The Geometry of Noncommutative Spheres and their Symmetries

Thesis submitted for the degree of Doctor Philosophiæ

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Prologue

Symmetries play a central role in both physics and mathematics. In physics, they can be found at the heart of practically any theory where they encode an invariance of the theory under certain transformations. As a simple example, one could think of the translational invariance of classical mechanics: the way in which an apple falls from a tree does not depend on the location of the garden. More interesting (but more confusing) are symmetries in Einstein's theory of special relativity. Here we find that the speed of light emitted by a torch of a person at rest coincides with the speed of light coming from a torch held by a person who is moving at constant velocity; the symmetries involved are given in terms of the so-called Lorentz transformations.

In mathematics, symmetries appear for example in the theory of group actions on some space. One could think here of the set of rotations in three dimensions acting on an ordinary sphere. Also, mathematics allows for a more general type of symmetries, known as quantum symmetries; they are described in quantum group theory. Such symmetries are supposed to act on so-called noncommutative spaces. Notice that on an ordinary space (think of the plane), we can choose coordinates which are ordinary numbers indicating the position on this space (say the (x, y)-coordinates on the plane). A noncommutative space can be described in a similar manner, with the only (but drastic) difference that the coordinates are not numbers anymore but abstract objects which in general do not even commute (in the sense that $x \cdot y \neq y \cdot x$). The description of such spaces forms the basic subject of Alain Connes' noncommutative geometry [27].

A beautiful synthesis between mathematics and physics is found in Yang-Mills theory. This theory forms the basis of the celebrated Standard Model of physics which provides a highly accurate description of interactions between particles at a subatomic scale. Symmetries arise in the form of Lie groups. For example, the Lie group $SU(2)$ lies at the heart of the theory of the weak interactions; it will be of central interest to us in what follows. Yang-Mills theory is defined in terms of a Yang-Mills action, expressing the energy of the configuration. We stress here the physical importance of finding the absolute minima of such an action; they are given by configurations called instantons.

The mathematical structure behind Yang-Mills theory is the theory of connections on principal bundles. The ideas of Yang-Mills theory culminated in Donaldson's construction of invariants of smooth four-dimensional manifolds [42] in which a central role is played by instantons.

We are interested in "quantum versions" of two different parts of this Yang-Mills theory. The first one is concerned with the symmetry alone, and considers the quantum symmetry group $SU_q(2)$ in the framework of Connes' noncommutative geometry. The second one leaves the symmetry group as it is and considers a formulation of $SU(2)$ Yang-Mills theory on noncommutative spaces, still in the same framework of noncommutative geometry. The motivation for this is two-fold. Firstly, there is the idea that (quantum) Yang-Mills theories on quantum spaces or with quantum symmetries behave –in some sense– better than on ordinary spaces.

Secondly, Alain Connes' noncommutative geometry has all the ingredients for the formulation of Yang-Mills theory. For instance, the choice of a certain slightly noncommutative space[1] allows for a derivation of the successful Standard Model of physics from basic principles [21, 22].

We construct in Part I a quantum version of the symmetry group $\mathsf{SU}(2)$ described above, and connect this quantum group with the noncommutative geometry of Connes. In this way, we describe the geometry of the quantum group $\mathsf{SU}_q(2)$ as a noncommutative space. A guiding principle is provided by imposing symmetry under certain transformations; there are two quantum symmetries and we impose invariance –or equivariance– under their action.

In Part II, we consider a formulation of $\mathsf{SU}(2)$ Yang-Mills theory on noncommutative spaces. In particular, we explore the geometry of a noncommutative principal bundle and define a Yang-Mills action on a four-dimensional noncommutative sphere. On the way, we encounter more quantum symmetries. We discuss a (infinitesimal) quantum version of the five dimensional rotation group acting as symmetries on the noncommutative four-sphere as well as the twisted conformal transformations. The latter gives rise to a family of infinitesimal instantons which are the minima of the Yang-Mills action.

This thesis consists for a great part of the articles [39, 93] (Part I) and [68, 69] (Part II). We added several remarks and considerations, together with some introductory material, collected in the appendix.

Acknowledgements

'Do you like Trieste?' This was one of the questions during my entrance exam of SISSA. My affirmative answer to this question at that time still stands, if not even more profoundly. During the past three years, it has been a pleasure to study, work and live here, thanks to the help and support of many people.

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[1] It will be the product of an ordinary space by a certain "internal" discrete space

Part I

The noncommutative spin geometry of quantum SU(2)

Chapter 1

Introduction

One of the basic motivations of noncommutative geometry is that one can describe a topological space by the C^*-algebra of continuous functions on it. See Appendix A.1 for an overview of the theory of C^*-algebras. The Gelfand transform allows to construct a Hausdorff space X from a commutative C^*-algebra A, in such a way that $A \simeq C_0(X)$.

Theorem 1.1. *There is an equivalence between the category of Hausdorff topological spaces and the category of commutative* C^**-algebras.*

Alain Connes developed this idea further in the 1980's, in that also the metric structure of a Riemannian manifold becomes encoded in dual objects. In this case, the topology of the manifold is described in terms of an algebra as above, whereas the geodesic distance can be derived from the Laplacian on the manifold. In the case of a Riemannian spin manifold, one usually works with the Dirac operator. The advantage of this over the Laplacian is that it is a first-order instead of a second-order differential operator.

This motivates the key idea of noncommutative geometry in that forgetting about the commutativity of the algebra allows to describe 'virtual' quantum spaces in a dual manner. One arrives at the following basic set of data [27].

Definition 1.2. *A* spectral triple $(\mathcal{A}, H, D)$ *consists of a* $*$*-algebra of bounded operators on a Hilbert space* H*, together with a self-adjoint operator* D *on* H *satisfying*

1. *The resolvent* $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, *is a compact operator*

2. *The commutator* $[D, a] = D \cdot a - a \cdot D$ *is a bounded operator for all* $a \in \mathcal{A}$.

The basic and motivating example is the canonical spectral triple associated to an n-dimensional Riemannian spin manifold M. It is defined by

- $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on M.

- $H = L^2(M, S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$.

- D, the Dirac operator associated with the Levi-Civita connection.

The topology of M is recovered by the C^*-completion of $\mathcal{A}$, which is the algebra of continuous functions on M. The geodesic distance between any two points on M is given by

$$d(p, q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)| : \|[D, f]\| \leq 1\} ,$$

whereas the Riemannian measure on M is given in terms of the so-called Dixmier trace [41], denoted Tr_ω. It is defined on the class of compact operators of order 1, i.e. with singular values $\mu_n(T)$ satisfying $\mu_n(T) = \mathcal{O}(\frac{1}{n})$. On such operators, it filters out the coefficient of the logarithmic divergence of the singular value sums $\sigma_N = \sum_{0 \leq k \leq N} \mu_k(T)$ (see Appendix A.2 for more details). It turns out that on a Riemannian spin manifold of dimension m, the compact operator $|D|^{-m}$ is in this class of operators and in fact, for a smooth function f on M we have

$$\int d\mu_g(x) f(x) = \mathrm{Tr}_\omega f |D|^{-m} .$$

The Connes' reconstruction theorem [29] provides a full duality between Riemannian spin manifolds and spectral triples for which the algebra $\mathcal{A}$ is commutative. It constructs a Riemannian spin manifold M from such a spectral triple, so that the latter coincides with the canonical triple on M. The spectral triple is subject to several conditions, first spelled out in [28], and completed later in [80].

The dictionary that translates spaces into algebras and metric structures into Dirac operators can be extended to many geometrical structures. We will note two more important cases, which will be of interest to us in what follows.

Firstly, the Serre-Swan theorem [91] encodes a vector bundle on a compact topological space X into a finite projective module over the algebra $C(X)$. A (right) module $\mathcal{E}$ over an algebra $\mathcal{A}$ is called finite projective if there exists a projection $p \in M_N(\mathcal{A})$, such that $\mathcal{E} \simeq p\mathcal{A}^N$ as right $\mathcal{A}$-modules.

Theorem 1.3. *There is an equivalence between the category of vector bundles on (compact) topological spaces* X *and the category of finite projective modules over* $C(X)$*.*

In view of this, we can think of a finite projective module $\mathcal{E}$ over an algebra $\mathcal{A}$ as describing a noncommutative vector bundle. The theory of characteristic classes for vector bundles on topological spaces is replaced by the so-called Chern-Connes pairing between the cyclic cohomology and K-theory, culminating in the Connes-Moscovici index theorem. The latter can be understood as a generalization of the Atiyah-Singer index theorem to the realm of noncommutative geometry.

Secondly, we consider Hopf algebras, which will be the translation of (Lie) groups. They arise from the following question: *How does one encode the group structure on a (Hausdorff) topological group* G *in terms of the* C^**-algebra* $C_0(G)$. Let us see what happens with the product, inverse and identity of the group on the level of the C^*-algebra $C(G)$, in the case of a compact Lie group G. For convenience we restrict to the algebra of *representative functions* $\mathcal{A} := \mathcal{F}(G)$. Let $\rho : G \to GL_n(\mathbb{C})$ be a continuous finite-dimensional representation of G on $\mathbb{C}^n$. The matrix elements ρ_{ij}, as ρ runs through all finite-dimensional representations of G, generate the subalgebra $\mathcal{F}(G)$ of $C(G)$; from the Peter-Weyl theorem it follows that $\mathcal{F}(G)$ is a dense $*$-subalgebra of $C(G)$. Also, the algebraic tensor product $\mathcal{F}(G) \otimes \mathcal{F}(G)$ can be identified with $\mathcal{F}(G \times G)$.

The multiplication of the group can be seen as a map $G \times G \to G$, given by $(g, h) \to gh$. Since dualization reverses arrows, this becomes a map $\Delta : A \to A \otimes A$ called the *coproduct* and given by

$$\Delta(f)(g, h) = f(gh),$$

where we used $\mathcal{F}(G) \otimes \mathcal{F}(G) \simeq \mathcal{F}(G \times G)$. The property of associativity on G becomes *coassociativity* on A:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{a}$$

stating that $f\big((gh)k\big) = f\big(g(hk)\big)$.

The unit $e \in G$ gives rise to a *counit*, as a map $\epsilon : A \to \mathbb{C}$, given by $\epsilon(f) = f(e)$ and the property $eg = ge = g$ becomes on the algebra level

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta, \tag{b}$$

which reads explicitly $f(ge) = f(eg) = f(g)$.

The inverse map $g \mapsto g^{-1}$, becomes the *antipode* $S : A \to A$, defined by $S(f)(g) = f(g^{-1})$. The property $gg^{-1} = g^{-1}g = e$, becomes on the algebra level:

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = 1_A \epsilon, \tag{c}$$

where $m : A \otimes A \to A$ denotes pointswise multiplication of functions in A.

Definition 1.4. *A* Hopf algebra A *is an algebra* A*, together with two algebra maps* $\Delta : A \otimes A \to A$ *(coproduct),* $\epsilon : A \to \mathbb{C}$ *(counit), and a bijective* $\mathbb{C}$*-linear map* $S : A \to A$ *(antipode), such that equations* (a)–(c) *are satisfied.*

In the category of Hopf algebras, there is also an analogue of the Gelfand transform. This is the Tannaka-Kreĭn duality giving an equivalence between the category of Lie groups to a certain subcategory of the category of commutative Hopf algebras. Hence, in view of this duality, a noncommutative Hopf algebra describes a virtual quantum group.

The rich interaction between classical differential geometry and Lie groups motivates the study of the interaction between the theory of spectral triples and Hopf algebras. Only quite recently, several examples have been constructed. In [38, 40, 77, 87], the noncommutative geometry of the "two-dimensional" spheres of Podleś [79] is described by several spectral triples, being generalized in [61] to quantum flag manifolds. A left-equivariant spectral triple on the quantum group $SU_q(2)$ was constructed in [20] and fully analyzed in [31]. However, this spectral triple does not have a good limit at the classical value of the deformation parameter.

We show how to successfully construct a (noncommutative) 3-dimensional spectral geometry on the manifold of the quantum group $SU_q(2)$, deforming the classical geometry of the "round" sphere $S^3 \simeq SU(2)$. This is done by building a 3^+-summable spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ which is equivariant with respect to a left and a right action of $\mathcal{U}_q(su(2))$. The geometry is isospectral to the classical case in the sense that the spectrum of the operator D is the same as that of the usual Dirac operator on S^3 with the "round" metric.

The possibility of such an isospectral deformation was suggested in [33] but subsequent investigations [47] seemed to rule out this deformation because some of the commutators $[D, x]$, with $x \in \mathcal{A}(SU_q(2))$, failed to extend to bounded operators, a property which is essential to the definition of a spectral triple.

These difficulties are overcome here by constructing on a Hilbert space of spinors $\mathcal{H}$ a spin representation of the algebra $\mathcal{A}(SU_q(2))$ which differs slightly from the one used in [47]. Our spin representation is determined by requiring that it be equivariant with respect to a left and a right action of $\mathcal{U}_q(su(2))$, a condition which is not present in the previous approach. The role of Hopf-algebraic equivariance in producing interesting spectral triples has already met with some success [20, 40, 17]; for a programmatic viewpoint, see [90].

In Chapter 2, we discuss the quantum group $SU_q(2)$ together with its symmetries and construct its left regular representation via equivariance. We then transfer that construction to spinors and consider a class of equivariant "Dirac" operators D on the Hilbert space of spinors. For such an operator D having a classical spectrum, that is, with eigenvalues depending linearly on "total angular momentum", we prove boundedness of the commutators $[D, x]$, for all $x \in \mathcal{A}(SU_q(2))$. In fact, this equivariant Dirac operator is essentially determined by a modified first-order condition, as is shown later on. Since the spectrum is classical, the deformation –from $SU(2)$ to $SU_q(2)$– is isospectral, and in particular the metric dimension of the spectral geometry is 3.

The new feature of the spin geometry of $SU_q(2)$ is the nature of the real structure J, whose existence and properties are addressed in Chapter 3. An equivariant J is constructed by suitably lifting to the Hilbert space of spinors $\mathcal{H}$ the antiunitary Tomita conjugation operator for the left regular representation of $\mathcal{A}(SU_q(2))$. However, this J is not the usual real structure, defined as the tensor product of the Tomita operator with a Pauli matrix; for if it were, the spectral triple (in particular the Dirac operator) would inherit equivariance under the co-opposite symmetry algebra $\mathcal{U}_{1/q}(su(2))$, forcing it to be trivial. Indeed, the equivariant J we shall use does not intertwine the spin representation of $\mathcal{A}(SU_q(2))$ with its commutant, and it is not possible to satisfy all the desirable properties of a real spectral triple as set forth in [28, 49]. This rupture was already observed in [38]; just as in that paper, we must also weaken the first-order requirement on D. In Section 3.2, we rescue the formalism by showing that the commutant and first-order properties nevertheless do hold, up to compact operators. In fact, we identify an ideal of certain trace-class operators containing all commutation defects; these defects vanish in the classical case. An appropriately modified first-order condition is given, which distinguishes Dirac operators with classical spectra.

Finally, in Chapter 4 we discuss the Connes-Moscovici local index formula for the noncommutative geometry on $SU_q(2)$. The introduction of a quantum cosphere bundle S^*_q turns out to simplify the discussion drastically, since it provides a systematic way of working modulo smoothing operators. We work out the dimension spectrum as well as the local cyclic cocycles yielding the index formula. As a simple example, we compute the Fredholm index of D coupled with the unitary representative of the generator of $K_1(\mathcal{A}(SU_q(2)))$.

Chapter 2

The quantum group $SU_q(2)$ and its symmetries

In this chapter, we construct a noncommutative version of the geometry of $S^3 \simeq SU(2)$ as a homogeneous space of $\mathrm{Spin}(4)$:

$$S^3 \simeq SU(2) \simeq \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)},$$

on quotienting out the diagonal $SU(2)$ subgroup of $\mathrm{Spin}(4)$. We thus realize $SU(2)$ as the base space of the principal spin bundle $\mathrm{Spin}(4) \to S^3$, with projection map $(g, h) \mapsto gh^{-1}$. The action of $\mathrm{Spin}(4)$ on $SU(2)$ is given by $(g, h) \cdot x := gxh^{-1}$, and the stabilizer of 1 is the diagonal $SU(2)$ subgroup. We may choose to regard this as a pair of commuting actions of $SU(2)$ on the base space $SU(2)$, apart from the nuance of switching one of them from a right to a left action via the group inversion map. Of course, there is the induced infinitesimal action of the Lie algebra $\mathfrak{spin}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$. We can equip $SU(2)$ with the 'round' metric, so that the action of $\mathfrak{spin}(4)$ is isometrical. The spinor bundle $S = \mathrm{Spin}(4) \times_{SU(2)} \mathbb{C}^2$ is parallelizable: $S \simeq SU(2) \times \mathbb{C}^2$, although one needs to specify the trivialization. The Dirac operator associated to the metric then acts as an unbounded self-adjoint operator on the Hilbert space $L^2(SU(2)) \otimes \mathbb{C}^2$. The isometries in $\mathfrak{spin}(4)$ can be represented on this Hilbert space by unitaries implementing the action of $\mathfrak{spin}(4)$ on $SU(2)$. By definition, the unitaries commute with the Dirac operator.

We will extend the above scheme to the case $q \neq 1$ by defining the quantum group $SU_q(2)$ in terms of a Hopf algebra, together with its symmetries encoded in two actions of the Hopf algebra $\mathcal{U}_q(\mathfrak{su}(2))$. We study the left regular representation of $SU_q(2)$ and find that it is equivariant (in some proper sense) with respect to the two actions of $\mathcal{U}_q(\mathfrak{su}(2))$. Then we introduce the spinor bundle on $SU_q(2)$ and define the Dirac operator that is invariant under the two actions of $\mathcal{U}_q(\mathfrak{su}(2))$.

2.1 Algebraic preliminaries

Definition 2.1. *Let q be a real number with $0 < q < 1$, and let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$-algebra generated by a and b, subject to the following commutation rules:*

$$\begin{gathered} ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b, \\ a^*a + q^2 b^*b = 1, \qquad aa^* + bb^* = 1. \end{gathered} \tag{2.1.1}$$

As a consequence, $a^*b = qba^*$ *and* $a^*b^* = qb^*a^*$. *This becomes a Hopf* $*$*-algebra under the coproduct*

$$\Delta a := a \otimes a - q\, b \otimes b^*,$$
$$\Delta b := b \otimes a^* + a \otimes b,$$

counit $\epsilon(a) = 1$, $\epsilon(b) = 0$, *and antipode* $Sa = a^*$, $Sb = -qb$, $Sb^* = -q^{-1}b^*$, $Sa^* = a$.

Remark 2.2. *Here we follow Majid's "lexicographic convention" [73, 72] (where, with* $c = -qb^*$, $d = a^*$, *a factor of* q *is needed to restore alphabetical order). Another much-used convention is related to ours by* $a \leftrightarrow a^*$, $b \leftrightarrow -b$*; see, for instance, [20, 31].*

Definition 2.3. *The Hopf* $*$*-algebra* $\mathcal{U} = \mathcal{U}_q(su(2))$ *is generated as an algebra by elements* e, f, k, *with* k *invertible, satisfying the relations*

$$ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef), \tag{2.1.2}$$

and its coproduct Δ *is given by*

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f.$$

Its counit ϵ, *antipode* S, *and star structure* * *are given respectively by*

$$\begin{aligned} &\epsilon(k) = 1, && Sk = k^{-1}, && k^* = k, \\ &\epsilon(f) = 0, && Sf = -qf, && f^* = e, \\ &\epsilon(e) = 0, && Se = -q^{-1}e, && e^* = f. \end{aligned}$$

There is an automorphism ϑ *of* $\mathcal{U}_q(su(2))$ *defined on the algebra generators by*

$$\vartheta(k) := k^{-1}, \quad \vartheta(f) := -e, \quad \vartheta(e) := -f. \tag{2.1.3}$$

Remark 2.4. *We recall that there is another convention for the generators of* $\mathcal{U}_q(su(2))$ *in widespread use: see [58], for instance. The handy compendium [60] gives both versions, denoting by* $\breve{U}_q(su(2))$ *the version which we adopt here. However, the parameter* q *of this paper corresponds to* q^{-1} *in [60], or alternatively, we keep the same* q *but exchange* e *and* f *of that book; the equivalence of these procedures is immediate from the above formulas* (2.1.2).

The older literature uses the convention which we follow here, with generators usually written as $K = k$, $X^+ = f$, $X^- = e$.

We employ the so-called "q-integers", defined for each $n \in \mathbb{Z}$ as

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{provided} \quad q \neq 1. \tag{2.1.4}$$

Definition 2.5. *There is a bilinear pairing between* $\mathcal{U}$ *and* $\mathcal{A}$, *defined on generators by*

$$\langle k, a \rangle = q^{\frac{1}{2}}, \quad \langle k, a^* \rangle = q^{-\frac{1}{2}}, \quad \langle e, -qb^* \rangle = \langle f, b \rangle = 1,$$

with all other couples of generators pairing to 0. It satisfies

$$\langle (Sh)^*, x \rangle = \overline{\langle h, x^* \rangle}, \quad \textit{for all} \quad h \in \mathcal{U},\ x \in \mathcal{A}. \tag{2.1.5}$$

We regard $\mathcal{U}$ as a subspace of the linear dual of $\mathcal{A}$ via this pairing. There are canonical left and right $\mathcal{U}$-module algebra structures on $\mathcal{A}$ [96] such that

$$\langle g, h \triangleright x\rangle := \langle gh, x\rangle, \quad \langle g, x \triangleleft h\rangle := \langle hg, x\rangle, \quad \textit{for all} \quad g, h \in \mathcal{U},\ x \in \mathcal{A}.$$

They are given by $h \triangleright x := (\mathrm{id} \otimes h)\,\Delta x$ and $x \triangleleft h := (h \otimes \mathrm{id})\,\Delta x$, or equivalently by

$$h \triangleright x := x_{(1)}\, \langle h, x_{(2)}\rangle, \qquad x \triangleleft h := \langle h, x_{(1)}\rangle\, x_{(2)}, \tag{2.1.6}$$

using the Sweedler notation $\Delta x =: x_{(1)} \otimes x_{(2)}$ with implicit summation.

The right and left actions of $\mathcal{U}$ on $\mathcal{A}$ are mutually commuting:

$$(h \triangleright a) \triangleleft g = (a_{(1)}\, \langle h, a_{(2)}\rangle) \triangleleft g = \langle g, a_{(1)}\rangle\, a_{(2)}\, \langle h, a_{(3)}\rangle = h \triangleright (\langle g, a_{(1)}\rangle\, a_{(2)}) = h \triangleright (a \triangleleft g),$$

and it follows from (2.1.5) that the star structure is compatible with both actions:

$$h \triangleright x^* = ((Sh)^* \triangleright x)^*, \quad x^* \triangleleft h = (x \triangleleft (Sh)^*)^*, \quad \text{for all} \quad h \in \mathcal{U},\ x \in \mathcal{A}.$$

On the generators, the left action is given explicitly by

$$\begin{aligned} &k \triangleright a = q^{\frac12} a, && k \triangleright a^* = q^{-\frac12} a^*, && k \triangleright b = q^{-\frac12} b, && k \triangleright b^* = q^{\frac12} b^*, \\ &f \triangleright a = 0, && f \triangleright a^* = -q b^*, && f \triangleright b = a, && f \triangleright b^* = 0, \\ &e \triangleright a = b, && e \triangleright a^* = 0, && e \triangleright b = 0, && e \triangleright b^* = -q^{-1} a^*, \end{aligned} \tag{2.1.7}$$

and the right action is likewise given by

$$\begin{aligned} &a \triangleleft k = q^{\frac12} a, && a^* \triangleleft k = q^{-\frac12} a^*, && b \triangleleft k = q^{\frac12} b, && b^* \triangleleft k = q^{-\frac12} b^*, \\ &a \triangleleft f = -q b^*, && a^* \triangleleft f = 0, && b \triangleleft f = a^*, && b^* \triangleleft f = 0, \\ &a \triangleleft e = 0, && a^* \triangleleft e = b, && b \triangleleft e = 0, && b^* \triangleleft e = -q^{-1} a. \end{aligned} \tag{2.1.8}$$

We remark in passing that since $\mathcal{A}$ is also a Hopf algebra, the left and right actions are linked through the antipodes:

$$S(Sh \triangleright x) = Sx \triangleleft h.$$

Indeed, it is immediate from (2.1.6) and the duality relation $\langle Sh, y\rangle = \langle h, Sy\rangle$ that

$$S(Sh \triangleright x) = S(x_{(1)})\, \langle Sh, x_{(2)}\rangle = S(x_{(1)})\, \langle h, S(x_{(2)})\rangle = (Sx)_{(2)}\, \langle h, (Sx)_{(1)}\rangle = Sx \triangleleft h.$$

As noted in [48], for instance, the invertible antipode of $\mathcal{U}$ serves to transform the right action $\triangleleft$ into a second left action of $\mathcal{U}$ on $\mathcal{A}$, commuting with the first. Here we also use the automorphism ϑ of (2.1.3), and define

$$h \cdot x := x \triangleleft S^{-1}(\vartheta(h)).$$

Indeed, it is immediate that

$$g \cdot (h \cdot x) = (x \triangleleft S^{-1}(\vartheta h)) \triangleleft S^{-1}(\vartheta g) = x \triangleleft (S^{-1}(\vartheta h) S^{-1}(\vartheta g)) = x \triangleleft (S^{-1}(\vartheta(gh)) = gh \cdot x,$$

i.e., it is a left action. We tabulate this action directly from (2.1.8):

$$\begin{array}{llll} k \cdot a = q^{\frac{1}{2}} a, & k \cdot a^* = q^{-\frac{1}{2}} a^*, & k \cdot b = q^{\frac{1}{2}} b, & k \cdot b^* = q^{-\frac{1}{2}} b^*, \\ f \cdot a = 0, & f \cdot a^* = qb, & f \cdot b = 0, & f \cdot b^* = -a, \\ e \cdot a = -b^*, & e \cdot a^* = 0, & e \cdot b = q^{-1} a^*, & e \cdot b^* = 0. \end{array} \tag{2.1.9}$$

We recall [60] that $\mathcal{A}$ has a vector-space basis consisting of matrix elements of its irreducible corepresentations, $\{ t^l_{mn} : 2l \in \mathbb{N},\ m, n = -l, \dots, l-1, l \}$, where

$$t^0_{00} = 1, \qquad t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} = a, \qquad t^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} = b.$$

The coproduct has the matricial form $\Delta t^l_{mn} = \sum_k t^l_{mk} \otimes t^l_{kn}$, while the product is given by

$$t^j_{rs} t^l_{mn} = \sum_{k=|j-l|}^{j+l} C_q \begin{pmatrix} j & l & k \\ r & m & r+m \end{pmatrix} C_q \begin{pmatrix} j & l & k \\ s & n & s+n \end{pmatrix} t^k_{r+m,s+n}, \tag{2.1.10}$$

where the $C_q(-)$ factors are q-Clebsch–Gordan coefficients [8, 59].

The Haar state on the C^*-completion $C(SU_q(2))$, which we shall denote by ψ, is faithful, and it is determined by setting $\psi(1) := 1$ and $\psi(t^l_{mn}) := 0$ if $l > 0$. (The Haar state is usually denoted by h, but here we use h for a generic element of $\mathcal{U}$ instead.) Let $\mathcal{H}_\psi = L^2(SU_q(2), \psi)$ be the Hilbert space of its GNS representation, denoted π_ψ; then the GNS map $\eta \colon C(SU_q(2)) \to \mathcal{H}_\psi$ is injective and satisfies

$$\|\eta(t^l_{mn})\|^2 = \psi((t^l_{mn})^* \, t^l_{mn}) = \frac{q^{-2m}}{[2l+1]}, \tag{2.1.11}$$

and the vectors $\eta(t^l_{mn})$ are mutually orthogonal. From the formula

$$C_q \begin{pmatrix} l & l & 0 \\ -m & m & 0 \end{pmatrix} = (-1)^{l+m} \frac{q^{-m}}{[2l+1]^{\frac{1}{2}}},$$

we see that the involution in $C(SU_q(2))$ is given by

$$(t^l_{mn})^* = (-1)^{2l+m+n} q^{n-m} \, t^l_{-m,-n}. \tag{2.1.12}$$

In particular, $t^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = -qb^*$ and $t^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} = a^*$, as expected.

An orthonormal basis of $\mathcal{H}_\psi$ is obtained by normalizing the matrix elements, using (2.1.11):

$$|lmn\rangle := q^m \, [2l+1]^{\frac{1}{2}} \, \eta(t^l_{mn}). \tag{2.1.13}$$

2.2 Equivariant representation of $\mathcal{A}(SU_q(2))$

Let $\mathcal{U}$ be a Hopf algebra and let $\mathcal{A}$ be a left $\mathcal{U}$-module algebra. A representation of $\mathcal{A}$ on a vector space V is called $\mathcal{U}$-equivariant if there is also an algebra representation of $\mathcal{U}$ on V, satisfying the following compatibility relation:

$$h(x\xi) = (h_{(1)} \triangleright x)(h_{(2)}\xi), \qquad h \in \mathcal{U},\ x \in \mathcal{A},\ \xi \in V,$$

where $\triangleright$ denotes the Hopf action of $\mathcal{U}$ on $\mathcal{A}$. If $\mathcal{A}$ is instead a right $\mathcal{U}$-module algebra, the appropriate compatibility relation is $x(h\xi) = h_{(1)}((x \triangleleft h_{(2)})\xi)$. Also, if $\mathcal{A}$ is an $\mathcal{U}$-bimodule algebra (carrying commuting left and right Hopf actions of $\mathcal{U}$), one can demand both of these conditions simultaneously for pair of representations of $\mathcal{A}$ and $\mathcal{U}$ on the same vector space V.

In the present case, it turns out to be simpler to consider equivariance under two commuting left Hopf actions, as exemplified in the previous section. We shall first work out in detail a construction of the regular representation of the Hopf algebra $\mathcal{A}(SU_q(2))$, showing how it is determined by its equivariance properties.

We begin with the known representation theory [60] of $\mathcal{U}_q(su(2))$. The irreducible finite dimensional representations σ_l of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and they are given by

$$\begin{aligned} \sigma_l(k)\,|lm\rangle &= q^m\,|lm\rangle, \\ \sigma_l(f)\,|lm\rangle &= \sqrt{[l-m][l+m+1]}\,|l, m+1\rangle, \\ \sigma_l(e)\,|lm\rangle &= \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle, \end{aligned} \tag{2.2.1}$$

where the vectors $|lm\rangle$, for $m = -l, -l+1, \dots, l-1, l$, form a basis for the irreducible $\mathcal{U}$-module V_l, and the brackets denote q-integers as in (2.1.4). Moreover, σ_l is a $*$-representation of $\mathcal{U}_q(su(2))$, with respect to the hermitian scalar product on V_l for which the vectors $|lm\rangle$ are orthonormal.

Remark 2.6. *The irreducible representations* (2.2.1) *coincide with those of* $\breve{U}_q(su(2))$ *in [60], after exchange of* e *and* f *(see Remark 2.4). Further results on the representation theory of* $\mathcal{U}_q(su(2))$ *are taken from [60, Chap. 3] without comment; in particular we use the* q*-Clebsch–Gordan coefficients found therein for the decomposition of tensor product representations. An alternative source for these coefficients is [8], although their* $q^{\frac{1}{2}}$ *is our* q*.*

Definition 2.7. *Let* λ *and* ρ *be mutually commuting representations of the Hopf algebra* $\mathcal{U}$ *on a vector space* V*. A representation* π *of the* $*$*-algebra* $\mathcal{A}$ *on* V *is* (λ, ρ)*-equivariant if the following compatibility relations hold:*

$$\begin{aligned} \lambda(h)\,\pi(x)\xi &= \pi(h_{(1)} \cdot x)\,\lambda(h_{(2)})\xi, \\ \rho(h)\,\pi(x)\xi &= \pi(h_{(1)} \triangleright x)\,\rho(h_{(2)})\xi, \end{aligned} \tag{2.2.2}$$

for all $h \in \mathcal{U}$*,* $x \in \mathcal{A}$ *and* $\xi \in V$*.*

We shall now exhibit an equivariant representation of $\mathcal{A}(SU_q(2))$ on the pre-Hilbert space which is the (algebraic) direct sum

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l.$$

The two $\mathcal{U}_q(su(2))$ symmetries λ and ρ will act on the first and the second leg of the tensor product respectively; both actions will be via the irreps (2.2.1). In other words,

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h) \qquad \text{on } V_l \otimes V_l.$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \dots, l-1, l$; these form an orthonormal basis for $V_l \otimes V_l$, for each fixed l. (As we shall see, this is consistent with our labelling (2.1.13) of the orthonormal basis of $\mathcal{H}_\psi$ in the previous section.) Also, we adopt a shorthand notation:

$$l^\pm := l \pm \tfrac{1}{2}, \quad m^\pm := m \pm \tfrac{1}{2}, \quad n^\pm := n \pm \tfrac{1}{2}.$$

Proposition 2.8. *A (λ, ρ)-equivariant $*$-representation π of $\mathcal{A}(SU_q(2))$ on the Hilbert space V of (2.8) must have the following form:*

$$\begin{aligned} \pi(a)\,|lmn\rangle &= A^+_{lmn}|l^+m^+n^+\rangle + A^-_{lmn}|l^-m^+n^+\rangle, \\ \pi(b)\,|lmn\rangle &= B^+_{lmn}|l^+m^+n^-\rangle + B^-_{lmn}|l^-m^+n^-\rangle, \\ \pi(a^*)\,|lmn\rangle &= \widetilde{A}^+_{lmn}|l^+m^-n^-\rangle + \widetilde{A}^-_{lmn}|l^-m^-n^-\rangle, \\ \pi(b^*)\,|lmn\rangle &= \widetilde{B}^+_{lmn}|l^+m^-n^+\rangle + \widetilde{B}^-_{lmn}|l^-m^-n^+\rangle, \end{aligned} \tag{2.2.3}$$

where the constants $A^\pm_{lmn}$ and $B^\pm_{lmn}$ are, up to phase factors depending only on l, given by

$$\begin{aligned} A^+_{lmn} &= q^{(-2l+m+n-1)/2}\left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]}\right)^{\frac{1}{2}}, \\ A^-_{lmn} &= q^{(2l+m+n+1)/2}\left(\frac{[l-m][l-n]}{[2l][2l+1]}\right)^{\frac{1}{2}}, \\ B^+_{lmn} &= q^{(m+n-1)/2}\left(\frac{[l+m+1][l-n+1]}{[2l+1][2l+2]}\right)^{\frac{1}{2}}, \\ B^-_{lmn} &= -q^{(m+n-1)/2}\left(\frac{[l-m][l+n]}{[2l][2l+1]}\right)^{\frac{1}{2}}, \end{aligned} \tag{2.2.4}$$

and the other coefficients are complex conjugates of these, namely,

$$\widetilde{A}^\pm_{lmn} = (A^\mp_{l^\pm m^- n^-})^\star, \qquad \widetilde{B}^\pm_{lmn} = (B^\mp_{l^\pm m^- n^+})^\star. \tag{2.2.5}$$

Proof. First of all, notice that hermiticity of π entails the relations (2.2.5). We now use the covariance properties (2.2.2). When $h = k$, they simplify to

$$\lambda(k)\,\pi(x)\,\xi = \pi(k \cdot x)\,\lambda(k)\,\xi, \qquad \rho(k)\,\pi(x)\,\xi = \pi(k \triangleright x)\,\rho(k)\,\xi. \tag{2.2.6}$$

Thus, for instance, when $x = a$ we find the relations

$$\begin{aligned} \lambda(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac{1}{2}}a)\big(q^m|lmn\rangle\big) = q^{m+\frac{1}{2}}\pi(a)\,|lmn\rangle, \\ \rho(k)\,\pi(a)\,|lmn\rangle &= \pi(q^{\frac{1}{2}}a)\big(q^n|lmn\rangle\big) = q^{n+\frac{1}{2}}\pi(a)\,|lmn\rangle, \end{aligned}$$

where we have invoked $k \cdot a = k \triangleright a = q^{\frac{1}{2}}a$. We conclude that $\pi(a)\,|lmn\rangle$ must lie in the closed span of the basis vectors $|l'm^+n^+\rangle$. A similar argument with $x = b$ in (2.2.6) shows that $\pi(b)$ increments n and decrements m by $\frac{1}{2}$, since $k \cdot b = q^{\frac{1}{2}}b$ while $k \triangleright b = q^{-\frac{1}{2}}b$. The analogous behaviour for $x = a^*$ and $x = b^*$ follows in the same way from (2.1.7) and (2.1.9).

Thus, $\pi(a)|lmn\rangle$ is a (possibly infinite) sum

$$\pi(a)\,|lmn\rangle = \textstyle\sum_{l'} C_{l'lmn}\,|l'm^+n^+\rangle, \tag{2.2.7}$$

where the sum runs over nonnegative half-integers $l' = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$.

Next, we call on (2.2.2) with $h = f$, $x = a$, to get

$$\lambda(f)\,\pi(a)\xi = \pi(f \cdot a)\,\lambda(k)\xi + \pi(k^{-1} \cdot a)\,\lambda(f)\xi = q^{-\frac{1}{2}}\pi(a)\,\lambda(f)\xi,$$

on account of (2.1.7). Consequently, $\lambda(f)^r\pi(a) = q^{-r/2}\pi(a)\,\lambda(f)^r$ for $r = 1, 2, 3, \dots$. On applying $\lambda(f)^r$ to both sides of (2.2.7), we obtain on the left hand side a multiple of $\pi(a)\,|l, m+r, n\rangle$, which vanishes for $m+r > l$; and on the right hand side we get $\sum_{l'} C_{l'lmn}\, D_{l'mr}\,|l', m^+{+}r, n^+\rangle$, where $D_{l'mr} \neq 0$ as long as $m + r + \frac{1}{2} \leq l'$. We conclude that $C_{l'lmn} = 0$ for $l' > l + \frac{1}{2}$, by linear independence of these summands.

In order to get a lower bound on the range of the index l' in (2.2.7), we consider the analogous expansion $\pi(a^*)\,|lmn\rangle = \sum_{l'} \widetilde{C}_{l'lmn}\,|l'm^-n^-\rangle$. Now $\lambda(e)^r\pi(a^*)\,|lmn\rangle = q^{r/2}\pi(a^*)\,\lambda(e)^r\,|lmn\rangle \propto \pi(a^*)\,|l, m-r, n\rangle$ vanishes for $m - r < -l$; while $\lambda(e)^r|l'm^-n^-\rangle = F_{l'mr}\,|l', m^- - r, n^-\rangle$ with $F_{l'mr} \neq 0$ for $m - r - \frac{1}{2} \geq -l'$. Again we conclude that $\widetilde{C}_{l'lmn} = 0$ for $l' > l + \frac{1}{2}$. However, since π is a $*$-representation, the matrix element $\langle l'm'n' \mid \pi(a) \mid lmn\rangle$ is the complex conjugate of $\langle lmn \mid \pi(a^*) \mid l'm'n'\rangle$, which vanishes for $l > l' + \frac{1}{2}$, so that the indices in (2.2.7) satisfy $l - \frac{1}{2} \leq l' \leq l + \frac{1}{2}$. Clearly, $l' = l$ is ruled out because $l - m$ and $l' - m \pm \frac{1}{2}$ must both be integers.

Therefore, $\pi(a)$ and also $\pi(a^*)$ have the structure indicated in (2.2.3). A parallel argument shows the corresponding result for $\pi(b)$ and $\pi(b^*)$.

The coefficients which appear in (2.2.4) may be determined by further application of the equivariance relations. Since $f \triangleright a = 0$ and $e \triangleright b = 0$, then by applying $\rho(f)$ and $\rho(e)$ to the first two relations of (2.2.3), we obtain the following recursion relations for the coefficients $A^\pm_{lmn}$, $B^\pm_{lmn}$:

$$\begin{aligned}
A^+_{lmn}[l+n+2]^{\frac{1}{2}} &= q^{-\frac{1}{2}} A^+_{lm,n+1}[l+n+1]^{\frac{1}{2}}, \\
A^-_{lmn}[l-n-1]^{\frac{1}{2}} &= q^{-\frac{1}{2}} A^-_{lm,n+1}[l-n]^{\frac{1}{2}}, \\
B^+_{lmn}[l-n+2]^{\frac{1}{2}} &= q^{\frac{1}{2}} B^+_{lm,n-1}[l-n+1]^{\frac{1}{2}}, \\
B^-_{lmn}[l+n-1]^{\frac{1}{2}} &= q^{\frac{1}{2}} B^-_{lm,n-1}[l+n]^{\frac{1}{2}}.
\end{aligned}$$

Then, applying $\lambda(f)$ to the same pair of equations, we further find that

$$\begin{aligned}
A^+_{lmn}[l+m+2]^{\frac{1}{2}} &= q^{-\frac{1}{2}} A^+_{l,m+1,n}[l+m+1]^{\frac{1}{2}}, \\
A^-_{lmn}[l-m-1]^{\frac{1}{2}} &= q^{-\frac{1}{2}} A^-_{l,m+1,n}[l-m]^{\frac{1}{2}}, \\
B^+_{lmn}[l+m+2]^{\frac{1}{2}} &= q^{-\frac{1}{2}} B^+_{l,m+1,n}[l+m+1]^{\frac{1}{2}}, \\
B^-_{lmn}[l-m-1]^{\frac{1}{2}} &= q^{-\frac{1}{2}} B^-_{l,m+1,n}[l-m]^{\frac{1}{2}}.
\end{aligned} \tag{2.2.8a}$$

These recursions are explicitly solved by

$$\begin{aligned}
A^+_{lmn} &= q^{(m+n)/2}[l+m+1]^{\frac{1}{2}}[l+n+1]^{\frac{1}{2}}\ a^+_l, \\
A^-_{lmn} &= q^{(m+n)/2}[l-m]^{\frac{1}{2}}[l-n]^{\frac{1}{2}}\ a^-_l, \\
B^+_{lmn} &= q^{(m+n)/2}[l+m+1]^{\frac{1}{2}}[l-n+1]^{\frac{1}{2}}\ b^+_l, \\
B^-_{lmn} &= q^{(m+n)/2}[l-m]^{\frac{1}{2}}[l+n]^{\frac{1}{2}}\ b^-_l,
\end{aligned} \tag{2.2.8b}$$

where $a_l^\pm$, $b_l^\pm$ depend only on l.

Once more, we apply the equivariance relations (2.2.2); this time, we use

$$\rho(e)\pi(a) = \pi(e \triangleright a)\rho(k) + \pi(k^{-1} \triangleright a)\rho(e) = \pi(b)\rho(k) + q^{-\frac{1}{2}}\pi(a)\rho(e). \tag{2.2.9}$$

Applied to $|lmn\rangle$, it yields an equation between linear combinations of $|l^+m^+n^-\rangle$ and $|l^-m^+n^-\rangle$; equating coefficients, we find

$$b_l^+ = q^l a_l^+, \qquad b_l^- = -q^{-l-1} a_l^-.$$

Furthermore, applying also to $|lmn\rangle$ the relation

$$\begin{aligned} \lambda(e)\pi(b) &= \pi(e \cdot b)\lambda(k) + \pi(k^{-1} \cdot b)\lambda(e) \\ &= q^{-1}\pi(a^*)\lambda(k) + q^{-\frac{1}{2}}\pi(b)\lambda(e), \end{aligned} \tag{2.2.10}$$

we get, after a little simplification and use of (2.2.5),

$$(a^-_{l+\frac{1}{2}})^\star = q^{2l+\frac{3}{2}}\, a_l^+.$$

It remains only to determine the parameters a_l^+. We turn to the algebra commutation relation $ba = qab$ and compare coefficients in the expansion of $\pi(b)\pi(a)\,|lmn\rangle = q\,\pi(a)\pi(b)\,|lmn\rangle$. Those of $|l+1, m+1, n\rangle$ and $|l-1, m+1, n\rangle$ already coincide; but from the $|l, m+1, n\rangle$ terms, we get the identity

$$q[2l+2]\,|a_l^+|^2 = [2l]\,|a^+_{l-\frac{1}{2}}|^2.$$

This can be solved immediately, to give

$$a_l^+ = \frac{C\zeta_l\, q^{-l}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}},$$

where C is a positive constant, and ζ_l is a phase factor which can be absorbed in the basis vectors $|lmn\rangle$; hereinafter we take $\zeta_l = 1$ (we comment on that choice at the end of the section).

Finally, from the relation $a^*a + q^2b^*b = 1$ we obtain

$$1 = \langle 000\,|\,\pi(a^*a + q^2b^*b)\,|\,000\rangle = |a_0^+|^2 + q^2|b_0^+|^2 = (1+q^2)C^2/[2] = q\,C^2,$$

and thus $C = q^{-\frac{1}{2}}$. We therefore find that

$$a_l^+ = \frac{q^{-l-\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}}, \qquad a_l^- = \frac{q^{l+\frac{1}{2}}}{[2l]^{\frac{1}{2}}[2l+1]^{\frac{1}{2}}},$$

$$b_l^+ = \frac{q^{-\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}[2l+2]^{\frac{1}{2}}}, \qquad b_l^- = -\frac{q^{-\frac{1}{2}}}{[2l]^{\frac{1}{2}}[2l+1]^{\frac{1}{2}}},$$

and substitution in (2.2.8b) yields the coefficients (2.2.4). □

It is easy to check that the formulas (2.2.3) give precisely the left regular representation π_ψ of $\mathcal{A}(SU_q(2))$. Indeed, that representation was implicitly given already by the product rule

(2.1.10). From [8, (3.53)] we obtain

$$\begin{aligned}
C_q\begin{pmatrix} \frac12 & l & l^+ \\ \frac12 & m & m^+ \end{pmatrix} &= q^{-\frac12(l-m)} \frac{[l+m+1]^{\frac12}}{[2l+1]^{\frac12}}, \\
C_q\begin{pmatrix} \frac12 & l & l^+ \\ -\frac12 & m & m^- \end{pmatrix} &= q^{\frac12(l+m)} \frac{[l-m+1]^{\frac12}}{[2l+1]^{\frac12}}, \\
C_q\begin{pmatrix} \frac12 & l & l^- \\ \frac12 & m & m^+ \end{pmatrix} &= q^{\frac12(l+m+1)} \frac{[l-m]^{\frac12}}{[2l+1]^{\frac12}}, \\
C_q\begin{pmatrix} \frac12 & l & l^- \\ -\frac12 & m & m^- \end{pmatrix} &= -q^{-\frac12(l-m+1)} \frac{[l+m]^{\frac12}}{[2l+1]^{\frac12}}.
\end{aligned} \tag{2.2.11}$$

By setting $j = r = s = \frac12$ in (2.1.10), we find

$$\pi_\psi(a)\eta(t^l_{mn}) = \sum_{\pm} C_q\begin{pmatrix} \frac12 & l & l^\pm \\ \frac12 & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac12 & l & l^\pm \\ \frac12 & n & n^+ \end{pmatrix} \eta(t^{l^\pm}_{m^+n^+}).$$

Taking the normalization (2.1.13) into account, this becomes

$$\begin{aligned}
\pi_\psi(a)|lmn\rangle &= q^{-\frac12} \frac{[2l+1]^{\frac12}}{[2l+2]^{\frac12}} C_q\begin{pmatrix} \frac12 & l & l^+ \\ \frac12 & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac12 & l & l^+ \\ \frac12 & n & n^+ \end{pmatrix} |l^+m^+n^+\rangle \\
&\quad + q^{-\frac12} \frac{[2l+1]^{\frac12}}{[2l]^{\frac12}} C_q\begin{pmatrix} \frac12 & l & l^- \\ \frac12 & m & m^+ \end{pmatrix} C_q\begin{pmatrix} \frac12 & l & l^- \\ \frac12 & n & n^+ \end{pmatrix} |l^-m^+n^+\rangle \\
&= q^{\frac12(-2l+m+n-1)} \frac{[l+m+1]^{\frac12}[l+n+1]^{\frac12}}{[2l+1]^{\frac12}[2l+2]^{\frac12}} |l^+m^+n^+\rangle \\
&\quad + q^{\frac12(2l+m+n+1)} \frac{[l-m]^{\frac12}[l-n]^{\frac12}}{[2l]^{\frac12}[2l+1]^{\frac12}} |l^-m^+n^+\rangle \\
&= \pi(a)|lmn\rangle.
\end{aligned}$$

A similar calculation, using (2.2.11) again, shows that $\pi(b) = \pi_\psi(b)$. Since a and b generate $\mathcal{A}$ as a $*$-algebra, we conclude that $\pi = \pi_\psi$. (It should be noted that π_ψ has already been exhibited in [20] in the same way, albeit with different convention for the algebra generators.)

The identification (2.1.13) embeds the pre-Hilbert space V densely in the Hilbert space $\mathcal{H}_\psi$, and the representation π_ψ extends to the GNS representation of $C(SU_q(2))$ on $\mathcal{H}_\psi$, as described by the Peter–Weyl theorem [60, 96]. In like manner, all other representations of $\mathcal{A}$ exhibited in this paper extend to C^*-algebra representations of $C(SU_q(2))$ on the appropriate Hilbert spaces.

The only lack of uniqueness in the proof of Proposition 2.8 involved the choice of the phase factors ζ_l; if Z is the linear operator on V which multiplies vectors in $V_l \otimes V_l$ by ζ_l, then Z commutes with each $\lambda(h)$ and $\rho(g)$, and extends to a unitary operator on $\mathcal{H}_\psi$. In other words, any (λ, ρ)-equivariant representation π extends to $\mathcal{H}_\psi$ and is unitarily equivalent to the left regular representation. The (standard) choice $\zeta_l = 1$ ensures that *all coefficients* $A^\pm_{lmn}$ *and* $B^\pm_{lmn}$ *are real*: it is indeed an extension of the Conden–Shortley phase convention [9].

2.3 The spin representation

The left regular representation π of $\mathcal{A}$, constructed in the previous section, can be amplified to $\pi' = \pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$. The representation theory of $\mathcal{U}$ (and the corepresentation theory of $\mathcal{A}$) follows the same pattern as for $q = 1$; only the Clebsch–Gordan coefficients need to be modified [59] when $q \neq 1$.

To fix notations, we take

$$W := V \otimes \mathbb{C}^2 = V \otimes V_{\frac{1}{2}},$$

and its Clebsch–Gordan decomposition is the (algebraic) direct sum

$$W = \left(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l\right) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j). \tag{2.3.1}$$

We rename the finite-dimensional spaces on the right hand side as

$$W = W_0^{\uparrow} \oplus \bigoplus_{2j \geq 1} W_j^{\uparrow} \oplus W_j^{\downarrow}, \tag{2.3.2}$$

where $W_j^{\uparrow} \simeq V_{j+\frac{1}{2}} \otimes V_j$ and $W_j^{\downarrow} \simeq V_{j-\frac{1}{2}} \otimes V_j$, so that

$$\begin{aligned} \dim W_j^{\uparrow} &= (2j+1)(2j+2), & &\text{for } j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots , \\ \dim W_j^{\downarrow} &= 2j(2j+1), & &\text{for } j = \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots . \end{aligned} \tag{2.3.3}$$

Definition 2.9. *We amplify the representation ρ of $\mathcal{U}$ on V to $\rho' = \rho \otimes \mathrm{id}$ on $W = V \otimes \mathbb{C}^2$. However, we replace λ on V by its tensor product with $\sigma_{\frac{1}{2}}$ on $\mathbb{C}^2$:*

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}).$$

It is straightforward to check that the representations λ' and ρ' on W commute, and that the representation π' of $\mathcal{A}$ on W is (λ', ρ')-equivariant:

$$\begin{aligned} \lambda'(h)\, \pi'(x)\psi &= \pi'(h_{(1)} \cdot x)\, \lambda'(h_{(2)})\psi, \\ \rho'(h)\, \pi'(x)\psi &= \pi'(h_{(1)} \triangleright x)\, \rho'(h_{(2)})\psi, \end{aligned} \tag{2.3.4}$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\psi \in W$.

To determine an explicit basis for W which is well-adapted to (λ', ρ')-equivariance, consider the following vectors in $V \otimes \mathbb{C}^2$:

$$\begin{aligned} c_{lm} |lmn\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + s_{lm} |l, m-1, n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \\ -s_{lm} |lmn\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + c_{lm} |l, m-1, n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \end{aligned}$$

where

$$c_{lm} := q^{-(l+m)/2} \frac{[l-m+1]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}, \qquad s_{lm} := q^{(l-m+1)/2} \frac{[l+m]^{\frac{1}{2}}}{[2l+1]^{\frac{1}{2}}}$$

are the q-Clebsch–Gordan coefficients corresponding to the above decomposition (2.3.1), satisfying $c_{lm}^2 + s_{lm}^2 = 1$. These are eigenvectors for $\lambda'(C_q)$, where $C_q := qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2 ef$

is the Casimir element of $\mathcal{U}$, with respective eigenvalues $q^{2l+2} + q^{-2l-2}$ and $q^{2l} + q^{-2l}$. Thus, to get a good basis, one should offset the index l by $\pm\frac{1}{2}$ (as is also suggested by the decomposition (2.3.2) of W).

For $j = l + \frac{1}{2}$, $\mu = m - \frac{1}{2}$, with $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$, let

$$|j\mu n{\downarrow}\rangle := C_{j\mu}\, |j^-\mu^+ n\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + S_{j\mu}\, |j^-\mu^- n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle; \tag{2.3.5a}$$

and for $j = l - \frac{1}{2}$, $\mu = m - \frac{1}{2}$, with $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$, let

$$|j\mu n{\uparrow}\rangle := -S_{j+1,\mu}\, |j^+\mu^+ n\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + C_{j+1,\mu}\, |j^+\mu^- n\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \tag{2.3.5b}$$

where the coefficients are now

$$C_{j\mu} := q^{-(j+\mu)/2} \frac{[j-\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \qquad S_{j\mu} := q^{(j-\mu)/2} \frac{[j+\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}. \tag{2.3.5c}$$

Notice that there are no $\downarrow$ vectors for $j = 0$. It is now straightforward, though tedious, to verify that these vectors are orthonormal bases for the respective subspaces $W_j^{\downarrow}$ and $W_j^{\uparrow}$.

The Hilbert space of spinors is $\mathcal{H} := \mathcal{H}_\psi \otimes \mathbb{C}^2$, which is just the completion of the algebraic direct sum (2.3.2). We may decompose it as $\mathcal{H} = \mathcal{H}^{\uparrow} \oplus \mathcal{H}^{\downarrow}$, where $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$ are the respective completions of $\bigoplus_{2j\geq 0} W_j^{\uparrow}$ and $\bigoplus_{2j\geq 1} W_j^{\downarrow}$.

Lemma 2.3.1. *The basis vectors* $|j\mu n{\uparrow}\rangle$ *and* $|j\mu n{\downarrow}\rangle$ *are joint eigenvectors for* $\lambda'(k)$ *and* $\rho'(k)$*, and* e, f *are represented on them as ladder operators:*

$$\begin{aligned} \lambda'(k)|j\mu n{\uparrow}\rangle &= q^{\mu}|j\mu n{\uparrow}\rangle, & \rho'(k)|j\mu n{\uparrow}\rangle &= q^{n}|j\mu n{\uparrow}\rangle, \\ \lambda'(k)|j\mu n{\downarrow}\rangle &= q^{\mu}|j\mu n{\downarrow}\rangle, & \rho'(k)|j\mu n{\downarrow}\rangle &= q^{n}|j\mu n{\downarrow}\rangle. \end{aligned} \tag{2.3.6a}$$

Moreover,

$$\begin{aligned} \lambda'(f)|j\mu n{\uparrow}\rangle &= [j-\mu]^{\frac{1}{2}}[j+\mu+1]^{\frac{1}{2}}|j,\mu+1,n{\uparrow}\rangle, \\ \lambda'(e)|j\mu n{\uparrow}\rangle &= [j+\mu]^{\frac{1}{2}}[j-\mu+1]^{\frac{1}{2}}|j,\mu-1,n{\uparrow}\rangle, \\ \lambda'(f)|j\mu n{\downarrow}\rangle &= [j-\mu]^{\frac{1}{2}}[j+\mu+1]^{\frac{1}{2}}|j,\mu+1,n{\downarrow}\rangle, \\ \lambda'(e)|j\mu n{\downarrow}\rangle &= [j+\mu]^{\frac{1}{2}}[j-\mu+1]^{\frac{1}{2}}|j,\mu-1,n{\downarrow}\rangle, \end{aligned} \tag{2.3.6b}$$

and

$$\begin{aligned} \rho'(f)|j\mu n{\uparrow}\rangle &= [j-n+\tfrac{1}{2}]^{\frac{1}{2}}[j+n+\tfrac{3}{2}]^{\frac{1}{2}}|j\mu,n+1,{\uparrow}\rangle, \\ \rho'(e)|j\mu n{\uparrow}\rangle &= [j+n+\tfrac{1}{2}]^{\frac{1}{2}}[j-n+\tfrac{3}{2}]^{\frac{1}{2}}|j\mu,n-1,{\uparrow}\rangle, \\ \rho'(f)|j\mu n{\downarrow}\rangle &= [j-n-\tfrac{1}{2}]^{\frac{1}{2}}[j+n+\tfrac{1}{2}]^{\frac{1}{2}}|j\mu,n+1,{\downarrow}\rangle, \\ \rho'(e)|j\mu n{\downarrow}\rangle &= [j+n-\tfrac{1}{2}]^{\frac{1}{2}}[j-n+\tfrac{1}{2}]^{\frac{1}{2}}|j\mu,n-1,{\downarrow}\rangle. \end{aligned} \tag{2.3.6c}$$

The representation π' can now be computed in the new spinor basis by conjugating the form of $\pi \otimes \mathrm{id}$ found in Proposition 2.8 by the basis transformation (2.3.5). However, it is more instructive to derive these formulas from the property of (λ', ρ')-equivariance. First, we introduce a handy notation.

Definition 2.10. *For* $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ *with* $\mu = -j, \ldots, j$ *and* $n = -j - \frac{1}{2}, \ldots, j + \frac{1}{2}$, *we juxtapose the pair of spinors*

$$|j\mu n\rangle\rangle := \begin{pmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{pmatrix},$$

with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ *or* $j = 0$. *Furthermore, a matrix with scalar entries,*

$$A = \begin{pmatrix} A_{\uparrow\uparrow} & A_{\uparrow\downarrow} \\ A_{\downarrow\uparrow} & A_{\downarrow\downarrow} \end{pmatrix},$$

is understood to act on $|j\mu n\rangle\rangle$ *by the rule:*

$$\begin{aligned} A|j\mu n\uparrow\rangle &= A_{\uparrow\uparrow}|j\mu n\uparrow\rangle + A_{\downarrow\uparrow}|j\mu n\downarrow\rangle, \\ A|j\mu n\downarrow\rangle &= A_{\downarrow\downarrow}|j\mu n\downarrow\rangle + A_{\uparrow\downarrow}|j\mu n\uparrow\rangle. \end{aligned} \tag{2.3.7}$$

Proposition 2.11. *The representation* $\pi' := \pi \otimes \mathrm{id}$ *of* $\mathcal{A}$ *is given by*

$$\begin{aligned} \pi'(a)\,|j\mu n\rangle\rangle &= \alpha^+_{j\mu n}|j^+\mu^+n^+\rangle\rangle + \alpha^-_{j\mu n}|j^-\mu^+n^+\rangle\rangle, \\ \pi'(b)\,|j\mu n\rangle\rangle &= \beta^+_{j\mu n}|j^+\mu^+n^-\rangle\rangle + \beta^-_{j\mu n}|j^-\mu^+n^-\rangle\rangle, \\ \pi'(a^*)\,|j\mu n\rangle\rangle &= \tilde\alpha^+_{j\mu n}|j^+\mu^-n^-\rangle\rangle + \tilde\alpha^-_{j\mu n}|j^-\mu^-n^-\rangle\rangle, \\ \pi'(b^*)\,|j\mu n\rangle\rangle &= \tilde\beta^+_{j\mu n}|j^+\mu^-n^+\rangle\rangle + \tilde\beta^-_{j\mu n}|j^-\mu^-n^+\rangle\rangle, \end{aligned} \tag{2.3.8}$$

where $\alpha^\pm_{j\mu n}$ *and* $\beta^\pm_{j\mu n}$ *are, up to phase factors depending only on* j, *the following triangular* 2×2 *matrices:*

$$\begin{aligned} \alpha^+_{j\mu n} &= q^{(\mu+n-\frac{1}{2})/2}[j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} q^{-j-\frac{1}{2}}\,\frac{[j+n+\frac{3}{2}]^{1/2}}{[2j+2]} & 0 \\ q^{\frac{1}{2}}\,\frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1]\,[2j+2]} & q^{-j}\,\frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1]} \end{pmatrix}, \\ \alpha^-_{j\mu n} &= q^{(\mu+n-\frac{1}{2})/2}[j-\mu]^{\frac{1}{2}} \begin{pmatrix} q^{j+1}\,\frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1]} & -q^{\frac{1}{2}}\,\frac{[j+n+\frac{1}{2}]^{1/2}}{[2j]\,[2j+1]} \\ 0 & q^{j+\frac{1}{2}}\,\frac{[j-n-\frac{1}{2}]^{1/2}}{[2j]} \end{pmatrix}, \\ \beta^+_{j\mu n} &= q^{(\mu+n-\frac{1}{2})/2}[j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} \frac{[j-n+\frac{3}{2}]^{1/2}}{[2j+2]} & 0 \\ -q^{-j-1}\,\frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1]\,[2j+2]} & q^{-\frac{1}{2}}\,\frac{[j-n+\frac{1}{2}]^{1/2}}{[2j+1]} \end{pmatrix}, \\ \beta^-_{j\mu n} &= q^{(\mu+n-\frac{1}{2})/2}[j-\mu]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}}\,\frac{[j+n+\frac{1}{2}]^{1/2}}{[2j+1]} & -q^{j}\,\frac{[j-n+\frac{1}{2}]^{1/2}}{[2j]\,[2j+1]} \\ 0 & -\frac{[j+n-\frac{1}{2}]^{1/2}}{[2j]} \end{pmatrix}, \end{aligned} \tag{2.3.9}$$

and the remaining matrices are the hermitian conjugates

$$\tilde\alpha^\pm_{j\mu n} = (\alpha^\mp_{j^\pm\mu^-n^-})^\dagger, \qquad \tilde\beta^\pm_{j\mu n} = (\beta^\mp_{j^\pm\mu^-n^+})^\dagger.$$

Proof. The proof of Proposition 2.8 applies with minor changes. From the analogues of (2.2.6) and the relations $\lambda'(f)\pi'(a) = q^{-\frac{1}{2}}\pi'(a)\,\lambda'(f)$ and $\lambda'(e)\pi'(a^*) = q^{\frac{1}{2}}\pi'(a^*)\,\lambda'(e)$, applied to the spinors $|j\mu n\rangle\rangle$, together with the formulas (2.3.6a) and (2.3.6b), we determine that $\pi'(a)$ has the indicated form, where the $\alpha^{\pm}_{j\mu n}$ are 2×2 matrices. The other cases of (2.3.8) are handled similarly.

To compute these matrices, we again use the commutation relations of $\lambda'(f)$ with $\pi'(a)$ and $\pi'(b)$ to establish recurrence relations, analogous to (2.2.8a), which yield

$$\alpha^+_{j\mu n} = q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12}\ A^+_{jn}, \qquad \alpha^-_{j\mu n} = q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12}\ A^-_{jn},$$
$$\beta^+_{j\mu n} = q^{(\mu+n-\frac12)/2}[j+\mu+1]^{\frac12}\ B^+_{jn}, \qquad \beta^-_{j\mu n} = q^{(\mu+n-\frac12)/2}[j-\mu]^{\frac12}\ B^-_{jn}.$$

The new matrices $A^{\pm}_{jn}$, $B^{\pm}_{jn}$ may be further refined by using commutation relations involving $\rho'(f)$ and $\rho'(e)$. For instance, $\rho'(f)\pi'(a) = q^{-\frac12}\pi'(a)\,\rho'(f)$ entails

$$\begin{pmatrix} [j-n+\frac12]^{\frac12}[j+n+\frac52]^{\frac12} & 0 \\ 0 & [j-n-\frac12]^{\frac12}[j+n+\frac32]^{\frac12} \end{pmatrix} A^+_{jn}$$
$$= A^+_{j,n+1} \begin{pmatrix} [j-n+\frac12]^{\frac12}[j+n+\frac32]^{\frac12} & 0 \\ 0 & [j-n-\frac12]^{\frac12}[j+n+\frac12]^{\frac12} \end{pmatrix}.$$

This yields four recurrence relations for the entries of A^+_{jn}, one of which has only the trivial solution; we conclude that

$$A^+_{jn} = \begin{pmatrix} [j+n+\frac32]^{\frac12}\, a^+_{j\uparrow\uparrow} & 0 \\ [j-n+\frac12]^{\frac12}\, a^+_{j\downarrow\uparrow} & [j+n+\frac12]^{\frac12}\, a^+_{j\downarrow\downarrow} \end{pmatrix},$$

where the $a^+_{j\updownarrow\updownarrow}$ are scalars depending only on j. In a similar fashion, we arrive at

$$A^-_{jn} = \begin{pmatrix} [j-n+\frac12]^{\frac12}\, a^-_{j\uparrow\uparrow} & [j+n+\frac12]^{\frac12}\, a^-_{j\uparrow\downarrow} \\ 0 & [j-n-\frac12]^{\frac12}\, a^-_{j\downarrow\downarrow} \end{pmatrix},$$

$$B^+_{jn} = \begin{pmatrix} [j-n+\frac32]^{\frac12}\, b^+_{j\uparrow\uparrow} & 0 \\ [j+n+\frac12]^{\frac12}\, b^+_{j\downarrow\uparrow} & [j-n+\frac12]^{\frac12}\, b^+_{j\downarrow\downarrow} \end{pmatrix},$$

$$B^-_{jn} = \begin{pmatrix} [j+n+\frac12]^{\frac12}\, b^-_{j\uparrow\uparrow} & [j-n+\frac12]^{\frac12}\, b^-_{j\uparrow\downarrow} \\ 0 & [j+n-\frac12]^{\frac12}\, b^-_{j\downarrow\downarrow} \end{pmatrix}.$$

The analogue of (2.2.9) leads quickly to the relations

$$\begin{aligned} b^+_{j\uparrow\uparrow} &= q^{j+\frac12}\, a^+_{j\uparrow\uparrow}, & b^+_{j\downarrow\uparrow} &= -q^{-j-\frac32}\, a^+_{j\downarrow\uparrow}, & b^+_{j\downarrow\downarrow} &= q^{j-\frac12}\, a^+_{j\downarrow\downarrow}, \\ b^-_{j\uparrow\uparrow} &= -q^{-j-\frac32}\, a^-_{j\uparrow\uparrow}, & b^-_{j\uparrow\downarrow} &= q^{j-\frac12}\, a^-_{j\uparrow\downarrow}, & b^-_{j\downarrow\downarrow} &= -q^{-j-\frac12}\, a^-_{j\downarrow\downarrow}. \end{aligned} \tag{2.3.10}$$

Next, from the analogue of (2.2.10) we get

$$(a^-_{j+\frac12,\uparrow\uparrow})^\star = q^{2j+2} a^+_{j\uparrow\uparrow}, \qquad (a^-_{j+\frac12,\uparrow\downarrow})^\star = -a^+_{j\downarrow\uparrow}, \qquad (a^-_{j+\frac12,\downarrow\downarrow})^\star = q^{2j+1} a^+_{j\downarrow\downarrow}.$$

The $a^+_{j\updownarrow\updownarrow}$ parameters may be determined from $\pi'(b)\pi'(a)\,|j\mu n\rangle\rangle = q\,\pi'(a)\pi'(b)\,|j\mu n\rangle\rangle$. The coefficients of $|j\pm1, \mu+1, n\rangle\rangle$ yield only the relation

$$[2j+1]\, a^+_{j+\frac12,\downarrow\downarrow} a^+_{j\downarrow\uparrow} = [2j+3]\, a^+_{j+\frac12,\downarrow\uparrow} a^+_{j\uparrow\uparrow}. \tag{2.3.11}$$

From the $|j, \mu+1, n\rangle\!\rangle$ terms, we obtain

$$B^-_{j^+n^+} A^+_{jn} + B^+_{j^-n^+} A^-_{jn} = q^{\frac{1}{2}} (A^-_{j^+n^-} B^+_{jn} + A^+_{j^-n^-} B^-_{jn}).$$

Comparison of the diagonal entries on both sides gives two more relations:

$$[2j+1]\, |a^+_{j\downarrow\uparrow}|^2 = q^{2j+1}\big([2j+1]\, |a^+_{j-\frac{1}{2},\uparrow\uparrow}|^2 - q[2j+3]\, |a^+_{j\uparrow\uparrow}|^2\big),$$
$$[2j+1]\, |a^+_{j-\frac{1}{2},\downarrow\uparrow}|^2 = q^{2j}\big(q[2j+1]\, |a^+_{j\downarrow\downarrow}|^2 - [2j-1]\, |a^+_{j-\frac{1}{2},\downarrow\downarrow}|^2\big).$$

Finally, the expectation of $\pi'(a^*a + q^2 b^* b) = 1$ in the vector states for $|j\mu n\uparrow\rangle$ and $|j\mu n\downarrow\rangle$ leads to the relations

$$q^{2j}[2j+1]^2 |a^+_{j-\frac{1}{2},\uparrow\uparrow}|^2 = 1, \qquad q^{2j}[2j+1]^2 |a^+_{j\downarrow\downarrow}|^2 = 1.$$

Thus all coefficients are now determined, up to a few j-dependent phases:

$$a^+_{j\uparrow\uparrow} = \zeta_j \frac{q^{-j-\frac{1}{2}}}{[2j+2]}, \qquad a^+_{j\downarrow\uparrow} = \eta_j \frac{q^{\frac{1}{2}}}{[2j+1]\,[2j+2]}, \qquad a^+_{j\downarrow\downarrow} = \xi_j \frac{q^{-j}}{[2j+1]}, \tag{2.3.12}$$

with $|\zeta_j| = |\eta_j| = |\xi_j| = 1$. The relation (2.3.11) also implies $\zeta_{j+\frac{1}{2}}\eta_j = \eta_{j+\frac{1}{2}}\xi_j$. As before, we may reset these phases to 1 by redefining $|j\mu n\uparrow\rangle$ and $|j\mu n\downarrow\rangle$, without breaking the (λ', ρ')-equivariance. Substituting (2.3.12) back in previous formulas then gives (2.3.9). □

As already mentioned, formulas (2.3.9) for the matrices $\alpha^\pm_{j\mu n}$ and $\beta^\pm_{j\mu n}$ could have been obtained also from a direct but tedious computation using equations (2.3.5) and their inverses.

Remark 2.12. *Were we to consider a representation of $\mathcal{A}$ that need not be (λ', ρ')-equivariant, we could as well have defined our spinor space, like in [47], as $\mathbb{C}^2 \otimes V$, instead of $V \otimes \mathbb{C}^2$. The Clebsch–Gordan decomposition of $\mathbb{C}^2 \otimes V$ would be that of equation (2.3.1), but the q-Clebsch–Gordan coefficients appearing in (2.3.5a) and (2.3.5b) would be different due to the rule for exchanging the first two columns in q-Clebsch–Gordan coefficients [60]:*

$$C_q\begin{pmatrix} j & l & m \\ r & s & t \end{pmatrix} = C_q\begin{pmatrix} l & j & m \\ -s & -r & -t \end{pmatrix},$$

which results in a substitution of q by q^{-1} in (2.3.5c).

However, this is not the correct lifting of the (λ, ρ)-equivariant representation π of $\mathcal{A}$ to a (λ', ρ')-equivariant representation of $\mathcal{A}$ on spinor space. We already noted that π' as defined by $\pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$ is (λ', ρ')-equivariant, directly from (λ, ρ)-equivariance of π. One checks, simply by working out both sides of equation (2.3.4), that the noncocommutativity of $\mathcal{U}_q(su(2))$ spoils (λ'', ρ'')-equivariance of the representation $\pi'' := \mathrm{id} \otimes \pi$ of $\mathcal{A}$ on the tensor product $\mathbb{C}^2 \otimes V$, where we now define $\rho'' := \mathrm{id} \otimes \rho$, and

$$\lambda''(h) := (\sigma_{\frac{1}{2}} \otimes \lambda)(\Delta h) = \sigma_{\frac{1}{2}}(h_{(1)}) \otimes \lambda(h_{(2)}).$$

2.4 The equivariant Dirac operator

Recall the central Casimir element $C_q = qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2 ef \in \mathcal{U}$. The symmetric operators $\lambda'(C_q)$ and $\rho'(C_q)$ on $\mathcal{H}$, initially defined with dense domain W, extend to selfadjoint operators on $\mathcal{H}$. The finite-dimensional subspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$ are their joint eigenspaces:

$$\lambda'(C_q)|j\mu n\uparrow\rangle = (q^{2j+1} + q^{-2j-1})\,|j\mu n\uparrow\rangle, \qquad \rho'(C_q)|j\mu n\uparrow\rangle = (q^{2j+2} + q^{-2j-2})\,|j\mu n\uparrow\rangle,$$
$$\lambda'(C_q)|j\mu n\downarrow\rangle = (q^{2j+1} + q^{-2j-1})\,|j\mu n\downarrow\rangle, \qquad \rho'(C_q)|j\mu n\downarrow\rangle = (q^{2j} + q^{-2j})\,|j\mu n\downarrow\rangle,$$

directly from (2.3.6).

Let D be a selfadjoint operator on $\mathcal{H}$ which commutes strongly with $\lambda'(C_q)$ and $\rho'(C_q)$; then the finite-dimensional subspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$ reduce D. We look for the general form of such a selfadjoint operator D which is moreover (λ', ρ')-invariant in the sense that it commutes with $\lambda'(h)$ and $\rho'(h)$, for each $h \in \mathcal{U}_q(su(2))$.

Lemma 2.4.1. *The subspaces* $W_j^{\uparrow}$ *and* $W_j^{\downarrow}$ *are eigenspaces for* D.

Proof. We may restrict to either the subspace $W_j^{\uparrow}$ or $W_j^{\downarrow}$. Since $\lambda'(k)$ and $\rho'(k)$ are required to commute with D and moreover have distinct eigenvalues on these subspaces, it follows that D has a diagonal matrix with respect to the basis $|j\mu n\uparrow\rangle$, respectively $|j\mu n\downarrow\rangle$. If we provisionally write $D|j\mu n\uparrow\rangle = d_{j\mu n}^{\uparrow}|j\mu n\uparrow\rangle$, then the vanishing of

$$[D, \lambda'(f)]\,|j\mu n\uparrow\rangle = (d_{j,\mu+1,n}^{\uparrow} - d_{j\mu n}^{\uparrow})\,[j-\mu]^{\frac{1}{2}}[j+\mu+1]^{\frac{1}{2}}\,|j, \mu+1, n\uparrow\rangle,$$

for $\mu = -j, \dots, j-1$, shows that $d_{j\mu n}^{\uparrow}$ is independent of μ; and $[D, \rho'(f)] = 0$ likewise shows that $d_{j\mu n}^{\uparrow}$ does not depend on n. The same goes for $d_{j\mu n}^{\downarrow}$, too. Thus we may write

$$D|j\mu n\uparrow\rangle = d_j^{\uparrow}\,|j\mu n\uparrow\rangle, \qquad D|j\mu n\downarrow\rangle = d_j^{\downarrow}\,|j\mu n\downarrow\rangle, \tag{2.4.1}$$

where $d_j^{\uparrow}$ and $d_j^{\downarrow}$ are real eigenvalues of D. The respective multiplicities are $(2j+1)(2j+2)$ and $2j(2j+1)$, in view of (2.3.3). □

One of the conditions for the triple $(\mathcal{A}, \mathcal{H}, D)$ to be a spectral triple, is boundedness of the commutators $[D, \pi'(x)]$ for $x \in \mathcal{A}$. This naturally imposes certain restrictions on the eigenvalues $d_j^{\uparrow}, d_j^{\downarrow}$ of the operator D.

For convenience, we recall the representation π' of a in the basis $|j\mu n\rangle\!\rangle$, written explicitly on $|j\mu n\uparrow\rangle$ and $|j\mu n\downarrow\rangle$ as in (2.3.7):

$$\pi'(a)|j\mu n\uparrow\rangle = \sum_{\pm} \alpha_{j\mu n\uparrow\uparrow}^{\pm}|j^{\pm}\mu^{+}n^{+}\uparrow\rangle + \alpha_{j\mu n\downarrow\uparrow}^{+}|j^{+}\mu^{+}n^{+}\downarrow\rangle,$$
$$\pi'(a)|j\mu n\downarrow\rangle = \sum_{\pm} \alpha_{j\mu n\downarrow\downarrow}^{\pm}|j^{\pm}\mu^{+}n^{+}\downarrow\rangle + \alpha_{j\mu n\uparrow\downarrow}^{-}|j^{-}\mu^{+}n^{+}\uparrow\rangle.$$

Then, a straightforward computation shows that

$$[D, \pi'(a)]\,|j\mu n\uparrow\rangle = \sum_{\pm} \alpha_{j\mu n\uparrow\uparrow}^{\pm}(d_{j^{\pm}}^{\uparrow} - d_j^{\uparrow})|j^{\pm}\mu^{+}n^{+}\uparrow\rangle + \alpha_{j\mu n\downarrow\uparrow}^{+}(d_{j^{+}}^{\downarrow} - d_j^{\uparrow})|j^{+}\mu^{+}n^{+}\downarrow\rangle,$$
$$[D, \pi'(a)]\,|j\mu n\downarrow\rangle = \sum_{\pm} \alpha_{j\mu n\downarrow\downarrow}^{\pm}(d_{j^{\pm}}^{\downarrow} - d_j^{\downarrow})|j^{\pm}\mu^{+}n^{+}\downarrow\rangle + \alpha_{j\mu n\uparrow\downarrow}^{-}(d_{j^{-}}^{\uparrow} - d_j^{\downarrow})|j^{-}\mu^{+}n^{+}\uparrow\rangle. \tag{2.4.2}$$

Recall that the standard Dirac operator $\not{D}$ on the sphere S^3, with the round metric, has eigenvalues $(2j+\frac{3}{2})$ for $j = 0, \frac{1}{2}, 1, \frac{3}{2}$, with respective multiplicities $(2j+1)(2j+2)$; and $-(2j+\frac{1}{2})$ for $j = \frac{1}{2}, 1, \frac{3}{2}$, with respective multiplicities $2j(2j+1)$: see [5, 55], for instance. Notice that its spectrum is symmetric about 0.

In [7] a "q-Dirac" operator D was proposed, which in our notation corresponds to taking $d_j^{\uparrow} = 2[2j+1]/(q+q^{-1})$ and $d_j^{\downarrow} = -d_j^{\uparrow}$; these are q-analogues of the classical eigenvalues of $\not{D} - \frac{1}{2}$. For this particular choice of eigenvalues, it follows directly from the explicit form (2.3.9) of the matrices $\alpha^{\pm}_{j\mu n}$ that then the right hand sides of (2.4.2) diverge, and therefore $[D, \pi'(a)]$ is unbounded. This was already noted in [33] and it was suggested that one should instead consider an operator D whose spectrum matches that of the classical Dirac operator. In fact, Proposition 3.8 below shows that this is essentially the only possibility for a Dirac operator satisfying a (modified) first-order condition.

Let us then consider any operator D given by (2.4.1) –that is, a bi-equivariant one– with eigenvalues of the following form:

$$d_j^{\uparrow} = c_1^{\uparrow} j + c_2^{\uparrow}, \qquad d_j^{\downarrow} = c_1^{\downarrow} j + c_2^{\downarrow}, \tag{2.4.3}$$

where $c_1^{\uparrow}$, $c_2^{\uparrow}$, $c_1^{\downarrow}$, $c_2^{\downarrow}$ are independent of j. For brevity, we shall say that the eigenvalues are "linear in j". On the right hand side of (2.4.2), the "diagonal" coefficients simplify to

$$\alpha^{\pm}_{j\mu n\uparrow\uparrow}(d^{\uparrow}_{j\pm} - d^{\uparrow}_j) = \tfrac{1}{2}\alpha^{\pm}_{j\mu n\uparrow\uparrow} c_1^{\uparrow}, \qquad \alpha^{\pm}_{j\mu n\downarrow\downarrow}(d^{\downarrow}_{j\pm} - d^{\downarrow}_j) = \tfrac{1}{2}\alpha^{\pm}_{j\mu n\downarrow\downarrow} c_1^{\downarrow}, \tag{2.4.4}$$

which can be uniformly bounded with respect to j –see expressions (2.3.9). For the off-diagonal terms, involving $\alpha^{+}_{j\mu n\downarrow\uparrow}$ and $\alpha^{-}_{j\mu n\uparrow\downarrow}$, the differences between the "up" and "down" eigenvalues are linear in j. Since $0 < q < 1$, it is clear that $[N] \sim (q^{-1})^{N-1}$ for large N, and thus $\alpha^{+}_{j\mu n\downarrow\uparrow} \sim q^{3j+n+\frac{3}{2}} \leq q^{2j+1}$ for large j. Similar easy estimates yield

$$\begin{aligned} \alpha^{+}_{j\mu n\downarrow\uparrow} &= O(q^{2j+1}), & \beta^{+}_{j\mu n\downarrow\uparrow} &= O(q^{2j+\frac{1}{2}}), \\ \alpha^{-}_{j\mu n\uparrow\downarrow} &= O(q^{2j}), & \beta^{-}_{j\mu n\uparrow\downarrow} &= O(q^{2j+\frac{1}{2}}), \quad \text{as } j \to \infty. \end{aligned} \tag{2.4.5}$$

We therefore arrive at

$$|\alpha^{+}_{j\mu n\downarrow\uparrow}(d^{\downarrow}_{j^+} - d^{\uparrow}_j - 1)| \leq C j q^{2j}, \qquad |\alpha^{-}_{j\mu n\uparrow\downarrow}(d^{\uparrow}_{j^-} - d^{\downarrow}_j - 1)| \leq C' j q^{2j}, \tag{2.4.6}$$

for some $C > 0$, $C' > 0$, independent of j; and similar estimates hold for the off-diagonal coefficients of $\pi'(b)$.

Proposition 2.13. *Let* D *be any selfadjoint operator with eigenspaces* $W_j^{\uparrow}$ *and* $W_j^{\downarrow}$*, and eigenvalues* (2.4.1)*. If the eigenvalues* $d_j^{\uparrow}$ *and* $d_j^{\downarrow}$ *are linear in* j *as in* (2.4.3)*, then* $[D, \pi'(x)]$ *is a bounded operator for all* $x \in \mathcal{A}$*.*

Proof. Since a and b generate $\mathcal{A}$ as a $*$-algebra, it is enough to consider the cases $x = a$ and $x = b$. For $x = a$ and any $\xi \in \mathcal{H}$, the relations (2.4.2) and (2.4.4), together with the Schwarz inequality, give the estimate

$$\|[D, \pi'(a)]\,\xi\|^2 \leq \tfrac{1}{4}\max\{(c_1^{\uparrow})^2, (c_1^{\downarrow})^2\}\,\|\pi'(a)\xi\|^2 + \|\xi\|^2\|\eta\|^2,$$

where η is a vector whose components are estimated by (2.4.6), which establishes finiteness of $\|\eta\|$ since $0 < q < 1$. Therefore, $[D, \pi'(a)]$ is norm bounded. In the same way, we find that $[D, \pi'(b)]$ is bounded. □

Now, if D is a selfadjoint operator as in Proposition 2.13, and if the eigenvalues of D satisfy (2.4.3) and, moreover,

$$c_1^{\downarrow} = -c_1^{\uparrow}, \qquad c_2^{\downarrow} = c_2^{\uparrow} - c_1^{\uparrow}, \tag{2.4.7}$$

then the spectrum of D coincides with that of the classical Dirac operator $\not{D}$ on the round sphere S^3, up to rescaling and addition of a constant. Thus, we can regard our spectral triple as an isospectral deformation of $(C^\infty(S^3), \mathcal{H}, \not{D})$, and in particular, its spectral dimension[1] is 3. We summarize our conclusions in the following theorem.

Theorem 2.14. *The triple* $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$*, where the eigenvalues of* D *satisfy* (2.4.3) *and* (2.4.7)*, is a* 3^+*-summable spectral triple.* □

At this point, it is appropriate to comment on the relation of our construction with that of [47]. There, a spinor representation is constructed by tensoring the left regular representation of $\mathcal{A}(SU_q(2))$ by $\mathbb{C}^2$ on the left. This spinor space is then decomposed into two subspaces, similar to our "up" and "down" subspaces, on which D acts diagonally with eigenvalues linear in the total spin number j. The corresponding decomposition of the representation π' of $\mathcal{A}(SU_q(2))$ on spinor space is obtained by using the appropriate Clebsch–Gordan coefficients. However, contrary to what we have established above, in [47] it is found that a certain commutator $[D, \pi'(x)]$ is an unbounded operator. In particular, the off-diagonal terms in the representation of [47] do not have the compact nature we encountered in (2.4.5). They can be bounded from below by a positive constant, which leads, when multiplied by a term linear in j, to an unbounded operator.

The origin of this notable contrast is the following. Since in [47] no condition of $\mathcal{U}_q(su(2))$-equivariance is imposed *a priori* on the representation of $\mathcal{A}(SU_q(2))$, the spinor space W could be identified either with $V \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes V$, according to convenience. However, as we noted in Remark 2.12, the choice of $\mathbb{C}^2 \otimes V$ is not allowed by the condition of (λ', ρ')-equivariance, because $\mathcal{U}_q(su(2))$ is not cocommutative. Indeed, repeating the construction of a spinor representation and Dirac operator on the spinor space $\mathbb{C}^2 \otimes V$ instead of $V \otimes \mathbb{C}^2$ –hence ignoring equivariance– results eventually in unbounded commutators.

[1] The spectral dimension (or $^+$-summability) of a spectral triple is the unique nonnegative integer n for which the partial sums σ_N of the eigenvalues of $|D|^{-n}$ satisfy $\sigma_N \sim \log N$ as $N \to \infty$. In the case of the canonical triple on a Riemannian spin manifold, this coincides with the ordinary notion of dimension.

Chapter 3

Algebraic properties of the spectral triple

In this chapter, we discuss some of Connes' seven axioms [28] in the case of the previously defined spectral triple on $SU_q(2)$. The axioms we are interested in are the so-called commutant property and the first-order condition and both involve the notion of a *real structure* on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Recall that a *real structure* on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (cf. Appendix A.2) to be an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$, such that $J^2 = \pm 1$, $JD = \pm DJ$, with the signs depending on the spectral dimension of the spectral triple. We impose the following conditions:

$$[a, Jb^* J^{-1}] = 0, \qquad \text{(commutant property)}$$
$$[[D, a], Jb^* J^{-1}] = 0, \qquad \text{(first-order condition)}$$

for all $a, b \in \mathcal{A}$.

We will discuss the real structure J on the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$. However, we shall see that by requiring equivariance of J it is not possible to satisfy the above two conditions. Nevertheless, we shall be able to satisfy these two conditions up to infinitesimal operators of any order.

3.1 Reality of $SU_q(2)$

3.1.1 The Tomita operator of the regular representation

On the GNS representation space $\mathcal{H}_\psi$, there is a natural involution $T_\psi \colon \eta(x) \mapsto \eta(x^*)$, with domain $\eta(C(SU_q(2)))$, which may be regarded as an unbounded (antilinear) operator on $\mathcal{H}_\psi$. The Tomita–Takesaki theory [92] shows that this operator is closable (we denote its closure also by T_ψ) and that the polar decomposition $T_\psi =: J_\psi \Delta_\psi^{1/2}$ defines both the positive "modular operator" Δ_ψ and the antiunitary "modular conjugation" J_ψ. It has already been noted by Chakraborty and Pal [19] that this J_ψ has a simple expression in terms of the matrix elements of our chosen orthonormal basis for $\mathcal{H}_\psi$. Indeed, it follows immediately from (2.1.12) and (2.1.13) that

$$T_\psi \, |lmn\rangle = (-1)^{2l+m+n} q^{m+n} \, |l, -m, -n\rangle.$$

One checks, using (2.2.3), that

$$T_\psi \pi(a) \, |000\rangle = \pi(a^*) \, |000\rangle, \qquad T_\psi \pi(b) \, |000\rangle = \pi(b^*) \, |000\rangle.$$

Since π is the GNS representation for the state ψ, this is enough to conclude that

$$T_\psi \eta(x) = \eta(x^*) \quad \text{for all} \quad x \in \mathcal{A}. \tag{3.1.1}$$

The adjoint antilinear operator, satisfying $\langle \eta \mid T_\psi^* \mid \xi \rangle = \langle \xi \mid T_\psi \mid \eta \rangle$, is given by $T_\psi^* |lmn\rangle = (-1)^{2l+m+n} q^{-m-n} |l,-m,-n\rangle$, and since $\Delta_\psi = T_\psi^* T_\psi$, we see that every $|lmn\rangle$ lies in DomΔ_ψ with $\Delta_\psi |lmn\rangle = q^{2m+2n} |lmn\rangle$. Consequently,

$$J_\psi |lmn\rangle = (-1)^{2l+m+n} |l,-m,-n\rangle. \tag{3.1.2}$$

It is clear that $J_\psi^2 = 1$ on $\mathcal{H}_\psi$.

Definition 3.1. *Let $\pi^\circ(x) := J_\psi \pi(x^*) J_\psi^{-1}$, so that π° is a $*$-antirepresentation of $\mathcal{A}$ on $\mathcal{H}_\psi$. Equivalently, π° is a $*$-representation of the opposite algebra $\mathcal{A}(SU_{1/q}(2))$. By Tomita's theorem [92], π and π° are commuting representations.*

As an example, we compute

$$\begin{aligned}
\pi^\circ(a)\,|lmn\rangle &= (-1)^{2l+m+n} J_\psi \pi(a^*)\,|l,-m,-n\rangle \\
&= (-1)^{2l+m+n} J_\psi \big(\widetilde{A}^+_{l,-m,-n}|l^+,-m^+,-n^+\rangle + \widetilde{A}^-_{l,-m,-n}|l^-,-m^+,-n^+\rangle\big) \\
&= \widetilde{A}^+_{l,-m,-n}|l^+m^+n^+\rangle + \widetilde{A}^-_{l,-m,-n}|l^-m^+n^+\rangle \\
&= A^-_{l^+,-m^+,-n^+}|l^+m^+n^+\rangle + A^+_{l^-,-m^+,-n^+}|l^-m^+n^+\rangle,
\end{aligned}$$

where, explicitly,

$$\begin{aligned}
A^-_{l^+,-m^+,-n^+} &= q^{(2l-m-n+1)/2}\left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]}\right)^{\frac12}, \\
A^+_{l^-,-m^+,-n^+} &= q^{-(2l+m+n+1)/2}\left(\frac{[l-m][l-n]}{[2l][2l+1]}\right)^{\frac12}.
\end{aligned}$$

A glance back at (2.2.4) shows that these coefficients are identical with those of $\pi(a)\,|lmn\rangle$, *after substituting* $q \mapsto q^{-1}$. A similar phenomenon occurs with the coefficients of $\pi^\circ(b)$. We find, indeed, that

$$\begin{aligned}
\pi^\circ(a)\,|lmn\rangle &= A^{\circ+}_{lmn}|l^+m^+n^+\rangle + A^{\circ-}_{lmn}|l^-m^+n^+\rangle, \\
\pi^\circ(b)\,|lmn\rangle &= B^{\circ+}_{lmn}|l^+m^+n^-\rangle + B^{\circ-}_{lmn}|l^-m^+n^-\rangle,
\end{aligned}$$

where

$$A^{\circ\pm}_{lmn}(q) = A^\pm_{lmn}(q^{-1}), \qquad B^{\circ\pm}_{lmn}(q) = q^{-1} B^\pm_{lmn}(q^{-1}). \tag{3.1.3}$$

We can now verify directly that the representations π and π° commute, without need to appeal to the theorem of Tomita. For instance,

$$\begin{aligned}
\langle l+1,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle &= A^{\circ+}_{l^+m^+n^+} A^+_{lmn} - A^+_{l^+m^+n^+} A^{\circ+}_{lmn} \\
&= Q\left(\frac{[l+m+1][l+m+2][l+n+1][l+n+2]}{[2l+1][2l+2]^2[2l+3]}\right)^{\frac12},
\end{aligned}$$

where

$$Q = q^{\frac12(2l^+-m^+-n^++1)} q^{\frac12(-2l+m+n-1)} - q^{\frac12(-2l^++m^++n^+-1)} q^{\frac12(2l-m-n+1)} = 0.$$

Likewise, $\langle l-1,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle = 0$, and one checks that the matrix element $\langle l,m+1,n+1 \mid [\pi(a),\pi^\circ(a)] \mid lmn\rangle$ vanishes, too.

The (λ,ρ)-equivariance of π is reflected in an analogous equivariance condition for π°. We now identify this condition explicitly.

Lemma 3.1.1. *The symmetry of the antirepresentation* π° *of* $\mathcal{A}$ *on* $\mathcal{H}_\psi$ *is given by the equivariance conditions:*

$$\lambda(h)\,\pi^\circ(x)\xi = \pi^\circ(\widetilde{h}_{(2)} \cdot x)\,\lambda(h_{(1)})\xi,$$
$$\rho(h)\,\pi^\circ(x)\xi = \pi^\circ(\widetilde{h}_{(2)} \triangleright x)\,\rho(h_{(1)})\xi, \tag{3.1.4}$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ *and* $\xi \in V$, *and* $h \mapsto \widetilde{h}$ *is the automorphism of* $\mathcal{U}$ *determined on generators by* $\widetilde{k} := k$, $\widetilde{f} := q^{-1}f$, *and* $\widetilde{e} := qe$.

Proof. We work only on the dense subspace V. From (2.2.1) and (3.1.2), we get at once

$$J_\psi\lambda(k)^*J_\psi^{-1} = \lambda(k^{-1}), \quad J_\psi\lambda(f)^*J_\psi^{-1} = -\lambda(f), \quad J_\psi\lambda(e)^*J_\psi^{-1} = -\lambda(e), \tag{3.1.5}$$

and identical relations with ρ instead of λ. Write α for the antiautomorphism of $\mathcal{U}$ determined by $\alpha(k) := k^{-1}$, $\alpha(f) := -f$, and $\alpha(e) := -e$; so that $J_\psi\lambda(h)^*J_\psi^{-1} = \lambda(\alpha(h))$ for $h \in \mathcal{U}$, and similarly with ρ instead of λ.

Next, the first relation of (2.2.2) is equivalent to

$$\pi(x)\,\lambda(Sh) = \lambda(Sh_{(1)})\,\pi(h_{(2)} \cdot x). \tag{3.1.6}$$

Indeed, the left hand side can be expanded as

$$\pi(x)\,\lambda(\epsilon(h_{(1)})\,Sh_{(2)}) = \lambda(Sh_{(1)}\,h_{(2)})\,\pi(x)\,\lambda(Sh_{(3)}) = \lambda(Sh_{(1)})\,\pi(h_{(2)} \cdot x)\,\lambda(h_{(3)})\,\lambda(Sh_{(4)})$$

on applying (2.2.2); and the rightmost expression equals the right hand side of (3.1.6). Taking hermitian adjoints and conjugating by J_ψ, we get

$$\lambda(\alpha(Sh))\,\pi^\circ(x) = \pi^\circ(h_{(2)} \cdot x)\,\lambda(\alpha(Sh_{(1)})).$$

It remains only to note that $S\alpha = \alpha S$ is an automorphism of $\mathcal{U}$, whose inverse is the map $h \mapsto \widetilde{h}$ above; and to repeat the argument with ρ instead of λ, changing only the left action of $\mathcal{U}$ in concordance with (2.2.2). □

An independent check of (3.1.4) is afforded by the following argument. We may ask which antirepresentations π° of $\mathcal{H}_\psi$ satisfy these equivariance conditions. It suffices to run the proof of Proposition 2.8, *mutatis mutandis*, to determine the possible form of such a π° on the basis vectors $|lmn\rangle$. For instance, (2.2.9) is replaced by

$$\rho(e)\pi^\circ(a) = \pi^\circ(\widetilde{e} \triangleright a)\rho(k^{-1}) + \pi^\circ(\widetilde{k} \triangleright a)\rho(e) = q\,\pi^\circ(b)\rho(k^{-1}) + q^{\frac{1}{2}}\pi^\circ(a)\rho(e).$$

One finds that all formulas in that proof are reproduced, except for changes in the powers of q that appear; and, apart from the aforementioned phase ambiguities, one recovers precisely the form of π° given by (3.1.3).

Before proceeding, we indicate also the symmetry of the Tomita operator T_ψ, analogous to (3.1.5) above. Combining (3.1.1) with (2.2.2), and recalling that $\eta(x) = \pi(x)\,|000\rangle$, we find that for generators h of $\mathcal{U}$,

$$T_\psi\lambda(h)\pi(x)\,|000\rangle = \pi(x^* \triangleleft \vartheta(h)^*)\,|000\rangle.$$

On the other hand,

$$\lambda(\vartheta^{-1}S(\vartheta(h^*)))T_\psi\pi(x)\,|000\rangle = \pi(x^* \triangleleft \vartheta(h)^*)\,|000\rangle.$$

One checks easily on generators that $\vartheta^{-1}S(\vartheta(h)^*) = S(h)^*$. Since the vector $|000\rangle$ is separating for the GNS representation, we conclude that

$$T_\psi\,\lambda(h)\,T_\psi^{-1} = \lambda(Sh)^*.$$

Similarly, we find that

$$T_\psi\,\rho(h)\,T_\psi^{-1} = \rho(Sh)^*.$$

In other words, the antilinear involutory automorphism $h \mapsto (Sh)^*$ of the Hopf $*$-algebra $\mathcal{U}$ is implemented by the Tomita operator for the Haar state of the dual Hopf $*$-algebra $\mathcal{A}$. This is a known feature of quantum-group duality in the C*-algebra framework; for this and several other implementations by spatial operators, see [74].

3.1.2 The real structure on spinors

We are now ready to come back to spinors. Notice that J_ψ does not appear explicitly in the equivariance conditions (3.1.4) for the right regular representation π° of $\mathcal{A}$ on $\mathcal{H}_\psi$. Thus, we are now able to construct the "right multiplication" representation of $\mathcal{A}$ on spinors from its symmetry alone, and to deduce the conjugation operator J on spinors after the fact.

Proposition 3.2. *Let π'° be an antirepresentation of $\mathcal{A}$ on $\mathcal{H} = \mathcal{H}_\psi \oplus \mathcal{H}_\psi$ satisfying the following equivariance conditions:*

$$\begin{aligned}\lambda'(h)\,\pi'^\circ(x)\xi &= \pi'^\circ(\widetilde{h}_{(2)} \cdot x)\,\lambda'(h_{(1)})\xi,\\ \rho'(h)\,\pi'^\circ(x)\xi &= \pi'^\circ(\widetilde{h}_{(2)} \triangleright x)\,\rho'(h_{(1)})\xi.\end{aligned} \tag{3.1.7}$$

Then, up to some phase factors depending only on the index j *in the decomposition* (2.3.2), π'° *is given on the spinor basis by*

$$\begin{aligned}\pi'^\circ(a)\,|j\mu n\rangle\rangle &= \alpha^{\circ+}_{j\mu n}|j^+\mu^+n^+\rangle\rangle + \alpha^{\circ-}_{j\mu n}|j^-\mu^+n^+\rangle\rangle,\\ \pi'^\circ(b)\,|j\mu n\rangle\rangle &= \beta^{\circ+}_{j\mu n}|j^+\mu^+n^-\rangle\rangle + \beta^{\circ-}_{j\mu n}|j^-\mu^+n^-\rangle\rangle,\\ \pi'^\circ(a^*)\,|j\mu n\rangle\rangle &= \widetilde{\alpha}^{\circ+}_{j\mu n}|j^+\mu^-n^-\rangle\rangle + \widetilde{\alpha}^{\circ-}_{j\mu n}|j^-\mu^-n^-\rangle\rangle,\\ \pi'^\circ(b^*)\,|j\mu n\rangle\rangle &= \widetilde{\beta}^{\circ+}_{j\mu n}|j^+\mu^-n^+\rangle\rangle + \widetilde{\beta}^{\circ-}_{j\mu n}|j^-\mu^-n^+\rangle\rangle,\end{aligned} \tag{3.1.8}$$

where $\alpha^{\circ\pm}_{j\mu n}$ and $\beta^{\circ\pm}_{j\mu n}$ are the triangular 2×2 matrices, given by $\alpha^{\circ\pm}_{j\mu n}(q) = \alpha^{\pm}_{j\mu n}(q^{-1})$ and $\beta^{\circ\pm}_{j\mu n}(q) = q^{-1}\beta^{\pm}_{j\mu n}(q^{-1})$, with $\alpha^{\pm}_{j\mu n}$ and $\beta^{\pm}_{j\mu n}$ given by (2.3.9).

Proof. We retrace the steps of the proof of Proposition 2.11, *mutatis mutandis.* Since $\widetilde{k}\cdot a = k\cdot a = q^{\frac12}a$, the relations involving $\lambda'(k)$ and $\rho'(k)$ are unchanged. We quickly conclude that π'° must have the form (3.1.8), and it remains to determine the coefficient matrices.

The commutation relations of $\lambda'(f)$ with $\pi'^\circ(a)$ and $\pi'^\circ(b)$ give:

$$\begin{aligned}\alpha^{\circ+}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j+\mu+1]^{\frac12}\ A^{\circ+}_{jn}, & \alpha^{\circ-}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j-\mu]^{\frac12}\ A^{\circ-}_{jn},\\ \beta^{\circ+}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j+\mu+1]^{\frac12}\ B^{\circ+}_{jn}, & \beta^{\circ-}_{j\mu n} &= q^{-\frac12(\mu+n-\frac12)}[j-\mu]^{\frac12}\ B^{\circ-}_{jn}.\end{aligned}$$

The matrices $A^{\circ\pm}_{jn}$, $B^{\circ\pm}_{jn}$ may be determined, as before, by the commutation relations involving $\rho'(f)$ and $\rho'(e)$. One finds that the n-dependent factors such as $[j+n+\frac32]^{\frac12}$ and so on, are the same as the respective entries of $A^{\pm}_{jn}$, $B^{\pm}_{jn}$; let $a^{\circ+}_{j\uparrow\uparrow}$, etc., be the remaining factors which depend on j only. Then (2.3.10) is replaced by

$$\begin{aligned} b^{\circ+}_{j\uparrow\uparrow} &= q^{-j-\frac{3}{2}}\,a^{\circ+}_{j\uparrow\uparrow}, & b^{\circ+}_{j\downarrow\uparrow} &= -q^{j+\frac{1}{2}}\,a^{\circ+}_{j\downarrow\uparrow}, & b^{\circ+}_{j\downarrow\downarrow} &= q^{-j-\frac{1}{2}}\,a^{\circ+}_{j\downarrow\downarrow}, \\ b^{\circ-}_{j\uparrow\uparrow} &= -q^{j+\frac{1}{2}}\,a^{\circ-}_{j\uparrow\uparrow}, & b^{\circ-}_{j\uparrow\downarrow} &= q^{-j-\frac{1}{2}}\,a^{\circ-}_{j\uparrow\downarrow}, & b^{\circ-}_{j\downarrow\downarrow} &= -q^{j-\frac{1}{2}}\,a^{\circ-}_{j\downarrow\downarrow}. \end{aligned}$$

Next, we find

$$(a^{\circ-}_{j+\frac{1}{2},\uparrow\uparrow})^{\star} = q^{-2j-2}a^{\circ+}_{j\uparrow\uparrow}, \qquad (a^{\circ-}_{j+\frac{1}{2},\uparrow\downarrow})^{\star} = -a^{\circ+}_{j\downarrow\uparrow}, \qquad (a^{\circ-}_{j+\frac{1}{2},\downarrow\downarrow})^{\star} = q^{-2j-1}a^{\circ+}_{j\downarrow\downarrow}.$$

Since π'° is an antirepresentation, $ab = q^{-1}ba$ implies $\pi'^{\circ}(b)\pi'^{\circ}(a) = q^{-1}\,\pi'^{\circ}(a)\pi'^{\circ}(b)$. The matrix elements of both sides lead to three relations:

$$[2j+1]\,a^{\circ+}_{j+\frac{1}{2},\downarrow\downarrow}a^{\circ+}_{j\downarrow\uparrow} = [2j+3]\,a^{\circ+}_{j+\frac{1}{2},\downarrow\uparrow}a^{\circ+}_{j\uparrow\uparrow}, \tag{3.1.9}$$

which is formally identical to (2.3.11), and

$$\begin{aligned} [2j+1]\,|a^{\circ+}_{j\downarrow\uparrow}|^2 &= q^{-2j-1}\big([2j+1]\,|a^{\circ+}_{j-\frac{1}{2},\uparrow\uparrow}|^2 - q^{-1}[2j+3]\,|a^{\circ+}_{j\uparrow\uparrow}|^2\big), \\ [2j+1]\,|a^{\circ+}_{j-\frac{1}{2},\downarrow\uparrow}|^2 &= q^{-2j}\big(q^{-1}[2j+1]\,|a^{\circ+}_{j\downarrow\downarrow}|^2 - [2j-1]\,|a^{\circ+}_{j-\frac{1}{2},\downarrow\downarrow}|^2\big). \end{aligned}$$

Finally, the relation $aa^* + bb^* = 1$ yields $\pi'^{\circ}(a^*)\pi'^{\circ}(a) + \pi'^{\circ}(b^*)\pi'^{\circ}(b) = 1$; its diagonal matrix elements gives the last two relations:

$$q^{-2j}[2j+1]^2|a^{\circ+}_{j-\frac{1}{2},\uparrow\uparrow}|^2 = 1, \qquad q^{-2j}[2j+1]^2|a^{\circ+}_{j\downarrow\downarrow}|^2 = 1.$$

All coefficients are now determined except for their phases:

$$a^{\circ+}_{j\uparrow\uparrow} = \zeta^{\circ}_j\,\frac{q^{j+\frac{1}{2}}}{[2j+2]}, \quad a^{\circ+}_{j\downarrow\uparrow} = \eta^{\circ}_j\,\frac{q^{-\frac{1}{2}}}{[2j+1]\,[2j+2]}, \quad a^{\circ+}_{j\downarrow\downarrow} = \xi^{\circ}_j\,\frac{q^{j}}{[2j+1]}, \tag{3.1.10}$$

and (3.1.9) also entails the phase relations $\zeta^{\circ}_j\eta^{\circ}_{j+\frac{1}{2}} = \eta^{\circ}_j\xi^{\circ}_{j+\frac{1}{2}}$. Once more, we choose all phases to be $+1$ by convention. Substituting (3.1.10) back in previous formulas, we find

$$\alpha^{\circ\pm}_{j\mu n}(q) = \alpha^{\pm}_{j\mu n}(q^{-1}), \qquad \beta^{\circ\pm}_{j\mu n}(q) = q^{-1}\beta^{\pm}_{j\mu n}(q^{-1}). \tag{3.1.11}$$

in perfect analogy with (3.1.3). □

Definition 3.3. *The conjugation operator* J *is the antilinear operator on* $\mathcal{H}$ *which is defined explicitly on the orthonormal spinor basis by*

$$\begin{aligned} J\,|j\mu n\uparrow\rangle &:= i^{2(2j+\mu+n)}\,|j,-\mu,-n,\uparrow\rangle, \\ J\,|j\mu n\downarrow\rangle &:= i^{2(2j-\mu-n)}\,|j,-\mu,-n,\downarrow\rangle. \end{aligned} \tag{3.1.12}$$

It is immediate from this presentation that J *is antiunitary and that* $J^2 = -1$*, since each* $4j \pm 2(\mu+n)$ *is an odd integer.*

Proposition 3.4. *The invariant operator* D *of Section 2.4 commutes with the conjugation operator* J*:*

$$JDJ^{-1} = D. \tag{3.1.13}$$

Proof. This is clear from the diagonal form of both D and J on their common eigenspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$, given by the respective equations (2.4.1) and (3.1.12). □

Proposition 3.4 is a minimal requirement for $(\mathcal{A}(SU_q(2)), \mathcal{H}, D, J)$ to constitute a real spectral triple. However, here is where we part company with the axiom scheme for real spectral triples proposed in [28]. Indeed, the conjugation operator J that we have defined by (3.1.12) is *not* the modular conjugation J_ψ of (3.1.2) lifted to the spinor representation of $\mathcal{A}$. That conjugation operator is $J_\psi \otimes \sigma_3$ with $\sigma_3 = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ the Pauli matrix acting on $V_{\frac{1}{2}} = \mathbb{C}^2$. On the other hand, the conjugation operator J we defined above takes the following form in terms of the basis $|lmn\rangle \otimes |\frac{1}{2}, \pm\frac{1}{2}\rangle$:

$$
\begin{aligned}
J|lmn\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle &= (-1)^{2l-m-n}(-i)\bigg(\frac{[l-m+1]+[l+m]}{[2l+1]}|l,-m,-n\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle \\
&\quad + \left(q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}\right) \frac{[l+m]^{\frac{1}{2}}[l-m+1]^{\frac{1}{2}}}{[2l+1]}|l,-m+1,-m\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\bigg), \\
J|lmn\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle &= (-1)^{2l-m-n} i\bigg(\frac{[l+m+1]+[l-m]}{[2l+1]}|l,-m,-n\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \\
&\quad + \left(q^{l+\frac{1}{2}} - q^{-l-\frac{1}{2}}\right) \frac{[l-m]^{\frac{1}{2}}[l+m+1]^{\frac{1}{2}}}{[2l+1]}|l,-m-1,-m\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle\bigg).
\end{aligned}
$$

In the limit $q \to 1$, the diagonal terms will disappear so that J coincides with $J_\psi \otimes \sigma_3$ in that case.

Notice that the map $\pi'(a) \mapsto J\pi'(a)^* J^{-1}$ for a *generic antiunitary operator* J defines a right action of $a \in \mathcal{A}(\mathrm{SU}_q(2))$ on the Hilbert space $\mathcal{H}$. One can rephrase this by saying that it defines a representation of the opposite algebra $\mathcal{A}(\mathrm{SU}_q(2))^\circ$. The latter is defined to be $\mathcal{A}(SU_q(2))$ as a vector space but with product given by $a \circ b = ba$. From the commutation relations of $\mathrm{SU}_q(2)$ (2.1.1), we conclude that $\mathcal{A}(\mathrm{SU}_q(2))^\circ \simeq \mathcal{A}(\mathrm{SU}_{1/q}(2))$.

If we impose the commutation relation analogous to (3.1.13) for the conjugation operator $J_\psi \otimes \sigma_3$, this would force D to be equivariant under the corresponding symmetry of $U_{1/q}(su(2))$, denoted by (λ'', ρ'') in our earlier Remark 2.12. It is not hard to check that this extra equivariance condition would force D to be merely a scalar operator, thereby negating the possibility of an equivariant 3^+-summable real spectral triple based on $\mathcal{A}(SU_q(2))$ with the modular conjugation operator. This result is consonant with the “no-go theorem” of Schmüdgen [86] for nontrivial commutator representations of Woronowicz differential calculi on $SU_q(2)$.

The remedy that we propose here is to modify J, in keeping with the symmetry of the spinor representation, to a non-Tomita conjugation operator. We shall see, however, that the expected properties of real spectral triples do hold “up to compact perturbations”.

It should be noted that J satisfies the analogue of (3.1.5) for the representations λ' and ρ':

$$
\begin{aligned}
J\lambda'(k)J^{-1} &= \lambda'(k^{-1}), \qquad & J\lambda'(e)J^{-1} &= -\lambda'(f), \\
J\rho'(k)J^{-1} &= \rho'(k^{-1}), \qquad & J\rho'(e)J^{-1} &= -\rho'(f),
\end{aligned}
\tag{3.1.14}
$$

which follows directly from the definition (3.1.12) and the relations (2.3.6).

Proposition 3.5. *The antiunitary operator J intertwines the left and right spinor representations:*

$$
J\,\pi'(x^*)\,J^{-1} = \pi'^\circ(x), \quad \textit{for all} \quad x \in \mathcal{A}. \tag{3.1.15}
$$

Proof. It follows directly from the proof of Lemma 3.1.1, using the relations (3.1.14) instead of (3.1.5), that the antirepresentation $x \mapsto J\,\pi'(x^*)\,J^{-1}$ complies with the equivariance conditions (3.1.7). By Proposition 3.2, it coincides with π'° up to an equivalence obtained by resetting the phase factors in (3.1.10). It remains only to check that $\zeta_j^\circ = \eta_j^\circ = \xi_j^\circ = 1$ for the aforementioned antirepresentation. This check is easily effected by calculating $J\,\pi'(a^*)\,J^{-1}$ directly on the basis vectors $|j\mu n{\uparrow}\rangle$. We compute

$$
\begin{aligned}
&J\pi'(a^*)J^{-1}\,|j\mu n{\uparrow}\rangle = i^{-2(2j-\mu-n)} J\pi'(a^*)\,|j,-\mu,-n,\uparrow\rangle \\
&= i^{-2(2j-\mu-n)} J\big(\tilde\alpha^+_{j,-\mu,-n,\uparrow\uparrow}\,|j^+,-\mu^+,-n^+{\uparrow}\rangle + \tilde\alpha^+_{j,-\mu,-n,\downarrow\uparrow}\,|j^+,-\mu^+,-n^+{\downarrow}\rangle \\
&\qquad + \tilde\alpha^-_{j,-\mu,-n,\uparrow\uparrow}\,|j^-,-\mu^+,-n^+{\uparrow}\rangle\big) \\
&= \tilde\alpha^+_{j,-\mu,-n,\uparrow\uparrow}\,|j^+\mu^+n^+{\uparrow}\rangle - \tilde\alpha^+_{j,-\mu,-n,\downarrow\uparrow}\,|j^+\mu^+n^+{\downarrow}\rangle + \tilde\alpha^-_{j,-\mu,-n,\uparrow\uparrow}\,|j^-\mu^+n^+{\uparrow}\rangle \\
&= \alpha^-_{j^+,-\mu^+,-n^+,\uparrow\uparrow}\,|j^+\mu^+n^+{\uparrow}\rangle - \alpha^-_{j^+,-\mu^+,-n^+,\downarrow\uparrow}\,|j^+\mu^+n^+{\downarrow}\rangle + \alpha^+_{j^-,-\mu^+,-n^+,\uparrow\uparrow}\,|j^-\mu^+n^+{\uparrow}\rangle \\
&= q^{-\frac12(\mu+n-\frac12)}\Bigg(q^{j+\frac12}\,\frac{[j+\mu+1]^{\frac12}[j+n+\frac32]^{\frac12}}{[2j+2]}\,|j^+\mu^+n^+{\uparrow}\rangle \\
&\qquad + q^{-\frac12}\,\frac{[j+\mu+1]^{\frac12}[j-n+\frac12]^{\frac12}}{[2j+1][2j+2]}\,|j^+\mu^+n^+{\downarrow}\rangle + q^{-j-1}\,\frac{[j-\mu]^{\frac12}[j-n+\frac12]^{\frac12}}{[2j+1]}\,|j^-\mu^+n^+{\uparrow}\rangle\Bigg) \\
&= \alpha^{\circ+}_{j\mu n\uparrow\uparrow}\,|j^+\mu^+n^+{\uparrow}\rangle + \alpha^{\circ+}_{j\mu n\downarrow\uparrow}\,|j^+\mu^+n^+{\downarrow}\rangle + \alpha^{\circ-}_{j\mu n\uparrow\uparrow}\,|j^-\mu^+n^+{\uparrow}\rangle \\
&= \pi'^\circ(a)\,|j\mu n{\uparrow}\rangle,
\end{aligned}
$$

where the $\alpha^{\circ\pm}_{j\mu n}$ coefficients are taken according to (3.1.11).

In the same way, one finds that $J\pi'(b^*)J^{-1}\,|j\mu n{\uparrow}\rangle = \pi'^\circ(b)\,|j\mu n{\uparrow}\rangle$, again using (3.1.11) for $\beta^{\circ\pm}_{j\mu n}$; and similar calculations show that both sides of (3.1.15) coincide on the basis vector $|j\mu n{\downarrow}\rangle$. (These four calculations, taken together, afford a direct proof of (3.1.15) without need to consider the symmetries of J.) □

3.2 Commutant property and first-order condition

In this section, we discuss the properties of the real spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D, J)$, in particular the commutant property and the first-order condition. We will see that these are only satisfied up to infinitesimals of arbitrary order, quite similarly to [38]. Recall that a compact operator T is called an infinitesimal of order α if its singular values μ_j satisfy $\mu_j = \mathcal{O}(j^{-\alpha})$.

We can simplify our discussion somewhat by replacing the spinor representation π' of $\mathcal{A} = \mathcal{A}(SU_q(2))$ of Proposition 2.11 by a so-called approximate representation $\underline{\pi}'\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, such that $\pi'(x) - \underline{\pi}'(x)$ is a compact operator for each $x \in \mathcal{A}$. In other words, although $\underline{\pi}'$ need not preserve the algebra relations of $\mathcal{A}$, the mappings π' and $\underline{\pi}'$ have the same image in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, that is, they define the same $*$-homomorphism of $\mathcal{A}$ into the Calkin algebra.

We denote by L_q the positive trace-class operator given by

$$L_q|j\mu n\rangle\rangle := q^j\,|j\mu n\rangle\rangle \quad \text{for} \quad j \in \tfrac12\mathbb{N},$$

and let $\mathcal{K}_q$ the two-sided ideal of $\mathcal{B}(\mathcal{H})$ generated by L_q; it is contained in the ideal of trace-class operators. In fact, an element in $\mathcal{K}_q$ is an infinitesimal of arbitrary high order.

Proposition 3.6. *The following equations define a mapping* $\underline{\pi}'\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *on generators, which is a* $*$*-representation modulo* $\mathcal{K}_q$*, and is approximate to the spin representation* π' *of Proposition 2.11 in the sense that* $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ *for each* $x \in \mathcal{A}$*:*

$$\begin{aligned}
\underline{\pi}'(a)\,|j\mu n\rangle\!\rangle &= \underline{\alpha}^+_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \underline{\alpha}^-_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle,\\
\underline{\pi}'(b)\,|j\mu n\rangle\!\rangle &= \underline{\beta}^+_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \underline{\beta}^-_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle,\\
\underline{\pi}'(a^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde\alpha}^+_{j\mu n}|j^+\mu^-n^-\rangle\!\rangle + \underline{\tilde\alpha}^-_{j\mu n}|j^-\mu^-n^-\rangle\!\rangle,\\
\underline{\pi}'(b^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde\beta}^+_{j\mu n}|j^+\mu^-n^+\rangle\!\rangle + \underline{\tilde\beta}^-_{j\mu n}|j^-\mu^-n^+\rangle\!\rangle,
\end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned}
\underline{\alpha}^+_{j\mu n} &:= \sqrt{1-q^{2j+2\mu+2}} \begin{pmatrix} \sqrt{1-q^{2j+2n+3}} & 0 \\ 0 & \sqrt{1-q^{2j+2n+1}} \end{pmatrix},\\
\underline{\alpha}^-_{j\mu n} &:= q^{2j+\mu+n+\frac12} \sqrt{1-q^{2j-2\mu}} \begin{pmatrix} q\sqrt{1-q^{2j-2n+1}} & 0 \\ 0 & \sqrt{1-q^{2j-2n-1}} \end{pmatrix},\\
\underline{\beta}^+_{j\mu n} &:= q^{j+n-\frac12} \sqrt{1-q^{2j+2\mu+2}} \begin{pmatrix} q\sqrt{1-q^{2j-2n+3}} & 0 \\ 0 & \sqrt{1-q^{2j-2n+1}} \end{pmatrix},\\
\underline{\beta}^-_{j\mu n} &:= -q^{j+\mu}\sqrt{1-q^{2j-2\mu}} \begin{pmatrix} \sqrt{1-q^{2j+2n+1}} & 0 \\ 0 & \sqrt{1-q^{2j+2n-1}} \end{pmatrix},
\end{aligned} \tag{3.2.2}$$

and

$$\underline{\tilde\alpha}^\pm_{j\mu n} = \underline{\alpha}^\mp_{j^\pm\mu^-n^-}, \qquad \underline{\tilde\beta}^\pm_{j\mu n} = \underline{\alpha}^\mp_{j^\pm\mu^-n^+}. \tag{3.2.3}$$

Proof. First of all, we claim that the defining relations (2.1.1) are preserved by $\underline{\pi}'$ modulo the ideal $\mathcal{K}_q$ of $\mathcal{B}(\mathcal{H})$, that is, $\underline{\pi}'(b)\underline{\pi}'(a) - q\,\underline{\pi}'(a)\underline{\pi}'(b) \in \mathcal{K}_q$, and so on. Indeed, it can be verified by a direct but tedious check on the spinor basis that $\underline{\pi}'(b)\underline{\pi}'(a) - q\,\underline{\pi}'(a)\underline{\pi}'(b) = (L_q)^4 A$ where A is a bounded operator; the same is true for each of the other relations listed in (2.1.1).

It is well known, and easily checked from (2.1.1), that $\mathcal{A}$ is generated as a vector space by the products $a^k b^l b^{*m}$ and $b^l b^{*m} a^{*n}$, for $k, l, m, n \in \mathbb{N}$. We may thus define $\underline{\pi}'(x)$ for any $x \in \mathcal{A}$ by extending (3.2.1) multiplicatively on such products, and then extending further by linearity. With this convention, we conclude that

$$\underline{\pi}'(xy) - \underline{\pi}'(x)\underline{\pi}'(y) \in \mathcal{K}_q \quad \text{for all} \quad x, y \in \mathcal{A}. \tag{3.2.4}$$

The defining formulas also entail that $\underline{\pi}'(x)^* = \underline{\pi}'(x^*)$ for each $x \in \mathcal{A}$.

If $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ and $\pi'(y) - \underline{\pi}'(y) \in \mathcal{K}_q$, then

$$\pi'(xy) - \underline{\pi}'(x)\underline{\pi}'(y) = \pi'(x)\big(\pi'(y) - \underline{\pi}'(y)\big) + \big(\pi'(x) - \underline{\pi}'(x)\big)\underline{\pi}'(y) \in \mathcal{K}_q,$$

and therefore $\pi'(xy) - \underline{\pi}'(xy)$ lies in $\mathcal{K}_q$ also; thus, it suffices to verify this property in the cases $x = a, b$.

On comparing the coefficients (3.2.2) with the corresponding ones of $\pi'(a)$ and $\pi'(b)$ from equation (2.3.9), we get, for instance,

$$\begin{aligned} \alpha^{+}_{j\mu n\uparrow\uparrow} - \underline{\alpha}^{+}_{j\mu n\uparrow\uparrow} &= \frac{q^{4j+4}\sqrt{1-q^{2j+2\mu+2}}\sqrt{1-q^{2j+2n+3}}}{1-q^{4j+4}} = q^{4j+4}\,\alpha^{+}_{j\mu n\uparrow\uparrow}, \\ \alpha^{+}_{j\mu n\downarrow\downarrow} - \underline{\alpha}^{+}_{j\mu n\downarrow\downarrow} &= \frac{q^{4j+2}\sqrt{1-q^{2j+2\mu+2}}\sqrt{1-q^{2j+2n+1}}}{1-q^{4j+2}} = q^{4j+2}\,\alpha^{+}_{j\mu n\downarrow\downarrow}. \end{aligned} \tag{3.2.5a}$$

and similarly,

$$\alpha^{-}_{j\mu n\uparrow\uparrow} - \underline{\alpha}^{-}_{j\mu n\uparrow\uparrow} = q^{4j+2}\,\alpha^{-}_{j\mu n\uparrow\uparrow}, \qquad \alpha^{-}_{j\mu n\downarrow\downarrow} - \underline{\alpha}^{-}_{j\mu n\downarrow\downarrow} = q^{4j}\,\alpha^{-}_{j\mu n\downarrow\downarrow}. \tag{3.2.5b}$$

We estimate the off-diagonal terms, using the inequalities $q^{\pm\mu} \leq q^{-j}$, $q^{\pm n} \leq q^{-j-\frac{1}{2}}$ and $[N]^{-1} < q^{N-1}$:

$$\begin{aligned} |\alpha^{+}_{j\mu n\downarrow\uparrow}| &= q^{(\mu+n+\frac{1}{2})/2}\,\frac{[j+\mu+1]^{\frac{1}{2}}\,[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]\,[2j+2]} \leq \frac{q^{-2j-2}}{[2j+1]\,[2j+2]} < q^{2j-1}, \\ |\alpha^{-}_{j\mu n\uparrow\downarrow}| &= q^{(\mu+n+\frac{1}{2})/2}\,\frac{[j-\mu]^{\frac{1}{2}}\,[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j]\,[2j+1]} \leq \frac{q^{-2j-1}}{[2j]\,[2j+1]} < q^{2j-2}. \end{aligned}$$

On account of (3.2.5) and analogous relations for the coefficients of $\underline{\pi}'(b)$, we find that

$$\begin{aligned} \pi'(a) - \underline{\pi}'(a) &\equiv T\pi'(a)T \mod \mathcal{K}_q, \\ \pi'(b) - \underline{\pi}'(b) &\equiv T\pi'(b)T \mod \mathcal{K}_q, \end{aligned}$$

where T is the operator defined by

$$T\,|j\mu n\rangle\!\rangle := \begin{pmatrix} q^{2j+\frac{3}{2}} & 0 \\ 0 & q^{2j+\frac{1}{2}} \end{pmatrix} |j\mu n\rangle\!\rangle = \begin{pmatrix} q^{\frac{3}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix} (L_q)^2\,|j\mu n\rangle\!\rangle. \tag{3.2.6}$$

Clearly, $T \in \mathcal{K}_q$, so that by boundedness of $\pi'(x)$ it follows that $\pi'(x) - \underline{\pi}'(x) \in \mathcal{K}_q$ for $x = a, b$. □

Using the conjugation operator J, we can also define an approximate antirepresentation of $\mathcal{A}$ by $\underline{\pi}'^{\circ}(x) := J\underline{\pi}'(x)J^{-1}$. It is immediate that $\pi'^{\circ}(x) - \underline{\pi}'^{\circ}(x) \in \mathcal{K}_q$, with π'° as defined in Proposition 3.2. Explicitly, we can write

$$\begin{aligned} \underline{\pi}'^{\circ}(a)\,|j\mu n\rangle\!\rangle &= \underline{\alpha}^{\circ+}_{j\mu n}|j^+\mu^+n^+\rangle\!\rangle + \underline{\alpha}^{\circ-}_{j\mu n}|j^-\mu^+n^+\rangle\!\rangle, \\ \underline{\pi}'^{\circ}(b)\,|j\mu n\rangle\!\rangle &= \underline{\beta}^{\circ+}_{j\mu n}|j^+\mu^+n^-\rangle\!\rangle + \underline{\beta}^{\circ-}_{j\mu n}|j^-\mu^+n^-\rangle\!\rangle, \\ \underline{\pi}'^{\circ}(a^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde{\alpha}}^{\circ+}_{j\mu n}|j^+\mu^-n^-\rangle\!\rangle + \underline{\tilde{\alpha}}^{\circ-}_{j\mu n}|j^-\mu^-n^-\rangle\!\rangle, \\ \underline{\pi}'^{\circ}(b^*)\,|j\mu n\rangle\!\rangle &= \underline{\tilde{\beta}}^{\circ+}_{j\mu n}|j^+\mu^-n^+\rangle\!\rangle + \underline{\tilde{\beta}}^{\circ-}_{j\mu n}|j^-\mu^-n^+\rangle\!\rangle, \end{aligned}$$

where

$$\underline{\alpha}^{\circ\pm}_{j\mu n} = \underline{\tilde{\alpha}}^{\pm}_{j,-\mu,-n}, \quad \underline{\tilde{\alpha}}^{\circ\pm}_{j\mu n} = \underline{\alpha}^{\pm}_{j,-\mu,-n}, \quad \underline{\beta}^{\circ\pm}_{j\mu n} = -\underline{\tilde{\beta}}^{\pm}_{j,-\mu,-n}, \quad \underline{\tilde{\beta}}^{\circ\pm}_{j\mu n} = -\underline{\beta}^{\pm}_{j,-\mu,-n}.$$

It turns out that the approximate representations $\underline{\pi}'$ and $\underline{\pi}'^{\circ}$ almost commute, in the following sense.

Proposition 3.7. *For each* $x, y \in \mathcal{A}$*, the commutant* $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)]$ *lies in* $\mathcal{K}_q$*.*

Proof. In view of our earlier remarks on the almost-multiplicativity of $\underline{\pi}'$, and thus also of $\underline{\pi}'^{\circ}$, it is enough to check this for the cases $x, y = a, a^*, b, b^*$. We omit the detailed calculation, which we have performed with a symbolic computer program. In each case, the commutator $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)]$ decomposes as a direct sum of operators in the subspaces $W_j^{\uparrow}$ and $W_j^{\downarrow}$ separately, in view of (3.2.2) and (3.1.12), and the explicit calculation shows that for each pair of generators x, y, we obtain $[\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)] = (L_q)^2 A$ where A is a bounded operator. □

If we further impose the first-order condition up to compact operators in the ideal $\mathcal{K}_q$, it turns out that this (almost) determines the Dirac operator.

Proposition 3.8. *Up to rescaling, adding constants, and adding elements of* $\mathcal{K}_q$*, there is only one operator* D *of the form* (2.4.1) *which satisfies the first order condition modulo* $\mathcal{K}_q$*, that is, each* $[D, \underline{\pi}'(y)]$ *is bounded, and*

$$[\underline{\pi}'^{\circ}(x), [D, \underline{\pi}'(y)]] \in \mathcal{K}_q \quad \textit{for all} \quad x, y \in \mathcal{A}. \tag{3.2.7}$$

This operator D *has eigenvalues that are linear in* j*.*

Proof. Suppose first that D is an equivariant selfadjoint operator of the type considered in Section 2.4, with eigenvalues linear in j; that is, D is determined by (2.4.1) and (2.4.3). Since each operator appearing in (3.2.7) decomposes into a pair of operators on the "up" and "down" spinor subspaces, it is clear that the nested commutators are independent of the parameters $c_2^{\uparrow}$ and $c_2^{\downarrow}$; and that $c_1^{\uparrow}$ and $c_1^{\downarrow}$ are merely scale factors on both subspaces. Again we take x and y to be generators: explicit calculations show that in each case, $[\underline{\pi}'^{\circ}(x), [D, \underline{\pi}'(y)]] = (L_q)^2 B$ with B a bounded operator.

To prove the converse, assume only that D satisfies the equivariance condition (2.4.1), and that $[D, \underline{\pi}'(a)]$ and $[D, \underline{\pi}'(b)]$ are bounded.

We may decompose $\underline{\pi}'(a) = \underline{\pi}'(a)^+ + \underline{\pi}'(a)^-$ according to whether the index j in (3.2.1) is raised or lowered; and similarly for $\underline{\pi}'(b)$, $\underline{\pi}'^{\circ}(a)$, and $\underline{\pi}'^{\circ}(b)$. Proposition 3.7 shows that, modulo $\mathcal{K}_q$:

$$\begin{aligned}
\underline{\pi}'(a)^+\underline{\pi}'^{\circ}(a)^+ &\equiv \underline{\pi}'^{\circ}(a)^+\underline{\pi}'(a)^+, \\
\underline{\pi}'(a)^-\underline{\pi}'^{\circ}(a)^- &\equiv \underline{\pi}'^{\circ}(a)^-\underline{\pi}'(a)^-, \\
\underline{\pi}'(a)^+\underline{\pi}'^{\circ}(a)^- + \underline{\pi}'(a)^-\underline{\pi}'^{\circ}(a)^+ &\equiv \underline{\pi}'^{\circ}(a)^+\underline{\pi}'(a)^- + \underline{\pi}'^{\circ}(a)^-\underline{\pi}'(a)^+.
\end{aligned}$$

By (3.2.2), the operators $\underline{\pi}'(a)$ and $\underline{\pi}'(b)$, as well as D, are diagonal for the decomposition $\mathcal{H} = \mathcal{H}^{\uparrow} \oplus \mathcal{H}^{\downarrow}$. On the subspace $\mathcal{H}^{\uparrow}$, we obtain

$$\begin{aligned}
&[[D, \underline{\pi}'(a)], \underline{\pi}'^{\circ}(a)]\,|j\mu n{\uparrow}\rangle \\
&= \big(D\underline{\pi}'(a)\underline{\pi}'^{\circ}(a) + \underline{\pi}'^{\circ}(a)\underline{\pi}'(a)D - \underline{\pi}'(a)D\underline{\pi}'^{\circ}(a) - \underline{\pi}'^{\circ}(a)D\underline{\pi}'(a)\big)\,|j\mu n{\uparrow}\rangle \\
&= \Big((d_{j+1}^{\uparrow} + d_j^{\uparrow} - 2d_{j^+}^{\uparrow})\,\underline{\pi}'(a)^+\underline{\pi}'^{\circ}(a)^+ + (d_{j-1}^{\uparrow} + d_j^{\uparrow} - 2d_{j^-}^{\uparrow})\,\underline{\pi}'(a)^-\underline{\pi}'^{\circ}(a)^- \\
&\qquad + 2d_j^{\uparrow}\,(\underline{\pi}'(a)^+\underline{\pi}'^{\circ}(a)^- + \underline{\pi}'(a)^-\underline{\pi}'^{\circ}(a)^+) - d_{j^+}^{\uparrow}\,(\underline{\pi}'(a)^-\underline{\pi}'^{\circ}(a)^+ + \underline{\pi}'^{\circ}(a)^-\underline{\pi}'(a)^+) \\
&\qquad - d_{j^-}^{\uparrow}\,(\underline{\pi}'(a)^+\underline{\pi}'^{\circ}(a)^- + \underline{\pi}'^{\circ}(a)^+\underline{\pi}'(a)^-) + R\Big)\,|j\mu n{\uparrow}\rangle,
\end{aligned} \tag{3.2.8}$$

where $R \in \mathcal{K}_q$. On the subspace $\mathcal{H}^{\downarrow}$, we get the precisely analogous expression with the arrows reversed.

In order that the expression on the right hand side of (3.2.8) comes from an element of $\mathcal{K}_q$ applied to $|j\mu n\uparrow\rangle$, and likewise for $|j\mu n\downarrow\rangle$, it is necessary and sufficient that the scalars

$$w_j^{\uparrow} := d_{j+1}^{\uparrow} + d_j^{\uparrow} - 2d_{j^+}^{\uparrow}, \qquad w_j^{\downarrow} := d_{j+1}^{\downarrow} + d_j^{\downarrow} - 2d_{j^+}^{\downarrow} \tag{3.2.9}$$

satisfy $w_j^{\uparrow} = O(q^j)$ and $w_j^{\downarrow} = O(q^j)$ as $j \to \infty$.

In the particular case where $w_j^{\uparrow} = 0$ and $w_j^{\downarrow} = 0$ for all j, equation (3.2.9) gives elementary recurrence relations for $d_j^{\uparrow}$ and $d_j^{\downarrow}$, whose solutions are precisely the expressions (2.4.3) that are linear in j, namely,

$$d_j^{\uparrow} = c_1^{\uparrow} j + c_2^{\uparrow}, \qquad d_j^{\downarrow} = c_1^{\downarrow} j + c_2^{\downarrow}.$$

The general case gives a pair of perturbed recurrence relations, that may be treated by generating function methods [50]; their solutions differ from the linear case by terms that are $O(q^j)$ as $j \to \infty$. Thus, the corresponding operator D differs from one whose eigenvalues are linear in j by an element of $\mathcal{K}_q$. □

We finish by summarizing the implications of the above Propositions 3.6, 3.7 and 3.8 for the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D, J)$, where $\mathcal{A}(SU_q(2))$ acts on $\mathcal{H}$ via the spinor representation π'.

The representations π' and π'° do not commute, since the conjugation operator J differs from the Tomita conjugation for π'. However, we do obtain commutation "up to infinitesimals of arbitrary order"; since $[\pi'^{\circ}(x), \pi'(y)] \equiv [\underline{\pi}'^{\circ}(x), \underline{\pi}'(y)] \bmod \mathcal{K}_q$, Proposition 3.7 entails the analogous result for the exact representations:

$$[\pi'^{\circ}(x), \pi'(y)] \in \mathcal{K}_q \quad \text{for all} \quad x, y \in \mathcal{A}.$$

To examine the first-order property, we note first if $x, y \in \mathcal{A}$ and $[D, \pi'(y) - \underline{\pi}'(y)]$ lies in $\mathcal{K}_q$, then

$$\begin{aligned} [\pi'^{\circ}(x), [D, \pi'(y)]] &= \big[\underline{\pi}'^{\circ}(x) + (\pi'^{\circ}(x) - \underline{\pi}'^{\circ}(x)),\ [D, \underline{\pi}'(y) + (\pi'(y) - \underline{\pi}'(y))]\big] \\ &\equiv [\underline{\pi}'^{\circ}(x), [D, \underline{\pi}'(y)]] \equiv 0 \bmod \mathcal{K}_q. \end{aligned} \tag{3.2.10}$$

Since D commutes with the positive operator T defined in (3.2.6), we find in the case of a generator $y = a$, a^*, b or b^*, that

$$[D, \pi'(y) - \underline{\pi}'(y)] = [D, T\pi'(y)T] = T[D, \pi'(y)]T,$$

which lies in $\mathcal{K}_q$ since $[D, \pi'(y)]$ is bounded, by Proposition 2.13. Thus, $[D, \underline{\pi}'(y)]$ is bounded, too –as required by Proposition 3.8. The general case of $[D, \pi'(y) - \underline{\pi}'(y)] \in \mathcal{K}_q$ then follows from (3.2.4). Thus (3.2.10) holds for general $x, y \in \mathcal{A}$. Combining that with Proposition 3.8, we arrive at the following characterization of our spectral triple over $\mathcal{A}(SU_q(2))$.

Theorem 3.9. *The real spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D, J)$ defined here, with $\mathcal{A}(SU_q(2))$ acting on $\mathcal{H}$ via the spinor representation π', satisfies both the commutant property and the first order condition up to compact operators:*

$$\left.\begin{aligned} [\pi'^{\circ}(x), \pi'(y)] &\in \mathcal{K}_q, \\ [\pi'^{\circ}(x), [D, \pi'(y)]] &\in \mathcal{K}_q, \end{aligned}\right. \qquad \textit{for all} \quad x, y \in \mathcal{A}(SU_q(2)).$$

In [54] it was argued that there are obstructions to the construction of a "deformed spectral triples" for SU(2). Such a deformed spectral triple [1] for a (pseudo)Riemannian manifold M is defined as a triple $(\mathcal{A}, V, D)$ where $\mathcal{A}$ is an algebra, V an $\mathcal{A}$-bimodule (not necessarily a Hilbert space) and D a linear operator in V, such that $\mathcal{A}$, V and D reduce *modulo* $\hbar$ to $C^\infty(M)$, $\Gamma(M, \mathcal{S})$ and the Dirac operator on $\mathcal{S}$, respectively, for a spinor bundle $\mathcal{S} \to M$. Furthermore, one imposes the conditions:

1. For any $a \in \mathcal{A}$, $[D, a]$ commutes with the right multiplication of $\mathcal{A}$ on V.
2. The construction of noncommutative differential forms from $\mathcal{A}$ and D (cf. Appendix A.3) gives a deformation of 1-forms on M, that is, $\Omega^1_D(\mathcal{A})/\hbar\Omega^1_D(\mathcal{A}) \simeq \Omega^1_{\mathrm{dR}}(M)$.

With this general definition, it was proved that it is impossible to construct a deformed spectral triple on $SU(2)$. Theorem 3.9 above shows a way to overcome this obstruction by relaxing the first condition, together with the condition of V to be an $\mathcal{A}$-bimodule, to hold only up to operators in the ideal $\mathcal{K}_q$. Moreover, the differential calculus constructed from $\mathcal{A}(SU_q(2))$ and D is not necessarily finite dimensional; therefore, it might not be a deformation of the de Rham differential calculus on M.

[1] For the more precise definition, we refer to [54]

Chapter 4

The local index formula for $SU_q(2)$

We now discuss the Connes-Moscovici local index formula [34] in the case of our spectral triple on $SU_q(2)$. We refer to Appendix A.7 for more details on the general statement of the local index formula.

The treatment closely follows the discussion in [31] on the general theory of Connes-Moscovici applied to the "singular" (in the sense of not admitting a commutative limit) spectral triple that was constructed in [20]. It turns out that most of our results coincide with the ones found therein.

The main idea of that paper is to construct a (quantum) cosphere bundle S_q^* on $SU_q(2)$, that considerably simplifies the computations concerning the local index formula. Essentially, with the operator derivation δ defined by $\delta(T) := |D|T - T|D|$, one considers an operator x in the algebra $\mathcal{B} = \bigcup_{n=0}^{\infty} \delta^n(\mathcal{A})$ up to smoothing operators; these give no contribution to the residues appearing in the local cyclic cocycle giving the local index formula. The removal of the irrelevant smoothing operators is accomplished by introducing a symbol map from $SU_q(2)$ to the cosphere bundle S_q^*. The latter is defined by its algebra $C^\infty(S_q^*)$ of "smooth functions" which is, by definition, the image of a map

$$\rho : \mathcal{B} \to C^\infty(D_{q+}^2 \times D_{q-}^2 \times S^1)$$

where $D_{q\pm}^2$ are two quantum disks. One finds that an element x in the algebra $\mathcal{B}$ can be determined up to smoothing operators by $\rho(x)$.

In our present case, the cosphere bundle coincides with the one obtained in [31]; the same being true for the dimension spectrum. Indeed, using this much simpler form of operators up to smoothing ones, it is not difficult to compute the dimension spectrum and obtain simple expressions for the residues appearing in the local index formula. We find that the dimension spectrum is simple and given by the set $\{1, 2, 3\}$.

The cyclic cohomology of the algebra $\mathcal{A}(SU_q(2))$ has been computed explicitly in [74] where it was found to be given in terms of a single generator. We express this element in terms of a single local cocycle similarly to the computations in [31]. But contrary to the latter, we get an extra term involving $P|D|^{-3}$ which drops in [31], being traceclass for the case considered there. Here $P = \frac{1}{2}(1 + F)$ with $F = \mathrm{Sign} D$, the sign of the operator D.

Finally as a simple example, we compute the Fredholm index of D coupled with the unitary representative of the generator of $K_1(\mathcal{A}(SU_q(2)))$.

It turns out that working modulo infinitesimals of arbitrary order like before, simplifies the discussion drastically. Moreover, we make the following choice for our Dirac operator

$$D|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle.$$

whose spectrum (with multiplicity!) coincides with that of the classical Dirac operator of the sphere S^3 equipped with the round metric (indeed, the spin geometry of the 3-sphere can now be recovered by taking $q = 1$). Indeed, apart from the issue of their signs, the particular constants that appear in (2.4.3) are fairly immaterial: $c_2^{\uparrow}$ and $c_2^{\downarrow}$ do not affect the index calculations later on while $c_1^{\uparrow}$ and $|c_1^{\downarrow}|$ yield scaling factors on some noncommutative integrals.

4.1 Regularity and the cosphere bundle

Let us first establish regularity for the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$. Recall [16, 34, 49] (cf. Appendix A.2) that this means that the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ should lie within the smooth domain $\bigcap_{n=0}^{\infty} \mathrm{Dom}\,\delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$.

We let $D = F|D|$ be the polar decomposition of D where $|D| := (D^2)^{\frac{1}{2}}$ and $F = \mathrm{Sign}D$. Explicitly, we see that

$$F|j\mu n\rangle\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |j\mu n\rangle\rangle, \qquad |D|\,|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & 2j+\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle.$$

Clearly, $P^{\uparrow} := \frac{1}{2}(1+F)$ and $P^{\downarrow} := \frac{1}{2}(1-F) = 1 - P^{\uparrow}$ are the orthogonal projectors whose range spaces are $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$, respectively.

In the following we will denote by a_+ and a_- the operators on $\mathcal{H}$ that add up to give $\pi'(a)$ in obvious notation, and similarly for b_+ and b_-.

Proposition 4.1. *The triple* $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ *is a regular spectral triple.*

Proof. Since $2j + \frac{3}{2} = 2j^+ + \frac{1}{2}$ and $2j + \frac{1}{2} = 2j^- + \frac{3}{2}$ and due to the triangular forms of the matrices in (2.3.8), the off-diagonal terms vanish in the 2×2-matrix expressions for $\delta(a_+)$ and $\delta(a_-)$. Indeed one finds,

$$\delta(a_+)|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{5}{2} & 0 \\ 0 & 2j+\frac{3}{2} \end{pmatrix} a_+|j\mu n\rangle\rangle - a_+ \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & 2j+\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle,$$
$$\delta(a_-)|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{1}{2} & 0 \\ 0 & 2j-\frac{1}{2} \end{pmatrix} a_-|j\mu n\rangle\rangle - a_- \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & 2j+\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle.$$

In both cases we obtain

$$\delta(a_+) = P^{\uparrow} a_+ P^{\uparrow} + P^{\downarrow} a_+ P^{\downarrow}, \qquad \delta(a_-) = -P^{\uparrow} a_- P^{\uparrow} - P^{\downarrow} a_- P^{\downarrow}.$$

Replacing a by b, the same triangular matrix structure leads to

$$\delta(b_+) = P^{\uparrow} b_+ P^{\uparrow} + P^{\downarrow} b_+ P^{\downarrow}, \qquad \delta(b_-) = -P^{\uparrow} b_- P^{\uparrow} - P^{\downarrow} b_- P^{\downarrow}.$$

Thus $\delta(\pi(a)) = \delta(a_+) + \delta(a_-)$ is bounded, with $\|\delta(\pi(a))\| \leq \|\pi(a)\|$; and likewise for $\pi(b)$. Next, $\delta([D, a_+]) = [D, \delta(a_+)]$, so that

$$\delta([D, a_+])|j\mu n\rangle\rangle = \begin{pmatrix} 2j+\frac{5}{2} & 0 \\ 0 & -2j-\frac{3}{2} \end{pmatrix} \delta(a_+)|j\mu n\rangle\rangle - \delta(a_+) \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle,$$

since all matrices appearing are diagonal. This, together with the analogous calculation for $\delta([D, a_-])$, shows that

$$\delta([D, a_+]) = P^\uparrow a_+ P^\uparrow - P^\downarrow a_+ P^\downarrow, \qquad \delta([D, a_-]) = P^\uparrow a_- P^\uparrow - P^\downarrow a_- P^\downarrow. \tag{4.1.1}$$

A similar argument for b gives

$$\delta([D, b_+]) = P^\uparrow b_+ P^\uparrow - P^\downarrow b_+ P^\downarrow, \qquad \delta([D, b_-]) = P^\uparrow b_- P^\uparrow - P^\downarrow b_- P^\downarrow. \tag{4.1.2}$$

Combining (4.1.1), (4.1.1), and the analogous relations with a replaced by b, we see that both $\mathcal{A}$ and $[D, \mathcal{A}]$ lie within Domδ. An easy induction shows that they also lie within Domδ^k for $k = 2, 3, \ldots$. □

This proposition continues to hold if we replace $\mathcal{A}(SU_q(2))$ by a suitably completed algebra, which is stable under the holomorphic function calculus (cf. Appendix A.2).

Let $\Psi^0(\mathcal{A})$ be the algebra generated by $\delta^k(\mathcal{A})$ and $\delta^k([D, \mathcal{A}])$ for all $k \geq 0$ (the notation suggests that, in the spirit of [34] one thinks of it as an "algebra of pseudodifferential operators of order 0"). Since, for instance,

$$P^\uparrow \pi(a) P^\uparrow = \tfrac{1}{2}\delta^2(\pi(a)) + \tfrac{1}{2}\delta([D, \pi(a)]),$$
$$P^\uparrow a_+ P^\uparrow = \tfrac{1}{2}P^\uparrow \pi(a) P^\uparrow + \tfrac{1}{2}P^\uparrow \delta(\pi(a)) P^\uparrow,$$

we see that $\Psi^0(\mathcal{A})$ is in fact generated by the diagonal-corner operators $P^\uparrow a_\pm P^\uparrow$, $P^\downarrow a_\pm P^\downarrow$, $P^\uparrow b_\pm P^\uparrow$, $P^\downarrow b_\pm P^\downarrow$, together with the other-corner operators $P^\downarrow a_+ P^\uparrow$, $P^\uparrow a_- P^\downarrow$, $P^\downarrow b_+ P^\uparrow$, and $P^\uparrow b_- P^\downarrow$. Following [31], let $\mathcal{B}$ be the algebra generated by all $\delta^n(\mathcal{A})$ for $n \geq 0$. It is a subalgebra of $\Psi^0(\mathcal{A})$ and it is generated by the diagonal operators

$$\widetilde{a}_\pm := \pm\delta(a_\pm) = P^\uparrow a_\pm P^\uparrow + P^\downarrow a_\pm P^\downarrow, \qquad \widetilde{b}_\pm := \pm\delta(b_\pm) = P^\uparrow b_\pm P^\uparrow + P^\downarrow b_\pm P^\downarrow, \tag{4.1.3}$$

and by the off-diagonal operators $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$ and $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$.

In Proposition 3.6 we introduced an approximate representation of $\mathcal{A}(SU_q(2))$ such that the operators $\underline{\pi}'(x) - \pi'(x)$ are given by sequences of rapid decay (i.e., in the ideal $\mathcal{K}_q$), and hence are elements in OP$^{-\infty}$ (as defined in Appendix A.2). Therefore, we can replace π' by $\underline{\pi}'$ when dealing with the local cocycle in the local index theorem in the next section. If we write $\underline{\pi}'(a) = \underline{a}_+ + \underline{a}_-$, it is not difficult to see that

$$[|D|, \underline{\pi}'(a)] = \underline{a}_+ - \underline{a}_-, \qquad [D, \underline{\pi}'(a)] = F(\underline{a}_+ - \underline{a}_-),$$
$$[|D|, \underline{\pi}'(b)] = \underline{b}_+ - \underline{b}_-, \qquad [D, \underline{\pi}'(b)] = F(\underline{b}_+ - \underline{b}_-),$$

and also that F commutes with $\underline{a}_\pm$ and $\underline{b}_\pm$. The operators $\underline{a}_\pm$ and $\underline{b}_\pm$ have a simpler expression if we use the following relabelling of the orthonormal basis of $\mathcal{H}$,

$$v^j_{xy\uparrow} := |j, x-j, y-j-\tfrac{1}{2}, \uparrow\rangle \quad \text{for} \quad x = 0, \ldots, 2j;\ y = 0, \ldots, 2j+1,$$
$$v^j_{xy\downarrow} := |j, x-j, y-j+\tfrac{1}{2}, \downarrow\rangle \quad \text{for} \quad x = 0, \ldots, 2j;\ y = 0, \ldots, 2j-1. \tag{4.1.4}$$

We again employ the pairs of vectors

$$v^j_{xy} := \begin{pmatrix} v^j_{xy\uparrow} \\ v^j_{xy\downarrow} \end{pmatrix},$$

where the lower component is understood to be zero if $y = 2j$ or $2j+1$, or if $j = 0$. The simplification is that on these vector pairs, all the 2×2 matrices in (3.2.1) become scalar matrices,[1]

$$\begin{aligned} \underline{a}_+ v^j_{xy} &= \sqrt{1-q^{2x+2}}\sqrt{1-q^{2y+2}}\, v^{j^+}_{x+1,y+1}, \\ \underline{a}_- v^j_{xy} &= q^{x+y+1}\, v^{j^-}_{xy}, \\ \underline{b}_+ v^j_{xy} &= q^y \sqrt{1-q^{2x+2}}\, v^{j^+}_{x+1,y}, \\ \underline{b}_- v^j_{xy} &= -q^x \sqrt{1-q^{2y}}\, v^{j^-}_{x,y-1}. \end{aligned} \tag{4.1.5}$$

These formulas coincide with those found in [31, Sec. 6] up to a doubling of the Hilbert space and the change of conventions $a \leftrightarrow a^*$, $b \leftrightarrow -b$. Indeed, since the spin representation is isomorphic to a direct sum of two copies of the regular representation, the formulas in (4.1.5) exhibit the same phenomenon for the approximate representations.

In [31] Connes constructs a "cosphere bundle" using the regular representation of $\mathcal{A}(SU_q(2))$. In view of (4.1.5), the same cosphere bundle may be obtained directly from the spin representation by adapting that construction, as we now proceed to do. In what follows, we use the algebra $\mathcal{A} = \mathcal{A}(SU_q(2))$, but we could as well replace it with its completion $C^\infty(SU_q(2))$, which is closed under holomorphic functional calculus (see Section 4.1.1).

We recall two well-known infinite dimensional representations $\pi_\pm$ of $\mathcal{A}(SU_q(2))$ by bounded operators on the Hilbert space $\ell^2(\mathbb{N})$. On the standard orthonormal basis $\{\epsilon_x : x \in \mathbb{N}\}$, they are given by

$$\pi_\pm(a)\,\epsilon_x := \sqrt{1-q^{2x+2}}\,\epsilon_{x+1}, \qquad \pi_\pm(b)\,\epsilon_x := \pm q^x\,\epsilon_x. \tag{4.1.6}$$

We may identify the Hilbert space $\mathcal{H}$ spanned by all $v^j_{xy\uparrow}$ and $v^j_{xy\downarrow}$ with the subspace $\mathcal{H}'$ of $\ell^2(\mathbb{N})_x \otimes \ell^2(\mathbb{N})_y \otimes \ell^2(\mathbb{Z})_{2j} \otimes \mathbb{C}^2$ determined by the parameter restrictions in (4.1.4). Thereby, we get the correspondence

$$\begin{aligned} \underline{a}_+ &\leftrightarrow \pi_+(a) \otimes \pi_-(a) \otimes V \otimes 1_2, \\ \underline{a}_- &\leftrightarrow -q\,\pi_+(b) \otimes \pi_-(b^*) \otimes V^* \otimes 1_2, \\ \underline{b}_+ &\leftrightarrow -\pi_+(a) \otimes \pi_-(b) \otimes V \otimes 1_2, \\ \underline{b}_- &\leftrightarrow -\pi_+(b) \otimes \pi_-(a^*) \otimes V^* \otimes 1_2, \end{aligned} \tag{4.1.7}$$

where V is the unilateral shift operator $\epsilon_{2j} \mapsto \epsilon_{2j+1}$ in $\ell^2(\mathbb{Z})$. This again, apart from the 2×2 identity matrix 1_2, coincides with the formula (204) in [31], up to the aforementioned exchange of the generators.

The shift V in the action of the operators $\underline{a}_\pm$ and $\underline{b}_\pm$ on $\mathcal{H}$ can be encoded using the $\mathbb{Z}$-grading coming from the one-parameter group of automorphisms $\gamma(t)$ generated by $|D|$,

$$\gamma(t) = \begin{pmatrix} \gamma_{\uparrow\uparrow}(t) & \gamma_{\uparrow\downarrow}(t) \\ \gamma_{\downarrow\uparrow}(t) & \gamma_{\downarrow\downarrow}(t) \end{pmatrix}, \quad \text{where} \quad \begin{cases} \gamma_{\uparrow\uparrow}(t) : P^\uparrow T P^\uparrow \mapsto P^\uparrow e^{it|D|} T e^{-it|D|} P^\uparrow, \\ \gamma_{\uparrow\downarrow}(t) : P^\uparrow T P^\downarrow \mapsto P^\uparrow e^{it|D|} T e^{-it|D|} P^\downarrow, \\ \gamma_{\downarrow\uparrow}(t) : P^\downarrow T P^\uparrow \mapsto P^\downarrow e^{it|D|} T e^{-it|D|} P^\uparrow, \\ \gamma_{\downarrow\downarrow}(t) : P^\downarrow T P^\downarrow \mapsto P^\downarrow e^{it|D|} T e^{-it|D|} P^\downarrow, \end{cases}$$

[1] We simplified (3.2.1) a little further by using the equality $1 - \sqrt{1-q^\alpha} \le q^\alpha$.

for any operator T on $\mathcal{H}$. On the subalgebra of "diagonal" operators $T = P^{\uparrow}TP^{\uparrow} + P^{\downarrow}TP^{\downarrow}$, the compression $\gamma_{\uparrow\uparrow} \oplus \gamma_{\downarrow\downarrow}$ detects the shift of j of the restrictions of T to $\mathcal{H}^{\uparrow}$ and $\mathcal{H}^{\downarrow}$ respectively. For example, $\gamma_{\uparrow\uparrow}(t) \oplus \gamma_{\downarrow\downarrow}(t) : a_{\pm} \mapsto e^{\pm it}a_{\pm}$, so that the $\mathbb{Z}$-grading encodes the correct shifts $j \to j \pm \frac{1}{2}$ in the formulas for $a_{\pm}$; and likewise for $b_{\pm}$.

From equation (4.1.6) it follows that $b - b^* \in \ker \pi_{\pm}$, and so the representations $\pi_{\pm}$ are not faithful on $\mathcal{A}(SU_q(2))$. We define two algebras $\mathcal{A}(D^2_{q\pm})$ to be the corresponding quotients,

$$0 \to \ker \pi_{\pm} \to \mathcal{A}(SU_q(2)) \xrightarrow{r_{\pm}} \mathcal{A}(D^2_{q\pm}) \to 0. \tag{4.1.8}$$

We elaborate a little on the structure of the algebras $\mathcal{A}(D^2_{q\pm})$. For convenience, we shall omit the quotient maps $r_{\pm}$ in this discussion. Then $b = b^*$ in $\mathcal{A}(D^2_{q\pm})$, and from the defining relations (2.1.1) of $\mathcal{A}(SU_q(2))$, we obtain

$$ba = q\,ab, \qquad a^*b = q\,ba^*,$$
$$a^*a + q^2b^2 = 1, \qquad aa^* + b^2 = 1.$$

These algebraic relations define two isomorphic quantum 2-spheres $S^2_{q+} \simeq S^2_{q-} =: S^2_q$ which have a classical subspace S^1 given by the characters $b \mapsto 0$, $a \mapsto \lambda$ with $|\lambda| = 1$. A substitution $q \mapsto q^2$, followed by $b \mapsto q^{-2}b$ shows that S^2_q is none other than the equatorial Podleś sphere [79]. Thus, the above quotients of $\mathcal{A}(SU_q(2))$ with respect to $\ker \pi_{\pm}$ either coincide with $\mathcal{A}(S^2_q)$ or are quotients of it. Now, from (4.1.6) one sees that the spectrum of $\pi_{\pm}(b)$ is either real positive or real negative, depending on the $\pm$ sign. Hence, the algebras $\mathcal{A}(D^2_{q+})$ and $\mathcal{A}(D^2_{q+})$ describe the two hemispheres of S^2_q and may be thought of as quantum disks, thus justifying the notation $D_{q\pm}$.

There is a symbol map $\sigma\colon \mathcal{A}(D^2_{q\pm}) \to \mathcal{A}(S^1)$ that maps these "noncommutative disks" to their common boundary S^1, which is the equator of the equatorial Podleś sphere S^2_q. Explicitly, the symbol map is given as a $*$-homomorphism on the generators of $\mathcal{A}(D^2_{q,\pm})$ by

$$\sigma(r_{\pm}(a)) := u; \qquad \sigma(r_{\pm}(b)) := 0, \tag{4.1.9}$$

where u is the unitary generator of $\mathcal{A}(S^1)$.

Recall the algebra $\mathcal{B}$ defined around (4.1.3) with generators $\widetilde{a}_{\pm}$, $\widetilde{b}_{\pm}$ and $P^{\downarrow}a_+P^{\uparrow} + P^{\uparrow}a_-P^{\downarrow}$, $P^{\downarrow}b_+P^{\uparrow} + P^{\uparrow}b_-P^{\downarrow}$. The following result emulates Proposition 4 of [31] and establishes the correspondence (4.1.7). Proposition 3.6 is crucial to its proof.

Proposition 4.2. *There is a $*$-homomorphism*

$$\rho : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}) \otimes \mathcal{A}(S^1) \tag{4.1.10}$$

defined on generators by

$$\rho(\widetilde{a}_+) := r_+(a) \otimes r_-(a) \otimes u, \qquad \rho(\widetilde{a}_-) := -q\, r_+(b) \otimes r_-(b^*) \otimes u^*,$$
$$\rho(\widetilde{b}_+) := -r_+(a) \otimes r_-(b) \otimes u, \qquad \rho(\widetilde{b}_-) := -r_+(b) \otimes r_-(a^*) \otimes u^*.$$

while the off-diagonal operators $P^{\downarrow}a_+P^{\uparrow} + P^{\uparrow}a_-P^{\downarrow}$ and $P^{\downarrow}b_+P^{\uparrow} + P^{\uparrow}b_-P^{\downarrow}$ are declared to lie in the kernel of ρ.

Proof. First note that the j-dependence of the operators in $\mathcal{B}$ is taken care of by the factor u. Thus, it is enough to show that the following prescription,

$$\rho_1(\tilde{a}_+) := \pi_+(a) \otimes \pi_-(a), \qquad \rho_1(\tilde{a}_-) := -q\,\pi_+(b) \otimes \pi_-(b^*),$$
$$\rho_1(\tilde{b}_+) := -\pi_+(a) \otimes \pi_-(b), \qquad \rho_1(\tilde{b}_-) := -\pi_+(b) \otimes \pi_-(a^*),$$

together with $\rho_1(P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow) = \rho_1(P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow) := 0$, defines a $*$-homomorphism $\rho_1 : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-})$. In the notation, we have replaced the representations $\pi_\pm$ of $\mathcal{A}(SU_q(2))$ by corresponding faithful representations of $\mathcal{A}(D^2_{q\pm})$ (omitting the maps $r_\pm$).

We define a map $\Pi : \mathcal{H} \to (\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})) \otimes \mathbb{C}^2$, which simply forgets the j-index on the basis vectors v^j_{xy}:

$$\Pi : v^j_{xy} = \begin{pmatrix} v^j_{xy\uparrow} \\ v^j_{xy\downarrow} \end{pmatrix} \mapsto \epsilon_{xy} := \begin{pmatrix} \epsilon_{xy\uparrow} \\ \epsilon_{xy\downarrow} \end{pmatrix},$$

where $\epsilon_{xy\uparrow} := \epsilon_x \otimes \epsilon_y$ and $\epsilon_{xy\downarrow} := \epsilon_x \otimes \epsilon_y$ in the two respective copies of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ in its tensor product with $\mathbb{C}^2$.

For any operator T in $\mathcal{B}$, we define the map ρ_1 by

$$\rho_1(T)\epsilon_{xy} = \lim_{j\to\infty} \Pi(T v^j_{xy}).$$

This map is well-defined, since T is a polynomial in the generators of $\mathcal{B}$. Each such generator shifts the indices x, y, j by $\pm\frac{1}{2}$, with a coefficient matrix that can be bounded uniformly in x, y and j (cf. the paragraph above (2.4.5)) so that the limit $j \to \infty$ exists.

We slightly extend the domain of the map ρ_1 by adjoining the principal ideal $\mathcal{K}_q$ to it. The limit is well-defined and in fact, $\rho_1(\mathcal{K}_q) = 0$. Indeed, for any $\delta > 0$, there exists a J such that for $j > J$ we have $\|L_q v^j_{xy\uparrow}\| = \|L_q v^j_{xy\downarrow}\| < q^J < \delta$.

From this, it follows that the off-diagonal operators $P^\downarrow a_+ P^\uparrow + P^\uparrow a_- P^\downarrow$ and $P^\downarrow b_+ P^\uparrow + P^\uparrow b_- P^\downarrow$ are in the kernel of ρ_1 as they are elements in $\mathcal{K}_q$. Moreover, we can replace $\tilde{a}_\pm$ and $\tilde{b}_\pm$ by $\underline{a}_\pm$ and $\underline{b}_\pm$, respectively, since their differences lie in $\mathcal{K}_q$. Since the coefficients in the definition of $\underline{a}_\pm$ and $\underline{b}_\pm$ (equation (4.1.5)) are j-independent, we conclude that ρ_1 is of the desired form. For example, we compute:

$$\begin{aligned} \rho_1(\tilde{a}_+)\epsilon_{xy} = \rho_1(\underline{a}_+)\epsilon_{xy} &= \lim_{j\to\infty} \sqrt{1-q^{2x+2}}\sqrt{1-q^{2y+2}}\,\Pi(v^{j^+}_{x+1,y+1}) \\ &= \sqrt{1-q^{2x+2}}\sqrt{1-q^{2y+2}}\,\epsilon_{x+1,y+1} = (\pi_+(a) \otimes \pi_-(a) \otimes 1_2)\epsilon_{xy}. \end{aligned}$$

Since a product of the operators $\underline{a}_\pm$ and $\underline{b}_\pm$ still does not contain j-dependent coefficients, ρ_1 respects the multiplication in $\mathcal{B}$. By linearity of the limit, ρ_1 is an algebra map. □

Definition 4.3. *The* cosphere bundle on $SU_q(2)$ *is defined as the range of the map* ρ *in* $\mathcal{A}(D^2_{q+}) \otimes A(D^2_{q-}) \otimes \mathcal{A}(S^1)$ *and is denoted by* $\mathcal{A}(S^*_q)$.

Note that S^*_q coincides with the cosphere bundle defined in [31, 30], where it is regarded as a noncommutative space over which $D^2_{q+} \times D^2_{q-} \times S^1$ is fibred.

The symbol map ρ rectifies the correspondence (4.1.7). Denote by Q the orthogonal projector on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ with range $\mathcal{H}'$, which is the Hilbert subspace previously identified with $\mathcal{H}$ just before (4.1.7). Using (4.1.7) in combination with Proposition 4.2, we conclude that

$$T - Q(\rho(T) \otimes 1_2)Q \in \mathrm{OP}^{-\infty} \quad \text{for all} \quad T \in \mathcal{B}. \tag{4.1.11}$$

Here, the action of $\rho(T)$ on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ is determined by regarding $\ell^2(\mathbb{Z})$ as the Hilbert space of square-summable Fourier series on S^1.

4.1.1 Smooth algebras

Thus far, we have employed finitely generated algebras $\mathcal{A}(X)$, where $X = SU_q(2)$, $D^2_{q\pm}$, S^1 or S^2_q. In each case, we can enlarge them to algebras $C^\infty(X)$ by replacing polynomials in the generators (given in a prescribed order) by series with coefficients of rapid decay: this is clear when $X = S^1$, where smooth functions have rapidly decaying Fourier series. Using the symbol maps (4.1.8), (4.1.9) and (4.1.10) we can "pullback" this smooth structure to the noncommutative spaces $SU_q(2), D^2_{q\pm}$ and S^2_q using the following technical Lemma taken from [31]. Suppose that (B, H, D) is a spectral triple and let $\rho : B \to C$ be a morphism of C^*-algebras.

Lemma 4.4. *Let $\mathcal{C} \subset C$ be a subalgebra stable under holomorphic calculus together with a linear map $\lambda : \mathcal{C} \to \mathcal{L}(\mathcal{H})$ such that $\lambda(1) = 1$, $\lambda(c) \in \mathrm{OP}^0$ for all $c \in \mathcal{C}$ and $\lambda(a)\lambda(b) - \lambda(ab) \in \mathrm{OP}^{-\infty}$ for all $a, b \in \mathcal{C}$.*
The algebra $\mathcal{B} := \{x \in B | x \in \mathrm{OP}^0, \rho(x) \in \mathcal{C}, x - \lambda(\rho(x)) \in \mathrm{OP}^{-\infty}\} \subset B$ is a subalgebra stable under holomorphic function calculus.

Proof. Let $x \in \mathcal{B}$ be invertible in B. Let $a = \rho(x) \in \mathcal{C}$; then since $\mathcal{C}$ is stable under holomorphic calculus its inverse $b = \rho(x^{-1})$ belongs to $\mathcal{C}$. Also, since $x \in \mathrm{OP}^0$, we have that $x^{-1} \in \mathrm{OP}^0$. We want to establish that $x^{-1} - \lambda(b) \in \mathrm{OP}^{-\infty}$. Since $ab = 1$, we have by the above properties of λ that $\lambda(a)\lambda(b) - 1 \in \mathrm{OP}^{-\infty}$. Using the fact that $\mathrm{OP}^{-\infty}$ is a two-sided ideal in OP^0, we obtain from $x - \lambda(a) \in \mathrm{OP}^{-\infty}$ that $x\lambda(b) - 1 \in \mathrm{OP}^{-\infty}$, by multiplying by $\lambda(b)$ on the right. Multiplying this on the left by $x^{-1} \in \mathrm{OP}^0$, we get that $x^{-1} - \lambda(b) \in \mathrm{OP}^{-\infty}$. □

Thus, the algebras $C^\infty(X)$ for the above noncommutative spaces are closed under holomorphic functional calculus. All foregoing and upcoming results apply, mutatis mutandis, to the regular spectral triple $(C^\infty(SU_q(2)), \mathcal{H}, D)$.

4.2 The dimension spectrum

We again follow [31] for the computation of the dimension spectrum. We define three linear functionals $\tau_0^\uparrow$, $\tau_0^\downarrow$ and τ_1 on the algebras $\mathcal{A}(D^2_{q\pm})$. Since their definitions for both disks D^2_{q+} and D^2_{q-} are identical, we shall omit the $\pm$ for notational convenience.

For $x \in \mathcal{A}(D^2_q)$ we define,

$$\tau_1(x) := \frac{1}{2\pi}\int_{S^1} \sigma(x),$$
$$\tau_0^\uparrow(x) := \lim_{N\to\infty} \mathrm{Tr}_N\, \pi(x) - (N + \tfrac{3}{2})\tau_1(x),$$
$$\tau_0^\downarrow(x) := \lim_{N\to\infty} \mathrm{Tr}_N\, \pi(x) - (N + \tfrac{1}{2})\tau_1(x),$$

where σ is the symbol map (4.1.9), and Tr_N is the truncated trace

$$\mathrm{Tr}_N(T) := \sum_{k=0}^{N} \langle \epsilon_k | T \epsilon_k \rangle.$$

The definition of the two different maps $\tau_0^\uparrow$ and $\tau_0^\downarrow$ is suggested by the constants $\frac{3}{2}$ and $\frac{1}{2}$ appearing in our choice of the Dirac operator; it will simplify some residue formulas later on. We find that

$$\begin{aligned}\mathrm{Tr}_N(\pi(a)) &= (N+\tfrac{3}{2})\tau_1(a) + \tau_0^\uparrow(a) + O(N^{-k})\\ &= (N+\tfrac{1}{2})\tau_1(a) + \tau_0^\downarrow(a) + O(N^{-k}) \quad \text{for all} \quad k>0.\end{aligned}$$

Let us denote by r the restriction homomorphism from $\mathcal{A}(D^2_{q+})\otimes A(D^2_{q-})\otimes\mathcal{A}(S^1)$ onto the first two legs of the tensor product. In particular, we will use it as a map

$$r : \mathcal{A}(S_q^*) \to \mathcal{A}(D^2_{q+})\otimes A(D^2_{q-}).$$

In the following, we adopt the notation [34]:

$$⨍ T := \mathrm{Res}_{z=0}\, T|D|^{-z}.$$

Theorem 4.5. *The dimension spectrum of the spectral triple* $(\mathcal{A}(SU_q(2)),\mathcal{H},D)$ *is simple and given by* $\{1,2,3\}$*; the corresponding residues are*

$$\begin{aligned}⨍ T|D|^{-3} &= 2(\tau_1\otimes\tau_1)\big(r\rho(T)^0\big),\\ ⨍ T|D|^{-2} &= \big(\tau_1\otimes(\tau_0^\uparrow+\tau_0^\downarrow) + (\tau_0^\uparrow+\tau_0^\downarrow)\otimes\tau_1\big)\big(r\rho(T)^0\big),\\ ⨍ T|D|^{-1} &= (\tau_0^\uparrow\otimes\tau_0^\downarrow + \tau_0^\downarrow\otimes\tau_0^\uparrow)\big(r\rho(T)^0\big),\end{aligned}$$

with $T\in\Psi^0(\mathcal{A})$.

Proof. If we identify $\mathcal{H}'\subset\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{Z})\otimes\mathbb{C}^2$ with $\mathcal{H}$ as above, the one-parameter group of automorphisms $\gamma(t)$ induces a $\mathbb{Z}$-grading on $\mathcal{A}(S_q^*)$, in its representation on $\mathcal{H}'$. We denote by $\rho(T)^0$ the degree-zero part of the diagonal operator $\rho(T)$, for $T\in\mathcal{B}$. For the calculation of the dimension spectrum we need to find the poles of the zeta function $\zeta_T(z) := \mathrm{Tr}(T|D|^{-z})$ for all $T\in\Psi^0(\mathcal{A})$. From our discussion of the generators of $\Psi^0(\mathcal{A})$, we see that we only need to adjoin $P^\uparrow\mathcal{B}$ to $\mathcal{B}$.

In the zeta function $\zeta_T(z)$ for $T\in\mathcal{B}$, we can replace T by $Q(\rho(T)\otimes 1_2)Q$ since their difference is a smoothing operator by (4.1.11). The operator $Q(\rho(T)\otimes 1_2)Q$ commutes with the projector $P^\uparrow$ so we can first calculate

$$\begin{aligned}\mathrm{Tr}(P^\uparrow Q(\rho(T)\otimes 1_2)Q\,|D|^{-z}) &= \sum_{2j=0}^{\infty}(2j+\tfrac{3}{2})^{-z}(\mathrm{Tr}_{2j}\otimes\mathrm{Tr}_{2j+1})(P^\uparrow Q(\rho(T)\otimes 1_2)Q)\\ &= (\tau_1\otimes\tau_1)(r\rho(T)^0)\,\zeta(z-2)\\ &\quad + (\tau_1\otimes\tau_0^\downarrow + \tau_0^\uparrow\otimes\tau_1)(r\rho(T)^0)\,\zeta(z-1)\\ &\quad + (\tau_0^\uparrow\otimes\tau_0^\downarrow)(r\rho(T)^0)\,\zeta(z) + f_\uparrow(z),\end{aligned}$$

where $f_\uparrow(z)$ is holomorphic in $z \in \mathbb{C}$. Similarly,

$$\begin{aligned}\mathrm{Tr}(P^\downarrow Q(\rho(T)\otimes 1_2)Q\,|D|^{-z}) &= \sum_{2j=0}^{\infty}(2j+\tfrac{3}{2})^{-z}(\mathrm{Tr}_{2j}\otimes\mathrm{Tr}_{2j+1})(P^\downarrow Q(\rho(T)\otimes 1_2)Q)\\ &= (\tau_1\otimes\tau_1)(r\rho(T)^0)\,\zeta(z-2)\\ &\quad + (\tau_1\otimes\tau_0^\uparrow+\tau_0^\downarrow\otimes\tau_1)(r\rho(T)^0)\,\zeta(z-1)\\ &\quad + (\tau_0^\downarrow\otimes\tau_0^\uparrow)(r\rho(T)^0)\,\zeta(z)+f_\downarrow(z),\end{aligned}$$

where $f_\downarrow(z)$ is holomorphic in z. Since $\zeta(z)$ has a simple pole at $z=1$, we see that the zeta function ζ_T has simple poles at 1, 2 and 3. □

From the above proof, we derive the following formulas which will be used later on:

$$\begin{aligned}&⨍ P^\uparrow T|D|^{-3} = (\tau_1\otimes\tau_1)\big(r\rho(T)^0\big),\\ &⨍ P^\uparrow T|D|^{-2} = \big(\tau_1\otimes\tau_0^\downarrow+\tau_0^\uparrow\otimes\tau_1\big)\big(r\rho(T)^0\big),\\ &⨍ P^\uparrow T|D|^{-1} = (\tau_0^\uparrow\otimes\tau_0^\downarrow)\big(r\rho(T)^0\big),\end{aligned} \tag{4.2.1}$$

with T any element in $\Psi^0(\mathcal{A})$.

4.3 Local index formula in 3 dimensions

We begin by discussing the local cyclic cocycles giving the local index formula, in the general case when the spectral triple $(\mathcal{A},\mathcal{H},D)$ has simple discrete dimension spectrum not containing 0 and bounded above by 3.

Let us recall that with a general (odd) spectral triple $(\mathcal{A},\mathcal{H},D)$ there comes a Fredholm index of the operator D as an additive map $\varphi : K_1(\mathcal{A}) \to \mathbb{Z}$ defined as follows. If $F = \mathrm{Sign} D$ and P is the projector $P = \frac{1}{2}(1+F)$ then

$$\varphi([u]) = \mathrm{Index}\ (PuP), \tag{4.3.1}$$

with $u \in \mathrm{Mat}_r(\mathcal{A})$ a unitary representative of the K_1 class (the operator PuP is automatically Fredholm). The above map is computed by pairing $K_1(\mathcal{A})$ with "nonlocal" cyclic cocycles χ_n given in terms of the operator F and of the form

$$\chi_n(a_0,\dots,\alpha_n) = \lambda_n\,\mathrm{Tr}(a_0\,[F,a_1]\dots[F,a_n]),\quad \text{for all}\quad a_j\in\mathcal{A}, \tag{4.3.2}$$

where λ_n is a suitable normalization constant. The choice of the integer n is determined by the degree of summability of the Fredholm module $(\mathcal{H},F)$ over $\mathcal{A}$; any such module is declared to be p-summable if the commutator $[F,a]$ is an element in the p-th Schatten ideal $\mathcal{L}^p(\mathcal{H})$, for any $a\in A$. The minimal n in (4.3.2) needs to be taken such that $n\geq p$.

On the other hand, the Connes–Moscovici local index theorem [34] expresses the index map in terms of a local cocycle ϕ_{odd} in the (b,B) bicomplex of $\mathcal{A}$ which is a local representative of the cyclic cohomology class of χ_n (the cyclic cohomology Chern character). The cocycle ϕ_{odd} is given in terms of the operator D and is made of a finite number of terms $\phi_{\mathrm{odd}} = (\phi_1,\phi_3,\dots)$; the pairing of the cyclic cohomology class $[\phi_{\mathrm{odd}}] \in HC^{\mathrm{odd}}(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index (4.3.1) of D with coefficients in $K_1(\mathcal{A})$. The components of the cyclic cocycle $\phi_{\rm odd}$ are explicitly given in [34]; we shall presently give them for our case.

We know from Proposition 4.1 that our spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A} = \mathcal{A}(SU_q(2))$ has metric dimension equal to 3. As for the corresponding Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A} = \mathcal{A}(SU_q(2))$, it is 1-summable since all commutators $[F, \pi(x)]$, with $x \in \mathcal{A}$, are off-diagonal operators given by sequences of rapid decay. Hence each $[F, \pi(x)]$ is trace-class and we need only the first Chern character $\chi_1(a_0, a_1) = \mathrm{Tr}(a_0 [F, a_1])$, with $a_1, a_2 \in \mathcal{A}$ (we shall omit discussing the normalization constant for the time being and come back to it in the next section). An explicit expression for this cyclic cocycle on the PBW-basis of $SU_q(2)$ was obtained in [74].

The local cocycle has two components, $\phi_{\rm odd} = (\phi_1, \phi_3)$, the cocycle condition $(b+B)\phi_{\rm odd} = 0$ reading $B\phi_1 = 0$, $b\phi_1 + B\phi_3 = 0$, $b\phi_3 = 0$ (see Appendix A.6); it is explicitly given by

$$\phi_1(a_0, a_1) = ⨍ a_0 [D, a_1] |D|^{-1} - \frac{1}{4} ⨍ a_0 \nabla([D, a_1]) |D|^{-3} + \frac{1}{8} ⨍ a_0 \nabla^2([D, a_1]) |D|^{-5},$$
$$\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} ⨍ a_0 [D, a_1] [D, a_2] [D, a_3] |D|^{-3},$$

where $\nabla(T) := [D^2, T]$ for any operator T on $\mathcal{H}$. Under the assumption that $[F, a]$ is traceclass for each $a \in \mathcal{A}$, these expressions can be rewritten as follows:

$$\phi_1(a_0, a_1) = ⨍ a_0 \delta(a_1) F|D|^{-1} - \frac{1}{2} ⨍ a_0 \delta^2(a_1) F|D|^{-2} + \frac{1}{4} ⨍ a_0 \delta^3(a_1) F|D|^{-3},$$
$$\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} ⨍ a_0 \delta(a_1) \delta(a_2) \delta(a_3) F|D|^{-3}. \qquad (4.3.3)$$

We now quote Proposition 2 of [31], referring to that paper for its proof.

Proposition 4.6. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete simple dimension spectrum not containing 0 and bounded above by 3. If $[F, a]$ is trace-class for all $a \in \mathcal{A}$, then the Chern character χ_1 is equal to $\phi_{\rm odd} - (b + B)\phi_{\rm ev}$ where the cochain $\phi_{\rm ev} = (\phi_0, \phi_2)$ is given by*

$$\phi_0(a) := \mathrm{Tr}(Fa |D|^{-z})\big|_{z=0},$$
$$\phi_2(a_0, a_1, a_2) := \frac{1}{24} ⨍ a_0 \delta(a_1) \delta^2(a_2) F|D|^{-3}.$$

The absence of 0 in the dimension spectrum is needed for the definition of ϕ_0. The cochain $\phi_{\rm ev} = (\phi_0, \phi_2)$ was named η-cochain in [31]. In components, the equivalence of the characters means that

$$\phi_1 = \chi_1 + b\phi_0 + B\phi_2, \qquad \phi_3 = b\phi_2.$$

The following general result, in combination with the above proposition, shows that χ_1 can be given (up to coboundaries) in terms of one single (b, B)-cocycle ψ_1.

Proposition 4.7. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete simple dimension spectrum not containing 0 and bounded above by 3. Assume that $[F, a]$ is trace class for all $a \in \mathcal{A}$, and set $P := \frac{1}{2}(1 + F)$. Then, the local Chern character $\phi_{\rm odd}$ is equal to $\psi_1 - (b + B)\phi'_{\rm ev}$ where*

$$\psi_1(a_0, a_1) := 2 ⨍ a_0 \delta(a_1) P|D|^{-1} - ⨍ a_0 \delta^2(a_1) P|D|^{-2} + \frac{2}{3} ⨍ a_0 \delta^3(a_1) P|D|^{-3},$$

and $\phi'_{\mathrm{ev}} = (\phi'_0, \phi'_2)$ *is given by*

$$\phi'_0(a) := \mathrm{Tr}(a\,|D|^{-z})\big|_{z=0},$$
$$\phi'_2(a_0, a_1, a_2) := -\frac{1}{24}\fint a_0\,\delta(a_1)\,\delta^2(a_2)F|D|^{-3}.$$

Proof. One needs to verify the following equalities between cochains in the (b, B) bicomplex:

$$\phi_1 + b\phi'_0 + B\phi'_2 = \psi_1,$$
$$\phi_3 + b\phi'_2 = 0.$$

The second equality follows from a direct computation of $b\phi'_2$ and comparing with equation (4.3.3). Note that this identity proves that ψ_1 is indeed a cyclic cocycle. One also shows that

$$B\phi'_2(a_0, a_1) = \frac{1}{12}\fint a_0\,\delta^3(a_1)F|D|^{-3}.$$

Then, using the asymptotic expansion [34]:

$$|D|^{-z}a \sim \sum_{k\geq 0}\binom{-z}{k}\delta^k(a)\,|D|^{-z-k}$$

modulo very low powers of $|D|$, one computes

$$b\phi'_0(a_0, a_1) = \fint a_0\,\delta(a_1)|D|^{-1} - \frac{1}{2}\fint a_0\,\delta^2(a_1)|D|^{-2} + \frac{1}{3}\fint a_0\,\delta^3(a_1)|D|^{-3},$$

and it is now immediate that $\phi_1 + b\phi'_0 + B\phi'_2$ gives the cyclic cocycle ψ_1. □

Remark 4.8. *The term involving* $P|D|^{-3}$ *would vanish if the latter were traceclass, which is the case in [31] (this is the statement that the metric dimension of the projector* P *is* 2*).*

Combining these two propositions, it follows that the cyclic 1-cocycles χ_1 and ψ_1 are related as:

$$\chi_1 = \psi_1 - b\beta, \tag{4.3.4}$$

where $\beta(a) = 2\,\mathrm{Tr}(Pa\,|D|^{-z})\big|_{z=0}$.

4.4 The pairing between $\mathbf{HC}^1$ and $\mathbf{K}_1$

In this section, we shall calculate the value of the index map (4.3.1) when U is the unitary operator representing the generator of $K_1(\mathcal{A}(SU_q(2)))$,

$$\varphi([U]) = \mathrm{Index}\,(P^{\uparrow}UP^{\uparrow}) := \dim\ker\,P^{\uparrow}UP^{\uparrow} - \dim\ker\,P^{\uparrow}U^*P^{\uparrow},$$

with

$$U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}, \tag{4.4.1}$$

acting on the doubled Hilbert space $\mathcal{H}\otimes\mathbb{C}^2$ via the representation $\pi\otimes 1_2$. One expects this index to be nonzero, since the K-homology class of $(\mathcal{A}, \mathcal{H}, D)$ is non-trivial. This has been remarked also in [18], where our spectral triple is decomposed in terms of the spectral triple constructed in [20].

We first compute the above index directly, which is possible due to the simple nature of this particular example. A short computation shows that the kernel of the operator $P^\uparrow U^* P^\uparrow$ is trivial, whereas the kernel of $P^\uparrow U P^\uparrow$ contains only elements proportional to the vector

$$\begin{pmatrix} |0,0,-\frac{1}{2},\uparrow\rangle \\ -q^{-1}|0,0,\frac{1}{2},\uparrow\rangle \end{pmatrix},$$

leading to $\varphi([U]) = \text{Index}\ (P^\uparrow U P^\uparrow) = 1$.

Recall that for $\mathcal{A} = \mathcal{A}(SU_q(2))$, our Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}(SU_q(2))$ is 1-summable. From the previous section we know that Index $(P^\uparrow U P^\uparrow)$ can be computed using the local cyclic cocycle ψ_1, see eqn. (4.3.4). To prepare for this index computation via ψ_1, we recall the following lemma [27, IV.1.γ], which fixes the normalization constant in front of χ_1. For completeness we recall the proof.

Lemma 4.4.1. *Let* $(\mathcal{H}, F)$ *be a* 1*-summable Fredholm module over* $\mathcal{A}$ *with* $P = \frac{1}{2}(1+F)$*; let* $u \in \mathrm{Mat}_r(\mathcal{A})$ *be unitary with a suitable* r*. Then* PuP *is a Fredholm operator on* $P\mathcal{H}$ *and*

$$\text{Index}\ (PuP) = -\tfrac{1}{2}\,\mathrm{Tr}(u^*[F,u]) = -\tfrac{1}{2}\chi_1(u^*,u).$$

Proof. We claim that Pu^*P is a parametrix for PuP, that is, an inverse modulo compact operators on $P\mathcal{H}$. Indeed, since $P - u^*Pu = -\frac{1}{2}u^*\,[F,u]$ is traceclass by 1-summability, by composing it from both sides with P it follows that $P - Pu^*PuP$ is traceclass. Therefore,

$$\text{Index}\ (PuP) = \mathrm{Tr}(P - Pu^*PuP) - \mathrm{Tr}(P - PuPu^*P),$$

and the identities $P - Pu^*PuP = -\frac{1}{2}Pu^*\,[F,u]P$ and $[F,u]\,u^* + u\,[F,u^*] = 0$, together with $[F,[F,u]] = 0$, imply the statement. □

Thus, the index of $P^\uparrow U P^\uparrow$, for the U of (4.4.1) is given, up to an overall $-\frac{1}{2}$ factor, by

$$\psi_1(U^{-1},U) = 2\fint U^*_{kl}\,\delta(U_{lk})P^\uparrow|D|^{-1} - \fint U^*_{kl}\,\delta^2(U_{lk})P^\uparrow|D|^{-2} + \frac{2}{3}\fint U^*_{kl}\,\delta^3(U_{lk})P^\uparrow|D|^{-3},$$

with summation over $k,l = 0,1$ understood. We compute this expression using equation (4.2.1). First note that since the entries of U are generators of $\mathcal{A}(SU_q(2))$, we see from (4.1.1) and (4.1.1) that $\rho(\delta^2(U_{kl})) = \rho(U_{kl})$, a relation that simplifies the above formula. We compute the degree 0 part of $\rho(U^*_{kl}\,\delta(U_{lk}))$ with respect to the grading coming from $\gamma(t)$ –the only part that contributes to the trace– using the algebra relations of $\mathcal{A}(D^2_{q\pm})$,

$$\rho(U^*_{kl}\,\delta(U_{lk}))^0 = 2(1-q^2)\,1 \otimes r_-(b)^2.$$

Using the basic equalities

$$\begin{aligned} \tau_1(1) &= 1, & \tau_0^\uparrow(1) &= -\tau_0^\downarrow(1) = -\tfrac{1}{2}, \\ \tau_1(r_\pm(b)^n) &= 0, & \tau_0^\uparrow(r_\pm(b)^n) &= \tau_0^\downarrow(r_\pm(b)^n) = (\pm 1)^n(1-q^n)^{-1}, \end{aligned}$$

we find that

$$\psi_1(U^{-1},U) = 2(1-q^2)(2\tau_0^\uparrow \otimes \tau_0^\downarrow + \tau_1 \otimes \tau_1)\big(1 \otimes r_-(b)^2\big) - (\tau_1 \otimes \tau_0^\downarrow + \tau_0^\uparrow \otimes \tau_1)\big(1 \otimes 1\big) = -2.$$

Taking the proper coefficients, we finally obtain

$$\text{Index}\ (P^\uparrow U P^\uparrow) = -\tfrac{1}{2}\psi_1(U^{-1},U) = 1.$$

Part II

The geometry of gauge fields on toric noncommutative manifolds

Chapter 1

Introduction

The ADHM construction [2, 3] of instantons in Yang-Mills theory was the starting point of a new interplay between physics and mathematics. On the physics side –as instantons are gauge potentials that are the minima of the Yang-Mills action– they can be used to obtain an approximation for the path integral

$$Z(t) = \int \mathcal{D}[A] e^{-tS[A]} ,$$

if $t << 1$. Here the (formal) integral is taken over the space of all gauge potentials. As $t \to 0$, the path integral is essentially modelled on the integral over the moduli space of instantons. In mathematics, these ideas culminated in Donaldson's construction of invariants of smooth four-dimensional manifolds [42] (cf. [43]). In fact, Witten showed in [95] that as t tends to 0 the path integral $Z(t)$ recovers the Donaldson invariants.

In the same period, another field in mathematical physics was founded by Alain Connes. His noncommutative geometry [26] (cf. [27]) provides a mathematical framework that incorporates Yang-Mills gauge field theories into the realm of quantum spaces. A quantum space is understood as the virtual space underlying a noncommutative algebra in the sense of the Gelfand-Naimark theorem. The latter gives a (categorical) equivalence between unital commutative C^*–algebras and compact Hausdorff topological spaces. This means that all geometrical notions on a topological space can be translated into properties of the C^*–algebra of continuous functions on it. Once this has been achieved, one drops the commutativity to describe a virtual 'quantum space' dual to the noncommutative algebra. Even more, noncommutative geometry as invented by Connes, provides a description of noncommutative Riemannian spin structures. Here the Riemannian "metric" becomes encoded in the Dirac operator and its spectrum, whereas the noncommutative topological space is described by the algebra of functions in the previously described manner.

Classically, Yang-Mills gauge theory is described by the theory of principal bundles and connections on them. Recall that $P \to X$ is a principal G–bundle on X with G a Lie group, if it is a fibre bundle with typical fibre G and G acts freely and transitively on P.

Definition 1.1. *A* connection one-form *on* P *is defined as a one-form* ω *taking values in the Lie algebra* $\mathfrak{g}$ *of* G *satisfying*

(i) $\omega(A^\#) = A$*;*

(ii) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$*;*

where $A^{\#}$ *is the fundamental vector field associated to* $A \in \mathfrak{g}$*,*

$$A^{\#}f(p) = \tfrac{\partial}{\partial t} f(p \exp(tA))|_{t=0},$$

acting on a smooth function f *on* X*,* R_g^* *is the induced action of* G *on the one-forms* $\Omega^1(P)$ *on* P *and*

$$\left(\mathrm{Ad}_{g^{-1}}\omega\right)_p(Y) := g^{-1}\omega_p(Y)g; \qquad (Y \in TP).$$

The horizontal subspace HP *is defined by the kernel of* ω *in the tangent bundle* TP*:*

$$H_pP = \{Y \in T_pP : \omega(Y) = 0\} \ .$$

We define a covariant derivative associated to ω on this principal bundle as follows. Let $\phi \in \Omega^r(P) \otimes V$ with V a vector space. Let Y^H denote the projection of $Y \in TP$ onto the horizontal subspace HP. The covariant derivative of ϕ is then defined as:

$$\begin{aligned} &D : \Omega^r(P) \otimes V \to \Omega^{r+1}(P) \otimes V \\ &D\phi(Y_1, \dots, Y_{r+1}) = d_P\phi(Y_1^H, \dots, Y_{r+1}^H) \end{aligned}$$

where d_P is the exterior derivative on P.

Definition 1.2. *The* curvature Ω *of* ω *is the covariant derivative of* ω*:*

$$\Omega = D\omega$$

A more explicit form of the curvature is given in terms of Cartan's structure equation:

$$\Omega(X, Y) = d_P\omega(X, Y) + [\omega(X), \omega(Y)]$$

which is usually written as $\Omega = d_P\omega + \omega \wedge \omega$.

If ρ is a (finite-dimensional) representation of G on the vector space V, then the associated bundle to P by V is defined to be the vector bundle $E := P \times_G V$ having typical fibre V. The space of continuous sections $\Gamma(E)$ can be given as the collection of G-equivariant maps from P to V:

$$C_G(P, V) := \{\phi \in C(P, V) := C(P) \otimes V : \phi(p \cdot g) = \rho_g(\phi(p))\} \ .$$

A *connection* (or *covariant derivative*) on E is defined as a map ∇ from $\Gamma(E)$ to $\Gamma(E) \otimes_{C(X)} \Omega^1(X)$ by

$$\nabla(\phi) = d_P\phi + \omega\phi.$$

There is the following equivalent description of connections, in terms of local charts. Choose a local section of $P \to X$ and define A to be the pull-back of ω under this section. Then A is a (locally defined) one-form on X, taking values in $\mathfrak{g}$, and is called *gauge potential.* Its curvature F, becomes in terms of A,

$$F = dA + A \wedge A.$$

It turns out that F is in fact a two-form taking values in the adjoint bundle $\mathrm{ad}(P) := P \times_G \mathfrak{g}$, where G acts on $\mathfrak{g}$ in the adjoint representation.

Another important concept in physics is a gauge transformation. Mathematically speaking, a gauge transformation is a section of the bundle of automorphisms of E. More precisely, the infinite dimensional group $\mathcal{G}$ of gauge transformations consists of sections of the bundle $P \times_G G$ where G acts on itself by conjugation. A gauge transformation f acts on a connection like

$$\nabla \mapsto f^{-1}\nabla f,$$

inducing the familiar transformation rule for the connection one-form A,

$$A \mapsto f^{-1}Af + f^{-1}df,$$

together with

$$F \mapsto f^{-1}Ff.$$

for its curvature. Connections have many invariants under gauge transformations, as constructed for example by Chern-Weil theory. Clearly, the Chern characters define gauge invariants:

$$\mathrm{ch}_n(\nabla) = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n \mathrm{tr}\, F^n.$$

as do the Chern numbers c_n defined as $\int_X \mathrm{ch}_n$, called topological charges in physics. Of course, the most important invariant in physics is the Yang-Mills action:

$$S = \|F\|^2 := \int_X \mathrm{tr}(F \wedge *F)$$

where $*$ is the Hodge star operator. If we decompose $F = F_+ \oplus F_-$ into its selfdual and anti-selfdual part, i.e. $*F_\pm = \pm F_\pm$, we can relate this action to the second Chern number. In fact, if $c_2(\nabla) = k \in \mathbb{Z}$

$$\begin{aligned} S &= \|F_+\| + \|F_-\| \\ 8\pi^2 k &= \|F_+\| - \|F_-\| \end{aligned}$$

from which we deduce the lower bound $S \geq 8\pi^2|k|$. Note that equality holds if $*F = \pm F$. Connections with selfdual or anti-selfdual curvature are called instantons; they are absolute minima of the Yang-Mills action.

Let us illustrate the above structure by an example. Consider the principal Hopf fibration $S^7 \to S^4$ with structure group $SU(2)$, and let E be the associated vector bundle by the fundamental representation: $E = S^7 \times_{SU(2)} \mathbb{C}^2$. In this case, an instanton is a connection on the $\mathrm{rank}_\mathbb{C}$ two vector bundle on S^4 –which we can define by a gauge potential A– having selfdual curvature. Let us give an example of the latter on a local chart with coordinates $\{\zeta^\mu, \zeta^{\mu*}\}$, coming from stereographical projection. The basic instanton is given by [6]

$$A := \frac{1}{1+|\zeta|^2}\Big((\zeta_1 d\zeta_1^* - \zeta_1^* d\zeta_1 - \zeta_2 d\zeta_2^* + \zeta_2^* d\zeta_2)\sigma_3 \\ + 2(\zeta_1 d\zeta_2^* - \bar{\lambda}\zeta_2^* d\zeta_1)\sigma_+ + 2(\zeta_2 d\zeta_1^* - \lambda\zeta_1^* d\zeta_2)\sigma_-\Big),$$

with corresponding gauge curvature (field strength), $F = dA + A^2$,

$$F = \frac{1}{(1+|\zeta|^2)^2}\Big((d\zeta_1 d\zeta_1^* - d\zeta_2^* d\zeta_2)\sigma_3 + 2(d\zeta_1 d\zeta_2^*)\sigma_+ + 2(d\zeta_2 d\zeta_1^*)\sigma_-\Big).$$

Here $\sigma_3, \sigma_\pm$ are the generators of the Lie algebra $su(2)$.

It turns out [2] that acting with the conformal group $SL(2, \mathbb{H})$ of S^4 on the basic instanton gauge potential, one can obtain different instantons. Indeed, the conformal group leaves both the (anti-)selfdual equation $*F = \pm F$ and the Yang-Mills action invariant. There is a five-parameter family of instantons up to gauge transformations. On the local chart $\mathbb{R}^4$ of S^4, these five parameters correspond to one scaling ρ and four "translations" of the basic instanton. This forms the five-dimensional moduli space of charge 1 instantons.

In order to describe Yang-Mills theory on a noncommutative space we need the noncommutative analogue of a vector bundle. The Serre-Swan theorem [91] indicates that this analogue is given by the finitely generated projective modules over the algebra A describing the quantum space. They showed that any finitely generated projective $C(X)$–module is isomorphic to the $C(X)$–module of sections on a vector bundle E for a compact manifold X. Also the notion of a connection on a vector bundle can be generalized to noncommutative geometry [27] (see Appendix A.4).

Principal bundles are incorporated more recently in noncommutative geometry [15, 52]. Apparently, the structure group G should be replaced by a quantum group and P and X by a quantum space. Such a quantum group is then the virtual dual to a Hopf algebra. However, several examples [11] have shown that this definition of noncommutative principal bundle is not general enough. Therefore, one was led to replace the Hopf algebra by merely a coalgebra. More precisely, a principal bundle generalizes to a principal coalgebra extension [13] and a principal Hopf-Galois extension in the case that the coalgebra is a Hopf algebra. Again, there are the notions of connections and associated vector bundles. Recently, a noncommutative version of Chern-Weil theory for principal coalgebra extensions has been developed by Brzeziński and Hajac [14].

We develop Yang-Mills theory on toric noncommutative manifolds which were introduced in [33] (see also [32]). These noncommutative spaces M_θ are defined as deformations of a Riemannian manifold M carrying an action of $\mathbb{T}^n$: this torus is deformed to a noncommutative tori $\mathbb{T}^n_\theta$ [81] with θ a matrix of deformation parameters. We start by recalling their construction and derive a simplified form of the Connes-Moscovici local index formula [34] on these noncommutative spaces.

In Chapter 3, we focus on two such noncommutative manifolds and construct a noncommutative principal Hopf fibration $S^7_{\theta'} \to S^4_\theta$ with structure group $SU(2)$, starting with the algebras $\mathcal{A}(S^4_\theta), \mathcal{A}(S^7_{\theta'})$ of polynomials on them. The algebra $\mathcal{A}(S^7_{\theta'})$ carries an action of $SU(2)$ by automorphisms and we identify the subalgebra consisting of invariants under this action with $\mathcal{A}(S^4_\theta)$. This gives a one-parameter family of Hopf fibrations, where θ' is expressed in terms of θ.

We construct the $\mathcal{A}(S^4_\theta)$-bimodules associated to all finite-dimensional representations V of $SU(2)$ as the collection of "equivariant maps from $S^7_{\theta'}$ to V" with respect to the action of $SU(2)$, and define connections on them. We prove that these modules are finite projective by explicit construction of projections. This allows for a computation of the indices of Dirac operators having coefficients in these noncommutative vector bundles. Finally, we establish that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a faithfully flat Hopf-Galois extension in the sense of [14], therefore, it can be considered as a noncommutative principal bundle.

In Chapter 4, we develop Yang-Mills theory on S^4_θ by defining a Yang-Mills action functional in terms of the curvature of a connection and derive that the minima of this action are given by connections with (anti-)selfdual curvature: such connections are called instantons. Starting with the basic instanton given in [33], gauge non-equivalent instantons are obtained by acting on it by twisted infinitesimal conformal transformations, encoded in the Hopf algebra $\mathcal{U}_\theta(so(5,1))$.

The Hopf subalgebra $\mathcal{U}_\theta(\mathsf{so}(5))$ is made of twisted infinitesimal symmetries under which the basic instanton is invariant. This leads to a collection of (infinitesimal) instantons. Finally we prove, by using an index theoretical argument as in [4], that this collection is in fact the complete set of (infinitesimal) charge 1 instantons.

In Chapter 5, we sketch how to generalize Yang-Mills theory from S^4_θ to any four-dimensional toric noncommutative manifold M_θ. Let $P \to M$ be a G-principal bundle, where G is a semisimple Lie group. We assume that the action of the torus $\mathbb{T}^2$ on M can be lifted to P, in such a way that this lifted action commutes with the action of G on P. This allows for the definition of the two algebras $C^\infty(M_\theta)$ and $C^\infty(P_\theta)$ as toric noncommutative manifolds. The inclusion $C^\infty(M_\theta) \subset C^\infty(P_\theta)$ can be understood as a noncommutative principal bundle: $C^\infty(P_\theta)$ carries an action of G by automorphisms in such a way that $C^\infty(M_\theta)$ forms the subalgebra consisting of elements in $C^\infty(P_\theta)$ that are invariant under the action of G.

We define the associated vector bundles $P_\theta \times_G V$ for all finite-dimensional representations V of G as $C^\infty(M_\theta)$-bimodules of G-equivariant maps from P_θ to V; these modules are again finite projective. Finally, we define a Yang-Mills action functional and find that its minima are given by instantons, i.e. connections with selfdual or anti-selfdual curvature.

Chapter 2

Toric noncommutative manifold

In this chapter, we will recall the construction of the noncommutative manifolds M_θ as introduced in [33] and elaborated in [32]. Essentially, these θ-deformations are a natural extension of the noncommutative torus (for a review see [82]) to (compact) Riemannian manifolds carrying an action of the n-torus $\mathbb{T}^n$.

We will first consider the cases of planes and spheres, and will then move on to the general case of a Riemannian manifold carrying an action of $\mathbb{T}^n$. We then discuss the local index formula of Connes and Moscovici [34] which simplifies drastically in the case of M_θ.

2.1 Noncommutative spherical manifolds

For $\lambda_{\mu\nu} = e^{2\pi i\theta_{\mu\nu}}$, where $\theta_{\mu\nu}$ is an anti-symmetric real-valued matrix, the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ of polynomial functions on the noncommutative $2n$-plane is defined to be the unital $*$-algebra generated by $2n$ elements $z_\mu, z^*_\mu (\mu = 1, \ldots, n)$ with relations

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu; \quad z^*_\mu z_\nu = \lambda_{\nu\mu} z_\nu z^*_\mu; \quad z^*_\mu z^*_\nu = \lambda_{\mu\nu} z^*_\nu z^*_\mu,$$

The involution $*$ is defined by putting $(z_\mu)^* = z^*_\mu$. For $\theta = 0$ one recovers the commutative $*$-algebra of complex polynomial functions on $\mathbb{R}^{2n}$.

Let $\mathcal{A}(S^{2n-1}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by the two-sided ideal generated by the central element $\sum_\mu z_\mu z^*_\mu - 1$. We will denote the images of z_μ under the quotient map again by z_μ.

A key role in what follows is played by the action of the abelian group $\mathbb{T}^n$ on $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by automorphisms. For $s = (s_\mu) \in \mathbb{T}^n$, the $*$-automorphism σ_s is defined on the generators by $\sigma_s(z_\mu) = e^{2\pi i s_\mu} z_\mu$. Clearly, $s \mapsto \sigma_s$ is a group-homomorphism from $\mathbb{T}^n \to \mathrm{Aut}(\mathcal{A}(\mathbb{R}^{2n}_\theta))$. In the special case that $\theta = 0$, we see that σ is induced by a smooth action of $\mathbb{T}^n$ on the manifold $\mathbb{R}^{2n}$. Since the ideal generating $\mathcal{A}(S^{2n-1}_\theta)$ is invariant under the action of $\mathbb{T}^n$, σ induces a group-homomorphism from $\mathbb{T}^n$ into the group of automorphisms on the quotient $\mathcal{A}(S^{2n-1}_\theta)$ as well.

We continue by defining the unital $*$-algebra $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ of polynomial functions on the noncommutative $(2n+1)$-plane which is given by adjoining a central self-adjoint generator z_0 to the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$, i.e. $z^*_0 = z_0$ and $z_0 z_\mu = z_\mu z_0$ $(\mu = 1, \ldots, n)$. The action of the group $\mathbb{T}^n$ is extended trivially by $\sigma_s(z_0) = z_0$. Let $\mathcal{A}(S^{2n}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ by the ideal generated by the central element $\sum z_\mu z^*_\mu + z_0^2 - 1$. As before, we will denote the canonical images of z_μ and z_0 again by z_μ and z_0, respectively. Since $\mathbb{T}^n$ leaves this ideal invariant, it induces an action by $*$-automorphisms on the quotient $\mathcal{A}(S^{2n}_\theta)$.

We will now construct a differential calculus on $\mathbb{R}^m_\theta$. For $m = 2n$, the complex unital associative graded $*$-algebra $\Omega(\mathbb{R}^{2n}_\theta)$ is generated by $2n$ elements z_μ, z^*_μ of degree 0 and $2n$ elements dz_μ, dz^*_μ of degree 1 with relations:

$$dz_\mu dz_\nu + \lambda_{\mu\nu} dz_\nu dz_\mu = 0; \quad dz^*_\mu dz_\nu + \lambda_{\nu\mu} dz_\nu dz^*_\mu = 0; \quad dz^*_\mu dz^*_\nu + \lambda_{\mu\nu} dz^*_\nu dz^*_\mu = 0;$$
$$z_\mu dz_\nu = \lambda_{\mu\nu} dz_\nu z_\mu; \quad z^*_\mu dz_\nu = \lambda_{\nu\mu} dz_\nu z^*_\mu; \quad z^*_\mu dz^*_\nu = \lambda_{\mu\nu} dz^*_\nu z^*_\mu. \tag{2.1.1}$$

There is a unique differential d on $\Omega(\mathbb{R}^{2n}_\theta)$ such that $d : z_\mu \mapsto dz_\mu$. The involution $\omega \mapsto \omega^*$ for $\omega \in \Omega(\mathbb{R}^{2n}_\theta)$ is the graded extension of $z_\mu \mapsto z^*_\mu$, i.e. it is such that $(d\omega)^* = d\omega^*$ and $(\omega_1\omega_2)^* = (-1)^{p_1 p_2}\omega^*_2\omega^*_1$ for $\omega_i \in \Omega^{p_i}(\mathbb{R}^{2n}_\theta)$.
For $m = 2n + 1$, we adjoin to $\Omega(\mathbb{R}^{2n}_\theta)$ one generator z_0 of degree 0 and one generator dz_0 of degree 1 such that

$$z_0 dz_0 = dz_0 z_0; \quad z_0\omega = \omega z_0; \quad dz_0\omega = (-1)^{|\omega|}\omega dz_0.$$

We extend the differential d and the graded involution $\omega \mapsto \omega^*$ of $\Omega(\mathbb{R}^{2n}_\theta)$ to $\Omega(\mathbb{R}^{2n+1}_\theta)$ by setting $z^*_0 = z_0$ and $(dz_0)^* = dz_0$, so that $(dz_0)^* = dz_0$.

The differential calculi $\Omega(S^m_\theta)$ on the noncommutative spheres S^m_θ are defined to be the quotients of $\Omega(\mathbb{R}^{m+1}_\theta)$ by the differential ideals generated by the central elements $\sum_\mu z_\mu z^*_\mu - 1$ and $\sum z_\mu z^*_\mu + z_0^2 - 1$, for $m = 2n - 1$ and $m = 2n$ respectively.

The action of $\mathbb{T}^n$ by $*$-automorphisms on $\mathcal{A}(M_\theta)$ can be easily extended to the differential calculi $\Omega(M_\theta)$, for $M_\theta = \mathbb{R}^m_\theta$ and $M = S^m_\theta$, by imposing $\sigma_s \circ d = d \circ \sigma_s$.

2.2 Toric noncommutative manifolds

More generally, we have the following construction of noncommutative manifolds [33]. Suppose M is a compact spin Riemannian manifold of dimension m equipped with a smooth action of the n-torus $\mathbb{T}^n$ by isometries. Let D be the Dirac operator on the Hilbert space of spinors $\mathcal{H} := L^2(M, \mathcal{S})$ and denote by σ_s the action of $\mathbb{T}^n$ by automorphisms on the algebra $C^\infty(M)$ of smooth functions on M. It is induced from the action on M by

$$\sigma_s(f)(x) = f(s^{-1} \cdot x).$$

There is a double cover $p : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a representation of $\widetilde{\mathbb{T}}^n$ on $\mathcal{H}$ by unitary operators $U(s), s \in \widetilde{\mathbb{T}}^n$ such that

$$U(s)DU(s)^{-1} = D,$$
$$U(s)fU(s)^{-1} = \sigma_{p(s)}(f),$$

for all $f \in C^\infty(M)$ acting on $\mathcal{H}$ via the representation π, given by pointwise multiplication. These unitary operators induce a grading on $\mathcal{B}(\mathcal{H})$ as follows. Recall that an element $T \in \mathcal{B}(\mathcal{H})$ is called smooth for the action of $\widetilde{\mathbb{T}}^n$, if the map $\widetilde{\mathbb{T}}^n \ni s \mapsto \alpha_s(T) := U(s)TU(s)^{-1}$ is smooth for the norm topology. It can be expanded as $T = \sum T_r$ with $r = (r_1, r_2, \dots, r_n)$ a multi-index, and with each T_r of homogeneous degree r under the action of $\widetilde{\mathbb{T}}^n$, i.e.

$$\alpha_s(T_r) = e^{2\pi i r \cdot s} T_r \quad (s \in \widetilde{\mathbb{T}}^n).$$

From its very definition, α_s coincides on $\pi(C^\infty(M)) \subset \mathcal{B}(\mathcal{H})$ with the automorphism $\sigma_{p(s)}$. Then, let $(p_1, p_2, \ldots, p_n)$ be the infinitesimal generators of the action of $\widetilde{\mathbb{T}}^n$ so that we can write $U(s) = \exp 2\pi i s \cdot p$. For $T \in \mathcal{B}(\mathcal{H})$ we define a twisted representation on $\mathcal{H}$ by

$$L_\theta(T) := \sum_r T_r U(\tfrac{1}{2} r_\mu \theta_{\mu 1}, \ldots, \tfrac{1}{2} r_\mu \theta_{\mu n}) = \sum_r T_r \exp\left\{\pi i \sum_\mu r_\mu \theta_{\mu\nu} p_\nu\right\} \tag{2.2.1}$$

with θ an $n \times n$ anti-symmetric matrix. Notice that since D is of degree 0 we have $L_\theta([D, a]) = [D, L_\theta(f)]$ for $f \in C^\infty(M)$. There is still an action of $\mathbb{T}^n$ on $L_\theta(C^\infty(M))$ by $\alpha_s(L_\theta(f)) = U(s) L_\theta(f) U(s)^{-1} = L_\theta(\sigma_{p(s)}(f))$ since $\widetilde{\mathbb{T}}^n$ is abelian; we will denote this action again by σ. We will denote $L_\theta(C^\infty(M))$ by $C^\infty(M_\theta)$.

The map L_θ can also be understood as a representation of the algebra $C^\infty(M)$ equipped with a so-called star product. If $f_r, g_{r'}$ are two functions in $C^\infty(M)$ homogeneous of degree r and r' respectively, we define the star product $\times_\theta$ by

$$f_r \times_\theta g_{r'} = f_r \ \sigma_{r \cdot \theta}(g_{r'}) \equiv e^{2\pi i r \cdot \theta \cdot r'} f_r g_{r'} \tag{2.2.2}$$

where $r \cdot \theta = (r_\mu \theta_{\mu 1}, \ldots, r_\mu \theta_{\mu n}) \in \mathbb{T}^n$. This product is then extended linearly to all functions in $C^\infty(M)$. By the very definition of $\times_\theta$ we have

$$L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g),$$

proving that the algebra $C^\infty(M)$ equipped with the product $\times_\theta$ is isomorphic to the algebra $L_\theta(C^\infty(M))$. Thus, we can understand L_θ as a *quantization map* from $C^\infty(M)$ to $C^\infty(M_\theta)$. It will play a key role in what follows, allowing us to extend differential geometrical techniques from M to the noncommutative space M_θ.

Recall that a noncommutative spin geometry [28, 49] (see also Appendix A.2) is given by an algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$ together with an unbounded self-adjoint operator D. This so-called spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has to satisfy several properties, of which the most essential are that $[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$ and that the resolvent of D is a compact operator. We call such a spin geometry m^+-summable if m is the unique nonnegative integer for which the partial sums σ_N of the eigenvalues of $|D|^{-m}$ satisfy $\sigma_N \sim \log N$ as $N \to \infty$.

The triples $(C^\infty(M_\theta), \mathcal{H}, D)$ reviewed above provide a wide class of examples of noncommutative spin geometries [33]. It is m^+-summable, which follows directly from the classical case. Indeed, since the noncommutative geometry is an isospectral deformation of the classical Riemannian geometry of M, the spectrum of the operator D coincides with that of the Dirac operator D on M. In particular, there is a noncommutative integral in terms of the Dixmier trace [41] (cf. Appendix A.2):

$$⨍ f := \mathrm{Tr}_\omega\big(f |D|^{-m}\big)$$

with $f \in C^\infty(M_\theta)$ understood in its representation on $\mathcal{H}$.

The following Lemma [46] gives a drastic simplification of this noncommutative integral.

Lemma 2.1. *If* $f \in C^\infty(M)$ *then*

$$⨍ L_\theta(f) = \int_M f d\nu$$

Proof. Any element $f \in C^\infty(M)$ is given as an infinite sum of functions that are homogeneous under the action of $\mathbb{T}^n$. Let us therefore assume that f is homogeneous of degree k so that

$\sigma_s(L_\theta(f)) = L_\theta(\sigma_s(f)) = e^{2\pi i k\cdot s}L_\theta(f)$. From the tracial property of the noncommutative integral and the invariance of D under the action of $\mathbb{T}^n$, we see that

$$\mathrm{Tr}_\omega\big(\sigma_s(L_\theta(f))|D|^{-m}\big) = \mathrm{Tr}_\omega\big(U(s)L_\theta(f)U(s)^{-1}|D|^{-m}\big) = \mathrm{Tr}_\omega(f|D|^{-m}).$$

In other words, $e^{2\pi i k\cdot s}\mathrm{Tr}_\omega(L_\theta(f)|D|^{-m}) = \mathrm{Tr}_\omega(L_\theta(f)|D|^{-m})$ from which we infer that this trace vanishes if $k \neq 0$. If $k = 0$, then $L_\theta(f) \equiv f$ leading to the desired result. □

2.2.1 Description of M_θ in terms of fixed point algebras

A different (but equivalent) approach to these noncommutative manifolds M_θ was introduced in [32]. It identifies $C^\infty(M_\theta)$ as a certain fixed point subalgebra of $C^\infty(M) \otimes C^\infty(\mathbb{T}^n_\theta)$ where $C^\infty(\mathbb{T}^n_\theta)$ is the algebra of smooth functions on the noncommutative torus. This identification allows one to extend the techniques from commutative differential geometry on M to the noncommutative space M_θ.

Let us recall the definition of the noncommutative n-torus $\mathbb{T}^n_\theta$ [81, 82]. The unital $*$-algebra $\mathcal{A}(\mathbb{T}^n_\theta)$ of polynomial functions on $\mathbb{T}^n_\theta$ is generated by n unitary elements U^μ with relations

$$U^\mu U^\nu = \lambda^{\mu\nu} U^\nu U^\mu, \quad (\mu, \nu = 1, \ldots, n)$$

with $\lambda^{\mu\nu} = e^{2\pi i\theta_{\mu\nu}}$ as before. There is a natural action of $\mathbb{T}^n$ on $\mathcal{A}(\mathbb{T}^n_\theta)$ by $*$-automorphisms given by $\tau_s(U^\mu) = e^{2\pi i s_\mu}U^\mu$ with $s = (s_\mu) \in \mathbb{T}^n$. The locally convex $*$-algebra $C^\infty(\mathbb{T}^n_\theta)$ of smooth functions on the noncommutative torus $\mathbb{T}^n_\theta$ is defined as follows [25]. There are the following seminorms $\mathcal{A}(\mathbb{T}^n_\theta)$:

$$|u|_r := \sup_{r_1+\cdots+r_n\leq r} \|X_1^{r_1}\cdots X_n^{r_n}(u)\| \ ,$$

where $\|\cdot\|$ is the C^*-norm and X_μ are the infinitesimal generators of the action of $\mathbb{T}^n$ on $\mathbb{T}^n_\theta$. The completion of $\mathcal{A}(\mathbb{T}^n_\theta)$ with respect to the locally convex topology generated by these seminorms is denoted by $C^\infty(\mathbb{T}^n_\theta)$ and turns out to be a nuclear Fréchet space.

Due to the latter remark and nuclearity of $C^\infty(M)$, we can unambiguously define the completed tensor product $C^\infty(M)\overline{\otimes}C^\infty(\mathbb{T}^n_\theta)$. We define $\big(C^\infty(M)\overline{\otimes}C^\infty(\mathbb{T}^n_\theta)\big)^{\sigma\otimes\tau^{-1}}$ to be the fixed point subalgebra of $C^\infty(M)\overline{\otimes}C^\infty(\mathbb{T}^n_\theta)$ consisting of functions f in this tensor product that are invariant under the diagonal action of $\mathbb{T}^n$, i.e. such that $\sigma_s\otimes\tau_{-s}(f) = f$ for all $s \in \mathbb{T}^n$. This defines by duality the noncommutative manifold M_θ by setting $C^\infty(M_\theta) := \big(C^\infty(M)\overline{\otimes}C^\infty(\mathbb{T}^n_\theta)\big)^{\sigma\otimes\tau^{-1}}$. As the notation suggests, the algebra $C^\infty(M_\theta)$ is isomorphic to the algebra $L_\theta(C^\infty(M))$ defined above, as one easily checks.

For the Dirac operator one has the following construction. Let $\mathcal{S}$ be a spin bundle over M and D the Dirac operator on $\Gamma(M, \mathcal{S})$, the $C^\infty(M)$-module of smooth sections of $\mathcal{S}$. The action of the group $\mathbb{T}^n$ on M does not lift directly to the spinor bundle. Rather, there is a double cover $p : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a group-homomorphism $\widetilde{s} \to V_{\widetilde{s}}$ of $\widetilde{\mathbb{T}}^n$ into $\mathrm{Aut}(\mathcal{S})$ covering the action of $\mathbb{T}^n$ on M:

$$V_{\widetilde{s}}(f\psi) = \sigma_{p(s)}(f)V_{\widetilde{s}}(\psi),$$

for $f \in C^\infty(M)$ and $\psi \in \Gamma(M, \mathcal{S})$. According to [32], the proper notion of smooth sections $\Gamma(M_\theta, \mathcal{S})$ of a spinor bundle on M_θ is given by the subalgebra of $\Gamma(M, \mathcal{S})\widehat{\otimes}C^\infty(\mathbb{T}^n_{\theta/2})$ made of elements which are invariant under the diagonal action $V \otimes \widetilde{\tau}^{-1}$ of $\widetilde{\mathbb{T}}^n$. Here $\widetilde{s} \mapsto \widetilde{\tau}_{\widetilde{s}}$ is the canonical action of $\widetilde{\mathbb{T}}^n$ on $\mathcal{A}(\mathbb{T}^n_{\theta/2})$. Since the Dirac operator D will commute with $V_{\widetilde{s}}$ one can restrict $D \otimes \mathrm{id}$ to the fixed point subalgebra $\Gamma(M_\theta, \mathcal{S})$.

Next, let $L^2(M,\mathcal{S})$ be the space of square integrable spinors on M and let $L^2(\mathbb{T}^n_{\theta/2})$ be the completion of $C^\infty(\mathbb{T}^n_{\theta/2})$ in the norm $f \mapsto \|f\| = \tau(f^*f)^{1/2}$, with τ the usual trace on $C^\infty(\mathbb{T}^n_{\theta/2})$. The diagonal action $V \otimes \widetilde{\tau}^{-1}$ of $\widetilde{\mathbb{T}}^n$ extends to $L^2(M,\mathcal{S}) \otimes L^2(\mathbb{T}^n_{\theta/2})$ (where it becomes $U \otimes \tau$ and we define $L^2(M_\theta,\mathcal{S})$ to be the fixed point Hilbert subspace. If D also denotes the closure of the Dirac operator on $L^2(M,\mathcal{S})$, we denote the operator $D \otimes \mathrm{id}$ on $L^2(M,\mathcal{S}) \otimes L^2(\mathbb{T}^n_{\theta/2})$ when restricted to $L^2(M_\theta,\mathcal{S})$ by D. Again the triple $(C^\infty(M_\theta), L^2(M_\theta,\mathcal{S}), D)$ is a m^+-summable noncommutative spin geometry.

2.2.2 Vector bundles on M_θ

We recall the construction of noncommutative vector bundles on M_θ [32], i.e. finite projective modules (cf. Appendix A.4) over $C^\infty(M_\theta)$. They were obtained as fixed point submodules of $\Gamma(M,E) \otimes C^\infty(\mathbb{T}^n_\theta)$ under some diagonal action of the torus $\mathbb{T}^n$. However, we will give an equivalent description in terms of a type of $*$-product.

Let E be a vector bundle on M, carrying an action V of $\mathbb{T}^n$ by automorphisms, such that it covers the action of $\mathbb{T}^n$ on M:

$$V_s(f\psi) = \sigma_s(f)V_s(\psi) \tag{2.2.3}$$

for $f \in C^\infty(M)$ and $\psi \in \Gamma(M,E)$. Such vector bundles are called σ-*equivariant vector bundles.*

The $C^\infty(M_\theta)$-bimodule $\Gamma(M_\theta,E)$ is defined as the vector space $\Gamma(M,E)$ but with the bimodule structure given by

$$f \triangleright_\theta \psi = \sum_k f_k V_{k\cdot\theta}(\psi) \tag{2.2.4}$$

$$\psi \triangleleft_\theta f = \sum_k V_{-k\cdot\theta}(\psi) f_k \tag{2.2.5}$$

where $f = \sum_k f_k$ with $f_k \in C^\infty(M)$ homogeneous of degree k and ψ is a smooth section of E. One easily checks that this is indeed an action of $C^\infty(M_\theta)$, using the explicit expression for the star product (2.2.2) and the above equation (2.2.3). Moreover, this action satisfies the equivariance condition (2.2.3) for both the left and right action of $C^\infty(M_\theta)$.

Although we defined the above left and right actions with respect to an action of $\mathbb{T}^n$ on E, the same construction can be done for vector bundles carrying instead an action of the double cover $\widetilde{\mathbb{T}}^n$. We have already seen an example of this in the case of the spinor bundle, where we defined a left action of $C^\infty(M_\theta)$ in terms of (2.2.1).

The $C^\infty(M_\theta)$-bimodule $\Gamma(M_\theta,E)$ is finite projective [32] and still carries an action of $\mathbb{T}^n$ by V. In fact, all finite projective modules on $C^\infty(M_\theta)$ are of this type, due to the fact that the category of σ-equivariant finite projective module over $C^\infty(M)$ is equivalent to the category of σ-equivariant finite projective modules over $C^\infty(M_\theta)$ [56]. This clearly reflects the analogue for an action of $\mathbb{T}^n$ of the result by Rieffel in [83] that the K-groups of a C^*-algebra deformed by an action of $\mathbb{R}^n$ are isomorphic to the K-groups of the original C^*-algebra.

Definition 2.2. *Let* E *be a vector bundle on* M*. The* $C^\infty(M)$*-module* $\Gamma(M,E)$ *is said to have the* homogeneous decomposition property *if any section* $\phi \in \Gamma(M,E)$ *can be decomposed as* $\phi = \sum_r \phi_r$ *where* $\phi_r \in \Gamma(M,E)$ *is homogeneous of degree* r *under the action of* $\mathbb{T}^n$*, i.e.* $V_s(\phi_r) := e^{2\pi s\cdot r}\phi_r$.

Lemma 2.3. *1. If* $E \simeq F$ *as* σ*-equivariant vector bundles, then* $\Gamma(M_\theta,E) \simeq \Gamma(M_\theta,F)$ *as* $C^\infty(M_\theta)$*-bimodules.*

2. *Suppose* E *and* F *are* σ*-equivariant vector bundles on* M *such that* $\Gamma(M,E)$ *and* $\Gamma(M,F)$ *satisfy the homogeneous decomposition property for the actions* V^E, V^F *of* $\mathbb{T}^n$*, respectively. Then* $E\otimes F$ *is a* σ*-equivariant vector bundle for the diagonal action of* $\mathbb{T}^n$ *and we have,*

$$\Gamma(M_\theta, E\otimes F)\simeq \Gamma(M_\theta,E)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,F),$$

as both left and right $C^\infty(M_\theta)$*-modules.*

Proof. 1. follows from the very definition of $\Gamma(M_\theta,E)$. For *2.*, note first that $\Gamma(M_\theta,E\otimes F)\simeq \Gamma(M,E)\otimes_{C^\infty(M)}\Gamma(M,F)$ as σ-equivariant modules if the action of $\mathbb{T}^n$ on the tensor product $\Gamma(M,E)\otimes_{C^\infty(M)}\Gamma(M,F)$ is defined as the diagonal action $V^E_t\otimes V^F_t$, $t\in\mathbb{T}^n$. Consequently, the left and right action of $C^\infty(M_\theta)$ on $\Gamma(M_\theta,E\otimes F)$ takes the following form on the tensor product:

$$f\triangleright_\theta(\phi\otimes_{C^\infty(M)}\psi)=\sum_k f_k V^E_{k\theta}(\phi)\otimes_{C^\infty(M)}V^F_{k\theta}(\psi),$$

$$(\phi\otimes_{C^\infty(M)}\psi)\triangleleft_\theta f=\sum_k V^E_{-k\theta}(\phi)\otimes_{C^\infty(M)}V^F_{-k\theta}(\psi)f_k.$$

Let us construct an explicit isomorphism of the left and right $C^\infty(M_\theta)$-modules $\Gamma(M_\theta,E\otimes F)$ and $\Gamma(M_\theta,E)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,F)$. Let us start with the left module structure. Since $\Gamma(M_\theta,E)=\Gamma(M,E)$ as vector spaces, it makes sense to define a linear map by

$$T:\Gamma(M_\theta,E)\otimes\Gamma(M_\theta,F)\to\Gamma(M,E)\otimes\Gamma(M,F)$$
$$\phi_r\otimes\psi_s\mapsto\phi_r\otimes V^F_{r\theta}(\psi_s)$$

on homogeneous sections ϕ_r and ψ_s of degree r and s respectively, i.e. such that $V^E_t(\phi_r)=e^{2\pi i t\cdot r}\phi_r$ and $V^F_t(\psi_s)=e^{2\pi i t\cdot s}\psi_s$ for $t\in\mathbb{T}^n$. The map T is extended linearly on generic elements $\phi=\sum_r\phi_r$ and $\psi=\sum_s\psi_s$.

Clearly, the above map is an isomorphism of vector spaces. We check that it maps the ideal $I_\theta=\{\phi\triangleleft_\theta f\otimes\psi-\phi\otimes f\triangleright_\theta\psi\}$ isomorphically to the ideal $I_0=\{f\cdot\phi\otimes\psi-\phi\otimes f\cdot\psi\}$ where $\cdot$ denotes pointwise multiplication. We have for $f\in C^\infty(M_\theta)$ homogeneous of degree k:

$$T(\phi_r\triangleleft_\theta f\otimes\psi_s-\phi_r\otimes f\triangleright_\theta\psi_s)=e^{2\pi(k+r)\theta(k+s)}(f\cdot\phi_r\otimes\psi_s-\phi_r\otimes f\cdot\psi_s).$$

Hence T becomes an isomorphisms from $\Gamma(M_\theta,E)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,F)$ to $\Gamma(M,E)\otimes_{C^\infty(M)}\Gamma(M,F)$ *as vector spaces.* From the definition of T, one checks that it is in fact a left $C^\infty(M_\theta)$-module map, i.e.

$$T((f\triangleright_\theta\phi)\otimes_{C^\infty(M_\theta)}\psi)=f\triangleright_\theta T(\phi\otimes_{C^\infty(M_\theta)}\psi)$$

Similarly, there is an isomorphism of right $C^\infty(M)$-modules

$$T':\Gamma(M_\theta,E)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,F)\to\Gamma(M,E)\otimes_{C^\infty(M)}\Gamma(M,F)$$
$$\phi_r\otimes_{C^\infty(M_\theta)}\psi_s\mapsto V^E_{-s\theta}\phi_r\otimes_{C^\infty(M)}\psi_s.$$

□

Corollary 2.4. *Let* E *and* F *be vector bundles on* M *such that* $\Gamma(M,E)$ *and* $\Gamma(M,F)$ *satisfy the homogeneous decomposition property. Then we have the isomorphism,*

$$\Gamma(M_\theta,E)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,F)\simeq\Gamma(M_\theta,F)\otimes_{C^\infty(M_\theta)}\Gamma(M_\theta,E),$$

of both left and right $C^\infty(M_\theta)$ *modules.*

Proof. Classically, such an isomorphism exists as it is given by the tensor flip. Since the action of $\mathbb{T}^n$ on $E\otimes F$ is diagonal, the tensor flip commutes with the action of $\mathbb{T}^n$. □

2.2.3 Differential calculus on M_θ

A differential calculus on M_θ is constructed as follows [32]. Let $(\Omega(M), d)$ be the usual differential calculus on M, with d the exterior derivative. We extend the map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$ to $\Omega(M)$ by imposing it to commute with d. The image $L_\theta(\Omega(M))$ will be denoted by $\Omega(M_\theta)$. Equivalently, one could define $\Omega(M_\theta)$ to be $\Omega(M)$ as a vector space but equipped with the star product (2.2.2) extended to $\Omega(M)$ under the condition that it commutes with d.

Note that this is concordance with the previous section, when $\Omega(M)$ is considered as the $C^\infty(M)$-bimodule of sections of the cotangent bundle. Indeed, since the action of $\mathbb{T}^n$ on M is isometrical, it commutes with d. Therefore, the action σ_s on $C^\infty(M)$ can be extended to $\Omega(M)$, and gives $\Omega(M_\theta)$ the structure of a $C^\infty(M_\theta)$-bimodule with the left and right action given in (2.2.4) and (2.2.5).

A similar argument allows to construct a Hodge star operator on $\Omega(M_\theta)$. Classically, the Hodge star operator is a map $* : \Omega^p(M_\theta) \to \Omega^{m-p}(M_\theta)$ depending only on the conformal class of the metric on M. Since $\mathbb{T}^n$ acts by isometries, it leaves the conformal structure invariant and therefore, it commutes with $*$. Since $\Omega(M_\theta)$ coincides with $\Omega(M)$ as vector spaces, we can define a Hodge star operator $*_\theta$ on $\Omega(M_\theta)$ as the classical operator $*$, resulting in a map $*_\theta : \Omega^p(M_\theta) \to \Omega^{m-p}(M_\theta)$. In terms of the quantization map L_θ this becomes $*_\theta L_\theta(\omega) = L_\theta(*\omega)$ for $\omega \in \Omega(M)$.

Remark 2.5. *We can also define* $\Omega(M_\theta)$ *as a fixed point algebra. There is an action* σ *of* $\mathbb{T}^n$ *on* $\Omega(M)$ *which allows to define* $\Omega(M_\theta)$ *by* $\left(\Omega(M)\overline{\otimes}C^\infty(\mathbb{T}^n_\theta)\right)^{\sigma\otimes\tau^{-1}}$. *Since the exterior derivative* d *on* $\Omega(M)$ *commutes with the isometrical action of* $\mathbb{T}^n$ *on* M, *we can define the differential* d_θ *as* $d \otimes \mathrm{id}$ *in terms of the fixed point algebra. The Hodge star operator takes the form* $*_\theta = * \otimes \mathrm{id}$.

An inner product on $\Omega(M_\theta)$ can be constructed as follows. Since $*_\theta$ maps $\Omega^p(M_\theta)$ to $\Omega^{m-p}(M_\theta)$, we can define for $\alpha, \beta \in \Omega^p(M_\theta)$

$$(\alpha, \beta)_2 = \fint *_\theta(\alpha^* *_\theta \beta), \tag{2.2.6}$$

since $*_\theta(\alpha^* *_\theta \beta)$ is an element in $C^\infty(M_\theta)$.

Lemma 2.6. *The formal adjoint* d^* *of* d *with respect to the inner product* $(\cdot,\cdot)_2$ *(i.e. so that* $(d^*\alpha, \beta)_2 = (\alpha, d\beta)_2$*), is given on* $\Omega^p(M_\theta)$ *by*

$$d^* = (-1)^{m(p+1)+1} *_\theta d *_\theta$$

Proof. Just as in the classical case, this follows from Stokes theorem on M_θ:

$$\fint *_\theta(dL_\theta(\omega)) = 0,$$

for $\omega \in \Omega^{m-1}(M)$. This can be derived from the classical case, using Lemma 2.1 and the fact that $*_\theta dL_\theta(\omega) = L_\theta(*d\omega)$ □

In [27], an inner product on differential forms was defined in terms of Connes' differential calculus $\Omega_D(C^\infty(M_\theta))$ on $C^\infty(M_\theta)$. The $C^\infty(M_\theta)$-bimodule $\Omega^p_D(C^\infty(M_\theta))$ of p-forms is made of classes of operators of the form

$$\omega = \sum_j a_0^j[D, a_1^j] \cdots [D, a_p^j], \quad a_i^j \in C^\infty(M_\theta),$$

modulo the sub-bimodule of operators

$$\Big\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{p-1}^j] : \; b_i^j \in C^\infty(M_\theta), \; b_0^j[D, b_1^j] \cdots [D, b_{p-1}^j] = 0 \Big\}.$$

The exterior differential d_D is given by

$$d_D \Big[\sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j] \Big] = \sum_j [D, a_0^j][D, a_1^j] \cdots [D, a_p^j].$$

For two p-forms ω_1, ω_2, an inner product can be defined by

$$(\omega_1, \omega_2)_D = ⨍ \omega_1^* \omega_2. \tag{2.2.7}$$

The following result is taken from [32].

Lemma 2.7. *1. The differential calculus* $\Omega_D(C^\infty(M_\theta))$ *is isomorphic to* $\Omega(M_\theta)$ *defined above.*

2. Under this isomorphism, the inner product $(\cdot, \cdot)$ *coincides with* $(\cdot, \cdot)_D$.

Proof. 1. follows from the fact that $\Omega_D(C^\infty(M)) \simeq \Omega(M)$ as σ-equivariant bimodules over $C^\infty(M)$ whereas *2.* follows from the classical case using Lemma 2.1. □

2.3 Local index formula on toric noncommutative manifolds

In the case of the toric noncommutative manifolds introduced in [33] (cf. Section 2.2), the local index formula of Connes and Moscovici [34] simplifies drastically. We refer to that paper or Appendix A.7 for the general form of the local index formula.

Theorem 2.8. *For a projection* $p \in M_N(C^\infty(M_\theta))$*, we have*

$$\text{Index } D_p = \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}\left(\gamma p |D|^{-2z}\right) + \sum_{k \geq 1} c_k \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma \big(p - \frac{1}{2}\big)[D, p]^{2k} |D|^{-2(k+z)}\right)$$

where $c_k = (k-1)!/(2k)!$.

Proof. As noted before, the twist L_θ commutes with the action α_s of $\widetilde{\mathbb{T}}^n$ on an operator T. In fact, if T is homogeneous of degree r, then $L_\theta(T)$ is of degree r:

$$\alpha_s(L_\theta(T)) = U(s)\; TU(r')\; U(s)^{-1} = U(s)\; T\; U(s)^{-1} U(r') = e^{2\pi i s_\mu r_\mu} L_\theta(T),$$

with $r'_\nu = r_\mu \theta_{\mu\nu}$ so that $r' \in \widetilde{\mathbb{T}}^n$.

We write the cocycles ϕ^{2k} that define the local index formula in (A.7.1) in terms of the twist L_θ:

$$\phi^{2k}(L_\theta(f^0), L_\theta(f^1), \ldots, L_\theta(f^{2k}) = \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma L_\theta(f^0 \times_\theta [D, f^1]^{(\alpha_1)} \cdots \times_\theta [D, f^{2k}]^{(\alpha_{2k})}) |D|^{-2(|\alpha|+k+z)}\right)$$

where we extended the $\times_\theta$-product to $C^\infty(M_\theta) \bigcup [D, C^\infty(M_\theta)]$ which can be done unambiguously since D is of degree 0. Suppose now that $f^0, \ldots, f^{2k} \in C^\infty(M)$ are homogeneous of degree

$r^0, \dots, r^{2k}$, respectively, under the action of $\mathbb{T}^n$, so that the operator $f^0 \times_\theta [D, f^1] \cdots \times_\theta [D, f^{2k}]$ is a homogeneous element of degree r (which can be expressed in terms of the r^i). It is in fact a multiple of $f^0[D, f^1] \cdots [D, f^{2k}]$ by working out the $\times_\theta$-product. Forgetting about this factor –which is a power of λ– we obtain from (2.2.1)

$$L_\theta(f^0[D, f^1] \cdots [D, f^{2k}]) = f^0[D, f^1] \cdots [D, f^{2k}] U(r_\mu \theta_{\mu 1}, \dots, r_\mu \theta_{\mu n}).$$

Each term in the local index formula for $(C^\infty(M_\theta), \mathcal{H}, D)$ then takes the form

$$\operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})} |D|^{-2(|\alpha|+k+z)} U(s)\right),$$

for $s_\nu = r_\mu \theta_{\mu\nu}$ so that $s \in \widetilde{\mathbb{T}}^n$. The appearance of $U(s)$ here, is a consequence of the close relation with the index formula for a $\mathbb{T}^n$-equivariant Dirac spectral triple on M. In [24], Chern and Hu considered an even dimensional compact spin manifold M on which a (connected compact) Lie group G acts by isometries. The equivariant Chern character was defined as an equivariant version of the JLO-cocycle, the latter being an element in equivariant entire cyclic cohomology. The essential point is that they obtained an explicit formula for the above residues. In the case of the previous $\mathbb{T}^n$-action on M, one gets

$$\begin{aligned} &\operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})} |D|^{-2(|\alpha|+k+z)} U(s)\right) \\ &\qquad = \Gamma(|\alpha| + k) \lim_{t \to 0} t^{|\alpha|+k} \operatorname{tr}\left(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})} e^{-tD^2} U(s)\right) \end{aligned}$$

for every $s \in \widetilde{\mathbb{T}}^n$; moreover, this limit vanishes when $|\alpha| \neq 0$ (Thm 2 in [24]). □

Chapter 3

The Hopf fibration on S^4_θ

We now focus on two noncommutative spheres S^4_θ and $S^7_{\theta'}$, starting from the algebras $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on them. The latter algebra carries an action of the (classical) group $SU(2)$ by automorphisms in such a way that its invariant elements are exactly the polynomials on S^4_θ. The anti-symmetric 2×2 matrix θ is given by a single real number also denoted by θ. On the other hand, the requirements that $SU(2)$ acts by automorphisms and that S^4_θ makes the algebra of invariant functions, give the matrix θ' in terms of θ as well. This yields a one-parameter family of noncommutative Hopf fibrations. Moreover, there is an inclusion of the differential calculi $\Omega(S^4_\theta) \subset \Omega(S^7_{\theta'})$, as defined in Section 2.2.3.

For each irreducible representation $V^{(n)} := \mathrm{Sym}^n(\mathbb{C}^2)$ of $SU(2)$ we construct the noncommutative vector bundles $E^{(n)}$ associated to the fibration $S^7_{\theta'} \to S^4_\theta$. These bundles are described by the $C^\infty(S^4_\theta)$-bimodules of 'equivariant maps from $S^7_{\theta'}$ to $V^{(n)}$'. As expected, these modules are finite projective and we construct explicitly the projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ such that these modules are isomorphic to the image of $p_{(n)}$ in $\mathcal{A}(S^4_\theta)^{4^n}$. In the special case of the defining representation, we recover the basic instanton projection on the sphere S^4_θ constructed in [33]. Then, one defines connections $\nabla = p_{(n)}d$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$. The corresponding connection one-form A turns out to be valued in a representation of the Lie algebra $su(2)$. By using the projection $p_{(n)}$, the Dirac operator with coefficients in the noncommutative vector bundle $E^{(n)}$ is given by $D_{p_{(n)}} := p_{(n)} D p_{(n)}$. We compute its index by using the very simple form of the local index theorem of Connes and Moscovici [34] in the case of toric noncommutative manifolds as obtained in Theorem 2.8.

Finally, we show that the fibration $S^7_{\theta'} \to S^4_\theta$ is a 'not-trivial principal bundle with structure group $SU(2)$'. This means that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft Hopf-Galois extension [62, 75]; in fact, it is a principal extension [14]. We find an explicit form of the so-called strong connection [51] which induces connections on the associated bundles $E^{(n)}$ as maps from $\Gamma(S^4_\theta, E^{(n)})$ to $\Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$, where $\Omega^*_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ is the universal differential calculus on $\mathcal{A}(S^4_\theta)$ (cf. Appendix A.3). We show that these connections coincide with the Grassmann connections $\nabla = p_{(n)}d$ on $\Omega(S^4_\theta)$.

3.1 Construction of the fibration $S^7_{\theta'} \to S^4_\theta$

3.1.1 Classical Hopf fibration

We review the classical construction of the instanton bundle on S^4 [2] taking the approach of [66]. We generalize slightly and construct complex vector bundles on S^4 associated to all finite-dimensional irreducible representations of $SU(2)$.

We start by recalling the Hopf fibration $\pi: S^7 \to S^4$. Let

$$\begin{aligned} S^7 &:= \{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{C}^4 : |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 = 1\}, \\ S^4 &:= \{z = (z_1, z_2, z_0) \in \mathbb{C}^2 \oplus \mathbb{R} : z_1^* z_1 + z_2^* z_2 + z_0^2 = 1\}, \\ SU(2) &:= \{w \in GL(2, \mathbb{C}) : w^* w = w w^* = 1, \det w = 1\} \\ &= \left\{ w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} : w^1 \overline{w}^1 + w^2 \overline{w}^2 = 1 \right\}. \end{aligned}$$

The space S^7 carries a right $SU(2)$-action:

$$\begin{gathered} S^7 \times SU(2) \to S^7, \\ \big((\psi_1, -\psi_2^*, \psi_3, -\psi_4^*), w\big) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}. \end{gathered}$$

It might seem unnatural to define this action in this way, mixing the ψ's and ψ^*'s. However, this is only a labelling which is more convenient for the left action of $Spin(5)$ on S^7 as we will see later on (cf. equation (4.2.6)). The Hopf map is defined as a map $\pi(\psi) \mapsto (z)$ where

$$\begin{aligned} z_0 &= \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4, \\ z_1 &= 2(\psi_1 \psi_3^* + \psi_2^* \psi_4), \quad z_2 = 2(-\psi_1^* \psi_4 + \psi_2 \psi_3^*), \end{aligned}$$

and one computes $z_1^* z_1 + z_2^* z_2 + z_0^2 = (\sum_a \psi_a^* \psi_a)^2 = 1$.

The finite-dimensional irreducible representations of $SU(2)$ are labeled by a positive integer n with $n+1$-dimensional representation space $V^{(n)} \simeq \mathrm{Sym}^n(\mathbb{C}^2)$. The space of smooth $SU(2)$-equivariant maps from S^7 to $V^{(n)}$ is defined by

$$C^\infty_{SU(2)}(S^7, V^{(n)}) := \{\phi : S^7 \to V^{(n)} : \phi(\psi \cdot w) = w^{-1} \cdot \phi(\psi)\}. \tag{3.1.1}$$

It forms the $C^\infty(M)$-module of smooth sections of the associated vector bundle $S^7 \times_{SU(2)} V^{(n)} \to S^4$. We will now construct projections $p_{(n)}$ as $N \times N$ matrices taking values in $C^\infty(S^4)$), such that $\Gamma^\infty(S^4, E^{(n)}) := p_{(n)} C^\infty(S^4)^N$ is isomorphic to $C^\infty_{SU(2)}(S^7, V^{(n)})$ as right $C^\infty(S^4)$-modules. As the notation suggests, $E^{(n)}$ is the vector bundle over S^4 associated with the corresponding representation. Let us first recall the case $n = 1$ from [66] and then use this to generate the vector bundles for any n. The $SU(2)$-equivariant maps from S^7 to $V^{(1)} \simeq \mathbb{C}^2$ are of the form

$$\phi_{(1)}(\psi) = \begin{pmatrix} \psi_1^* \\ -\psi_2 \end{pmatrix} f_1 + \begin{pmatrix} \psi_2^* \\ \psi_1 \end{pmatrix} f_2 + \begin{pmatrix} \psi_3^* \\ -\psi_4 \end{pmatrix} f_3 + \begin{pmatrix} \psi_4^* \\ \psi_3 \end{pmatrix} f_4, \tag{3.1.2}$$

where $f_1, \dots, f_4$ are smooth functions that are invariant under the action of $SU(2)$, i.e. they are functions on the base space S^4.

A nice description of the equivariant maps is given in terms of ket-valued functions $|\xi\rangle$ on S^7, which are then elements in the free module $\mathcal{E} := \mathbb{C}^N \otimes C^\infty(S^7)$. The $C^\infty(S^7)$-valued hermitian structure (cf. Appendix A.4) on $\mathcal{E}$ given by $\langle \xi, \eta \rangle = \sum_b \xi_b^* \eta_b$ allows one to associate dual elements $\langle \xi | \in \mathcal{E}^*$ to each $|\xi\rangle \in \mathcal{E}$ by $\langle \xi | (\eta) := \langle \xi, \eta \rangle$, $\forall \eta \in \mathcal{E}$.

If we define $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{A}(S^7)^4$ by

$$|\psi_1\rangle = (\psi_1, \psi_2, \psi_3, \psi_4)^t; \quad |\psi_2\rangle = (-\psi_2^*, \psi_1^*, -\psi_4^*, \psi_3^*)^t,$$

with t denoting transposition, the equivariant maps in (3.1.2) are given by

$$\phi_{(1)}(\psi) = \begin{pmatrix} \langle\psi_1|f\rangle \\ \langle\psi_2|f\rangle \end{pmatrix},$$

where $|f\rangle \in (C^\infty(S^4))^4 := \mathbb{C}^4 \otimes C^\infty(S^4)$. Since $\langle\psi_k|\psi_l\rangle = \delta_{kl}$ as is easily seen, we can define a projection in $M_4(C^\infty(S^4))$ by

$$p_{(1)} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|.$$

Indeed, by explicit computation we find a matrix with entries in $C^\infty(S^4)$ which is the limit of the projection (3.2.3) for $\theta = 0$. Denoting the right $C^\infty(S^4)$-module $p_{(1)}(C^\infty(S^4))^4$ by $\Gamma(S^4, E^{(1)})$, we have

$$\begin{aligned} \Gamma(S^4, E^{(1)}) &\simeq C^\infty_{SU(2)}(S^7, \mathbb{C}^2) \\ \sigma_{(1)} = p_{(1)}|f\rangle &\leftrightarrow \phi_{(1)} = \begin{pmatrix} \langle\psi_1|f\rangle \\ \langle\psi_2|f\rangle \end{pmatrix}. \end{aligned}$$

For the general case, we note that the $SU(2)$-equivariant maps from S^7 to $V^{(n)}$ are of the form

$$\phi_{(n)}(\psi) = \begin{pmatrix} \langle\phi_1|f\rangle \\ \vdots \\ \langle\phi_{l+1}|f\rangle \end{pmatrix}, \tag{3.1.3}$$

where $|f\rangle \in C^\infty(S^4)^{4^n}$ and $|\phi_k\rangle$ is the completely symmetrized form of the tensor product $|\psi_1\rangle^{\otimes n-k+1} \otimes |\psi_2\rangle^{\otimes k-1}$ for $k = 1, \ldots, n+1$, normalized to have norm 1. For example, for the adjoint representation $n = 2$, we have

$$\begin{aligned} |\phi_1\rangle &:= |\psi_1\rangle \otimes |\psi_1\rangle, \\ |\phi_2\rangle &:= \frac{1}{\sqrt{2}}\big(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle\big), \\ |\phi_3\rangle &:= |\psi_2\rangle \otimes |\psi_2\rangle. \end{aligned}$$

Since in general, $\langle\phi_k|\phi_l\rangle = \delta_{kl}$, the matrix-valued function

$$p_{(n)} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in M_{4^n}(C^\infty(S^4))$$

defines a projection whose entries are in $C^\infty(S^4)$, since each entry $\sum_k |\phi_k\rangle_a\langle\phi_k|_b$ is $SU(2)$-invariant (cf. below formula (3.2.6)). We conclude that

$$\begin{aligned} p_{(n)}(C^\infty(S^4)^{4^n}) &\simeq C^\infty_{SU(2)}(S^7, V^{(n)}) \\ \sigma_{(n)} = p_{(n)}|f\rangle &\leftrightarrow \phi_{(n)} = \begin{pmatrix} \langle\phi_1|f\rangle \\ \vdots \\ \langle\phi_{n+1}|f\rangle \end{pmatrix}. \end{aligned}$$

3.1.2 The noncommutative spheres S^4_θ and $S^7_{\theta'}$

With θ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere S^4_θ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, subject to relations

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu^* z_\mu, \quad z_\mu^* z_\nu^* = \lambda_{\mu\nu} z_\nu^* z_\mu^*, \quad \mu, \nu = 0, 1, 2, \tag{3.1.4}$$

with deformation parameters given by

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi i\theta}, \quad \lambda_{j0} = \lambda_{0j} = 0, \quad j = 1, 2, \tag{3.1.5}$$

and together with the spherical relation $\sum_\mu z^*_\mu z_\mu = 1$. For $\theta = 0$ one recovers the $*$-algebra of complex polynomial functions on the usual sphere S^4.

The differential calculus $\Omega(S^4_\theta)$ is generated as a graded differential $*$-algebra by the elements z_μ, z^*_μ of degree 0 and elements dz_μ, dz^*_μ of degree 1 satisfying the relations:

$$\begin{aligned} &dz^\mu dz^\nu + \lambda^{\mu\nu} dz^\nu dz^\mu = 0; \qquad && d\bar{z}^\mu dz^\nu + \lambda^{\nu\mu} dz^\nu d\bar{z}^\mu = 0; \\ &z^\mu dz^\nu = \lambda^{\mu\nu} dz^\nu z^\mu; && \bar{z}^\mu dz^\nu = \lambda^{\nu\mu} dz^\nu \bar{z}^\mu. \end{aligned} \tag{3.1.6}$$

with $\lambda^{\mu\nu}$ as before. There is a unique differential d on $\Omega(S^4_\theta)$ such that $d : z_\mu \mapsto dz_\mu$ and the involution on $\Omega(S^4_\theta)$ is the graded extension of $z_\mu \mapsto z^*_\mu$.

With $\lambda'_{ab} = e^{2\pi i\theta'_{ab}}$ and (θ'_{ab}) a real antisymmetric matrix, the algebra $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on the sphere $S^7_{\theta'}$ is generated by elements ψ_a, ψ^*_a, $a = 1, \ldots, 4$, subject to relations

$$\psi_a\psi_b = \lambda'_{ab}\psi_b\psi_a, \quad \psi_a\psi^*_b = \lambda'_{ba}\psi^*_b\psi_a, \quad \psi^*_a\psi^*_a = \lambda'_{ab}\psi^*_b\psi^*_a, \tag{3.1.7}$$

and with the spherical relation $\sum_a \psi^*_a\psi_a = 1$. At $\theta = 0$ it is the $*$-algebra of complex polynomial functions on the sphere S^7. As before, a differential calculus $\Omega(S^7_{\theta'})$ can be defined to be generated by the elements ψ_a, ψ^*_a of degree 0 and elements $d\psi_a, d\psi^*_a$ of degree 1 satisfying relations similar to the one in (3.1.6).

3.1.3 Hopf fibration

Firstly, we remind that while there is a θ-deformation of the manifold $S^3 \simeq SU(2)$, to a sphere S^3_θ, on the latter there is no compatible group structure so that there is no θ-deformation of the group $SU(2)$ [32]. Therefore, we must choose the matrix $\theta'_{\mu\nu}$ in such a way that the noncommutative 7-sphere $S^7_{\theta'}$ carries a classical $SU(2)$ action, which in addition is such that the subalgebra of $\mathcal{A}(S^7_{\theta'})$ consisting of $SU(2)$-invariant polynomials is exactly $\mathcal{A}(S^4_\theta)$.

The action of $SU(2)$ on the generators of $\mathcal{A}(S^7_{\theta'})$ is simply defined by

$$\alpha_w : (\psi_1, -\psi^*_2, \psi_3, -\psi^*_4) \mapsto (\psi_1, -\psi^*_2, \psi_3, -\psi^*_4) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \qquad w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}. \tag{3.1.8}$$

Here w^1 and w^2, satisfying $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$, are the coordinates on $SU(2)$. By imposing that the map $w \mapsto \alpha_w$ embeds $SU(2)$ in $\mathrm{Aut}(\mathcal{A}(S^7_{\theta'}))$ we find that $\lambda'_{12} = \lambda'_{34} = 1$ and $\lambda'_{14} = \lambda'_{23} = \overline{\lambda'_{24}} = \overline{\lambda'_{13}} =: \lambda'$.

The subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_{\theta'})$ can be found in the following way. From the diagonal nature of the action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ and the above formulæ for λ'_{ab}, we see that the action of $SU(2)$ commutes with the action of $\mathbb{T}^2$ on $\mathcal{A}(S^7_{\theta'})$. Since $\mathcal{A}(S^7_{\theta'})$ coincides with $\mathcal{A}(S^7)$ as vector spaces, we see that the subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7_{\theta'})$ is completely determined by the classical subalgebra of $SU(2)$-invariant elements in $\mathcal{A}(S^7)$. From Section 3.1.1 we can conclude that

$$\mathrm{Inv}_{SU(2)}(\mathcal{A}(S^7_\theta)) = \mathbb{C}[\, 1, \psi_1\psi^*_3 + \psi^*_2\psi_4, -\psi^*_1\psi_4 + \psi_2\psi^*_3, \psi_1\psi^*_1 + \psi^*_2\psi_2 \,]$$

modulo the relations in the algebra $\mathcal{A}(S^7_{\theta'})$. We identify

$$
\begin{aligned}
z_0 &= \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 \\
&= 2(\psi_1^*\psi_1 + \psi_2^*\psi_2) - 1 = 1 - 2(\psi_3^*\psi_3 + \psi_4^*\psi_4), \\
z_1 &= 2(\mu\psi_3^*\psi_1 + \psi_2^*\psi_4) = 2(\psi_1\psi_3^* + \psi_2^*\psi_4), \\
z_2 &= 2(-\mu\psi_4\psi_1^* + \psi_2\psi_3^*) = 2(-\psi_1^*\psi_4 + \psi_2\psi_3^*).
\end{aligned} \tag{3.1.9}
$$

and compute that $z_1z_1^* + z_2z_2^* + z_0^2 = 1$. By imposing commutation rules $z_1z_2 = \lambda z_2z_1$ and $z_1z_2^* = \bar{\lambda} z_2^*z_1$, we infer that $\lambda'_{14} = \lambda'_{23} = \overline{\lambda'_{24}} = \overline{\lambda'_{13}} = \sqrt{\lambda} =: \mu$ on $S^7_{\theta'}$. We conclude that $\mathrm{Inv}_{SU(2)}(\mathcal{A}(S^7_{\theta'})) = \mathcal{A}(S^4_\theta)$ for $\lambda'_{ab} = e^{2\pi i \theta'_{ab}}$ of the following form:

$$
\lambda'_{ab} = \begin{pmatrix} 1 & 1 & \bar{\mu} & \mu \\ 1 & 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 & 1 \\ \bar{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = \sqrt{\lambda}, \qquad \text{or} \qquad \theta'_{ab} = \frac{\theta}{2}\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}. \tag{3.1.10}
$$

The relations (3.1.9) can be also expressed in the form,

$$
z_\mu = \sum_{ab} \psi_a^*(\gamma_\mu)_{ab}\psi_b, \qquad z_\mu^* = \sum_{ab} \psi_a^*(\gamma_\mu^*)_{ab}\psi_b,
$$

with γ_μ the following twisted 4×4 Dirac matrices:

$$
\gamma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad \gamma_1 = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = 2\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1.11}
$$

Note that γ_0 is the usual grading given by

$$
\gamma_0 = -\frac{1}{4}[\gamma_1, \gamma_1^*][\gamma_2, \gamma_2^*].
$$

These matrices satisfy the following relations of the twisted Clifford algebra [32]:

$$
\begin{aligned}
\gamma_\mu\gamma_\nu + \lambda_{\mu\nu}\gamma_\nu\gamma_\mu &= 0, \\
\gamma_\mu\gamma_\nu^* + \lambda_{\nu\mu}\gamma_\nu^*\gamma_\mu &= 4\delta_{\mu\nu}.
\end{aligned} \tag{3.1.12}
$$

The action σ of $\mathbb{T}^2$ on S^4_θ is given as follows. For $s \in \mathbb{T}^2$ it maps $z_i \to e^{2\pi i s_i} z_i$ while leaving z_0 invariant. This action can be lifted to $S^7_{\theta'}$ in the following way. There is a double cover $\widetilde{\mathbb{T}}^2$ of $\mathbb{T}^2$ coming from the spin cover $\mathrm{Spin}(5)$ of SO(5), given explicitly by the map $p : (s_1, s_2) \mapsto (s_1 + s_2, -s_1 + s_2)$. Then $\widetilde{\mathbb{T}}^2$ acts on the ψ_a's as:

$$
\widetilde{\sigma} : \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i s_1}\ \psi_1 \\ e^{-2\pi i s_1}\ \psi_2 \\ e^{-2\pi i s_2}\ \psi_3 \\ e^{2\pi i s_2}\ \psi_4 \end{pmatrix} \tag{3.1.13}
$$

The classical counterpart of equation (3.1.9) shows that $\widetilde{\sigma}$ is indeed a lifting to S^7 of the action of $\mathbb{T}^2$ on S^4. Of course, this is inherent to the construction of the Hopf fibration $S^7_{\theta'} \to S^4_\theta$ in that it is a deformation of the classical Hopf fibration $S^7 \to S^4$ with respect to an action of $\mathbb{T}^2$, as one can easily see from the block-form of λ' in (3.1.10).

3.2 Associated modules

There is a nice description of the instanton projection constructed in [33] in terms of ket-valued polynomials on $S^7_{\theta'}$. The latter are elements in the right $\mathcal{A}(S^7_{\theta'})$-module $\mathcal{E} := \mathbb{C}^4 \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^4$ with a $\mathcal{A}(S^7_{\theta'})$-valued hermitian structure (cf. Appendix A.4) given by $\langle \xi, \eta \rangle = \sum_b \xi^*_b \eta_b$. To any $|\xi\rangle \in \mathcal{E}$ one associates its dual $\langle \xi| \in \mathcal{E}^*$ by setting $\langle \xi|(\eta) := \langle \xi, \eta\rangle$, $\forall \eta \in \mathcal{E}$.

Similarly to the classical case (see Section 3.1.1), we define a 2×4 matrix Ψ in terms of two ket-valued polynomials $|\psi_1\rangle$ and $|\psi_2\rangle$ by

$$\Psi = (|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}. \tag{3.2.1}$$

Then $\Psi^*\Psi = \mathbb{I}_2$ so that the 4×4-matrix,

$$p = \Psi\Psi^* = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|,$$

is a projection, $p^2 = p = p^*$, with entries in $\mathcal{A}(S^4_\theta)$. The action (3.1.8) becomes

$$\alpha_w(\Psi) = \Psi w, \tag{3.2.2}$$

from which the invariance of the entries of p follows at once. Explicitly one finds

$$p = \frac{1}{2}\begin{pmatrix} 1+z_0 & 0 & z_1 & -\bar{\mu} z_2^* \\ 0 & 1+z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\mu z_2 & \bar{\mu} z_1 & 0 & 1-z_0 \end{pmatrix}. \tag{3.2.3}$$

The projection p is easily seen to be equivalent to the projection describing the instanton on S^4_θ constructed in [33]. Indeed, if one defines

$$|\widetilde{\psi}_1\rangle = (\psi_1, \psi_2, \psi_3, \mu\psi_4)^t, \quad |\widetilde{\psi}_2\rangle = (-\psi_2^*, \psi_1^*, -\psi_4^*, \mu\psi_3^*)^t,$$

one obtains after a substitution $z_2 \mapsto -\bar{\lambda} z_2^*$ exactly the projection obtained therein, that is,

$$\widetilde{p} = \frac{1}{2}\begin{pmatrix} 1+z_0 & 0 & z_1 & -\bar{\lambda} z_2^* \\ 0 & 1+z_0 & z_2 & -z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\lambda z_2 & -z_1 & 0 & 1-z_0 \end{pmatrix}$$

Notice the following form of the projection in terms of the Dirac matrices defined in (3.1.11) [32].

Lemma 3.1. *The matrices* $\widetilde{\gamma}_0 := \gamma_0$, $\widetilde{\gamma}_1 := \bar{\mu}\gamma_1^t$ *and* $\widetilde{\gamma}_2 := \mu\gamma_2^t$ *satisfy the relations*

$$\begin{aligned} \widetilde{\gamma}_\mu \widetilde{\gamma}_\nu + \lambda_{\nu\mu} \widetilde{\gamma}_\nu \widetilde{\gamma}_\mu &= 0, \\ \widetilde{\gamma}_\mu \widetilde{\gamma}^*_\nu + \lambda_{\mu\nu} \widetilde{\gamma}^*_\nu \widetilde{\gamma}_\mu &= 4\delta_{\mu\nu}, \end{aligned} \tag{3.2.4}$$

and the above projection (3.2.3) *can expressed as*

$$p = \tfrac{1}{2}\left(1 + \widetilde{\gamma}_0 z_0 + \widetilde{\gamma}_i z_i + \widetilde{\gamma}_i^* z_i^*\right).$$

Note the difference of (3.2.4) with (3.1.12) in the exchange of $\lambda_{\mu\nu}$ by $\lambda_{\nu\mu}$.

We will denote the image of p in $\mathcal{A}(S^4_\theta)^4$ by $\Gamma(S^4_\theta, E) = p\mathcal{A}(S^4_\theta)^4$ which is clearly a right $\mathcal{A}(S^4_\theta)$-module. Another description of the module $\Gamma(S^4_\theta, E)$ comes from considering "equivariant maps" from $S^7_{\theta'}$ to $\mathbb{C}^2$ [44, 45], similar to (3.1.1). The defining left representation of $SU(2)$ on $\mathbb{C}^2$ is given by $SU(2) \times \mathbb{C}^2 \to \mathbb{C}^2; (w, v) \mapsto w \cdot v$. The collection of equivariant maps from $S^7_{\theta'}$ to $\mathbb{C}^2$, denoted by $A(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$, consists of elements $\phi \in \mathcal{A}(S^7_{\theta'}) \otimes \mathbb{C}^2$, such that

$$(\alpha_w \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \rho(w)^{-1})(\phi). \tag{3.2.5}$$

Compare with its classical analogue in equation (3.1.1). It is a right $\mathcal{A}(S^4_\theta)$-module (it is in fact a $\mathcal{A}(S^4_\theta)$-bimodule) since multiplication by an element in $\mathcal{A}(S^4_\theta)$ does not affect the equivariance condition (3.2.5).

Since $SU(2)$ acts classically on $\mathcal{A}(S^7_{\theta'})$, one sees from (3.2.2) that the equivariant maps are given by elements of the form $\phi := \Psi^* f$ for some $f \in \mathcal{A}(S^4_\theta) \otimes \mathbb{C}^4$ (summation understood). In terms of the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$, we can write $\phi = \sum_k \langle \psi_k | f \rangle \otimes e_k$ for $|f\rangle = |f_1, f_2, f_3, f_4\rangle^t$, with $f_a \in \mathcal{A}(S^4_\theta)$ (cf. Section 3.1.1). We then have the following isomorphism

$$\begin{array}{rcl} \Gamma(S^4_\theta, E) & \simeq & \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2 \\ \sigma = p|f\rangle & \leftrightarrow & \phi = \Psi^* f = \sum \langle \psi_k | f \rangle \otimes e_k \end{array}$$

More generally, one can define the right $\mathcal{A}(S^4_\theta)$-module $\Gamma(S^4_\theta, E^{(n)})$ associated with any irreducible representation $\rho_{(n)} : SU(2) \to \mathrm{GL}(V^{(n)})$, with $V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$ for a positive integer n. The module of $SU(2)$-equivariant maps from $S^7_{\theta'}$ to $V^{(n)}$ is defined as

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} V := \left\{ f \in \mathcal{A}(S^7_{\theta'}) \otimes V : (\alpha_w \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho_{(n)}(w)^{-1})(f) \right\}.$$

It is easy to see that these maps are of the form $\phi_{(n)} = \sum_k \langle \phi_k | f \rangle \otimes e_k$ on the basis $\{e_1, \ldots, e_{n+1}\}$ of $V^{(n)}$ where now $|f\rangle \in \mathcal{A}(S^4_\theta)^{4^n}$ and

$$|\phi_k\rangle = \frac{1}{a_k} |\psi_1\rangle^{\otimes(n-k+1)} \otimes_S |\psi_2\rangle^{\otimes(k-1)} \quad (k = 1, \ldots, n+1),$$

with $\otimes_S$ denoting symmetrization and a_k are suitable normalization constants. These vectors $|\phi_k\rangle \in \mathbb{C}^{4^n} \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^{4^n}$ are orthogonal (with the natural hermitian structure), and with $a_k^2 = \binom{n}{k-1}$ they are also normalized. Then

$$p_{(n)} := |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in \mathrm{Mat}_{4^n}(\mathcal{A}(S^4_\theta)) \tag{3.2.6}$$

defines a projection $p^2 = p = p^*$. That its entries are in $\mathcal{A}(S^4_\theta)$ and not in $\mathcal{A}(S^7_{\theta'})$ is easily seen. Indeed, much as it happens for the vector Ψ in equation (3.2.2), for every $i = 1, \ldots, 4^n$, the vector $\big(|\phi_1\rangle_i, |\phi_2\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big)$ transforms under the action of $SU(2)$ to the vector $\big(|\phi_1\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big) \cdot \rho_{(n)}(w)$ so that each entry $\sum_k |\phi_k\rangle_i \langle\phi_k|_j$ of $p_{(n)}$ is $SU(2)$-invariant and hence an element in $\mathcal{A}(S^4_\theta)$. With this we proved the following.

Proposition 3.2. *The right $\mathcal{A}(S^4_\theta)$-module of equivariant maps $\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} V^{(n)}$ is isomorphic to the right $\mathcal{A}(S^4_\theta)$-module $\Gamma(S^4_\theta, E^{(n)}) := p_{(n)}(\mathcal{A}(S^4_\theta)^{4^n})$ with the isomorphism given explicitly by:*

$$\begin{array}{rcl} \Gamma(S^4_\theta, E^{(n)}) & \simeq & \mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} V^{(n)} \\ \sigma_{(n)} = p_{(n)}|f\rangle & \leftrightarrow & \phi_{(n)} = \sum_k \langle \phi_k | f \rangle \otimes e_k. \end{array}$$

One can easily lift this whole construction to the smooth level by replacing polynomial algebras by their smooth completions as defined in 2.2. One proves that the $C^\infty(S^4_\theta)$-module $\Gamma^\infty(S^4_\theta, E^{(n)})$ defined by $p_{(n)}(C^\infty(S^4_\theta))^{4^n}$ is isomorphic to $C^\infty(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} V^{(n)}$.

With the projections $p_{(n)}$ one associates (Grassmann) connections on the right $C^\infty(S^4_\theta)$-modules $\Gamma(S^4_\theta, E^{(n)})$ in a canonical way:

$$\nabla = p_{(n)} \circ d : \Gamma(S^4_\theta, E^{(n)}) \to \Gamma(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta) \tag{3.2.7}$$

where $(\Omega^*(S^4_\theta), d)$ is the differential calculus defined in the previous section. An expression for these connections as acting on coequivariant maps can be obtained using the above isomorphism and results in:

$$\nabla(\phi_k) = d(\phi_k) + A_{kl}\phi_l \tag{3.2.8}$$

where $A_{kl} = \langle \phi_k | d\phi_l \rangle \in \Omega^1(S^7_{\theta'})$. The corresponding matrix A is called the connection one-form; it is clearly anti-hermitian, and it is valued in the derived representation space, $\rho'_n : su(2) \to \mathrm{End}(V^{(n)})$, of the Lie algebra $su(2)$.

3.2.1 Properties of the associated modules

Let us now discuss some properties of the associated modules, like hermitian structures and the structure of the algebra of endomorphisms on them, as defined in Appendix A.4.

Let ρ be any representation of SU(2) on an n-dimensional vector space V. The $C^\infty(S^4_\theta)$ bimodule associated to V is defined by

$$\mathcal{E} := C^\infty(S^7_{\theta'}) \boxtimes_\rho V := \left\{ f \in C^\infty(S^7_{\theta'}) \otimes V : (\alpha_w \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(w)^{-1})(f) \right\}.$$

As we have proved above, the module $\mathcal{E}$ is a finite projective $C^\infty(S^4_\theta)$ module. Note that the choice of a projection for a finite projective module requires the choice of one of the two (left or right) module structures. Similarly, the definition of a hermitian structure requires the choice of a left or right module structure. In the following, we will always work with the right structure for the associated modules. There is a natural (right) Hermitian structure on $\mathcal{E}$, defined in terms of the inner product of V as:

$$\langle f, g \rangle := \sum_i \bar{f}_i g_i. \tag{3.2.9}$$

where we denoted $f = \sum_i f_i \otimes e_i, g = \sum_i g_i \otimes e_i$ for a basis $\{e_i\}_{i=1}^n$ in V. One quickly checks that $\langle f, g \rangle$ is an element in $C^\infty(S^4_\theta)$, and that $\langle \ , \ \rangle$ satisfies all conditions of a right Hermitian structure.

Remark 3.3. *The bimodules $C^\infty(S^7_{\theta'}) \boxtimes_\rho V$ are also of the type described in Section 2.2.2. Indeed, the associated vector bundle $E = S^7 \times_\rho V$ on S^4 carries an action of $\mathbb{T}^2$ induced from its action on S^7, which is obviously σ-equivariant. By the very definition of $C^\infty(S^7_\theta)$ and $\Gamma(S^4_\theta, E)$ it then follows that $C^\infty(S^7_{\theta'}) \boxtimes_\rho V \simeq \Gamma(S^4_\theta, E)$. Moreover, the $C^\infty(S^4)$-modules $\Gamma(S^4, E)$ posses the homogeneous decomposition property of Definition 2.2. Indeed, from the explicit form of the $|\phi_k\rangle$ in equation (3.1.3), we see that there is a basis $\{e_i\}_{i=1}^{4^n}$ of the module $\Gamma(S^4, E^{(n)})$ given by:*

$$e_i := \begin{pmatrix} \langle \phi_1 |_i \\ \vdots \\ \langle \phi_{l+1} |_i \end{pmatrix}$$

which are homogeneous under the action of $\widetilde{\mathbb{T}}^2$. A generic element $\sum_i e_i f_i$ with $f_i \in C^\infty(S^4)$ can then clearly be decomposed into homogeneous elements.

The *dual module of* $\mathcal{E}$ is defined by

$$\mathcal{E}' := \{\phi : \mathcal{E} \to C^\infty(S^4_\theta) : \phi(fa) = \phi(f)a, \quad a \in C^\infty(S^4_\theta)\} ,$$

which is isomorphic to $\mathcal{E}$ with the isomorphism given by the map $f \mapsto \langle f, \cdot\rangle$. The representation ρ on V induces a dual representation ρ' on V' by

$$(\rho'(w)v')(v) := v'(\rho(w)^{-1}v); \qquad (\forall v' \in V', v \in V) ,$$

and we have

$$\mathcal{E}' \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\rho'} V' := \{\phi \in C^\infty(S^7_{\theta'}) \otimes V' : (\alpha_w \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \rho'(w)^{-1})(\phi)\} .$$

Let $L(V)$ denote the space of linear maps on V, so that $L(V) = V \otimes V'$. The adjoint action of SU(2) on $L(V)$ is defined as the tensor product representation $\mathrm{ad} := \rho \otimes \rho'$ on $V \otimes V'$. We define

$$C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V) := \{T \in C^\infty(S^7_{\theta'}) \otimes L(V) : (\alpha_w \otimes \mathrm{id})(T) = (\mathrm{id} \otimes \mathrm{ad}(w)^{-1})(T)\} ,$$

and write $T = T_{ij} \otimes e_{ij}$ with respect to the basis $\{e_{ij}\}$ of $L(V)$ induced from the basis of V.

Proposition 3.4. *There is an isomorphism of algebras* $\mathrm{End}(\mathcal{E}) \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V)$. [1]

Proof. Recall that $\mathcal{E} \otimes_{C^\infty(S^4_\theta)} \mathcal{E}' \subset \mathrm{End}(\mathcal{E})$ densely (in the operator norm, cf. [65]). We define a map from $\mathrm{End}(\mathcal{E})$ to $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V)$ by

$$f \otimes f' \mapsto f_i f'_j \otimes e_{ij}$$

On the other hand, $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V)$ acts on $\mathcal{E}$ in the following way:

$$(T, f) \mapsto T_{ij} f_j \otimes e_i,$$

which is clearly a right $C^\infty(S^4_\theta)$-linear map with image in $\mathcal{E}$. Hence, $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V) \subset \mathrm{End}(\mathcal{E})$. □

We see that the algebra of endomorphisms of $\mathcal{E}$ can be understood as the space of sections of the noncommutative vector bundle associated to the adjoint representation on $L(V)$, exactly as it happens in the classical case. In particular, we have a Hermitian structure on $\mathrm{End}(\mathcal{E})$ defined by (3.2.9). For the skew-hermitian endomorphisms we have the following result.

Corollary 3.5. *We have* $\mathrm{End}^s(\mathcal{E}) \simeq C^\infty_{\mathbb{R}}(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} u(n)$*, where* $C^\infty_{\mathbb{R}}(S^4_\theta)$ *denotes the algebra of self-adjoint elements in* $C^\infty(S^7_{\theta'})$ *whereas* $u(n)$ *consists of skew-adjoint matrices in* $M_n(\mathbb{C}) \simeq L(V)$.

Proof. Note that the involution $T \mapsto T^*$ in $\mathrm{End}(\mathcal{E})$ reads in components $T_{ij} \mapsto \overline{T_{ji}}$ so that $\mathrm{End}^s(\mathcal{E})$ is given by elements $X \in C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(V)$ satisfying $\overline{X_{ji}} = -X_{ij}$. Since any element in $C^\infty(S^7_{\theta'})$ can be written as the sum of two self-adjoint elements, $X_{ij} = X^{\mathfrak{R}}_{ij} + iX^{\mathfrak{I}}_{ij}$, we can write

$$X = \sum_i X^{\mathfrak{I}}_{ii} \otimes ie_{ii} + \sum_{i \neq j} X^{\mathfrak{R}}_{ij} \otimes (e_{ij} - e_{ji}) + X^{\mathfrak{I}}_{ij} \otimes (ie_{ij} + ie_{ji}) = \sum_a X_a \otimes \sigma^a,$$

where X_a are generic elements in $C^\infty_{\mathbb{R}}(S^7_{\theta'})$ and σ^a are the generators of $u(n)$ (for $a = 1, \ldots, n^2$). □

[1] We suppressed the subscript $C^\infty(S^4_\theta)$ from End and will continue to do so.

Example 3.6. *Let us return to the instanton bundle* $\mathcal{E} = C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$. *In this case,* $\mathrm{End}(\mathcal{E}) \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} M_2(\mathbb{C}))$. *Since* $M_2(\mathbb{C})$ *decomposes into the adjoint representation* $\mathfrak{su}(2)$ *and the trivial representation* $\mathbb{C}$ *and because* $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{id}} \mathbb{C} \simeq C^\infty(S^4_\theta)$, *we conclude that*

$$\mathrm{End}(\mathcal{E}) \simeq \Gamma(\mathrm{ad}(S^7_{\theta'})) \oplus C^\infty(S^4_\theta), \tag{3.2.10}$$

where we have set $\Gamma(\mathrm{ad}(S^7_{\theta'})) := C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} \mathfrak{su}(2)$. *The latter* $C^\infty(S^4_\theta)$*-bimodule will be understood as the space of (complex) sections of the adjoint bundle. It is the complexification of the traceless skew-hermitian endomorphism* $C^\infty_{\mathbb{R}}(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} \mathfrak{su}(2)$.

3.2.2 Index of twisted Dirac operators

In this section, we shall compute explicitly the index of the Dirac operator with coefficients in the bundles $E^{(n)}$, that is the index of the operator of $D_{p_{(n)}} := p_{(n)}(D \otimes \mathbb{I}_{4^n})p_{(n)}$. We will compute this index using the special form of the Connes Moscovici local index formula, as we derived in Theorem 2.8. In our case of interest, the index of the Dirac operator on S^4_θ with coefficients in some noncommutative vector bundle determined by $e \in K_0(C(S^4_\theta))$, we obtain

$$\begin{aligned}\text{Index } D_e = \langle \phi^*, \mathrm{ch}_*(e) \rangle &= \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}\left(\gamma \pi_D(\mathrm{ch}_0(e))|D|^{-2z}\right) \\ &\quad + \frac{1}{2!} \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma \pi_D(\mathrm{ch}_1(e))|D|^{-2-2z}\right) \\ &\quad + \frac{1}{4!} \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma \pi_D(\mathrm{ch}_2(e))|D|^{-4-2z}\right)\end{aligned}$$

Here π_D is the representation of the universal differential calculus (cf. Appendix A.3) given by

$$\pi_D : \Omega^p_{\mathrm{un}}(\mathcal{A}(S^4_\theta)) \to \mathcal{B}(\mathcal{H}), \quad a^0 \delta a^1 \cdots \delta a^p \mapsto a^0[D, a^1] \cdots [D, a^p].$$

Let us examine at which quotients of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ this representation π_D is well-defined. Unfortunately, π_D is not well-defined on the quotient $\Omega(S^4_\theta)$ defined in the previous section. For example already $[D, \alpha][D, \alpha] \neq 0$ whereas $d\alpha d\alpha = 0$ in $\Omega(S^4_\theta)$. This was already noted in [32] and in fact

$$\Omega(S^4_\theta) \simeq \pi_D\big(\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))\big)/\pi_D(\delta J_0)$$

where $J_0 := \{\omega \in \Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta)) | \pi_D(\omega) = 0\}$ are the so-called 'junk-forms' [27]. We will avoid a discussion on junk-forms and introduce instead a different quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$. We define $\Omega_D(S^4_\theta)$ to be $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ modulo the relations

$$\begin{aligned}&\alpha\delta\beta - \lambda(\delta\beta)\alpha = 0, \quad (\delta\alpha)\beta - \lambda\beta\delta\alpha = 0, \\ &\alpha\delta\beta^* - \bar\lambda(\delta\beta^*)\alpha = 0, \quad (\delta\alpha^*)\beta - \bar\lambda\beta\delta\alpha^* = 0, \\ &a\delta x - (\delta x)a = 0, \qquad \forall a \in \mathcal{A}(S^4_\theta),\end{aligned}$$

avoiding the second order relations that define $\Omega(S^4_\theta)$. Using the splitting homomorphism one proves that the above relations are in the kernel of π_D, for instance, $\alpha[D, \beta] - \lambda[D, \beta]\alpha = 0$ so that π_D is well-defined on $\Omega_D(S^4_\theta)$. The differential calculus $\Omega_D(S^7_{\theta'})$ is the quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))$ by only the relations in (2.1.1) of order one, that is by the relations:

$$z^i \delta z^j = \lambda^{ij}(\delta z^j) z^i; \quad z^i \delta \bar z^j = \lambda^{ji}(\delta \bar z^j) z^i.$$

Lemma 3.7. *The following formulæ hold for the images under* π_D *of the Chern characters of* $p_{(n)}$*:*

$$\begin{aligned}\pi_D(\mathrm{ch}_0(p_{(n)})) &= n+1;\\ \pi_D(\mathrm{ch}_1(p_{(n)})) &= 0;\\ \pi_D(\mathrm{ch}_2(p_{(n)})) &= \frac{1}{6}n(n+1)(n+2)\pi_D(\mathrm{ch}_2(p_{(1)}));\end{aligned}$$

up to the coefficients $\mu_k = (-1)^k \frac{(2k)!}{k!}$.

Proof. Recall that the projections $p_{(n)}$ were defined by $p_{(n)} = \sum_k |\phi_k\rangle\langle\phi_k|$, where $|\phi_k\rangle$ with $k = 1, \dots, n+1$, is given by

$$|\phi_k\rangle = \frac{1}{a_k}|\psi_1\rangle^{\otimes(n-k+1)} \otimes_S |\psi_2\rangle^{\otimes(k-1)}, \quad a_k^2 = \binom{n}{k-1}.$$

Before we start the computation of the Chern characters, we state the computation rules in $\Omega_D(S^7_{\theta'})$. Firstly, from the very definition of the vectors $|\phi_k\rangle$ and the inner product in $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{E}$, we can express, for any $k = 1, \dots, n+1$,

$$\langle\phi_k|\delta\phi_{k-1}\rangle = \sqrt{(n-k)(k+1)}\langle\psi_2|\delta\psi_1\rangle \tag{3.2.11}$$

$$\langle\phi_k|\delta\phi_{k+1}\rangle = \sqrt{(n-k-1)(k+2)}\langle\psi_1|\delta\psi_2\rangle \tag{3.2.12}$$

$$\begin{aligned}\langle\phi_k|\delta\phi_k\rangle &= (n-k-1)\langle\psi_1|\delta\psi_1\rangle + (k+1)\langle\psi_2|\delta\psi_2\rangle\\ &= (n-2k-2)\langle\psi_1|\delta\psi_1\rangle,\end{aligned} \tag{3.2.13}$$

by using the relation $\langle\psi_2|\delta\psi_2\rangle = -\langle\psi_1|\delta\psi_1\rangle$. The previous are in fact the only nonzero expressions for $\langle\phi_k|\delta\phi_l\rangle$. If we apply δ to these equations, we obtain expressions for $\langle\delta\phi_k|\delta\phi_l\rangle$ in terms of ψ_1 and ψ_2. From this, we deduce the following result that will be central in the computation of the Chern characters.

Lemma 3.8. *The following relations hold in* $\Omega_D(S^7_{\theta'})$*:*

$$\begin{aligned}\sum_{k,l=1}^{n+1}\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle &= \frac{1}{6}n(n+1)(n+2)\sum_{r,s=1}^{2}\langle\psi_r|\delta\psi_s\rangle\langle\psi_s|\delta\psi_r\rangle,\\ \sum_{k,l,m=1}^{n+1}\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle &= \frac{1}{6}n(n+1)(n+2)\sum_{r,s,t=1}^{2}\langle\psi_r|\delta\psi_s\rangle\langle\psi_s|\delta\psi_t\rangle\langle\psi_t|\delta\psi_r\rangle.\end{aligned}$$

Of course, there will be similar formulæ for $\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle$, etc.

The zeroth Chern character is easy to compute:

$$\mathrm{ch}_0(p_{(n)}) = \mathrm{tr}(p_{(n)}) = \sum_k \langle\phi_k|\phi_k\rangle = n+1.$$

In the computation of $\mathrm{ch}_1(p_{(n)})$ we use the relation $\langle\delta\phi_k|\phi_l\rangle = -\langle\phi_k|\delta\phi_l\rangle$, which follows from applying the derivation δ to $\langle\phi_k|\phi_l\rangle = \delta_{kl}$ and the fact that in $\Omega_D(S^7_{\theta'})$, $\langle\phi_k|\delta\phi_l\rangle$ commutes with any element in $\mathcal{A}(S^7_{\theta'})$, in particular with $\langle\phi_m|_i$. Thus,

$$\begin{aligned}
\mathrm{ch}_1(p) &= \sum |\phi_k\rangle\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m| + \sum |\phi_k\rangle\langle\delta\phi_k|\delta\phi_m\rangle\langle\phi_m| \\
&-\frac{1}{2}\sum\Big(|\delta\phi_k\rangle\langle\phi_k|\delta\phi_l\rangle\langle\phi_l| + |\delta\phi_k\rangle\langle\delta\phi_k| + |\phi_k\rangle\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l| + |\phi_k\rangle\langle\delta\phi_k|\phi_l\rangle\langle\delta\phi_l|\Big) \\
&= \frac{1}{2}\sum_{m=1}^{n+1}\big(\langle\delta\phi_m|\delta\phi_m\rangle - |\delta\phi_m\rangle\langle\delta\phi_m|\big)
\end{aligned}$$

By using equation (3.2.13) and its analogue for $|\delta\phi_m\rangle\langle\delta\phi_m|$, $m = 1, \ldots, n+1$,

$$|\delta\phi_m\rangle\langle\delta\phi_m| = (k+1)\langle\psi_1|\delta\psi_1\rangle + (n-k-1)\langle\psi_2|\delta\psi_2\rangle,$$

we find that

$$\mathrm{ch}_1(p_{(n)}) = \frac{1}{2}n(n+1)\big(\langle\psi_1|\delta\psi_1\rangle + \langle\psi_2|\delta\psi_2\rangle\big) = \frac{1}{2}n(n+1)\mathrm{ch}_1(p_{(1)}).$$

Note that this equation holds in the differential subalgebra $\Omega_D(S^4_\theta)$. Since $\mathrm{ch}_1(p_{(1)})$ was shown to vanish in [33], we proved the vanishing of the first Chern character in $\Omega_D(S^4_\theta)$. The vanishing of $\mathrm{ch}_1(p_{(1)})$ can also be seen from the explicit form of $|\psi_1\rangle$ and $|\psi_2\rangle$.

A slightly more involved computation in $\Omega_D(S^7_{\theta'})$ shows that

$$\begin{aligned}
\mathrm{ch}_2(p_{(n)}) \quad = \quad & \frac{1}{2}\sum\big\{\delta\big(\langle\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle\big) + \langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_m\rangle\langle\phi_m|\delta\phi_k\rangle \\
& +\langle\delta\phi_k|\delta\phi_l\rangle\langle\delta\phi_l|\delta\phi_k\rangle + \delta\big(\langle\delta\phi_k|\delta\phi_l\rangle\langle\phi_l|\delta\phi_k\rangle\big)\big\}.
\end{aligned}$$

And by using Lemma 3.8 we finally get

$$\mathrm{ch}_2(p_{(n)}) = \frac{1}{6}n(n+1)(n+2)\mathrm{ch}_2(p_{(1)}),$$

as an element in $\Omega^4_D(S^4_\theta)$. □

Combining this with the simple form of the index formula in Theorem 2.8 while taking the proper coefficients, we find that

$$\mathrm{Index}\ D_{p_{(n)}} = \frac{1}{4!}\frac{4!}{2!}\frac{1}{6}n(n+1)(n+2)\operatorname*{Res}_{z=0}\mathrm{tr}\left(\gamma\pi_D(\mathrm{ch}_2(p_{(1)}))|D|^{-4-2z}\right)$$

where for the vanishing of the first term, we used the fact that Index $D = 0$, since the first Pontrjagin class on S^4 vanishes. Thm I.2 in [34] allows one to express the residue as a Dixmier trace. Combining this with $\pi_D(\mathrm{ch}_2(p_{(1)})) = 3\gamma$ (as computed in [33]), we obtain

$$3 \cdot \operatorname*{Res}_{z=0}\mathrm{tr}(|D|^{-4-2z}) = 6 \cdot \mathrm{Tr}_\omega(|D|^{-4}) = 2$$

since the Dixmier trace of $|D|^{-m}$ on the m-sphere equals $8/m!$ (cf. for instance [49, 65]). This combines to give:

Proposition 3.9. *The index of the Dirac operator on S^4_θ with coefficients in $E^{(n)}$ is given by:*

$$\mathrm{Index}\ D_{p_{(n)}} = \frac{1}{6}n(n+1)(n+2).$$

□

Note that this coincides with the classical result.

3.3 The structure of the noncommutative principal bundle

In this section, we apply the general theory of Hopf-Galois extensions [62, 75] to the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$. Such extensions can be understood as noncommutative principal bundles. We will first dualize the construction of the previous section, i.e. replace the action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ by a coaction of $\mathcal{A}(SU(2))$. Then, we will recall some definitions involving Hopf-Galois extensions and principality ([14]) of such extensions. We show that $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft (i.e. not-trivial) principal Hopf-Galois extension and compare the connections on the associated bundles, induced from the strong connection, with the Grassmann connection defined in Section 3.1.

The action of $SU(2)$ on $\mathcal{A}(S^7_{\theta'})$ by automorphisms can be easily dualized to a coaction $\Delta_R : \mathcal{A}(S^7_{\theta'}) \to \mathcal{A}(S^7_{\theta'}) \otimes \mathcal{A}(SU(2))$, where now $\mathcal{A}(SU(2))$ is the unital complex $*$-algebra generated by $w^1, \overline{w}^1, w^2, \overline{w}^2$ with relation $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$. Clearly, $\mathcal{A}(SU(2))$ is a Hopf algebra with comultiplication

$$\Delta : \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \mapsto \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \otimes \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix},$$

antipode $S(w^1) = \overline{w}^1, S(w^2) = -w^2$ and counit $\epsilon(w^1) = \epsilon(\overline{w}^1) = 1, \epsilon(w^2) = \epsilon(\overline{w}^2) = 0$. The coaction of $\mathcal{A}(SU(2))$ on $\mathcal{A}(S^7_{\theta'})$ is given by

$$\Delta_R : (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \otimes \begin{pmatrix} w^1 & w^2 & 0 & 0 \\ -\overline{w}^2 & \overline{w}^1 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -\overline{w}^2 & \overline{w}^1 \end{pmatrix}.$$

The algebra of coinvariants in $\mathcal{A}(S^7_{\theta'})$, which consists of elements $p \in \mathcal{A}(S^7_{\theta'})$ satisfying $\Delta_R(p) = p \otimes 1$, can be identified with $\mathcal{A}(S^4_\theta)$ for the particular values of θ'_{ij} found before, in the same way as in Sect. 3.1.

The associated modules $\Gamma(S^4_\theta, E^{(n)})$ are described in the following way. Given an irreducible corepresentation of $\mathcal{A}(SU(2))$, $\rho_{(n)} : V^{(n)} \to \mathcal{A}(SU(2)) \otimes V^{(n)}$ with $V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$, we denote $\rho_{(n)}(v) = v_{(0)} \otimes v_{(1)}$. Then, the module of coequivariant maps $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$ consists of maps $\phi : V^{(n)} \to \mathcal{A}(S^7_{\theta'})$ satisfying

$$\phi(v_{(1)}) \otimes S v_{(0)} = \Delta_R \phi(v); \quad v \in \mathbb{C}^2.$$

Again, such maps are $\mathbb{C}$-linear maps of the form $\phi_{(n)}(e_k) = \langle \phi_k | f \rangle$ on the basis $\{e_1, \ldots, e_{n+1}\}$ of $V^{(n)}$ in the notation of the previous section. Also, Proposition 3.2 above translates straightforwardly into the isomorphism $\mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \simeq p_{(n)}(\mathcal{A}(S^4_\theta))^{4^n}$ for the projections defined in equation (3.2.6).

Before we proceed, recall that for an algebra P and a subalgebra $B \subset P$, $P \otimes_B P$ denotes the quotient of the tensor product $P \otimes P$ by the ideal generated by expressions $p \otimes bp' - pb \otimes p'$, for $p, p' \in P, b \in B$.

Definition 3.10. *Let H be a Hopf algebra and P a right H-comodule algebra, i.e. such that the coaction $\Delta_R : P \to P \otimes H$ is an algebra map. Let B denote the algebra of coinvariants, $B := \mathrm{Coinv}_{\Delta_R}(P) := \{p \in P : \Delta_R(p) = p \otimes 1\}$. One says that $B \hookrightarrow P$ is a* Hopf-Galois extension *if the canonical map*

$$\chi : P \otimes_{\mathcal{B}} P \to P \otimes H; \quad p' \otimes_B p \mapsto p' \Delta_R(p) = p' p_{(0)} \otimes p_{(1)} \tag{3.3.1}$$

is bijective.

We use Sweedler-like notation for the coaction: $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$. The canonical map is left P-linear and right H-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left P-modules and right H-comodules. It is also clear that P is both a left and a right B-module.

Classically, the notion of Hopf-Galois extension corresponds to freeness of the action of a Lie group G on a manifold P. Indeed, freeness can be translated into bijectivity of the map

$$\widetilde{\chi} : P \times G \to P \times_G P, \quad (p, g) \mapsto (p, p \cdot g),$$

where $P \times_G P$ denotes the fibred direct product consisting of elements (p, p') with the same image under the quotient map $P \to P/G$.

For a Hopf algebra H which is cosemisimple, surjectivity of the canonical map (3.3.1) implies its bijectivity [88]. Moreover, in order to prove surjectivity of χ, it is enough to prove that for any generator h of H, the element $1 \otimes h$ is in the image of the canonical map. Indeed, if $\chi(g_k \otimes_{\mathcal{B}} g'_k) = 1 \otimes g$ and $\chi(h_l \otimes_{\mathcal{B}} h'_l) = 1 \otimes h$ for $g, h \in H$, then $\chi(g_k h_l \otimes_{\mathcal{B}} h'_l g'_k) = g_k h_l \chi(1 \otimes_{\mathcal{B}} h'_l g'_k) = 1 \otimes hg$, using the fact that the canonical map restricted to $1 \otimes_B P$ is a homomorphism. Extension to all of $P \otimes_{\mathcal{B}} P$ then follows from left P-linearity of χ. It would also be easy to write down an explicit expression for the inverse of the canonical map. Indeed, one has $\chi^{-1}(1 \otimes hg) = g_k h_l \otimes_B h'_l g'_k$ in the above notation so that the general form of the inverse follows again from left P-linearity.

Proposition 3.11. *The inclusion* $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ *is a Hopf-Galois extension.*

Proof. Since $\mathcal{A}(SU(2))$ is cosemisimple, we can rely for a proof of this statement on the previous remarks. On the other hand, it is straightforward to check that in terms of the ket-valued polynomials defined in (3.2.1) we have

$$\chi(\textstyle\sum_a \langle\psi_1|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_a) = 1 \otimes w^1; \qquad \chi(\displaystyle\sum_a \langle\psi_1|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_a) = 1 \otimes w^2;$$

$$\chi(\textstyle\sum_a \langle\psi_2|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_a) = -1 \otimes \overline{w}^2; \qquad \chi(\displaystyle\sum_a \langle\psi_2|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_a) = 1 \otimes \overline{w}^1.$$

□

In the definition of a principal bundle in differential geometry there is much more than the requirement of bijectivity of the canonical map. It turns out that our 'structure group' being $H = \mathcal{A}(SU(2))$ which, besides being cosemisimple has also bijective antipode, all additional desired properties follows from the surjectivity of the canonical map which we have just established. We refer to [85, 14] for the full fledged theory while giving only the basic definitions that we shall need.

For our purposes, a better algebraic translation of the notion of a principal bundle is encoded in the requirement that the extension $B \subset P$, besides being Hopf-Galois, is also faithfully flat. We recall [63] that a right module P over a ring R is said to be *faithfully flat* if the functor $P \otimes_R \cdot$ is exact and faithful on the category ${}_R\mathcal{M}$ of left R-modules. Flatness means that the functor associates exact sequences of abelian groups to exact sequences of R-modules and the functor is faithful if it is injective on morphisms. Equivalently one could state that a right module P over a ring R is faithfully flat if a sequence $M' \to M \to M''$ in ${}_R\mathcal{M}$ is exact if and only if $P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$ is exact.

As mentioned, from the fact that $H = \mathcal{A}(SU(2))$ is both cosemisimple and has also bijective antipode, the faithful flatness of $\mathcal{A}(S^7_{\theta'})$ as a right (as well as left) $\mathcal{A}(S^4_\theta)$-module follows from the surjectivity of the canonical map ([88], Th. I).

One says that a principal Hopf-Galois extension is *cleft* if there exists a (unital) convolution-invertible colinear map $\phi : H \to P$, called a *cleaving map* [36, 85]. Classically, this notion is close (although not equivalent) to triviality of a principal bundle [37]. In [15] (cf. [52]) it is shown that if a principal Hopf-Galois extension is cleft, its associated modules are trivial, i.e. isomorphic to the free module B^N for some N. In our case, we can conclude the following.

Proposition 3.12. *The Hopf-Galois extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is not cleft.*

Proof. This is a simple consequence of the nontriviality of the Chern characters of the projection $p_{(n)}$ as seen in Sect. 3.2.2. Indeed, this implies that the associated modules are nontrivial. □

Summing up what we have shown up to now, we have the following.

Theorem 3.13. *The inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a not-cleft faithfully flat $\mathcal{A}(SU(2))$-Hopf-Galois extension.*

An important consequence is the existence of a so-called *strong connection* [51, 36]. In fact, the existence of such a connection could be used to give a more intuitive definition of 'principality of an extension' [14]. Let us first recall that if H is cosemisimple and has a bijective antipode, then a H-Hopf-Galois extension $B \hookrightarrow P$ is *equivariantly projective*, that is, there exists a left B-linear right H-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = \mathrm{id}_P$ [85]. Such a map characterizes a strong connection.

Definition 3.14. *Let $B \hookrightarrow P$ be a H-Hopf-Galois extension. A* strong connection one-form *is a map $\omega : H \to \Omega^1_{\mathrm{un}}P$ satisfying*

1. $\bar{\chi} \circ \omega = 1 \otimes (\mathrm{id} - \epsilon)$, *(fundamental vector field condition)*
2. $\Delta_{\Omega^1_{\mathrm{un}}(P)} \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}_R$, *(right adjoint colinearity)*
3. $\delta p - p_{(0)}\omega(p_{(1)}) \in (\Omega^1_{\mathrm{un}}B)P$, $\forall p \in P$, *(strongness condition).*

Here $\Delta_R : P \to P \otimes H$, $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$, is extended to $\Delta_{\Omega^1_{\mathrm{un}}(P)}$ on $\Omega^1_{\mathrm{un}}P \subset P \otimes P$ in a natural way by

$$\Delta_{\Omega^1_{\mathrm{un}}(P)}(p' \otimes p) \mapsto p'_{(0)} \otimes p_{(0)} \otimes p'_{(1)}p_{(1)},$$

and $\mathrm{Ad}_R(h) = h_{(2)} \otimes S(h_{(1)})h_{(3)}$ is the right adjoint coaction of H. Finally, the map $\bar{\chi} : P \otimes P \to P \otimes H$ is defined like the canonical map as $\bar{\chi}(p' \otimes p) = p'p_{(0)} \otimes p_{(1)}$.

As shown in [14] (cf. [12, 53]), a strong connection can always be given by a map $\ell : H \to P \otimes P$ satisfying

$$\begin{aligned} \ell(1) &= 1 \otimes 1, \\ \bar{\chi}(\ell(h)) &= 1 \otimes h, \\ (\ell \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta_R) \circ \ell, \\ (\mathrm{id} \otimes \ell) \circ \Delta &= (\Delta_L \otimes \mathrm{id}) \circ \ell, \end{aligned} \tag{3.3.2}$$

where $\Delta_L : P \to H \otimes P$, $p \mapsto S^{-1}p_{(1)} \otimes p_{(0)}$. Then, one defines the connection one-form by

$$\omega : h \mapsto \ell(h) - \epsilon(h)1 \otimes 1.$$

Indeed, if one writes $\ell(h) = h^{\langle 1\rangle} \otimes h^{\langle 2\rangle}$ (summation understood) and applies $\mathrm{id} \otimes \epsilon$ to the second formula in (3.3.2), one has $h^{\langle 1\rangle} h^{\langle 2\rangle} = \epsilon(h)$. Therefore,

$$\omega(h) = h^{\langle 1\rangle} \delta h^{\langle 2\rangle}$$

where $\delta : P \to \Omega^1_{\mathrm{un}} P,\ p \mapsto 1 \otimes p - p \otimes 1$. Equivariant projectivity of $B \hookrightarrow P$ follows by taking as splitting of the multiplication the map $s : P \to B \otimes P,\ p \to p_{(0)} \ell(p_{(1)})$.

For later use, we prove the following Lemma, analogous to the strongness condition 3. above.

Lemma 3.15. *Let ω be a strong connection one-form on a H-Hopf-Galois extension $B \hookrightarrow P$ with the antipode of H invertible. Then*

$$\delta p + \omega(S^{-1} p_{(1)}) p_{(0)} \in P\Omega^1_{\mathrm{un}} B, \quad \forall p \in P.$$

Proof. By writing ω in terms of ℓ it follows that $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ reduces to the expression $-p \otimes 1 + l(S^{-1} p_{(1)}) p_{(0)}$. From the second property of ℓ in (3.3.2), it follows that this expression is in the kernel of $\overline{\chi}$. Since χ is an isomorphism, $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ is in the ideal generated by expressions of the form $p \otimes bp' - pb \otimes p'$. In other words, it is an element in $P\Omega^1_{\mathrm{un}}(B)P$. Finally, it is not difficult to show that

$$(\mathrm{id} \otimes \Delta_R)\big(\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}\big) = \big(\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}\big) \otimes 1$$

from which we conclude that $\delta p + \omega(S^{-1} p_{(1)}) p_{(0)}$ is in fact in $P\Omega^1_{\mathrm{un}}(B)$. □

In our case, the existence of a strong connection follows from [85]. However, we will write an explicit expression in terms of the inverse of the canonical map. If we denote the latter when lifted to $P \otimes P$ by τ it follows that $\ell(h) = \tau(1 \otimes h)$ satisfies the same recursive relation found before for χ^{-1} (proof of Proposition 3.11 above): if $\ell(h) = h_l \otimes h'_l$ and $\ell(g) = g_k \otimes g'_k$, then

$$\ell(hg) = g_k h_l \otimes h'_l g'_k. \tag{3.3.3}$$

It turns out that in our case the map $\ell : H \to P \otimes P$ defined in this way defines a strong connection.

Proposition 3.16. *On the Hopf-Galois extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$, the following formulæ on the generators of $\mathcal{A}(SU(2))$,*

$$\begin{array}{ll} \ell(w^1) = \sum_a \langle\psi_1|_a \otimes |\psi_1\rangle_a; & \ell(w^2) = \sum_a \langle\psi_1|_a \otimes |\psi_2\rangle_a; \\ \ell(\overline{w}^2) = -\sum_a \langle\psi_2|_a \otimes |\psi_1\rangle_a; & \ell(\overline{w}^1) = \sum_a \langle\psi_2|_a \otimes |\psi_2\rangle_a. \end{array} \tag{3.3.4}$$

define a strong connection.

Proof. We extend the expressions (3.3.4) to all of $\mathcal{A}(SU(2))$ by giving recursive relations, using formula (3.3.3). Recall the usual vector basis $\{r^{klm} : k \in \mathbb{Z}, m, n \geq 0\}$ in $\mathcal{A}(SU(2))$ given by

$$r^{klm} := \begin{cases} (-1)^n (w^1)^k (w^2)^m (\overline{w}^2)^n & k \geq 0, \\ (-1)^n (w^2)^m (\overline{w}^2)^n (\overline{w}^1)^{-k} & k < 0. \end{cases} \tag{3.3.5}$$

The recursive expressions on this basis are explicitly given by

$$\begin{aligned}
\ell(r^{k+1,mn}) &= \psi_1^*\ell(r^{kmn})\psi_1 + \psi_2^*\ell(r^{kmn})\psi_2 + \psi_3^*\ell(r^{kmn})\psi_3 + \psi_4^*\ell(r^{kmn})\psi_4, \quad k \geq 0,\\
\ell(w^{k-1,mn}) &= \psi_2\ell(r^{kmn})\psi_2^* + \psi_1\ell(r^{kmn})\psi_1^* + \psi_4\ell(r^{kmn})\psi_4^* + \psi_3\ell(r^{kmn})\psi_3^*, \quad k < 0,\\
\ell(w^{k,m+1,n}) &= -\psi_1^*\ell(r^{kmn})\psi_2^* + \psi_2^*\ell(r^{kmn})\psi_1^* - \psi_3^*\ell(r^{kmn})\psi_4^* + \psi_4^*\ell(r^{kmn})\psi_3^*,\\
\ell(w^{km,n+1}) &= -\psi_2\ell(r^{kmn})\psi_1 + \psi_1\ell(r^{kmn})\psi_2 - \psi_4\ell(r^{kmn})\psi_3 + \psi_3\ell(r^{kmn})\psi_4, \qquad (3.3.6)
\end{aligned}$$

while setting $\ell(1) = 1 \otimes 1$. In essentially the same manner as was done in [11] (although much simpler in our case) we prove that ℓ defined by the above recursive relations indeed satisfies all conditions of a strong connection. □

The strong connection on the extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ induces connections on the associated modules in the following way [52]. For $\phi \in \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'}))$, we set

$$\nabla_\omega(\phi)(v) \mapsto \delta\phi(v) + \omega(v_{(0)})\phi(v_{(1)}).$$

Using the right adjoint colinearity of ω and a little algebra one shows that $\nabla_\omega(\phi)$ satisfies the following coequivariance condition

$$\nabla_\omega(\phi)(v_{(1)}) \otimes Sv_{(0)} = \Delta_{\Omega^1_{\mathrm{un}}(P)}\big(\nabla_\omega(\phi)(v)\big)$$

so that

$$\nabla_\omega : \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \to \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))).$$

In fact, from Lemma 3.15 it follows that ∇_ω is a map

$$\nabla_\omega : \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \to \mathrm{Hom}^{\rho_{(n)}}(V^{(n)}, \mathcal{A}(S^7_{\theta'})) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1_{\mathrm{un}}(\mathcal{A}(S^4_\theta)).$$

This allows one to compare it to the Grassmann connection of equation (3.2.7). It turns out that the connection one-form ω coincides with the connection one-form A of equation (3.2.8), on the quotient $\Omega^1(S^7_{\theta'})$ of $\Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))$. More precisely, let $\{e^{(n)}_k\}$ be a basis of $V^{(n)}$, and $e^{(n)}_{kl}$ the corresponding matrix coefficients of $\mathcal{A}(SU(2))$ in the representation $\rho_{(n)}$. An explicit expression for $\omega(e^{(n)}_{kl})$ can be obtained from equations (3.3.6); for example $\omega(e^{(1)}_{kl}) = \langle\psi_k|\delta\psi_l\rangle$, $k, l = 1, 2$.

By using these and formulæ (3.2.11)-(3.2.13), one shows that

$$\pi(\omega(e^{(n)}_{kl})) = A^{(n)}_{kl} = \langle\phi_k|d\phi_l\rangle,$$

where $\pi : \Omega_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'})) \to \Omega(S^7_{\theta'})$ is the quotient map.

Chapter 4

Gauge theory on S^4_θ

In this chapter, we put the noncommutative instanton discussed above in the setting of a noncommutative Yang-Mills theory on S^4_θ. The Hopf subalgebra $\mathcal{U}_\theta(so(5))$ is made of twisted infinitesimal symmetries under which the basic instanton is invariant. We construct a collection of (infinitesimal) gauge-nonequivalent instantons, by acting with a twisted conformal symmetry on the basic instanton associated canonically with the noncommutative instanton bundle constructed previously. A completeness argument on this collection is provided using an index theoretical argument, similar to [4]. The dimension of the "tangent" of the moduli space can be computed as the index of a twisted Dirac operator and it turns out to be equal to its classical value which is five.

4.1 Yang-Mills theory on S^4_θ

We introduce the Yang-Mills action functional on S^4_θ together with its equations of motion. We will see that instantons naturally arise as the local minima of this action.

Let $\mathcal{E} = \Gamma(S^4_\theta, E)$ for some σ-equivariant vector bundle E on S^4, so that there exist a projection $p \in M_N(C^\infty(S^4_\theta))$ such that $\mathcal{E} \simeq p(C^\infty(S^4_\theta)^N$. Recall from Appendix A.4 that a connection ∇ on $\mathcal{E} = \Gamma(S^4_\theta, E)$ for some vector bundle E on S^4, is a map from $\mathcal{E}$ to $\mathcal{E} \otimes \Omega(S^4_\theta)$. The Yang-Mills action functional is defined in terms of the curvature of a connection on $\mathcal{E}$, which is an element in $\mathrm{End}_{C^\infty(S^4_\theta)}(\mathcal{E}, \mathcal{E} \otimes \Omega^2(S^4_\theta))$. Equivalently, it is an element in $\mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$ of degree 2. We define an inner product on the latter algebra as follows. An element $T \in \mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$ of degree k can be understood as an element in $pM_N(\Omega^k(S^4_\theta))p$, since $\mathcal{E} \otimes \Omega(S^4_\theta)$ is a finite projective module over $\Omega(S^4_\theta)$. A trace over internal indices together with the inner product defined in (2.2.6), defines the inner product $(\cdot,\cdot)_2$ on $\mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$. In particular, we can make the following definition.

Definition 4.1. *The* Yang-Mills action *for a connection* ∇ *on* $\mathcal{E}$ *with curvature* F *is defined by*

$$\mathrm{YM}(\nabla) = \big(F, F\big)_2 = \int\!\!\!\!\!- \mathrm{tr} *_\theta (F *_\theta F).$$

In [27] (cf. [65]) a Yang-Mills action was introduced on Connes' differential forms (see above) taking values in the endomorphisms of $\mathcal{E}$, using instead the inner product (2.2.7). However, in Lemma 2.7 we showed that both inner products coincide which allows us to take the following result from [27, VI.1]. Recall from Appendix A.4 that a gauge transformation is given by a unitary endomorphism of $\mathcal{E}$.

Lemma 4.2. *The Yang-Mills action is gauge invariant, positive and quartic.*

In physics, the Yang-Mills equations are obtained from the Yang-Mills action by a variational principle. Let us describe how this principle works in our case. We consider a linear perturbation $\nabla_t = \nabla + t\alpha$ of a connection ∇ on $\mathcal{E}$ by an element $\alpha \in \mathrm{End}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega^1(S^4_\theta))$. The curvature F_t of ∇_t is readily computed as $F_t = F + t[\nabla, \alpha] + \mathcal{O}(t^2)$. If we suppose that ∇ is an extremum of the Yang-Mills action, this linear perturbation should not affect the action. In other words, we should have

$$\frac{\partial}{\partial t}\bigg|_{t=0} \mathrm{YM}(\nabla_t) = 0.$$

If we substitute the explicit formula for F_t, we obtain

$$\big([\nabla, \alpha], F\big)_2 + \overline{\big([\nabla, \alpha], F\big)_2} = 0,$$

using the fact that $(\cdot,\cdot)_2$ defines a complex scalar product on $\mathrm{End}(\mathcal{E}, \mathcal{E} \otimes \Omega(S^4_\theta))$. Positive definiteness of this scalar product implies that $(F_t, F_t) = \overline{(F_t, F_t)}$, which when differentiated with respect to t, setting $t = 0$ afterwards, yields $\big([\nabla, \alpha], F\big)_2 = \overline{\big([\nabla, \alpha], F\big)_2}$; hence, $([\nabla, \alpha], F)_2 = 0$. Using the fact that α was arbitrary, we derive the following equations of motion

$$[\nabla^*, F] = 0,$$

where the adjoint of $[\nabla, \cdot]$ is defined with respect to the inner product $(\ , \ )_2$, i.e.

$$\big([\nabla^*, \alpha], \beta\big)_2 = \big(\alpha, [\nabla, \beta]\big)_2$$

for $\alpha \in \mathrm{End}(\mathcal{E}, \mathcal{E} \otimes \Omega^3(S^4_\theta))$ and $\beta \in \mathrm{End}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$. From Lemma 2.6, it follows that $[\nabla^*, F] = *_\theta[\nabla, *_\theta F]$, so that the equations of motion can also be written as the more familiar *Yang-Mills equations*:

$$[\nabla, *_\theta F] = 0. \tag{4.1.1}$$

Note that connections with a selfdual or anti-selfdual curvature $*_\theta F = \pm F$ are special solutions of the Yang-Mills equation. Indeed, in this case the latter equation follows directly from the Bianchi identity, Proposition A.12, stating that $[\nabla, F] = 0$. We call such connections instantons on S^4_θ.

We will now establish a connection between the Yang-Mills action functional and the so-called topological action [27, VI.3] on S^4_θ. Suppose $\mathcal{E}$ is a finite projective module over $C^\infty(S^4_\theta)$ defined by a projection $p \in M_N(C^\infty(S^4_\theta))$. The topological action for $\mathcal{E}$ is given by a pairing between the class of p in K-theory and the cyclic cohomology of $C^\infty(S^4_\theta)$. For computational purposes, we give the following definition in terms of the curvature of a connection on $\mathcal{E}$

Definition 4.3. *Let ∇ be a connection on $\mathcal{E}$ with curvature F. The* topological action *is given by*

$$\mathrm{Top}(\mathcal{E}) = (F, *_\theta F)_2 = ⨍ *_\theta \operatorname{tr}(F^2)$$

where the trace is taken over internal indices.

Let us show that this does not depend on the choice of a connection on $\mathcal{E}$. Since two connections differ by an element α in $\mathrm{End}_{C^\infty(S^4_\theta)}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$, we have to establish that $(F', *_\theta F')_2 = (F, *_\theta F)_2$ where F' is the curvature of $\nabla' := \nabla + \alpha$. By definition of the inner product $(\cdot,\cdot)_2$ we then have

$$\begin{aligned}(F', *_\theta F')_2 &= (F, *_\theta[\nabla, \alpha])_2 + ([\nabla, \alpha], *_\theta F)_2 \\ &= (F, [\nabla^*, *_\theta \alpha])_2 + ([\nabla^*, *_\theta \alpha], F)_2\end{aligned}$$

which vanishes due to the Bianchi identity $[\nabla, F] = 0$, Proposition A.12.

The Hodge star operator $*_\theta$ splits $\Omega^2(S^4_\theta)$ into a selfdual and anti-selfdual space,

$$\Omega^2(S^4_\theta) = \Omega^2_+(S^4_\theta) \oplus \Omega^2_-(S^4_\theta).$$

This decomposition is orthogonal with respect to the inner product $(\cdot,\cdot)_2$ which follows from the property $(\alpha, \beta)_2 = \overline{(\beta, \alpha)_2}$, so that we can write the Yang-Mills action functional as

$$\mathrm{YM}(\nabla) = \big(F_+, F_+\big)_2 + \big(F_-, F_-\big)_2.$$

Comparing this with the topological action,

$$\mathrm{Top}(\mathcal{E}) = \big(F_+, F_+\big)_2 - \big(F_-, F_-\big)_2,$$

we see that $\mathrm{YM}(\nabla) \geq \mathrm{Top}(\mathcal{E})$, with equality holding if and only if $*_\theta F = \pm F$. We conclude that the instantons correspond to the absolute minima of the Yang-Mills action functional.

4.2 Construction of SU(2)-instantons on S^4_θ

In this section, we construct a set of charge 1 SU(2) instantons on S^4_θ, by acting with a twisted infinitesimal conformal symmetry on the basic instanton on S^4_θ constructed in [33]. We will find a five-parameter family of infinitesimal instantons. Then we show that the 'tangent space' of the moduli space of irreducible instantons at the basic instanton is five-dimensional, proving that this set is complete. Here, one has to be careful with the notion of tangent space to the moduli space. As will be discussed elsewhere [67], it turns out that the moduli space is a noncommutative space given as the quantum quotient space of the deformed conformal group $SL_\theta(2, \mathbb{H})$ by the deformed gauge group $\mathrm{Sp}_\theta(2)$. It turns out that the basic instanton of [33] is the only classical point in this moduli space of instantons. We perturb this connection ∇_0 linearly by sending $\nabla_0 \mapsto \nabla_0 + t\alpha$ where $t \in \mathbb{R}$ and $\alpha \in \mathrm{End}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$. In order for this new connection still to be an instanton, we have to impose the selfdual equation on its curvature. After deriving this equation with respect to t, setting $t = 0$ afterwards, we obtain the linearized selfdual equation to be fulfilled by α. It is in this sense that we are considering the tangent space to the moduli space of instantons at the origin ∇_0.

Let us start with a technical lemma that simplifies the discussion.

Lemma 4.4. *There is the following isomorphism of right* $C^\infty(S^4_\theta)$*-modules:*

$$\mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta) \simeq \Omega(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \mathcal{E}.$$

Consequently, $\mathrm{End}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta)) \simeq \Omega(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \mathrm{End}(\mathcal{E})$.

Proof. This follows from Corollary 2.4 and Remark 3.3, together with the observation that the $C^\infty(S^4)$-module $\Omega(S^4)$ has a $\mathbb{T}^2$-homogeneous basis. □

We let $\nabla_0 = p \circ d$ be the canonical connection on the projective module $\mathcal{E} = \Gamma(S^7_{\theta'} \otimes_{SU(2)} \mathbb{C}^2) = p(C^\infty(S^4_\theta))^4$, with the projection p of (3.2.3). It can be written on $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2 \simeq \mathcal{E}$ as (cf. (3.2.8))

$$\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega^1(S^4_\theta)$$
$$(\nabla_0 f)_i = df_i + \omega_{ij} f_j,$$

with $\omega = \Psi^\dagger d\Psi$ a 2×2-matrix with entries in $\Omega^1(S^7_{\theta'})$, satisfying $\overline{\omega_{ij}} = \omega_{ji}$ and $\sum_i \omega_{ii} = 0$. We will refer to ω as the gauge potential.

Remark 4.5. *Note here that the entries ω_{ij} commute with all elements in $C^\infty(S^7_{\theta'})$. Indeed, from (3.2.1) we see that the elements in ω_{ij} are $\mathbb{T}^2$-invariant and hence central (as one forms) in $\Omega(S^7_{\theta'})$. Since also d commutes with the action of $\mathbb{T}^2$, we conclude that the connection ∇ commutes with the action of $\mathbb{T}^2$.*

Remark 4.6. *In Section 3.2, we constructed projections $p_{(n)}$ for all modules $\Gamma(S^7_{\theta'} \times_{SU(2)} \mathbb{C}^n)$ over $C^\infty(S^4_\theta)$ associated to the irreducible representations $\mathbb{C}^n$ of $SU(2)$. The induced Grassmann connections $\nabla_0^{(n)} := p_{(n)} d$ were written as $d + \omega_{(n)}$ when acting on $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^n$, with $\omega_{(n)}$ an $n \times n$ matrix with values in $\Omega^1(S^7_{\theta'})$. A similar argument as above then shows that all $\omega_{(n)}$ have entries that are central (as one forms) in $\Omega(S^7_{\theta'})$. In particular, this holds for the adjoint bundle associated to the adjoint representation on $su(2) \simeq \mathbb{C}^3$ (as representation spaces), from which we conclude that $\nabla_0^{(2)}$ coincides with $[\nabla_0, \cdot]$ (since this is the case if $\theta = 0$).*

The curvature $F_0 = \nabla_0^2 = d\omega + \omega^2$ of ∇_0 is an element of $\mathrm{End}(\mathcal{E}) \otimes_{C^\infty(S^4_\theta)} \Omega^2(S^4_\theta)$ that satisfies the selfdual equation $*_\theta F_0 = F_0$ [1, 32], hence this connection is an instanton.

We aim at constructing all connections ∇ on $\mathcal{E}$ whose curvature satisfies this selfdual equation. We can write any such connection in terms of the canonical connection as in equation (A.4.4), i.e. $\nabla = \nabla_0 + \alpha$ with α a one-form valued endomorphism of $\mathcal{E}$. We are particularly interested in $SU(2)$-instantons, so we impose that α is traceless and skew-hermitian. Here the trace is taken in the second leg of $\mathrm{End}(\mathcal{E}) \simeq P \boxtimes_{\mathrm{ad}} M_2(\mathbb{C})$. When complexified, this gives an element $\alpha \in \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_{\theta'})) =: \Omega^1(\mathrm{ad}(S^4_\theta))$ (cf. Example 3.6).

Moreover, we impose the following irreducibility condition on the instanton connections. As usual, a connection on $\mathcal{E}$ is called *irreducible* if it can not be written as the sum of two other connections on $\mathcal{E}$. We are interested only in the irreducible instanton connections on the module $\mathcal{E}$.

4.2.1 Infinitesimal conformal transformations

The noncommutative sphere S^4_θ can be realized as a quantum homogeneous space of the quantum orthogonal group: $SO_\theta(5)$ [94, 32]. In other words, $\mathcal{A}(S^4_\theta)$ can be obtained as the subalgebra of $\mathcal{A}(SO_\theta(5))$ consisting of the elements that are coinvariant under the natural coaction of $SO_\theta(4)$ on $SO_\theta(5)$. For our purposes, it turns out to be more convenient to take a dual point of view and consider an *action* instead of a coaction [89]. We obtain a twisted symmetry action of $U_\theta(so(5))$ on S^4_θ and its lift to $S^7_{\theta'}$ and will see that the above instanton ∇_0 is invariant under this infinitesimal quantum symmetry.

Different instantons are obtained by a twisted symmetry action of $U_\theta(so(5,1))$. Classically, $so(5,1)$ is the conformal Lie algebra consisting of the infinitesimal diffeomorphisms leaving the conformal structure invariant. We construct the Hopf algebra $U_\theta(so(5,1))$ by adding 5 generators to $U_\theta(so(5))$ and describe its action on S^4_θ together with its lift to $S^7_{\theta'}$. The induced action of $\mathcal{U}_\theta(so(5,1))$ on $\Omega(M_\theta)$ leaves the conformal structure invariant. We are then ready to act with $U_\theta(so(5,1))$ on ∇_0 which eventually results in a five-parameter family of instantons.

Let us start with the construction of the twisted symmetry $U_\theta(so(5))$. The eight roots of the Lie algebra $so(5)$ are two-component vectors $r = (r_1, r_2)$ of the form $r = (\pm 1, \pm 1), r = (0, \pm 1), r = (\pm 1, 0)$. There are corresponding generators E_r of $so(5)$ together with two mutually commuting generators H_1, H_2 of the Cartan subalgebra. The Lie brackets are

$$[H_1, H_2] = 0, \quad [H_j, E_r] = r_j E_r, \quad [E_{-r}, E_r] = r_1 H_1 + r_2 H_2, \quad [E_r, E_{r'}] = N_{r,r'} E_{r+r'}, \tag{4.2.1}$$

with $N_{r,r'} = 0$ if $r + r'$ is not a root. The universal enveloping algebra $U(so(5))$ is the algebra generated by elements $\{H_j, E_r\}$ modulo relations given by the previous Lie brackets[1]. The twisted universal enveloping algebra $U_\theta(so(5))$ is generated as above (i.e. one does not change the algebra structure) but is endowed with a twisted coproduct, $\Delta_\theta : U_\theta(so(5)) \to U_\theta(so(5)) \otimes U_\theta(so(5))$ which, on the generators E_r, H_j, reads

$$\begin{aligned} \Delta_\theta(E_r) &= E_r \otimes \lambda^{-r_1 H_2} + \lambda^{-r_2 H_1} \otimes E_r, \\ \Delta_\theta(H_j) &= H_j \otimes \mathbb{I} + \mathbb{I} \otimes H_j. \end{aligned} \tag{4.2.2}$$

This coproduct allows to represent $U_\theta(so(5))$ as an algebra of twisted derivations on both S^4_θ and $S^7_{\theta'}$ as we shall see below. With counit and antipode given by

$$\begin{aligned} &\varepsilon(E_r) = \varepsilon(H_j) = 0, \\ &S(E_r) = -\lambda^{r_2 H_1} E_r \lambda^{r_1 H_2}, \quad S(H_j) = -H_j, \end{aligned} \tag{4.2.3}$$

the algebra $U_\theta(so(5))$ becomes an Hopf algebra [23]; at the classical value of the deformation parameter, $\theta = 0$, one recovers the Hopf algebra structure of $U(so(5))$.

We are ready for the representation of $U_\theta(so(5))$ on S^4_θ. For convenience, we introduce "partial derivatives", ∂_μ and ∂^*_μ with the usual action on the generators of the algebra $\mathcal{A}(S^4_\theta)$: $\partial_\mu(z_\nu) = \delta_{\mu\nu}$, $\partial_\mu(z^*_\nu) = 0$, and $\partial^*_\mu(z^*_\nu) = \delta_{\mu\nu}$, $\partial^*_\mu(z_\nu) = 0$. Then, the action of $U_\theta(so(5))$ on $\mathcal{A}(S^4_\theta)$ is given by the following operators,

$$\begin{aligned} H_1 &= z_1\partial_1 - z^*_1\partial^*_1, & H_2 &= z_2\partial_2 - z^*_2\partial^*_2 \\ E_{+1,+1} &= z_2\partial^*_1 - z_1\partial^*_2, & E_{+1,-1} &= z^*_2\partial^*_1 - z_1\partial_2\,, \\ E_{+1,0} &= \tfrac{1}{\sqrt{2}}(2z_0\partial^*_1 - z_1\partial_0), & E_{0,+1} &= \tfrac{1}{\sqrt{2}}(2z_0\partial^*_2 - z_2\partial_0)\,, \end{aligned} \tag{4.2.4}$$

and $E_{-r} = (E_r)^*$, with the obvious meaning of the adjoint.

Remark 4.7. *Note that the operators H_1 and H_2 are the infinitesimal generators of the action of $\mathbb{T}^2$ on S^4_θ as defined above equation* (3.1.13).

[1] There are additional Serre relations; they generate an ideal that needs to be quotiented out. This is not problematic and we shall not dwell upon this point here.

These operators (not the partial derivatives!) are extended to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations via the coproduct (4.2.2),

$$\begin{aligned} E_r(ab) &= m\Delta_\theta(E_r)(a \otimes b) = E_r(a)\lambda^{-r_1 H_2}(b) + \lambda^{-r_2 H_1}(a)E_r(b), \\ H_j(ab) &= m\Delta_\theta(H_j)(a \otimes b) = H_j(a)b + aH_j(b), \end{aligned} \tag{4.2.5}$$

for any two elements $a, b \in \mathcal{A}(S^4_\theta)$. With these twisted rules, one readily checks compatibility with the commutation relations (3.1.4) of $\mathcal{A}(S^4_\theta)$. We can also write this twisted derivation using the quantization map L_θ defined above as follows. For $a \in \mathcal{A}(S^4)$ a polynomial on S^4 and $T \in \mathcal{U}_\theta(so(5))$ we define a twisted action by

$$T \cdot L_\theta(a) = L_\theta(t \cdot a)$$

where t is the classical limit ($\theta = 0$) of T and $t \cdot a$ is the classical action of $\mathcal{U}(so(5))$ on $\mathcal{A}(S^4)$. One checks that both of these definitions of the twisted action coincide (cf. [89]). With these twisted rules, one readily checks compatibility with the commutation relations (3.1.4) of $\mathcal{A}(S^4_\theta)$.

The representation of $U_\theta(so(5))$ on S^4_θ given in (4.2.4) is the fundamental vector representation. When lifted to S^7_θ one gets the fundamental spinor representation: as we see from the quadratic relations among corresponding generators, as given in (3.1.9), the lifting amounts to take the "square root" representation. The action on $U_\theta(so(5))$ on $\mathcal{A}(S^7_\theta)$ is constructed by requiring twisted derivation properties via the coproduct (4.2.5) so as to reduce to the action (4.2.4) on $\mathcal{A}(S^4_\theta)$ when using the defining quadratic relations (3.1.9). This action can be given as the action of matrices Γ's on the ψ's,

$$\psi_a \mapsto \sum_b \Gamma_{ab}\psi_b; \qquad \psi^*_a \mapsto \sum_b \widetilde{\Gamma}_{ab}\psi^*_b \tag{4.2.6}$$

with the matrices $\Gamma = \{H_j, E_r\}$ given explicitly by,

$$\begin{aligned} H_1 &= \tfrac{1}{2}\begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& 1 \end{pmatrix}, & H_2 &= \tfrac{1}{2}\begin{pmatrix} -1 &&& \\ & 1 && \\ && -1 & \\ &&& 1 \end{pmatrix}, \\ E_{+1,+1} &= \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}, & E_{+1,-1} &= \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ -\mu & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \\ E_{+1,0} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} \\ \begin{smallmatrix} \mu & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, & E_{0,+1} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & \begin{smallmatrix} 0 & \overline{\mu} \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \end{aligned} \tag{4.2.7}$$

and $\widetilde{\Gamma} := \sigma\Gamma\sigma^{-1}$ with

$$\sigma := \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}.$$

Furthermore, we have $E_{-r} = (E_r)^*$. Notice also that for $\theta = 0$, $\widetilde{\Gamma} = -\Gamma^t$. There is the following beautiful correspondence between these matrices and the twisted Dirac matrices introduced in (3.1.11):

$$\begin{aligned} \tfrac{1}{4}[\gamma_1^*, \gamma_1] &= 2H_1 & \tfrac{1}{4}[\gamma_2^*, \gamma_2] &= 2H_2 \\ \tfrac{1}{4}[\gamma_1, \gamma_2] &= (\mu + \overline{\mu})E_{+1,+1} & \tfrac{1}{4}[\gamma_1, \gamma_2^*] &= (\mu + \overline{\mu})E_{+1,-1} \\ \tfrac{1}{4}[\gamma_1, \gamma_0] &= \sqrt{2}E_{+1,0} & \tfrac{1}{4}[\gamma_2, \gamma_0] &= \sqrt{2}\overline{\mu}E_{0,+1} \end{aligned}$$

Remark 4.8. *Compare the form of the matrices* H_1 *and* H_2 *with the lifted action* $\widetilde{\sigma}$ *of* $\widetilde{\mathbb{T}}^2$ *on* S^7_θ *as defined in* (3.1.13). *One checks that* $\widetilde{\sigma}_s$ *is given by* $e^{\pi i((s_1+s_2)H_1+(-s_1+s_2)H_2)}$ *acting on the spinor* (ψ_a).

Remark 4.9. *The matrices* γ_i *(cf.* (3.1.11)*) satisfy the following relations under conjugation by* σ*:*

$$(\sigma\gamma_0\sigma^{-1})^t = \gamma_0; \qquad (\sigma\gamma_1\sigma^{-1})^t = \gamma_1\lambda^{H_2}; \qquad (\sigma\gamma_2\sigma^{-1})^t = \gamma_2\lambda^{H_1}. \tag{4.2.8}$$

With the twisted rules (4.2.5) for the action on products, one checks compatibility of the above action with the commutation relations (3.1.7) of $\mathcal{A}(S^7_\theta)$. In fact, because of the form of $\widetilde{\Gamma}$ and the property $\Psi_{a2} = \sigma_{ab}\psi^*_b$ for the second column of Ψ, we have that $U_\theta(so(5))$ acts on Ψ by left matrix multiplication by Γ, and by right matrix multiplication on Ψ^* by the matrix transpose $\widetilde{\Gamma}^t$ as follows

$$\Psi_{ai} \mapsto \sum_b \Gamma_{ab}\Psi_{bi}, \qquad \Psi^*_{ia} \mapsto \sum_a \Psi^*_{ib}\widetilde{\Gamma}_{ab}.$$

Proposition 4.10. *The instanton gauge potential* ω *is invariant under the twisted action of* $U_\theta(so(5))$.

Proof. From the above observations, the gauge potential transforms as:

$$\omega = \Psi^* d\Psi \mapsto \Psi^*\big(\widetilde{\Gamma}^t\lambda^{-r_1 H_2} + \lambda^{r_2 H_1}\Gamma\big)d\Psi.$$

where $\lambda^{-r_i H_j}$ is understood in its representation (4.2.7) on $S^7_{\theta'}$. Direct computation for $\Gamma = \{H_j, E_r\}$ shows that $\widetilde{\Gamma}^t\lambda^{-r_1 H_2} + \lambda^{r_2 H_1}\Gamma = 0$, which finishes the proof. □

The conformal Lie algebra $so(5,1)$ consists of the generators of $so(5)$ together with dilation and the so-called special conformal transformations. On $\mathbb{R}^4$ they are given on a local chart $\{x_\mu\}_{\mu=1,\ldots,4}$ by the operators $\sum_\mu x_\mu \partial/\partial x_\mu$ and $2x_\mu \sum_\nu x_\nu \partial/\partial x_\nu - \sum_\nu x_\nu^2(\partial/\partial x_\nu)$, respectively [70].

In the definition of $\mathcal{U}_\theta(so(5,1))$ we do not change the algebra structure (i.e. one takes the relations of $\mathcal{U}(so(5,1))$), as we did in the case of $\mathcal{U}_\theta(so(5))$. We thus define $U_\theta(so(5,1))$ as the algebra $U_\theta(so(5))$ with five generators H_0, F_r, $r = (\pm1,0),(0,\pm1)$ adjoined, subject to the relations of $\mathcal{U}_\theta(so(5))$ of equation (4.2.1) together with the (classical) relations:

$$[H_0, H_i] = 0, \quad [H_j, F_r] = r_j F_r, \quad [H_0, F_r] = \sqrt{2}E_r, \quad [H_0, E_r] = \tfrac{1}{\sqrt{2}}F_r,$$

whenever $r = (\pm1,0),(0,\pm1)$, and

$$\begin{aligned} &[F_{-r}, F_r] = 2r_1H_1 + 2r_2H_2, && [E_r, F_{r'}] = \widetilde{N}_{r,r'}F_{r+r'},\\ &[F_r, F_{r'}] = N_{r,r'}E_{r+r'} && [E_{-r}, F_r] = \sqrt{2}H_0, \end{aligned}$$

with $N_{r,r'}$ as before and $\widetilde{N}_{r,r'} = 0$ if $r + r'$ is a root of $so(5,1)$ but not of $so(5)$. Although the algebra structure is unchanged, the Hopf algebra structure of $U_\theta(so(5,1))$ gets twisted. This structure is given by equations (4.2.2) and (4.2.3) together with

$$\begin{aligned} &\Delta_\theta(F_r) = F_r \otimes \lambda^{-r_1 H_2} + \lambda^{-r_2 H_1} \otimes F_r, && S(F_r) = -\lambda^{r_2 H_1}F_r\lambda^{r_1 H_2},\\ &\Delta_\theta(H_0) = H_0 \otimes 1 + 1 \otimes H_0, && S(H_0) = -H_0, \end{aligned}$$

and $\varepsilon(F_r) = \varepsilon(H_0) = 0$. The representation of $U_\theta(so(5,1))$ on S^4_θ is given by (4.2.4) together with

$$\begin{aligned} H_0 &= \partial_0 - z_0(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*), \\ F_{1,0} &= 2\partial_1^* - z_1(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + \bar{\lambda} z_2\partial_2 + \lambda z_2^*\partial_2^*), \\ F_{0,1} &= 2\partial_2^* - z_2(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*), \end{aligned}$$

and $F_{-r} = (F_r)^*$. The introduction of the λ's in $F_{1,0}$ is necessary for the algebra structure of $\mathcal{U}_\theta(so(5,1))$ (as dictated by the above Lie brackets) to be unchanged with respect to $\mathcal{U}(so(5,1))$. Since H_0 and F_r are quadratic in the z's, one has to be careful when deriving the above Lie brackets and use the twisted rules (4.2.5). For instance, on the generator z_2, we have

$$\begin{aligned} [E_{-1,-1}, F_{1,0}](z_2) &= E_{-1,-1}(-\bar{\lambda} z_1 z_2) + F_{1,0}(z_1^*) \\ &= -\bar{\lambda}(E_{-1,-1}(z_1)\lambda^{H_2}(z_2) + \lambda^{H_1}(z_1)E_{-1,-1}(z_2)) + F_{1,0}(z_1^*) \\ &= -z_2^* z_2 + z_1 z_1^* + 2 - z_1 z_1^* = F_{0,-1}(z_2) \end{aligned}$$

Again, the operators H_0, F_r are extended to the whole of $\mathcal{A}(S^4_\theta)$ by the analogue of (4.2.5) in the case of F_r and H_0, i.e.

$$\begin{aligned} F_r(ab) &= F_r(a)\lambda^{-r_1 H_2}(b) + \lambda^{-r_2 H_1}(a)F_r(b), \\ H_0(ab) &= H_0(a)b + aH_0(b). \end{aligned}$$

for any two elements $a, b \in \mathcal{A}(S^4_\theta)$. Equivalently, $\mathcal{U}_\theta(so(5,1))$ is defined to act by

$$T \cdot L_\theta(a) = L_\theta(t \cdot a)$$

for $T \in \mathcal{U}_\theta(so(5,1))$ deforming $t \in \mathcal{U}(so(5,1))$ and $a \in \mathcal{A}(S^4)$.

Lemma 4.11. *1. The twisted action of the Hopf algebra $U_\theta(so(5,1))$ on $\mathcal{A}(S^4_\theta)$ can be extended to the differential calculus $(\Omega(S^4_\theta), \mathrm{d})$ by defining it to commute with the exterior derivative:*

$$T \cdot \mathrm{d}\omega = \mathrm{d}(T \cdot \omega) \ .$$

for $T \in U_\theta(so(5,1))$, $\omega \in \Omega(S^4_\theta)$.

2. Under this twisted action, the Hopf algebra $U_\theta(so(5,1))$ leaves the Hodge $*_\theta$-structure of $\Omega(S^4_\theta)$ invariant:

$$T \cdot (*_\theta \omega) = *_\theta(T \cdot \omega) \ ,$$

Proof. 1. is Lemma 3 in [89] and *2.* follows from the fact that $T(L_\theta(f)) = L_\theta(t \cdot f)$ for $f \in \mathcal{A}(S^4)$ and $t \in U(so(5,1))$ the classical limit $(\theta = 0)$ of $T \in U_\theta(so(5,1))$. Then from the fact that d commutes with the action of $U_\theta(so(5,1))$, we find that the latter algebra leaves the Hodge $*_\theta$-structure of $\Omega(S^4_\theta)$ invariant since $U(so(5,1))$ leaves the Hodge $*$-structure of $\Omega(S^4)$ invariant. □

Thus, the Hopf algebra $\mathcal{U}_\theta(so(5,1))$ consists of the infinitesimal twisted conformal transformations on S^4_θ.

In the same manner as before, the action of $U_\theta(so(5,1))$ on S^4_θ can be lifted to $S^7_{\theta'}$. It can be written as in (4.2.6) in terms of matrices Γ's on the ψ's:

$$\psi_a \mapsto \sum_b \Gamma_{ab}\psi_b; \qquad \psi_a^* \mapsto \sum_b \widetilde{\Gamma}_{ab}\psi_b^*$$

where in addition to (4.2.7) we have the following matrices $\Gamma = \{H_0, F_r\}$

$$H_0 = \tfrac{1}{2}(-z_0\mathbb{I}_4 + \gamma_0),$$
$$F_{1,0} = \tfrac{1}{2}(-z_1\lambda^{-H_2} + \gamma_1),$$
$$F_{0,1} = \tfrac{1}{2}(-z_2 + \lambda^{-H_1}\gamma_2),$$

$F_{-r} = (F_r)^*$ and $\widetilde{\Gamma} = \sigma\Gamma\sigma^{-1}$. Notice the reappearance of the twisted Dirac matrices γ_μ and γ_μ^* of (3.1.11) in the above expressions.

As before, $U_\theta(so(5,1))$ acts on Ψ by left matrix multiplication by Γ and on Ψ^* by $\widetilde{\Gamma}$, i.e.

$$\Psi_{ai} \mapsto \sum_b \Gamma_{ab}\Psi_{bi}, \qquad \Psi^*_{ia} \mapsto \sum_a \widetilde{\Gamma}_{ab}\Psi^*_{ib}.$$

Here we have to be careful with the ordering between $\widetilde{\Gamma}$ and Ψ^* in the second term since the $\widetilde{\Gamma}$'s involve the (not-central) z's. There are the following useful commutation relations between the z's and Ψ:

$$\begin{aligned} z_1\Psi_{ai} &= (\lambda^{-H_2})_{ab}\Psi_{bi}z_1, & z_2\Psi_{ai} &= (\lambda^{-H_1})_{ab}\Psi_{bi}z_2, \\ z_1\Psi^*_{ia} &= \Psi^*_{ib}(\lambda^{-H_2})_{ba}z_1, & z_2\Psi^*_{ia} &= \Psi^*_{ib}(\lambda^{-H_1})_{ba}z_2. \end{aligned} \tag{4.2.9}$$

with λ^{-H_j} understood as 4×4 matrices.

Proposition 4.12. *The instanton gauge potential* $\omega = \Psi^* d\Psi$ *transforms under* $U_\theta(so(5,1))$ *as* $\omega \mapsto \omega + \delta\omega_i$*, where*

$$\begin{aligned} \delta\omega_0 &:= H_0(\omega) = -z_0\omega - \tfrac{1}{2}dz_0\mathbb{I}_2 + \Psi^*\ \gamma_0\ d\Psi, \\ \delta\omega_1 &:= F_{+1,0}(\omega) = -z_1\omega - \tfrac{1}{2}dz_1\ \mathbb{I}_2 + \Psi^*\ \gamma_1\ d\Psi, \\ \delta\omega_2 &:= F_{0,+1}(\omega) = -z_2\omega - \tfrac{1}{2}dz_2\ \mathbb{I}_2 + \Psi^*\ \gamma_2\ d\Psi, \\ \delta\omega_3 &:= F_{-1,0}(\omega) = -\omega\bar{z}_1 - \tfrac{1}{2}d\bar{z}_1\ \mathbb{I}_2 + \Psi^*\ \gamma_1^*\ d\Psi, \\ \delta\omega_4 &:= F_{0,-1}(\omega) = -\omega\bar{z}_2 - \tfrac{1}{2}d\bar{z}_2\ \mathbb{I}_2 + \Psi^*\ \gamma_2^*\ d\Psi, \end{aligned}$$

with γ_μ, γ_μ^* *the twisted* 4×4 *Dirac matrices defined in* (3.1.11).

Proof. The action of H_0 on the instanton gauge potential $\omega = \Psi^* d\Psi$ takes the following form

$$H_0(\omega) = H_0(\Psi^*)d\Psi + \Psi^* d(H_0(\Psi)) = \Psi^*(-z_0\mathbb{I}_4 + \gamma_0)d\Psi - \tfrac{1}{2}dz_0\Psi^*\Psi,$$

since z_0 is central. Direct computation results in the above expression for $\delta\omega_0$. Instead, the twisted action of F_r on ω takes the form,

$$F_r : \omega_{ij} \mapsto \sum_{a,b,c} \widetilde{\Gamma}_{ab}\Psi^*_{ib}(\lambda^{-r_1 H_2})_{ac}d\Psi_{cj} + (\lambda^{r_2 H_1})_{ab}\Psi^*_{ib}\Gamma_{ac}d\Psi_{cj} + (\lambda^{r_2 H_1})_{ab}\Psi^*_{ib}(d\Gamma_{ac})\Psi_{cj}\ ,$$

where we used the fact that $\widetilde{H}_j = \sigma H_j\sigma^{-1} = -H_j$. Let us consider the case $r = (+1, 0)$. Firstly, note that the complex numbers $(\lambda^{-H_2})_{ac}$ commute with Ψ^*_{ib} so that from the definition of Γ and $\widetilde{\Gamma}$, and using (4.2.9), we obtain for the first two terms:

$$-z_1(\Psi^* d\Psi)_{ij} + \tfrac{1}{2}\Psi^*_{ib}(\sigma\gamma_1\sigma^{-1})_{cb}(\lambda^{-H_2})_{cd}d\Psi_{dj} + \tfrac{1}{2}\Psi^*_{ib}(\gamma_1)_{bc}d\Psi_{cj}.$$

The first term forms the matrix $-z_1\omega$ whereas the second two terms combine to give $\Psi^*\gamma_1 d\Psi$, due to relation (4.2.8). Finally, the term $\Psi^*_{ib}(d\Gamma_{ac})\Psi_{cj}$ reduces to $-\frac{1}{2}dz_1\Psi^*_{ib}\Psi_{bj} = -\frac{1}{2}dz_1\mathbb{I}_2$ using equation (4.2.9). The formulas for $r = (-1, 0)$ and $r = (0, \pm 1)$ are established in like manner. □

At first sight, the above infinitesimal gauge potentials $\delta\omega_i$ do not seem to satisfy $\overline{(\delta\omega_i)_{kl}} = (\delta\omega_i)_{lk}$ and $\sum_k (\delta\omega_i)_{kk} = 0$. i.e. they are not $su(2)$-gauge potentials. However, this is only due to the fact that the generators F_r and H_0 are the deformed analogues of the generators of the complexified Lie algebra $so(5,1) \otimes_{\mathbb{R}} \mathbb{C}$. One recovers $su(2)$-gauge potentials by acting with the real generators $\frac{1}{2}(F_r + F_r^*)$, $\frac{1}{2i}(F_r - F_r^*)$ and H_0. One checks that the resulting gauge potentials are $\delta\omega_0$, $\frac{1}{2}(\delta\omega_1 + \delta\omega_3)$, $\frac{1}{2i}(\delta\omega_1 - \delta\omega_3)$, $\frac{1}{2}(\delta\omega_2 + \delta\omega_4)$ and $\frac{1}{2i}(\delta\omega_2 - \delta\omega_4)$, which are traceless skew-hermitian matrices with values in $\Omega^1(S^7_{\theta'})$.

The above transformation of the gauge potential ω under the twisted symmetry $\mathcal{U}_\theta(so(5,1))$ induces a natural transformation of the connection ∇_0 to $\nabla_{t,i} := \nabla_0 + t\delta\omega_i$ for $i = 0, \dots, 4$ and $t \in \mathbb{R}$. Let us see if these new connections are (infinitesimal) instantons, i.e. if their curvature is selfdual. We start by writing $\nabla_{t,i}$ in terms of the Grassmann connection on $\mathcal{E} \simeq p\big(\mathcal{A}(S^4_\theta)\big)^4$. Using the explicit isomorphism (Proposition 3.2) between this module and the module of equivariant maps $\mathcal{A}(S^7_\theta) \boxtimes_\rho \mathbb{C}^2$, we find that $\nabla_{t,i} = pd + t\delta\alpha_i$ with $\delta\alpha_0 = p\gamma_0(dp)p - \frac{1}{2}\Psi dz_0 \Psi^*$ and

$$\delta\alpha_1 = p\gamma_1(dp)p - \tfrac{1}{2}\Psi dz_1\Psi^*, \qquad \delta\alpha_3 = p\gamma_1^*(dp)p - \tfrac{1}{2}\Psi dz_1^*\Psi^*,$$
$$\delta\alpha_2 = p\gamma_2(dp)p - \tfrac{1}{2}\Psi dz_2\Psi^*, \qquad \delta\alpha_4 = p\gamma_2^*(dp)p - \tfrac{1}{2}\Psi dz_2^*\Psi^*,$$

Note that $\delta\alpha_i$ are 4×4 matrices with entries in the one-forms $\Omega^1(S^4_\theta)$ satisfying $p\delta\alpha_i = \delta\alpha_i p = p\delta\alpha_i p = \delta\alpha_i$, as expected from the general theory on connections on modules in Appendix A.4). Indeed, using relations (4.2.9) one can move the dz's to the left of Ψ at the cost of some μ's, yielding expression like $dz_i\, p \in M_4(\Omega^1(S^4_\theta))$.

The curvature $F_{t,i}$ of the connection $\nabla_{t,i}$ is given by (cf. equation (A.4.5))

$$F_{t,i} = F_0 + tpd(\delta\alpha_i) + \mathcal{O}(t^2).$$

In order to check selfduality of this curvature (modulo t^2), we will express the curvature in terms of the projection p and consider $F_{t,i}$ as a two-form valued endomorphism on $\mathcal{E} \simeq p\big(\mathcal{A}(S^4_\theta)\big)^4$.

Proposition 4.13. *The curvatures $F_{t,i}$ of the connections $\nabla_{t,i}$ ($i = 0, \dots, 4$) are given by $F_{t,i} = F_0 + t\delta F_i + \mathcal{O}(t^2)$, where $F_0 = pdpdp$ and $\delta F_0 = -2z_0F_0$,*

$$\delta F_1 = -2z_1\lambda^{H2}F_0; \qquad \delta F_3 = -2z_1^*\lambda^{-H2}F_0;$$
$$\delta F_2 = -2z_2\lambda^{H1}F_0; \qquad \delta F_4 = -2z_2^*\lambda^{-H1}F_0.$$

Proof. A small computation yields the following expression for $\delta F_i = pd(\delta\alpha_i)$ as an endomorphism on $\mathcal{E}$ taking values in $\Omega^2(S^4_\theta)$:

$$\delta F_i = p(dp)\gamma_i(dp)p - p\gamma_i(dp)(dp)p,$$

where we introduced $\gamma_3 = \gamma_1^*$ and $\gamma_4 = \gamma_2^*$, and used the fact that $p(dp)p = 0$. We then use the crucial property $p(dp\gamma_i + \gamma_i dp)(dp)p = 0$ for all $i = 0, \dots, 4$ to deduce that $\delta F_i = -2p\gamma_i dpdpp$. This can be expressed as $\delta F_i = -2p\gamma_i p dp dp$ using the property $dp = (dp)p + pdp$. Finally, $p\gamma_i p = \Psi(\Psi^*\gamma_i\Psi)\Psi^*$, so that the result follows from the definition of the z's in terms of the Dirac matrices (cf. above equation (3.1.11)) together with the commutation relations between them and the matrix Ψ in equation (4.2.9) □

Corollary 4.14. *The connections $\nabla_{t,i}$ are (infinitesimal) instantons, i.e. $*_\theta F_{t,i} = F_{t,i}$ modulo t^2.*

Proof. This follows directly from the above expressions for δF_i and selfduality of F_0. It also follows from Lemma 4.11 stating that $\mathcal{U}_\theta(\mathrm{so}(5,1))$ acts by conformal transformation therefore leaving the selfdual equation $*_\theta F_0 = F_0$ for the basic instanton ∇_0 invariant. □

Let us now establish that the obtained connections $\nabla_{t,i}$ are not gauge equivalent to ∇_0. Recall that an infinitesimal gauge transformation is given by $[\nabla_0, X]$ for $X \in \Gamma(\mathrm{ad}(S^7_{\theta'}))$; we want to show that $\delta\omega_i$ is orthogonal to $[\nabla_0, X]$ for any such X, i.e. that

$$([\nabla_0, X], \delta\omega_i)_2 = 0,$$

with the natural inner product on $\Omega^1(\mathrm{ad}(S^7_{\theta'}) := \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_{\theta'}))$. From Remark 4.6, we see that this means that $(\nabla^{(2)}_0(X), \delta\omega_i)_2 = (X, (\nabla^{(2)}_0)^*(\delta\omega^i))_2$ should vanish for all X. Now $\delta\omega^i$ coincides with its classical counterpart since $\Gamma(\mathrm{ad}(S^7_{\theta'}))$ is isomorphic to $\Gamma(\mathrm{ad}(S^7))$ *as vector spaces* and also $\nabla^{(2)}_0$ coincides with its classical counterpart (Remark 4.6). Hence, $(\nabla^{(2)}_0)^*(\delta\omega^i) = 0$ because it vanishes in the classical case [4].

4.2.2 Local expressions

In this section, we will obtain "local expressions" for the instantons on S^4_θ in the following sense. We define the algebra $\mathcal{A}(\mathbb{R}^4_\theta)$ of polynomials on the 4-plane $\mathbb{R}^4_\theta$ as the $*$-algebra generated by ζ_1, ζ_2 satisfying

$$\zeta_1\zeta_2 = \lambda\zeta_2\zeta_1; \qquad \zeta_1\zeta_2^* = \bar{\lambda}\zeta_2^*\zeta_1.$$

with λ as above. In the case $\theta = 0$ ($\lambda = 1$), one recovers the $*$-algebra of polynomials on the usual 4-plane $\mathbb{R}^4$. Again, it is possible to define this algebra as the fixed point algebra as before $\left(\mathcal{A}(\mathbb{R}^4) \otimes \mathcal{A}(\mathbb{T}^2_\theta)\right)^{\sigma\otimes\tau^{-1}}$, where the torus $\mathbb{T}^2$ acts as $\sigma_s : \zeta_i \mapsto e^{2\pi i s_i}\zeta_i$. In particular, we can define the algebra $C^\infty(\mathbb{R}^4_\theta)$ of smooth functions on $\mathbb{R}^4_\theta$.

In this smooth algebra, the element $(1+|\zeta|^2)^{-1}$ with $|\zeta| := \zeta_1^*\zeta_1 + \zeta_2^*\zeta_2$ clearly exists. Let us define the elements $\tilde{z}_\mu$ ($\mu = 1, 2, 3$) by

$$\tilde{z}_i = 2\zeta_i(1+|\zeta|^2)^{-1}; \qquad \tilde{z}_0 = (1-|\zeta|^2)(1+|\zeta|^2)^{-1} \tag{4.2.10}$$

One sees that the $\tilde{z}_\mu$ satisfy the same relations as the generators $z_\mu$of $\mathcal{A}(S^4_\theta)$ (cf. (3.1.4)). The difference is that the classical point $z_0 = -1, z_j = z_j^* = 0$ of S^4_θ is not in the spectrum of $\tilde{z}_\mu$. We interpret the noncommutative plane $\mathbb{R}^4_\theta$ as a "chart" of the noncommutative 4-sphere S^4_θ and the above equation (4.2.10) as the inverse stereographical projection. In fact, one can cover S^4_θ by two such charts with domain $\mathbb{R}^4_\theta$, with transition on $\mathbb{R}^4_\theta\backslash\{0\}$, where $\{0\}$ is the classical point $\zeta_j = \zeta_j^* = 0$ of $\mathbb{R}^4_\theta$.

There is a differential calculus $(\Omega(\mathbb{R}^4_\theta), \mathrm{d})$ on $\mathbb{R}^4_\theta$, defined as in Section 2.2. Explicitly, $\Omega(\mathbb{R}^4_\theta)$ is the graded $*$-algebra generated by the elements ζ^μ of degree 0 and $d\zeta^\mu$ of degree 1 with relations:

$$d\zeta^\mu d\zeta^\nu + \lambda^{\mu\nu} d\zeta^\nu d\zeta^\mu = 0; \quad d\bar{\zeta}^\mu d\zeta^\nu + \lambda^{\nu\mu} d\zeta^\nu d\bar{\zeta}^\mu = 0;$$
$$\zeta^\mu d\zeta^\nu = \lambda^{\mu\nu} d\zeta^\nu \zeta^\mu; \quad \bar{\zeta}^\mu d\zeta^\nu = \lambda^{\nu\mu} d\zeta^\nu \bar{\zeta}^\mu.$$

There is a unique differential d on $\Omega(\mathbb{R}^4_\theta)$ such that $\mathrm{d} : \zeta^\mu \mapsto d\zeta^\mu$. The involution $\omega \mapsto \omega^*$ for $\omega \in \Omega(\mathbb{R}^4_\theta)$ is the graded extension of $\zeta^\mu \mapsto \bar{\zeta}^\mu$, i.e. it is such that $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$ and $(\omega_1\omega_2)^* = (-1)^{p_1p_2}\omega_2^*\omega_1^*$ for $\omega_i \in \Omega^{p_i}(\mathbb{R}^4_\theta)$.

Moreover, there is a Hodge star operator $*_\theta$ mapping $\Omega^p(\mathbb{R}^4_\theta)$ to $\Omega^{4-p}(\mathbb{R}^4_\theta)$, and is obtained from the classical Hodge star operator as before. In terms of the standard Riemannian metric on $\mathbb{R}^4$, we have the following useful formulas for $*_\theta$ on two-forms:

$$*_\theta d\zeta_1 d\zeta_2 = -d\zeta_1 d\zeta_2; \qquad *_\theta d\zeta_1 d\zeta_1^* = -d\zeta_2 d\zeta_2^*; \qquad *_\theta d\zeta_1 d\zeta_2^* = d\zeta_1 d\zeta_2^* \tag{4.2.11}$$

which are the same as the formulas for $*$ on $\mathbb{R}^4$ (remember that we do not change the metric in an isospectral deformation).

Since the stereographical projection from S^4 onto $\mathbb{R}^4$ is a conformal map commuting with the action of $\mathbb{T}^2$, it makes sense to investigate the form of the instanton connections on S^4_θ obtained in Proposition 4.12 on the local chart $\mathbb{R}^4_\theta$. As in [64], we first introduce a "local section" of the principal bundle $S^7_\theta \to S^4_\theta$ on the local chart of S^4_θ defined in (4.2.10). Let $u = (u_1, u_2)$ be a complex spinor of modulus one, $u_1^* u_1 + u_2^* u_2 = 1$, and define

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \rho \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \rho \begin{pmatrix} \zeta_1^* & \zeta_2^* \\ -\mu\zeta_2 & \bar{\mu}\zeta_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{4.2.12}$$

Here ρ is a central element in $C^\infty(\mathbb{R}^4_\theta)$ such that $\rho^2 = (1 + |\zeta|^2)^{-1}$ and the commutations rules of the u_j's with the ζ_k's are dictated by those of the ψ_j:

$$u_1 \zeta_j = \mu \zeta_j u_1\,, \quad u_2 \zeta_j = \bar{\mu} \zeta_j u_2\,, \quad j = 1, 2\,.$$

The right action of SU(2) rotates the vector u while mapping to the "same point' of S^4_θ, which, from the choice in (4.2.12) is found to be

$$2(\psi_1\psi_3^* + \psi_2^*\psi_4) = \tilde{z}_1, \quad 2(-\psi_1^*\psi_4 + \psi_2\psi_3^*) = \tilde{z}_2, \quad 2(\psi_1^*\psi_1 + \psi_2^*\psi_2) - 1 = \tilde{z}_0,$$

which is in the local chart (4.2.10), as expected.

By writing the unit vector u as an SU(2) matrix, $u = \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix}$, we can write

$$\Psi = \rho \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathcal{Z} \end{pmatrix} \begin{pmatrix} u \\ u \end{pmatrix}, \qquad \text{with } \mathcal{Z} = \begin{pmatrix} \zeta_1^* & \zeta_2^* \\ -\mu\zeta_2 & \bar{\mu}\zeta_1 \end{pmatrix}.$$

One can check by direct computation that the gauge potential takes the form

$$\omega = \rho^{-1} d\rho \; u^* u + \rho^2 u^* \mathcal{Z}^* d\mathcal{Z} u + u^* du$$

and that its curvature $F = d\omega + \omega^2$ satisfies:

$$uFu^* = \rho^4 d\mathcal{Z}^* d\mathcal{Z} = \frac{1}{2(1+|\zeta|^2)^2} \begin{pmatrix} d\zeta_1 d\zeta_1^* - d\zeta_2 d\zeta_2^* & 2 d\zeta_1 d\zeta_2^* \\ 2 d\zeta_2 d\zeta_1^* & -d\zeta_1 d\zeta_1^* - d\zeta_2 d\zeta_2^* \end{pmatrix}.$$

It is immediate from (4.2.11) that this curvature is selfdual, as expected.

The explicit local expressions for the transformed gauge potentials and their curvature can be obtained in a similar manner. Let us work out the local expression for $\delta\omega_0$, being the most transparent one. A direct computation shows that

$$\delta\omega_0 = -2\rho d\rho u^* u - 2\rho^4 u^* \mathcal{Z}^* d\mathcal{Z} u,$$

which implies for the transformed curvature:

$$F_{t,0} = F_0 + 2t(1 - 2\rho^2) F_0.$$

It is clear that this rescaled curvature still satisfies the selfdual equation; it is also in concordance with Proposition 4.13, since $\tilde{z}_0 = 2\rho^2 - 1$.

4.2.3 Moduli space of instantons

We will closely follow the infinitesimal construction of instantons in [4]. This will eventually result in the dimension of the 'tangent space' to the moduli space of instantons on S^4_θ; it will show that the collection of instantons constructed above is in fact the complete set of infinitesimal instantons on S^4_θ.

Let us start by considering the following family of connections on S^4_θ,

$$\nabla_t = \nabla_0 + t\alpha$$

where $\alpha \in \Omega^1(\mathrm{ad}(S^7_{\theta'}))$. For ∇_t to be an instanton, we have to impose the selfdual equation $*_\theta F_t = F_t$ on the curvature $F_t = F_0^2 + t[\nabla, \alpha] + \mathcal{O}(t^2)$ of ∇_t. This leads, when differentiated with respect to t, setting $t = 0$ afterwards, to the following *linearized selfdual equation*

$$P_-[\nabla_0, \alpha] = 0,$$

with $P_- := \frac{1}{2}(1 - *_\theta)$ the projection onto the anti-selfdual 2-forms. Here $[\nabla_0, \alpha]$ is an element in $\Omega^2(\Gamma(\mathrm{ad}(S^7_{\theta'}))) := \Omega^2(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma(\mathrm{ad}(S^7_{\theta'}))$ by Proposition A.13 and because

$$[\nabla_0, \alpha]_{ij} = d\alpha_{ij} + \omega_{ik}\alpha_{kj} - \alpha_{ik}\omega_{kj}$$

has vanishing trace. This is due to the fact that $\omega_{ik}\alpha_{kj} = \alpha_{kj}\omega_{ik}$.

If the family were obtained from an infinitesimal gauge transformation, then we would have had $\alpha = [\nabla_0, X]$, for some $X \in \Gamma(\mathrm{ad}(S^7_{\theta'}))$. Indeed, $[\nabla_0, X]$ is an element in $\Omega^1(\mathrm{ad}(S^7_{\theta'}))$, for the same reasons as before. Now $P_-[\nabla_0, [\nabla_0, X]] = [P_- F_0, X] = 0$, since F_0 is selfdual. Hence, we have defined an element in the first cohomology group H^1 of the so-called *selfdual complex*:

$$0 \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{d_0} \Omega^1(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{d_1} \Omega^2_-(\mathrm{ad}(S^7_{\theta'})) \to 0$$

where $d_0 = [\nabla_0, \cdot]$ and $d_1 := P_-[\nabla_0, \cdot]$. Note that these operators are Fredholm operators, so that the cohomology groups of the complex are finite dimensional. The complex can be replaced by a single Fredholm operator

$$d_0^* + d_1 : \Omega^1(\mathrm{ad}(S^7_{\theta'})) \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \oplus \Omega^2_-(\mathrm{ad}(S^7_{\theta'})) \tag{4.2.13}$$

where d_0^* is the adjoint of d_0 with respect to the inner product (2.2.6).

Our goal is to compute $h^1 = \dim H^1$ by calculating the alternating sum $h^0 - h^1 + h^2$ from the index of this Fredholm operator. The vanishing of h^0 and h^2 follows from the following observation. By definition, H^0 consists of the covariant constant elements in $\Gamma(\mathrm{ad}(S^7_{\theta'}))$. Since $[\nabla_0, \cdot]$ commutes with the action of $\mathbb{T}^2$ and coincides with $\nabla_0^{(2)}$ on $\Gamma(\mathrm{ad}(S^7_{\theta'}))$ (cf. Remark 4.6), we see that $[\nabla_0, T] = \nabla_0^{(2)}(T) = 0$ coincides with the corresponding classical equation. Since classically there are no covariant constant elements in $\Gamma(\mathrm{ad}(S^7))$ for an irreducible selfdual connection on $\mathcal{E}$, we conclude that $h^0 = 0$. A completely analogous argument for the kernel of the operator d_1^* shows that also $h^2 = 0$.

4.2.4 Dirac operator associated to the complex

The Fredholm operator $d_0^* + d_1$ defined in (4.2.13) can be replaced by a Dirac operator on the spinor bundle $\mathcal{S}$ with coefficients in the adjoint bundle. For this, we need the following lemma, which is a straightforward modification of its classical analogue [4]. Recall that the $\mathbb{Z}^2$-grading

γ_5 induces a decomposition of the spinor bundle $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$. Note that $\mathcal{S}^-$ coincides classically with the charge -1 (anti-)instanton bundle. Indeed, the Levi-Civita connection has anti-selfdual curvature, when lifted to the spinor bundle and restricted to negative chirality spinors. Also, $\mathcal{S}^+$ coincides with the charge 1 instanton bundle. Note that then Remark 3.3 implies that the $C^\infty(S^4)$-modules $\Gamma(S^4, \mathcal{S}^\pm)$ have the homogeneous decomposition property of Definition 2.2.

Lemma 4.15. *There are the following isomorphisms of right* $C^\infty(S^4_\theta)$*-modules:*

$$\Omega^1(S^4_\theta) \simeq \Gamma(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^-) \simeq \Gamma(S^4_\theta, \mathcal{S}^+) \otimes_{C^\infty(S^4_\theta)} \Gamma(S^4_\theta, \mathcal{S}^-)$$
$$\Omega^0(S^4_\theta) \oplus \Omega^2_-(S^4_\theta) \simeq \Gamma(S^4_\theta, \mathcal{S}^- \otimes \mathcal{S}^-) \simeq \Gamma(S^4_\theta, \mathcal{S}^-) \otimes_{C^\infty(S^4_\theta)} \Gamma(S^4_\theta, \mathcal{S}^-)$$

Proof. Since this is true in the classical case, we have to establish equivariance of the isomorphisms under the action of $\mathbb{T}^2$. The result then follows from Lemma 2.3. By definition, the action of $\mathbb{T}^2$ on S^4 is lifted to an action of $\widetilde{\mathbb{T}}^2$ on the spinor bundle $\mathcal{S}$ such that it coincides on the endomorphism bundle $\mathrm{End}(\mathcal{S}) \simeq \mathcal{S}^+ \otimes \mathcal{S}^-$ with the underlying action of $\mathbb{T}^2$ on the cotangent bundle Λ^1. A completely analogous statement holds for the second isomorphism. □

Let us for the moment forget about the adjoint bundle $\mathrm{ad}(S^7_{\theta'})$. The operator $d^* + P_- d$ can be understood as a map from $\Omega^1(S^4) \to \Omega^0(S^4) \oplus \Omega^2_-(S^4)$, since $\Omega(S^4_\theta) \simeq \Omega(S^4)$ as vector spaces and both d and $*$ commute with the action of $\mathbb{T}^2$ (see Section 2.2.3). Under the above isomorphisms, this operator is replaced by the Dirac operator with coefficients in $\mathcal{S}^-$ [4], i.e.

$$D' : \Gamma(S^4, \mathcal{S}^+ \otimes \mathcal{S}^-) \to \Gamma(S^4, \mathcal{S}^- \otimes \mathcal{S}^-).$$

If we take into account the twisting by the adjoint bundle, we see that this involves merely a multiplication by the projection $p_{(2)}$ defining the adjoint bundle $\mathrm{ad}(S^7_{\theta'})$. Hence, the operator $d_0^* + d_1$ is replaced by the following Dirac operator:

$$\mathcal{D} : \Gamma(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})) \to \Gamma(S^4_\theta, \mathcal{S}^- \otimes \mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})),$$

with coefficients in the vector bundle $\mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})$ on S^4_θ.

Let us now compute the index of this Dirac operator using the Connes-Moscovoci local index formula. It is given by the following pairing:

$$\mathrm{Index}\, \mathcal{D} = \langle \phi^*, \mathrm{ch}(\mathcal{S}^- \otimes \mathrm{ad}(S^7_\theta)) \rangle = \langle \phi^*, \mathrm{ch}(\mathcal{S}^-) \cdot \mathrm{ch}(\mathrm{ad}(S^7_\theta)) \rangle.$$

Recall the following representation of the Chern characters as operators on the Hilbert space $\mathcal{H}$:

$$\pi_D(\mathrm{ch}_k(e)) = (-1)^k \frac{(2k)!}{k!} \sum (\pi(e_{i_0 i_1}) - \frac{1}{2}\delta_{i_0 i_1})[D, \pi(e_{i_1 i_2})] \cdots [D, \pi(e_{i_{2k} i_0})].$$

In Section 3.2.2, we computed the image of the Chern characters under π_D of all modules associated to the noncommutative principal bundle $S^7_{\theta'} \to S^4_\theta$. In particular, we found for the adjoint bundle that

$$\pi_D\big(\mathrm{ch}_0(\mathrm{ad}(S^7_{\theta'}))\big) = 3, \qquad \pi_D\big(\mathrm{ch}_1(\mathrm{ad}(S^7_{\theta'}))\big) = 0, \qquad \pi_D\big(\mathrm{ch}_2(\mathrm{ad}(S^7_{\theta'}))\big) = 4(3\gamma_5).$$

The Chern character of the spinor bundle $\mathcal{S}^-$ can be computed as follows. Note that $\mathcal{S}^-$ coincides classically with the charge -1 instanton bundle. Indeed, the Levi-Civita connection has anti-selfdual curvature, when lifted to the spinor bundle and restricted to negative chirality spinors. We conclude from $\mathbb{T}^2$-equivariance that then also $\Gamma(S^4_\theta, \mathcal{S}^-)$ is isomorphic to the basic (anti-)instanton bundle $\Gamma(S^7_{\theta'} \times_{SU(2)} \mathbb{C}^2)$ on S^4_θ. It then follows from [33] (cf. Section 3.2.2) that

$$\pi_D\big(\mathrm{ch}_0(\mathcal{S}^-)\big) = 2, \qquad \pi_D\big(\mathrm{ch}_1(\mathcal{S}^-)\big) = 0, \qquad \pi_D\big(\mathrm{ch}_2(\mathcal{S}^-)\big) = -3\gamma_5.$$

Combining both Chern characters and using the local index formula on S^4_θ, we find that

$$\mathrm{Index}\, \mathcal{D} = 6 \operatorname*{Res}_{z=0} z^{-1} \mathrm{tr}(\gamma_5 |D|^{-2z}) + 0 + \tfrac{1}{2}(2 \cdot 4 - 3 \cdot 1) \operatorname*{Res}_{z=0} \mathrm{tr}(3\gamma_5^2 |D|^{-4-2z}) = 5,$$

similar to the computation done in Section 3.2.2.

For the moduli space of instantons on S^4_θ, this implies the following.

Theorem 4.16. *The tangent space at the base point ∇_0 to the moduli space of (irreducible) SU(2)-instantons on S^4_θ is five-dimensional.*

Chapter 5

Towards Yang-Mills theory on M_θ

As alluded to before, let us now describe how the just constructed Yang-Mills theory on S^4_θ can be generalized to any four-dimensional toric noncommutative manifold M_θ.

Suppose $P \to M$ is a principal G bundle on M, where G is a semisimple Lie group. We assume that M is a four-dimensional Riemannian manifold equipped with an isometrical action σ of the torus $\mathbb{T}^2$. For the construction to work, we assume that this action can be lifted to an action $\widetilde{\sigma}$ of a cover $\widetilde{\mathbb{T}}^2 \to \mathbb{T}^2$ on P, while it commutes with the action of G. As in Section 2.2, we define the noncommutative algebras $C^\infty(P_\theta)$ and $C^\infty(M_\theta)$ as the vector spaces $C^\infty(P)$ and $C^\infty(M)$ with star products defined with respect to the action of $\widetilde{\mathbb{T}}^2$ and $\mathbb{T}^2$ respectively, like in (2.2.2). The action α of G on the algebra $C^\infty(P)$ defined by

$$\alpha_g(f)(p) = f(g^{-1} \cdot p)$$

induces an action of G by automorphisms on the algebra $C^\infty(P_\theta)$, because the action of $\widetilde{\mathbb{T}}^2$ commutes with the action of G on P. This means that also the inclusion $C^\infty(M) \subset C^\infty(P)$ as G-invariant element in $C^\infty(P)$ extends to an inclusion $C^\infty(M_\theta) \subset C^\infty(P_\theta)$ of G-invariant element in $C^\infty(P_\theta)$. Notice that the action of G translates trivially into a coaction of the Hopf algebra $C^\infty(G)$ on $C^\infty(P_\theta)$.

Proposition 5.1. *The inclusion* $C^\infty(M_\theta) \hookrightarrow C^\infty(P_\theta)$ *is a principal* $C^\infty(G)$ *extension.*

Proof. As in Section 3.3, it is enough to establish surjectivity of the canonical map

$$\begin{aligned} \chi : C^\infty(P_\theta) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta) &\to C^\infty(P_\theta) \otimes C^\infty(G); \\ f' \otimes_{C^\infty(M_\theta)} f &\mapsto f'\Delta_R(f) = f' f_{(0)} \otimes f_{(1)} \end{aligned}$$

Principality then follows from the cosemisimplicity of the Hopf algebra $C^\infty(G)$ [88]. Note that in the classical case, the canonical map from $\chi^{(0)} : C^\infty(P) \otimes_{C^\infty(M)} C^\infty(P) \to C^\infty(P) \otimes C^\infty(G)$ is bijective by the very definition of a principal bundle. Moreover, there is an isomorphism of vector spaces:

$$\begin{aligned} T : C^\infty(P_\theta) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta) &\to C^\infty(P) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta) \\ f' \otimes_{C^\infty(M_\theta)} f &\mapsto \sum f'_r \otimes_{C^\infty(M)} \widetilde{\sigma}_{r\theta}(f) \end{aligned}$$

where $f' = \sum_r f'_r$ is the homogeneous decomposition of f' under the action of $\widetilde{\mathbb{T}}^2$. Compare with the proof of Lemma 2.3. We claim that the canonical map is given as the composition

$\chi = \chi^{(0)} \circ T$, hence, it is bijective. Indeed,

$$\begin{aligned}\chi^{(0)} \circ T\big(f' \otimes_{C^\infty(M_\theta)} f\big) &= \sum_r f'_r \widetilde{\sigma}_{r\theta}(f_{(0)}) \otimes f_{(1)} \\ &= f' \times_\theta f_{(0)} \otimes f_{(1)} \\ &= \chi(f' \otimes_{C^\infty(M_\theta)} f)\end{aligned}$$

since the action of $\widetilde{\mathbb{T}}^2$ on $C^\infty(P_\theta)$ commutes with the coaction of $C^\infty(G)$. □

We define noncommutative associated bundles as before by setting

$$\mathcal{E} = C^\infty(P_\theta) \boxtimes_\rho V := \big\{f \in C^\infty(P_\theta) \otimes V | (\alpha_g \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(g)^{-1})(f)\big\}$$

for a representation ρ of G on V. These $C^\infty(M_\theta)$-bimodules are finite projective since they are of the form of the modules defined in Section 2.2.2 (cf. Remark 3.3). Moreover, the corresponding classical $C^\infty(M)$-modules $C^\infty(P) \boxtimes_\rho V$ have the homogeneous decomposition property of Definition 2.2. Indeed, let $f \in C^\infty(P) \boxtimes_\rho V$ and write in a basis of V:

$$f = \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix},$$

so that $f^i \in C^\infty(P)$ satisfies $\alpha_g(f^i) = \rho(g)_{ij} f^j$. Suppose that $f^i = \sum_r f^i_r$ as a sum of $\widetilde{\mathbb{T}}^2$-homogeneous elements in $C^\infty(P)$, i.e. so that $\sigma_t(f^i_r) = e^{2\pi i t \cdot r} f^i_r$ for $t \in \widetilde{\mathbb{T}}^2$. Since $\alpha_g \circ \sigma_t = \sigma_t \circ \alpha_g$, we find that

$$\alpha_g(\sum_r e^{2\pi i t\cdot r} f^i_r) = \sum_r \rho(g)_{ij} e^{2\pi i t \cdot r} f^j_r.$$

for $t \in \widetilde{\mathbb{T}}^2$ and $g \in G$. By linear independence of the exponentials, we derive $\alpha_g(f^i_r) = \rho(g)_{ij} f^j_r$, and conclude that $C^\infty(P) \boxtimes_\rho V$ has the homogeneous decomposition property.

Moreover, Proposition 3.4 generalizes to the statement that $\mathrm{End}(\mathcal{E}) \simeq C^\infty(P_\theta) \boxtimes_{\mathrm{ad}} L(V)$, where ad is the adjoint representation of G on $L(V)$. Also, one identifies the adjoint bundle as the module arising from the adjoint representation of G on $\mathfrak{g} \subset L(V)$, i.e. $\Gamma(\mathrm{ad}(P_\theta)) := C^\infty(P_\theta) \boxtimes_{\mathrm{ad}} \mathcal{G}$.

For the Yang-Mills action, we again define an inner product on $\mathrm{End}_{C^\infty(M_\theta)}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega(M_\theta))$ for a (right) finite projective $C^\infty(M_\theta)$-module $\mathcal{E}$ as in Chapter 4 and define the Yang-Mills action for a connection ∇ on $\mathcal{E}$ in terms of its curvature F by

$$\mathrm{YM}(\nabla) = (F, F)_2$$

This is a gauge invariant, positive and quartic functional. The derivation of the Yang-Mills equations (4.1.1) on S^4_θ does not rely on the specific properties of S^4_θ and continues to hold on M_θ. The same is true for the topological action, and $\mathrm{YM}(\nabla) \geq \mathrm{Top}(\mathcal{E})$ with equality if $*_\theta F = \pm F$. In other words, the minima of the Yang-Mills action are given by instanton connections.

The explicit construction of instanton connections on S^4_θ in Section 4.2.1 can of course not be generalized to a manifold M_θ. However, local expressions could in principle be obtained on a “local chart” $\mathbb{R}^4_\theta$ of M_θ, if $\mathbb{T}^2$ acts on the corresponding local chart $\mathbb{R}^4$ of M.

The generalization of the infinitesimal construction of instantons on S^4_θ to any toric noncommutative manifold M_θ remains still to be understood. A crucial point here is to establish that the commutator $[\nabla_0, \cdot]$ for a ‘base point’ instanton connection ∇_0 on $\mathcal{E}$ defines a connection on the adjoint bundle $P_\theta \times_G \mathfrak{g}$.

Epilogue

We considered several noncommutative spheres, together with the symmetries they describe or the symmetries they carry.

In the first part, we discussed the quantum group $SU_q(2)$ and constructed a noncommutative spin geometry on it. A central guiding principle was the invariance or equivariance with respect to two quantum symmetries, given by $\mathcal{U}_q(su(2))$. An interesting phenomenon occurred involving the real structure; we found that two conditions of Connes' noncommutative spin geometry, namely the commutant property and first-order condition, are only satisfied *up to infinitesimals of arbitrary order.*

In the second part, we discussed two spheres as the central ingredients of a noncommutative $SU(2)$ Hopf fibration $S^7_{\theta'} \to S^4_\theta$. We constructed a Yang-Mills theory on the space S^4_θ, and understood instantons as the minima of the Yang-Mills action functional. By an action of twisted conformal symmetries on the Hopf fibration $S^7_{\theta'} \to S^4_\theta$ we constructed all infinitesimal charge 1 instantons on S^4_θ. A completeness argument is provided by computing the index of a certain twisted Dirac operator, which gives the dimension of the tangent to the moduli space.

An interesting open problem involving the spin geometry of $SU_q(2)$ is the computation of the spectral action on $SU_q(2)$. In general, let $(\mathcal{A}, \mathcal{H}, D, J)$ be a real spectral triple and set $D_A := D + A + JAJ^*$ where $A \in \Omega^1_D(\mathcal{A})$. Then Connes' spectral action is given by

$$S(D, A) = \mathrm{tr}_{\mathcal{H}} \left(\chi \left(\frac{D_A^2}{\Lambda^2} \right) \right)$$

where $\mathrm{tr}_{\mathcal{H}}$ is the usual trace in the Hilbert space $\mathcal{H}$, Λ a "cut off parameter" and χ a suitable function which cuts off all eigenvalues of D_A^2 larger than Λ^2. For a treatment of the spectral action principle, we refer to [29] (see also [65]) and to [21, 22] for the derivation of the Standard Model Lagrangian from the spectral action. It would be interesting to compute this spectral action in the case of the abovely constructed almost real spectral triple on $SU_q(2)$. The main problem in this computation is the understanding of the "adjoint representation" of $\Omega^1_D(\mathcal{A})$ on $\mathcal{H}$ defined by $A + JAJ^*$ in the case that J does not fulfill the commutant property and the first-order condition. A deeper understanding of this might also shed light on the apparent failure of Poincaré duality [19].

Concerning Part II, it would be interesting to develop the set of ADHM data describing instantons on the noncommutative plane $\mathbb{R}^4_\theta$ (as in Section 2.1) and confront it with the ADHM data on the Moyal plane, as described by Nekrasov and Schwarz in [76]. Here the complex coordinates $\zeta^\mu, \zeta^{\mu*}$ satisfy $[\zeta^\mu, \zeta^{\mu*}] = \theta$. This would possibly lead to more insight into the global structure of the moduli space, and could be extended to instantons of higher charge.

Finally, it remains to be understood how noncommutative instantons serve in quantum Yang-Mills theory on toric noncommutative manifolds. However, since even a classical mathematical definition of quantum Yang-Mills theory does presently not exist – this is in fact one of the Clay Mathematics Institute Millennium Prize Problems – this seems to be a formidable task.

Appendix A

Some concepts from noncommutative geometry

We recall some of the basic concepts that appear in Connes' noncommutative geometry. For more details, the reader should consult his book [27] and the books [49] and [65]. Also motivation for the definitions below can be found in these books as well as in the text.

A.1 C^*-algebras

Let A be an algebra over $\mathbb{C}$. We say that this is a normed algebra if it is equipped (as a vector space) with a norm satisfying the following multiplicative property

$$\|ab\| \leq \|a\|\|b\| \tag{A.1.1}$$

for all $a, b \in A$. A Banach algebra is a normed algebra which is complete in the norm topology.

Definition A.1. *A C^*-algebra is a Banach algebra A, which is also a $*$-algebra such that for all $a \in A$ one has*

$$\|a^*a\| = \|a\|^2.$$

It follows from this definition that $\|a^*\| = \|a\|$ for all $a \in A$.

Example A.2. *The algebra $C(X)$ of continuous functions on a Hausdorff topological space forms a C^*-algebra equipped with the supremum norm*

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Definition A.3. *A Hilbert space $\mathcal{H}$ is a vector space with a hermitian inner product, which is complete in the associated norm. If $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$, the associated norm is given by $\|\chi\| := \langle \chi, \chi \rangle^{1/2}$ where $\chi \in \mathcal{H}$.*

The algebra $\mathcal{B}(\mathcal{H})$ of bounded operators consists of linear operators T on $\mathcal{H}$ for which there is a constant $C > 0$ such that $\|T\chi\| \leq C\|\chi\|$. The operator norm of a bounded operator is given by

$$\|T\| := \sup_{\chi \in \mathcal{H}} \left\{ \|T\chi\|_{\mathcal{H}} : \|\chi\| \leq 1 \right\}.$$

The involution on $\mathcal{B}(\mathcal{H})$ is given by the adjoint with respect to the inner product on $\mathcal{H}$, i.e. for $T \in \mathcal{B}(\mathcal{H})$, its adjoint is given by the unique operator $T^* \in \mathcal{B}(\mathcal{H})$ satisfying $\langle T\chi, \psi \rangle = \langle \chi, T^*\psi \rangle$ for all $\chi, \psi \in \mathcal{H}$.

It is not difficult to check that $\mathcal{B}(\mathcal{H})$ is a C*-algebra. The contrary is also true, i.e. any C*-algebra can be realized as a subalgebra of $\mathcal{B}(\mathcal{H})$; the GNS-construction (see for example the textbooks [78, 57]) allows to construct a Hilbert space $\mathcal{H}$ from a C*-algebra A in such a way that A becomes isomorphic to a subalgebra of $\mathcal{B}(\mathcal{H})$.

We also define the C*-algebra of compact operators.

Definition A.4. *An operator T on $\mathcal{H}$ is said to be compact if for every $\epsilon > 0$, there exists a finite dimensional subspace $E \subset \mathcal{H}$ such that $\|T|_{E^\perp}\| < \epsilon$.*

The set $\mathcal{K}(\mathcal{H})$ will denote the set of all compact operators on the Hilbert space $\mathcal{H}$. One can easily verify that it is a C*-algebra equipped with the operator norm; it is in fact a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$. There is the following notion of order of a compact operator.

Definition A.5. *For any $\alpha \in \mathbb{R}^+$, a compact operator $T \in \mathcal{K}(\mathcal{H})$ is said to be an infinitesimal of order α if its singular values $\mu_n(T)$ satisfy $\mu_n(T) = \mathcal{O}(n^{-\alpha})$.*

Another useful concept is a pre-C*-algebra which is defined as follows.

Definition A.6. *A subalgebra B of a unital Banach algebra A is said to be* stable under holomorphic function calculus *if whenever $b \in B$ is invertible in A, $b^{-1} \in B$.*
A pre-C*-algebra *is a subalgebra of a C*-algebra that is stable under holomorphic function calculus.*

An example of a pre-C*-algebra is provided by the algebra $C^\infty(M)$ of smooth functions on a manifold M, which is a subalgebra of the C*-algebra $C(M)$ of continuous functions on M.

A.2 Noncommutative spin geometries

The basic ingredient of a noncommutative spin geometry is a spectral triple.

Definition A.7. *A* spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *consists of a $*$-algebra of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator D on $\mathcal{H}$ satisfying*

1. *The resolvent $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, is a compact operator*
2. *The commutator $[D, a] = D \cdot a - a \cdot D$ is a bounded operator for all $a \in \mathcal{A}$.*

The triple is said to be even *if there is a $\mathbb{Z}_2$ grading of $\mathcal{H}$, namely an operator Γ on $\mathcal{H}$ with $\Gamma = \Gamma^*$ and $\Gamma^2 = 1$, such that*

$$\begin{aligned} \Gamma D + D\Gamma &= 0, \\ \Gamma a - a\Gamma &= 0, \qquad \textit{for all } a \in \mathcal{A}. \end{aligned}$$

If such a grading does not exist, the triple is said to be odd.

The basic example is the commutative spin geometry of a Riemannian spin manifold given by the triple

- $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on M.
- $\mathcal{H} = L^2(M, S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$.
- D, the Dirac operator associated with the Levi-Civita connection.

If the manifold is even dimensional, there is a grading defined by $\Gamma := -\gamma^1\gamma^2\cdots\gamma^{\dim M}$, where γ^i are the Dirac matrices satisfying $\{\gamma^i, \gamma^j\} = \delta^{ij}$.

A noncommutative spin geometry is a spectral triple satisfying some additional properties, as defined in [29]. They are needed for Connes' reconstruction theorem, providing an equivalence between Riemannian spin manifolds and spectral triples satisfying these properties for which the algebra is commutative. We will spell out only the properties used in the text which are finite summability, regularity and reality (including the commutant property and first condition).

A spectral triple is said to be n^+-*summable* (or to have spectral dimension n) if the compact operator $|D|^{-n}$ is an infinitesimal of order 1 (cf. above). A noncommutative integral is then provided by the *Dixmier trace* [41] $\mathrm{tr}_\omega \colon \mathcal{K}(\mathcal{H}) \to \mathbb{C}$, constructed in such a way that

1. Compact operators of order 1 are in the domain of the Dixmier trace.
2. Compact operators of order > 1 have vanishing Dixmier trace.

It is defined as

$$\mathrm{tr}_\omega(T) = \mathrm{Lim}_\omega \frac{1}{\ln N} \sum_0^{N-1} \mu_n(T) \tag{A.2.1}$$

where T is a compact operator and Lim_ω a generalization of the usual limit procedure, in order to obtain finite results, even for divergent (but bounded) series $\sum \mu_n(T)$. If T is a compact operator of order 1, then the partial sums of singular values $\sigma_N = \sum_{0\le k\le N} \mu_k(T)$ satisfy $\sigma_N \sim C \ln N$ as $N \to \infty$ and the Dixmier trace filters out the coefficients C. In this way, we can define a noncommutative integral on the algebra $\mathcal{A}$ by $\mathrm{tr}_\omega(a|D|^{-n})$. In the case of the canonical spectral triple $(C^\infty(M), L^2(M,S), D)$ on a Riemannian spin manifold, it reduces to the ordinary Riemannian integral: $\mathrm{tr}_\omega(f|D|^{-n}) = \int_M f d\nu$ for a smooth function f on M and $d\nu$ the Riemannian measure.

A spectral triple is said to be *real* if there exists an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$, such that $J^2 = \pm 1,\ JD = \pm DJ$, with the signs depending on the spectral dimension of the spectral triple. We impose the following conditions:

$$[a, Jb^*J^{-1}] = 0, \qquad \text{(commutant property)}$$
$$[[D,a], Jb^*J^{-1}] = 0, \qquad \text{(first-order condition)}$$

for all $a, b \in \mathcal{A}$. The map $a \to Ja^*J^{-1}$ defines a right action of $\mathcal{A}$ on $\mathcal{H}$ and the commutant property states that $\mathcal{H}$ is an $\mathcal{A}$-bimodule, whereas the first-order condition states that the left action of $[D, \mathcal{A}]$ on $\mathcal{H}$ commutes with the right action of $\mathcal{A}$.

The real structure J is usually related to Tomita-Takesaki theory [92], which states that for a weakly closed $*$-algebra $\mathcal{M}$ of operators on a Hilbert space $\mathcal{H}$, which admits a cyclic and separating vector[1], there exists an antilinear isometric involution $J : \mathcal{H} \to \mathcal{H}$, which conjugates $\mathcal{M}$ onto its commutant

$$\mathcal{M}' := \{S \in \mathcal{B}(\mathcal{H}) : ST = TS, \forall T \in \mathcal{M}\},$$

i.e. $JMJ^* = M'$.

[1] Note that a vector $\psi \in \mathcal{H}$ is cyclic for $\mathcal{M}$ if $\mathcal{M}\psi$ is dense in $\mathcal{H}$. It is called separating if for any $T \in \mathcal{M}$, $T\psi = 0$ implies $T = 0$.

A.2.1 Regularity and abstract differential calculus

A spectral triple is called *regular* (or *smooth*) if the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ lies within the smooth domain $\bigcap_{n=0}^{\infty} \mathrm{Dom}\delta^n$ of the operator derivation $\delta(T) := |D|T - T|D|$. This condition permits to introduce the analogue of Sobolev spaces $\mathcal{H}^s := \mathrm{Dom}(1+D^2)^{s/2}$ for $s \in \mathbb{R}_\theta$. Let $\mathcal{H}^\infty := \bigcap_{s\geq 0} \mathcal{H}^s$, which is a core for $|D|$. Then $T\colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ has *analytic order* $\leq k$ if T extends to a bounded operator from $\mathcal{H}^{k+s}$ to $\mathcal{H}^s$ for all $s \geq 0$. It turns out that $\mathcal{A}(\mathcal{H}^\infty) \subset \mathcal{H}^\infty$.

Assume that $|D|$ is invertible –which is a generic case of the D used in this paper (for a careful treatment of the noninvertible case, see [16]). The space OP^α of *operators of order* $\leq \alpha$ consists of those $T\colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ such that

$$|D|^{-\alpha} T \in \bigcap_{n=1}^{\infty} \mathrm{Dom}\delta^n.$$

(Operators of order α have analytic order α). In particular, $\mathrm{OP}^0 = \bigcap_{n=1}^{\infty} \mathrm{Dom}\delta^n$, the algebra of operators of order ≤ 0 includes $A \cup [D, \mathcal{A}]$ and their iterated commutators with $|D|$. Moreover, $[D^2, \mathrm{OP}^\alpha] \subset \mathrm{OP}^{\alpha+1}$ and $\mathrm{OP}^{-\infty} := \bigcap_{\alpha\leq 0} \mathrm{OP}^\alpha$ is a two-sided ideal in OP^0.

The algebra structure can be read off in terms of an *asymptotic expansion*: $T \sim \sum_{j=0}^{\infty} T_j$ whenever T and each T_j are operators from $\mathcal{H}^\infty$ to $\mathcal{H}^\infty$; and for each $m \in \mathbb{Z}$, there exists N such that for all $M > N$, the operator $T - \sum_{j=1}^{M} T_j$ has analytic order $\leq m$. For instance, for complex powers of $|D|$ (defined by the Cauchy formula) there is a binomial expansion:

$$[|D|^z, T] \sim \sum_{k=1}^{\infty} \binom{z}{k} \delta^k(T)\, |D|^{z-k}.$$

We define the dimension spectrum as follows.

Definition A.8. *The* dimension spectrum *of a regular spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is the subset* $\Sigma \subset \mathbb{C}$ *of singularities of the meromorphic functions*

$$\zeta_b(z) = \mathrm{tr}(b|D|^{-z})$$

where b *is an element in the algebra generated by* $\delta^k(\mathcal{A})$ *and* $\delta^k([D, \mathcal{A}])$ *for all* $k \geq 0$.

A.3 Noncommutative differential forms

Let $\mathcal{A}$ be an algebra with unit over $\mathbb{C}$. The universal differential algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is the graded algebra generated by $a \in \mathcal{A}$ of degree 0 and symbols δa, $a \in \mathcal{A}$ of degree 1, such that

$$\delta(ab) = (\delta a)b + a\delta b \quad \delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b; \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

We can write $\Omega_{\mathrm{un}}(\mathcal{A})$ as a direct sum of subspaces $\Omega^p_{\mathrm{un}}(\mathcal{A})$ generated by linear combinations of $a_0\delta a_1 \cdots \delta a_p$. Furthermore, there is the isomorphism of vector spaces

$$\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes p} \simeq \Omega^p_{\mathrm{un}}(\mathcal{A}), \tag{A.3.1}$$

where $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}\mathbb{I}$. The operator δ is defined on $\Omega_{\mathrm{un}}(\mathcal{A})$ by

$$\delta(a_0\delta a_1 \cdots \delta a_p) = \delta a_0 \delta a_1 \cdots \delta a_p,$$
$$\delta(\delta a_1 \cdots \delta a_p) = 0.$$

By construction, the algebra $\Omega_{\mathrm{un}}(\mathcal{A})$ is also a $\mathcal{A}$-bimodule. As the name suggests, the universal differential algebra satisfies the following universal property.

Proposition A.9. *Let* (Ω, d) *be a graded differential algebra and let* ρ *be a morphism of unital algebras. Then, there exists a unique extension of* ρ *to a morphism of graded differential algebras* $\widetilde{\rho} : \Omega_{\mathrm{un}}(\mathcal{A}) \to \Omega$ *such that* $\widetilde{\rho} \circ \delta = d \circ \widetilde{\rho}$.

An example of a frequently used differential calculus in the text and more generally, in noncommutative geometry, is Connes' differential calculus [27]. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. The $\mathcal{A}$-bimodule $\Omega^p_D(\mathcal{A})$ of Connes' differential p-forms is made of classes of operators of the form

$$\omega = \sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j], \quad a_i^j \in \mathcal{A},$$

modulo the sub-bimodule of operators

$$\Big\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{p-1}^j] : \ b_i^j \in \mathcal{A}, \ b_0^j [D, b_1^j] \cdots [D, b_{p-1}^j] = 0 \Big\}.$$

The exterior differential d_D is given by

$$d_D \Big[\sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j] \Big] = \sum_j [D, a_0^j][D, a_1^j] \cdots [D, a_p^j].$$

In the case of the canonical triple $(C^\infty(M), \mathcal{H}, D)$ of a Riemannian spin manifold M, this differential calculus is isomorphic to the de Rham differential calculus.

A.4 Modules and connections

We recall some basic definitions on modules and connections thereon. We derive a general Bianchi identity for the curvature of such connections and link with gauge theory.

A.4.1 Modules

Let $\mathcal{A}$ be an algebra over the complex numbers $\mathbb{C}$.

Definition A.10. *A* right module $\mathcal{E}$ *is a vector space over* $\mathbb{C}$ *that carries a right representation of* $\mathcal{A}$*, i.e. there is a map* $\mathcal{E} \times \mathcal{A} \ni (\eta, a) \to \eta a$ *such that*

$$\begin{aligned} \eta(ab) &= (\eta a)b, \\ \eta(a + b) &= \eta a + \eta b, \\ (\eta + \xi)a &= \eta a + \xi a, \end{aligned}$$

for any $\eta, \xi \in \mathcal{E}$ *and* $a, b \in \mathcal{A}$.

There is the natural notion of a morphism of (right) $\mathcal{A}$-modules as linear maps that respect this structure. Thus, a morphism between two (right) $\mathcal{A}$-modules $\mathcal{E}$ and F is a linear map $\rho : \mathcal{E} \to \mathcal{F}$ that is also right $\mathcal{A}$-linear:

$$\rho(\eta a) = \rho(\eta) a; \qquad \forall \eta \in \mathcal{E}, a \in \mathcal{A}.$$

Left modules and morphisms of left modules are defined similarly. A *bimodule* over an algebra $\mathcal{A}$ is both a left and a right $\mathcal{A}$-module such that the left and right action of $\mathcal{A}$ commute:

$$(a\eta)b = a(\eta b); \qquad \forall \eta \in \mathcal{E}, a, b \in \mathcal{A}.$$

Given a right $\mathcal{A}$-module, we define its *dual module* $\mathcal{E}'$ as the collection of all morphisms from $\mathcal{E}$ into $\mathcal{A}$, where $\mathcal{A}$ is seen as the trivial right $\mathcal{A}$-module; in other words:

$$\mathcal{E}' := \left\{\phi : \mathcal{E} \to \mathcal{A} \mid \phi(\eta a) = \phi(\eta)a, \ \ \eta \in \mathcal{E}, a \in \mathcal{A}\right\} .$$

Definition A.11. *A right $\mathcal{A}$-module $\mathcal{E}$ is said to be* finite projective *if there exists an idempotent* $p = p^2 \in M_N(\mathcal{A})$ *such that* $\mathcal{E} \simeq p\mathcal{A}^N$ *as right $\mathcal{A}$-modules.*

Here $M_N(\mathcal{A}) \simeq M_N(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$ denotes the algebra of $N \times N$ matrices with entries in $\mathcal{A}$ whereas $\mathcal{A}^N := \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$ which can be thought of as the set of N-dimensional vectors with entries in $\mathcal{A}$, and is clearly a right $\mathcal{A}$-module.

A.4.2 Connections

Let us suppose we have an algebra $\mathcal{A}$ with a differential calculus $(\Omega\mathcal{A} = \oplus_p \Omega^p\mathcal{A}, d)$. We now review the notion of a (gauge) connection on a (finite projective) module $\mathcal{E}$ over $\mathcal{A}$ with respect to the given calculus; we take a right module structure.

A *connection* on the right $\mathcal{A}$-module $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}\mathcal{A} ,$$

defined for any $p \geq 0$, and satisfying the Leibniz rule

$$\nabla(\omega\rho) = (\nabla\omega)\rho + (-1)^p \omega d\rho , \ \ \forall \ \omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} , \ \rho \in \Omega\mathcal{A} .$$

A connection is completely determined by its restriction

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} ,$$

which satisfies

$$\nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} da , \ \ \forall \ \eta \in \mathcal{E} , \ a \in \mathcal{A} ,$$

and which is extended by the Leibniz rule. It is again the latter property that implies that the composition,

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+2}\mathcal{A} ,$$

is $\Omega\mathcal{A}$-linear. Indeed, for any $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A}$, $\rho \in \Omega\mathcal{A}$,

$$\begin{aligned}\nabla^2(\omega\rho) &= \nabla\left((\nabla\omega)\rho + (-1)^p \omega d\rho\right) \\ &= (\nabla^2\omega)\rho + (-1)^{p+1}(\nabla\omega)d\rho + (-1)^p(\nabla\omega)d\rho + \omega d^2\rho \\ &= (\nabla^2\omega)\rho .\end{aligned}$$

The restriction of ∇^2 to $\mathcal{E}$ is the *curvature*

$$F : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A} , \tag{A.4.1}$$

of the connection. It is $\mathcal{A}$-linear, $F(\eta a) = F(\eta)a$ for any $\eta \in \mathcal{E}, a \in \mathcal{A}$, and satisfies

$$\nabla^2(\eta \otimes_{\mathcal{A}} \rho) = F(\eta)\rho , \ \ \forall \ \eta \in \mathcal{E} , \ \rho \in \Omega\mathcal{A} .$$

Thus, $F \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A})$, the collection of (right) $\mathcal{A}$-linear endomorphisms of $\mathcal{E}$, taking values in the two-forms $\Omega^2\mathcal{A}$.

In order to have the notion of a Bianchi identity we need some generalization. We let $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ be the collection of all $\Omega\mathcal{A}$-linear endomorphisms of $\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$. This forms an algebra under decomposition, as one can easily check. The curvature F can be thought of as an element of $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$. There is a well defined map

$$\begin{aligned} [\nabla, \ . \,] \ : \ & \mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \longrightarrow \mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \\ [\nabla, \mathsf{T}] := & \nabla \circ \mathsf{T} - (-1)^{|\mathsf{T}|} \, \mathsf{T} \circ \nabla \ . \end{aligned} \tag{A.4.2}$$

where T is of order $|\mathsf{T}|$ with respect to the $\mathbb{Z}^2$-grading of $\Omega\mathcal{A}$. Indeed, for any $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p \mathcal{A}$, $\rho \in \Omega\mathcal{A}$,

$$\begin{aligned} [\nabla, \mathsf{T}](\omega\rho) &= \nabla(\mathsf{T}(\omega\rho)) - (-1)^{|\mathsf{T}|} \, \mathsf{T}(\nabla(\omega\rho)) \\ &= \nabla\big(\mathsf{T}(\omega)\rho\big) - (-1)^{|\mathsf{T}|} \, \mathsf{T}\big((\nabla\omega)\rho + (-1)^p \omega d\rho\big) \\ &= \big(\nabla(\mathsf{T}(\omega))\big)\rho + (-1)^{p+|\mathsf{T}|} \, \mathsf{T}(\omega) d\rho - (-1)^{|\mathsf{T}|} \, \mathsf{T}(\nabla\omega)\rho - (-1)^{p+|\mathsf{T}|} \, \mathsf{T}(\omega) d\rho \\ &= \big(\nabla(\mathsf{T}(\omega)) - (-1)^{|\mathsf{T}|} \, \mathsf{T}(\nabla\omega)\big)\rho \\ &= \big([\nabla, \mathsf{T}](\omega)\rho\big) \ . \end{aligned}$$

Notice that $[\nabla, \cdot]$ acts as a graded derivation on the algebra of $\Omega\mathcal{A}$-linear endomorphisms, i.e. $[\nabla, \mathsf{S} \circ \mathsf{T}] = [\nabla, \mathsf{S}] \circ \mathsf{T} + (-1)^{|\mathsf{S}|} \mathsf{S}[\nabla, \mathsf{T}]$.

Proposition A.12.
The curvature F *satisfies the* Bianchi identity*,*

$$[\nabla, \mathsf{F}] = 0 \ .$$

Proof. Since $\mathsf{F} \in \mathrm{End}^0_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$, the map $[\nabla, \mathsf{F}]$ makes sense. Furthermore,

$$[\nabla, \mathsf{F}] = \nabla \circ \nabla^2 - \nabla^2 \circ \nabla = \nabla^3 - \nabla^3 = 0 \ .$$

□

To our knowledge and rather surprisingly, there is no presence in the literature of the notion of a noncommutative Bianchi identity. The one given in [71] or [65] works only when the algebra $\mathcal{A}$ is commutative.

Connections always exist on a projective module. On the free module $\mathcal{E} = \mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A} \simeq \mathcal{A}^N$, a connection is given by the operator

$$\nabla_0 = \mathbb{I} \otimes d : \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p \mathcal{A} \longrightarrow \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1} \mathcal{A} \ .$$

With the canonical identification $\mathbb{C}^N \otimes_{\mathbb{C}} \Omega\mathcal{A} = (\mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A}) \otimes_{\mathcal{A}} \Omega\mathcal{A} \ \simeq \ (\Omega\mathcal{A})^N$, one thinks of ∇_0 as acting on $(\Omega\mathcal{A})^N$ as the operator $\nabla_0 = (d, d, \cdots, d)$ (N-times).
For a generic projective module $\mathcal{E}$ one has a canonical inclusion map, $\lambda : \mathcal{E} \to \mathcal{A}^N$, which identifies $\mathcal{E}$ as a direct summand of the free module $\mathcal{A}^N$ and a canonical idempotent $p : \mathcal{A}^N \to \mathcal{E}$ which allows to identify $\mathcal{E} = p\mathcal{A}^N$. Using these maps as well as their natural extension to $\mathcal{E}$-valued forms, on $\mathcal{E}$ a connection ∇_0 is given by the composition

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^p \mathcal{A} \ \stackrel{\lambda}{\longrightarrow} \ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p \mathcal{A} \ \stackrel{\mathbb{I} \otimes d}{\longrightarrow} \ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1} \mathcal{A} \ \stackrel{p}{\longrightarrow} \ \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1} \mathcal{A}$$

(we have also used canonical identifications for the free module). This connection is called the *Levi-Civita* or *Grassmann connection* and is explicitly given by

$$\nabla_0 = p \circ (\mathbb{I} \otimes d) \circ \lambda$$

although one simply indicates it by

$$\nabla_0 = pd. \tag{A.4.3}$$

In fact, the existence of a connection on the module $\mathcal{E}$ is completely equivalent to it being projective [35]. Furthermore, the space $C(\mathcal{E})$ of all connections on $\mathcal{E}$ is an affine space modeled on $\mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. Indeed, if ∇_1, ∇_2 are two connections on $\mathcal{E}$, their difference is $\mathcal{A}$-linear,

$$(\nabla_1 - \nabla_2)(\eta a) = ((\nabla_1 - \nabla_2)(\eta))a \ , \quad \forall\, \eta \in \mathcal{E} \ , \ a \in \mathcal{A} \ ,$$

so that $\nabla_1 - \nabla_2 \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. Thus, any connection can be written as

$$\nabla = pd + \alpha \ , \tag{A.4.4}$$

where α is any element in $\mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. The "matrix of 1-forms" α as in (A.4.4) is called the *gauge potential* of the connection ∇. The corresponding curvature F of ∇ is given by

$$F = pdpdp + pd\alpha + \alpha^2 \ . \tag{A.4.5}$$

Let us now suppose that the algebra $\mathcal{A}$ is involutive with involution denoted by *. We shall also extend this to the whole of $\Omega\mathcal{A}$ by requiring that $(da)^* = da^*$ for any $a \in \mathcal{A}$. A *Hermitian structure* on the module $\mathcal{E}$ is a map $\langle\cdot,\cdot\rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{A}$ with the properties

$$\begin{aligned}
&\langle \eta a, \xi \rangle = a^* \langle \xi, \eta \rangle \ , \\
&\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle \ , \\
&\langle \eta, \eta \rangle \geq 0 \ , \langle \eta, \eta \rangle = 0 \iff \eta = 0 \ ,
\end{aligned}$$

for any $\eta, \xi \in \mathcal{E}$ and $a \in \mathcal{A}$ (an element $a \in \mathcal{A}$ is positive if it is of the form $a = b^* b$ for some $b \in \mathcal{A}$). The Hermitian structure is naturally extended to a linear map from $\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A} \times \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$ to $\Omega\mathcal{A}$ by

$$\langle \eta \otimes_{\mathcal{A}} \omega, \xi \otimes_{\mathcal{A}} \rho \rangle \ = (-1)^{|\eta||\omega|} \omega^* \langle \eta, \xi \rangle \rho \ , \quad \forall\, \eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A} \ , \ \omega, \rho \in \Omega\mathcal{A}. \tag{A.4.6}$$

A connection ∇ on $\mathcal{E}$ and a Hermitian structure $\langle\cdot,\cdot\rangle$ on $\mathcal{E}$ are said to be compatible if the following condition is satisfied [27],

$$\langle \nabla\eta, \xi \rangle + \langle \eta, \nabla\xi \rangle = d \langle \eta, \xi \rangle \ , \quad \forall\, \eta, \xi \in \mathcal{E} \ .$$

It follows directly from the Leibniz rule and (A.4.6) that this extends to

$$\langle \nabla\eta, \xi \rangle + (-1)^{|\eta|} \langle \eta, \nabla\xi \rangle = d \langle \eta, \xi \rangle \ , \quad \forall\, \eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A} \ . \tag{A.4.7}$$

Compatible connections always exist. Indeed, any Hermitian structure on $\mathcal{E} = p\mathcal{A}^N$ can be written as $\langle \eta, \xi \rangle = \sum_{j=1}^{N} \eta_j^* \xi_j$ with $\eta = p\eta = (\eta_1, \cdots, \eta_N)$ and the same for ξ. Then the Grassmann connection (A.4.3) is easily seen to be compatible,

$$d \langle \eta, \xi \rangle = \langle \nabla_0 \eta, \xi \rangle + \langle \eta, \nabla_0 \xi \rangle \ .$$

For a general connection (A.4.4), the compatibility with the Hermitian structure reduces to

$$\langle \alpha\eta, \xi \rangle + \langle \eta, \alpha\xi \rangle = 0 \ , \quad \forall \ \eta, \xi \in \mathcal{E} \ ,$$

which just says that the gauge potential is skew-hermitian,

$$\alpha^* = -\alpha \ .$$

We still use the symbol $\mathsf{C}(\mathcal{E})$ to denote the space of compatible connections on $\mathcal{E}$.

Let $\mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ denote the space of skew-hermitian elements in $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ with respect to the Hermitian structure (A.4.6), i.e. those elements $\mathsf{T} \in \mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ satisfying

$$\langle \mathsf{T}\eta, \xi \rangle + \langle \eta, \mathsf{T}\xi \rangle = 0; \qquad \forall \eta, \xi \in \mathcal{E}.$$

Proposition A.13. *The map* $[\nabla, \ . \ ]$ *defined in* (A.4.2) *restricted to* $\mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ *defines a derivation*

$$[\nabla, \ . \ ] \ : \ \mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \longrightarrow \mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$$

Proof. Suppose that T is an element in $\mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ of order $|\mathsf{T}|$. Since $[\nabla, \mathsf{T}]$ is $\Omega\mathcal{A}$-linear, it is enough to show that

$$\langle [\nabla, \mathsf{T}]\eta, \xi \rangle + \langle \eta, [\nabla, \mathsf{T}]\xi \rangle = 0; \qquad \forall \eta, \xi \in \mathcal{E}.$$

For this, note that T satisfies

$$\langle \mathsf{T}\eta, \xi \rangle + (-1)^{|\eta||\mathsf{T}|} \langle \eta, \mathsf{T}\xi \rangle = 0,$$

for $\eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$. This yields together with equation (A.4.7):

$$\begin{aligned} \langle [\nabla, \mathsf{T}]\eta, \xi \rangle + \langle \eta, [\nabla, \mathsf{T}]\xi \rangle &= \langle \nabla \mathsf{T}\eta, \xi \rangle - (-1)^{|\mathsf{T}|} \langle \mathsf{T}\nabla\eta, \xi \rangle + \langle \eta, \nabla \mathsf{T}\xi \rangle - (-1)^{|\mathsf{T}|} \langle \eta, \mathsf{T}\nabla\xi \rangle \\ &= \langle \nabla \mathsf{T}\eta, \xi \rangle - \langle \nabla\eta, \mathsf{T}\xi \rangle + \langle \eta, \nabla \mathsf{T}\xi \rangle - (-1)^{|\mathsf{T}|} \langle \mathsf{T}\eta, \nabla\xi \rangle \\ &= \mathrm{d}\big(\langle \mathsf{T}\eta, \xi \rangle + \langle \eta, \mathsf{T}\xi \rangle \big) \\ &= 0. \end{aligned}$$

□

A.4.3 Gauge transformations

An $\mathcal{A}$-linear map $\mathsf{T} : \mathcal{E} \to \mathcal{E}$ is said to be adjointable if it admits an adjoint, i.e. there exists an $\mathcal{A}$-linear map $\mathsf{T}^* : \mathcal{E} \to \mathcal{E}$

$$\langle \mathsf{T}^*\eta, \xi \rangle = \langle \eta, \mathsf{T}\xi \rangle \ , \quad \forall \ \eta, \xi \in \mathcal{E} \ .$$

The collection $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of all $\mathcal{A}$-linear adjointable maps is an algebra with involution; its elements are also called endomorphisms of $\mathcal{E}$. The group $\mathcal{U}(\mathcal{E})$ of unitary endomorphisms of $\mathcal{E}$ is given by

$$\mathcal{U}(\mathcal{E}) := \{ u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \mid uu^* = u^*u = \mathrm{id}_{\mathcal{E}} \} \ .$$

This group plays the role of the *infinite dimensional group of gauge transformations*. It naturally acts on compatible connections by

$$(u, \nabla) \mapsto \nabla^u := u^* \nabla u \ , \quad \forall \ u \in \mathcal{U}(\mathcal{E}), \ \nabla \in \mathsf{C}(\mathcal{E}) \ , \tag{A.4.8}$$

where u^* is really $u^* \otimes \mathrm{id}_{\Omega\mathcal{A}}$; this will always be understood in the following. Then the curvature transforms in a covariant way

$$(u, F) \mapsto F^u = u^* F u \ ,$$

since, evidently, $F^u = (\nabla^u)^2 = u^*\nabla u u^* \nabla u^* = u^* \nabla^2 u = u^* F u$.
As for the gauge potential, one has the usual affine transformation

$$(u, \alpha) \mapsto \alpha^u := u^* p d u + u^* \alpha u \ . \tag{A.4.9}$$

Indeed, for any $\eta \in \mathcal{E}$,

$$\begin{aligned} \nabla^u(\eta) &= u^*(pd + \alpha)u\eta = u^* pd(u\eta) + u^*\alpha u \eta \\ &= u^* p u d\eta + u^* p (du)\eta + u^* \alpha u \eta \\ &= pd\eta + (u^* p du + u^* \alpha u)\eta \quad \text{using} \quad up = pu \\ &= (pd + \alpha^u)\eta \ , \end{aligned}$$

which yields (A.4.9) for the transformed potential.

We now describe the 'tangent' of the gauge group $\mathcal{U}(\mathcal{E})$ as the vector space of infinitesimal gauge transformations. For $X \in \mathrm{End}(\mathcal{E})$ we define a family $\{u_t\}_{t\in\mathbb{R}}$ of elements in $\mathcal{U}(\mathcal{E})$ by $u_t = 1 + tX$, so that $X = (\partial u_t / \partial t)_{t=0}$. Unitarity of u_t becomes $(1 + t(X + X^*) + \mathcal{O}(t^2)) = 1$. If we take derivatives with respect to t, putting $t = 0$ afterwards, we find $X = -X^*$. In other words, for u_t to be a gauge transformation, X should be a skew-hermitian endomorphisms of $\mathcal{E}$. In this way, we understand $\mathrm{End}^s_{\mathcal{A}}(\mathcal{E})$ as the set of *infinitesimal gauge transformations*. Note that this is a real vector space, and that its complexification $\mathrm{End}^s_{\mathcal{A}}(\mathcal{E}) \otimes_{\mathbb{R}} \mathbb{C}$ can be identified with $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

The action of an infinitesimal gauge transformation on a connection can be derived as follows. Let the above gauge transformation u_t act on ∇ as in (A.4.8). Since $(\partial(u_t \nabla u_t^*)/\partial t)_{t=0} = [\nabla, X]$, we conclude that an element $X \in \mathrm{End}^s_{\mathcal{A}}(\mathcal{E})$ acts infinitesimally on a connection ∇ by $[\nabla, X]$.

A.5 K-theory of C*-algebras

We briefly review some of the notions of K-theory of C*-algebras, while referring to [10] and [84] for more details.

The group $K_0(\mathcal{A})$ of a unital algebra $\mathcal{A}$ is defined as the so-called Grothendieck group of certain equivalence classes of projections $p = p^2 = p^*$ in $M_\infty(\mathcal{A}) := \bigcup_{n=1}^\infty M_n(\mathcal{A})$. There is the following Murray-von Neumann equivalence relation between two projections $p \in M_n(\mathcal{A})$ and $q \in M_m(\mathcal{A})$:

$p \sim_0 q$ if there exists an $m \times n$ matrix $u \in M_{mn}(\mathcal{A})$ s.t. $p = u^* u, q = u u^*$.

The collection of equivalence classes of projections in $M_\infty(\mathcal{A})$ is an abelian semigroup, with addition given by

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right] .$$

Definition A.14. *The* Grothendieck group $G(S)$ *of an abelian semigroup* $(S, +)$ *is the collection of equivalence classes* $[(x, y)]$ *for* $x, y \in S$ *where* $(x, y) \sim (x', y')$ *if there exists a* $z \in S$ *such that* $x + y' + z = x' + y + z$. *The addition in* $G(S)$ *is induced from* S *by* $(x, y) + (x', y') = (x + x', y + y')$ *and is well-defined, whereas* $0 = [(x, x)]$ *is the identity element.*

Notice that the construction of the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$ is a special case of this construction.

Definition A.15. *The group* $K_0(\mathcal{A})$ *is the Grothendieck group of the abelian semigroup consisting of Murray-von Neumann equivalence classes of projections in* $M_\infty(\mathcal{A})$*.*

It is useful to think of an element in $K_0(\mathcal{A})$ as the formal difference between two Murray-von Neumann classes of projections, $[p]-[q]$. Compare this with the expression of an integer as the difference between two natural numbers.

The group $K_1(\mathcal{A})$ is defined as follows. Let $\mathcal{U}_\infty(\mathcal{A}) = \bigcup_{n=1}^\infty \mathcal{U}_n(\mathcal{A})$ with $\mathcal{U}_n(\mathcal{A})$ the algebra of unitary elements in $M_n(\mathcal{A})$. We define a binary operation $\oplus$ on $\mathcal{U}_\infty(\mathcal{A})$ by

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

Two unitary elements $u \in \mathcal{U}_n(\mathcal{A})$, $v \in \mathcal{U}_m(\mathcal{A})$ are equivalent, write $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus \mathbb{I}_{k-n}$ is homotopy equivalent to $v \oplus \mathbb{I}_{k-m}$ in $\mathcal{U}_k(\mathcal{A})$.

Definition A.16. *The group* $K_1(\mathcal{A})$ *is the group consisting of equivalence classes of unitaries* $u \in \mathcal{U}_\infty(\mathcal{A})$ *with addition given by* $[u] + [v] = [u \oplus v]$*.*

A.6 Cyclic cohomology

We summarize here the cyclic cohomology of an algebra $\mathcal{A}$, referring to [71, 27] for more details.

A cyclic n-cochain on an algebra $\mathcal{A}$ is an element $\varphi \in C^n_\lambda(\mathcal{A})$, the collection of $(n+1)$-linear functionals on $\mathcal{A}$ which in addition are cyclic, $\lambda\varphi = \varphi$, with

$$\lambda\varphi(a_0, a_1, \dots, a_n) = (-1)^n \varphi(a_n, a_0, \dots, a_{n-1}).$$

There is a cochain complex $(C^\bullet_\lambda(\mathcal{A}) = \bigoplus_n C^n_\lambda(\mathcal{A}), b)$ with (Hochschild) coboundary operator $b\colon C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ defined by

$$b\varphi(a_0, a_1, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \varphi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \dots, a_n).$$

The cyclic cohomology $HC^\bullet(\mathcal{A})$ of the algebra $\mathcal{A}$ is the cohomology of this complex,

$$HC^n(\mathcal{A}) := H^n(C^\bullet_\lambda(\mathcal{A}), b).$$

Equivalently, $HC^\bullet(\mathcal{A})$ can be described [27, 49] by using the second filtration of a (b, B) bicomplex of arbitrary (i.e., noncyclic) cochains on $\mathcal{A}$. Here the operator B decreases the degree $B\colon C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$, and is defined as $B = NB_0$, with

$$(B_0\varphi)(a_0, \dots, a_{n-1}) := \varphi(1, a_0, \dots, a_{n-1}) - (-1)^n \varphi(a_0, \dots, a_{n-1}, 1)$$

$$(N\psi)(a_0, \dots, a_{n-1}) := \sum_{j=0}^{n-1} (-1)^{(n-1)j} \psi(a_j, \dots, a_{n-1}, a_0, \dots, a_{j-1}).$$

It is straightforward to check that $B^2 = 0$ and that $bB + Bb = 0$; thus $(b+B)^2 = 0$. By putting together these two operators, one gets a bicomplex $(C^\bullet(\mathcal{A}), b, B)$ with $C^{p-q}(\mathcal{A})$ in bidegree

(p, q). To a cyclic n-cocycle one associates the (b, B) cocycle φ, $(b + B)\varphi = 0$, having only one nonvanishing component $\varphi_{n,0}$ given by $\varphi_{n,0} := (-1)^{\lfloor n/2 \rfloor}\psi$.

Dually, there is the following construction of *cyclic homology.* We let $C_\bullet(\mathcal{A})$ be the complex consisting of chains over the algebra $\mathcal{A}$, that is in degree n, $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$. There is the obvious pairing between $C^\bullet(\mathcal{A})$ and $C_\bullet(\mathcal{A})$ and one defines the Hochschild operator $b : C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A})$ and the boundary operator $B : C_n(\mathcal{A}) \to C_{n+1}(\mathcal{A})$, by dualizing the above maps on $C^\bullet(\mathcal{A})$. They satisfy $b^2 = 0, B^2 = 0, bB + Bb = 0$; thus $(b + B)^2 = 0$. We define the cyclic homology $HC^\bullet(\mathcal{A})$ as the second filtration of the (b, B) bicomplex of chains.

A.6.1 Pairing of cyclic cohomology with K-theory

There is a pairing between the cyclic cohomology and the K-theory of an algebra $\mathcal{A}$ [27]. For $p \in K_0(\mathcal{A})$, the Chern character is defined as an even cycle $\mathrm{ch}(p) = \sum_k \mathrm{ch}_k(p)$ with for $k = 0$,

$$\mathrm{ch}_0(p) := \mathrm{tr}(p);$$

whereas for $k \neq 0$

$$\mathrm{ch}_k(p) := (-1)^k \frac{(2k)!}{k!} \sum (p_{i_0 i_1} - \frac{1}{2}\delta_{i_0 i_1}) \otimes p_{i_1 i_2} \otimes p_{i_1 i_2} \otimes \cdots \otimes p_{i_{2k} i_0}.$$

The pairing between $K_0(\mathcal{A})$ and cyclic cohomology is then given by the natural pairing between (even) cycles and cocycles:

$$\langle [\phi^{even}], [p] \rangle = \sum_k \phi^{2k}(\mathrm{ch}_k(p)).$$

Note that due to the isomorphism (A.3.1), the Chern character can also be defined by

$$\mathrm{ch}_k(p) := (-1)^k \frac{(2k)!}{k!} \langle (p - \frac{1}{2})(\delta p)^{2k} \rangle \in \Omega^{2k}_{\mathrm{un}}(\mathcal{A}).$$

In the odd case, we have the following definition of the Chern character. For $u \in K_1(\mathcal{A})$ we define an odd cycle $\mathrm{ch}(u) = \sum_k \mathrm{ch}_k(u)$ with

$$\mathrm{ch}_k(u) := \sum_{k \geq 0} (-1)^k k! u^{-1}_{i_0 i_1} \otimes u_{i_1 i_2} \otimes \cdots \otimes u^{-1}_{i_{2k} i_{2k+1}} \otimes u_{i_{2k+1} i_0}.$$

The pairing between $K_1(\mathcal{A})$ and cyclic cohomology is then given as

$$\langle [\phi^{odd}], [u] \rangle = \sum_k \phi^{2k+1}(\mathrm{ch}_k(u))$$

A.7 The local index formula of Connes and Moscovici

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple, as defined above. The local index formula of Connes and Moscovici [34] expresses the index of twisted Dirac operators in terms of cocycles in the (b, B) bicomplex, which are easier to compute. Recall that a Fredholm operator in a Hilbert space is an operator with a finite dimensional kernel and cokernel.

We will be interested in the indices of the following two Fredholm operators. Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is even. If $p \in M_N(\mathcal{A})$ is a projection (so that $[p] \in K_0(\mathcal{A})$), then the operator $D_p = p(D \otimes \mathbb{I}_N)p$ is a Fredholm operator on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^N$. This follows from the fact that D_p is essentially a finite dimensional extension of the elliptic operator D. We are interested in the index of this so-called twisted Dirac operator D_p.

Theorem A.17 (even case). *(a) An even cocycle* $\phi^{\text{even}} = \sum_{k\geq 0} \phi^{2k}$ *in the* (b, B) *bicomplex of* $\mathcal{A}$, *is defined by the following formulæ. For* $k = 0$,

$$\phi^0(a) := \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}(\gamma a |D|^{-2z});$$

whereas for $k \neq 0$

$$\phi^{2k}(a^0, \ldots, a^{2k}) := \sum_\alpha c_{k,\alpha} \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma a^0 [D, a^1]^{(\alpha_1)} \cdots [D, a^{2k}]^{(\alpha_{2k})} |D|^{-2(|\alpha|+k+z)}\right) \quad \text{(A.7.1)}$$

where

$$c_{k,\alpha} = (-1)^{|\alpha|}\Gamma(k + |\alpha|)\big(\alpha!(\alpha_1 + 1)(\alpha_1 + \alpha_2 + 2)\cdots(\alpha_1 + \cdots + \alpha_{2k} + 2k)\big)^{-1}$$

and $T^{(j)}$ *denotes the j'th iteration of the derivation* $T \mapsto [D^2, T]$.

(b) The index is given by the natural pairing between cyclic cohomology and K-theory:

$$\text{Index } D_p = \langle [\phi^{\text{even}}], [p] \rangle.$$

In the case that $(\mathcal{A}, \mathcal{H}, D)$ is an odd spectral triple, we take a unitary $u \in M_N(\mathcal{A})$ (defining an element $[u] \in K_1(\mathcal{A})$) and define $D_u = (P \otimes I_N)u(P \otimes \mathbb{I}_N)$, where $P = \frac{1}{2}(1 + \mathrm{Sign} D)$. Again, D_u is a Fredholm operator on $\mathcal{H} \otimes \mathbb{I}_N$ and we are interested in the index of D_u.

Theorem A.18 (odd case). *(a) An odd cocycle* $\phi^{\text{odd}} = \sum_{k\geq 0} \phi^{2k+1}$ *in the* (b, B) *bicomplex of* $\mathcal{A}$, *is defined by the following formulæ:*

$$\phi^{2k+1}(a^0, \ldots, a^{2k+1}) := \sum_\alpha c_{k,\alpha} \operatorname*{Res}_{z=0} \operatorname{tr}\left(a^0 [D, a^1]^{(\alpha_1)} \cdots [D, a^{2k+1}]^{(\alpha_{2k+1})} |D|^{-2(|\alpha|+k+z)+1}\right)$$

where

$$c_{k,\alpha} = (-1)^{|\alpha|}\Gamma(k + |\alpha| + \tfrac{1}{2})\big(\alpha!(\alpha_1 + 1)(\alpha_1 + \alpha_2 + 2)\cdots(\alpha_1 + \cdots + \alpha_{2k+1} + 2k + 1)\big)^{-1}$$

and $T^{(j)}$ *denotes the j'th iteration of the derivation* $T \mapsto [D^2, T]$.

(b) The index is given by the natural pairing between cyclic cohomology and K-theory:

$$\text{Index } D_u = \langle [\phi^{\text{odd}}], [u] \rangle.$$

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