is complete. □

Remark 58. The parameter α can be used to arbitrarily increase the speed of convergence of the proposed algorithm, e.g., by letting each agent choose the same $\alpha \gg 1$. However, to select a specific (e.g., instance-dependent) value of α, some form of distributed coordination or agreement (e.g., distributed consensus [106]) is required before the execution of the proposed algorithm.

We now characterize the structure of the particular global minimum of $g(\cdot)$ computed by the proposed algorithm.

Theorem 59. *Let us consider a connected bidirectional graph* $G = \{V, E^d \bigcup E^r\}$ *with* n *nodes, where* E^d *and* E^r *reflect, respectively, the difference and ratio information available to the agents. Let the agents execute the synchronous update rule in Eq. (8.10), with initial condition* $x_i(0) > 0$ *and* $\alpha > 0$. *Without loss of generality, let* λ_i *be the* i*-th smallest eigenvalue of the Hessian matrix* H *of* $g(\cdot)$ *and let* $\boldsymbol{z}_i$ *be the corresponding eigenvector such that the set* $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ *represents an orthonormal basis for* H. *The state of the agents asymptotically converges to*

$$\boldsymbol{x}_{eq} = \begin{cases} \left(\boldsymbol{z}_1^T \boldsymbol{x}(0)\right) \boldsymbol{z}_1 + \sum_{i=2}^n \frac{1}{\lambda_i} \left(\boldsymbol{z}_i^T \boldsymbol{\delta}\right) \boldsymbol{z}_i, & \text{if } \lambda_1 = 0, \\ \sum_{i=1}^n \frac{1}{\lambda_i} \left(\boldsymbol{z}_i^T \boldsymbol{\delta}\right) \boldsymbol{z}_i, & \text{otherwise.} \end{cases} \tag{8.12}$$

Proof. Theorem 59
As noted in Remark 52 and Lemma 55, the Hessian matrix H is symmetric and it has at most one eigenvalue equal to zero, while all other eigenvalues are positive. Hence, we diagonalize H by writing $H = Z\Lambda Z^{-1}$, where Λ is a

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diagonal $n \times n$ matrix with $\Lambda_{ii} = \lambda_i$ and $Z = [\boldsymbol{z}_1, \dots, \boldsymbol{z}_n]$; since $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_n\}$ represents an orthonormal basis, we have that it holds $Z^{-1} = Z^T$. The state of the agents at time t is given by $\boldsymbol{x}(t) = e^{-\alpha H t}\boldsymbol{x}(0) + \int_0^t e^{-\alpha H(t-\tau)}\alpha\boldsymbol{\delta} d\tau$ and can be rearranged as

$$\boldsymbol{x}(t) = Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{x}(0) + \alpha Ze^{-\alpha\Lambda t}Z^{-1}\int_0^t Ze^{\alpha\Lambda\tau}Z^{-1}\boldsymbol{\delta} d\tau$$

Let us define $\boldsymbol{\eta} = Z^{-1}\boldsymbol{\delta}$, so that it holds $\eta_i = \boldsymbol{z}_i^T\boldsymbol{\delta}$ for all $i \in \{1, \dots, n\}$. Notice that $H\boldsymbol{x}_{eq} = \boldsymbol{\delta}$; hence, when $\lambda_1 = 0$ it holds $\eta_1 = \boldsymbol{z}_1^T\boldsymbol{\delta} = \boldsymbol{z}_1^T H\boldsymbol{x}_{eq} = 0$. Since $e^{\alpha\Lambda\tau}$ is diagonal and $Z^{-1} = Z^T$, it holds

$$\begin{aligned}\boldsymbol{x}(t) &= Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{x}(0) + \alpha Ze^{-\alpha\Lambda t}Z^{-1}\int_0^t \sum_{i=1}^n \boldsymbol{z}_i e^{\alpha\lambda_i\tau}\eta_i d\tau \\ &= Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{x}(0) + \alpha\sum_{i=1}^n Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{z}_i\eta_i\int_0^t e^{\alpha\lambda_i\tau}d\tau.\end{aligned}$$

Let $\boldsymbol{e}_i$ be the i-th vector in the canonical base in $\mathbb{R}^n$; since $Z^{-1} = Z^T$, we have that

$$Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{z}_i = Ze^{-\alpha\Lambda t}\boldsymbol{e}_i = Ze^{-\alpha\lambda_i t}\boldsymbol{e}_i = e^{-\alpha\lambda_i t}Z\boldsymbol{e}_i = e^{-\alpha\lambda_i t}\boldsymbol{z}_i;$$

hence, it holds

$$\boldsymbol{x}(t) = Ze^{-\alpha\Lambda t}Z^{-1}\boldsymbol{x}(0) + \alpha\sum_{i=1}^n \eta_i e^{-\alpha\lambda_i t}\boldsymbol{z}_i\int_0^t e^{\alpha\lambda_i\tau}d\tau.$$

At this point we notice that, when $\lambda_1 = 0$, it holds $\eta_1 = 0$; hence,

$$\boldsymbol{x}(t) = \sum_{i=1}^n e^{-\alpha\lambda_i t}\left(\boldsymbol{z}_i^T\boldsymbol{x}(0)\right)\boldsymbol{z}_i + \sum_{i=2}^n\left(\boldsymbol{z}_i^T\boldsymbol{\delta}\right)\frac{1-e^{-\alpha\lambda_i t}}{\lambda_i}\boldsymbol{z}_i$$

and therefore $\lim_{t\to\infty}\boldsymbol{x}(t) = \boldsymbol{x}_{eq}$, where $\boldsymbol{x}_{eq}$ corresponds to the first case in Eq. (8.12). Conversely, when $\lambda_1 > 0$, we have that

$$\boldsymbol{x}(t) = \sum_{i=1}^n e^{-\alpha\lambda_i t}\left(\boldsymbol{z}_i^T\boldsymbol{x}(0)\right)\boldsymbol{z}_i + \sum_{i=1}^n\left(\boldsymbol{z}_i^T\boldsymbol{\delta}\right)\frac{1-e^{-\alpha\lambda_i t}}{\lambda_i}\boldsymbol{z}_i$$

and therefore $\lim_{t\to\infty}\boldsymbol{x}(t) = \boldsymbol{x}_{eq}$, where this time $\boldsymbol{x}_{eq}$ corresponds to the second case in Eq. (8.12). This completes our proof. □

Remark 60. Notice that, when $\lambda_1 = 0$ the set of global minima of $g(\cdot)$ correspond to a subspace of $\mathbb{R}^n$ of dimension equal to one. Conversely, when $\lambda_1 > 0$ the problem admits a unique global minimum. In particular,

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as shown in Eq. (8.12), in the first case the solutions coincide with an affine space of the eigenspace spanned by $\boldsymbol{z}_1$ (the particular value computed by the agents depends on the initial condition $\boldsymbol{x}(0)$ and on the complete sets of eigenvalues and eigenvectors of H), while in the latter case the solution is unique, and it depends on all the eigenvalues and eigenvectors of H but is independent on the initial condition.

Remark 61. Note that, although Eq. (8.12) provides a closed-form solution for the global minima of $g(\cdot)$, its structure depends on the entire set of eigenvalues and eigenvectors of H; a distributed algorithm to compute such information has a remarkably higher computational burden for the agents (e.g., see [125]) with respect to the proposed algorithm, thus justifying the adoption of our approach in a distributed computing scenario.

8.6 Simulation Results

In this section we provide numerical evidence to corroborate the theoretical findings. Let us take into account two small scale instances such that $g(\boldsymbol{x}^*) = 0$, i.e., such that the available information is perfectly consistent. Specifically, we consider two graphs with $|V| = 5$ nodes and $|E| = 12$ edges (i.e., six pairs of bidirectional edges); the graphs and the available differences/ratios are reported, respectively, in Figure 8.1(a) and Figure 8.1(c). Let us now discuss the first example. Note that the information associated to the example in Figure 8.1(a) satisfies the necessary and sufficient condition in Proposition 56, hence it can be shown that $\boldsymbol{g}(\cdot)$ has a unique global minimum at $\boldsymbol{x}^* = \sum_{i=1}^{n} \frac{1}{\lambda_i} \left(\boldsymbol{z}_i^T \boldsymbol{\delta}\right) \boldsymbol{z}_i = [1, 2, 2, 8, 1]^T$, thus numerically corroborating Eq. (8.12). Figure 8.1(b) shows the evolution of the proposed distributed algorithm when $\alpha = 1$; it can be noted that the state $x_i(t)$ of each agent v_i effectively converges to x_i^*.
Let us now discuss the example in Figure 8.1(c); Figure 8.1(d) shows that, for $\boldsymbol{x}(0) = [0.5768, 0.0259, 0.4465, 0.6463, 0.5212]^T$, the state of the agents converges to an $\boldsymbol{x}_{eq}$ corresponding to the first case in Eq. (8.12), numerically validating Eq. (8.12). It can be easily shown that for any $\epsilon \in \mathbb{R}$ the vector $\boldsymbol{x}^*(\epsilon) = [1 + \epsilon, 2 + \epsilon, 8 + \epsilon, 4 + \epsilon/2, 1 + \epsilon/8]^T$ is a global minimum, since it holds $g(\boldsymbol{x}^*(\epsilon)) = 0$; note that, as demonstrated in Theorem 59, the particular value of ϵ associated to the asymptotic solution found depends on the initial condition $\boldsymbol{x}(0)$.

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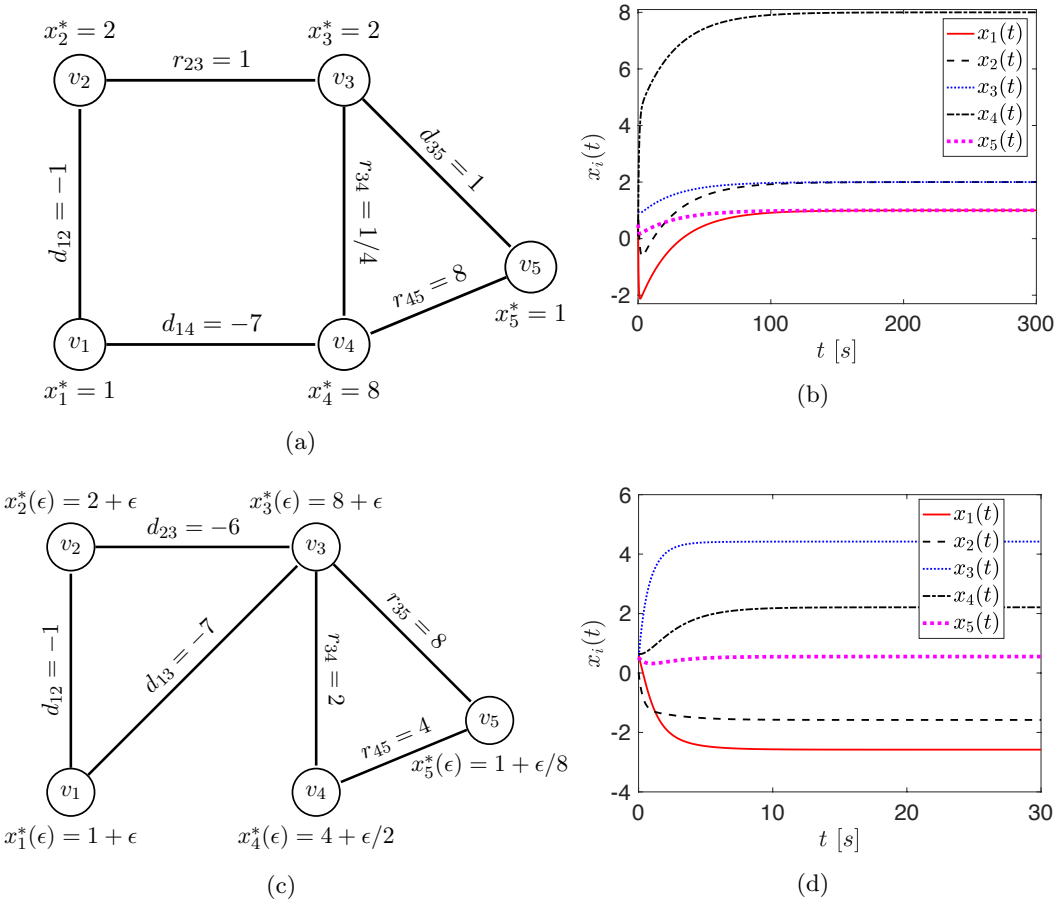

Figure 8.1: Examples with $|V| = 5$ nodes and $|E| = 12$ edges (i.e., six pairs of bidirectional edges). In panel 8.1(a) the condition in Proposition 56 is satisfied and there is a unique global minimum for $g(\cdot)$. In panel 8.1(c) the condition in Proposition 56 is violated and there are several global minima for $g(\cdot)$ (we show them as a function of the parameter ϵ). Panels 8.1(b) and 8.1(d) show the evolution of the proposed algorithm for $\alpha = 1$, considering the instance in panels 8.1(a) and 8.1(c), respectively.

In order to assess the effect of perturbations on the available information, and to compare with standard formation control and AHP approaches, in Figure 8.2 we consider a graph where $|V| = 100$ nodes are sampled uniformly at random in the unit square $[0, 1]^2$ and a pair of nodes v_i, v_j is connected by an edge provided that their Euclidean distance is smaller than $\rho = 0.2$; the resulting graph, reported in Figure 8.2(a), has $|E| = 1038$ links, i.e. 519 distinct pairs. Moreover, we consider a scenario where the utility of the i-th agent is $x_i^* = \frac{2i}{n(n+1)}$ (so that, overall, it holds $\mathbf{1}_n^T \boldsymbol{x}^* = 1$) and we

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partition the links of E into the sets E^d (black solid lines in Figure 8.2(a)) and E^r (blue dotted lines in Figure 8.2(a)), which correspond to difference and ratio information, respectively. Specifically, in order to guarantee that the problem can be solved based on just differences or ratios, we first calculate two edge-disjoint spanning trees over G and we assign their links to E^d and E^r, respectively; then, we randomly partition the remaining links in E, assigning them to the sets E^d and E^r with equal probability; as a result we obtain $|E^d| = 554$ (277 distinct pairs) and $|E^r| = 484$ (242 distinct pairs). In order to evaluate the effectiveness of the proposed methodology, we consider multiplicative errors affecting the available ratios and additive perturbations affecting the available differences. In more detail, we consider ratios affected by log-normal random perturbations, as typically done in the AHP literature [100], i.e., we set $r_{ij} = exp(\mathcal{N}(0,\sigma))x_i^*/x_j^*$, where $\mathcal{N}(0,\sigma)$ is a normal random number with zero average and standard deviation σ. Then, we select random additive perturbations for the difference information which are comparable to the magnitude of the multiplicative ones. To this end, we observe that if $x_i^*/x_j^* = r_{ij}e^\sigma$, then $x_i^* - x_j^* = (r_{ij}e^\sigma - 1)x_j^* = x_i^*e^\sigma - x_j^*$; therefore, if we seek for a perturbation γ such that $x_i^* - x_j^* = d_{ij} + \gamma$ we have that $\gamma = x_i^*e^\sigma - x_j^* - d_{ij} = x_i^*(e^\sigma - 1)$. For the above reason, we set $d_{ij} = x_i^* - x_j^* + \mathcal{N}(0, x_i^*(e^\sigma - 1))$. In Figure 8.2(b) we compare the results achieved by considering only difference information via formation control (blue dashed line), only ratios via the AHP approach in [100] (green dotted line) and the performance of the proposed algorithm when we consider both differences and ratios (red solid lines); for all curves we show the results in terms of average and standard deviation over $M = 100$ runs with the same choice of σ. Specifically, we plot against σ the *Kendall's Tau Distance* [126] τ between the nominal and perturbed ranking of the agents; such a distance is such that $\tau \in [0,1]$, where $\tau = 0$ means that the ranking is the same and $\tau = 1$ means that the rankings are in reverse order. As shown by Figure 8.2(b), the proposed approach is remarkably more robust to the perturbations; for instance, $\tau \leq 0.01$ for $\sigma \leq 0.1$, while using only differences or ratios we get $\tau \approx 0.07$ and $\tau \approx 0.04$, respectively. The difference in the result of the three approaches widens as σ grows, and for $\sigma = 0.3$ we have that the proposed approach yields $\tau \approx 0.07$, while the cases of using only differences and ratios yield $\tau \approx 0.31$ and $\tau \approx 0.13$, respectively.

8.7 Conclusions and Future Work

In this paper we develop a novel distributed decision making technique that endows a network of agents with the capability to compute a quantity that represents their own utility or importance, based on the knowledge of pairwise relative information of heterogeneous nature, i.e., the differences

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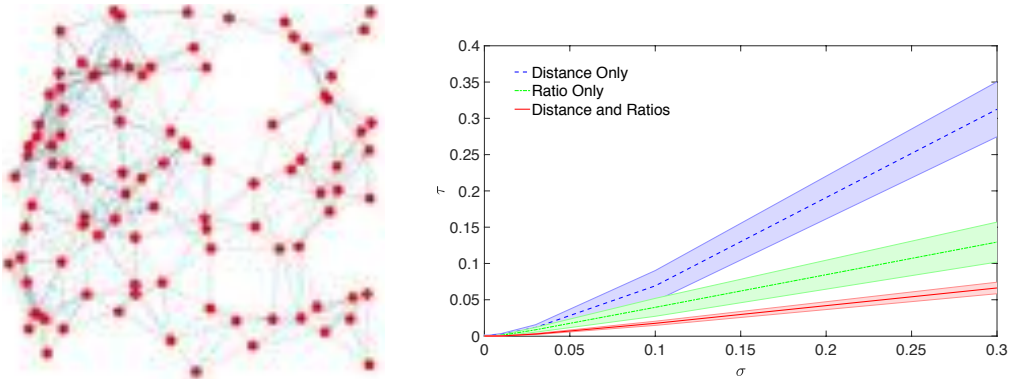

(a) (b)

Figure 8.2: (a) graph considered in the simulation. (b) comparison of the proposed approach (red line) with formation control (blue dashed line) and AHP (green dotted line) for growing perturbations.

and ratios of the utilities of a node with respect to its neighbors. Specifically, we frame the problem in terms of a least-squares minimization problem and we characterize the structure of the global minima of such problem, providing a necessary and sufficient condition that guarantees the existence of a unique solution. Moreover, we develop a distributed continuous-time algorithm that lets the agents asymptotically find a global minimum. Future work will follow four main directions: (i) extending the framework to directed graphs; (ii) introducing constraints in the formulation; (iii) including in the framework different typologies of nonlinear functions describing the relative information available; (iv) extending the approach to wireless sensor network localization, considering a scenario where some sensors are able to measure distances while other sensors are able to estimate of the ratio between their distance from pairs of neighbors.

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Appendix Part II

We recall some basic notions and definitions relating to the topics of Chapters 7 and 8.

Kendall's Correlation Index

Let $\boldsymbol{a} \in \mathbb{R}^n$ and let $\boldsymbol{b} \in \mathbb{R}^n$ be a permutation of the elements in $\boldsymbol{a}$. Given two pairs of values (a_i, b_i) and (a_j, b_j), we say they are *concordant* if both $a_i > a_j$ and $b_i > b_j$ or if both $a_i < a_j$ and $b_i < b_j$; similarly the pairs are *discordant* if $a_i > a_j$ and $b_i < b_j$ or if $a_i < a_j$ and $b_i > b_j$. If $a_i = a_j$ or $b_i = b_j$ the pairs are neither concordant nor discordant. The *Kendall's correlation index* [126] τ is defined as $\tau = \frac{|\mathcal{C}|-|\mathcal{P}|}{n(n-1)/2}$, where $\mathcal{C}$ and $\mathcal{P}$ are the sets of concordant and discordant pairs (a_i, b_i) and (a_j, b_j), respectively. We point out that τ can be regarded as the degree of shuffling of $\boldsymbol{b}$ with respect to $\boldsymbol{a}$; in fact, for τ equal to one the two vectors are identical, while for $\tau = -1$ $\boldsymbol{b}$ is in reverse order with respect to $\boldsymbol{a}$ and for $\tau \approx 0$ the two vectors are independent.

Graph theory and Markov Chains Definitions

Let $G = \{V, E\}$ be a *graph* with n nodes $V = \{v_1, \ldots, v_n\}$ and e edges $E \subseteq V \times V$, where $(v_i, v_j) \in E$ captures the existence of a link from node v_i to node v_j. A graph is said to be *undirected* if $(v_i, v_j) \in E$ whenever $(v_j, v_i) \in E$, and is said to be *directed* otherwise. In the following we will consider only undirected graphs. A graph is *connected* if for each pair of nodes v_i, v_j there is a path over G that connects them. Let the neighborhood $\mathcal{N}_i$ of a node v_i be the set of nodes v_j that are connected to v_i via an edge (v_j, v_i) in E. The *degree* d_i of a node v_i is the number of its incoming edges[4], i.e., $d_i = |\mathcal{N}_i|$. The *adjacency matrix* A of a graph $G = \{V, E\}$ with n nodes is the $n \times n$ matrix such that $A_{ij} = 1$ if $(v_j, v_i) \in E$ and $A_{ij} = 0$ otherwise; moreover, the $n \times n$ *degree matrix* D is an $n \times n$ diagonal matrix such that $D_{ii} = d_i$, for all $i \in \{1, \ldots, n\}$. The $n \times n$ *Laplacian matrix* of G is $L = D - A$;

[4] Over undirected graphs, for each node v_i the number of its incoming and outgoing edges coincide.

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since, by construction, the rows of L sum to zero, it can be noted that L is singular. The *diameter* δ of a graph G is the length of the longest among the minimum paths between any pair of nodes. We say an $n \times n$ matrix W *has the same structure* as a graph $G = \{V, E\}$ with n nodes if $A_{ij} = 0$ implies $W_{ij} = 0$. It can be shown that, in the case of undirected graphs, the fact W is irreducible corresponds to the fact that W has the same structure as a connected graph G.

Markov Chains

Let a graph $G = \{V, E\}$ with n nodes; a (time-homogeneous) Markov chain [127, 128] is a dynamic system in the form $\boldsymbol{p}(k+1) = H^T \boldsymbol{p}(k)$, where $\boldsymbol{p}(0) \in \mathbb{R}^n$ is a *probability distribution*, i.e., it has just nonnegative entries and satisfies $\mathbf{1}_n^T \boldsymbol{p}(0) = 1$, and the *transition probability matrix* H is such that $H_{ij} \in [0, 1]$ for all $i, j \in \{1, \dots, n\}$ and $H\mathbf{1}_n = \mathbf{1}_n$. The notation $\mathbf{1}_n$ denotes a vector with n components, each equal to 1 A Markov chain is said to be irreducible if H is irreducible, i.e., if H has the same structure as a connected graph G. For each state variable i, the associated *period* is $h = \gcd\{n \in \mathbb{N}_+ \,|\, (H^n)_{ii} > 0\}$, where gcd is the greatest common divisor. From the definition, it follows that a sufficient condition for the period of the i-th variable to be $h = 1$ is that $H_{ii} > 0$. A Markov chain is said to be *aperiodic* if all the state variables have period equal to one.

Remark 62. Note that, if the Markov chain is irreducible, then all the states have the same period (see, for instance [127, 128] and references therein). Therefore, a sufficient aperiodicity condition is that matrix H has at least one nonzero diagonal entry.

Remark 63. If a Markov chain is irreducible but not aperiodic, and H has a vector $\boldsymbol{f}$ as its left dominant eigenvector, i.e., $H^T \boldsymbol{f} = \boldsymbol{f}$, it can be easily noted that for any $\alpha \in (0, 1)$ the matrix $H^* = \alpha H + (1 - \alpha) I$ is aperiodic and it holds $(H^*)^T \boldsymbol{f} = \alpha H^T \boldsymbol{f} + (1 - \alpha)\boldsymbol{f} = \alpha \boldsymbol{f} + (1 - \alpha)\boldsymbol{f} = \boldsymbol{f}$, i.e., H^* has the same left dominant eigenvector as the original matrix H.

A Markov chain is said to have a *limiting distribution* $\boldsymbol{p}_\infty$ if, for all probability distributions $\boldsymbol{p}(0)$ it holds $\lim_{k\to\infty} (H^T)^k \boldsymbol{p}(0) = \boldsymbol{p}_\infty$. Note that, in general, a Markov chain might not have a limiting distribution; a necessary and sufficient condition for its existence is that the Markov chain is irreducible and aperiodic.

The following remark shows that, in the general case when the initial condition is not a probability distribution vector, any Markov chain is *sum-preserving*.

Remark 64. Let us consider a Markov chain with transition probability matrix H, a vector $\boldsymbol{p}(0) \in \mathbb{R}^n$ with non-negative entries and such that

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$\mathbf{1}_n^T \boldsymbol{p}(0) \notin \{0, 1\}$, and a vector $\boldsymbol{p}'(0) = \boldsymbol{p}(0)/\mathbf{1}_n^T \boldsymbol{p}(0)$. Moreover, let $\boldsymbol{p}(k)$ and $\boldsymbol{p}'(k)$ be the state of the Markov chain when the initial condition is $\boldsymbol{p}(0)$ and $\boldsymbol{p}'(0)$, respectively. At each time step k it holds

$$\boldsymbol{p}(k) = (M^T)^k \boldsymbol{p}(0) = \left(\mathbf{1}^T \boldsymbol{p}(0)\right) (M^T)^k \boldsymbol{p}'(0) = \left(\mathbf{1}^T \boldsymbol{p}(0)\right) \boldsymbol{p}'(k),$$

and since $\boldsymbol{p}'(k)$ is a distribution for all times k, it follows that $\mathbf{1}^T \boldsymbol{p}(k) = \mathbf{1}^T \boldsymbol{p}(0), \quad \forall k = 0, 1, \ldots$.

A consequence of the above remark is that, if a Markov chain has a limiting distribution $\boldsymbol{p}_\infty$, then in the general case where $\boldsymbol{p}(0)$ is not a distribution vector the state asymptotically converges to a vector $\widehat{\boldsymbol{p}}_\infty = (\mathbf{1}^T \boldsymbol{p}(0))\boldsymbol{p}_\infty$, i.e., to a vector in the span of the limiting distribution $\boldsymbol{p}_\infty$ with $\mathbf{1}^T \widehat{\boldsymbol{p}}_\infty = \mathbf{1}^T \boldsymbol{p}(0)$.

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