Università di Pisa

Ph. D. Program in Mathematics
XXVIII cycle

Ph. D. Thesis

Perturbed Hamiltonians in one dimension: analysis for linear and nonlinear Schrödinger problems

By

Anna Rita Giammetta

Supervisor: Prof. Vladimir Georgiev

Coordinator: Prof. Rita Pardini

Academic Year 2015/2016

Introduction

Partial differential equations (PDEs) are mathematical equations that involve an unknown function ψ depending on different independent variables, for instance the time variable $t \in \mathbb{R}$ and the spacial variable $x \in \mathbb{R}$, and also its partial derivatives $(\partial_t \psi,\ \partial_x \psi,\ \partial_{xx}\psi,\ \dots)$. Nevertheless the first trace of partial differential process dates back to 17th century ([9]), the real study of PDEs dates from the 18th century with the works of Euler, d'Alambert, Lagrange and Laplace with applications on the mechanics of continua. Through the 19th and the 20th century, thanks to the advance of the functional analysis and the theory of the self-adjoint linear operators due to Fourier, Poincaré, Hilbert, Banach and Sobolev, the analysis of PDEs played an important role also within mathematics itself. In particular, this mathematical setting allows to reformulate many propagation phenomena in physics, chemistry, biology in the common abstract language of theoretic operators. In [8] one can find an excursus on the development of the PDEs through 18th and 20th century (models connecting with PDEs, methods and technique developed to solve problems in PDEs field).

In this work we are interested into the analysis of a particular class of PDEs, the *dispersive* PDEs. These equations were introduced to model waves propagation affected by dispersive phenomena, i.e. the solution (wave phenomenon) spreads out spatially over time when no boundary conditions are imposed ([67], [66]). Let ψ be a complex valued function. A linear dispersive PDE in one dimension (for the sake of simplicity) has the following structure

$$\partial_t \psi(t,x) = iP(-i\partial_x)\psi(t,x),\ (t,x) \in \mathbb{R}\times\mathbb{R}$$

where $P(-i\partial_x)$ is a linear differential operator with symbol $-P(\xi)$. The dispersive character of the equation is captured in the relation $\tau = P(\xi)$, where τ is the time Fourier variable. If such a relation is nonlinear the equation is said dispersive.

The linear Schrödinger (LS) equation

$$i\partial_t\psi - \mathcal{H}_0\psi = 0,\ \psi(0) = \psi_0,\ (t,x) \in \mathbb{R}\times\mathbb{R}^n,$$

is the classical model to explain and to study the dispersive equations. Here $\mathcal{H}_0 = -\Delta$ denotes the Laplace operator on $\mathbb{R}^n$. This equation is the quantum mechanics model to describe the evolution of a free particle in a non relativistic regime when the system is initially prepared in the state ψ_0. The study of the linear evolution is strictly connected with the analysis of the free Schrödinger operator $\mathcal{H}_0$ and the free Schrödinger propagator $e^{-it\mathcal{H}_0}$. It is well known that $\mathcal{H}_0$ admits a self-adjoint realization in $L^2(\mathbb{R})$ and hence the Stone's theorem guarantees that the unique solution is given by $e^{-it\mathcal{H}_0}\psi_0$. Moreover, since we are on the whole space, the natural tool to study the propagator is the Fourier analysis. From the kernel expression of the propagator $e^{-it\mathcal{H}_0}$

$$\frac{1}{(4\pi i t)^{n/2}} e^{-\frac{|x-y|^2}{4it}}$$

we deduce the dispersive estimate

$$\left\|e^{-it\mathcal{H}_0}\psi_0\right\|_{L^\infty(\mathbb{R}^n)} \leq C\frac{1}{t^{n/2}}\|\psi_0\|_{L^1(\mathbb{R}^n)},\ \psi_0 \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n).$$

Interpolating this decay with the L^2-conservation law we get the $L^p - L^{p'}$ dispersive estimates

$$\left\|e^{-it\mathcal{H}_0}\psi_0\right\|_{L^p(\mathbb{R}^n)} \leq C\frac{1}{t^{n(1/p-1/2)}}\|\psi_0\|_{L^{p'}(\mathbb{R}^n)}.$$

The Strichartz estimates follow from dispersive estimates via TT^* argument (non endpoint case)

$$\left\|e^{-it\mathcal{H}_0}\psi_0\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \leq C\|\psi_0\|_{L^2(\mathbb{R}^n)},\ 2 \leq r \leq \infty,$$

where the pair (q, r) is admissible with respect to the scaling of the equation. The Strichartz estimates are a fundamental tool to solve the nonlinear equation (one can see Section 2 in [13] for a complete treatment of LS equation and we quote [16] for a recent survey on Strichartz estimates for the Schrödinger equation).

Recently many authors studied potential type perturbation of the free Hamiltonian (one can see [56], [43], [42], [44], [74], [75], [76], [71], [72], [77], [18], [17]). These kind of equations occur to model the evolution of the quantum particle in presence of a potential V. Mathematically, this means that in the model equation, the free Hamiltonian $\mathcal{H}_0$ is replaced by a perturbed one, $\mathcal{H} = \mathcal{H}_0 + V$, and hence the equation becomes

$$i\partial_t \psi - \mathcal{H}\psi = 0, \;\; \psi(0, x) = \psi_0,$$

where V is a real valued function. In the following we will call the equation above the perturbed LS (LSP) equation. An important point is to understand whether or not the decay and Strichartz estimates can be extended to the LSP problem. We can reasonably expect that certain conditions on spectral scenario and decay of the potential have to be considered. Here we shall recall some standard relevant assumptions. A significant decay restriction required is $V \in L^1_\gamma(\mathbb{R}^n)$ (short range perturbation) where

$$L^1_\gamma(\mathbb{R}^n) = \left\{ f \mid \int (\sqrt{1+|x|^2})^\gamma |f(x)|\, dx < \infty \right\}, \;\; \gamma \geq 1.$$

This decay guarantees that the operator $\mathcal{H}$ admits a self-adjoint realization in $L^2(\mathbb{R})$, moreover the absolutely continuous spectrum of $\mathcal{H}$ is $[0, \infty)$ and its point spectrum consists of a finite number of negative eigenvalues. We immediately note that, if possibly bound states are presents, it is necessary to remove them considering the ortogonal projection onto the continuous subspace of $\mathcal{H}$, $P_{ac}(\mathcal{H})$. Indeed, the bound states generate stationary solutions that cannot verify any decay estimates. Finally, from the spectral view point, we shall also require that the perturbed Hamiltonian $\mathcal{H}$ has no resonances, since a resonance state can be interpreted as an approximation in a suitable norm of a bound state.

The problem to generalize the dispersive estimates for potentials contained in the above specified class was extensively studied in any dimension [44], [74], [75] ($n \geq 3$), [76] ($n = 2$), [1], [71], [72] ($n = 1$) (one can see [62] for a survey on the main techniques used under suitable regularity and decay assumptions on the potential in different dimensions).

In this thesis we are interested in the one dimensional case: $\mathcal{H}_0 = -\partial_x^2$ and $\mathcal{H} = -\partial_x^2 + V(x)$, where $x \in \mathbb{R}$. First of all we observe that in a perturbative regime the Fourier analysis is naturally replaced by the spectral theory. As we can see in [72], [71], [1], [18], the main goal is the analysis of the kernel of the perturbed resolvent $(\tau^2 - \mathcal{H})^{-1}$ in terms of the spectral measure. Indeed, in one dimension, we have the following explicit formula for the resolvent

$$(\tau^2 - \mathcal{H})^{-1}(x, y) = \frac{T(\tau)}{2i\tau} m_-(x, \tau) m_+(y, \tau) e^{-i\tau(x-y)}, \;\; x < y,$$

in terms of the modified Just functions $m_\pm(x, \tau)$ and of the transmission coefficient $T(\tau)$. Here the modified Jost functions are defined as

$$m_\pm(x, \tau) = e^{\mp i x \tau} f_\pm(x, \tau),$$

$$-\frac{d^2}{dx^2} f_\pm + V f_\pm = \tau^2 f_\pm, \quad f_\pm(x, \tau) \approx e^{\pm i x \tau} \text{ as } x \to \pm\infty,$$

$T(\tau) f_+(x, \tau)$ describes the incoming plane wave sent from $-\infty$ and $T(\tau) f_-(x, \tau)$ describes the incoming plane wave sent from $+\infty$ (one can see [18] or Chapter 2 of this thesis for the rigorous definition of $T(\tau)$). In [1] the problem of establishing decay estimates for the perturbed problem is reduced to the problem to prove the L^p-continuity of the wave operators $W_\pm$. Here the wave operators are defined by the strong limits

$$W_\pm = s - \lim_{t \to \pm\infty} P_{ac}(\mathcal{H}) e^{it\mathcal{H}} e^{-it\mathcal{H}_0}$$

and they intertwine $\mathcal{H}_0$ and the absolutely continuous part of $\mathcal{H}$, $P_{ac}(\mathcal{H})\mathcal{H}$, via the following relation

$$W_\pm f(\mathcal{H}_0) W_\pm^* = f(\mathcal{H}) P_{ac}(\mathcal{H}), \;\; f \in L^\infty_{loc}(\mathbb{R}^n).$$

In [1] it is proved that if the potential $V \in L^1_3(\mathbb{R})$ and $T(0) = 0$ (zero is not a resonance) the wave operators $W_\pm$ are bounded in L^p for $1 < p < \infty$. The proof is based on the resolvent estimates obtained by standard properties of the modified Jost functions and of the transmission coefficient ([19]) combined with the explicit representation of the perturbed resolvent kernel. This result is improved first in [72] ($V \in L^1_\gamma(\mathbb{R})$, $\gamma > 3/2$) and then in [18] ($V \in L^1_1(\mathbb{R})$). The more relaxed hypotheses on the decay of the potential follows improving the results in [19] and using Fourier analysis arguments. We note that the L^p-continuity of the wave operators and the intertwining property immediately imply the decay estimates for LSP. Similar estimates are also proved in Sobolev spaces $W^{k,p}(\mathbb{R})$ [71] using additional decay assumptions on the weak $k-$derivatives of the potential.

Most of this analysis is motivated by the interest in understanding the asymptotic behaviour for the nonlinear equation. In this respect we introduce the nonlinear problem. The nonlinear Schrödinger equation (NLS)

$$i\partial_t\psi - \mathcal{H}_0\psi = \pm|\psi|^{p-1}\psi,\ \psi(0) = \psi_0,$$

with $p > 1$, is one of the universal model used to describe the evolution of a wave packet in a weakly nonlinear dispersive media. Here we focus our analysis on the cubic NLS ($p = 3$) in one dimension. The cubic nonlinearity comes from by a simplification process in the description of the N-particles Schrödinger equation when the interaction potential behaves like a δ function ([66], [51]). This equation occurs to model several phenomena in quantum optics. Special attention is also paid to the perturbed NLS (NLSP)

$$i\partial_t\psi - \mathcal{H}\psi = \pm|\psi|^2\psi,\ \psi(0) = \psi_0,$$

where the potential V is a small perturbation in the sense above specified. Actually we can work with more general nonlinearities but here, for the sake of simplicity, we consider just the pure power nonlinearity.

For cubic NLS in one dimension in absence of potential, $V = 0$, global well posedness results in $H^s(\mathbb{R})$ for small data solutions have been established for any $s \geq 0$. The case $s > 1/2$ follows from fixed point argument combined with the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, while for $s \leq 1/2$ it is crucial to use the Strichartz estimates in the auxiliary Sobolev ($H^s_p(\mathbb{R})$) and Besov ($B^s_p(\mathbb{R})$) spaces. Similarly, if the potential V satisfies certain decay and spectral assumptions, the generalization of Strichartz estimates for the perturbed propagator allows to get the analogous well posedness results for NLSP (one can see Chapter 4 in [13]).

Once the solution exists globally in time, it is natural to ask whether or not this solution has a linear behaviour. So, the first problem to address to have answers in this direction is to understand if the solution ψ of NLS (or NLSP) satisfies the decay estimate

$$\|\Psi(t)\|_{L^\infty(\mathbb{R})} \leq C\frac{1}{t^{1/2}}\|\psi_0\|_{H^s(\mathbb{R})},$$

for small initial data. The problem of establishing the decay estimate and scattering (existence of solutions that for large time behave like the solution of LS) for NLS has a large amount of literature [54], [64], [5], [55], [38]. What it is proved is that for powers $3 < p < 5$ the dispersive effect dominates the equation and the solutions behave like the free ones. By contrast, in the case $1 < p \leq 3$ the nonlinearity modifies the asymptotics of the solutions. Indeed, in this last case the zero solution represents the unique solution asymptotically free. Despite this result, in the cubic case, the dispersive estimate holds and it is possible to construct a modified scattering profile that takes into account the long range interaction and describes the asymptotics of the solutions. For this reason the cubic power is called *critical* for the scattering. In particular, the decay estimate is proved for cubic NLS for small initial data in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ ([55]) and in weighted Sobolev spaces $H^{s,s}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2(\mathbb{R}, \langle x\rangle^s\, dx)$ for $s > 1/2$ ([38]). Their approach is based on the use of the generators of the pseudoconformal transform.

Analogous problems for the NLSP equation have also been recently studied [10], [11], [17], [12], [28]. In particular, in [10] and [11] local and global well posedness for NLSP equation are discussed for smooth and possibly unbounded potential in various weighted Sobolev spaces. The problem to generalize decay estimates and scattering results for one dimensional NLSP is addressed in [17]. The authors adapt the approach used in [54] to get analogous results for the perturbed equation. A scattering subcritical nonlinearity is considered, i.e. $3 < p < 5$. It is shown that for potential V sufficiently smooth, such that $\sigma(\mathcal{H}) = [0, +\infty)$ and for $\mathcal{H}$ without zero resonance ($T(0) = 0$), the NLSP problem is globally well posed in $H^{s,s}(\mathbb{R})$ for $s > 1/2$ and for small data solutions. Moreover, it is proved that the solutions have a free asymptotic profile. The main point in this work is to prove the equivalence between classical homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$ and the ones generated by the perturbed Hamiltonian, $\dot{H}^s_V(\mathbb{R})$, when $0 \leq s < 1/2$. To this end the analysis of the Paley-Littlewood operators $\varphi(\sqrt{\mathcal{H}})$ and the condition of no zero resonance in terms of the transmission coefficient will play a fundamental role. From the equivalence of the Sobolev spaces $\dot{H}^s(\mathbb{R})$ and $\dot{H}^s_V(\mathbb{R})$ it is possible to deduce the fractional Leibniz rule for the perturbed Hamiltonian. This tool will be essential to estimate the nonlinear terms.

The aim of this thesis is to present some results concerning with questions related on one side to LSP, we want to study the continuity of wave operators in homogeneous Besov and Sobolev spaces, and, on the other side, concerning with NLSP, we want to prove well posedness and decay estimates in the critical case.

The thesis is divided into three main parts. The first part (Chapters 1, 2, 3) discusses problem strictly connected with the perturbed Hamiltonian and with the LSP: The sectorial properties and the spectral scenario (modes and resonances) of the perturbed Hamiltonian are studied. Motivated by the works [72], [1], [18] we study the continuity of the wave operators in homogeneous Sobolev and Besov spaces. The second part (Chapter 4) examines one dimensional NLSP with cubic nonlinearity. Motivated by the analysis in [17] the main goal is to generalize the results obtained in [55] and [38] in presence of short range potentials. The last part (Appendix A, B, C) is a collection of some fundamental tools and known results used in the thesis. Moreover all the spaces introduced in the thesis are defined. In order to keep the presentation of the material short and readable, the third part is not fully rigorous.

In the following we give a more detailed description of the contents in the various chapters.

The first chapter is devoted to the study of the sectorial properties of the perturbed Hamiltonian, $\mathcal{H} = \mathcal{H}_0 + V$, when the potential $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1$ and the Hamiltonian $\mathcal{H}$ has neither point spectrum, (1.3.2) nor zero resonance (one can see Definition 1.1.7 for the free case, Definition 1.3.6 for the perturbed one and Proposition 1.3.10 for a characterization). It is shown that these assumptions guarantee that the perturbed Hamiltonian $\mathcal{H}$ is a sectorial operator in L^p spaces with $1 < p \leq \infty$ (Theorem 1.4.6). In particular this property allows to define the fractional powers of the perturbed Hamiltonian, $\mathcal{H}^{-\alpha}$, $\alpha \in (0, 1)$ by means of the Balakrishnan representation (1.4.15). The chapter is organized as follows: In Section 1.1 we briefly recall some classical arguments on functional calculus, zero resonance and sectorial properties for the Laplace operator $\mathcal{H}_0$. In Section 1.2 we collect some known results to understand the spectral scenario of the Hamiltonian $\mathcal{H}$ in presence of short range perturbations. In Section 1.3 we introduce the notion of no zero resonance for the perturbed Hamiltonian in terms of poles of the resolvent or equivalently in terms of L^∞ solutions of the equation $\mathcal{H}u = 0$ (see Proposition 1.3.10). Finally, in the last section we establish L^p estimates for the resolvent of the perturbed Hamiltonian. The main result of the chapter is summarized in the statement of the Theorem 1.4.6.

Chapter 2 considers the case of perturbed Hamiltonian $\mathcal{H}$ on the real line, where the potential V verifies the decay hypothesis $V \in L^1_\gamma(\mathbb{R})$ and neither modes nor resonances are allowed for $\mathcal{H}$. The main goal is to study how the classical homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$, $s \in (0, 1)$, $1 < p < \infty$, are transformed under the action of the wave operators. In particular we show that, under the necessary restriction $s < n/p$ (in our case we consider $n = 1$), the homogeneous spaces generated from the perturbed Hamiltonian are equivalent to the classical ones if $V \in L^1_\gamma(\mathbb{R})$ with $\gamma > 1 + 1/p$. The plan of the chapter is the following: In Section 2.1 we first recall some notions and estimates concerning with the Jost functions, the transmission coefficient and the reflection coefficients ([72], [19]). Then we prove some improved estimates for the functions just mentioned (one can see Lemma 2.1.3 and Lemma 2.1.9). These estimates will turn to be crucial in order to establish the equivalence of the norms. To this argument is devoted the Section 2.2. At first we provide a counterexample to show that the requirement $s < n/p$ is necessary to guarantee the equivalence of the norms. Then, the functional calculus for the perturbed Hamiltonian combined with the improved estimates got for the Jost functions, the transmission and the reflection coefficients allow us to get appropriate kernel estimates of the operator $\varphi(\sqrt{\mathcal{H}}/2^j)$, where $j \in \mathbb{Z}$ and φ is the Paley-Littlewood localization function. The equivalence of the norms is established in Theorem 2.2.4 and as an immediate consequence we have the continuity of the wave operators in homogeneous Besov spaces (Corollary 2.2.5). From the continuity of the wave operators and the splitting property (2.2.1) we also get the Strichartz estimates in $\dot{B}^s_p(\mathbb{R})$ for the perturbed Schrödinger propagator.

The Chapter 3 follows the same spirit of the previous chapter replacing the homogeneous Besov spaces with the Sobolev ones. In this chapter we remove the technical assumption that the Hamiltonian $\mathcal{H}$ has no modes. This requires to work with the absolutely continuous part of the Hamiltonian $\mathcal{H}_{ac} = P_{ac}(\mathcal{H})\mathcal{H}$. The main goal is to show the equivalence of the homogeneous Sobolev spaces generated by the Hamiltonian $\mathcal{H}_{ac}$ and the classical ones $\dot{H}^s_p(\mathbb{R})$. This result is established in Theorem 3.1.2 under the conditions $s < 1/p$ and $V \in L^1_\gamma(\mathbb{R})$ with $\gamma = 1+s$. From the equivalence of the norms we can deduce the Hardy inequality for the perturbed Hamiltonian $\mathcal{H}_{ac}$ (see inequality (3.1.2)), the fractional Leibnitz rule (see Corollary 3.1.6) and the continuity of the wave operators in these spaces. Here we outline the contents of the various sections: In Section 3.1 and Section 3.2 the study of the perturbed Sobolev spaces is motivated and the main results are exposed. The Section 3.3 and Section 3.4 contain the proof of the main results of the chapter. Finally, in Section 3.5 a counterexample of the equivalence of the norms in the case $s = n/p$ is given.

In the last chapter we study the asymptotic behaviour of the solutions of perturbed NLS with cubic nonlinearity on the real line

$$i\partial_t\psi - \mathcal{H}\psi = \pm\psi|\psi|^2.$$

The presence of the potential V breaks the symmetries of the equation. In order to preserve at least the reflection symmetry we consider even real-valued potential and odd initial data. Finally, we suppose that $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, and that the Hamiltonian $\mathcal{H}$ has nor modes neither resonances. Since to characterize the potential without zero resonance is rather tricky, inspired by the work [45], we require that a small perturbation of the potential V is in the image of the Miura map (see relation (4.1.12)) and that $V \neq 0$ almost everywhere. In particular these last assumptions guarantee the absence of the resonance at zero. The assumptions specified above and the additional assumption of smallness on initial data guarantee the globally well-posedness of the NLSP problem in exams in weighted Sobolev spaces $H^{s,s}(\mathbb{R})$, when $1/2 < s < 3/4$. Moreover it is shown that the solution verifies the L^∞ decay estimates (one can see Theorem 4.1.1). The sections are organised as follows: In Section 4.1 we expose the motivations and the state of the art of the problem. Then we discuss in details the assumptions for the potential and for the initial data. Finally we state the main result (Theorem 4.1.1) and we sketch the fundamental ideas to prove it. The structure of the potentials connected with the Miura map is analysed in Section 4.2. In particular we show that if a small perturbation of the potential V lives in the image of the Miura map (see (4.1.12)) and $V \neq 0$ a. e. then the perturbed Hamiltonian $\mathcal{H}$ has not zero resonance (Lemma 4.2.2). Moreover, these information are crucial to establish the equivalence of the standard Sobolev spaces $H^1(\mathbb{R})$ and the perturbed ones $H^1_V(\mathbb{R})$ when we consider their restrictions on the odd functions (see Lemma 4.2.3). The Section 4.3 and Section 4.4 are addressed to the construction of the modified scattering profile for the perturbed problem. Indeed, to prove the main Theorem 4.1.1 we first transform the original global problem (see (4.1.15)) into a new local one (see (4.1.18)) by means of the pseudoconformal transformation. Then we construct the modified scattering profile using an approach based on the two parameters groups, similar groups (see Definition 4.4.1) and splitting generators (see Definition 4.4.2). Once we have the expression of the solutions of the perturbed problem (see (4.4.11)) in terms of the two parameters groups, the fundamental step will be to define a leading part for the scattering profile ((4.4.16)) and then prove a priori bounds for the leading term and the remainder. In Section 4.5 we establish the equivalence between the classical Sobolev norms and the ones generated by the generator of the two parameters group (see Lemma 4.5.2). The Section 4.5 and Section 4.6 are devoted to establish a priori estimates for the two parameters group. The fundamental result of this section is Lemma 4.6.2. The estimates established are the keys to control the H^s and L^∞ norms for the leading and the remainder term of the scattering modified profile. These estimates and the complete proof of the Theorem 4.1.1 can be found in Section 4.7.

Finally, in the Appendix A we first define the functional spaces and the relative norms used in this work. Then we recall classical Sobolev embeddings. In Appendix B the standard well-posedness results for NLS is briefly exposed. In Appendix C we prove some modification of the Gronwall inequalities on the real line. For the convenience of an inexpert reader we suggest to find relevant and exaustive material from the references quoted in the appendices.

For a good understanding of this work it is required a knowledge of classical functional analysis and basic PDEs theory.

Acknowledgements

The author would like to sincerely thank her advisor, Prof. Vladimir Georgiev, for suggesting the problems examined in the Ph.D. thesis and for having introduced her in many stimulating conversations with several experts in the PDEs field. In particular the author is grateful to Prof. Atanas Stefanov for the collaboration in the preparation of papers connected with this thesis. Moreover, it is a pleasure to thank Prof. Svetlana Roudenko and Prof. Nicola Visciglia for the interesting discussions concerning some questions addressed in this work. Many thanks are due to the anonymous referees for their critical remarks and for their valuable modifications suggested.

The author would also like to thank the coordinators of the Ph.D. program, Prof. Fabrizio Broglia and Prof. Rita Pardini, her advisor Prof. Vladimir Georgiev, for having authorised the financial supports in order to participate to schools, workshops and scientific programs abroad. All these experiences have contributed to a deeper professional (and not only professional) growth of the author.

The author would like to express particular gratitude to Ph.D. Daniele Angella for having enthusiastically (i.e. with *priscio*) involved her into the popularization of mathematics and to Prof. Giuseppe Buttazzo for having introduced her in the teaching of mathematics and for having shared with her his illuminating perspectives in this direction. The author is greatly indebted to Ph.D. Sandra Lucente for having strongly encouraged her to apply for the Ph.D. in mathematics.

Finally, the author would like to warmly thank her family, Giuseppe (the *Negro*), all the colleagues and the friends for having supported her with *love*, *pizza* and *very expensive ice creams* during the Ph.D. years.

Contents

Chapter 1

Sectorial Hamiltonians without zero resonance in one dimension

In this chapter we recall the concept of resonance and the sectorial estimates for the free Laplacian in one dimension. Then we introduce the notion of resonance for the one dimensional Laplace operator perturbed with a short range potential $V \in L^1_a(\mathbb{R})$, $a > 1$. We prove that the absence of eigenvalues and resonances for the perturbed Hamiltonian $\mathcal{H} = -\partial_x^2 + V$, combining with the decay hypothesis $V \in L^1_a(\mathbb{R})$, guarantee that the perturbed Hamiltonian $\mathcal{H}$ is a sectorial operator in $L^p(\mathbb{R})$, with $1 < p \leq \infty$.

1.1 Laplace operator in one dimension

In this Section we collect some known results for Laplace operator on the real line. One can find a detailed treatment in [58] for the Spectral Theorem and in [39] for the Sectorial properties.

1.1.1 Functional calculus and spectral measure for Laplace operator

We consider the operator $\mathcal{H}_0 = -\partial_x^2$ with domain $D(\mathcal{H}_0)$ the Schwartz functions $\mathcal{S}(\mathbb{R})$. The Spectral Theorem for unbounded self-adjoint operator on Hilbert spaces applied to the case of free Laplacian establishes that the operator $(\mathcal{H}_0, D(\mathcal{H}_0))$ is essentially self-adjoint with respect to the standard scalar product of $L^2(\mathbb{R})$ and by means of the Fourier transform

$$\mathcal{F} \colon L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_\xi),$$

$$\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\psi(x)\,dx,\ \psi \in L^2(\mathbb{R}),$$

it is unitary equivalent to the multiplication operator $M_{|\cdot|^2}$ on $L^2(\mathbb{R}_\xi, d\xi)$, where $M_{|\cdot|^2}\phi(\xi) = |\xi|^2\phi(\xi)$, for any $\phi \in L^2(\mathbb{R}_\xi, d\xi)$. Moreover, given f a bounded borel function, $f\colon \mathbb{R}_\xi \to \mathbb{R}$ defined almost everywhere, we can define the functional calculus as follows

$$f(\mathcal{H}_0)\psi = \mathcal{F}^{-1}M_{f(|\xi|^2)}\mathcal{F}\psi,\ \psi \in L^2(\mathbb{R}).$$

The functional calculus allows to build the spectral measure and express the Schrödinger group $e^{it\mathcal{H}_0}$ in terms of it. The Spectral Theorem in the projection valued measure form applied to the operator $(\mathcal{H}_0, D(\mathcal{H}_0))$ establishes that:

Theorem 1.1.1. *The operator $(\mathcal{H}_0, D(\mathcal{H}_0))$ is essentially self-adjoint with respect to the standard scalar product of $L^2(\mathbb{R})$. Let $B(\mathbb{R})$ denote the Borel sets on $\mathbb{R}$, and let $\mathcal{L}(L^2(\mathbb{R}))$ be the set of the bounded operators on $L^2(\mathbb{R})$. Hence, there exists a unique projection valued measure (PVM)*

$$E\colon B(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R})),$$

such that the following correspondence is verified:

$$\mathcal{H}_0 = \int_{\mathbb{R}} \lambda dE(\lambda).$$

Moreover, if $f \in \mathcal{B}(\mathbb{R})$, i.e. if f is a bounded Borel function, we have that

$$f(\mathcal{H}_0) = \int_{\mathbb{R}} f(\lambda)dE(\lambda).$$

Remark 1.1.2. Fixed an initial configuration $\psi \in L^2(\mathbb{R})$, the above theorem and the Riesz-Markov representation theorem imply that there exists a unique Baire measure E_ψ such that

$$(\psi, f(\mathcal{H}_0)\psi) = \int_{\mathbb{R}} f(\lambda)\,dE_\psi(\lambda). \tag{1.1.1}$$

The measure E_ψ is called *spectral measure* associated to the state ψ. If we consider $\Omega \in B(\mathbb{R})$ and

$f = \chi_\Omega$ the characteristic function, then we can interpret the spectral measure μ_ψ

$$\Omega \mapsto (\psi, \chi_\Omega(\mathcal{H}_0)\psi),$$

as a tool for quantum measurements. Indeed, this measure represents the probability that the quantum observable $\mathcal{H}_0$, takes values in $\Omega \subset \sigma(\mathcal{H}_0)$, if the system is prepared in the state ψ. Here $\sigma(\mathcal{H}_0)$ denotes the spectrum of $\mathcal{H}_0$. (One can see sections VII.2 and VIII.5 in [58] for a proof of the spectral theorems and [53] for the physical interpretation of the spectral theorems and the more general aspects of the *quantum probability theory.*)

The Stone formula provides an explicit representation of the spectral measure associated with the self-adjoint operator $(\mathcal{H}_0, D(\mathcal{H}_0))$ in terms of its resolvent.

Proposition 1.1.3 (Stone formula)**.** *Let $a, b \in \mathbb{R}$, $a \leq b$. Let $(A, D(A))$ be a self-adjoint operator on the Hilbert space H. Then the following formula holds*

$$\lim_{\epsilon \to 0^+} \frac{1}{2i\pi} \int_a^b R(\lambda - i\epsilon, A) - R(\lambda + i\epsilon, A)\, d\lambda = \frac{1}{2}\left(E(\{a\}) + E(\{b\})\right) + E((a,b)), \tag{1.1.2}$$

where the operator $R(z, A) = (z - A)^{-1}$ denotes the resolvent operator.

Proof. We first compute the following limit

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{1}{\lambda - i\epsilon - t} - \frac{1}{\lambda + i\epsilon - t}\, d\lambda.$$

We have that

$$\begin{aligned} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{1}{\lambda - i\epsilon - t} - \frac{1}{\lambda + i\epsilon - t}\, d\lambda &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{2i\epsilon}{(\lambda - t)^2 + \epsilon^2} d\lambda \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\pi\epsilon} \int_a^b \frac{1}{1 + \left(\frac{\lambda - t}{\epsilon}\right)^2}\, d\lambda \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{\frac{a-t}{\epsilon}}^{\frac{b-t}{\epsilon}} \frac{1}{1 + \lambda^2}\, d\lambda. \end{aligned}$$

The last integrand is uniformly bounded and the integral converges pointwise. So, computing this integral and via the spectral theorem

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b R(\lambda - i\epsilon, A) - R(\lambda + i\epsilon, A)\, d\lambda = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{1}{\lambda - i\epsilon - t} - \frac{1}{\lambda + i\epsilon - t}\, dE(\lambda)$$

we get the Stone formula (1.1.2). □

On the other side we can also get an explicit representation formula for the kernel of the resolvent of the free Laplace operator:

Proposition 1.1.4. *Let z be in the resolvent set $\rho(\mathcal{H}_0) = \mathbb{C} \setminus [0, +\infty)$. Then, the resolvent operator of the Laplacian can be written as an integral operator:*

$$R(z, \mathcal{H}_0)u(x) = \int_{\mathbb{R}} R(z, \mathcal{H}_0)(x, y)u(y)\, dy$$

where $u \in L^2(\mathbb{R})$ and

$$R(z, \mathcal{H}_0)(x, y) = \frac{e^{i|x-y|\sqrt{z}}}{2i\sqrt{z}}.$$

Notice that $\sqrt{z}$ is the analytic branch of $z^{1/2}$ with branch cut $[0, \infty)$ such that the map

$$z \in \{z \in \mathbb{C} \setminus [0, \infty)\} \implies \Im\sqrt{z} > 0. \tag{1.1.3}$$

is a well-defined analytic diffeomorphism.

Proof. Let $z \in \rho(\mathcal{H}_0)$. We consider the problem $(z - \mathcal{H}_0)f = u$. Our goal is to write f as an integral operator. Applying the Fourier transform and then the anti Fourier transform, we have that

$$R(z, \mathcal{H}_0)(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i(x-y)\xi}}{z - \xi^2}\, d\xi.$$

Hence, we need to compute the following integral

$$\int_{\mathbb{R}} \frac{e^{ix\xi}}{z - \xi^2}\, d\xi.$$

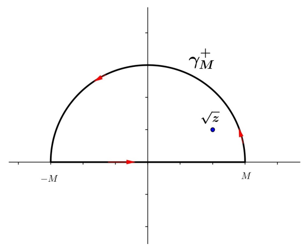
We consider the holomorphic function

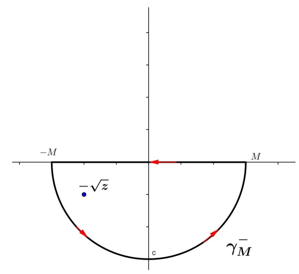
$$F(\alpha) = \frac{e^{ix\alpha}}{z - \alpha^2}, \;\; \alpha \in \mathbb{C} \setminus \{\pm\sqrt{z}\}, \; \Im\sqrt{z} > 0.$$

Let $M > 0$. We consider the closed curves $\gamma_M^{\pm}$ defined as in the figure below.

By the Cauchy's residue theorem we have that

$$\int_{\gamma_M^{\pm}} F(\alpha)\, d\alpha = 2\pi i Res(F, \pm\sqrt{z}) = 2\pi i \frac{\mp e^{\pm ix\sqrt{z}}}{2\sqrt{z}}.$$





We first consider the case $x \geq 0$ and the path γ_M^+. It follows that

$$\int_{\gamma_M^+} F(\alpha)\, d\alpha = \int_{-M}^{M} \frac{e^{ix\xi}}{z-\xi^2}\, d\xi + \int_0^{\pi} \frac{e^{ixM(cos\theta + i\sin\theta)}}{z - M^2 e^{2i\theta}} M e^{i\theta}\, d\theta.$$

Passing to the limit $M \to +\infty$ we prove that

$$2\pi i \frac{-e^{ix\sqrt{z}}}{2\sqrt{z}} = \int_{\mathbb{R}} \frac{e^{ix\xi}}{z-\xi^2}\, d\xi.$$

If $x < 0$ we consider the path γ_M^- and similarly we get

$$2\pi i \frac{e^{-ix\sqrt{z}}}{2\sqrt{z}} = -\int_{\mathbb{R}} \frac{e^{ix\xi}}{z-\xi^2}\, d\xi.$$

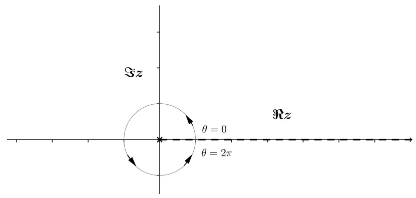
This completes the proof. □

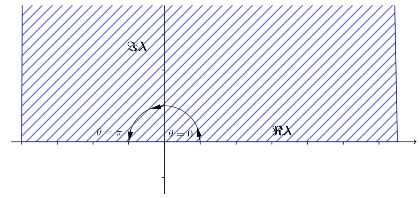
1.1.2 Meromorphic extension of the free resolvent and resonances

In this section we define the concept of *quantum resonance* for the free Laplace operator. In physics, the resonances are related to the existence of meta-stable states (one can see [80] and references therein for a physical description of resonances). In mathematics, the resonances can be seen as a generalization of the eigenvalues. Broadly speaking, the eigenvalues of a certain operator are the poles of its resolvent operator in a suitable space while the resonances will be defined as poles of the meromorphic continuation of the resolvent in a larger space.

In order to rigorous define the resonances, we firstly reparametrize the complex plane with the following change of variable:

$$z \mapsto \lambda^2,\ \Im\lambda > 0.$$





Hence, we consider the resolvent operator on the complex half plane $\lambda \in \mathbb{C}$, and $\Im\lambda > 0$:

$$R(\lambda^2, \mathcal{H}_0) = (\lambda^2 - \mathcal{H}_0)^{-1} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

It is a bounded operator on $L^2(\mathbb{R})$, meromorphic on $\mathbb{C}_+$ and continuous on the real axis, except possibly at zero. Moreover, as showed in Proposition 1.1.4, we have the following explicit formula for the kernel of the resolvent

$$R(\lambda^2, \mathcal{H}_0)(x, y) = \frac{e^{i\lambda|x-y|}}{2i\lambda},$$

where $\lambda \in \mathbb{C}$, $\Im\lambda > 0$.

This representation of the resolvent suggests us to define a meromorphic extension to the entire complex plane, $\lambda \in \mathbb{C}$. In order to do this, we consider the cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ and we define the cut-off resolvent $\varphi R(\lambda^2, \mathcal{H}_0)\varphi$. The kernel of this operator is explicitly given by

$$\varphi(x)\frac{e^{i\lambda|x-y|}}{2i\lambda}\varphi(y),$$

with $\lambda \in \mathbb{C}, \Im\lambda > 0$. Hence, it is natural to define the resolvent for $\lambda \in \mathbb{C}$ and $\Im\lambda < 0$, as follows

$$\varphi R(-\lambda^2, \mathcal{H}_0)\varphi.$$

The operator $\varphi R(-\lambda^2, \mathcal{H}_0)\varphi$ is meromorphic in $\Im\lambda < 0$. We note that, if we consider the operator defined as $\varphi R(\lambda^2, \mathcal{H}_0)\varphi$ on the positive complex half plane and as $\varphi R(-\lambda^2, \mathcal{H}_0)\varphi$ on the negative one, we have a discontinuity on the real line.

Proposition 1.1.5. *Let $\lambda > 0$. Then we have the following jump discontinuity on the real axis at the point λ:*

$$\lim_{\epsilon \to 0^+} \varphi R((\lambda + i\epsilon)^2, \mathcal{H}_0)\varphi - \varphi R(-(\lambda - i\epsilon)^2, \mathcal{H}_0)\varphi = \varphi M(\lambda)\varphi$$

where

$$M(\lambda)(x,y) = \frac{e^{i|x-y|\lambda}}{2i\lambda} + \frac{e^{-i|x-y|\lambda}}{2i\lambda} = \frac{\cos(|x-y|\lambda)}{i\lambda}.$$

Proposition 1.1.6. *The cut off resolvent*

$$\varphi R(\lambda^2, \mathcal{H}_0)\varphi$$

admits a unique meromorphic continuation as a compact operator valued function to $\mathbb{C}$*. Moreover, the meromorphic continuation of the cut-off resolvent has an isolated pole of order one at* $\lambda = 0$*.*

Proof. We consider the following extension of the resolvent

$$\tilde{R}_\varphi(\lambda^2, \mathcal{H}_0) = \begin{cases} \varphi R(\lambda^2, \mathcal{H}_0)\varphi, & \Im\lambda \geq 0, \\ \varphi R(-\lambda^2, \mathcal{H}_0)\varphi + \varphi M(\lambda)\varphi, & \Im\lambda < 0. \end{cases}$$

By definition, $\tilde{R}_\varphi(\lambda^2, \mathcal{H}_0)$ is continuous on the real line. Moreover, let $f \in L^2(\mathbb{R})$, we have that

$$\tilde{R}_\varphi(\lambda^2, \mathcal{H}_0)f(x) = \frac{\varphi(x)}{2i\lambda}\int_{\mathbb{R}} \varphi(y)f(y)\,dy + \varphi(x)\int_{\mathbb{R}} \frac{e^{i\lambda|x-y|}-1}{2i\lambda}\varphi(y)f(y)\,dy.$$

□

Definition 1.1.7. *Let* $z \in \mathbb{C}$*. We say that* z *is a quantum resonance point for the Hamiltonian* $\mathcal{H}_0$ *or equivalentely we say that the Hamiltonian* $\mathcal{H}_0$ *has a* resonance *in* z*, iff the meromorphic continuation of the cut-off resolvent has a pole in* z*.*

Remark 1.1.8. We note that the free Hamiltonian $\mathcal{H}_0$ has a resonance in zero.

One can find a more detailed description of resonances in terms of poles of the resolvent in [40] and references therein.

1.1.3 Sectorial properties for Laplace operator

In this section we recall some classical results on the sectorial operators. We quote Section 1 in [39] for the classical theory of sectorial operators and [3] for the fractional calculus defined on sectorial operators.

In the following we denote with $(A, D(A))$ a closed operator defined on the Banach space X. The reported results state that, if the spectrum $\sigma(A)$ is confined in a certain sector of the complex plane, and the resolvent operator satisfies suitable estimates, then we can define the functional calculus on A.

Definition 1.1.9. *Let $(A, D(A))$ be a closed densely defined linear operator on the Banach space X. A is called sectorial operator if there exist $\theta \in (0, \pi/2)$, $M \geq 1$, $a \in \mathbb{R}$ such that*

$$\text{(i)} \quad S_{a,\theta} = \{z \,|\, \theta \leq |\arg(z-a)| \leq \pi, z \neq a\} \subset \rho(A),$$
$$\text{(ii)} \quad \|R(z, A)\| \leq \frac{M}{|z-a|}, \; \forall z \in S_{a,\theta}.$$

Theorem 1.1.10. *Let $(A, D(A))$ be a sectorial operator. Then $-A$ is the infinitesimal generator of the analytic semigroup $(e^{-At})_{t\geq 0}$, where*

$$e^{-tA} = \frac{1}{2\pi i}\int_{\Gamma} (z+A)^{-1} e^{zt}\, dz$$

where Γ is a contour in the resolvent set $\rho(-A)$ with $\arg z \to \pm\theta$ as $|z| \to +\infty$ for some $\theta \in (\pi/2, \pi)$. The operator e^{-At} can be continued analytically into a sector $\{t \neq 0 : |\arg t| < \epsilon\}$. Moreover, if $\Re\sigma(A) > b$, then there exists a constant $C > 0$ such that for any $t > 0$ we have that

$$\|e^{-At}\| \leq Ce^{-at}, \quad \|Ae^{-At}\| \leq \frac{C}{t}e^{-at}.$$

Definition 1.1.11. *Let $(A, D(A))$ be a sectorial operator such that $\Re\sigma(A) > 0$. Then, for any $\alpha > 0$ we can define the fractional powers of the operator A as follows*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1} e^{-tA}\, dt.$$

Theorem 1.1.12. *Let $(A, D(A))$ be a sectorial operator on X with $\Re\sigma(A) > 0$. Then for any $\alpha > 0$, the fractional operator $A^{-\alpha}$ is a bounded linear operator on X such that $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ for any $\alpha, \beta > 0$. Moreover, if $0 < \alpha < 1$ we have the Balakrishnan representation:*

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^{\infty} z^{-\alpha}(z+A)^{-1}\, dz.$$

Corollary 1.1.13. *Let us consider $(\mathcal{H}_0, C_c^{\infty}(\mathbb{R}))$ and $z \in \mathbb{C}$ be such that $|\arg z| \leq \theta < \pi$. Then , for any $1 \leq p \leq \infty$, the following estimate holds*

$$\|R(z, \mathcal{H}_0)f\|_{L^p(\mathbb{R})} \leq \frac{C}{(\cos(\theta/2))^{3/2}}\frac{1}{|z|}\|f\|_{L^p(\mathbb{R})}.$$

Moreover, the free Hamiltonian $\mathcal{H}_0$ is a sectorial operator in the spaces $L^p(\mathbb{R})$, with $1 \leq p < \infty$, in the spaces of uniformly continuous bounded functions and in the space of continuous functions that vanish at infinity, with spectrum $\sigma(\mathcal{H}_0) = [0, \infty)$ in each case.

In [39] one can find a proof of the sectorial estimates for the free Laplacian for any dimensions.

1.2 Spectral properties for the Schrödinger operator on the real line

Let $V : \mathbb{R} \to \mathbb{R}$ be a given potential. We consider the Schrödinger operator in one dimension associated with the potential V,

$$\mathcal{H} = \mathcal{H}_0 + V(x),$$

defined on the space $C_c^\infty(\mathbb{R})$. We are interested in potentials V such that the Hamiltonian $\mathcal{H}$ can be considered as a perturbation of the free Hamiltonian $\mathcal{H}_0$. In particular this means that we are interested in potentials such that the spectral properties of $\mathcal{H}$ (eigenvalues and resonances), resemble those of $\mathcal{H}_0$.

Now we collect some results that describe the spectral scenario for the perturbed Hamiltonian $\mathcal{H}$ when the potential represents just a *small disorder*. In this case we will also say that V has short range effects. In the following we consider potentials bounded from below or that verify certain decay at infinity (one can see [15] and references therein for a more detailed tour on the spectral properties of the Schrödinger operator associated with short range potential).

Theorem 1.2.1 (Theorem 3.1 and Theorem 3.2, Chapter 3, [6])**.** *Let $V \in L^\infty_{loc}(\mathbb{R})$ be a locally bounded potential. Let $a \in \mathbb{R}$, such that*

$$\liminf_{|x| \to \infty} V(x) \geq a.$$

Then $\mathcal{H}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ with discrete spectrum contained in $(-\infty, a)$, and for any $a' < a$ we have that $\sigma(\mathcal{H}) \cap (-\infty, a')$ consists only of a finite number of eigenvalue with finite multiplicity. Moreover, if z is an eigenvalue with eigenvector u, i.e. $\mathcal{H}u = zu$, then the eigenvector u has exponentially decay. In particular the following estimate holds

$$|u(x)| \leq C_\epsilon e^{-\sqrt{\frac{a-z-\epsilon}{2}}|x|},$$

for any $\epsilon > 0$.

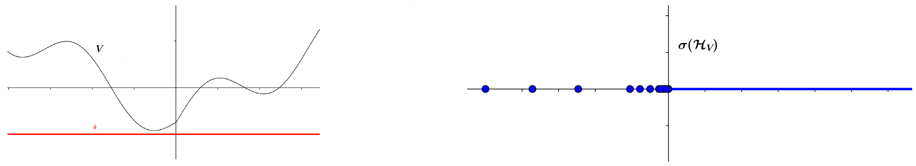


Figure 1.2.1: Potential bounded from below without decay at infinity.

The next result shows that if additional decay on the negative part of the potential is required then the spectrum of the perturbed operator resembles the spectrum of the unperturbed one except for a finite number of negative eigenvalues.

Theorem 1.2.2 (Theorem 5.3, Chapter 2, [6]). *Let $V \in L^\infty_{loc}(\mathbb{R})$ such that the following decay hypothesis are satisfied*

$$V(x) \geq a, \text{ and } \int_{\mathbb{R}} |x||V_-(x)|\, dx < \infty,$$

where $V_-(x) = \min(V(x), 0)$ is continuous and $a \in \mathbb{R}$. Then the number of the negative eigenvalues $N_-(\mathcal{H})$ is finite and the following bound is proved

$$N_-(\mathcal{H}) \leq 1 + \int_{\mathbb{R}} |x||V_-(x)|\, dx.$$

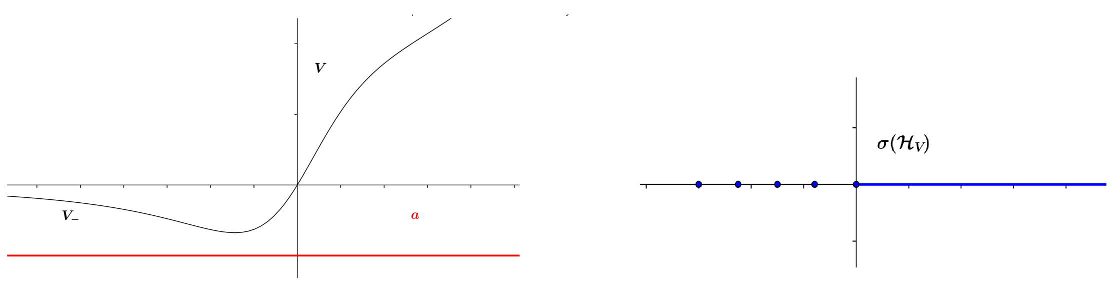


Figure 1.2.2: Example of a potential bounded from below and with decay at infinity on the negative part.

Finally, we state the following more general result:

Theorem 1.2.3 (Theorem 10.2, [68]). *Let $V \in L^\infty(\mathbb{R}) + L^2(\mathbb{R})$. Then $\mathcal{H}$ is a self-adjoint operator on $H^2(\mathbb{R})$ and essentially self-adjoint on $C_c^\infty(\mathbb{R})$. Moreover*

$$\sigma_{ess}(\mathcal{H}) = \sigma(\mathcal{H}) \smallsetminus \sigma_p(\mathcal{H}) = [0, \infty) = \sigma(\mathcal{H}_0).$$

In the following we focus our attention on particular potentials V that are space localized and that decay at infinity. In particular, we will consider $V \in L^1_\gamma(\mathbb{R})$ namely

$$\int_{\mathbb{R}} \langle x \rangle^\gamma |V(x)| dx < \infty,$$

with $\langle x \rangle^2 = 1 + x^2$, and $\gamma \geq 1$.

To this end, we state the following result that fix the general spectral scenario for the perturbed Hamiltonian $\mathcal{H}$ in examination:

Theorem 1.2.4. *Let $V \in L^1(\mathbb{R})$. Then $\mathcal{H}$ has only point spectrum in $(-\infty, 0)$, where 0 is the only possible accumulation point. Moreover $[0, \infty)$ is an essential support for the absolutely continuous spectrum. If $V \in L^1_1(\mathbb{R})$ then there are only finitely many bound states.*

The potentials $V \in L^1_1(\mathbb{R})$ can be interpreted as a *small disorder* that perturbs the free evolution of the quantum particle.

1.3 Zero resonance for perturbed Laplacian in one dimension

Let $V : \mathbb{R} \to \mathbb{R}$ be a time independent potential and we suppose that V verifies an infinity decay, namely following [72], [19] we require

$$\text{(H1)} \ V \in L^1_a(\mathbb{R}), \ a > 1. \tag{1.3.1}$$

We consider the perturbed Hamiltonian

$$\mathcal{H} = -\partial_x^2 + V = \mathcal{H}_0 + V$$

and we assume that the point spectrum $\sigma_p(\mathcal{H})$ is empty, i.e.

$$\text{(H2)} \ (\mathcal{H} - zI)\, u = 0, \ u \in L^2(\mathbb{R}), \ z \in \mathbb{C} \implies u = 0. \tag{1.3.2}$$

Now, we are going to introduce heuristically the notion of zero resonance for the perturbed Hamiltonian $\mathcal{H}$, following the heuristics for the free case.

The free Hamiltonian $\mathcal{H}_0$ has resolvent $R(z, \mathcal{H}_0) = (z - \mathcal{H}_0)^{-1}$ well defined in $\mathbb{C} \setminus [0, \infty)$ as integral operator with kernel

$$R(z, \mathcal{H}_0)(x, y) = \frac{e^{i\sqrt{z}|x-y|}}{2i\sqrt{z}}. \tag{1.3.3}$$

Here and below $\sqrt{z}$ is the analytic branch of $z^{1/2}$ with branch cut $[0,\infty)$, such that the map

$$z \in \{z \in \mathbb{C} \setminus [0,\infty)\} \mapsto \lambda = \sqrt{z} \in \{\Im\lambda > 0\} \tag{1.3.4}$$

is a well-defined analytic diffeomorphism. Let f be a function in a suitable Banach space. Rewriting the relation (1.3.3) as follows

$$R(\lambda^2, \mathcal{H}_0)f(x) = \frac{1}{2i\lambda}\int_{\mathbb{R}} f(y)\,dy + \int_{\mathbb{R}} \frac{e^{i\lambda|x-y|}-1}{2i\lambda} f(y)\,dy, \tag{1.3.5}$$

we interpret the resonance in zero as the simple pole in zero of the resolvent operator $R(\lambda^2, \mathcal{H}_0)$. We note that, if $\int_{\mathbb{R}} f(y)\,dy = 0$, the effects of the pole are nullified.

Now we turn to the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$, where the potential V satisfies (H1) and (H2). We want assumptions that guarantee that the spectral scenario (eigenvalues and resonances) of the operator $\mathcal{H}$ resembles the one of the free operator $\mathcal{H}_0$. Hence, at least informally, in addition to the hypotheses (H1) and (H2), we need to require that the resolvent operator $R(\lambda^2, \mathcal{H})$ has not any poles apart from the one generated by $R(\lambda^2, \mathcal{H}_0)$. If this last requirement is satisfied, we will say that

(H3) $\quad$ $\mathcal{H}$ has no zero resonance.

To formalize this last requirement, we firstly note that on the resolvent set $\mathbf{C} \setminus [0,\infty)$, the following identity is verified

$$R(\lambda^2, \mathcal{H})\left(I - VR(\lambda^2, \mathcal{H}_0)\right) = R(\lambda^2, \mathcal{H}_0). \tag{1.3.6}$$

Then, we introduce a projector P on the space of the integrable functions $L^1(\mathbb{R})$, such that if $f \in L^1(\mathbb{R})$ then $\int_{\mathbb{R}} Pf(y)\,dy = 0$. This projector removes the zero resonance generated by the free Hamiltonian $\mathcal{H}_0$. Applying the projector in (1.3.6) we have the following relation

$$R(\lambda^2, \mathcal{H})\left(I - VR(\lambda^2, \mathcal{H}_0)P\right)P = R(\lambda^2, \mathcal{H}_0)P. \tag{1.3.7}$$

Hence, it is clear that the problem to guarantee the absence of additional resonances in zero for the perturbed Hamiltonian $\mathcal{H}$ is strictly related to the problem of the invertibility of the operator

$$\left(I - VR(\lambda^2, \mathcal{H}_0)P\right) \tag{1.3.8}$$

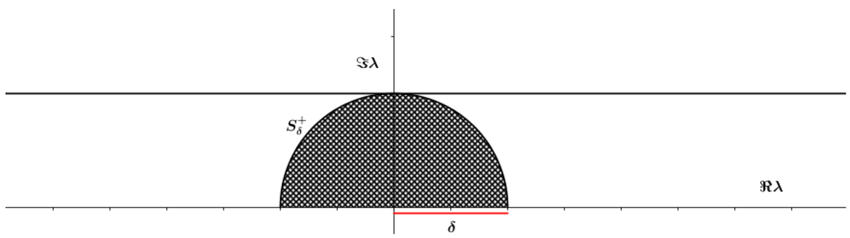
in appropriate Banach spaces.

In this section we will begin introducing a family of projectors so that it will be possible to give a first rigorous definition of the zero resonance for the perturbed Hamiltonian. Finally, we will connect the presence of resonances in zero to the existence of solutions $u \in L^\infty(\mathbb{R})$ of the equation $\mathcal{H}u = 0$.

Definition 1.3.1. *Let $\delta \in (0,1)$ and let*

$$S_\delta^+ = \{\lambda \in \mathbf{C}; |\lambda| < \delta, \Im\lambda > 0\}.$$

Let us consider the functions



$$e_0 \colon \mathbb{R} \times S_\delta^+ \to \mathbf{C},$$
$$e \colon \mathbb{R} \times S_\delta^+ \to \mathbf{C},$$

such that

(i) $e_0(\lambda) = e_0(\cdot, \lambda) \in C_c^\infty(\mathbb{R})$ *for any* $\lambda \in S_\delta^+$ *and the support of* $e_0(\lambda)$ *is independent from* λ*. Moreover* $\int_\mathbb{R} e_0(x,\lambda)\,dx = 1$ *for any* $\lambda \in S_\delta^+$*;*

(ii) $e_0(x) = e_0(x,\cdot)$ *is analytic in* S_δ^+ *and continuous in* $\overline{S_\delta^+}$ *for any* $x \in \mathbb{R}$*;*

(iii) $e(x,\lambda) = (\lambda^2 - \mathcal{H})e_0(x,\lambda)$, $\int_\mathbb{R} e(x,\lambda)\,dx = 1$ *for any* $\lambda \in S_\delta^+$ *and* $e_0(x,\lambda)$ *satisfies (i) and (ii).*

We define the set E_δ *as follows*

$$E_\delta = \left\{ e \colon \mathbb{R} \times S_\delta^+ \to \mathbf{C} \mid (iii) \text{ is satisfied} \right\}. \tag{1.3.9}$$

Lemma 1.3.2. *Let $V \in L_1^1(\mathbb{R})$. Then for any $\delta \in (0,1)$ we have that E_δ is not empty.*

Proof. Since the potential $V \in L_1^1(\mathbb{R})$, there exist $0 < a_1 < b_1 < a_2 < b_2 < \infty$ such that $b_1 - a_1 = b_2 - a_2$ and $\int_{a_1}^{b_1} V(x)dx \neq \int_{a_2}^{b_2} V(x)dx$. To construct e_0 as in (i) and (ii), we introduce the bump functions

$\varphi_1(x) = \varphi_1^V(x), \varphi_2(x) = \varphi_2^V(x) \in C_c^\infty(\mathbb{R})$, so that $\text{supp}\varphi_1 \subset [a_1, b_1]$, $\varphi_2(x) = \varphi_1(x - a_2 + a_1)$ and $\int_{\mathbb{R}} \varphi_1(x)\, dx = \int_{\mathbb{R}} \varphi_2(x)\, dx = 1$. We set

$$v_1 = \int_{\mathbb{R}} \varphi_1(x)V(x)dx \neq \int_{\mathbb{R}} \varphi_2(x)V(x)dx = v_2 \tag{1.3.10}$$

and we shall look for $e_0 = e_0(V)$ of the following form

$$e_0(x,\lambda) = c_1(\lambda)\varphi_1(x) + c_2(\lambda)\varphi_2(x),$$

where $c_1(\lambda)$ and $c_2(\lambda)$ satisfy the following relations

$$\int_{\mathbb{R}} e_0(x,\lambda)\, dx = 1 \implies c_1(\lambda) + c_2(\lambda) = 1, \tag{1.3.11}$$

$$\int_{\mathbb{R}} (\lambda^2 - \mathcal{H})e_0(x,\lambda)\, dx = 1 \implies c_1(\lambda)v_1 + c_2(\lambda)v_2 = \lambda^2 - 1. \tag{1.3.12}$$

The existence and uniqueness of solutions $c_1(\lambda)$ and $c_2(\lambda)$ is guaranteed by the relation

$$\det \begin{pmatrix} 1 & 1 \\ v_1 & v_2 \end{pmatrix} = \int_{\mathbb{R}} \varphi_2(x)V(x) - \int_{\mathbb{R}} \varphi_1(x)V(x) = v_2 - v_1 \neq 0 \tag{1.3.13}$$

that is true due to (1.3.10). Hence we have that

$$e_0(x,\lambda) = \frac{1 - \lambda^2 + v_2}{v_2 - v_1}\varphi_1(x) + \frac{\lambda^2 - 1 - v_1}{v_2 - v_1}\varphi_2(x).$$

By construction the requirements (i), (ii) and (iii) are satisfied for any $\delta \in (0,1)$. In particular, the condition $|\lambda| < \delta < 1$, guarantees that $|e_0(x,\lambda)| \leq M$, with $M > 0$ independent from λ. □

In the construction of the projector the weighted Lebesgue spaces play a fundamental role. Let $a \in [1,2]$. We define the following Banach spaces

$$L^1_a(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \mid \langle x\rangle^a f \in L^1(\mathbb{R}) \right\},$$

$$B^0_a(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \mid \langle x\rangle^a f \in L^1(\mathbb{R}), \int_{\mathbb{R}} f(y)\, dy = 0 \right\}.$$

Definition 1.3.3. *Let $\delta \in (0,1)$, $e \in E_\delta$ and $\lambda \in S_\delta^+$. Given $a \in [1,2]$, we define the projector*

$$P_{e(\lambda)} \colon L^1_a(\mathbb{R}) \to B^0_a(\mathbb{R})$$

as follows

$$P_{e(\lambda)} f(x) \colon = f(x) - e(x,\lambda) \int_{\mathbb{R}} f(y)\, dy. \tag{1.3.14}$$

For fixed $e \in E_\delta$ and for fixed $a \in (1,2]$ we will define the *zero resonance of a-order* using the projector (1.3.14). Then we will prove that the definition is independent from the choice of $e \in E_\delta$ and $a \in (1,2]$.

As mentioned in (1.3.7)-(1.3.8), to guarantee that zero is not a resonance, we need to establish the invertibility of the operator

$$(I - VR(\lambda^2, \mathcal{H}_0) P_{e(\lambda)}),$$

as λ goes to zero in a suitable functional space. In order to do this we prove the following estimates for the operators $R(\lambda^2, \mathcal{H}_0) P_{e(\lambda)}$, with $\lambda \in S_\delta^+$.

Lemma 1.3.4. *Let $a \in [1,2]$, $\delta > 0$ and $e \in E_\delta$. The following properties hold:*

(1) There exists a constant $C_e > 0$ so that for any $f \in L^1_a(\mathbb{R})$ and for any $\lambda \in S_\delta^+$ the below estimate is satisfied

$$\|R(\lambda^2, \mathcal{H}_0) P_{e(\lambda)}(f)\|_{L^\infty(\mathbb{R})} \le C_e \|f\|_{L^1_a(\mathbb{R})}; \tag{1.3.15}$$

(2) There exists a constant $C_e > 0$ such that for any $f \in L^1_a(\mathbb{R})$ and any couple $\lambda_1, \lambda_2 \in S_\delta^+$ we have

$$\| \left(R(\lambda_1^2; \mathcal{H}_0) P_{e(\lambda_1)} - R(\lambda_2^2; \mathcal{H}_0) P_{e(\lambda_2)} \right)(f) \|_{L^\infty(\mathbb{R})} \le C_e |\lambda_1 - \lambda_2|^{a-1} \|f\|_{L^1_a(\mathbb{R})}; \tag{1.3.16}$$

(3) Let $V \in L^1_a(\mathbb{R})$. Then the operator

$$K_{e(\lambda)} = VR(\lambda^2, \mathcal{H}_0) P_{e(\lambda)} \colon L^1_a(\mathbb{R}) \to L^1_a(\mathbb{R}), \tag{1.3.17}$$

is compact, analytic in S_δ^+ for any $a \in [1,2]$ and continuous in its closure $\overline{S_\delta^+}$ for $a \in (1,2]$.

Proof. We can use the relations (1.3.3) and (1.3.14) to derive the following relation

$$R(\lambda^2, \mathcal{H}_0) P_{e(\lambda)}(f)(x) = \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz \frac{\left(e^{i\lambda|x-y|} - e^{i\lambda|x-z|}\right)}{2i\lambda} f(y) e(z,\lambda). \tag{1.3.18}$$

Let $\alpha, \beta \geq 0$. Since in S_δ^+ we have that $\Im\lambda > 0$, we can use the following estimates:

$$|e^{i\lambda\alpha}| \leq 1, \ \ \forall \alpha \geq 0, \forall \lambda \in \mathbf{C}, \Im\lambda \geq 0; \tag{1.3.19}$$

$$\left|\frac{e^{i\lambda\alpha} - e^{i\lambda\beta}}{2i\lambda}\right| \leq \frac{|\alpha - \beta|}{2}, \ \ \forall \alpha, \beta \geq 0, \forall \lambda \in \mathbf{C}, \Im\lambda \geq 0. \tag{1.3.20}$$

The inequality (1.3.19) is trivial. Using this estimate and rewriting the left side in (1.3.20) as follows

$$\frac{\left(e^{i\lambda\alpha} - e^{i\lambda\beta}\right)}{2i\lambda} = \frac{1}{2}\int_\beta^\alpha e^{i\lambda\tau} d\tau, \tag{1.3.21}$$

we get (1.3.20).

In order to get (1.3.15) we consider the L^∞ norm of (1.3.18) and applying (1.3.20) we have

$$\left\|R(\lambda^2, \mathcal{H}_0)P_{e(\lambda)}f\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}\int_{\mathbb{R}} dy \int_{\mathbb{R}} dz \, ||x-y| - |x-z|| \, |f(y)||e(z,\lambda)|. \tag{1.3.22}$$

Hence, to complete the proof of (1.3.15) we note that, since $e(\lambda)$ verifies the property (iii) in Definition 1.3.1, then there exists a constant $M_e > 0$, such that $\operatorname{supp} e(\lambda) \subset [-M_e, M_e]$ and moreover, if $|z| \leq M_e$ we have that

$$||x-y| - |x-z|| \leq C \max(M_e, |y|) \leq C_e\langle y\rangle, \tag{1.3.23}$$

where $C_e > 0$.

To prove (1.3.16) we proceed similarly. Let us denote the left side in (1.3.16) as follows:

$$J(\lambda_1, \lambda_2)f = \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz \, \mathcal{L}_{\lambda_1,\lambda_2,x}(y,z)f(y),$$

where

$$\mathcal{L}_{\lambda_1,\lambda_2,x}(y,z) = \frac{\left(e^{i\lambda_1|x-y|} - e^{i\lambda_1|x-z|}\right)}{2i\lambda_1}e(z,\lambda_1) - \frac{\left(e^{i\lambda_2|x-y|} - e^{i\lambda_2|x-z|}\right)}{2i\lambda_2}e(z,\lambda_2)$$

Suppose $\Im\lambda_1 \leq \Im\lambda_2$. We have that

$$\mathcal{L}_{\lambda_1,\lambda_2,x}(y,z) = \frac{1}{2}\int_{|x-z|}^{|x-y|} e^{i\lambda_1\tau}\left(e(z,\lambda_1) - e(z,\lambda_2)\right) - e^{i\lambda_1\tau}(e^{i(\lambda_2-\lambda_1)\tau} - 1)e(z,\lambda_2)\, d\tau.$$

We can estimate the first addend above using the regularity property of $e(x,\cdot)$ and the inequalities (1.3.19) and (1.3.23). For the second addend we use the estimate (1.3.20) with $\beta = 0$. Then we integrate in τ. Operating this computation we get

$$\begin{aligned}\|J(\lambda_1,\lambda_2)f\|_{L^\infty(\mathbb{R})} \leq & C_e|\lambda_1-\lambda_2|\|f\|_{L^1_1(\mathbb{R})} + C_e|\lambda_1-\lambda_2|\|f\|_{L^1_2(\mathbb{R})} \\ \leq & C_e|\lambda_1-\lambda_2|\|f\|_{L^1_2(\mathbb{R})}.\end{aligned}$$

If $\Im\lambda_1 > \Im\lambda_2$ the proof is similar. So, the inequality (1.3.16) has been proved for $a=2$. In the case $a=1$ the proof follows by (1.3.15). Slightly modifying the proof of the Riesz-Thorin theorem we can use the complex interpolation in weighted Lebesgue spaces to deduce the (1.3.16) for any $a \in [1,2]$.

Finally, to complete the proof we have to discuss the compactness of the operator $K_{e(\lambda)}$ in $L^1_a(\mathbb{R})$, with $a \in [1,2]$. Since we are assuming $V \in L^1_a(\mathbb{R})$, the inequality (1.3.15) implies

$$K_{e(\lambda)} = VR(\lambda^2,\mathcal{H}_0)P_{e(\lambda)} \colon L^1_a(\mathbb{R}) \to L^1_a(\mathbb{R}).$$

At first we prove the compactness of the operator $K_{e(\lambda)}$ for fixed $\lambda \in S^+_\delta$. Then, we can pass to the limit on $\overline{S^+_\delta}$ thanks to the inequality (1.3.16).

Let $\lambda \in S^+_\delta$, $|\lambda| < \delta$, and let f_n be a bounded sequence in $L^1_a(\mathbb{R})$,

$$\|f_n\|_{L^1_a(\mathbb{R})} \leq M_a, \ \ \forall n \in \mathbb{N},$$

with $M_a > 0$. Combining the hypothesis $V \in L^1_a(\mathbb{R})$ and the inequality (1.3.15) we get

$$\|K_{e(\lambda)}f_n\|_{L^1_a(\mathbb{R})} \leq C_e M_a \|V\|_{L^1_a(\mathbb{R})}.$$

We put

$$R(\lambda^2,\mathcal{H}_0)P_{e(\lambda)}f_n = g_n \in L^\infty(\mathbb{R}).$$

Since

$$V \colon L^\infty(\mathbb{R}) \to L^1_a(\mathbb{R}),$$

there exists $N(\epsilon) > 0$ such that

$$\|Vg_n\|_{L^1_a(\Omega^c_\epsilon)} \leq \epsilon,$$

where $\Omega_\epsilon = \{x \in \mathbb{R} | |x| \leq N(\epsilon)\}$ and $\Omega_\epsilon^c = \mathbf{C} \setminus \Omega_\epsilon$. On the other side, we know that

$$P_{e(\lambda)} f_n = \lambda^2 g_n + \partial_{xx} g_n,$$

so, we can gain regularity and hence compactness for the sequence g_n in bounded domains. Indeed, we have that

$$\begin{aligned}\|\partial_{xx} g_n\|_{L^1(\Omega_\epsilon)} \leq & |\lambda|^2 \|g_n\|_{L^1(\Omega_\epsilon)} + \|P_{e(\lambda)} f_n\|_{L^1(\Omega_\epsilon)} \\ \leq & 2\delta^2 |N(\epsilon)| \|g_n\|_{L^\infty(\mathbb{R})} + M_a \\ \leq & C_e M_a |N(\epsilon)| + M_a.\end{aligned}$$

Hence, it is proved that $g_n \in W^{2,1}(\Omega_\epsilon)$. The proof follows by the compact embedding $W^{2,1}(\Omega_\epsilon) \hookrightarrow\hookrightarrow L^\infty(\Omega_\epsilon)$. The analiticity of the resolvent operator $R(\lambda^2, \mathcal{H}_0)$ and of the function $e(x, \cdot)$ imply the analiticity of the operator $VR(\lambda^2, \mathcal{H}_0) P_{e(\lambda)}$ in S_δ^+. The continuity in $\overline{S_\delta^+}$ follows by the inequality (1.3.16). □

The following Lemma will turn to be crucial to define rigorously the resononace at zero energy.

Lemma 1.3.5. *Let $V \in L_a^1(\mathbb{R})$ and let $a \in (1, 2]$. Suppose that there exist $\delta > 0$, $e_1 \in E_\delta$ such that $(I - K_{e_1(\lambda)})^{-1}$ there exists in $L_a^1(\mathbb{R})$ for any $\lambda \in S_\delta^+$. Then, for any $0 < \delta' < \delta$ and for any $e \in E_{\delta'}$ the operator $(I - K_{e(\lambda)})^{-1}$ exists in $L_a^1(\mathbb{R})$ for any $\lambda \in S_{\delta'}^+$.*

Proof. Suppose that there exist $0 < \delta' < \delta$, $e \in E_{\delta'}$ and $\lambda \in S_{\delta'}^+$ so that

$$\mathrm{Ker}\,(I - K_{e(\lambda)}) \neq \{0\}.$$

Hence, there exists $f = f_\lambda \in L_a^1(\mathbb{R})$, $f \neq 0$, such that

$$f = K_{e(\lambda)} f. \tag{1.3.24}$$

Let $\tilde{e} \in E_{\delta'}$, $\tilde{e} \neq e$. Our goal is to construct a function $g_\lambda \in L_a^1(\mathbb{R})$, $g_\lambda \neq 0$, such that

$$K_{\tilde{e}(\lambda)} g_\lambda = g_\lambda, \tag{1.3.25}$$

so, the proof will follow by contradiction.

By definition, we know that there exist e_0 and $\tilde{e}_0$ satisfying (ii) and (iii) such that

$$e(x,\lambda) - \tilde{e}(x,\lambda) = (\lambda^2 + \partial_x^2 - V)(e_0(x,\lambda) - \tilde{e}_0(x,\lambda)).$$

We can rewrite the equation above as follows

$$V\left(e_0(x,\lambda) - \tilde{e}_0(x,\lambda)\right) = \left(\lambda^2 + \partial_x^2\right)\left(e_0(x,\lambda) - \tilde{e}_0(x,\lambda)\right) - \left(e(x,\lambda) - \tilde{e}(x,\lambda)\right). \qquad (1.3.26)$$

We define

$$g_\lambda : \; = f + (f,1)V\left(e_0(x,\lambda) - \tilde{e}_0(x,\lambda)\right),$$

where $(f,1) = \int_{\mathbb{R}} f(x)\,dx$. Since $V \in L^1_a(\mathbb{R})$, we have that $g_\lambda \in L^1_a(\mathbb{R})$ for any $\lambda \in S^+_\delta$. Moreover, using (1.3.26) we have that

$$\int_{\mathbb{R}} V(x)\left(e_0(x,\lambda) - \tilde{e}_0(x,\lambda)\right)\,dx = 0, \qquad (1.3.27)$$

from which it follows that $\int_{\mathbb{R}} g_\lambda(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$, and in particular $g_\lambda \neq 0$. It remains to compute $K_{\tilde{e}(\lambda)} g_\lambda$. Using the definition of g_λ we have that

$$K_{\tilde{e}(\lambda)} g_\lambda = VR(\lambda^2, \mathcal{H}_0) P_{\tilde{e}(\lambda)}\left[f + (f,1)\left[V\left(e_0(\lambda) - \tilde{e}_0(\lambda)\right)\right]\right].$$

The relation (1.3.26), the definition of the projector combined with (1.3.27) and the properties

$$\textstyle\int_{\mathbb{R}} e(x,\lambda)\,dx = \int_{\mathbb{R}} e_0(x,\lambda)\,dx = 1, \qquad P_{e(\lambda)} e(\lambda) = 0,$$

lead to the following computation:

$$P_{\tilde{e}(\lambda)} g_\lambda = f - (f,1)\tilde{e}(\lambda) + (f,1)\left[(\lambda^2 + \partial_x^2)(e_0(\lambda) - \tilde{e}_0(\lambda)) - (e(\lambda) - \tilde{e}(\lambda))\right].$$

From the line above it follows that

$$P_{\tilde{e}(\lambda)} g_\lambda = P_{e(\lambda)} f + (f,1)(\lambda^2 + \partial_x^2)(e_0(\lambda) - \tilde{e}_0(\lambda)).$$

Hence, applying the operator $VR(\lambda^2, \mathcal{H}_0)$ and using (1.3.24) we have

$$K_{\tilde{e}(\lambda)} g_\lambda = VR(\lambda^2, \mathcal{H}_0) P_{e(\lambda)} f + (f,1) V(e_0(\lambda) - \tilde{e}_0(\lambda)) = g_\lambda.$$

In particular, since $e_1 \in E_\delta \subset E_{\delta'}$, we have proved that there exists $\lambda \in S^+_{\delta'} \subset S^+_\delta$ such that $(I - K_{e_1(\lambda)})$ is not invertible. $\square$

Thanks to Lemma 1.3.4 and Lemma 1.3.5 we are able to define the notion of zero resonance for the perturbed Hamiltonian $\mathcal{H}$.

Definition 1.3.6. *Let V be a potential such that the hypothesis (H1) is satisfied, namely $V \in L^1_a(\mathbb{R})$, $a > 1$. We say that the Hamiltonian $\mathcal{H} = -\partial_x^2 + V$ has zero resonance od $a-$order, with $a \in (1, 2]$, if there exist $\delta < 0$, $e \in E_\delta$ and $f \in L^1_a(\mathbb{R})$ with $f \neq 0$, such that*

$$\left(I - K_{e(0)}\right) f = 0. \tag{1.3.28}$$

Now we show that the assumption (H3), 0 is not a resonance, implies some consequences in terms of the $L^\infty(\mathbb{R})$ solutions of $\mathcal{H}u = 0$.

Proposition 1.3.7. *Let us suppose that V satisfies the hypothesis (H1). If the Hamiltonian $\mathcal{H}$ has a zero resonance of $a-$order with $a \in (1, 2]$, then there exists $u \in L^\infty(\mathbb{R})$, $u \neq 0$ solution of the equation*

$$\mathcal{H}u = 0.$$

Moreover, let $f \in L^1_a(\mathbb{R})$, $f \neq 0$, be a zero resonance state od $a-$order, i.e. there exist $\delta > 0$, $e \in E_\delta$ such that

$$(I - K_{e(0)})f = 0. \tag{1.3.29}$$

Then u has the following expression

$$u(x) = (\partial_x^2)^{-1} f_1 + (f, 1) e_0(x, 0), \tag{1.3.30}$$

with

$$f_1 = P_{e(0)} f \in L^1_a(\mathbb{R})$$

Proof. Let $f \in L^1_a(\mathbb{R})$, $f \neq 0$, such that (1.3.29) is verified. Using the definition of the projector we have that

$$f = P_{e(\lambda)} f + (f, 1) e(\lambda). \tag{1.3.31}$$

Applying $(I - K_{e(\lambda)})$ in the relation above, we get

$$0 = (I - K_{e(\lambda)})P_{e(\lambda)}f + (f, 1)e(\lambda). \tag{1.3.32}$$

We put

$$u_\lambda \colon = R(\lambda^2, \mathcal{H}_0)P_{e(\lambda)}f + (f, 1)e_0(\lambda).$$

The relation (1.3.32) and the resolvent identity imply that u_λ is a solution of the following problem

$$(\lambda^2 - \mathcal{H})u_\lambda = 0.$$

Moreover, by Lemma 1.3.4 and by the regularity properties of e_0 we have that

$$u \colon = R(0, \mathcal{H}_0)P_{e(0)}f + (f, 1)e_0(0) \in L^\infty(\mathbb{R}), \tag{1.3.33}$$

and

$$\mathcal{H}u = 0.$$

Now, to conclude the proof we have to show that $u \neq 0$. Suppose that $u = 0$. Then we have that

$$R(0, \mathcal{H}_0)P_{e(0)}f = -(f, 1)e_0(0).$$

So, we have that

$$R(0, \mathcal{H}_0)P_{e(0)}f(x) = \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz \frac{|x - y| - |x - z|}{2} f(y)e(z, 0),$$

but on the other side, $e_0(0)$ is in $C_c^\infty(\mathbb{R})$. □

Remark 1.3.8. Let $V \in L^1_a(\mathbb{R})$, $a \in (1, 2]$ and let $f \in L^1_b(\mathbb{R})$ be a zero resonance state of b-order with $b \in (1, a]$. Then $f \in L^1_a(\mathbb{R})$. Indeed, by formula (1.3.32) we have that

$$(I - K_{e(\lambda)})P_{e(\lambda)}f = -(f, 1)e(\lambda).$$

Since $e(\lambda) \in C_c^\infty(\mathbb{R})$, in particular $P_{e(\lambda)}f$ is $L^1_a(\mathbb{R})$. This prove that the zero resonance state f lives in $L^1_a(\mathbb{R})$.

Remark 1.3.9. We note that asymptotically the generalized eigenfunction u in (1.3.30) has the following structure

$$u(x) = c^* + O\left(\frac{1}{|x|^{a-1}}\right), \ |x| \to \infty,$$

where $c^* \in \mathbf{C}$. Indeed, using the formula (1.3.33) and the notations of the previous Lemma we have that

$$\begin{aligned} u(x) = & \int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0) \\ & + \int_{|y|<|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0) \\ & + (f,1)e_0(x,0), \end{aligned}$$

where $\Omega_e = [-M_e, M_e]$, with $M_e > 0$ such that $\operatorname{supp} e_0(\lambda) \subset \Omega_e$ for any $\lambda \in S_\delta^+$. Since we are interested in the behaviour for x large we can always consider $|x| \geq M_e$. We have that as $|x| \to +\infty$

$$\begin{gathered} e_0(x,0) = 0; \\ \int_{|y|<|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0) \to c_1 \in \mathbf{C}; \\ \int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0) \to 0. \end{gathered}$$

Now it remains to check the speed of convergence of the last term. We recall that $P_{e(0)}P_{e(0)} = P_{e(0)}$, hence

$$\int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0) = \int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} P_{e(0)}f(y)e(z,0).$$

Passing to the modulus we have that

$$\begin{aligned} | \int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} P_{e(0)}f(y)e(z,0) \ | \leq \\ \leq \int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y-x-z|}{2} |P_{e(0)}f(y)||e(z,0)|. \end{aligned}$$

Furthermore, using the property of e we have that

$$\int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y-x-z|}{2} |P_{e(0)}f(y)||e(z,0)| \leq C_e \int_{|y|\geq|x|} dy(1+|y|)|P_{e(0)}f(y)|.$$

Since we are in the case $|y| \geq |x|$, if we multiply and divide by $(1+|y|)^{a-1}$ in the integral on the right side we have the following estimate

$$\int_{|y|\geq|x|} dy(1+|y|)|P_{e(0)}f(y)| \leq \left(\frac{1}{(1+|x|)^{a-1}}\right) \|P_{e(0)}f\|_{L^1_a(\mathbb{R})}.$$

This prove that

$$\left|\int_{|y|\geq|x|} dy \int_{\Omega_e} dz \frac{|x-y|-|x-z|}{2} f(y)e(z,0)\right| \leq C_e \frac{1}{(1+|x|)^{a-1}} \|f\|_{L^1_a(\mathbb{R})}.$$

In the following proposition we give a characterization of the zero resonance for the perturbed Hamiltonian $\mathcal{H}$.

Proposition 1.3.10. *Let $V \in L^1_a(\mathbb{R})$, $a > 1$ and let $\mathcal{H} = \mathcal{H}_0 + V$ be the perturbed Hamiltonian. The following statement are equivalent:*

(a) There exist $b \in (1, a]$, $f \in L^1_b(\mathbb{R})$, $f \neq 0$ such that f is a zero resonance state of b-order;

(b) There exists $u \in L^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} V(x)u(x)\,dx = 0$ and $\mathcal{H}u = 0$;

(c) There exists $f \in L^1_a(\mathbb{R})$, $f \neq 0$, such that f is a zero resonance state od a-order.

Proof. The proposition (a) and (c) are equivalent thanks to the Remark 1.3.8. Hence, to prove the Proposition 1.3.10 we will prove that (b) and (c) are equivalent. We assume the statement (c). The Proposition 1.3.7 guarantees that in correspondence of a resonance state $f \in L^1_a(\mathbb{R})$ we can construct $u \in L^\infty(\mathbb{R})$ such that

$$u = (\partial_x^2)^{-1} P_{e(0)} f + (f, 1)e_0(0),$$

and

$$\mathcal{H}u = 0.$$

In particular we have that

$$Vu = \partial_x^2 u = P_{e(0)}f + (f, 1)\partial_x^2 e_0(0),$$

from which it follows that

$$\int_{\mathbb{R}} V(x)u(x)\,dx = 0.$$

Vice versa, if we suppose (b), let $\tilde{c} \in \mathbb{R}$ and let $\tilde{e}_0 \in C_c^\infty(\mathbb{R})$. We can define

$$f_1 = Vu - \tilde{c}\partial_x^2 \tilde{e}_0,$$

and we put

$$f:=V(\partial_x^2)^{-1}f_1.$$

By the inequalities (1.3.15), (1.3.16), the hypothesis $V\in L^1_a(\mathbb{R})$ and the fact that $\int_{\mathbb{R}} f_1=0$ we have $f\in L^1_a(\mathbb{R})$. Moreover, since $\partial_x^2 u=Vu$, and $u=(\partial_x^2)^{-1}f_1+\tilde{c}\tilde{e}_0$ we have that

$$f_1+\tilde{c}\partial_x^2\tilde{e}_0=V(\partial_x^2)^{-1}f_1+V\tilde{c}\tilde{e}_0,$$

and hence

$$f=f_1+\tilde{c}\partial_x^2\tilde{e}_0-V\tilde{c}\tilde{e}_0.$$

Finally we note that

$$K_{e(0)}f=V(\partial_x^2)^{-1}P_{e(0)}f=V(\partial_x^2)^{-1}f_1=f.$$

□

Now we extend the definition of resonances to the positive energies. Then we will prove that the hypothesis (H1) guarantees that the Hamiltonian $\mathcal{H}$ has not positive resonances.

Let $\alpha\in\mathbb{R}\setminus\{0\}$. The operator

$$R(\alpha^2,\mathcal{H}_0)=\left((\alpha+i0)^2-\mathcal{H}_0\right)^{-1}f=\lim_{\epsilon\to 0}\left((\alpha+i\epsilon)^2-\mathcal{H}_0\right)^{-1}f=\int_{\mathbb{R}}\frac{e^{i\alpha|x-y|}}{2i\alpha}f(y)\,dy$$

is well defined as operator from $L^1(\mathbb{R})\to L^\infty(\mathbb{R})$ and moreover the following estimate holds

$$\left\|R(\alpha^2,\mathcal{H}_0)f\right\|_{L^\infty(\mathbb{R})}\le\frac{C}{|\alpha|}\|f\|_{L^1(\mathbb{R})}.$$

Definition 1.3.11. *Let $\alpha^2>0$ and let $V\in L^1_a(\mathbb{R})$, $a>1$. We say that α^2 is a resonance for the perturbed Hamiltonian $\mathcal{H}$ if there exist $b\in(1,a]$, $f\in L^1_b(\mathbb{R})$, $f\neq 0$, such that*

$$\left[I-VR(\alpha^2,\mathcal{H}_0)\right]f=0.$$

The following Lemma guarantees that $\mathcal{H}$ does not admit positive resonance points.

Lemma 1.3.12. *Let $V\in L^1_a(\mathbb{R})$, $a>1$. Then $\mathcal{H}$ has no resonances in $(0,\infty)$.*

Proof. Suppose that there exists a positive resonance $\alpha^2>0$, i.e. there exists a no zero function

$f \in L^1_b(\mathbb{R})$, for some $b \in (1, a]$, such that $f = VR(\alpha^2, \mathcal{H}_0)f$. We set $u = R(\alpha^2, \mathcal{H}_0)f$. Then u solves the following equation

$$\alpha^2 u = \mathcal{H}u.$$

As in Remark 1.3.9, one can obtain the asymptotic expansions

$$u(x) = e^{i\alpha x}c_+ + O(|x|^{1-a}),\ u'(x) = i\alpha e^{i\alpha x}c_+ + O(|x|^{1-a}), \quad x \nearrow +\infty, \tag{1.3.34}$$

$$u(x) = e^{-i\alpha x}c_- + O(|x|^{1-a}),\ u'(x) = -i\alpha e^{i\alpha x}c_- + O(|x|^{1-a}), \quad x \searrow -\infty. \tag{1.3.35}$$

Let $f_\pm(x;\lambda)$ be the Jost functions, namely the solutions to

$$\lambda^2 f_\pm = \mathcal{H}f_\pm,$$

with $\Im\lambda > 0$, such that $f_\pm(x,\lambda) = e^{\pm i\lambda x}m_\pm(x,\lambda)$ and

$$\lim_{x\nearrow+\infty} |m_+(x;\lambda) - 1| + |m'_+(x;\lambda)| = 0, \quad \lim_{x\searrow-\infty} |m_-(x;\lambda) - 1| + |m'_-(x;\lambda)| = 0. \tag{1.3.36}$$

Let us denote the Wronskian as follows:

$$W(v_1, v_2) = v'_1(x)v_2(x) - v_1(x)v'_2(x),$$

where v_1, v_2 are two solutions to $\lambda^2 v = \mathcal{H}v$. Since $W(f_+, f_-) \neq 0$, we have

$$u = af_+ + bf_-.$$

We remember that the Wronskian is indipendent from x. Hence, using (1.3.34) and (1.3.36), and computing the Wronskian for $x \to +\infty$, we get $W(u, f_+) = 0$, that implies $b = 0$. Similarly, (1.3.35) and (1.3.36), for $x \to -\infty$, imply $W(u, f_-) = 0$. It follows that $a = 0$ and consequently $u = 0$.

This completes the proof of the lemma. □

We conclude this section noting that, if we suppose that the potential V satisfies the hypotheses (H1), (H2), (H3), then $\mathcal{H}$ has the same configuration in terms of eigenvalues and resonances of $\mathcal{H}_0$. Hence, we can expect to get for $R(z, \mathcal{H})$ estimates similar to the free case.

1.4 Estimates for the perturbed resolvent

The main goal of this section will be to get sectorial estimates for the perturbed resolvent operator. This result is presented in Theorem 1.4.6. At first we will prove sectorial estimates close to the origin and then away from the origin. The hypothesis that zero is not a resonance will turn to be fundamental.

The next results are preparatory for the perturbed resolvent estimates near the origin.

Lemma 1.4.1. *Let V be a potential such that the hypotheses (H1), (H2) and (H3) are satisfied. Then, there exists $\delta > 0$ such that for any $e \in E_\delta$ there exists $C_e > 0$ so that for any $\lambda \in \overline{S_\delta^+}$ the operator $\left(I - K_{e(\lambda)}\right)^{-1}$ exists in $L^1_b(\mathbb{R})$ and it satisfies the following inequality*

$$\| \left(I - K_{e(\lambda)}\right)^{-1} f\|_{L^1_b(\mathbb{R})} \leq C_e \|f\|_{L^1_b(\mathbb{R})}, \ b \in (1, a]. \tag{1.4.1}$$

Proof. Since we consider $V \in L^1_a(\mathbb{R})$ and $b \in (1, a]$, by the inequalities (1.3.15), (1.3.16) we deduce that

$$K_{e(\lambda)} \colon L^1_b(\mathbb{R}) \to L^1_b(\mathbb{R})$$

is a compact operator, continuous in $\overline{S_\delta^+}$ and

$$\|K_{e(\lambda)} f\|_{L^1_b(\mathbb{R})} \leq C_e \|V\|_{L^1_a(\mathbb{R})} \|f\|_{L^1_b(\mathbb{R})}.$$

Moreover, the assumption (H3) guarantees that there exist $\delta > 0$ such that for any $e \in E_\delta$ the operator $\left(I - K_{e(\lambda)}\right)$ is invertible. Hence we conclude that (1.4.1) is satisfied. □

Thanks to Lemma 1.3.5 and Lemma 1.4.1 we are able to derive a kind of limiting absorption principle for the perturbed Hamiltonian $\mathcal{H}$. This result is described in the following Lemma and it will be the key tool to establish the sectorial property of the perturbed Hamiltonian.

Lemma 1.4.2. *Let V be a potential such that (H1) is satisfied, i.e. $V \in L^1_a(\mathbb{R})$, with $a > 1$. Moreover, the Hamiltonian $\mathcal{H}$ satisfies the hypothesis (H2) and (H3). Then there exists $\delta > 0$ such that for any $e \in E_\delta$, for any $\lambda \in S_\delta^+$ the following resolvent estimate holds*

$$\left\|R(\lambda^2, \mathcal{H}) f\right\|_{L^\infty(\mathbb{R})} \leq C_e \|f\|_{L^1_a(\mathbb{R})}, \tag{1.4.2}$$

for any $f \in L^1_a(\mathbb{R})$.

Proof. The Lemma 1.4.1 shows that, assigned $f \in L^1_a(\mathbb{R})$, there exist $\delta > 0, e \in E_\delta, g_\lambda \in L^1_a(\mathbb{R})$ such that

$$f = (I - K_{e(\lambda)})g_\lambda,$$

for any $\lambda \in S^+_\delta$. We can rewrite g_λ as follows

$$g_\lambda(x) = P_{e(\lambda)}g_\lambda(x) + (g_\lambda, 1)e(\lambda, x). \tag{1.4.3}$$

We put

$$f_1(\lambda)(x) = f_1(x, \lambda) \colon = P_{e(\lambda)}g_\lambda(x).$$

Hence we have that $f_1(\lambda) \in B^0_a(\mathbb{R})$ for any $\lambda \in S^+_\delta$.

Applying the operator $(I - K_{e(\lambda)})$ in (1.4.3) and using that $K_{e(\lambda)}e(\lambda) = 0$ we get the following decomposition of the space $L^1_a(\mathbb{R})$:

$$f(x) = (I - K_{e(\lambda)})f_1(x, \lambda) + \left((I - K_{e(\lambda)})^{-1}f, 1\right) e(x, \lambda), \tag{1.4.4}$$

for any $x \in \mathbb{R}$, $\lambda \in S^+_\delta$. From the resolvent identity

$$R(\lambda^2, \mathcal{H})(I - K_{e(\lambda)})P_{e(\lambda)} = R(\lambda^2, \mathcal{H}_0)P_{e(\lambda)}$$

it follows that

$$R(\lambda^2, \mathcal{H})f = R(\lambda^2, \mathcal{H}_0)P_{e(\lambda)}f_1(\lambda) + \left((I - K_{e(\lambda)})^{-1}f, 1\right) e_0(\lambda). \tag{1.4.5}$$

Moreover, we know that

$$P_{e(\lambda)}(I - K_{e(\lambda)})^{-1}f = P_{e(\lambda)}g_\lambda = f_1(\lambda).$$

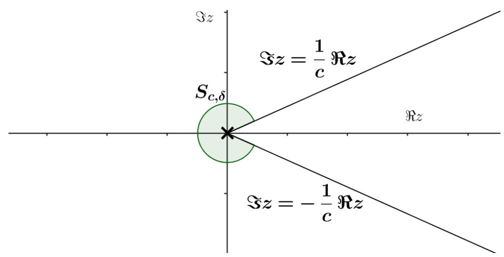
Now computing the L^∞ norm in (1.4.5), thanks to the estimates (1.3.15), (1.4.1) and to the hypotheses on e_0 we get

$$\|R(\lambda^2, \mathcal{H})f\|_{L^\infty(\mathbb{R})} \leq C_e\|f\|_{L^1_a(\mathbb{R})}.$$

$\square$

Now we prove the sectorial estimates near the origin.

Theorem 1.4.3. *Let V be a potential such that (H1) is satisfied. Moreover the Hamiltonian $\mathcal{H}$ satisfies* *the hypotheses (H2) and (H3). Then, for any* $1 < p \leq \infty$ *we have that there exists* $\delta > 0$*, such that for any* z *in the sector* $S_{c,\delta}$



$$S_{c,\delta} : = \{z \in \mathbf{C} \setminus [0,\infty) \mid \Re z \leq c|\Im z|, |z| \leq \delta\},$$

with $c > 0$*, we have that*

$$\|R(z, \mathcal{H})f\|_{L^p(\mathbb{R})} \leq \frac{C}{|z|}\|f\|_{L^p(\mathbb{R})}. \tag{1.4.6}$$

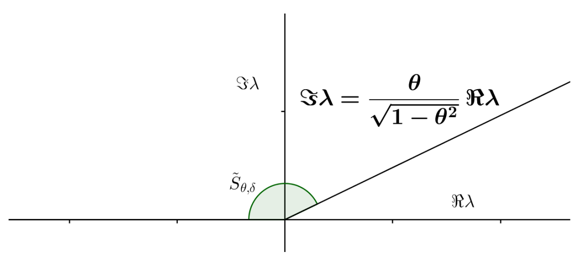
Proof. We use the diffeomorphism

$$z \mapsto \sqrt{z} = \lambda$$

with branch cut $[0, \infty)$, such that the map

$$z \in \{z \in \mathbf{C} \setminus [0,\infty)\} \mapsto \lambda = \sqrt{z} \in \{\Im\lambda > 0\}$$

is well defined and analytic. Using this transformation, we see that the sector $S_{c,\delta}$ is transformed into



$$\tilde{S}_{\theta,\delta} = \left\{\lambda \in \mathbf{C} \mid \Im\lambda > \theta|\lambda|, \theta = \theta(c) \in (0,1), |\lambda| < \sqrt{\delta}\right\}.$$

Hence, we have to prove that the inequality

$$\|R(\lambda^2, \mathcal{H})f\|_{L^p(\mathbb{R})} \leq \frac{C}{|\lambda|^2}\|f\|_{L^p(\mathbb{R})}, \tag{1.4.7}$$

holds for any $\lambda \in \tilde{S}_{\theta,\delta}$. To prove the inequality (1.4.7) we first consider the cases $p = 2$ and $p = \infty$.

If $p = 2$, the operator $\mathcal{H}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ with spectrum $\sigma(\mathcal{H}) = [0, \infty)$. Let $f \in L^2(\mathbb{R})$ and let dE_f be the spectral measure associated. The spectral theorem implies the following estimates

$$
\begin{aligned}
\|R(\lambda^2, \mathcal{H})f\|^2_{L^2(\mathbb{R})} = \big((\lambda^2 - \mathcal{H})^{-1} f, f\big) &\leq \int_0^\infty |\lambda^2 - \alpha|^{-2} \, dE_f \\
&\leq \frac{1}{|\Im\lambda|^4} \|f\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{1}{\theta^4 |\lambda|^4} \|f\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Now we consider the case $p = \infty$. The resolvent identity combined with the sectorial estimates for the free Hamiltonian and with Lemma 1.4.2 give us

$$
\begin{aligned}
\|R(\lambda^2, \mathcal{H})f\|_{L^\infty(\mathbb{R})} \leq& \|R(\lambda^2, \mathcal{H}_0)f\|_{L^\infty(\mathbb{R})} + \|R(\lambda^2, \mathcal{H})VR(\lambda^2, \mathcal{H}_0)f\|_{L^\infty(\mathbb{R})} \\
\leq& \frac{C(\theta, e, \|V\|_{L^1_b(\mathbb{R})})}{|\lambda|^2} \|f\|_{L^\infty(\mathbb{R})}.
\end{aligned}
$$

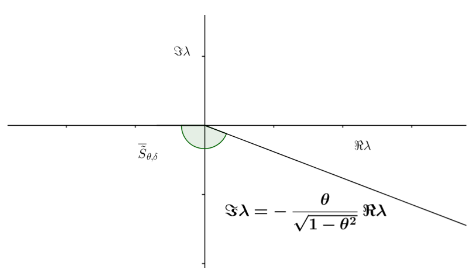
Applying the Riesz-Thorin interpolation theorem we get sectorial estimates

$$
\|R(\lambda^2, \mathcal{H})f\|_{L^p(\mathbb{R})} \leq \frac{C(\theta, e, \|V\|_{L^1_b(\mathbb{R})})}{|\lambda|^2} \|f\|_{L^p(\mathbb{R})},
$$

for $2 < p < \infty$. By duality argument we will prove the estimates for $p \in (1, 2)$. Indeed, let we consider the conjugate diffeomorphism

$$
\lambda \in \tilde{S}_{\theta,\delta} \mapsto \overline{\lambda} \in \overline{\tilde{S}_{\theta,\delta}}.
$$

We have that



$$
\|R(\overline{\lambda^2}, \mathcal{H})f\|_{L^p(\mathbb{R})} \leq \frac{C}{|\lambda|^2} \|f\|_{L^p(\mathbb{R})},
$$

for any $\lambda \in \overline{\tilde{S}_{\theta,\delta}}$, and $2 \leq p \leq \infty$. Let $p \in (1,2)$ and let p' be the conjugate exponent. Since $\mathcal{H}$ is a self-adjoint operator, the following relation holds

$$\left((\lambda^2 - \mathcal{H})^{-1} f, g\right) = \left(f, (\overline{\lambda^2} - \mathcal{H})^{-1} g\right),$$

where $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$. Therefore we get

$$\|R(\lambda^2, \mathcal{H}) f\|_{L^p(\mathbb{R})} = \sup_{\|g\|_{L^{p'}(\mathbb{R})}=1} \left|\left((\lambda^2 - \mathcal{H})^{-1} f, g\right)\right| \leq \frac{C}{|\lambda|^2} \|f\|_{L^p(\mathbb{R})}.$$

This completes the proof. □

Now it remains to prove the sectorial estimates away from the origin. In order to do this we will use that $\mathcal{H}$ cannot have positive resonances as proved in Lemma 1.3.12.

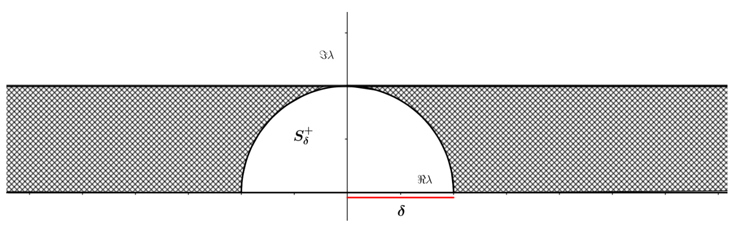
We define the operator

$$K_0(\lambda) = V(\lambda^2 - \mathcal{H}_0)^{-1} = V R(\lambda^2, \mathcal{H}_0),$$

and the sector

$$N_\delta^+ = \{\lambda \in \mathbf{C} \mid |\lambda| \geq \delta, \Im\lambda \in [0, \delta], \delta > 0\}.$$

As in the previous case, we need some preparatory lemmas to get resolvent estimates.



Lemma 1.4.4. *Suppose that the potential V satisfies (H1) and in addition the Hamiltonian $\mathcal{H}$ satisfies (H2) and (H3). Then, for any $b \in (1, a]$, there exist $C > 0$ and $\delta > 0$ such that for any $\lambda \in N_\delta^+$ the operator $(I - K_0(\lambda))^{-1} f$ exists in $L_b^1(\mathbb{R})$ and satisfies the following estimate*

$$\left\|(I - K_0(\lambda))^{-1} f\right\|_{L_b^1(\mathbb{R})} \leq C\|f\|_{L_b^1(\mathbb{R})}. \tag{1.4.8}$$

Proof. The key point in the proof is the estimate

$$\|\left(K_0(\alpha + i\varepsilon_1) - K_0(\alpha + i\varepsilon_2)\right) f\|_{L_b^1(\mathbb{R})} \leq C|\varepsilon_1 - \varepsilon_2|^{a-b}\|f\|_{L_b^1(\mathbb{R})} \tag{1.4.9}$$

valid for any $b \in [1, a]$, any $\alpha \in \mathbb{R}$ with $|\alpha| \geq \delta/\sqrt{2}$ and any $\varepsilon_1, \varepsilon_2 \in [0, \delta/\sqrt{2}]$. Using the relation 1.3.3, we get

$$K_0(\lambda)(f)(x) = V(x) \int_{\mathbb{R}} \frac{e^{i\lambda|x-y|}}{2i\lambda} f(y)dy. \tag{1.4.10}$$

We can proceed as in the proof of inequality (1.3.16). Let $\lambda_1 = \alpha + i\epsilon_1$, $\lambda_2 = \alpha + i\epsilon_2$, $0 < \bar{\delta} < \delta < |\lambda_i|$, with $i = 1, 2$, and let $\beta > 0$. Since $\epsilon_1, \epsilon_2 \geq 0$, we have that

$$\left| \frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} \right| \leq \frac{C}{\bar{\delta}^2} |\lambda_1 - \lambda_2| \langle \beta \rangle.$$

Indeed, we have that

$$\frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} = i \int_0^\beta e^{i\lambda_1 \tau} (1 - e^{i(\lambda_2 - \lambda_1)\tau}) \, d\tau + \frac{\lambda_2 - \lambda_1}{\lambda_2 \lambda_1}.$$

Integrating by part we get the following identity

$$\frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} = \left(\frac{e^{i\lambda_1 \beta}}{\lambda_1} \right) \left(1 - e^{i(\lambda_2 - \lambda_1)\beta} \right) + i \int_0^\beta \frac{e^{i\lambda_1 \tau}}{\lambda_1} (\lambda_2 - \lambda_1) e^{i(\lambda_2 - \lambda_1)\tau} \, d\tau + \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}.$$

Hence, passing to the modulus we have that

$$\begin{aligned} \left| \frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} \right| &\leq \frac{2}{\bar{\delta}} |\lambda_1 - \lambda_2| \beta + \frac{|\lambda_2 - \lambda_1|}{\bar{\delta}^2} \\ &\leq \frac{C}{\bar{\delta}^2} |\lambda_1 - \lambda_2| \langle \beta \rangle. \end{aligned} \tag{1.4.11}$$

On the other side we also have

$$\left| \frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} \right| \leq \frac{C}{\bar{\delta}^2}. \tag{1.4.12}$$

The inequalities (1.4.11) and (1.4.12) imply

$$\left| \frac{e^{i\lambda_1 \beta}}{\lambda_1} - \frac{e^{i\lambda_2 \beta}}{\lambda_2} \right| \leq \frac{C}{\bar{\delta}^2} |\lambda_1 - \lambda_2|^\theta \langle \beta \rangle^\theta,$$

for any $\theta \in [0,1]$. Choosing $\theta = a-b$, we have that

$$\begin{aligned}\|\left(K_0(\alpha+i\varepsilon_1)-K_0(\alpha+i\varepsilon_2)\right)f\|_{L^1_b(\mathbb{R})} \leq & \frac{C}{\overline{\delta}^2}|\varepsilon_1-\varepsilon_2|^{a-b}\int_{\mathbb{R}} dx\langle x\rangle^b|V(x)|\int_{\mathbb{R}} dy\langle x-y\rangle^{a-b}|f(y)| \\ \leq & \frac{C}{\overline{\delta}^2}|\varepsilon_1-\varepsilon_2|^{a-b}\int_{\mathbb{R}} dx\langle x\rangle^a|V(x)|\int_{\mathbb{R}} dy\frac{\langle x-y\rangle^{a-b}}{\langle x\rangle^{a-b}}|f(y)| \\ \leq & \frac{C}{\overline{\delta}^2}|\varepsilon_1-\varepsilon_2|^{a-b}\|V\|_{L^1_a(\mathbb{R})}\|f\|_{L^1_b(\mathbb{R})},\end{aligned}$$

for any $b \in [1,a]$. Finally we use that α^2 is not a resonance point to derive the invertibility of the operator $I - K_0(\lambda)$ as well as the estimate (1.4.8). This completes the proof of the Lemma. □

In particular we have the following Lemma:

Lemma 1.4.5. *There exists $\delta > 0$ so that for any $\lambda \in \{\lambda \in \mathbf{C}; \Im\lambda > 0,\ |\lambda| > \delta\}$ we have the estimate*

$$\|R(\lambda^2,\mathcal{H})f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^1_b(\mathbb{R})} \tag{1.4.13}$$

for $b \in [1,a]$.

Now we are in position to state and complete the proof of the main theorem of this chapter.

Theorem 1.4.6. *[Sectorial estimates] Suppose that the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V(x)$ satisfies the following assumptions:*

(H1) $V \in L^1_a(\mathbb{R})$, $a > 1$;

(H2) $\sigma_p(\mathcal{H}) = \emptyset$;

(H3) $\mathcal{H}$ has no zero resonance.

Then for any $1 < p \leq \infty$, the following resolvent estimate holds:

$$z \in \mathbf{C}\setminus[0,\infty), \Re z \leq c|\Im z|, \Longrightarrow \|R(z,\mathcal{H})f\|_{L^p(\mathbb{R})} \leq \frac{C}{|z|}\|f\|_{L^p(\mathbb{R})}. \tag{1.4.14}$$

Proof. The proof of Theorem 1.4.3 establishes this result for z close to the origin. If z is far from the origin, using Lemma 1.4.5 and hence the estimate (1.4.13) we can complete the proof in the case in which z is far from the origin. □

Remark 1.4.7. By Theorem 1.1.12 follows that we can define the fractional powers of the perturbed Hamiltonian $\mathcal{H}$ by means of the Balakrishnan representation

$$\mathcal{H}^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty z^{-\alpha}(z+\mathcal{H})^{-1}\,dz,\ \ \alpha \in (0,1). \tag{1.4.15}$$

Chapter 2

Perturbed Homogeneous Besov spaces: equivalent norms

In this chapter we consider 1-D Laplace operator with short range potential V and we study homogeneous Besov type spaces $\dot{B}^s_p(\mathbb{R})$ where $0 \leq s < 1/p$, $1 < p < \infty$. The main goal is to study how the classical homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ are transformed under the action of the wave operators. We present the plan of the chapter. In Section 2.1 we recall some classical results concerning with the Jost functions, the transmission and reflection coefficients. These results are mainly contained in the papers [72], [19]. In Lemma 2.1.2, Lemma 2.1.3 and Lemma 2.1.9 we establish some improved estimates for modified Jost functions, transmission and reflection coefficients (one can see also [27]). These estimates will be crucial to reach our goal. The Section 2.3 is devoted to the proof of the main results. In particular it is shown that non resonance assumption at zero and sufficiently decay of potential at infinity guarantee that the free Hamiltonian and the perturbed one generate equivalent Besov norms $\dot{B}^s_p(\mathbb{R})$, under the condition $s < 1/p$. Moreover some relevant counterexample that justify the requirement $s < 1/p$ are given.

2.1 Functional calculus for the perturbed Hamiltonian

We start giving an informal overview on the connection between the functional calculus for the perturbed Hamiltonian and the Jost functions, the transmission and the reflection coefficients. This connection will motivate the meticulous study of the aforementioned functions in the following.

The spectral Theorem 1.1.1 and the Stone formula (1.1.2) imply that for any $g \in L^\infty(0,\infty)$ we have

$$g(\mathcal{H}) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty g(\lambda) E_{a.c.}(d\lambda), \tag{2.1.1}$$

where

$$E_{a.c.}(d\lambda) = \left[(\lambda + \mathrm{i}0 + \partial_x^2 - V)^{-1} - (\lambda - \mathrm{i}0 + \partial_x^2 - V)^{-1}\right] d\lambda, \tag{2.1.2}$$

is the spectral measure. The representation formula for the kernel $(\lambda \pm \mathrm{i}0 - \mathcal{H})^{-1}(x,y)$, namely, the Green function of the operator $(\lambda \pm \mathrm{i}0 - \mathcal{H})$, is given by

$$(\lambda \pm \mathrm{i}0 - \mathcal{H})^{-1}(x,t) = \begin{cases} \frac{f_-(x,\pm\sqrt{\lambda}) f_+(t,\pm\sqrt{\lambda})}{w(\pm\sqrt{\lambda})}, & \text{if } x < t; \\ \frac{f_-(t,\pm\sqrt{\lambda}) f_+(x,\pm\sqrt{\lambda})}{w(\pm\sqrt{\lambda})}, & x \geq t. \end{cases} \tag{2.1.3}$$

Here, $\sqrt{\lambda}$ is the analytic branch of $\lambda^{1/2}$ with branch cut $[0,\infty)$, such that the map

$$\lambda \in \{\lambda \in \mathbb{C} \setminus [0,\infty)\} \mapsto \sqrt{\lambda} \in \{\Im \sqrt{\lambda} > 0\} \tag{2.1.4}$$

is well-defined analytic diffeomorphism. The functions $f_\pm(x,\sqrt{\lambda})$, known as the Just functions, are the solutions of the problem

$$-\frac{d}{dx^2} f_\pm + V f_\pm = \lambda f_\pm,$$

such that they verify the free asymptotic behaviour $f_\pm(x,\tau) \approx e^{\pm i\tau x}$ as $x \approx \pm\infty$. The Wronskian $w(\lambda)$ is defined by the relation

$$w(\lambda) := [f_-, f_+] = \partial_x f_+(x,\lambda) f_-(x,\lambda) - f_+(x,\lambda)\partial_x f_-(x,\lambda). \tag{2.1.5}$$

Operating in (2.1.3) the change of variable $\lambda = \tau^2$, the symmetry of the problem with respect to τ

$$-\frac{d}{dx^2} f_\pm + V f_\pm = \tau^2 f_\pm,$$

suggests to introduce the transmission and the reflection coefficients $T(\tau)$ and $R_\pm(\tau)$ defined later on. In particular, we could express the Wronskian in terms of the transmission coefficient by means of the following relation

$$\frac{1}{T(\tau)} = \frac{w(\tau)}{2\mathrm{i}\tau}. \tag{2.1.6}$$

Substituting the relations (2.1.3) and (2.1.6) in (2.1.1) we can connect the functional calculus for the operator $\mathcal{H}$ with the functions $f_{\pm}(x,\tau)$, $T(\tau)$ and $R_{\pm}(\tau)$. In particular, if $x<y$ we get

$$g(\mathcal{H})(x,y) = -\frac{1}{2\pi}\int_{\mathbf{R}} T(\tau)g(\tau^2)f_{-}(x,\tau)f_{+}(y,\tau)d\tau. \tag{2.1.7}$$

One can proceed similarly for $x \geq y$. Furthermore, we define the modified Jost functions $m_{\pm}(x,\tau)$ such that

$$f_{\pm}(x,\tau) = e^{\pm ix\tau}m_{\pm}(x,\tau) \tag{2.1.8}$$

and

$$\lim_{x\to\pm\infty} m_{\pm}(x,\tau) = 1, \tag{2.1.9}$$

that will turn to be crucial to simplify the problem of studying the properties of the Jost functions.

Finally, for any even function $\varphi(\tau) \in L^1(\mathbb{R})$, we can express the functional calculus for $\mathcal{H}$ as follows:

$$\varphi\left(\sqrt{\mathcal{H}}\right)(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\tau)\, T(\tau)m_{+}(y,\tau)m_{-}(x,\tau)e^{-\mathrm{i}\tau(x-y)}d\tau, \text{ for } x<y. \tag{2.1.10}$$

Hence, one of the main goal will be to investigate the properties of the functions $m_{\pm}(x,\tau), T(\tau), R_{\pm}(\tau)$ in order to get properties and estimates for the kernel (2.1.10).

2.1.1 An overview on Jost functions, transmission and reflection coefficients

We consider the Sturm-Liouville problem

$$-\frac{d}{dx^2}f(x,\tau) + V(x)f(x,\tau) = \tau^2 f(x,\tau), \tag{2.1.11}$$

with the limiting conditions

$$f_{+}(x,\tau) \approx e^{ix\tau}, \ x\to+\infty, \tag{2.1.12}$$

and

$$f_{-}(x,\tau) \approx e^{-ix\tau}, \ x\to-\infty. \tag{2.1.13}$$

The limiting conditions (2.1.12)-(2.1.13) require a free asymptotic behaviour, that is a natural requirement since we are considering short range potentials $V \in L^1_\gamma(\mathbb{R})$, with $\gamma \geq 1$. Moreover, we recall that we have also assumed $\sigma_p(\mathcal{H}) = \emptyset$ and $\mathcal{H}$ has no zero resonances, hence the spectral scenario is totally similar to the unperturbed one.

At first we prove formula (2.1.3). To this end, we want to write the general expression of the solution y of the following problem

$$\begin{cases} -\frac{d^2}{dx^2}y(x) + (V(x) - \tau^2)y(x) = F(x), \\ \alpha_1 y(-\infty) + \beta_1 y'(-\infty) = 0, \\ \alpha_2 y(+\infty) + \beta_2 y'(+\infty) = 0, \end{cases} \tag{2.1.14}$$

where $\alpha_1 = i\tau$, $\beta_1 = 1$, $\alpha_2 = -i\tau$, $\beta_2 = 1$. If we denote the solutions of the homogeneous problem with $f_-(x,\tau)$ and $f_+(x,\tau)$, then we can look for the solutions y of the form

$$y(x) = c_1(x)f_-(x,\tau) + c_2(x)f_+(x,\tau). \tag{2.1.15}$$

Proceeding with the variation of the parameters we get the following system

$$\begin{cases} c_1'(x)f_-(x,\tau) + c_2'(x)f_+(x,\tau) = 0, \\ c_1'(x)f_-'(x,\tau) + c_2'(x)f_+'(x,\tau) = -F(x). \end{cases}$$

Let us denote the Wronskian $w(x,\tau) = f_-(x,\tau)f_+'(x,\tau) - f_+(x,\tau)f_-'(x,\tau)$. The relation (2.1.11) implies that $\frac{d}{dx}w(x,\tau) = 0$. Hence, we can rename $w(x,\tau) = w(\tau)$. Solving the system above with respect to $c_1'(x)$, $c_2'(x)$ we get

$$c_1'(x) = \frac{f_+(x,\tau)F(x)}{w(\tau)},$$
$$c_2'(x) = -\frac{f_-(x,\tau)F(x)}{w(\tau)}.$$

By the limiting conditions in (2.1.14) follows respectively that $c_2(-\infty) = 0$ and $c_1(+\infty) = 0$. Then, integrating, we get

$$c_1(x) = -\int_x^{+\infty} \frac{f_+(t,\tau)F(t)}{w(\tau)}\,dt,$$
$$c_2(x) = -\int_{-\infty}^{x} \frac{f_-(t,\tau)F(t)}{w(\tau)}\,dt.$$

Substituting the expressions above in (2.1.15) we deduce the formula for the kernel of the operator

$(\tau^2 - \mathcal{H})^{-1}$:

$$(\tau^2 - \mathcal{H})^{-1}(x,t) = \begin{cases} \frac{f_+(t,\tau) f_-(x,\tau)}{w(\tau)} & x < t, \\ \frac{f_-(t,\tau) f_+(x,\tau)}{w(\tau)} & x \geq t. \end{cases}$$

Since the equation (2.1.11) is symmetric with respect to τ, it is evident that also $f_\pm(x,-\tau)$ are solutions of the problem (2.1.11) with limiting conditions (2.1.12), (2.1.13). Moreover, $f_+(x,\tau)$ and $f_+(x,-\tau)$ and respectively $f_-(x,\tau)$ and $f_-(x,-\tau)$ are independent solutions for $\tau \neq 0$

$$[f_+(x,\tau), f_+(x,-\tau)] = -2i\tau,$$

$$[f_-(x,\tau), f_-(x,-\tau)] = 2i\tau.$$

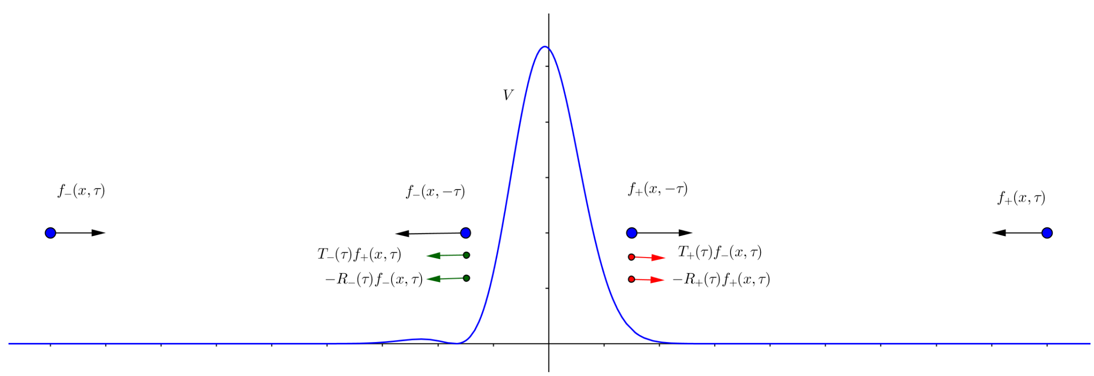
Hence, there exist unique functions $T_\pm(\tau)$ and $R_\pm(\tau)$ such that

$$f_+(x,-\tau) = T_+(\tau) f_-(x,\tau) - R_+(\tau) f_+(x,\tau), \tag{2.1.16}$$

$$f_-(x,-\tau) = T_-(\tau) f_+(x,\tau) - R_-(\tau) f_-(x,\tau), \tag{2.1.17}$$

for $\tau \neq 0$. By physical reasons the coefficients $T_\pm(\tau)$, $R_\pm(\tau)$ are called respectively transmission and reflection coefficients and they are defined by relations (2.1.16) and (2.1.17).

We note that $T_-(\tau) f_+(x,\tau)$ describes a plane wave sent in $-\infty$ that is the overlap of $e^{ix\tau}$ transmitted from $+\infty$ and $R_-(\tau)e^{-ix\tau}$ reflected from $-\infty$. Similarly, $T_+(\tau) f_-(x,\tau)$ represents a plan wave sent in $+\infty$ that is the overlap of the plane wave $e^{-ix\tau}$ transmitted from $-\infty$ and $R_+(\tau) f_+(x,\tau)$ reflected from $+\infty$.



Using the relations (2.1.16), (2.1.17) respectively when $x \approx +\infty$ and $x \approx -\infty$

$$f_-(x,\tau) = \frac{e^{-ix\tau}}{T_+(\tau)} + \frac{R_+(\tau)}{T_+(\tau)} e^{ix\tau},\ x \approx +\infty,$$
$$f_+(x,\tau) = \frac{e^{ix\tau}}{T_-(\tau)} + \frac{R_-(\tau)}{T_-(\tau)} e^{-ix\tau},\ x \approx -\infty,$$

we can compute the Wronskian $w(\tau)$ when x approaches to $+\infty$ or to $-\infty$ and we get the relation

$$w(\tau) = \frac{2i\tau}{T_+(\tau)} = \frac{2i\tau}{T_-(\tau)}.$$

It follows that $T(\tau) := T_-(\tau) = T_+(\tau)$. Moreover, we can easily recover the relations

$$R_+(\tau)T(-\tau) + R_-(-\tau)T(\tau) = 0,$$
$$\overline{T(\tau)} = T(-\tau), \overline{R_\pm(\tau)} = R_\pm(-\tau),$$
$$|T(\tau)|^2 + |R_\pm(\tau)|^2 = 1,$$

computing $[f_-(x,\tau), f_+(x,\tau)]$ and $[f_-(x,-\tau), f_+(x,\tau)]$. Hence, for any $\tau \neq 0$ the scattering matrix defined as

$$S(\tau) = \begin{pmatrix} T(\tau) & R_-(\tau) \\ R_+(\tau) & T(\tau) \end{pmatrix},$$

is an unitary matrix (one can see [19] to enter more in details in this direction).

To better understand the properties of the Jost functions $f_\pm(x,\tau)$, of the transmission and reflection coefficients $T(\tau)$ and $R_\pm(\tau)$ we are going to introduce the modified Jost functions $m_\pm(x,\tau)$ defined in (2.1.8) with limiting conditions (2.1.9).

Substituting (2.1.8) in (2.1.11) we get the following ordinary differential equations

$$\begin{cases} -\frac{d}{dx}\left(e^{\pm 2i\tau x}\frac{d}{dx}m_\pm(x,\tau)\right) + e^{\pm 2ix\tau}V(x)m_\pm(x,\tau) = 0, \\ \lim\limits_{x\to\pm\infty} m_\pm(x,\tau) = 1, \quad \lim\limits_{x\to\pm\infty} \frac{d}{dx}m_\pm(x,\tau) = 0. \end{cases}$$

Integrating we have

$$m_\pm(x,\tau) = 1 \pm \int_x^{\pm\infty} D(\pm(t-x),\tau)V(t)m_\pm(t,\tau)\,dt, \tag{2.1.18}$$

where

$$D(t,\tau) = \frac{e^{2i\tau t}-1}{2i\tau} = \int_0^t e^{2is\tau}\,ds. \tag{2.1.19}$$

The following subsections are devoted to the study of the functions $m_\pm(x,\tau)$, $T(\tau)$ and $R_\pm(\tau)$ and their derivatives $\partial_\tau^k m_\pm(x,\tau)$, $\partial_\tau^k T(\tau)$ and $\partial_\tau^k R_\pm(\tau)$ for $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1$ and $k \le \gamma - 1$.

2.1.2 Estimates for the Jost functions

Here we recall some preliminary estimates for the modified Just functions and their derivatives under the classical assumption $V \in L^1_2(\mathbb{R})$. Then, we improve these estimates in the general case $V \in L^1_\gamma(\mathbb{R})$ with $\gamma \ge 1$.

In the following we set $x_+ := \max\{0,x\}$, $x_- := \max\{0,-x\}$.

The next Lemma contains the main properties and estimates satisfied by the modified Jost functions $m_\pm(x,\tau)$ under the assumption $V \in L^1_2(\mathbb{R})$.

Lemma 2.1.1. *(see Lemma 1 p. 130 [19]) Assume $V \in L^1_2(\mathbb{R})$. Then we have the properties:*

a) For any $x \in \mathbb{R}$ the function

$$\tau \in \overline{\mathbb{C}_\pm} \mapsto m_\pm(x,\tau), \quad \mathbb{C}_\pm = \{\tau \in \mathbb{C}; \Im\tau \gtrless 0\} \tag{2.1.20}$$

is analytic in $\mathbb{C}_\pm$ and $C^1(\overline{\mathbb{C}_\pm})$;

b) There exist constants C_1 and $C_2 > 0$ such that for any $x, \tau \in \mathbb{R}$:

$$|m_\pm(x,\tau) - 1| \le C_1 \langle x_\mp\rangle \langle\tau\rangle^{-1}, \tag{2.1.21}$$

$$|\partial_\tau m_\pm(x,\tau)| \le C_2 \langle x\rangle^2. \tag{2.1.22}$$

Proof. Here, we just recall the main idea of the proof that will be helpful in the following. We consider $m_+(x,\tau)$ but there is the obvious analogous for $m_-(x,\tau)$.

We denote with

$$K_+^j(x,\tau) = \int_{x\le x_1\le \ldots \le x_j} dx_1 \ldots dx_j D(x_1 - x,\tau)\ldots D(x_j - x_{j-1},\tau)V(x_1)\ldots V(x_j), \tag{2.1.23}$$

and we prove that the iterates of the Volterra integral equations

$$m_+(x,\tau) = 1 + \sum_{j=1}^{+\infty} K_+^j(x,\tau)$$

converges. Indeed, considering the trivial estimate

$$|D(t,\tau)| \leq \min\left(t, \frac{1}{|\tau|}\right),$$

we have that

$$|K_+^j(x,\tau)| \leq \frac{\left(\int_x^\infty (t-x)|V(t)|\right)^j}{j!}, \quad |K_+^j(x,\tau)| \leq \frac{1}{|\tau|^j} \frac{\left(\int_x^\infty (t-x)|V(t)|\right)^j}{j!}.$$

Then, the Volterra integral converges and moreover the estimates of the terms K_+^j lead immediately to a rough version of the desired inequality. Indeed we have

$$|m_+(x,\tau) - 1| \leq e^{(1+|\min(0,x)|)\gamma(x)},$$

where $\gamma(x)$ is a bounded function. □

From now we suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma \geq 1$ and we derive some improved estimates.

Lemma 2.1.2. *Suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma \geq 1$. Then we have the following properties:*

a) There exists a constant $C > 0$ such that for any $x \in \mathbb{R}$, $\tau \in \overline{\mathbb{C}_\pm}$, we have

$$|m_\pm(x,\tau) - 1| \leq C\frac{\langle x_\mp \rangle}{\langle x_\pm \rangle^{\gamma-1}}; \tag{2.1.24}$$

b) There exists a constant $C > 0$ such that for any $x \in \mathbb{R}$, $\tau \in \overline{\mathbb{C}_\pm} \setminus \{0\}$, we have

$$|m_\pm(x,\tau) - 1| \leq C\frac{\langle x_\mp \rangle}{\langle x_\pm \rangle^{\gamma}|\tau|}; \tag{2.1.25}$$

c) Let $\sigma \in [0,1)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ we have

$$\|m_\pm(x,\tau) - 1\|_{C^{0,\sigma}(\mathbb{C}_\pm)} \leq C\frac{\langle x_\mp \rangle^{1+\sigma}}{\langle x_\pm \rangle^{\gamma-1-\sigma}}, \ \gamma > 1, \ 0 \leq \sigma \leq \gamma - 1; \tag{2.1.26}$$

d) Let $\sigma \in [0,1)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ we have

$$\|\tau(m_\pm(x,\tau)-1)\|_{C^{0,\sigma}(\mathbb{C}_\pm)} \leq C\frac{\langle x_\mp\rangle^{1+\sigma}}{\langle x_\pm\rangle^{\gamma-\sigma}},\ \gamma > 1. \tag{2.1.27}$$

Proof. We can fix for determinacy the sign $+$ in the left sides of the inequalities (2.1.24)-(2.1.27), since the argument is similar for the term m_-. We start proving the (2.1.24). The right side of (2.1.24) suggests to consider the quantity

$$v(x,\tau) = \frac{\langle x_+\rangle^{\gamma-1}}{\langle x_-\rangle}|m_+(x,\tau)-1|.$$

We plan to use the integral equation for $m_+(x,\tau)$ and to check inequalities of type

$$v(x) \leq a(x) + \int_x^\infty b(t)v(t)dt, \tag{2.1.28}$$

where $b \in L^1(\mathbb{R})$. Applying for $v(x)$ a Gronwall type inequality (see Lemma C.0.2), we can derive the a priori bound $v(x) \leq C(a(x), \|b\|_{L^1(\mathbb{R})})$.

The relations

$$m_+(x,\tau) - 1 = \int_x^{+\infty} D(t-x,\tau)V(t)m_+(t,\tau)dt, \tag{2.1.29}$$

$$|D(t-x,\tau)| \leq C\langle t-x\rangle \leq C(\langle t\rangle + \langle x_-\rangle),$$

imply the following estimate:

$$v(x,\tau) \leq C\int_x^{+\infty} \frac{\langle x_+\rangle^{\gamma-1}}{\langle x_-\rangle}\frac{\langle t-x\rangle}{\langle t\rangle^\gamma}\langle t\rangle^\gamma|V(t)|\left(|m_+(t,\tau)-1|+1\right)\,dt.$$

We set[1]

$$c_1 = \sup_{t\geq x}\frac{\langle x_+\rangle^{\gamma-1}\langle t-x\rangle\langle t_-\rangle}{\langle x_-\rangle\langle t\rangle^\gamma\langle t_+\rangle^{\gamma-1}} \in \mathbb{R}_+,\ \gamma \geq 1,$$

$$c_2 = \sup_{t\geq x}\frac{\langle x_+\rangle^{\gamma-1}\langle t-x\rangle}{\langle x_-\rangle\langle t\rangle^\gamma} \in \mathbb{R}_+,\ \gamma \geq 1$$

and we deduce that

$$v(x,\tau) \leq c_1\int_x^{+\infty}\langle t\rangle^\gamma|V(t)|v(t,\tau)\,dt + c_2\|V\|_{L^1_\gamma(\mathbb{R})}.$$

[1] To prove that the quantity c_1, c_2 are finite, we consider three different cases: $x < t < 0$, $0 < x < t$, $x < 0 < t$ separately. In the last case, we distinguish the behaviour for $x \approx t$, $|x| << |t|$ and $|t| << |x|$.

Now applying the Gronwall argument of Lemma C.0.2, we find (2.1.24).

We will follow the same idea to prove the other inequalities. Indeed, to get (2.1.25) we define

$$u(x,\tau) = |\tau| \frac{\langle x_+ \rangle^\gamma}{\langle x_- \rangle} |m_+(x,\tau) - 1|. \tag{2.1.30}$$

This time we quote the estimates

$$|D(t-x,\tau)| \le C \min\left(\langle t-x \rangle, \frac{1}{|\tau|} \right). \tag{2.1.31}$$

Hence, by the integral equation (2.1.29) and the estimates above follows

$$\begin{aligned} u(x,\tau) \le & \int_x^{+\infty} \frac{\langle x_+ \rangle^\gamma \langle t_- \rangle}{\langle x_- \rangle \langle t_+ \rangle^\gamma} |D(t-x,\tau)| |V(t)| u(t,\tau)\, d\tau \\ & + \int_x^{+\infty} \frac{\langle x_+ \rangle^\gamma}{\langle x_- \rangle} |\tau| |D(t-x,\tau)| |V(t)|\, d\tau. \end{aligned}$$

As before[2] we can set

$$c_1 = \sup_{t \ge x} \frac{\langle x_+ \rangle^\gamma \langle t-x \rangle \langle t_- \rangle}{\langle x_- \rangle \langle t \rangle^\gamma \langle t_+ \rangle^\gamma} \in \mathbb{R}_+, \ \gamma \ge 1,$$

$$c_2 = \sup_{t \ge x} \frac{\langle x_+ \rangle^\gamma}{\langle x_- \rangle \langle t \rangle^\gamma} \in \mathbb{R}_+, \ \gamma \ge 1$$

and via Gronwall argument we get $u(x,\tau) \le C(\|V\|_{L^1_\gamma(\mathbb{R})})$, i.e. (2.1.25).

Similarly to get the (2.1.26) we put

$$g^\sigma(x,\tau_1,\tau_2) = \frac{\langle x_+ \rangle^{\gamma-1-\sigma}}{\langle x_- \rangle^{\sigma+1}} \frac{|m_+(x,\tau_1) - m_+(x,\tau_2)|}{|\tau_1 - \tau_2|^\sigma}$$

and by the estimate

$$\frac{|D(t-x,\tau_1) - D(t-x,\tau_2)|}{|\tau_1 - \tau_2|^\sigma} \le C \langle t-x \rangle^{1+\sigma} \le C(\langle t \rangle^{1+\sigma} + \langle x_- \rangle^{1+\sigma}), \ \sigma \in (0,1)$$

we get

$$\begin{aligned} g^\sigma(x,\tau_1,\tau_2) \le & \int_x^{+\infty} \frac{\langle x_+ \rangle^{\gamma-1-\sigma} \langle t-x \rangle^{1+\sigma}}{\langle x_- \rangle^{\sigma+1}} |V(t)| |m_+(t,\tau_1)|\, dt + \\ & + \int_x^{+\infty} \frac{\langle x_+ \rangle^{\gamma-1-\sigma} \langle t-x \rangle \langle t_- \rangle^{\sigma+1}}{\langle x_- \rangle^{\sigma+1} \langle t_+ \rangle^{\gamma-1-\sigma}} |V(t)| g^\sigma(t,\tau_1,\tau_2)\, dt. \end{aligned}$$

[2] One can see footnote 1

Moreover, we can estimate $|m_+(t,\tau_1)|$ with (2.1.24).

If we consider $1<\gamma<2$ and $\sigma\le\gamma-1$ or $\gamma\ge 2$ and $\sigma\in(0,1)$, we have that the following quantities are finite[3]

$$c_1=\sup_{t\ge x}\frac{\langle x_+\rangle^{\gamma-1-\sigma}\langle t-x\rangle^{\gamma-1-\sigma}}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}}\in\mathbb{R}_+,$$

$$c_2=\sup_{t\ge x}\frac{\langle x_+\rangle^{\gamma-1-\sigma}\langle t-x\rangle\langle t_-\rangle^{1+\sigma}}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}\langle t_+\rangle^{\gamma-1-\sigma}}\in\mathbb{R}_+,$$

$$c_3=\sup_{t\ge x}\frac{\langle x_+\rangle^{\gamma-1-\sigma}\langle t-x\rangle^{1+\sigma}\langle t_-\rangle}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}\langle t_+\rangle^{\gamma-1}}\in\mathbb{R}_+.$$

Then we have $g^\sigma(x,\tau_1,\tau_2)\le C(\|V\|_{L^1_\gamma(\mathbb{R})})$.

Finally we prove the inequality (2.1.27) for any $\sigma\in(0,1)$. We rewrite (2.1.29) as

$$\begin{aligned}\tau\left(m_+(x,\tau)-1\right) &= \tau\int_x^{+\infty}D(t-x,\tau)V(t)dt+\\ &+\int_x^{+\infty}\tau D(t-x,\tau)V(t)\left(m_+(t,\tau)-1\right)dt.\end{aligned}\tag{2.1.32}$$

Setting now

$$h^\sigma(x,\tau)=\frac{\langle x_+\rangle^{\gamma-\sigma}}{\langle x_-\rangle^{\sigma+1}}\left\|\tau(m_+(x,\tau)-1)\right\|_{C^{0,\sigma}(\mathbb{C}_+)},$$

we can use the inequality

$$\|fg\|_{C^{0,\sigma}}\le C\left(\|f\|_{C^{0,\sigma}}\|g\|_{C^0}+\|f\|_{C^0}\|g\|_{C^{0,\sigma}}\right)$$

and arrive at the estimate

$$\begin{aligned}h^\sigma(x,\tau)\le&\underbrace{\int_x^\infty\frac{\langle x_+\rangle^{\gamma-\sigma}}{\langle x_-\rangle^{1+\sigma}}\left\|\tau D(t-x,\tau)\right\|_{C^{0,\sigma}_\tau(\mathbb{C}_+)}|V(t)|dt}_{I(x)}+\\ &+\underbrace{\int_x^\infty\frac{\langle x_+\rangle^{\gamma-\sigma}}{\langle x_-\rangle^{1+\sigma}}\left\|\tau D(t-x,\tau)\right\|_{C^{0,\sigma}_\tau(\mathbb{C}_+)}|V(t)|\|m_+(t,\tau)-1\|_{C^0_\tau(\mathbb{C}_+)}dt}_{II(x)}+\\ &+\underbrace{\frac{\langle x_+\rangle^{\gamma-\sigma}}{\langle x_-\rangle^{1+\sigma}}\int_x^\infty\left\|D(t-x,\tau)\right\|_{C^0_\tau(\mathbb{C}_+)}\frac{|V(t)|\langle t_-\rangle^{\sigma+1}h^\sigma(t.\tau)dt}{\langle t_+\rangle^{\gamma-\sigma}}}_{III(x)}.\end{aligned}$$

[3]One can see the footnote 3

We quote the inequalities

$$\|\tau^{1-k} D(t-x,\tau)\|_{C^{0,\sigma}(\mathbb{C}_+)} \leq C\langle t-x\rangle^{k+\sigma}, \quad k=0,1, \ \sigma \in [0,1). \tag{2.1.33}$$

For the term $I(x)$ we use the estimate (2.1.33) with $k=0$ and we note that

$$c_1 = \sup_{t \geq x} \frac{\langle x_+\rangle^{\gamma-\sigma}\langle t-x\rangle^{\sigma}}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}} < \infty,$$

for[4] $0 \leq \sigma \leq \gamma$, $\gamma \geq 1$. Hence,

$$I(x) \leq c_1 \|V\|_{L^1_\gamma(\mathbb{R})}.$$

In a similar way, for $II(x)$ we use the estimate (2.1.33) with $k=0$ combined with (2.1.24) and using the estimate

$$c_2 = \sup_{t \geq x} \frac{\langle x_+\rangle^{\gamma-\sigma}\langle t-x\rangle^{\sigma}\langle t_-\rangle}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}\langle t_+\rangle^{\gamma-1}} \in \mathbb{R}_+,$$

for $0 \leq \sigma < 1$, $\gamma \geq 1$, we arrive at

$$II(x) \leq c_2 \|V\|_{L^1_\gamma(\mathbb{R})}.$$

Finally, for $III(x)$ we use $\|D(t-x,\tau)\|_{C^0_\tau(\mathbb{C}_+)} \leq C\langle t-x\rangle$ and from

$$c_3 = \sup_{t \geq x} \frac{\langle x_+\rangle^{\gamma-\sigma}\langle t-x\rangle\langle t_-\rangle^{1+\sigma}}{\langle x_-\rangle^{1+\sigma}\langle t\rangle^{\gamma}\langle t_+\rangle^{\gamma-\sigma}} < \infty,$$

we deduce

$$III(x) \leq c_3 \int_x^\infty \langle t\rangle^{\gamma} |V(t)| h^{\sigma}(t,\tau) dt.$$

So, the application of Gronwall inequality implies $h^{\sigma}(x,\tau) \leq C$ and hence (2.1.27). This complete the proof. □

We can get similar estimates for the derivatives $\partial_\tau^k(m_\pm(x,\tau)-1)$. In particular the next result holds.

Lemma 2.1.3. *Suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma \geq 1$. Then we have the following properties:*

a) If $\gamma \geq 2$, then for any integer k, $1 \leq k \leq \gamma - 1$ the function in (2.1.20) *is $C^k(\overline{\mathbb{C}_\pm})$ and there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ and $\tau \in (\overline{\mathbb{C}_\pm})$ we have*

$$\left|\partial_\tau^k \left(m_\pm(x,\tau) - 1\right)\right| \leq C \frac{\langle x_\mp\rangle^{1+k}}{\langle x_\pm\rangle^{\gamma-1-k}}; \tag{2.1.34}$$

[4] the only case, when $\sigma \leq \gamma$ is necessary is the case $x < 0 < t$, $|x| \ll |t|$

b) For any integer k, $1 \leq k \leq \gamma$, then the function in (2.1.20) is $C^k(\overline{\mathbb{C}_\pm})$ and there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ and $\tau \in (\overline{\mathbb{C}_\pm} \setminus \{0\})$ we have

$$\left|\partial_\tau^k \left(m_\pm(x,\tau) - 1\right)\right| \leq C \frac{\langle x_\mp \rangle^{1+k}}{\langle x_\pm \rangle^{\gamma-k} |\tau|}; \tag{2.1.35}$$

c) If $\gamma > 2$, then for any integer k, $1 \leq k \leq \gamma - 1$ and for any $\sigma \in (0,1)$ such that $0 \leq \sigma \leq \gamma - 1 - k$, there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ we have

$$\|m_\pm(x,\tau) - 1\|_{C^{k,\sigma}(\mathbb{C}_\pm)} \leq C \frac{\langle x_\mp \rangle^{1+k+\sigma}}{\langle x_\pm \rangle^{\gamma-1-k-\sigma}}; \tag{2.1.36}$$

d) If $\gamma > 1$, then for any integer k, $1 \leq k \leq \gamma$ and for any $\sigma \in (0,1)$ such that $0 \leq \sigma \leq \gamma - k$, there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ we have

$$\|\tau(m_\pm(x,\tau) - 1)\|_{C^{k,\sigma}(\mathbb{C}_\pm)} \leq C \frac{\langle x_\mp \rangle^{1+k+\sigma}}{\langle x_\pm \rangle^{\gamma-k-\sigma}}. \tag{2.1.37}$$

Proof. The proof of this Lemma follows the same spirit of the proof of the previous one.

We prove the inequality (2.1.34) fixing the sign $+$ in the left side. The arguments are similar for the estimates involving m_-.

The right side in (2.1.34) suggests us to define

$$v^{(k)}(x) = \frac{\langle x_+ \rangle^{\gamma-1-k}}{\langle x_- \rangle^{k+1}} \left|\partial_\tau^k (m_+(x,\tau) - 1)\right|.$$

We intend to prove the (2.1.34), i.e.

$$v^{(k)}(x) \leq C(\|V\|_{L^1_\gamma(\mathbb{R})}),\ 0 \leq k \leq \gamma - 1, \tag{2.1.38}$$

by induction in k. The inequality above for $k = 0$ is already established in (2.1.24). Then we suppose that it holds for any $0 \leq k \leq \gamma - 1$ and so our goal will be to prove that

$$v^{(k+1)}(x) \leq C,$$

with $k+1 \leq \gamma - 1$. The key tools here will be to consider the following formula

$$\partial_\tau^{k+1} m_+(x,\tau) = \sum_{\ell=0}^{k+1} c_{k,\ell} \int_x^\infty \partial_\tau^{k+1-\ell} D(t-x,\tau) V(t) \partial_\tau^\ell m_+(t,\tau) dt \tag{2.1.39}$$

and to quote the following inequality

$$\left|\partial_\tau^k D(t,x)\right| \leq C \min\left\{ \langle t\rangle^{k+1}, \frac{\langle t\rangle^k}{|\tau|} \right\},\ k=0,1,2,\dots,\ \tau \in \mathbb{C}_+ \setminus \{0\}.$$

Then, from the boundness of the quantities[5]

$$c_1 = \sup_{t \geq x} \frac{\langle x_+\rangle^{\gamma-2-k} \langle t-x\rangle^{k+2}}{\langle x_-\rangle^{2+k} \langle t\rangle^\gamma} \in \mathbb{R}_+,$$

$$c_2 = \max_{0 \leq \ell \leq k+1} \sup_{t \geq x} \frac{\langle x_+\rangle^{\gamma-2-k} \langle t-x\rangle^{k+2-\ell} \langle t_-\rangle^{1+\ell}}{\langle x_-\rangle^{2+k} \langle t\rangle^\gamma \langle t_+\rangle^{\gamma-1-\ell}} \in \mathbb{R}_+,$$

combined with a Gronwall argument we get (2.1.34). We do not prove the inequalities (2.1.35), (2.1.36) and (2.1.37) to avoid the repetition of the same arguments. We just note that for the proof of the inequalities (2.1.36) and (2.1.37) we need also the following estimate

$$\|\tau^{1-k} D(t-x,\tau)\|_{C^{0,\sigma}(\mathbb{C}_+)} \leq C \langle t-x\rangle^{k+\sigma},\quad k=0,1,\ \sigma \in [0,1).$$

□

2.1.3 Estimates and expansions for the transmission and the reflection coefficients

In this section we study the transmission coefficient $T(\tau)$ and the reflection coefficients $R_\pm(\tau)$ defined by the formulas (2.1.16) and (2.1.17)

$$T(\tau) m_\mp(x,\tau) = R_\pm(\tau) e^{\pm 2 i \tau x} m_\pm(x,\tau) + m_\pm(x,-\tau), \tag{2.1.40}$$

[5] To prove that the quantity above are finite, we consider three different cases: $x < t < 0$, $0 < x < t$, $x < 0 < t$ separately. In the last case, we distinguish the behaviour for $x \approx t$, $|x| << |t|$ and $|t| << |x|$.

in terms of $m_\pm(x,\tau)$. In particular, as x goes to $-\infty$, by Lemma 2.1.1 follows that,

$$m_+(x,\tau) = e^{-2i\tau x}\int_{-\infty}^{+\infty}\frac{e^{2i\tau t}}{2i\tau}V(t)m_+(t,\tau) + \left(1 - \frac{1}{2i\tau}\int_{-\infty}^{+\infty}V(t)m_+(t,\tau)\right) + o(1).$$

By formula (2.1.40) and by limiting conditions on $m_-(x,\tau)$ as x goes to $-\infty$ we have

$$m_+(x,\tau) = \frac{R_-(\tau)}{T(\tau)}e^{-2i\tau x} + \frac{1}{T(\tau)} + o(1).$$

Hence, comparing the two last expressions we have that

$$\frac{R_-(\tau)}{T(\tau)} = \frac{1}{2i\tau}\int_{-\infty}^{+\infty} e^{2i\tau t}V(t)m_+(t,\tau)\,dt,$$

and

$$\frac{1}{T(\tau)} = 1 - \frac{1}{2i\tau}\int_{-\infty}^{+\infty} V(t)m_+(t,\tau)\,dt.$$

Similarly we can get

$$\frac{R_+(\tau)}{T(\tau)} = \frac{1}{2i\tau}\int_{-\infty}^{+\infty} e^{-2i\tau t}V(t)m_-(t,\tau)\,dt,$$

$$\frac{1}{T(\tau)} = 1 - \frac{1}{2i\tau}\int_{-\infty}^{+\infty} V(t)m_-(t,\tau)\,dt.$$

We summarize in the following Lemma some known results on the transmission and reflection coefficients proved in [19] and [72].

Lemma 2.1.4. *The transmission and reflection coefficients verify the following properties:*

a) $T, R_\pm \in C(\mathbb{R})$*;*

b) There exists $C_1, C_2 > 0$ *such that:*

$$|T(\tau) - 1| + |R_\pm(\tau)| \le C_1\langle\tau\rangle^{-1}, \tag{2.1.41}$$

$$|T(\tau)|^2 + |R_\pm(\tau)|^2 = 1; \tag{2.1.42}$$

c) If $T(0) = 0$, *(i.e. zero is not a resonance point), then for some* $\alpha \in \mathbb{C}\backslash\{0\}$ *and for some* $\alpha_+, \alpha_- \in \mathbb{C}$

$$T(\tau) = \alpha\tau + o(\tau), \quad 1 + R_\pm(\tau) = \alpha_\pm\tau + o(\tau), \tag{2.1.43}$$

for $\tau \in \mathbb{R}$, $\tau \to 0$.

In particular, (2.1.42), (2.1.43) follow from Sect.3 in [19] and (2.1.41) follows from Theorem 2.3 in [72].

We note that substituting (2.1.6) and (2.1.8) in (2.1.3) we get

$$\left(\tau^2 - \mathcal{H}\right)^{-1}(x,t) = e^{i\tau(t-x)}\frac{m_-(x,\tau)m_+(t,\tau)T(\tau)}{2i\tau},$$

if $x < t$. Similarly if $x \geq t$. Hence, the properties of $m_\pm(x,\tau)$ (one can see in particular Lemma 2 and Remark 5 in [19]) and the properties of the transmission coefficient $T(\tau)$ (one can see in particular property c) in Lemma 2.1.4) suggest the following definition.

Definition 2.1.5. *The origin is a resonance point for the hamiltonian* $\mathcal{H}$ *if and only if*

$$T(0) \neq 0.$$

Now we are going to use the assumption $V \in L^1_\gamma(\mathbb{R})$, $\gamma \geq 1$, to get from Lemma 2.1.2 and Lemma 2.1.3 similar bounds for the transmission and reflection coefficients.

Lemma 2.1.6. *Suppose* $V \in L^1_\gamma(\mathbb{R})$ *with* $\gamma \geq 1$ *and* $T(0) = 0$. *Then for any integer* k, $0 \leq k \leq \gamma - 1$ *we have:*

a) $T, R_\pm \in C^k(\mathbb{R})$;

b) There exists $C > 0$ *such that for any* $\tau \in \mathbb{R}$ *we have:*

$$\left|\frac{d^k}{d\tau^k}T(\tau)\right| + \left|\frac{d^k}{d\tau^k}R_\pm(\tau)\right| \leq C, \tag{2.1.44}$$

$$\left|\frac{d^k}{d\tau^k}\left[\tau\left(T(\tau)-1\right)\right]\right| + \left|\frac{d^k}{d\tau^k}\left[\tau R_\pm(\tau)\right]\right| \leq C. \tag{2.1.45}$$

Proof. The proof is based on the relations

$$\frac{\tau}{T(\tau)} = \tau - \frac{1}{2i}\int_{\mathbb{R}} V(t)m_+(t,\tau)dt, \quad \tau \in \mathbb{R}\setminus\{0\}, \tag{2.1.46}$$

$$R_\pm(\tau) = \frac{T(\tau)}{2i\tau}\int_{\mathbb{R}} e^{\mp 2it\tau}V(t)m_\mp(t,\tau)dt, \ \tau \in \mathbb{R}\setminus\{0\} \tag{2.1.47}$$

and the properties of the functions $m_\pm(t,\tau)$ from Lemma 2.1.3. Indeed, if we set

$$\Phi(\tau) = \frac{1}{2i}\int_{\mathbb{R}} V(t)m_+(t,\tau)dt,$$

it is not difficult to use the Lebesgue convergence theorem as well the uniform bounds of Lemma 2.1.3 and see that $\Phi(\tau) \in C^k(\mathbb{R})$ where k is as specified above according with Lemma 2.1.3 and

$$\sum_{k=0}^{2}\left|\frac{d^k}{d\tau^k}\Phi(\tau)\right| \leq C. \tag{2.1.48}$$

The relation (2.1.46) guarantees that

$$\tau = T(\tau)(\tau - \Phi(\tau)). \tag{2.1.49}$$

Moreover, we know that $T(\tau) \in C(\mathbb{R})$ and $\tau - \Phi(\tau) \in C^k(\mathbb{R})$. The relation (2.1.49) implies in particular that $\tau - \Phi(\tau) \neq 0$ for any $\tau \in \mathbb{R}\setminus\{0\}$, so $T \in C^k(\mathbb{R}\setminus\{0\})$. We have also the estimate

$$\left|\frac{d^k}{d\tau^k}T(\tau)\right| \leq C, \quad |\tau| \geq 1, \tag{2.1.50}$$

for any k integer as stated in Lemma 2.1.3. To study the differentiability of T near zero, we note that the assumption $T(0) = 0$ and (2.1.43) guarantee

$$\lim_{\tau\to 0}\frac{T(\tau)}{\tau} = \alpha \neq 0 \tag{2.1.51}$$

and hence

$$\Phi(0) = -\frac{1}{\alpha} \neq 0.$$

So we can deduce the differentiability (of class C^k) of

$$\frac{1}{\tau - \Phi(\tau)} = \frac{T(\tau)}{\tau}$$

near $\tau = 0$. In this way we can summarize the above argument into the regularity property

$$\frac{T(\tau)}{\tau} \in C^k(\mathbb{R}) \tag{2.1.52}$$

and using (2.1.50) we get

$$\left|\frac{d^k}{d\tau^k}T(\tau)\right| \leq C. \tag{2.1.53}$$

Further, we can use the relation

$$R_\pm(\tau) = \frac{T(\tau)}{2i\tau}\int_{\mathbb{R}} e^{\mp 2it\tau}V(t)m_\mp(t,\tau)dt \tag{2.1.54}$$

and observe that the inequality

$$\left|\frac{d^k}{d\tau^k}\Phi_1^\pm(\tau)\right| \leq C \tag{2.1.55}$$

with

$$\Phi_1^\pm(\tau) = \int_{\mathbb{R}} e^{\mp 2it\tau}V(t)m_\mp(t,\tau)dt$$

can be established following the proof of (2.1.48). This estimate and the relation (2.1.54) imply

$$\left|\frac{d^k}{d\tau^k}R_\pm(\tau)\right| \leq C.$$

For $|\tau|$ sufficiently large, we can use the relation

$$T(\tau) = \frac{1}{1-\tau^{-1}\Phi(\tau)} = 1 + \sum_{k=1}^{\infty}\left(\frac{\Phi(\tau)}{\tau}\right)^k \tag{2.1.56}$$

and note that the estimate (2.1.48) can be improved for $|\tau|$ sufficiently large as follows

$$\left|\langle\tau\rangle\frac{d^k}{d\tau^k}\Phi(\tau)\right| \leq C. \tag{2.1.57}$$

Indeed, we can use the estimates (2.1.35), using bounds involving the factor $|\tau|^{-1}$. In this way from (2.1.56) and (2.1.57) we get (2.1.45) for $T(\tau)$. □

Remark 2.1.7. It is easy to see that

$$\left|\frac{T(\tau)}{\tau}\right| + \left|\frac{R_\pm(\tau)+1}{\tau}\right| \leq C \tag{2.1.58}$$

and

$$\left\|\frac{T(\tau)}{\tau}\right\|_{C^{0,\sigma}(\mathbb{R})} + \left\|\frac{R_\pm(\tau)+1}{\tau}\right\|_{C^{0,\sigma}(\mathbb{R})} \leq C. \tag{2.1.59}$$

Indeed (2.1.58) follows from relations (2.1.46), (2.1.54), using the inequality (2.1.24) and the property (2.1.43) in Lemma 2.1.4. Similarly (2.1.59) follows from relations (2.1.46), (2.1.54), using the inequality (2.1.26) and the property (2.1.43) in Lemma 2.1.4.

In the same spirit of the Lemma before, we can establish the corresponding Hölder norm estimates for the transmission and the reflection coefficients.

Lemma 2.1.8. *Suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma > 1$ and $T(0) = 0$. Then for any $\sigma \in (0, \gamma-1]$ and $M \in (0, \infty)$ we have:*

a) $T, R_\pm \in C^{0,\sigma}(\mathbb{R})$;

b) For $M \in (0, 1)$ we have

$$\|\varphi(\cdot)T(M\cdot)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)\left(R_\pm(M\cdot) + 1\right)\|_{C^{0,\sigma}((1/2,2))} \le CM; \tag{2.1.60}$$

c) For $M \in [1, \infty)$ we have

$$\|\varphi(\cdot)\left(T(M\cdot) - 1\right)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)R_\pm(M\cdot)\|_{C^{0,\sigma}((1/2,2))} \le CM^{-1}., \tag{2.1.61}$$

where φ is a non negative cutoff, $\varphi \in C_0^\infty(\mathbb{R})$ and supp $\varphi \subseteq [1/2, 2]$.

Proof. The proof is based on the relations

$$\frac{\tau}{T(\tau)} = \tau - \frac{1}{2i}\int_{\mathbb{R}} V(t)m_+(t,\tau)dt, \quad \tau \in \mathbb{R}\setminus\{0\}, \tag{2.1.62}$$

$$R_\pm(\tau) = \frac{T(\tau)}{2i\tau}\int_{\mathbb{R}} e^{\mp 2it\tau}V(t)m_\mp(t,\tau)dt, \ \tau \in \mathbb{R}\setminus\{0\} \tag{2.1.63}$$

and the properties of the functions $m_\mp(t,\tau)$ from Lemma 2.1.2. Indeed, we can get the estimates

$$\left\|\frac{\varphi(\cdot)}{T(M\cdot)}\right\|_{C^{0,\sigma}([0,4])} + \left\|\frac{\varphi(\cdot)}{(R_\pm(M\cdot)+1)}\right\|_{C^{0,\sigma}([0,4])} \le CM^{-1} \tag{2.1.64}$$

first. Further, we can use the fact[6] that we can control the norm of the inverse of f in the subalgebra

[6] the problem to have norm-controlled inversion in smooth Banach algebra is well-known and some more general results and references can be found in [37]

$C^{0,\sigma}$ by the norm of f in $C^{0,\sigma}$ and the norm of $1/f$ in $C(T)$

$$\left\| \frac{\varphi(\cdot)}{f(\cdot)} \right\|_{C^{0,\sigma}([0,4])} \leq C \left\| \frac{\widetilde{\varphi}(\cdot)}{f(\cdot)} \right\|_{C^{0}([0,4])} + \frac{\|\widetilde{\varphi}(\cdot) f\|_{C^{0,\sigma}([0,4])}}{\|f(\cdot)\|^2_{C^{0}([0,4])}},$$

where $\widetilde{\varphi} \in C_0^\infty((0,\infty))$ has slightly larger support in $[1/2-\delta, 2+\delta]$ with $\delta > 0$ sufficiently small. Applying this estimate, the estimate (2.1.64) and the analogous one in C^0, all with φ replaced by a cut-off function with slightly larger support, we complete the proof. □

Lemma 2.1.9. *Suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma > 1$ and $T(0) = 0$. Then for any integer k, $0 \leq k < \gamma - 1$ and any $\sigma \in (0,1) \cap (0, \gamma - 1 - k]$ we have:*

a) $T, R_\pm \in C^{k,\sigma}(\mathbb{R})$;

b) There exists $C > 0$ such that for any $\tau \in \mathbb{R}$ we have:

$$\left\| \frac{d^k}{d\tau^k} T(\tau) \right\|_{C^{0,\sigma}(\mathbb{R})} + \left\| \frac{d^k}{d\tau^k} R_\pm(\tau) \right\|_{C^{0,\sigma}(\mathbb{R})} \leq C, \tag{2.1.65}$$

$$\left\| \frac{d^k}{d\tau^k} \left[\tau \left(T(\tau) - 1 \right) \right] \right\|_{C^{0,\sigma}(\mathbb{R})} + \left\| \frac{d^k}{d\tau^k} \left[\tau R_\pm(\tau) \right] \right\|_{C^{0,\sigma}(\mathbb{R})} \leq C. \tag{2.1.66}$$

2.2 Equivalence of homogeneous Besov norms

Here we consider the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$ where V is a short range potential. The wave operator methods have been used frequently in the study of the evolution flow generated by Hamiltonians that can be considered as perturbations of free Hamiltonians. The wave operators are defined by the strong limits

$$W_\pm = s - \lim_{t \to \pm\infty} e^{it\mathcal{H}} e^{-it\mathcal{H}_0}.$$

The existence of the wave operators $W_\pm$ is well known according to the results in [71], [1], [18], so $W_\pm$ are well defined operators in $L^p(\mathbb{R})$, $1 < p < \infty$. Moreover, the mapping properties for the case of Sobolev spaces $W^s_p(\mathbb{R}^n)$ are studied in [74] and [71] and they show examples of spaces invariant under the action of the wave operators. We recall that the results in [71] deal with short range assumptions that guarantee $W^k_p(\mathbb{R})$ boundedness of $W_\pm$. The $L^p(\mathbb{R})$ boundedness is studied in [18]. Here we analyse the mapping properties for the case of homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$.

The functional calculus for the perturbed operator $\mathcal{H}$ can be defined as follows

$$g(\mathcal{H}) = W_+ g(\mathcal{H}_0) W_+^* = W_- g(\mathcal{H}_0) W_-^* \tag{2.2.1}$$

for any function $g \in L^\infty_{loc}(\mathbb{R})$. So the wave operators map unperturbed Sobolev spaces in the perturbed ones

$$W_\pm : D(\mathcal{H}_0^{s/2}) \to D(\mathcal{H}^{s/2}).$$

In [74], [1], [71] it was proved the continuity of the wave operators on general Sobolev spaces and then the equivalence between perturbed and unperturbed Sobolev norms. The study of the dispersive properties of the evolution flow in some cases of short range perturbations shows (see [17]) that we have stronger equivalence between homogeneous Sobolev norms

$$\|\mathcal{H}^{s/2} f\|_{L^2(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2} f\|_{L^2(\mathbb{R}^n)}, \tag{2.2.2}$$

provided $s < n/2$. Our first goal is to show in Section 2.2.1 that the requirement $s < n/2$ is optimal at least for $n = 1, 2$. Then we study how classical homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ are transformed under the action of the wave operators.

We introduce a Paley-Littlewood partition of unity

$$1 = \sum_{j \in \mathbf{Z}} \varphi\left(\frac{t}{2^j}\right), \ t > 0$$

for an appropriate non-negative cutoff $\varphi \in C_0^\infty(\mathbf{R}_+)$, such that supp$\varphi \subseteq [1/2, 2]$.

The homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ for $1 \le p \le \infty$ and $s \ge 0$ can be defined as the closure of $S(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_p(\mathbb{R})} = \left(\sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f \right\|^2_{L^p(\mathbb{R})} \right)^{1/2}. \tag{2.2.3}$$

Similarly we can define the perturbed homogeneous Besov spaces $\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})$ associated with the perturbed Hamiltonian $\mathcal{H}$ as the closure of $S(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})} = \left(\sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^j}\right) f \right\|^2_{L^p(\mathbb{R})} \right)^{1/2}. \tag{2.2.4}$$

The splitting property (2.2.1) implies that

$$W_\pm : \dot{B}^s_p(\mathbb{R}) \to \dot{B}^s_{p,\mathcal{H}}(\mathbb{R}),\ \forall s \geq 0,\ 1 < p < \infty.$$

Then, once the equivalence of the homogeneous Besov norms is established

$$\|f\|^2_{\dot{B}^s_p(\mathbb{R})} \sim \|f\|^2_{\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})}, \tag{2.2.5}$$

it follows easily that the homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ are also invariant under the action of the wave operators $W_\pm$ under the natural restriction $0 \leq s < 1/p$.

To be more precise, we shall assume

$$V \in L^1_\gamma(\mathbb{R}),\ \gamma > 1 + 1/p,\ 1 < p < \infty, \tag{2.2.6}$$

$\sigma_p(\mathcal{H}) = \emptyset$ and 0 is not a resonance point for the perturbed Hamiltonian. Then, our approach to establish (2.2.5) is based on the study of the Paley-Littlewood localization operators

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right),\ \ \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right),\ \ \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^k}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^j}\right),\ j,k \in \mathbb{Z}.$$

The key point is to find an appropriate decomposition for the kernel of the operator $\varphi\left(\mathcal{H}/2^k\right)$ into a leading term involving similar estimates for the unperturbed Hamiltonian

$$\left|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)(x,y)\right| \leq \frac{C2^j}{\langle 2^j(x-y)\rangle^2},\ \ \forall j \in \mathbb{Z} \tag{2.2.7}$$

and a remainder satisfying better kernel estimates. We will treat differently the case of low energy (Lemma 2.2.2) and high energy (Lemma 2.2.3). Finally we need to get estimates of this kind

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)\right\|_{L^p(\mathbb{R})} \leq C\frac{1}{2^{|k-j|s}}\|f\|_{L^p(\mathbb{R})},$$

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^k}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^j}\right)\right\|_{L^p(\mathbb{R})} \leq C\frac{1}{2^{|k-j|s}}\|f\|_{L^p(\mathbb{R})},$$

to prove the equivalence (2.2.5).

2.2.1 Counterexample for the equivalence of homogeneous Besov spaces

In this section we consider the simplest case $p = 2$ and we shall prove that the equivalence property

$$\|(\mathcal{H}_0 + V)^{n/4}u\|_{L^2(\mathbb{R}^n)} \sim \|(\mathcal{H}_0)^{n/4}u\|_{L^2(\mathbb{R}^n)} \tag{2.2.8}$$

is not true for $n = 1, 2$. In particular we have the following result.

Theorem 2.2.1. *If* $n = 1, 2$, *and* $V(x)$ *is a positive potential such that*

$$\int_{\mathbb{R}^n} V^{n/2}(x)dx \leq C < \infty, \tag{2.2.9}$$

then (2.2.2) *with* $s = n/2$ *is not true.*

Proof. Let us suppose that the relation (2.2.8) holds. First, we show that

$$\|\mathcal{H}_0^z u\|^2_{L^2(\mathbb{R}^n)} \geq \|V^z u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.2.10}$$

with $\Re z = 0$ and $\Re z = 1/2$. Then, using the Stein interpolation Theorem (one can see [22]), we have that

$$\|\mathcal{H}_0^a u\|^2_{L^2(\mathbb{R}^n)} \geq \|V^a u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.2.11}$$

with $0 \leq a \leq 1/2$. It is easy to see that we have the property

$$\|\mathcal{H}_0^{\mathrm{i}b} u\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)},\ \forall\ b \in \mathbb{R},$$

and

$$\|V^{\mathrm{i}b} u\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R})},\ \forall\ b \in \mathbb{R},$$

so we have to check (2.2.11) only for $a = 1/2$. The equivalence of the norms (2.2.8) implies that

$$\begin{aligned}\|\mathcal{H}_0^{1/2} u\|_{L^2(\mathbb{R}^n)} \approx \|\left(-\Delta + V\right)^{1/2} u\|^2_{L^2(\mathbb{R}^n)} =& \langle(-\Delta + V)u, u\rangle_{L^2(\mathbb{R}^n)} \\ \geq& \langle Vu, u\rangle_{L^2(\mathbb{R}^n)} = \|V^{1/2}u\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

and we conclude that (2.2.11) is true. Then, assuming (2.2.8) is fulfilled and applying the proved inequality with $a = n/4 \leq 1/2$, i.e. $n \leq 2$, we get

$$\int_{\mathbb{R}^n} (V(x))^{n/2} |u(x)|^2 dx \leq C \|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}, \quad D = (-\Delta)^{1/2}. \tag{2.2.12}$$

Taking u in the Schwartz class $S(\mathbb{R}^n)$ of rapidly decreasing function, we can apply a rescaling argument. Indeed, considering the dilation

$$u_\lambda(x) = u(x\lambda),$$

we find

$$\|D^{n/2} u_\lambda\|^2_{L^2(\mathbb{R}^n)} = \underbrace{\|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}}_{\text{constant in } \lambda}$$

and

$$\lim_{\lambda \searrow 0} \int_{\mathbb{R}^n} V^{n/2}(x) |u_\lambda(x)|^2 dx = \left(\int_{\mathbb{R}^n} V^{n/2}(x) dx \right) |u(0)|^2.$$

In this way we deduce

$$|u(0)|^2 \left(\int_{\mathbb{R}^n} V^{n/2}(x) dx \right) \leq C \|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.2.13}$$

The homogeneous norm $\|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}$ is also invariant under translations, i.e. setting

$$u^{(\tau)}(x) = u(x + \tau),$$

we have

$$\widehat{u^{(\tau)}}(\xi) = e^{-\mathrm{i}\tau\xi} \widehat{u}(\xi)$$

and

$$\|D^{n/2} u^{(\tau)}\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^{n/2} \widehat{u^{(\tau)}}\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^{n/2} \widehat{u}\|^2_{L^2(\mathbb{R}^n)} = \|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}.$$

Applying (2.2.13) with $u^{(\tau)}$ in the place of u, we find

$$|u(\tau)|^2 \int_{\mathbb{R}^n} V^{n/2}(x) dx \leq C \|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)},$$

or equivalently

$$\|u\|^2_{L^\infty(\mathbb{R}^n)} \leq C_1 \|D^{n/2} u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.2.14}$$

where

$$C_1 = \frac{C}{\|V^{n/2}\|_{L^1(\mathbb{R}^n)}}.$$

The substitution $\phi = D^{n/2}u$ enables us to rewrite (2.2.14) as

$$\|I_{n/2}(\phi)\|^2_{L^\infty(\mathbb{R}^n)} \le C_1\|\phi\|^2_{L^2(\mathbb{R}^n)}, \tag{2.2.15}$$

where

$$I_\alpha(\phi)(x) = D^{-\alpha}(\phi)(x) = c\int_{\mathbb{R}^n} |x-y|^{-n+\alpha}\phi(y)dy,\ \alpha \in (0,n)$$

are the Riesz operators. It is easy to show that (2.2.15) leads to a contradiction. Indeed, taking

$$\phi_N(x) = \sum_{j=0}^{N} |x|^{-n/2}\ \mathbb{1}_{2^j \le |x| \le 2^{j+1}}(x),$$

with $N \ge 2$ sufficiently large and being $\mathbb{1}_A(x)$ the characteristic function of the set A, we can use the estimates

$$I_{n/2}(\phi_N)(0) \ge \left(\sum_{j=0}^{N}\int_{2^j}^{2^{j+1}} \frac{r^{n-1}dr}{r^n}\right) \ge CN$$

and

$$\|\phi_N\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=0}^{N}\int_{2^j}^{2^{j+1}} \frac{r^{n-1}dr}{r^n} \le C'N.$$

Hence, from (2.2.15) we deduce

$$CN^2 \le \|I_{n/2}(\phi_N)\|^2_{L^\infty(\mathbb{R}^n)} \le C_1\|\phi_N\|^2_{L^2(\mathbb{R}^n)} \le C_2N,$$

for any N sufficiently big and this is impossible. This completes the proof of the Theorem. □

2.2.2 Kernel estimates

In this section we establish an appropriate decomposition for the kernel of the operator $\varphi(\sqrt{\mathcal{H}}/M)$ where $M > 0$. As we have seen in (2.1.7), the kernel $\varphi(\sqrt{\mathcal{H}}/M)$ has the following representation

$$\varphi\left(\sqrt{\mathcal{H}}/M\right)(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} \varphi\left(\tau/M\right)T(\tau)f_+(y,\tau)f_-(x,\tau)d\tau, \quad \text{when} \quad x<y, \tag{2.2.16}$$

and

$$\varphi\left(\sqrt{\mathcal{H}}/M\right)(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} \varphi\left(\tau/M\right) T(\tau) f_-(y,\tau) f_+(x,\tau) d\tau, \quad \text{otherwise.} \tag{2.2.17}$$

In particular we will find a leading term involving $\varphi(\sqrt{\mathcal{H}_0}/M)$ and a remainder satisfying better estimates.

Using the classical result due to Weder [71] one can derive the following L^p estimate:

$$\left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f \right\|_{L^p(\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R})} \tag{2.2.18}$$

for $M > 0$, $f \in S(\mathbb{R})$, $1 < p < \infty$, as well the Bernstain inequality.

Here, using the improved estimates established in Lemma 2.1.2 and Lemma 2.1.9 we get the following low energy and high energy kernel estimates.

Lemma 2.2.2. *Suppose the condition* (2.2.6) *is fulfilled, the operator* $\mathcal{H}$ *has no point spectrum and* 0 *is not a resonance point for* $\mathcal{H}$. *If* φ *is an even non-negative function, such that* $\varphi \in C_0^\infty(\mathbf{R}\setminus\{0\})$, *then for any* $M \in (0,1]$ *and* $\sigma \in (0,1) \cap (0, \gamma - 1]$ *we have*

$$\left| \varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y) - K_M(x,y) \right| \le \tag{2.2.19}$$
$$\le CM \left(\sum_{\pm} \frac{1}{\langle M(x \pm y)\rangle^{\sigma}} \right) \left(\frac{1}{\langle x\rangle^{\gamma-\sigma}} + \frac{1}{\langle y\rangle^{\gamma-\sigma}} \right),$$

where

$$K_M(x,y) = c\int_{\mathbb{R}} e^{-i\tau(x-y)} \varphi\left(\frac{\tau}{M}\right) b(x,y,\tau)\, d\tau \tag{2.2.20}$$

with

$$b(x,y,\tau) = \begin{cases} T(\tau) & x < 0 < y, \\ (R_+(\tau)+1)e^{2i\tau x} - e^{2i\tau x} + 1 & 0 \le x < y, \\ (R_-(\tau)+1)e^{-2i\tau y} - e^{-2i\tau y} + 1 & x < y \le 0. \end{cases}$$

Lemma 2.2.3. *Suppose the condition* (2.2.6) *is fulfilled and the operator* $\mathcal{H}$ *has no point spectrum. If* φ *is an even non-negative function, such that* $\varphi \in C_0^\infty(\mathbf{R}\setminus\{0\})$, *then for any* $M \in [1,\infty)$, $\sigma \in (0,1) \cap (0, \gamma - 1]$

we have

$$\left|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y)-\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)(x,y)\right|\leq \tag{2.2.21}$$
$$\leq C\left(\sum_{\pm}\frac{1}{\langle M(x\pm y)\rangle^{\sigma}}\right)\left(\frac{1}{\langle x\rangle^{\gamma-\sigma}}+\frac{1}{\langle y\rangle^{\gamma-\sigma}}\right).$$

Proof of Lemma 2.2.2. We assume $x<y$ for determinacy and consider three cases:

$$x<0<y, \tag{Case A}$$

$$0\leq x<y, \tag{Case B}$$

$$x<y\leq 0. \tag{Case C}$$

In the Case A, we can use the representation

$$T(\tau)m_+(y,\tau)m_-(x,\tau)=T(\tau)+T(\tau)\underbrace{m_0^{rem,+}(y,\tau)}_{=a_1(y,\tau)}+$$

$$+T(\tau)\underbrace{m_0^{rem,-}(x,\tau)}_{=a_2(x,\tau)}+T(\tau)\underbrace{m_0^{rem,+}(y,\tau)m_0^{rem,-}(x,\tau)}_{=a_3(x,y,\tau)},$$

where

$$m_0^{rem,\pm}(x,\tau)=m_\pm(x,\tau)-1. \tag{2.2.22}$$

In this way, from (2.2.16), we have the representation

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y)=c\,\widehat{\varphi_M}(x-y)+c\sum_{j=1}^{3}I_M(a_j)(x,y), \tag{2.2.23}$$

where

$$I_M(a)(x,y)=M\int_{\mathbb{R}}\varphi(\tau)\,T(M\tau)a(x,y,M\tau)e^{-\mathrm{i}M\tau(x-y)}d\tau$$

and

$$\varphi_M(\tau)=T(\tau)\varphi\left(\frac{\tau}{M}\right).$$

The term $\widehat{\varphi_M}(x-y)$ is included in the leading term $K_M(x,y)$ defined in (2.2.20) since we have

$$\widehat{\varphi_M}(x-y) = \int_{\mathbb{R}} e^{-i\tau(x-y)} \varphi\left(\frac{\tau}{M}\right) T(\tau)\, d\tau.$$

To estimate the terms $I_M(a_j)(x,y)$ we are going to use the following fractional integration by parts estimate[7]

$$\left| \int_{\mathbb{R}} e^{\mathrm{i}\tau M\xi} g(\tau) d\tau \right| \leq \frac{C}{\langle M\xi \rangle^\sigma} \|g\|_{C^{0,\sigma}(\mathbb{R})}, \quad \forall \sigma \in (0,1). \tag{2.2.24}$$

Hence it follows that

$$|I_M(a_j)(x,y)| \leq C \frac{M}{\langle M(x-y)\rangle^\sigma} \left\| \varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_j(x,y,M\tau) \right\|_{C^{0,\sigma}(\mathbb{R})}. \tag{2.2.25}$$

Then, using the estimates proved in Lemma 2.1.2, combined with the following estimates for $T(\tau)$

$$\left\| \frac{T(\tau)}{\tau} \right\|_{C^{0,\sigma}(\mathbb{R})} + \left| \frac{T(\tau)}{\tau} \right| \leq C, \ \sigma \in (0,1) \cap (0, \gamma - 1]$$

we get

$$\left\| \varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_1(x,y,M\tau) \right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle y \rangle^\gamma} + \frac{M^\sigma}{\langle y \rangle^{\gamma-\sigma}} \right),$$

$$\left\| \varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_2(x,y,M\tau) \right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle x \rangle^\gamma} + \frac{M^\sigma}{\langle x \rangle^{\gamma-\sigma}} \right),$$

$$\left\| \varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_3(x,y,M\tau) \right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle x \rangle^{\gamma-1} \langle y \rangle^\gamma} + \frac{1}{\langle y \rangle^{\gamma-\sigma} \langle x \rangle^{\gamma-1}} + \frac{M^\sigma}{\langle y \rangle^\gamma \langle x \rangle^{\gamma-\sigma-1}} \right),$$

Turning back to (2.2.23) and using the estimates (2.2.25) together with Hölder estimates above, we obtain

$$\left| \varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) - c\widehat{\varphi_M}(x-y) \right| \leq C \frac{M}{\langle M(x-y)\rangle^\sigma} \left(\frac{1}{\langle y \rangle^{\gamma-\sigma}} + \frac{1}{\langle x \rangle^{\gamma-\sigma}} \right)$$

with $\sigma \in (0,1) \cap (0, \gamma-1]$, and $x < 0 < y$, i.e. we get (2.2.19) in the Case A.

In the Case B, since we have $x \geq 0$, we shall write $m_-(x,\tau)$ in terms of $m_+(x, \pm\tau)$. In order to do this, we can use the relation

$$T(\tau) m_-(x,\tau) = R_+(\tau) e^{2\mathrm{i}\tau x} m_+(x,\tau) + m_+(x,-\tau). \tag{2.2.26}$$

[7] here g is a compactly supported function in $C^{0,\sigma}(\mathbb{R})$ such that $0 \notin \mathrm{supp} g$.

Then we can write

$$T(\tau)m_+(y,\tau)m_-(x,\tau) =$$
$$(R_+(\tau)+1)e^{2\mathrm{i}\tau x}m_+(y,\tau)m_+(x,\tau)-$$
$$-e^{2\mathrm{i}\tau x}m_+(y,\tau)m_+(x,\tau)+m_+(y,\tau)m_+(x,-\tau).$$

Using the remainders introduced in (2.2.22) we can represent the kernel $\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y)$ as a sum of kernels of three types:

$$I_M(x,y) = \int_{\mathbb{R}} \varphi\left(\frac{\tau}{M}\right)\left((R_+(\tau)+1)\,e^{2\mathrm{i}\tau x}m_+(y,\tau)m_+(x,\tau)\right)e^{-\mathrm{i}\tau(x-y)}d\tau,$$
$$II_M(x,y) = M\widehat{\varphi}\,(M(x-y)) - M\widehat{\varphi}\,(M(x+y))$$
$$III_M(x,y) = \sum_{j=1}^{2} K_j(x,y;M),$$

where

$$K_1(x,y;M) = M\int_{\mathbb{R}} e^{-\mathrm{i}M\tau(x-y)}\varphi(\tau)b_1(x,y,M\tau)d\tau, \tag{2.2.27}$$

$$K_2(x,y;M) = -M\int_{\mathbb{R}} e^{\mathrm{i}M\tau(x+y)}\varphi(\tau)b_2(x,y,M\tau)d\tau, \tag{2.2.28}$$

and

$$b_1(x,y,M\tau) = m_0^{rem,+}(y,M\tau) + m_0^{rem,+}(x,-M\tau) + m_0^{rem,+}(y,M\tau)m_0^{rem,+}(x,-M\tau),$$

$$b_2(x,y,M\tau) = m_0^{rem,+}(y,M\tau) + m_0^{rem,+}(x,M\tau) + m_0^{rem,+}(y,M\tau)m_0^{rem,+}(x,M\tau).$$

As before, we firstly estimate the terms $I_M(x,y)$ and $III_M(x,y)$ with fractional integration by parts estimate (2.2.24) and then we use Lemma 2.1.2 combined with the estimates

$$\left\|\frac{R_\pm(\tau)+1}{\tau}\right\|_{C^{0,\sigma}(\mathbb{C}_\pm)} + \left|\frac{R_\pm(\tau)+1}{\tau}\right| \le C,\ \ \sigma\in(0,1)\cap(0,\gamma-1],$$

and the properties of the function φ to prove (2.2.19) in the Case B. Here

$$b(x,y,\tau) = (R_+(\tau)+1)e^{2i\tau x} - e^{2i\tau x} + 1$$

and $0 \leq x < y$.

In the Case C we follow the argument used in the Case B, but this time we replace (2.2.26) by

$$T(\tau)m_+(y,\tau) = R_-(\tau)e^{-2i\tau y}m_-(y,\tau) + m_-(y,-\tau), \tag{2.2.29}$$

to derive (2.2.19). This completes the proof of Lemma 2.2.3. □

Proof of Lemma 2.2.3. In the high energy domain $M > 1$ we can follow the proof of Lemma 2.2.2. Using the estimates

$$T(\tau) = 1 + O(\tau^{-1}),\ R(\tau) = O(\tau^{-1})$$

near $\tau \to \infty$ we can absorb the factor $M > 1$ that appears in

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y) - \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)(x,y) =$$
$$= M\int_{\mathbb{R}} \varphi(\tau)\left[T(\tau M)m_+(y,\tau M)m_-(x,\tau M) - 1\right]e^{-i\tau M(x-y)}\,d\tau.$$

Then, proceeding as in the proof of Lemma 2.2.2 we obtain the following estimate

$$\left|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y) - \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)(x,y)\right| \leq$$
$$\leq C\left(\sum_{\pm}\frac{1}{\langle M(x\pm y)\rangle^{\sigma}}\right)\left(\frac{1}{\langle x\rangle^{\gamma-\sigma}} + \frac{1}{\langle y\rangle^{\gamma-\sigma}}\right).$$

i.e. the inequality (2.2.21). □

2.2.3 Equivalence of the norms

In this section we prove the equivalence of the homogeneous Besov norms and hence the invariance of the homugeneous Besov spaces under the action of the wave operators.

Theorem 2.2.4. *Suppose*

$$V \in L^1_\gamma(\mathbb{R}),\quad \gamma > 1 + 1/p,\ 0 \leq s < 1/p,\quad p \in (1,\infty),$$

the operator $\mathcal{H}$ *has no point spectrum and* 0 *is not a resonance. Then we have*

$$\|f\|_{\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})} \sim \|f\|_{\dot{B}^s_p(\mathbb{R})}.$$

As immediate consequence we have the following result.

Corollary 2.2.5. *Suppose the assumptions of Theorem 2.2.4 are fulfilled. Then for any* $p \in (1, \infty)$, *any* $s \in [0, 1/p)$, *we have*

$$W_\pm : \dot{B}^s_p(\mathbb{R}) \to \dot{B}^s_p(\mathbb{R}).$$

In order to prove Theorem 2.2.4 we recall that the comparison of homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ and $\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})$ is closely connected with the definition and properties of fractional power of the Hamiltonians $\mathcal{H}$ and $\mathcal{H}_0$. As studied in Chapter 1, for sectorial operators A in $L^p(\mathbb{R})$ introduced in Definition 1.1.9 we can define for any $\sigma \in (0, 1)$ the fractional negative powers of A as follows

$$A^{-\sigma} = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty \lambda^{-\sigma}(\lambda + A)^{-1} d\lambda.$$

Sectorial properties of $\mathcal{H}$ are studied in Chapter 1 under the assumption that 0 is not a resonance for $\mathcal{H}$. In particular, if $s \in [0, 1)$ and $\sigma = 1 - s/2$, by Theorem 1.4.6, it follows that

$$\mathcal{H}^{s/2} = \mathcal{H}^{-\sigma}\mathcal{H} = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty \lambda^{-\sigma}\mathcal{H}(\lambda + \mathcal{H})^{-1} d\lambda. \tag{2.2.30}$$

As mentioned before, using the existence of the wave operators, its boundness in $L^p(\mathbb{R})$ and the splitting property, one can derive the following L^p estimate (partial case of Bernstein inequality)

$$\left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f \right\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})} \tag{2.2.31}$$

for $M > 0$ and $f \in S(\mathbb{R})$. Moreover, from Lemma 2.2.3 combined with Young convolution inequality we get the Bernstain inequality for $M > 1$

$$\left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f - \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right) f \right\|_{L^p(\mathbb{R})} \leq CM^{1/p-1/q-\delta}\|f\|_{L^p(\mathbb{R})}, \tag{2.2.32}$$

where $1 \leq p \leq q \leq \infty$ and $\delta > 0$, $\delta = s - \sigma$ according with the notations used in Lemma 2.2.3.

Now we are going to get some estimates for the operators

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right),\quad \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{\Lambda}\right),\quad M,\Lambda>0.$$

We start with the case of low energy domain $M \leq 1$.

Lemma 2.2.6. *Assume that* $V \in L^1_\gamma(\mathbb{R})$ *with*

$$\gamma > 1+1/p,\ 0<s<\frac{1}{p},\ 1<p<\infty.$$

Then for any even function $\varphi(\tau) \in C_0^\infty(\mathbb{R}\setminus 0)$ *there exists a constant* $C = C(\|V\|_{L^1_\gamma(\mathbb{R})})$ *so that for any pair of real positive numbers* Λ, M *such that* $0<\Lambda\leq M, M\leq 1$ *and for any* $f\in\mathcal{S}(\mathbb{R})$, *the following inequality holds:*

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right)f\right\|_{L^p_x(\mathbb{R})} \leq C\frac{\Lambda^s}{M^s}\|f\|_{L^p_x(\mathbb{R})}. \tag{2.2.33}$$

Proof. We can assume that the support of φ is in $[1/2,2]$. Firstly we note that if $\Lambda/4\leq M\leq 4\Lambda$, or $M/4\leq\Lambda\leq 4M$ the L^p boundness of the operators $\varphi(\sqrt{\mathcal{H}}/M)$ and $\varphi(\sqrt{\mathcal{H}_0}/\Lambda)$ imply the estimate (2.2.33). Now we prove the estimate in the remaining cases.

Since $V\in L^1_\gamma(\mathbb{R})$ with $\gamma>1+1/p$ and $0<s<1/p$, then $V\in L^1_{1+s}(\mathbb{R})$. Our first step is the proof of (2.2.33), assuming

$$\Lambda < M/4. \tag{2.2.34}$$

Our goal is to check the inequality

$$\left\|\int_{\mathbb{R}}\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y)f_\Lambda(y)dy\right\|_{L^p(\mathbb{R})}\leq C\left(\frac{\Lambda}{M}\right)^s\|f\|_{L^p(\mathbb{R})} \tag{2.2.35}$$

where $f_\Lambda = \varphi(\sqrt{\mathcal{H}_0}/\Lambda)f$.

We can apply the kernel estimate (2.2.19) from Lemma 2.2.2 so we get

$$\begin{aligned}&\left\|\int_{\mathbb{R}}\left[\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y)-K_M(x,y)\right]f_\Lambda(y)dy\right\|_{L^p(\mathbb{R})}\leq\\ \leq& CM\left\|\int_{\mathbb{R}}\frac{|f_\Lambda(y)|dy}{\langle M(x-y)\rangle^\sigma\langle x\rangle^{1+s-\sigma}}\right\|_{L^p(\mathbb{R})}+CM\left\|\int_{\mathbb{R}}\frac{|f_\Lambda(y)|dy}{\langle M(x-y)\rangle^\sigma\langle y\rangle^{1+s-\sigma}}\right\|_{L^p(\mathbb{R})},\end{aligned} \tag{2.2.36}$$

for any $\sigma\in(0,s]$. The terms in the right side of the inequality above can be evaluated using Hardy-

Sobolev estimates. To be more precise, the equivalence between the Lebesgue spaces $L^p(\mathbb{R})$ and the Lorentz ones $L^{p,p}(\mathbb{R})$ in the case $1 < p < \infty$ (one can see [7]) allows us to use the sharp inequalities in Lorentz spaces. Indeed, we recall that $|x|^{-1/\beta} \in L^{\beta,\infty}(\mathbb{R})$, for any $\beta \geq 1$. Using the relation

$$1 + \frac{1}{p} = \sigma + (1 + s - \sigma) + \left(\frac{1}{p} - s\right), \tag{2.2.37}$$

we are in position to apply Young and Hölder inequalities in Lorentz spaces to get

$$\left\| \int_{\mathbb{R}} \frac{|f_\Lambda(y)|dy}{|M(x-y)|^\sigma |y|^{1+s-\sigma}} \right\|_{L^{p,p}(\mathbb{R})} \leq C \frac{1}{M^\sigma} \left\| \frac{1}{|x|^\sigma} \right\|_{L^{\frac{1}{\sigma},\infty}(\mathbb{R})} \left\| \frac{1}{|y|^{1+s-\sigma}} \right\|_{L^{\frac{1}{1+s-\sigma},\infty}(\mathbb{R})} \|f_\Lambda\|_{L^{q,p}(\mathbb{R})}.$$

where

$$\frac{1}{q} = \frac{1}{p} - s.$$

This estimate can be combined with the Sobolev embedding in Lorentz spaces

$$\|f\|_{L^{q,p}(\mathbb{R})} \leq C \|D^s f\|_{L^{p,p}(\mathbb{R})}, \ \frac{1}{q} = \frac{1}{p} - s, \ 0 < s < 1/p \tag{2.2.38}$$

so that we obtain that the second term in the right side in (2.2.36) is bounded from

$$CM^{1-\sigma}\Lambda^s \|f\|_{L^p(\mathbb{R})} \leq C\Lambda^s \|f\|_{L^p(\mathbb{R})}.$$

One can proceed similarly to find

$$\left\| \int_{\mathbb{R}} \frac{|f_\Lambda(y)|dy}{|M(x-y)|^\sigma |x|^{1+s-\sigma}} \right\|_{L^p(\mathbb{R})} \leq C \frac{1}{M^\sigma} \Lambda^s \|f\|_{L^p(\mathbb{R})}.$$

At this point we have proved that, for $0 < \Lambda < M \leq 1$, the inequality

$$\begin{aligned} M \left\| \int_{\mathbb{R}} \frac{|f_\Lambda(y)|dy}{|M(x-y)|^\sigma |x|^{1+s-\sigma}} \right\|_{L^p(\mathbb{R})} + M \left\| \int_{\mathbb{R}} \frac{|f_\Lambda(y)|dy}{|M(x-y)|^\sigma |y|^{1+s-\sigma}} \right\|_{L^p(\mathbb{R})} \leq \\ \leq C \frac{M}{M^\sigma} \Lambda^s \|f\|_{L^p(\mathbb{R})}, \end{aligned} \tag{2.2.39}$$

holds, where $0 < \sigma \leq s$.

Now we go further to estimate the leading terms

$$A_{M,\Lambda}(f)(x) = \int_{\mathbb{R}} K_M(x,y) f_\Lambda(y) dy, \tag{2.2.40}$$

caracterized in (2.2.20). We start with the study of the kernel

$$K_M^{(1)}(x,y) = \mathbb{1}_{x>0}\mathbb{1}_{y>0} \int e^{-\mathrm{i}\tau(x-y)} \varphi\left(\frac{\tau}{M}\right) d\tau. \tag{2.2.41}$$

At first we look for the kernel $\tilde{K}_{M,\Lambda}(x,y)$, such that

$$A_{M,\Lambda}^{(1)}(f)(x) = \int \tilde{K}_{M,\Lambda}^{(1)}(x,y) f(y)\, dy$$

and then we will find suitable bounds for $|\tilde{K}_{M,\Lambda}^{(1)}(x,y)|$ in order to estimate $\|A_{M,\Lambda}^{(1)}(f)\|_{L^p(\mathbb{R})}$. We can neglect the characteristic function $\mathbb{1}_{x>0}$. On the other side, the presence of $\mathbb{1}_{y>0}$ and the integration in dy imply that

$$A_{M,\Lambda}^{(1)}(f)(x) = \iint e^{-\mathrm{i}\tau x} \varphi\left(\frac{\tau}{M}\right) \frac{1}{\tau-\xi} \varphi\left(\frac{\xi}{\Lambda}\right) \hat{f}(\xi)\, d\xi\, d\tau.$$

We note that $\tau - \xi \neq 0$ since we are considering the case $\Lambda < M/4$. By definition of the Fourier transform

$$\hat{f}(\xi) = c \int e^{-iy\xi} f(y)\, dy,$$

we get the expression of the kernel

$$\tilde{K}_{M,\Lambda}^{(1)}(x,y) = c \iint e^{-\mathrm{i}\tau x} e^{-iy\xi} \varphi\left(\frac{\tau}{M}\right) \varphi\left(\frac{\xi}{\Lambda}\right) \frac{1}{\tau-\xi}\, d\xi\, d\tau. \tag{2.2.42}$$

Operating the changes of variables $\tau \mapsto M\tau$ and $\xi \mapsto \Lambda\xi$ we obtain

$$\tilde{K}_{M,\Lambda}^{(1)}(x,y) = cM\Lambda \iint e^{-\mathrm{i}\tau M x} e^{-iy\Lambda\xi} \varphi(\tau)\, \varphi(\xi)\, \frac{1}{M\tau - \Lambda\xi}\, d\xi\, d\tau.$$

Integrating two times by parts in τ and then in ξ we find the following estimate

$$\begin{aligned} |\tilde{K}_{M,\Lambda}^{(1)}(x,y)| &\leq C \frac{M\Lambda}{\langle Mx\rangle^2 \langle \Lambda y\rangle^2} \iint \left| \partial_\xi^2 \partial_\tau^2 \left(\frac{\varphi(\tau)\, \varphi(\xi)}{M\tau - \Lambda\xi} \right) \right| d\xi\, d\tau \\ &\leq C \frac{\Lambda M}{\langle Mx\rangle^2 \langle \Lambda y\rangle^2} \frac{1}{\max(M,\Lambda)}. \end{aligned} \tag{2.2.43}$$

Now we can apply Hölder inequality to get

$$\|A^{(1)}_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \le C\frac{\Lambda}{M^{1/p}\Lambda^{1-1/p}}\|f\|_{L^p(\mathbb{R})}. \tag{2.2.44}$$

We can proceed similarly for the kernels

$$K^{(2)}_M(x,y) = \mathbb{1}_{x<0}\mathbb{1}_{y>0}\int e^{-\mathrm{i}\tau(x-y)}\varphi\left(\frac{\tau}{M}\right)T(\tau)\,d\tau \tag{2.2.45}$$

and

$$K^{(3)}_M(x,y) = \mathbb{1}_{\pm x>0}\mathbb{1}_{\pm y>0}\int e^{-\mathrm{i}\tau(x\pm y)}\varphi\left(\frac{\tau}{M}\right)(R_\pm(\tau)+1)\,d\tau \tag{2.2.46}$$

using the assumption $T(\tau)\sim\tau$, $(R_\pm(\tau)+1)\sim\tau$ near $\tau=0$ and fractional integration by parts.

Indeed, from the Theorem 2.3 in [72] we have that $T(\tau)$ is $C^1(\mathbb{R})$ and $R_\pm(\tau)\in C^{0,\alpha}(\mathbb{R})$ with $\alpha<\gamma-1$. Applying α integration by parts we have that

$$|\tilde{K}_{M,\Lambda}(x,y)| \le C\frac{\Lambda M}{\langle Mx\rangle^\alpha\langle\Lambda y\rangle^\alpha}\frac{1}{\max(M,\Lambda)}$$

and

$$\|A_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \le C\frac{\Lambda}{M^{1/p}\Lambda^{1-1/p}}\|f\|_{L^p(\mathbb{R})},$$

where we have chosen $\alpha>\max(1/p,1/p')$ thanks to the hypothesis $\gamma>1+1/p$. In conclusion the estimate (2.2.33) is checked and it holds whenever $0<\Lambda\le M\le 1$. □

Our proof of the equivalence of the high energy part of the homogeneous Besov norms (2.2.5) for the perturbed Hamiltonian and the corresponding unperturbed homogeneous Besov norms is based on (2.2.31). More precisely, we have the following estimates.

Lemma 2.2.7. *Assume that $V\in L^1_\gamma(\mathbb{R})$, $\gamma>1+1/p$, the operator $\mathcal{H}$ has no point spectrum and resonance at zero. Then for any even function $\varphi(\tau)\in C_0^\infty(\mathbb{R}\backslash 0)$ there exists a constant $C=C(\|V\|_{L^1_\gamma(\mathbb{R})})$ so that for any pair of real positive numbers Λ, M and for any $f\in\mathcal{S}(\mathbb{R})$, the following inequalities hold:*

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right)f\right\|_{L^p_x(\mathbb{R})} \le C\left(\frac{\Lambda}{M}\right)^{1/p}\|f\|_{L^p_x(\mathbb{R})},\quad \forall\ 0<\Lambda\le M, M\ge 1 \tag{2.2.47}$$

and

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right)f\right\|_{L^p_x(\mathbb{R})} \le C\left(\frac{M}{\Lambda}\right)^{1/p}\|f\|_{L^p_x(\mathbb{R})},\quad \forall\ \Lambda\ge M, M\ge 1, \tag{2.2.48}$$

with $1 < p < \infty$.

Proof. We shall prove in details (2.2.47), since the proof of (2.2.48) is similar. We take $p \in (1, \infty)$, $f, g \in S(\mathbb{R})$ and we set

$$f_\Lambda(x) = \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right) f.$$

We are considering the case

$$\Lambda \leq M, \ M \geq 1. \tag{2.2.49}$$

Now we use the relation

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f_\Lambda = M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\mathcal{H}_0^{s/2} f_\Lambda + M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\left(\mathcal{H}^{s/2} - \mathcal{H}_0^{s/2}\right) f_\Lambda,$$

where $s > 0$ will be chosen later on and

$$\varphi_1(\tau) = \varphi(\tau)\,\tau^{-s}. \tag{2.2.50}$$

Hence we have the representation formula

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f_\Lambda = M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(\mathcal{H}_0)^{s/2} f_\Lambda + G_{M,\Lambda}(f), \tag{2.2.51}$$

where

$$G_{M,\Lambda} = M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\left(\mathcal{H}^{s/2} - \mathcal{H}_0^{s/2}\right)\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right). \tag{2.2.52}$$

By (2.2.31), we can write

$$\left\| M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\mathcal{H}_0^{s/2} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq \frac{C}{M^s}\left\|\mathcal{H}_0^{s/2} f_\Lambda\right\|_{L^p(\mathbb{R})} \leq \frac{C\Lambda^s}{M^s}\|f\|_{L^p(\mathbb{R})}$$

so we have the estimate

$$\left\| M^{-s}\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\mathcal{H}_0^{s/2} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq \frac{C\Lambda^s}{M^s}\|f\|_{L^p(\mathbb{R})}. \tag{2.2.53}$$

The operator $\mathcal{H}^{s/2} - \mathcal{H}_0^{s/2}$, entering in the right side of (2.2.52) can be substituted by

$$C\int_0^\infty \lambda^{-1+s/2}\left[\mathcal{H}(\lambda+\mathcal{H})^{-1} - \mathcal{H}_0(\lambda+\mathcal{H}_0)^{-1}\right] d\lambda =$$

$$= C\int_0^\infty \lambda^{s/2}(\lambda+\mathcal{H})^{-1}V(\lambda+\mathcal{H}_0)^{-1}d\lambda$$

due to (2.2.30). Hence

$$\|G_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \leq \frac{C}{M^s}\int_0^\infty \lambda^{s/2}h(\lambda,\Lambda,M)d\lambda, \tag{2.2.54}$$

where

$$h(\lambda,\Lambda,M) = \left\|\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(\lambda+\mathcal{H})^{-1}V(\lambda+\mathcal{H}_0)^{-1}f_\Lambda\right\|_{L^p(\mathbb{R})}.$$

By (2.2.32) and the standard estimate

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{\Lambda}\right)(\lambda+\mathcal{H}_0)^{-1}f\right\|_{L^q(\mathbb{R})} \leq \frac{C\Lambda^{1/p-1/q}}{\lambda+\Lambda^2}\|f\|_{L^p(\mathbb{R})}, \tag{2.2.55}$$

we can write

$$h(\lambda,\Lambda,M) = \left\|\varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(\lambda+\mathcal{H})^{-1}V(\lambda+\mathcal{H}_0)^{-1}f_\Lambda\right\|_{L^p(\mathbb{R})} \leq$$

$$\leq \frac{CM^{1-1/p}}{\lambda+M^2}\left\|V(\lambda+\mathcal{H}_0)^{-1}f_\Lambda\right\|_{L^1(\mathbb{R})} \leq$$

$$\leq \frac{CM^{1-1/p}}{\lambda+M^2}\left\|(\lambda+\mathcal{H}_0)^{-1}f_\Lambda\right\|_{L^\infty(\mathbb{R})} \leq \frac{CM^{1-1/p}\Lambda^{1/p}}{(\lambda+M^2)(\lambda+\Lambda^2)}\|f\|_{L^p(\mathbb{R})}.$$

So, we derive from (2.2.54) the inequality

$$\|G_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \leq CM^{1-1/p-s}\Lambda^{1/p}\int_0^\infty \frac{\lambda^{s/2}d\lambda}{(\lambda+M^2)(\lambda+\Lambda^2)}\|f\|_{L^p(\mathbb{R})}. \tag{2.2.56}$$

The following estimate

$$\int_0^\infty \frac{\lambda^{s/2}d\lambda}{(\lambda+M^2)(\lambda+\Lambda^2)} \leq \int_0^\infty \frac{\lambda^{s/2-1}d\lambda}{(\lambda+M^2)} = CM^{s-2} \tag{2.2.57}$$

and (2.2.56) imply

$$\|G_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \leq CM^{-1-1/p}\Lambda^{1/p}\|f\|_{L^p(\mathbb{R})}. \tag{2.2.58}$$

Taking $s = 1/p$, via identity (2.2.51) and inequality (2.2.53) we obtain

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)f_\Lambda\right\|_{L^p(\mathbb{R})} \leq \frac{C\Lambda^{1/p}}{M^{1/p}}\|f_\Lambda\|_{L^p(\mathbb{R})} + \frac{C\Lambda^{1/p}}{M^{1+1/p}}\|f\|_{L^p(\mathbb{R})} \leq \frac{C\Lambda^{1/p}}{M^{1/p}}\|f\|_{L^p(\mathbb{R})},$$

for $M \geq 1$.

We can proceed similarly in the case $\Lambda \geq M$, $M \geq 1$. Indeed we can use the relation

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f_\Lambda = M^s \varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right) \mathcal{H}_0^{-s/2} f_\Lambda + M^s \varphi_1\left(\frac{\sqrt{\mathcal{H}}}{M}\right)\left(\mathcal{H}^{-s/2} - \mathcal{H}_0^{-s/2}\right) f_\Lambda,$$

where

$$\varphi_1(\tau) = \varphi(\tau)\,\tau^s.$$

We can write $\mathcal{H}^{-s/2} - \mathcal{H}_0^{-s/2}$ and $\mathcal{H}_0^{-s/2}$ via (2.2.3). Then operating computations similar to the previous case and using $M \geq 1$, we get

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right) f_\Lambda\right\|_{L^p(\mathbb{R})} \leq C\frac{M^s}{\Lambda^s}\|f\|_{L^p(\mathbb{R})},$$

for any $s \in (0,1)$. In particular it holds for $s = 1/p$. □

Corollary 2.2.8. *Assume that* $V \in L^1_\gamma(\mathbb{R})$ *with*

$$\gamma > 1 + 1/p,\ 0 < s < \frac{1}{p},\ 1 < p < \infty,$$

and assume that the operator $\mathcal{H}$ *has no point spectrum and no resonance at zero. Then for any even function* $\varphi(\tau) \in C_0^\infty(\mathbb{R} \setminus 0)$ *there exists a constant* $C = C(\|V\|_{L^1_\gamma(\mathbb{R})})$ *so that for any pair of real positive numbers* Λ, M *such that* $0 < \Lambda \leq M$, $M \leq 1$ *and for any* $f \in \mathcal{S}(\mathbb{R})$, *the following inequality holds:*

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{\Lambda}\right) f\right\|_{L^p_x(\mathbb{R})} \leq C\frac{\Lambda^s}{M^s}\|f\|_{L^p_x(\mathbb{R})}. \tag{2.2.59}$$

Proof. By Lemma 2.2.2 we have that

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{\Lambda}\right) = K_\Lambda + \mathrm{Rem}_\Lambda,$$

where the kernel $K_\Lambda(x,y)$ is defined in (2.2.20) and the kernel of the remainder $\mathrm{Rem}_\Lambda(x,y)$ satisfies the estimate (2.2.19). We first estimate the remainder term. By (2.2.19) we have

$$\left\| \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right) \mathrm{Rem}_\Lambda f \right\|_{L^p_x(\mathbb{R})} \leq$$
$$\leq C\Lambda \left\| \int_{\mathbb{R}} \left(\sum_{\pm} \frac{1}{\langle \Lambda(x \pm y) \rangle^\sigma} \right) \left(\frac{1}{\langle x \rangle^{\gamma-\sigma}} + \frac{1}{\langle y \rangle^{\gamma-\sigma}} \right) |f(y)|\, dy \right\|_{L^p(\mathbb{R})},$$

where $\sigma \in (0,1) \cap (0, \gamma - 1]$ will be choose small enough. Applying Hölder and Young inequalities in Lorentz spaces with the following indices relation

$$\frac{1}{p} + 1 = \sigma + (1-\sigma) + \frac{1}{p},$$

we get

$$\left\| \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right) \mathrm{Rem}_\Lambda f \right\|_{L^p(\mathbb{R})} \leq C \frac{\Lambda}{\Lambda^\sigma} \|f\|_{L^p(\mathbb{R})} \leq C \frac{\Lambda^s}{M^s} \|f\|_{L^p(\mathbb{R})}.$$

Now we turn to estimate the leading term. We consider

$$K_\Lambda(x,y) = c \mathbb{1}_{x>0} \mathbb{1}_{y>0} \int_{\mathbb{R}} e^{-i\tau(x-y)} \varphi\left(\frac{\tau}{\Lambda}\right) d\tau$$

since we can proceed similarly for the other terms defined in (2.2.20). We look for the kernel $\tilde{K}_{M,\Lambda}(x,y)$ such that

$$\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right) \left(\int_{\mathbb{R}} K_\Lambda(\cdot, y) f(y)\, dy \right)(x) = \int_{\mathbb{R}} \tilde{K}_{M,\Lambda}(x,y) f(y)\, dy.$$

We put

$$h(x) = \int dy \int d\tau\, e^{-i(x-y)\tau} \varphi\left(\frac{\tau}{\Lambda}\right) f(y) \mathbb{1}_{y>0}$$

and

$$g(x) = c \mathbb{1}_{x>0} h(x).$$

Using the notation above we have that

$$\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right) g(x) = c \int e^{ix\xi} \varphi\left(\frac{\xi}{M}\right) \hat{g}(\xi)\, d\xi$$

and

$$\hat{h}(\eta) = c \varphi\left(\frac{\eta}{\Lambda}\right) \int dy e^{iy\eta} f(y) \mathbb{1}_{y>0}.$$

Hence we deduce

$$\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)g(x)=\int dy\underbrace{\int d\xi\int d\eta e^{ix\xi}e^{iy\eta}\varphi\left(\frac{\xi}{M}\right)\varphi\left(\frac{\eta}{\Lambda}\right)\frac{1}{\xi-\eta}}_{\tilde{K}_{M,\Lambda}(x,y)}f(y)\mathbb{1}_{y>0}.$$

Then, as in (2.2.42), integrating by parts we get

$$\left|\tilde{K}_{M,\Lambda}(x,y)\right|\leq C\frac{M\Lambda}{\langle Mx\rangle^2\langle\Lambda y\rangle^2}\frac{1}{\max(M,\Lambda)}.$$

We note that we are considering the case $0<\Lambda<M\leq 1$. Then, using Hölder inequality combined with a scaling argument we get

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)g\right\|_{L^p(\mathbb{R})}\leq C\frac{\Lambda}{M^{1/p}\Lambda^{1-1/p}}\|f\|_{L^p(\mathbb{R})}.$$

□

Corollary 2.2.9. *Assume that $V\in L^1_\gamma(\mathbb{R})$, $\gamma>1+1/p$, the operator $\mathcal{H}$ has no point spectrum and no resonance at zero. Then for any even function $\varphi(\tau)\in C_0^\infty(\mathbb{R}\backslash 0)$ there exists a constant $C=C(\|V\|_{L^1_\gamma(\mathbb{R})})$ so that for any pair of real positive numbers Λ, M and for any $f\in\mathcal{S}(\mathbb{R})$, the following inequalities hold:*

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{\Lambda}\right)f\right\|_{L^p_x(\mathbb{R})}\leq C\left(\frac{\Lambda}{M}\right)^{1/p}\|f\|_{L^p_x(\mathbb{R})},\quad\forall\ 0<\Lambda\leq M, M\geq 1 \tag{2.2.60}$$

and

$$\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{M}\right)\varphi\left(\frac{\sqrt{\mathcal{H}}}{\Lambda}\right)f\right\|_{L^p_x(\mathbb{R})}\leq C\left(\frac{M}{\Lambda}\right)^{1/p}\|f\|_{L^p_x(\mathbb{R})},\quad\forall\ \Lambda\geq M, M\geq 1, \tag{2.2.61}$$

with $1<p<\infty$.

Proof. The proof of the inequalities (2.2.60) and (2.2.61) follows repeating the same arguments of Lemma 2.2.7 and replacing $\mathcal{H}$ with $\mathcal{H}_0$ and vice versa. □

Now we turn to the proof of the main theorem.

Proof of Theorem 2.2.4. We have to prove the equivalence of the norms in (2.2.3) and (2.2.4), i.e.

$$\sum_{k=-\infty}^{\infty}2^{2ks}\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)f\right\|^2_{L^p(\mathbb{R})}\sim\sum_{j=-\infty}^{\infty}2^{2js}\left\|\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)f\right\|^2_{L^p(\mathbb{R})}. \tag{2.2.62}$$

We set

$$a_k = \left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) f \right\|_{L^p(\mathbb{R})}, \quad b_j = \left\| \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f \right\|_{L^p(\mathbb{R})}.$$

Using the Paley-Littlewood partition

$$f = \sum_{j=-\infty}^{\infty} f_j = \sum_{j=-\infty}^{\infty} \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f,$$

we take $\psi(\tau) \in C_0^\infty(\mathbf{R}_+)$ such that $\psi(\tau) = 1$ on the support of φ. Then we can use the identity

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) f = \sum_{j=-\infty}^{\infty} \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) \psi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f_j. \tag{2.2.63}$$

We distinguish the two cases $k \geq 0$ and $k < 0$.

Let $k \geq 0$ be fixed. We can apply Lemma 2.2.7 and we obtain that

$$a_k = \left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) f \right\|_{L^p(\mathbb{R})} \leq C \sum_{j=-\infty}^{\infty} 2^{-|k-j|(1/p)} \|f_j\|_{L^p(\mathbb{R})} = C \sum_{j=-\infty}^{\infty} 2^{-|k-j|(1/p)} b_j.$$

From this we deduce that

$$\left\|2^{ks} a_k\right\|_{\ell^2_{k\geq 0}} \leq C \left\|2^{js} b_j\right\|_{\ell^2_j(\mathbb{Z})}. \tag{2.2.64}$$

Indeed we have

$$\left\|2^{ks} a_k\right\|_{\ell^2_{k\geq 0}(\mathbb{Z})} \leq C \left\| \sum\nolimits_{j\in\mathbb{Z}} 2^{-|j-k|(1/p)} 2^{-(j-k)s} 2^{js} \|f_j\|_{L^p(\mathbb{R})} \right\|_{\ell^2_{k\geq 0}}. \tag{2.2.65}$$

Using the discrete Young inequality combined with the estimate

$$\left\| 2^{-|n|(1/p)-ns} \right\|_{\ell^1_n(\mathbb{Z})} \leq C, \tag{2.2.66}$$

with $0 < s < 1/p$, we get the inequality (2.2.64).

Let $k < 0$ be fixed. Then we write

$$2^{ks} a_k \leq 2^{ks} \sum_{j\leq k} \left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) \psi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f_j \right\|_{L^p(\mathbb{R})} +$$
$$+ 2^{ks} \sum_{j\geq k} \left\| \varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right) \psi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f_j \right\|_{L^p(\mathbb{R})}.$$

Now we estimate the $\ell^2_{k\leq 0}$ norm of the two addends above. We can estimate the first addend as in the case $k > 0$ using the inequality (2.2.59) and the index s' such that $0 < s < s' < 1/p$. Then we can proceed as in (2.2.65), (2.2.66) replacing $1/p$ with s'. For the second addend the prove of the estimate is simpler. Indeed, using (2.2.31) we have

$$2^{ks}\sum_{j\geq k}\left\|\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)\psi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)f_j\right\|_{L^p(\mathbb{R})}\leq C\sum_{j\geq k}2^{ks}2^{-js}2^{js}\|f_j\|_{L^p(\mathbb{R})}. \tag{2.2.67}$$

Since we are considering the case $j \geq k$ and $k < 0$, we can estimate the right side above with the sum

$$C\sum_{j\in\mathbb{Z}}2^{-|k-j|s}2^{js}\|f_j\|_{L^p(\mathbb{R})}.$$

Now, computing the ℓ^2_k norm and applying the discrete Young inequality we complete the proof of the estimate

$$\left\|2^{ks}a_k\right\|_{\ell^2_k(\mathbb{Z})}\leq C\left\|2^{js}b_j\right\|_{\ell^2_j(\mathbb{Z})}. \tag{2.2.68}$$

To prove that

$$\left\|2^{js}b_j\right\|_{\ell^2_j(\mathbb{Z})}\leq C\left\|2^{ks}a_k\right\|_{\ell^2_j(\mathbb{Z})}, \tag{2.2.69}$$

we use Corollary 2.2.8 and Corollary 2.2.9. Indeed, if we write

$$\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)f=\sum_{k=-\infty}^{\infty}\varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right)\psi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)f_k,$$

where

$$f_k=\varphi\left(\frac{\sqrt{\mathcal{H}}}{2^k}\right)f,$$

as before we can distinguish the case $j \geq 0$ and $j < 0$. Computations similar to the ones used to prove (2.2.68) conclude the proof. $\square$

Chapter 3

Hardy inequality and fractional Leibnitz rule for perturbed Hamiltonians on the line

In this chapter we consider the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$ on the real line, where V is a short range potential, $V \in L^1_\gamma(\mathbb{R})$, $\gamma \geq 1$. We assume that the Hamiltonian $\mathcal{H}$ has no zero resonances and that the point spectrum of $\mathcal{H}$ consists of just real negative numbers with absolutely continuous part $[0, \infty)$. Our main goal will be to prove that the perturbed homogeneous Sobolev norms generated by the absolutely continuous part of the Hamiltonian $\mathcal{H}_{ac} = P_{ac}(\mathcal{H})\mathcal{H}$ are equivalent to the classical ones.

The plan of the chapter is the following. In Section 3.1 we motivate the study of the problems and we expose the main results. In Section 3.2 we outline the idea that we follow to prove the main results of the chapter. The Section 3.3 and Section 3.4 are devoted to the proof of the equivalence of the norms. Finally, in Section 3.5 we provide a counterexample that shows that some requirements on the relationship between the regularity index, the integrability index and the dimension of the homogeneous Sobolev spaces are necessary to guarantee the equivalence of the norms.

3.1 Motivation, assumptions and main results

The uncertainty principle in quantum mechanics is frequently associated with Hardy type inequality

$$\| \, |x|^{-s} f\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R}^n)}, \quad s \in [0, n/p), \tag{3.1.1}$$

where $\mathcal{H}_0 = -\Delta$ is the free Hamiltonian in $\mathbb{R}^n, n \geq 1$. The presence of a perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V(x)$ with a short range real-valued perturbation potential $V(x)$ leads to the natural question to verify if Hardy type inequality is true for this perturbed Hamiltonian. The appearance of eigenvectors of $\mathcal{H}$ is an obstacle to have Hardy type inequality or to establish existence and completeness of the wave operators in the whole $L^p(\mathbb{R}^n)$ space, so it is natural to look for estimates of type

$$\| \, |x|^{-s} f\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R}^n)}, \quad s \in [0, n/p), \tag{3.1.2}$$

where $\mathcal{H}_{ac}$ is the absolutely continuous part of the perturbed Hamiltonian and f is in the domain of $\mathcal{H}_{ac}$. Our key goal in this work is to study the equivalence of the following homogeneous Sobolev norms

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})} \sim \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}, \tag{3.1.3}$$

since this equivalence property (in the case $n = 1$) shows that (3.1.1) implies (3.1.2).

Another motivation to study the equivalence property (3.1.3) is connected with the necessity to generalize so called fractional Leibnitz rule, used as a basic tool in rigorous analysis of local well-posedness of nonlinear dispersive equations, to the case of fractional Hamiltonians of type $\mathcal{H}_{ac}^{s/2}$. To be more precise, the following estimate is known as fractional Leibnitz rule or Kato-Ponce estimate (one can see [36])

$$\|\mathcal{H}_0^{s/2}(fg)\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_0^{s/2} f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})} + C\|f\|_{L^{p_3}(\mathbb{R})}\|\mathcal{H}_0^{s/2} g\|_{L^{p_4}(\mathbb{R})}, \tag{3.1.4}$$

where the parameters $s, p, p_j, j = 1, \ldots, 4$, satisfy

$$s > 0, \quad 1 < p, p_1, p_2, p_3, p_4 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

The estimate can be considered as natural homogeneous version of the non-homogeneous inequality of type (3.1.4) involving Bessel potentials $(1 - \mathcal{H}_0)^{s/2}$ in the place of $\mathcal{H}_0^{s/2}$, obtained by Kato and Ponce in [48] (estimates of type (3.1.4) are also called Kato-Ponce estimates). More general domains for parameters can be found in [34]. A more precise estimate can be deduced when $0 < s < 1$. In particular, Kenig, Ponce, and Vega [50] obtained the estimate

$$\|\mathcal{H}_0^{s/2}(fg) - f\mathcal{H}_0^{s/2}g - g\mathcal{H}_0^{s/2}f\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_0^{s_1/2}f\|_{L^{p_1}(\mathbb{R})}\|\mathcal{H}_0^{s_2/2}g\|_{L^{p_2}(\mathbb{R})}, \tag{3.1.5}$$

provided

$$0 < s = s_1 + s_2 < 1, \quad s_1, s_2 \geq 0,$$

and

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{3.1.6}$$

Therefore, one can pose the question to find appropriate short range assumptions on the perturbed Hamiltonian so that the fractional Leibnitz rule (3.1.4) or the more precise bilinear estimate (3.1.5) are valid for this perturbed Hamiltonian. This problem can be solved again by using (3.1.3).

We can make another interpretation of (3.1.3) in terms of the invariance of homogeneous Sobolev spaces $\dot{H}^s_p(\mathbb{R})$ with norms

$$\|f\|_{\dot{H}^s_p(\mathbb{R})} = \|\mathcal{H}_0^{s/2}f\|_{L^p(\mathbb{R})}$$

under the action of the wave operators

$$W_\pm = s - \lim_{t\to\pm\infty} P_{ac}(\mathcal{H})e^{it\mathcal{H}}e^{-it\mathcal{H}_0},$$

where $s - \lim$ means strong limit. The existence and completeness of the wave operators in standard Hilbert space (typically Lebesgue space L^2) in case of short range perturbations is well known (see [52], [58], [41] and the references therein). The functional calculus for the absolutely continuous part $\mathcal{H}_{ac} = P_{ac}(\mathcal{H})\mathcal{H}$ of the perturbed non-negative operator $\mathcal{H}$ can be introduced with a relation involving $W_\pm$

$$g(\mathcal{H}_{ac}) = W_+ g(\mathcal{H}_0)W_+^* = W_- g(\mathcal{H}_0)W_-^*, \tag{3.1.7}$$

for any function $g \in L^\infty_{loc}(0,\infty)$. Moreover, the wave operators map unperturbed Sobolev spaces in the perturbed ones,

$$W_\pm : D(\mathcal{H}_0^{s/2}) \to D(\mathcal{H}_{ac}^{s/2})$$

and we have

$$W_\pm : \dot{H}^s_p(\mathbb{R}) \to \dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R}),\ \forall s \geq 0,\ 1 < p < \infty,$$

where $\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})$ is the perturbed homogeneous Sobolev space generated by the Hamiltonian $\mathcal{H}_{ac}$. More precisely, $\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})$ is the homogeneous Sobolev spaces associated with the absolutely continuous part $\mathcal{H}_{ac}$ of the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$. This is the closure of functions $f \in S(\mathbb{R})$ orthogonal[1] to the eigenvectors of $\mathcal{H}$ with respect to the norm

$$\|f\|_{\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})} = \left\| \mathcal{H}_{ac}^{s/2} f \right\|_{L^p(\mathbb{R})}. \tag{3.1.8}$$

The equivalence property (3.1.3) implies that the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R})$ is invariant under the action of the wave operators $W_\pm$ for $0 \leq s < 1/p$.

The study of the dispersive properties of the evolution flow in some cases of short range perturbed Hamiltonians $\mathcal{H}$ shows (see [17], [29]) that homogeneous Sobolev norms for perturbed and unperturbed Hamiltonians are equivalent

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^2(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2} f\|_{L^2(\mathbb{R}^n)}, \tag{3.1.9}$$

provided $s < n/2$. Our goal is to extend this equivalence to the case

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R}^n)}, \tag{3.1.10}$$

with $s < n/p$.

First, we shall show that the requirement $s < n/p$ is optimal, i.e. we shall prove the following result:

Theorem 3.1.1. *If $n \geq 1$ and $V(x)$ is defined as follows*

$$V(x) = \frac{1}{1 + |x|^3}, \tag{3.1.11}$$

then (3.1.3) *with $s = n/p \leq 2$ is not true.*

Next we turn to the proof of (3.1.3) for the case $n = 1$ and we shall describe the assumptions on the potential V.

We shall assume that the potential $V : \mathbb{R} \to \mathbb{R}$ is a real-valued potential, $V \in L^1(\mathbb{R})$ and V is decaying sufficiently rapidly at infinity, namely following [72] we require $V \in L^1_\gamma(\mathbb{R})$, with $\gamma \geq 1$. The

[1] the precise definition of eigenvectors is given below in (4.1.8)

key assumption is to suppose that $\mathcal{H}$ has no zero resonance using Definition 2.1.5. This definition is expressed in terms of the transmission coefficient

$$T(0) = 0.$$

Moreover the point spectrum of $\mathcal{H}$ consists of real numbers $\lambda \in (-\infty, 0]$, such that

$$\mathcal{H}f - \lambda f = 0, \ f \in L^2(\mathbb{R}), \tag{3.1.12}$$

and absolutely continuous part $[0, \infty)$. We shall denote by $L^2_{pp}(\mathbb{R})$ the linear space generated by the eigenvectors f in (3.1.12). This is finite dimensional space and its orthogonal complement in L^2 is the invariant subspace, where the perturbed Hamiltonian $\mathcal{H}$ is absolutely continuous.

The key tool to prove the Hardy inequality and the fractional Leibnitz rule (3.1.5) is the following estimate.

Theorem 3.1.2. *Suppose*

$$V \in L^1_\gamma(\mathbb{R}), \ \gamma > 1, \ s = \gamma - 1 < 1/p, \ p \in (1, \infty)$$

and the perturbed Hamiltonian $\mathcal{H}$ has no resonance at the origin. Then there exists a positive constant $C = C(s,p) > 0$ so that we have

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2})f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})},$$

for $1/p - 1/q = s$ and $f \in S(\mathbb{R})$.

It is natural to use a Paley-Littlewood localization associated with the perturbed Hamiltonian. Here and below $\varphi(\tau) \in C_0^\infty(\mathbb{R} \setminus 0)$ is a non-negative even function, such that

$$\sum_{j \in \mathbb{Z}} \varphi\left(\frac{\tau}{2^j}\right) = 1 \ , \quad \forall \ \tau \in \mathbb{R} \setminus 0 \tag{3.1.13}$$

and

$$\varphi\left(\frac{\tau}{2^k}\right)\varphi\left(\frac{\tau}{2^\ell}\right) = 0, \ \forall \ k, \ell \in \mathbb{Z}, \ |k - \ell| \geq 2. \tag{3.1.14}$$

We set

$$\pi_k^{ac} = \varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{2^k}\right), \ \pi_k^0 = \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^k}\right). \tag{3.1.15}$$

We have the following equivalent norm (see [79])

$$\|f\|_{\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})} \sim \left\| \left(\sum_{k=-\infty}^{\infty} 2^{2ks} \left|\pi_k^{ac} f\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}. \tag{3.1.16}$$

Our approach to prove Theorem 3.1.2 is based on establishing estimate of the type.

Lemma 3.1.3. *If the assumptions of Theorem 3.1.2 are fulfilled, then for any $s \in (0, 1/p)$ and $q \in (1, \infty)$ defined by*

$$\frac{1}{p} - \frac{1}{q} = s$$

we have

$$\left\| \left\| 2^{ks} \left(\pi_k^{ac} - \pi_k^0\right) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}. \tag{3.1.17}$$

Indeed if this estimate is verified, via (3.1.16), combined with (3.1.13) and (3.1.15) we can deduce the assertion of Theorem 3.1.2. Therefore, the estimate (3.1.17) is the key point in the proof of Theorem 3.1.2.

Corollary 3.1.4. *If the assumptions of Theorem 3.1.2 are fulfilled, then the equivalence property* (3.1.3) *holds.*

Proof. The results in [19], [71], [1], [18], [79] imply the existence and continuity of the wave operators in L^p, $1 < p < \infty$, so one can deduce Bernstein inequality

$$\|\pi_k^{ac} f\|_{L^q(\mathbb{R})} \leq C(2^k)^{1/p-1/q}\|f\|_{L^p(\mathbb{R})}, \quad 1 \leq p \leq q \leq \infty, \ k \in \mathbb{Z} \tag{3.1.18}$$

and via the equivalence property (3.1.16) we deduce the Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \tag{3.1.19}$$

From the estimate of Theorem 3.1.2 now we can write

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2}) f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})},$$

so we have

$$\|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})} \leq C \|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}.$$

The opposite estimate can be deduced in the same way from Theorem 3.1.2 and from the "free" Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \tag{3.1.20}$$

This completes the proof. □

Theorem 3.1.2 has also the following simple consequences.

Corollary 3.1.5. *If the assumptions of Theorem 3.1.2 are fulfilled, then the Hardy inequality* (3.1.2) *holds.*

Corollary 3.1.6. *If the assumptions of Theorem 3.1.2 are fulfilled, then we have the fractional Leibnitz rule, i.e.*

$$\|\mathcal{H}_{ac}^{s/2}(fg) - f\mathcal{H}_{ac}^{s/2} g - g\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})} \leq C \|\mathcal{H}_{ac}^{s_1/2} f\|_{L^{p_1}(\mathbb{R})} \|\mathcal{H}_{ac}^{s_2/2} g\|_{L^{p_2}(\mathbb{R})}, \tag{3.1.21}$$

provided

$$0 < s = s_1 + s_2 < 1, \quad s_1, s_2 \geq 0,$$

and

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{3.1.22}$$

3.2 Idea to prove the key Lemma 3.1.3

Our main tool to study the kernel

$$\varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{M}\right)(x, y)$$

is the following representation of the kernel as filtered Fourier transform

$$\mathcal{F}_{\varphi,M}(a)(\xi) = \int \varphi\left(\frac{\tau}{M}\right) a(\tau) e^{-i\xi\tau} d\tau \tag{3.2.1}$$

of symbols $a(\tau)$ represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{A} = \{ \ 1, \ T(\tau), \ R_{\pm}(\tau) \ \}, \tag{3.2.2}$$

or more generally of symbols involving functions $a(x,\tau)$ represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{B} = \{\widetilde{m_{\pm}}(x,\tau), \ T(\tau)\widetilde{m_{\pm}}(x,\tau), \ R_{\pm}(\tau)\widetilde{m_{\pm}}(x,\tau) \ \}, \tag{3.2.3}$$

where $\widetilde{m_{\pm}}(x,\tau) = m_{\pm}(x,\tau) - 1$, $m_{\pm}$ are modified Jost functions, while $T, R_{\pm}$ are the transmission and reflection coefficients.

It is simple to establish that the kernel $\varphi(\sqrt{\mathcal{H}_{ac}}/M)(x,y)$ can be decomposed as follows (one can see Section 2.2.2 in the previous chapter):

Lemma 3.2.1. *If φ is an even non-negative function, such that $\varphi \in C_0^\infty(\mathbf{R} \setminus \{0\})$, then for any $M > 0$ we have*

$$\varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{M}\right)(x,y) = K_M^0(x,y) + \widetilde{K}_M(x,y), \tag{3.2.4}$$

where $K_M^0(x,y)$ can be represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(a)(\epsilon_3 x + \epsilon_4 y) \tag{3.2.5}$$

and the term $\widetilde{K}_M(x,y)$ is represented as sum of the terms

$$\begin{gathered}\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_1(x,\cdot))(\epsilon_3 x + \epsilon_4 y) + \mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_2(y,\cdot))(\epsilon_3 x + \epsilon_4 y) + \\ + \mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_3(x,\cdot)b_4(y,\cdot))(\epsilon_3 x + \epsilon_4 y),\end{gathered} \tag{3.2.6}$$

where $\epsilon_i = \pm 1$, for $i = 1, \ldots, 4$, $a(\tau)$ represents a linear combination with constant coefficients of functions in the set $\mathcal{A}$ in (3.2.2) *and b_i, for $i = 1, \ldots, 4$, are linear combinations with constant coefficients of functions in the set $\mathcal{B}$ in* (3.2.3).

Remark 3.2.2. We shall call the term $K_M^0(x,y)$ the leading one, with the following exact representation

$$K_M^0(x,y) = c\int_{\mathbb{R}} e^{-\mathrm{i}\tau(x-y)}\varphi\left(\frac{\tau}{M}\right)\alpha(x,y,\tau)\,d\tau \tag{3.2.7}$$

with symmetric kernel $\alpha(x,y,\tau)=\alpha(y,x,\tau)$ and

$$\alpha(x,y,\tau)=\begin{cases} T(\tau) & x<0<y,\\ (R_+(\tau)+1)e^{2i\tau x}-e^{2i\tau x}+1 & 0\le x<y,\\ (R_-(\tau)+1)e^{-2i\tau y}-e^{-2i\tau y}+1 & x<y\le 0. \end{cases}$$

The term $\widetilde{K}_M(x,y)$ will be called the remainder one. In Lemma 3.2.1 to simplify the notation we neglected the symbolism $a^{\pm}$, $b_i^{\pm}$.

A priori estimates for the remainder term are obtained using the estimates of the filtered Fourier transform which will be established in Lemma 3.3.4 and Lemma 3.3.5.

Lemma 3.2.3. *Suppose $V\in L^1_\gamma(\mathbb{R})$, $\gamma\ge 1+s$, $s\in(0,1)$, the operator $\mathcal{H}$ has no point spectrum and 0 is not a resonance point for $\mathcal{H}$. If φ is an even non-negative function, such that $\varphi\in C_0^\infty(\mathbf{R}\setminus\{0\})$, then for any $p\in(1,1/s)$, any $M\in(0,\infty)$ and for any $b^{\pm}(x,\tau)$, $b_1^{\pm}(x,\tau)$, $b_2^{\pm}(x,\tau)$ in the set (3.2.3) we have*

$$\left\|\int_{\mathbb{R}} \mathbb{1}_{\pm x>0}\mathcal{F}_{\varphi,M}(b^{\pm}(x,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})}+ \tag{3.2.8}$$
$$+\left\|\int_{\mathbb{R}} \mathbb{1}_{\pm y>0}\mathcal{F}_{\varphi,M}(b^{\pm}(y,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})}\le \frac{C}{\langle M\rangle}\|f\|_{L^q(\mathbb{R})},$$

and

$$\left\|\int_{\mathbb{R}} \mathbb{1}_{\pm x>0}\mathbb{1}_{\pm y>0}\mathcal{F}_{\varphi,M}(b_1^{\pm}(x,\cdot)b_2^{\pm}(y,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})}\le \frac{C}{\langle M\rangle}\|f\|_{L^q(\mathbb{R})}, \tag{3.2.9}$$

where $\frac{1}{q}=\frac{1}{p}-s$.

According with the notation introduced in (3.1.15), we set

$$\pi^{ac}_{\le k}=\sum_{j\le k}\pi^{ac}_j,\quad \pi^{ac}_{\ge k}=\sum_{j\ge k}\pi^{ac}_j. \tag{3.2.10}$$

$$f_k=\pi^{ac}_k f,\quad f_{\le k}=\sum_{j\le k}\pi^{ac}_j f,\quad f_{\ge k}=\sum_{j\ge k}\pi^{ac}_j f,\quad f_{k_1,k_2}=\sum_{k_1\le j\le k_2}\pi^{ac}_j f$$

and respectively f^0_k, $f^0_{\le k}$, $f^0_{\ge k}$, $f^0_{k_1,k_2}$ defined as before replacing π^{ac}_j with π^0_j.

Hence, the decomposition (3.2.4) can be rewritten as follows

$$\pi^{ac}_k=I_k+(\pi^{ac}_k-I_k),$$

where the operator I_k represents the operators involved in the leading kernel and $(\pi_k^{ac} - I_k)$ is the remainder term.

To prove Lemma 3.1.3 we will establish the following inequalities:

$$\left\| \left\| 2^{ks} \left(\pi_k^{ac} - I_k \right) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}, \tag{3.2.11}$$

$$\left\| \left\| 2^{ks} \left(I_k - \pi_k^0 \right) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}, \tag{3.2.12}$$

with $1/p = 1/q + s$ and I_k are the operators

$$I_k(f)(x) = \int_{\mathbb{R}} K_{2^k}^0(x, y) f(y) dy$$

with kernels representing the leading term (3.2.5) in the expansion of Lemma 3.2.1 of π_k.

3.3 Estimates of the filtered Fourier transform of $m_{\pm} - 1$

Given a bump function $\varphi \in C_0^\infty(\mathbb{R})$, we define the corresponding filtered Fourier transform as in (3.2.1). We shall distinguish two different cases. If the bump function $\varphi \in C_0^\infty((0, \infty))$ is such that (3.1.13) and (3.1.14) are satisfied, then we can assert that $\varphi(\tau/M)$ has a support with $\tau \sim M$.

The integral equation

$$m_{\pm}(x, \tau) - 1 = \int_x^\infty D(t - x, \tau) V(t) m_+(t, \tau) \, dt,$$

can be rewritten as

$$\widetilde{m_+}(x, \tau) = \int_x^\infty \int_0^{t-x} e^{2i\tau y} V(t) dy dt + \int_x^\infty \int_0^{t-x} e^{2i\tau y} V(t) \widetilde{m_+}(t, \tau) dy dt, \tag{3.3.1}$$

where

$$\widetilde{m_+}(x, \tau) = m_+(x, \tau) - 1.$$

If we assume that $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1 + s$, then the assertion of Lemma 2.1.2 guarantees that $\widetilde{m_+}(x, \tau)$ is in $L^1_{x>0}(\mathbb{R})$. Applying the filtered Fourier transform and setting

$$g_M(\xi; x) = \int_{\mathbb{R}} e^{-i\tau\xi} \widetilde{m_+}(x, \tau) \varphi\left(\frac{\tau}{M}\right) d\tau = \mathcal{F}_{\varphi, M}(\widetilde{m_+}(x, \cdot))(\xi),$$

we get

$$g_M(\xi;x) = \underbrace{M\int_x^\infty \int_0^{t-x} V(t)\widehat{\varphi}(M(\xi-2y))dydt +}_{a_M(\xi;x)} \tag{3.3.2}$$
$$+\int_x^\infty \int_0^{t-x} V(t) g_M(\xi-2y;t)dydt.$$

We have the following pointwise estimates.

Lemma 3.3.1. *If $\varphi \in C_0^\infty(\mathbb{R})$, satisfies (3.1.13), (3.1.14) and $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, $s \in (0,1)$, then for $M \in (0,1)$ the filtered Fourier transform*

$$\mathcal{F}_{\varphi,M}\left(\widetilde{m_\pm}(x,\cdot)\right)(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi}\widetilde{m_\pm}(x,\tau)\varphi\left(\frac{\tau}{M}\right)d\tau$$

satisfies the pointwise estimates:

- *one can find functions*

$$F_M^\pm(\xi) \in L^1(\mathbb{R}), \quad \|F_M^\pm\|_{L^1(\mathbb{R})} \le C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})},$$

  so that

$$\mathbb{1}_{\{\pm x>0\}}\langle x\rangle^s \left|\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)\right| \le F_M^\pm(\xi). \tag{3.3.3}$$

Proof. We choose the sign + in (3.3.3) for determinacy. To prove (3.3.3) we set

$$G_M(\xi;x) = \mathbb{1}_{\{x>0\}} \sup_{\eta<\xi} |g_M(\eta;x)|\langle x\rangle^s,$$

where $g_M(\xi;x)$ is the Filtered Fourier transform of the remainder $\widetilde{m_+}(x,\tau) = m_+(x,\tau)-1$, satisfying the integral equation (3.3.2). The function

$$F_M(\xi) = M\int_0^\infty \langle t\rangle^s |V(t)| \int_0^t |\widehat{\varphi}(M(\xi-2y))|dydt, \tag{3.3.4}$$

satisfies

$$F_M(\xi) \in L^1(\mathbb{R}), \quad \|F_M\|_{L^1(\mathbb{R})} \le \|V\|_{L^1_\gamma(\mathbb{R})}\|\widehat{\varphi}\|_{L^1(\mathbb{R})}. \tag{3.3.5}$$

Moreover, since we are considering the case $x > 0$ we get easily the following estimates

$$|\mathbb{1}_{x>0}\langle x\rangle^s a_M(\xi;x)| \le F_M(\xi),$$

where $a_M(\xi;x)$ is defined in (3.3.2). Hence, coming back to $G_M(\xi;x)$ and recalling (3.3.2) we have

$$G_M(\xi;x) \le F_M(\xi) + \int_x^\infty \langle t\rangle |V(t)| G_M(\xi;t)dt, \ \forall x > 0. \tag{3.3.6}$$

Applying the Gronwall lemma we get

$$G_M(\xi;x) \le C F_M(\xi),$$

where C is a positive constant depending on $\|V\|_{L^1_1(\mathbb{R})}$ and $F_M(\xi)$ satisfies (3.3.4) and (3.3.5). This completes the proof. □

If $M \ge 1$ and φ satisfying (3.1.13) and (3.1.14), then we can improve the results of Lemma 3.3.1. Indeed, the term $a_M(\xi;x)$ in (3.3.2) can be rewritten as follows

$$a_M(\xi;x) = M \int_x^\infty dt \int_{\mathbb{R}} d\tau V(t) e^{-i\tau M\xi} \varphi(\tau) \frac{e^{2iM\tau(t-x)} - 1}{2iM\tau}.$$

Hence we have that

$$|\mathbb{1}_{x>0}\langle x\rangle^s a_M(\xi;x)| \le F_M^{(1)}(\xi),$$

where

$$F_M^{(1)}(\xi) = \int_x^\infty \langle t\rangle^{s+1} |V(t)| |\hat{\varphi}(M\xi)|\, dt \tag{3.3.7}$$

and

$$\|F_M^{(1)}(\xi)\|_{L^1(\mathbb{R})} \le \frac{1}{M} \|V\|_{L^1_{s+1}(\mathbb{R})} \|\hat{\varphi}\|_{L^1(\mathbb{R})}.$$

Proceeding as in the proof of Lemma 3.3.1 we get the following result.

Lemma 3.3.2. *If φ satisfies* (3.1.13) *and* (3.1.14) *and $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1 + s$, $s \in (0,1)$, then for $M \in (0,\infty)$ the filtered Fourier transform*

$$\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} \left(\widetilde{m_\pm}(x,\tau)\right) \varphi\left(\frac{\tau}{M}\right) d\tau$$

satisfies the pointwise estimates:

- *one can find functions*

$$F_M^\pm(\xi) \in L^1(\mathbb{R}), \quad \|F_M^\pm\|_{L^1(\mathbb{R})} \leq \frac{1}{\langle M\rangle} C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})},$$

so that

$$\mathbb{1}_{\{\pm x>0\}}\langle x\rangle^s \left|\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)\right| \leq F_M^\pm(\xi). \tag{3.3.8}$$

One can use a Wiener type argument and deduce estimates for $T(\tau), R_\pm(\tau)+1$.

Lemma 3.3.3. *(see [18], [79]) If* $\varphi \in C_0^\infty(\mathbb{R})$ *obeys* (3.1.13), (3.1.14) *and* $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, $s \in (0,1)$, *then for* $M \in (0,\infty)$ *the filtered Fourier transforms*

$$\mathcal{F}_{\varphi,M}(T(\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} T(\tau)\varphi\left(\frac{\tau}{M}\right) d\tau$$

and

$$\mathcal{F}_{\varphi,M}(R_\pm(\cdot)+1)(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi}(R_\pm(\tau)+1)\varphi\left(\frac{\tau}{M}\right) d\tau$$

are in $L^1(\mathbb{R})$ *and the following inequality are satisfied*

$$\|\mathcal{F}_{\varphi,M}(T(\cdot))(\xi)\|_{L^1(\mathbb{R})} + \|\mathcal{F}_{\varphi,M}(R_\pm(\cdot)+1)(\xi)\|_{L^1(\mathbb{R})} \leq C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})}, \ M \in (0,1),$$
$$\|\mathcal{F}_{\varphi,M}(T(\cdot)-1)(\xi)\|_{L^1(\mathbb{R})} + \|\mathcal{F}_{\varphi,M}R_\pm(\cdot)(\xi)\|_{L^1(\mathbb{R})} \leq \frac{1}{\langle M\rangle}C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})}, \ M > 1.$$

Turning to the estimates (3.3.3), we see that

$$a(x,\xi) = \mathbb{1}_{\{\pm x>0\}}\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)$$

satisfies estimate

$$|a(x,\xi)| \leq a_1(x)a_2(\xi), \ a_1 \in L^{1/s,\infty}(\mathbb{R}), \ a_2 \in L^1(\mathbb{R}), \tag{3.3.9}$$

where $a_1(x) = \langle x\rangle^{-s}$. Lemma 3.3.3 guarantees that

$$b(\xi) = \mathcal{F}_{\varphi,M}(T(\cdot))(\xi) \in L^1(\mathbb{R}).$$

Since

$$\mathbb{1}_{\{\pm x>0\}}\mathcal{F}_{\varphi,M}(T(\cdot)(\widetilde{m_\pm}(x,\cdot)))(\xi) = a(x,\cdot) * b(\cdot)(\xi),$$

we see that

$$|a(x,\cdot) * b(\cdot)(\xi)| \le a_1(x)\underbrace{a_2 * |b|}_{\widetilde{a_2}}(\xi),\ a_1 \in L^{1/s,\infty}(\mathbb{R}),\ \widetilde{a_2} \in L^1(\mathbb{R}),$$

since

$$L^1 * L^1 \subset L^1$$

due to the Young inequality.

The above inclusion actually can be modified in a suitable way for our a priori estimates as follows

$$\left(L^1 \cap L^\infty\right) * \left(L^1 \cap L^\infty\right) \subset \left(L^1 \cap L^\infty\right). \tag{3.3.10}$$

This observation leads to the following result.

Lemma 3.3.4. *If $\varphi \in C_0^\infty(\mathbb{R})$, $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, $s \in (0,1)$, and $a^\pm(x,\tau)$ is any function in the set*

$$\left\{\widetilde{m_\pm}(x,\tau),\ T(\tau)\widetilde{m_\pm}(x,\tau),\ (R_\pm(\tau)+1)\widetilde{m_\pm}(x,\tau)\right\}, \tag{3.3.11}$$

then for $M \in (0,\infty)$ the filtered Fourier transform

$$\mathcal{F}_{\varphi,M}(a^\pm(x,\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} a^\pm(x,\tau)\varphi\left(\frac{\tau}{M}\right) d\tau$$

satisfies the pointwise estimates:

$$\mathbb{1}_{\{\pm x>0\}}\left|\mathcal{F}_{\varphi,M}(a^\pm(x,\cdot))(\xi)\right| \le f_1(x) f_2^{(M)}(\xi), \tag{3.3.12}$$

where

$$f_1(x) \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ f_2^{(M)}(\xi) \in L^1(\mathbb{R})$$

and $\|f_2^{(M)}\|_{L^1(\mathbb{R})} \le C/\langle M\rangle$.

Finally we consider products of type $a^\pm(x,\tau)b^\pm(y,\tau)$, where a, b are in the set (3.3.11) and we have the following estimates.

Lemma 3.3.5. *If $\varphi \in C_0^\infty(\mathbb{R})$ is a bump function satisfying* (3.1.13), (3.1.14), *$V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$,*

$s \in (0,1)$, *then for* $M \in (0,\infty)$ *the filtered Fourier transform of* $a^{\pm}(x,\tau)b^{\pm}(y,\tau)$ *satisfies the pointwise estimate:*

$$\mathbb{1}_{\pm x>0}\mathbb{1}_{\pm y>0}\left|\mathcal{F}_{\varphi,M}(a^{\pm}(x,\cdot)b^{\pm}(y,\cdot))(\xi)\right| \le f_1(x)f_2^{(M)}(\xi)f_3(y), \tag{3.3.13}$$

where

$$f_1, f_3 \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ f_2^{(M)}(\xi) \in L^1(\mathbb{R}),\ \|f_2^{(M)}\|_{L^1(\mathbb{R})} \le \frac{C}{\langle M\rangle}$$

with some constant $C > 0$ *independent of* M.

Now we can proceed with the proof of Lemma 3.2.1.

Proof of Lemma 3.2.1. To fix the idea and to simplify the notation we consider the case involving $b^+(y,\tau) = b(y,\tau)$. We separate two cases: $M \in (0,1]$ and $M \ge 1$. For $M \in (0,1]$ our first step is to prove

$$\left\|\int_{\mathbb{R}} \mathbb{1}_{y>0}\mathcal{F}_{\varphi,M}(b(y,\cdot))(x \pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})}. \tag{3.3.14}$$

We use the pointwise estimate (3.3.12) so we can write

$$\mathbb{1}_{y>0}\left|\mathcal{F}_{\varphi,M}(b(y,\cdot))(x \pm y)\right| \le B_1^{(M)}(x \pm y)B_2(y),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}),\quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \le C,\quad B_2 \in L^{1/s,\infty}(\mathbb{R})$$

and (3.3.14) follows from Young inequality

$$\left\|B_1^{(M)} * (B_2 f)\right\|_{L^p_x(\mathbb{R})} \le C\|B_1^{(M)}\|_{L^1(\mathbb{R})}\|B_2 f\|_{L^p(\mathbb{R})}, \tag{3.3.15}$$

and the Hölder estimate

$$\|B_2 f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})},\quad B_2 \in L^{1/s,\infty}(\mathbb{R}),\ \frac{1}{q} = \frac{1}{p} - s. \tag{3.3.16}$$

Similarly, to prove

$$\left\|\int_{\mathbb{R}} \mathbb{1}_{x>0}\mathcal{F}_{\varphi,M}(b(x,\cdot))(x \pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})} \tag{3.3.17}$$

we use the pointwise estimate (3.3.12) again, so we can write

$$\mathbb{1}_{x>0}\,|\mathcal{F}_{\varphi,M}(b(x,\cdot))(x\pm y)| \leq B_1^{(M)}(x\pm y)B_2(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq C, \; B_2 \in L^{1/s,\infty}(\mathbb{R}).$$

This time we have to estimate the term

$$\left\| B_2(B_1^{(M)} * f) \right\|_{L^p_x(\mathbb{R})}$$

so first we apply Hölder estimate (3.3.16) and then the Young convolution inequality.

Finally, the estimate (3.2.9) follows from (3.3.13) since we have

$$\mathbb{1}_{x>0}\mathbb{1}_{y>0}\,|\mathcal{F}_{\varphi,M}(b_1(x,\cdot)b_2(y,\cdot))(x\pm y)| \leq B_1^{(M)}(x\pm y)B_2(y)B_3(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq C, \quad B_2(y), B_3(x) \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

This completes the proof for the case $M \in (0,1]$. For $M \geq 1$, we note that, we can also use the fact that we have better estimate

$$\|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq CM^{-1}$$

and we can prove (3.2.8) and (3.2.9) assuming $V \in L^1_1(\mathbb{R})$ only. This completes the proof. □

3.4 Equivalence of homogeneous Sobolev norms

In this section we are going to prove Lemma 3.1.3.

Proof of the inequality (3.2.11). The relation (3.2.6) guarantees that

$$\pi_k^{ac}(f)(x) - I_k(f)(x)$$

can be represented as a sum of remainder terms of the form

$$\sum_{\epsilon_1,\dots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_1(x,\cdot))(\epsilon_3 x+\epsilon_4 y) f(y)\, dy+$$
$$+\sum_{\epsilon_1,\dots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_2(y,\cdot))(\epsilon_3 x+\epsilon_4 y) f(y)\, dy+$$
$$+\sum_{\epsilon_1,\dots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_3(x,\cdot) b_4(y,\cdot))(\epsilon_3 x+\epsilon_4 y) f(y)\, dy,$$

such that the estimates of Lemma 3.2.3 imply

$$\|(\pi_k^{ac}-I_k)\, f\|_{L^p(\mathbb{R})} \le \frac{C}{\langle 2^k\rangle}\|f\|_{L^q(\mathbb{R})},$$

with

$$\frac{1}{q}=\frac{1}{p}-s.$$

Using the inequalities

$$\left\| \|2^{ks}\,(\pi_k^{ac}-I_k)\, f\|_{\ell^2_k} \right\|_{L^p_x(\mathbb{R})} \le \left\| \|2^{ks}\,(\pi_k^{ac}-I_k)\, f\|_{\ell^1_k} \right\|_{L^p_x(\mathbb{R})} \le$$

$$\le \left\| \|2^{ks}\,(\pi_k^{ac}-I_k)\, f\|_{L^p_x(\mathbb{R})} \right\|_{\ell^1_k} \le \left\| \frac{2^{ks}}{\langle 2^k\rangle} \right\|_{\ell^1_k} \|f\|_{L^q_x(\mathbb{R})},$$

we deduce (3.2.11). This completes the proof. □

Proof of Lemma 3.1.3. Our main goal is to establish the following estimate

$$\left\| \|2^{ks}(\pi_k^{ac}-\pi_k^0) f\|_{\ell^2_k} \right\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})}, \tag{3.4.1}$$

with $1/q = 1/p - s$.

We start proving that

$$\left\| \|2^{ks}(\pi_k^{ac}-\pi_k^0) f\|_{\ell^2_{k\le 0}} \right\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})}. \tag{3.4.2}$$

In particular, it will be enough to prove

$$\left\| \|2^{ks}(I_k-\pi_k^0) f\|_{\ell^2_{k\le 0}} \right\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})},$$

since the estimate (3.2.11) has been just established above.

Using the decomposition

$$f = \sum_{j\in\mathbb{Z}} f_j^0,$$

we have that

$$\left(I_k - \pi_k^0\right) f = \left(I_k - \pi_k^0\right) f_{k-2,k+2}^0. \tag{3.4.3}$$

Indeed, if follows from

$$\left(I_k - \pi_k^0\right) f_{\leq k-2}^0(x) = c \int\int e^{i(x+y)\tau} \varphi\left(\frac{\tau}{2^k}\right) \alpha(x,y,\tau) f_{\leq k-2}^0(y)\, d\tau\, dy = 0$$

and

$$\left(I_k - \pi_k^0\right) f_{\geq k+2}^0(x) = c \int\int e^{i(x+y)\tau} \varphi\left(\frac{\tau}{2^k}\right) \alpha(x,y,\tau) f_{\geq k+2}^0(y)\, d\tau\, dy = 0,$$

where $\alpha(x,y,\tau)$ has been defined in Remark 3.2.2. Moreover, the expression of the leading term shows that the kernel $\left(I_k - \pi_k^0\right)(x,y)$ can be also represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x>0} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(a)(\epsilon_3 x + \epsilon_4 y),$$

where $\epsilon_j = \pm 1, j = 1, \ldots, 4$, according with Remark 3.2.2, and the symbol a is a linear combination with constant coefficients of functions in the set $\mathcal{A}$ defined in (3.2.2).

For simplicity we consider the case $a = 1$, $\epsilon_j = 1$, $\forall j = 1, \ldots, 4$, and we shall estimate the term

$$\int \mathbb{1}_{x>0} \mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{M}\right) d\tau.$$

Then, we can proceed similarly for the other terms.

Integrating by parts and using Lemma 2.1.2, we get

$$\begin{aligned} \left\| 2^{ks} \int\int \mathbb{1}_{x>0} \mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y)\, d\tau\, dy \right\|_{\ell^2_{k\leq 0}} &\leq \\ \leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x>0} \mathbb{1}_{y>0}}{\langle 2^k(x+y)\rangle^{1+s}} f_k^0(y)\, dy \right\|_{\ell^2_{k\leq 0}} dy& \\ \leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x>0} \mathbb{1}_{y>0}}{\langle 2^k(x+y)\rangle^{1+s}} \right\|_{\ell^\infty_{k\leq 0}} \left\| f_k^0 \right\|_{\ell^2_{k\leq 0}} dy.& \end{aligned}$$

From the trivial inequality

$$\left\| \frac{2^{k(s+1)}}{\langle 2^k x\rangle^{1+s}} \right\|_{\ell^\infty_{k\leq 0}} \leq \frac{C}{|x|^{1+s}}$$

combined with the Young inequality in Lorentz spaces we have

$$\left\| \left\| 2^{ks} \int\int \mathbb{1}_{x>0}\mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y)\, d\tau\, dy \right\|_{\ell^2_{k\leq 0}} \right\|_{L^p(\mathbb{R})} \leq C \left\| \|f_k^0(y)\|_{\ell^2_{k\leq 0}} \right\|_{L^q(\mathbb{R})},$$

with $1/q = 1/p - s$ and $0 < s < 1/p$.

The case $k \geq 0$ follows similarly using the estimate

$$\left|(\pi_k^{ac} - \pi_k^0) f(x)\right| \leq C \int \frac{f(y)}{\langle 2^k(x \pm y)\rangle^s} \left(\frac{1}{\langle x\rangle} + \frac{1}{\langle y\rangle}\right) dy.$$

This complete the proof. □

3.5 Counterexample for equivalence of homogeneous Sobolev spaces

In this section we consider the case $p \in [n/2, \infty) \cap (1, \infty)$ and we shall prove Theorem 3.1.1, therefore we shall show that the equivalence property

$$\|(\mathcal{H}_0 + V)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \sim \|(\mathcal{H}_0)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \tag{3.5.1}$$

is not true for $n \in \mathbb{N}$.

Proof of Theorem 3.1.1. Let us suppose that the relation (3.5.1) holds. Choosing positive potential

$$V(x) = \frac{1}{1 + |x|^3},$$

we can apply the heat kernel estimate obtained in [78], i.e.

$$\frac{C_1 e^{-c_1|x-y|^2/4t}}{t^{n/2}} \leq e^{-t\mathcal{H}}(x, y) \leq \frac{C_2 e^{-c_2|x-y|^2/4t}}{t^{n/2}}. \tag{3.5.2}$$

This estimate and the relation

$$\mathcal{H}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t\mathcal{H}} dt$$

imply

$$\left|(\mathcal{H}_0 + V)^{-1} u(x)\right| \leq C \left|(\mathcal{H}_0)^{-1} u(x)\right|$$

so taking the L^p norm and using a duality argument, we can write

$$\|V(\mathcal{H}_0 + V)^{-1} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.5.3}$$

so we have

$$\|Vg\|_{L^p(\mathbb{R}^n)} \leq C\|(\mathcal{H}_0 + V)g\|_{L^p(\mathbb{R}^n)}. \tag{3.5.4}$$

Interpolation argument and the assumption $p \geq n/2$ combined with the equivalence property (3.5.1) lead to

$$\int_{\mathbb{R}^n} (V(x))^{n/2} |u(x)|^p dx \leq C\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)}. \tag{3.5.5}$$

Taking u in the Schwartz class $S(\mathbb{R}^n)$ of rapidly decreasing function, we can apply a rescaling argument. Indeed, considering the dilation

$$u_\lambda(x) = u(x\lambda),$$

we find

$$\|\mathcal{H}_0^{n/(2p)} u_\lambda\|^p_{L^p(\mathbb{R}^n)} = \underbrace{\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^2(\mathbb{R}^n)}}_{\text{constant in } \lambda}$$

and

$$\lim_{\lambda \searrow 0} \int_{\mathbb{R}^n} V^{n/2}(x) |u_\lambda(x)|^p dx = \left(\int_{\mathbb{R}^n} V^{n/2}(x) dx\right) |u(0)|^p.$$

In this way we deduce

$$|u(0)|^p \left(\int_{\mathbb{R}^n} V^{n/2}(x) dx\right) \leq C\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)}. \tag{3.5.6}$$

The homogeneous norm

$$\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)}$$

is also invariant under translations, i.e. setting

$$u^{(\tau)}(x) = u(x + \tau),$$

we have

$$\widehat{u^{(\tau)}}(\xi) = e^{\mathrm{i}\tau\xi}\widehat{u}(\xi)$$

and

$$\|\mathcal{H}_0^{n/(2p)} u^{(\tau)}\|^p_{L^p(\mathbb{R}^n)} = \|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)},$$

so applying (3.5.6) with $u^{(\tau)}$ in the place of u, we find

$$|u(\tau)|^p \int_{\mathbb{R}^n} V^{n/2}(x)dx \le C\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)},$$

or equivalently

$$\|u\|^p_{L^\infty(\mathbb{R}^n)} \le C_1\|\mathcal{H}_0^{n/(2p)} u\|^p_{L^p(\mathbb{R}^n)}, \tag{3.5.7}$$

where

$$C_1 = \frac{C}{\|V^{n/2}\|_{L^1(\mathbb{R}^n)}}.$$

The substitution $\phi = \mathcal{H}_0^{n/(2p)} u$ enables us to rewrite (3.5.7) as

$$\|I_{n/p}(\phi)\|^p_{L^\infty(\mathbb{R}^n)} \le C_1\|\phi\|^p_{L^p(\mathbb{R}^n)}, \tag{3.5.8}$$

where

$$I_\alpha(\phi)(x) = \mathcal{H}_0^{-\alpha/2}(\phi)(x) = c\int_{\mathbb{R}^n} |x-y|^{-n+\alpha}\phi(y)dy,\ \alpha \in (0,n)$$

are the Riesz operators.

It is easy to show that (3.5.8) leads to a contradiction. Indeed, taking

$$\phi_N(x) = \sum_{j=0}^{N} \underbrace{|x|^{-n/p}\ \mathbb{1}_{2^j \le |x| \le 2^{j+1}}(x)}_{\chi_j(x)},$$

with $N \ge 2$ sufficiently large and being $\mathbb{1}_A(x)$ the characteristic function of the set A. Since the functions χ_j have almost disjoint supports and they are non-negative, for almost every $x \in \mathbb{R}$ we have

$$\sum_{j=1}^{N} \chi_j^p(x) = \left(\sum_{j=1}^{N} \chi_j(x)\right)^p.$$

so

$$\|\phi_N\|^p_{L^p(\mathbb{R}^n)} = \sum_{j=0}^{N} \int_{2^j}^{2^{j+1}} \frac{r^{n-1}dr}{r^n} \leq C'N.$$

Further, we can use the estimates

$$I_{n/p}(\phi_N)(0) \geq \left(\sum_{j=0}^{N} \int_{2^j}^{2^{j+1}} \frac{r^{n-1}dr}{r^n} \right) \geq CN.$$

Hence, from (3.5.8) we deduce

$$CN^p \leq \|I_{n/p}(\phi_N)\|^p_{L^\infty(\mathbb{R}^n)} \leq C_1\|\phi_N\|^p_{L^p(\mathbb{R}^n)} \leq C_2N,$$

for any N sufficiently big and this is impossible. This completes the proof of the Theorem. □

Chapter 4

On gauge invariant NLS with short range potential

In this Chapter we consider the 1D Schrödinger equation with a gauge invariant nonlinearity and with Hamiltonian with short range potential without zero resonances. The aim will be to prove that the quantity $\sup_{t>0}(1+t)^{1/2}\|\psi(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is bounded if the initial data are chosen small enough in a suitable norm.

4.1 Introduction

We consider the nonlinear Schrödinger (NLS) equation with gauge invariant nonlinearity

$$\mathrm{i}\partial_t\psi - \mathcal{H}\psi = \psi F(|\psi|^2), \tag{4.1.1}$$

where the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V(x)$ can be considered as a real valued potential perturbation of the free hamiltonian $\mathcal{H}_0 = -\partial_x^2$ on the real line $x \in \mathbb{R}$. We will explain the properties of the nonlinearity function F later on. For the moment we can keep in mind the classical semilinear Schrödinger equation with pure power nonlinearity. We are interested in the asymptotic behaviour in time of the solution ψ of (4.1.1) when the initial data are suitably chosen. It is well-known that, in the case of free Hamiltonian, the cubic nonlinearity

$$\mathrm{i}\partial_t\psi - \mathcal{H}_0\psi = \pm\psi|\psi|^2, \tag{4.1.2}$$

is critical for the scattering in one dimension. At least heuristically, we can guess that the cubic case is critical. Indeed, let us consider the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t\psi - \mathcal{H}_0\psi = \pm|\psi|^{p-1}\psi \\ \psi(0) = \psi_0, \end{cases}$$

and its integral formulation

$$\psi(t) = e^{-\mathrm{i}t\mathcal{H}_0}\psi_0 \mp \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)\mathcal{H}_0}\psi|\psi|^{p-1}\,ds, \tag{4.1.3}$$

where $1 < p < 5$. We note that, the free L^∞ estimate could be verified also in the nonlinear case if the following estimate holds

$$\int_{t/2}^t \frac{ds}{s^{p/2}} \leq C\frac{1}{t^{1/2}}.$$

Hence, in one dimension, if the exponent p is greater than 3 we can expect asymptotic decay close to the free one. Moreover, we also expect that $p = 3$ is a threshold exponent in the study of the asymptotic behaviour of the solution, and in this case we will call this exponent *critical* for the scattering.

We are going to focus our attention to the one dimensional case. For the other dimensions it is possible to find the state of the art of the problem in [17] and references therein. It is well known that, if we consider initial data of small size in a suitable Sobolev norm, for $3 < p < 5$, the solution of the problem (4.1.3) verifies the decay estimate

$$\|\psi(t)\|_{L^\infty(\mathbb{R})} \leq C,$$

one can see [54]. On the other hand, in [64] and [5] one can find a proof that, if $1 < p \leq 3$, then the zero solution is the only one asymptotically free. In [17] and references therein one can find a more complete list of the literature connected with the problem of the decay of the solutions and the existence of the scattering operators for NLS. We also quote [55], where the existence and the form of the scattering operator are obtained and [38] in which the completeness of the scattering operator and the decay estimate are proved. In presence of a perturbed operator, $\mathcal{H}$, there is a narrower literature. Indeed, as far as we are concerned, for $p > 3$, the problems of establishing decay estimates and of proving the scattering of the small data solutions were addressed in [17]. If the potential verifies suitable decay hypotheses and there are neither eigenvalues nor resonances, then the solutions scatter if the initial data are small in the energy-variance space. Our goal will be to prove decay estimates in critical regime with perturbed Hamiltonians.

Now we come back to the problem (4.1.1). We consider $F : \mathbb{R} \to \mathbb{R}$ such that

$$F \in C^1(\mathbb{R}), \quad F(u) = Cu + O(u^q) \;\; 1 < q < 2, \text{ for } 0 < u < 1, \tag{4.1.4}$$

with $C \in \mathbb{R} \setminus \{0\}$, so that we can consider as typical example

$$F(|\psi|^2) = C|\psi|^2 + C_1|\psi|^{2q}, \; 1 < q < 2, \; C_1 \in \mathbb{R},$$

when the nonlinearity is gauge invariant, but the scale invariance of (4.1.2) is broken. The presence of the potential V brakes also the translation invariance of the NLS.

In this chapter we consider

$$V : \mathbb{R} \to \mathbb{R}, \tag{4.1.5}$$

i.e. real valued potential, that is even function

$$V(x) = V(-x). \tag{4.1.6}$$

This assumption enables us to preserve at least the reflection symmetry, i.e. if the initial data of the problem (4.1.1) are odd functions

$$\psi(0, x) = \psi_0(x) \in H^s_{odd}(\mathbb{R}) = \{f \in H^s(\mathbb{R}); f(-x) = -f(x)\},$$

then the solution flow preserves this symmetry, i.e.

$$\psi(t, x) = -\psi(t, -x)$$

for any time interval, where the solution flow of (4.1.1) is well defined. We deal with potentials decaying sufficiently rapidly at infinity, namely following [72] and the previous chapter we require

$$V \in L^1_\gamma(\mathbb{R}), \; \gamma = 1 + s. \tag{4.1.7}$$

Our next spectral assumptions concern the self-adjoint operator

$$\mathcal{H} = \mathcal{H}_0 + V(x),$$

namely we require that

$$\sigma_p(\mathcal{H}) = \emptyset. \tag{4.1.8}$$

Finally, we assume that 0 is not a resonance for $\mathcal{H}$. This practically means that V is of generic type, i.e. the transmission coefficient $T(\tau)$ defined in [19] satisfies

$$T(0) = 0. \tag{4.1.9}$$

Characterizing such potentials is rather tricky. We note that the assumption (4.1.7) is equivalent to $\mathcal{H} \geq 0$, if we add the requirements that zero is neither eigenvalue nor resonance for $\mathcal{H}$. A reasonable characterization appeared only recently in [45]. It is shown that for potentials with reasonable decay at $\pm\infty$, such as (4.1.7), we have

$$\mathcal{H} = -\partial_x^2 + V \geq 0 \Longleftrightarrow V(x) = w'(x) + w^2(x) = Miura(w). \tag{4.1.10}$$

In other words, V generates a non-negative Schrödinger operator, if and only if V is in the image of the Miura map, that it will be denoted with $Image(Miura)$. In fact, we observe that (4.1.10) implies in particular that

$$\mathcal{H} = (\partial_x + w)(-\partial_x + w) = MM^*. \tag{4.1.11}$$

For technical reasons, instead of assuming (4.1.10), we make a slightly more general assumption (one can see Lemma 4.2.2 for a proof), namely for some $\epsilon > 0$, we require that $(1+\epsilon)V \in Image(Miura)$, that is

$$(1+\epsilon)V(x) = w'(x) + w^2(x). \tag{4.1.12}$$

We will show below that this is slightly more general property than just $V \in Image(Miura)$. Indeed, the condition $V \in Image(Miura)$ guarantees the absence of negative eigenvalues whereas the condition (4.1.12) implies (4.1.10), see Lemma 4.2.2. Moreover, in Lemma 4.2.2 we also show that, under the additional assumption, $V \neq 0$ *a.e.* (and still in tandem with (4.1.12)), we can guarantee the absence of resonance at zero. It gives the equivalence of the norms of the twisted Sobolev spaces $H^1_V(\mathbb{R})$ with the standard $H^1(\mathbb{R})$ (we note that in Chapter 3 we established the equivalence of homogeneous Sobolev norms in the case $s < 1/p$, $1 < p \leq \infty$). Finally, regarding the non-resonance condition (4.1.9), this is equivalently defined by saying that the equation $\mathcal{H}[f] = 0$ does not have bounded solutions (it follows from Proposition 1.3.10 in Chapter 1, relation (2.1.10) and Lemma 2 in [19]). The connection between the Miura map and the solutions of $\mathcal{H}[f] = 0$ is described in [45], Lemma 3.1. In particular the following relation between the resonance state f and the generating function ω is established:

$$\omega(x) = \frac{d}{dx}\left(\ln f(x)\right).$$

The non-resonance condition is essentially requiring that the generating function w in (4.1.10) does not have an integrable decay.

Our goal is to study the asymptotic behavior of small data solutions to the Cauchy problems, under suitable smallness assumptions on the initial data.

In the following, for simplicity, we consider a time translation and we assume the initial data is given at $t = 1$, ψ_1.

The following is the main result of this chapter.

Theorem 4.1.1. *Let $s \in (1/2, 3/4)$. Let V be a potential such that* (4.1.6)*,* (4.1.7) *and* (4.1.12) *are satisfied and $V \neq 0 a.e.$ Then one can find constants $C > 0$ and $\delta > 0$ so that whenever*

$$e^{\frac{\mathrm{i}x^2}{4}}\psi_1 \in H^s_{odd}(\mathbb{R}), \quad \|e^{\frac{\mathrm{i}x^2}{4}}\psi_1\|_{H^s_{odd}(\mathbb{R})} \leq \varepsilon, \tag{4.1.13}$$

the unique global solution $\psi \in C([0,\infty); H^s_{odd}(\mathbb{R})) \cap L^\infty(\mathbb{R}; L^\infty(\mathbb{R}))$ to the Cauchy problem (4.1.1) *exists and moreover it satisfies*

$$\sup_{t>0}(1+t)^{1/2}\|\psi(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\varepsilon. \tag{4.1.14}$$

In the following of this section we are going to give the main idea of the proof.

For simplicity of the explanation we consider only the case

$$F(u) = u.$$

We can define $\psi(t)$ as a solution to the integral equation

$$\psi(t) = e^{-\mathrm{i}(t-1)\mathcal{H}}\psi_1 \mp \mathrm{i}\int_1^t e^{-\mathrm{i}(t-s)\mathcal{H}}\psi(s)|\psi(s)|^2 ds, \ t > 1. \tag{4.1.15}$$

If the initial data are in the weighted Sobolev spaces $H^{s,s}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2(\mathbb{R}, \langle x\rangle^s\, dx)$ with norm small enough and $V = 0$, then it is proved in [38] that the solutions live in the space $C(\mathbb{R}; H^{s,s}(\mathbb{R}))$ and moreover the free decay estimate (4.1.14) holds. In [38] is also proved the existence of modified scattering states. Our main goal will be to control the decay of the L^∞-norm of the solution provided we have small initial data as stated in (4.1.13).

In the following we will outline the main ideas of the proof.

We first operate the transform

$$(t, \psi) \implies (T, \Psi),$$

where

$$t = \frac{1}{T}, \quad \Psi(T, x) = \overline{\psi\left(\frac{1}{T}, x\right)}. \tag{4.1.16}$$

Then, we can rewrite the solution (4.1.15) as follows

$$\Psi(T) = e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-1\right)}\overline{\psi_1} \pm \mathrm{i}\int_T^1 e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-\frac{1}{S}\right)}\Psi(S)|\Psi(S)|^2\frac{dS}{S^2}. \tag{4.1.17}$$

In other words, we passed from the set of threes, time, space and real part of the wave function

$$(t, x, \Re\psi(t, x)),$$

to the set of threes, frequency, space and real part of the wave function

$$(T, x, \Re\Psi(T, x)).$$

Next, the key point is the construction of an appropriate isometry for the spacial distance,

$$B(T)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

so that

$$\Phi(T) = B(T)\Psi(T),$$

satisfies the integral equation

$$\Phi(T) = U(T, 1)B(1)\overline{\psi_1} \pm \mathrm{i}\int_T^1 U(T, S)\Phi(S)|\Phi(S)|^2\frac{dS}{S}, \tag{4.1.18}$$

and the two-parameter group $U(T,S)$ has property very close to the two-parameter group $e^{i(T-S)\mathcal{H}}$.

To be more precise, we choose

$$U(T,S) = B(T)e^{\mathrm{i}\mathcal{H}/T}e^{-\mathrm{i}\mathcal{H}/S}B^*(S), \tag{4.1.19}$$

where $B(T)$ is defined by

$$B(T) = M(T)\sigma_T, \tag{4.1.20}$$

with

$$M(T)f(x) = e^{\mathrm{i}\frac{x^2}{4T}}f(x), \quad \sigma_T(f)(x) = \frac{1}{T^{1/2}}f\left(\frac{x}{T}\right). \tag{4.1.21}$$

The introduction of the isometry $B(T)$, transforms the set of threes, frequency, space and real part of the wave function into the set of threes, frequency, momentum and the real part of the backward wave function

$$(T, X, \Re\Phi(T, X)).$$

In this way, the proof of Theorem 4.1.1 is reduced to the proof of the following estimate.

Theorem 4.1.2. *Suppose the conditions* (4.1.6), (4.1.7) *and* (4.1.12)*are fulfilled. Hence the operator* $\mathcal{H}$ *has no point spectrum and* 0 *is not a resonance for* $\mathcal{H}$. *Let* $\Phi(T)$ *be the solution to the integral equation* (4.1.18) *with initial data* $\phi_1(x) = e^{\frac{ix^2}{4}}\overline{\psi_1}(x)$ *and let* (4.1.13) *be verified. Then the following decay estimate holds*

$$\|\Phi(T,\cdot)\|_{L^\infty(\mathbb{R})} \le C\varepsilon. \tag{4.1.22}$$

To prove this a priori bound we are going to introduce a leading term for the asymptotic wave profile. Indeed we are interested in establishing the leading part of the backward wave function.

To be more precise, since we looking for $B(T)$ such that the property of $U(T,S)$ are very close to the one of $e^{-i\mathcal{H}(T-S)}$, we can define the leading term when $T \approx 0$ as specified in the following lines. If the existence of $U(0,S)$, with $S \in (0,1)$, can be justified, we set

$$\Phi_0 = U(0,1)B(1)\overline{\psi_1} \pm \mathrm{i}\int_0^1 \left[U(0,S) - I\right]\Phi(S)|\Phi(S)|^2\frac{dS}{S}. \tag{4.1.23}$$

Then, we define the leading term of the asymptotic wave profile as

$$\Phi_{lead}(T) = \Phi_0 \pm i\int_T^1 |\Phi|^2\Phi_{lead}(S)\frac{dS}{S},$$

namely,

$$\Phi_{lead}(T) = \Phi_0 e^{\pm i\Theta(T)}, \tag{4.1.24}$$

where

$$\Theta(T) = \int_T^1 |\Phi(S)|^2 \frac{dS}{S}. \tag{4.1.25}$$

To prove the Theorem 4.1.1, and in particular to control the L^∞ norm of the solution, we are going to establish the following a priori bounds:

$$\sup_{T\in[0,1]} \left(T^{\theta/N} \|\Phi_{lead}\|_{H^{s-\theta}(\mathbb{R})} \right) + \|\Phi_{lead}\|_{L^\infty([0,1]\times\mathbb{R})} \leq C\varepsilon, \tag{4.1.26}$$

and

$$\sup_{T\in[0,1]} \left(T^{\theta/N} \|\Phi(T) - \Phi_{lead}(T)\|_{H^{s-\theta}(\mathbb{R})} \right) + \|\Phi(T) - \Phi_{lead}(T)\|_{L^\infty([0,1]\times\mathbb{R})} \leq C\varepsilon, \tag{4.1.27}$$

where $\theta \in [0,1]$ is such that $1/2 < s - \theta < 3/4$, N will be chosen big enough and we are supposing that the condition (4.1.13) on the initial data is verified.

4.2 Spectral assumptions and Hardy type inequality

In this section we want to establish the equivalence between the norms $\|\sqrt{\mathcal{H}_0}\|_{L^2(\mathbb{R})}$ and $\|\sqrt{\mathcal{H}}\|_{L^2(\mathbb{R})}$ for a wide class of potentials.

Now, with a variational approach, we will show that the absence of eigenvalues and zero resonances for the operator $\mathcal{H}$, give us information on the whole family of operators $\mathcal{H}_0 + gV$ as g is close to 1. This information will be crucial to get an Hardy type inequality.

Lemma 4.2.1. *Suppose that the operator $\mathcal{H}$ satisfies the assumptions* (4.1.8), (4.1.9) *and*

$$V \in L^1_\gamma(\mathbb{R}),$$

with $\gamma > 1$. Then there exists $\delta_0 > 0$, so that

$$\|\nabla f\|^2_{L^2(\mathbb{R})} + (1+\delta_0) \int_{\mathbb{R}} V(x)|f(x)|^2 dx \geq 0, \ \forall f \in H^1(\mathbb{R}). \tag{4.2.1}$$

Proof. Define the functional

$$E_\delta(f) = \frac{1}{2}\|\nabla f\|^2_{L^2(\mathbb{R})} + \frac{(1+\delta)}{2}\int_{\mathbb{R}} V(x)|f(x)|^2 dx, \quad f \in H^1(\mathbb{R}).$$

We argue by contradiction. Suppose the assertion of the Lemma is not true. Then we can find a sequence

$$f_n \in H^1(\mathbb{R}), \ \|f_n\|^2_{L^2(\mathbb{R})} = 1,$$

so that

$$E_{1/n}(f_n) < 0.$$

Then the problem

$$I_\delta = \inf_{f\in H^1(\mathbb{R}),\ \|f\|_{L^2(\mathbb{R})}=1} E_\delta(f) \tag{4.2.2}$$

has negative minimum. Let $g_n \in H^1(\mathbb{R})$ be the minimum with constraint $\|g_n\|_{L^2(\mathbb{R})} = 1$. Hence we have that

$$I_{1/n} = E_{1/n}(g_n) < 0.$$

Rewriting the line above as follows

$$E_{1/n}(g_n) = I_{1/n}\|g_n\|_{L^2(\mathbb{R})},$$

and computing the first variation of the functional we get

$$-\partial_x^2 g_n + \left(1+\frac{1}{n}\right)V(x)g_n = 2I_{1/n}g_n, \quad \|g_n\|_{L^2(\mathbb{R})} = 1.$$

We can renormalize this relation as

$$\left(\mu_n - \partial_x^2\right)h_n + \left(1+\frac{1}{n}\right)V(x)h_n = 0, \quad \int_{\mathbb{R}} V(x)|h_n(x)|^2 dx = 1, \tag{4.2.3}$$

taking h_n proportional to g_n. Here and below, taking suitable monotone subsequences, we can assume the sequence

$$\mu_n = -2I_{1/n} > 0$$

is decreasing (thanks to the renormalization) and convergent, so we have

$$\mu_n = -2I_{1/n} \searrow \mu \geq 0. \tag{4.2.4}$$

If $\mu > 0$, then we can use (4.2.3) and see that

$$\mu_n \|h_n\|^2_{L^2(\mathbb{R})} + \|\partial_x h_n\|^2_{L^2(\mathbb{R})} = O(1), \quad \int_{\mathbb{R}} V(x)|h_n(x)|^2 dx = 1$$

as $n \to \infty$. So taking again subsequence, we can assure the weak convergence

$$h_n \rightharpoonup h^*$$

and the renormalization condition gives us

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}} V(x)|h_n(x)|^2 dx = \int_{\mathbb{R}} V(x)|h^*(x)|^2 dx.$$

In this way we find a nontrivial function $h^* \in H^1(\mathbb{R})$ so that

$$\left(\mu - \partial_x^2\right) h^* + V(x)h^* = 0$$

and this contradicts the assumption (4.1.8).

Therefore, it remains to consider only the case $\mu = 0$ in (4.2.4). In this case we can use the fact that (4.2.3) implies that

$$\left(\mu_n - \partial_x^2\right) h_n + V(x)h_n = -\frac{1}{n} V(x)h_n$$

and we are in position to use the assumption that zero is not a resonance and apply the resolvent estimate obtained in Lemma 1.4.2

$$\left\|(\lambda^2 - \mathcal{H})^{-1} f\right\|_{L^\infty(\mathbb{R})} \leq C \left\|\langle x\rangle^\gamma f\right\|_{L^1(\mathbb{R})} \;\; \forall \lambda \in \mathbb{C}, \Im\lambda > 0, \tag{4.2.5}$$

where $\gamma > 1$. So, choosing $\lambda = i\sqrt{\mu_n}$ we get

$$\|h_n\|_{L^\infty(\mathbb{R})} \leq \frac{C}{n} \left\|\langle x\rangle^\gamma V h_n\right\|_{L^1(\mathbb{R})} \leq \frac{C}{n} \left\|\langle x\rangle^\gamma V\right\|_{L^1(\mathbb{R})} \|h_n\|_{L^\infty(\mathbb{R})}.$$

Since $V \in L^1_\gamma(\mathbb{R})$, we can take n sufficiently large and deduce

$$\|h_n\|_{L^\infty(\mathbb{R})} = 0$$

that is in contradiction with the normalization condition

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}} V(x)|h_n(x)|^2 dx.$$

$\square$

Remark 4.2.1. The connection between the creation and annihilation of resonance at zero and the point spectrum of the family of operators

$$\mathcal{H}_0 + gV$$

as $g \to 1$, is studied in [57]. The result established in Theorem 3 in [57] requires however exponential decay of the potential, while the assumptions of the last Lemma just require the weaker assumption $V \in L^1_\gamma(\mathbb{R})$, with $\gamma > 1$. In particular, we have proved that, if $V \in L^1_\gamma(\mathbb{R})$, with $\gamma > 1$ and the Hamiltonian has nor zero resonances neither eigenvalues, then there exists $\delta_0 > 0$ such that

$$-\Delta + (1+\delta_0)V \geq 0.$$

Conversely, now we discuss the structure of the potentials connected with the class $Image(Miura)$

$$V(x) = w'(x) + w(x)^2, \tag{4.2.6}$$

its relations to the Hardy inequality and how it is connected with the positivity of the operator $\mathcal{H}$. In particular, we will establish an equivalence between the standard Sobolev space $H^1(\mathbb{R})$ and the twisted Sobolev space $H^1_V(\mathbb{R})$.

In this way we choose the potentials that are in the image of the Miura transform. More precise information for the image of the Miura transform can be found in [45].

We can infer the equivalence of the norms above supposing that a small perturbation of the potential lives in $Image(Miura)$.

Lemma 4.2.2. *Let $q > 0$ and $qV \in Image(Miura)$. Then for every $p : 0 < p < q$, $pV \in Image(Miura)$, and under the additional assumption that $V(x) \neq 0$ a.e., one can infer from $(1+\epsilon)V \in Image(Miura)$*

that $\mathcal{H} = \mathcal{H}_0 + V$ does not support resonances at zero.

In particular, our assumption (4.1.12) guarantees that $V \in Image(Miura)$, which by (4.1.10) implies that $\mathcal{H} = \mathcal{H}_0 + V \geq 0$.

Proof. We have that there exists w, so that

$$qV(x) = w'(x) + w^2.$$

Thus,

$$pV = \frac{pw'(x)}{q} + \frac{p^2 w^2(x)}{q^2} + w^2(x)\left(\frac{p}{q} - \frac{p^2}{q^2}\right) = Miura\left(\frac{pw}{q}\right) + w^2(x)\left(\frac{p}{q} - \frac{p^2}{q^2}\right).$$

But $\frac{p}{q} - \frac{p^2}{q^2} > 0$, so we have (in operator sense)

$$-\partial_x^2 + pV = \left[-\partial_x^2 + Miura\left(\frac{pw}{q}\right)\right] + w^2(x)\left(\frac{p}{q} - \frac{p^2}{q^2}\right) \geq -\partial_x^2 + Miura\left(\frac{pw}{q}\right) \geq 0, \tag{4.2.7}$$

where in the last step, we have used (4.1.10) in that $-\partial_x^2 + Miura(W) \geq 0$. Using again the other direction of the equivalence (4.1.10), we conclude $pV \in Image(Miura)$.

Let $V \neq 0$ *a.e.* and $(1+\epsilon)V \in Image(Miura)$. Assume for a contradiction that $f \neq 0$, $f \in L^\infty(\mathbb{R})$, is a resonance function for $\mathcal{H}$, that is $\mathcal{H}[f] = 0$. We take a positive cutoff function $\mathcal{H} : \varphi(x) = 1, |x| < 1$, $\varphi(x) = 0, |x| > 2$ and we put $\varphi_N(x) = \varphi(x/N)$. We can evaluate for any large $N >> 1$, $\langle H(f\varphi_N), f\varphi_N\rangle$. Since f is a resonance, we must have

$$\lim_{N\to\infty} \langle H(f\varphi_N), f\varphi_N\rangle = 0.$$

Following the line of reasoning of (4.2.7) however (with $p = 1, q = 1+\epsilon$), we have

$$\begin{aligned} \langle \mathcal{H}_V(f\varphi_N), f\varphi_N\rangle &\geq \langle [-\partial_{xx} + Miura((1+\epsilon)^{-1}w)](f\varphi_N), f\varphi_N\rangle + \\ + \quad \epsilon(1+\epsilon)^{-2}\langle w^2 f\varphi_N, f\varphi_N\rangle &\geq \frac{\epsilon}{(1+\epsilon)^2}\int_{\mathbb{R}} w^2(x)|f(x)|^2|\varphi_N|^2 dx \geq 0. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}} w^2(x)|f(x)|^2 dx = \lim_{N\to\infty}\int_{\mathbb{R}} w^2(x)|f(x)|^2|\varphi_N(x)|^2 dx = 0.$$

Hence we have that $\omega = 0$ a.e. on $\text{supp}\{f\}$. On the other side, since f is a resonance we have that

$$\int |\nabla f|^2 \, dx + \int \omega' |f|^2 \, dx = 0.$$

The line above implies that ω' is negative a.e. on $\text{supp}\{f\}$. This is in contradiction with the statement $\omega = 0$ a.e. on $\text{supp}\{f\}$. □

Our next result establishes the equivalence of the Sobolev spaces $H^1_V(\mathbb{R}) \cap L^2_{odd}(\mathbb{R})$ and $H^1_{odd}(\mathbb{R})$, under the assumptions put forward in Theorem 4.1.1.

Lemma 4.2.3. *Assume that V satisfies* (4.1.7) *and* (4.1.12). *Then, there exists a constant C, so that for all odd Schwartz functions f,*

$$\frac{1}{C}\|f'\|_{L^2(\mathbb{R})} \leq \|\sqrt{\mathcal{H}} f\|_{L^2(\mathbb{R})} \leq C\|f'\|_{L^2(\mathbb{R})}. \tag{4.2.8}$$

In other words, $\|f\|_{H^1_V(\mathbb{R}) \cap L^2_{odd}(\mathbb{R})} \sim \|f\|_{H^1(\mathbb{R})}$.

Proof. We have

$$\|\sqrt{\mathcal{H}} f\|^2_{L^2(\mathbb{R})} = \langle \mathcal{H} f, f \rangle = \int_{\mathbb{R}} |f'(x)|^2 dx + \int_{\mathbb{R}} V(x)|f(x)|^2 dx.$$

The right-hand side inequality in (4.2.8) is a direct consequence of the Hardy inequality. Indeed, by (4.1.7), $|V(x)| \leq C_1 |x|^{-2}$. Thus, by the standard Hardy's inequality, we have

$$\int_{\mathbb{R}} V(x)|f(x)|^2 dx \leq C_1 \int_{\mathbb{R}} \frac{|f(x)|^2}{|x|^2} dx \leq C_2 \int_{\mathbb{R}} |f'(x)|^2 dx.$$

whenever $f(0) = 0$ (which is of course satisfied if f is odd to begin with). This shows

$$\|\sqrt{\mathcal{H}} f\|_{L^2(\mathbb{R})} \leq \sqrt{C_2 + 1}\|f'\|_{L^2(\mathbb{R})}.$$

For the left-hand side inequality of (4.2.8), we argue from $-\partial_x^2 + (1+\epsilon)V \geq 0$, which is a consequence of our assumption (4.1.12). This implies

$$0 \leq \langle (-\partial_x^2 + (1+\epsilon)V) f, f \rangle = \|f'\|^2_{L^2(\mathbb{R})} + (1+\epsilon) \int_{\mathbb{R}} V(x)|f(x)|^2 dx.$$

Since we have

$$(1+\epsilon)\|\sqrt{\mathcal{H}}f\|^2_{L^2(\mathbb{R})} = \epsilon\|f'\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})} + (1+\epsilon)\int_{\mathbb{R}} V(x)|f(x)|^2 dx,$$

follows that

$$\|f'\|^2_{L^2(\mathbb{R})} \le \left(1+\frac{1}{\epsilon}\right)\|\sqrt{\mathcal{H}}f\|^2_{L^2(\mathbb{R})} \le (1+C_2)\left(1+\frac{1}{\epsilon}\right)\|f'\|^2_{L^2(\mathbb{R})}.$$

Dividing by $(1+1/\epsilon)$ we get (4.2.8). □

4.3 Heuristic idea to define modified scattering profile.

In this section, we want to define heuristically the modified scattering profile of the unperturbed problem, using an approach that involves the two parameters groups.

The flow map associated with cubic NLS is determined by the group $e^{-\mathrm{i}t\mathcal{H}}$ with generator $-\mathrm{i}\mathcal{H}$. How anticipated in the introduction to this chapter, the most important step is to define a family of unitary operators

$$B(T)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{4.3.1}$$

so that the new two parameters group

$$U(T,S) = B(T)e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-\frac{1}{S}\right)}B^*(S),\ 0 < T, S \le 1$$

has properties very close to the original group

$$e^{-\mathrm{i}\mathcal{H}(T-S)}.$$

The same question is meaningful for the unperturbed Hamiltonian $\mathcal{H}_0$ and can be posed in the following way. We are looking for a continuous family of unitary operators (4.3.1), so that the generator of

$$U_0(T,S) = B(T)e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}B^*(S),\ 0 < T, S \le 1 \tag{4.3.2}$$

is exactly $-\mathrm{i}\mathcal{H}_0$, and hence we can write

$$\underbrace{B(T)e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}B^*(S)}_{U_0(T,S)} = e^{-\mathrm{i}\mathcal{H}_0(T-S)}. \tag{4.3.3}$$

One can see that the generator of this two parameters group $U_0(T,S)$ is $\mathrm{i}\Delta$, provided

$$B(T) = M(T)\sigma_T,$$

where

$$M(T)f(x) = e^{\mathrm{i}\frac{x^2}{4T}}f(x), \quad \sigma_T(f)(x) = \frac{1}{T^{1/2}}f\left(\frac{x}{T}\right). \tag{4.3.4}$$

Indeed, taking $f \in D(\mathcal{H}_0)$ and using (4.3.2) we see that

$$\underbrace{U_0(T,1)f}_{\Phi(T)} = B(T)\,\underbrace{e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-1\right)}B^*(1)f}_{\Psi(T)}. \tag{4.3.5}$$

Hence

$$\Psi(T) = e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-1\right)}B^*(1)f$$

formally solves the Cauchy problems

$$\partial_T\Psi(T) = -\frac{\mathrm{i}}{T^2}\mathcal{H}_0\Psi(T), \quad \Psi(1) = B^*(1)f. \tag{4.3.6}$$

Setting

$$\Phi(T) = U_0(T,1)f = M(T)\sigma_T\Psi(T),$$

we obtain (the detailed proof is given in Lemma 4.4.5 below)

$$\partial_T\Phi(T) = \mathrm{i}\Delta\Phi(T), \quad \Phi(1) = f. \tag{4.3.7}$$

This observation proves (4.3.3).

The solution to the Cauchy problem

$$\begin{cases} \partial_t\psi = -\mathrm{i}\mathcal{H}_0\psi(x) \mp \mathrm{i}\psi|\psi|^2, & (t,x) \in \mathbb{R}\times\mathbb{R} \\ \psi(1,x) = \psi_1(x), \end{cases} \tag{4.3.8}$$

can be rewritten as a solution to the integral equation

$$\psi(t) = e^{-\mathrm{i}(t-1)\mathcal{H}_0}\psi_1 \mp \mathrm{i}\int_1^t e^{-\mathrm{i}(t-s)\mathcal{H}_0}\psi(s)|\psi(s)|^2 ds. \tag{4.3.9}$$

We can make the simple transform

$$(t, \psi) \implies (T, \Psi)$$

where

$$t = \frac{1}{T}, \quad \Psi(T, x) = \overline{\psi\left(\frac{1}{T}, x\right)} \tag{4.3.10}$$

and we can rewrite (4.3.9) as follows

$$\Psi(T) = e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-1\right)}\overline{\psi_1} \pm \mathrm{i}\int_T^1 e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}\Psi(S)|\Psi(S)|^2\frac{dS}{S^2}. \tag{4.3.11}$$

Now setting

$$\Phi(T) = B(T)\Psi(T),$$

we can recover $\Psi(T)$ by the aid of the relation

$$\Psi(T) = B^*(T)\Phi(T), B^*(T) = M^*(T)\sigma_T^*,$$

where

$$M^*(T)f(x) = e^{-\mathrm{i}\frac{x^2}{4T}}f(x), \quad \sigma_T^*(f)(x) = T^{1/2}f(Tx).$$

To this end we use the relation

$$\Psi(S)|\Psi(S)|^2 = \left[B^*(S)\Phi(S)|B^*(S)\Phi(S)|^2\right] = SB^*(S)\left[\Phi(S)|\Phi(S)|^2\right] \tag{4.3.12}$$

and via (4.3.3) we find

$$\Phi(T) = e^{-\mathrm{i}\mathcal{H}_0(T-1)}B(1)\left(\overline{\psi_1}\right) \pm \mathrm{i}\int_T^1 e^{-\mathrm{i}\mathcal{H}_0(T-S)}\Phi(S)|\Phi(S)|^2\frac{dS}{S}. \tag{4.3.13}$$

The free group $U_0(T, S)$ satisfies the estimates

$$\left\|(1-\Delta)^{\alpha/2}\left[U_0(T,S) - I\right]g\right\|_{L^2(\mathbb{R})} \leq C|T-S|^{\theta/2}\left\|(1-\Delta)^{(\alpha+\theta)/2}g\right\|_{L^2(\mathbb{R})} \tag{4.3.14}$$

for any $\theta \in (0,1)$ and for any $T, S \in (0,1]$.

Hence, the integral formulation (4.3.13) suggests us to define the leading term of the asymptotic profile as follows. Setting

$$\Phi_0 = e^{\mathrm{i}\mathcal{H}_0} B(1)\left(\overline{\psi_1}\right) \pm \mathrm{i} \int_0^1 \left[e^{\mathrm{i}\mathcal{H}_0 S} - I\right] \Phi(S)|\Phi(S)|^2 \frac{dS}{S},$$

we can define the leading term $\Phi_{lead}(T)$ of the solution $\Phi(T)$ as the solution of the integral equation

$$\Phi_{lead}(T) = \Phi_0 \pm \mathrm{i} \int_T^1 \Phi_{lead}(S)|\Phi(S)|^2 \frac{dS}{S}. \tag{4.3.15}$$

If we choose $\theta \in [0,1]$ sufficiently small such that

$$3/4 > \alpha = s - \theta > 1/2,$$

then, using (4.3.14) we can prove the estimates (4.1.27) and (4.1.26) provided the initial data satisfy the following smallness condition

$$\|e^{-ix^2/4}\psi_1\|_{H^s(\mathbb{R})} \le \epsilon.$$

The presence of the norm $\|e^{-\mathrm{i}x^2/4}\psi_1\|_{H^s(\mathbb{R})}$ in the estimates above, due to the introduction of the unitary operators $B(T)$, explains the use of weighted Sobolev spaces $H^{s,a}(\mathbb{R})$ equipped with norm

$$\|f\|_{H^{s,a}(\mathbb{R})} = \|\langle x\rangle^a f\|_{L^2(\mathbb{R})} + \|(1-\Delta)^{s/2} f\|_{L^2(\mathbb{R})},\ s \ge 0, a \ge 0.$$

4.4 Modified profile for the perturbed Hamiltonian

Now, following the free case, we want to construct the modified scattering profile for the perturbed problem. As before, we define the two parameters group

$$U(T,S) = B(T) e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T} - \frac{1}{S}\right)} B^*(S),\ 0 < T, S \le 1,$$

where

$$B(T) = M(T)\sigma_T,$$

and $M(T), \sigma_T$ are defined according to (4.3.4).

The generator of the two parameters group $U(T,S)$ is

$$\mathrm{i}\Delta(T),$$

where

$$\Delta(T) = -\frac{1}{T^2}\sigma_T \mathcal{H}\sigma_T^* = \Delta - \frac{1}{T^2}V\left(\frac{x}{T}\right). \tag{4.4.1}$$

Indeed, taking $f \in L^2(\mathbb{R})$, so that $B^*(1)f \in D(\mathcal{H})$, we see that

$$\underbrace{U(T,1)f}_{\Phi(T)} = B(T)\underbrace{e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-1\right)}B^*(1)f}_{\Psi(T)}, \tag{4.4.2}$$

Hence,

$$\Psi(T) = e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-1\right)}B^*(1)f,$$

solves the Cauchy problem

$$\partial_T\Psi(T) = -\frac{\mathrm{i}}{T^2}\mathcal{H}_V\Psi(T), \quad \Psi(1) = B^*(1)f. \tag{4.4.3}$$

Setting

$$\Phi(T) = U(T,1)f = M(T)\sigma_T\Psi(T),$$

we obtain (the detailed proof is given in Lemma 4.4.5 below)

$$\partial_T\Phi(T) = \mathrm{i}\Delta(T)\Phi(T), \quad \Phi(1) = f. \tag{4.4.4}$$

Using the fractional calculus for the perturbed Hamiltonian $\mathcal{H}$, one can define the fractional powers of $(-\Delta(T))^{s/2}$ and $(1-\Delta(T))^{s/2}$ as follows

$$(-\Delta(T))^{s/2} = \frac{1}{T^s}\sigma_T\mathcal{H}^{s/2}\sigma_T^*, \quad (1-\Delta(T))^{s/2} = \frac{1}{T^s}\sigma_T(T^2+\mathcal{H})^{s/2}\sigma_T^*. \tag{4.4.5}$$

Of special importance is the following estimate (for detailed proof see Lemma 4.5.2)

$$C^{-1}\|(1-\Delta(T))^{s/2}f\|_{L^2(\mathbb{R})} \le \|(1+\mathcal{H}_0)^{s/2}f\|_{L^2(\mathbb{R})} \le C\|(1-\Delta(T))^{s/2}f\|_{L^2(\mathbb{R})} \tag{4.4.6}$$

for any $s \in [0,1]$.

Turning to the Cauchy problem

$$\begin{cases} \partial_t \psi = -\mathrm{i}\mathcal{H}\psi(x) \mp \mathrm{i}\psi|\psi|^2, & (t,x) \in \mathbb{R}\times\mathbb{R} \\ \psi(1,x) = \psi_1(x), \end{cases} \tag{4.4.7}$$

we follow the same steps already done for the free Hamiltonian. So, we rewrite (4.4.7) as follows

$$\psi(t) = e^{-\mathrm{i}(t-1)\mathcal{H}}\psi_1 \mp \mathrm{i}\int_1^t e^{-\mathrm{i}(t-s)\mathcal{H}}\psi(s)|\psi(s)|^2 ds. \tag{4.4.8}$$

The transformation

$$(t,\psi) \implies (T,\Psi)$$

defined by

$$t = \frac{1}{T}, \quad \Psi(T,x) = \overline{\psi\left(\frac{1}{T},x\right)}. \tag{4.4.9}$$

leads to the integral equation

$$\Psi(T) = e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-1\right)}\overline{\psi_1} \pm \mathrm{i}\int_T^1 e^{\mathrm{i}\mathcal{H}\left(\frac{1}{T}-\frac{1}{S}\right)}\Psi(S)|\Psi(S)|^2\frac{dS}{S^2}. \tag{4.4.10}$$

Further, we defined

$$\Phi(T) = B(T)\Psi(T),$$

and from (4.3.11) we find $\Psi(T)$

$$\Psi(T) = B^*(T)\Phi(T), B(T)^* = M^*(T)\sigma_T^*.$$

This observation leads to the integral equation

$$\Phi(T) = U(T,1)\phi_1 \pm \mathrm{i}\int_T^1 U(T,S)\Phi(S)|\Phi(S)|^2\frac{dS}{S}, \tag{4.4.11}$$

where

$$\phi_1(x) = B(1)\overline{\psi_1}(x) = e^{\frac{\mathrm{i}x^2}{4}}\overline{\psi_1}(x). \tag{4.4.12}$$

The perturbed group $U(T,S)$ satisfies the estimates

$$\left\| (1-\Delta(T_1))^{\alpha/2} \left[U(T_1,T_2) - I \right] g \right\|_{L^2(\mathbb{R})} \leq C|T_1-T_2|^{\theta/8} \left\| (1-\Delta)^{(\alpha+\theta)/2} g \right\|_{L^2(\mathbb{R})} \tag{4.4.13}$$

provided g is an odd function and $0 < T_2, T_1 \leq 1$, $\alpha \in [0,3/4)$, $\theta \in [0,1]$ sufficiently small. One can see the hypotheses in Lemma 4.6.2 for the sharp constrains on θ and α.

If we can justify the existence of the strong limits

$$U(0,1)f = \lim_{\varepsilon \to 0} U(\varepsilon,1)f, \tag{4.4.14}$$

and

$$\int_0^1 [U(0,S)-I]\Phi(S)|\Phi(S)|^2 \frac{dS}{S} = \lim_{\varepsilon\to 0} \int_\varepsilon^1 [U(\varepsilon,S)-I]\Phi(S)|\Phi(S)|^2 \frac{dS}{S}, \tag{4.4.15}$$

then, in the same spirit of the free case, we can define

$$\Phi_0 = U(0,1)\phi_1 \pm \mathrm{i} \int_0^1 [U(0,S)-I]\Phi(S)|\Phi(S)|^2 \frac{dS}{S}.$$

By means of Φ_0, we can go further and we define the leading term $\Phi_{lead}(T)$ of the solution $\Phi(T)$ as in (4.3.15), i.e.

$$\Phi_{lead}(T) = \Phi_0 \pm \mathrm{i} \int_T^1 \Phi_{lead}(S)|\Phi(S)|^2 \frac{dS}{S}. \tag{4.4.16}$$

By the line above we find

$$\Phi_{lead}(T) = \Phi_0 e^{\pm \mathrm{i}\Theta(T)}, \tag{4.4.17}$$

where $\Theta(T)$ is defined in (4.1.25).

Our goal is to prove some a priori bounds (4.1.26), (4.1.27), for the leading term Φ_{lead} defined in (4.4.17), and for the remainder

$$\Phi_{rem}(T) = \Phi(T) - \Phi_{lead}(T).$$

Taking the difference between the equation (4.4.11) and the equation (4.4.16), we find

$$\Phi_{rem}(T) = [U(T,1)-U(0,1)]\phi_1 \pm \mathrm{i}[U(T,0)-I] \int_T^1 U(0,S)\Phi(S)|\Phi(S)|^2 \frac{dS}{S} \mp$$

$$\mp \mathrm{i} \int_0^T [U(0,S)-I]\Phi(S)|\Phi(S)|^2 \frac{dS}{S} \pm \mathrm{i} \int_T^1 \Phi_{rem}(S)|\Phi(S)|^2 \frac{dS}{S}.$$

This equation suggests to consider the linear operator

$$L : f \times F \in H^s_{odd}(\mathbb{R}) \times C([0,1]; H^s_{odd}(\mathbb{R})) \to C([0,1]; H^s_{odd}(\mathbb{R})),$$

defined as follows

$$\begin{aligned} L(f,F)(T) = \\ [U(T,1) - U(0,1)]f \pm \mathrm{i}[U(T,0) - I] \int_T^1 U(0,S)F(S)\frac{dS}{S} \mp \\ \mp \mathrm{i} \int_0^T [U(0,S) - I]F(S)\frac{dS}{S}. \end{aligned} \tag{4.4.18}$$

We can rewrite the equation for the remainder term as follows:

$$\Phi_{rem}(T) = L(\phi_1, \Phi|\Phi|^2)(T) \pm \mathrm{i} \int_T^1 \Phi_{rem}(S)|\Phi(S)|^2 \frac{dS}{S},$$

where ϕ_1 is defined in (4.4.12).

Hence, we can express the remainder by the identity

$$\Phi_{rem}(T) = G(T) \pm \mathrm{i} \int_T^1 e^{\pm \mathrm{i}(\Theta(T) - \Theta(S))} G(S)|\Phi|^2 \frac{dS}{S}, \tag{4.4.19}$$

where

$$G(T) = L(\phi_1, \Phi|\Phi|^2)(T).$$

Our next step is to choose, as in the free case, the parameter $\theta \in [0,1]$ sufficiently small such that we can select $\alpha = s - \theta \in (1/2, 3/4)$.

Using (4.4.13) we can show the estimates (4.1.27) and (4.1.26), provided the initial data satisfy

$$\|e^{-ix^2/4}\psi_1\|_{H^s_{odd}(\mathbb{R})} \le \epsilon.$$

4.4.1 Similar Orbits and Splitting Generators

In this section we will give a rigorous justification of the formulas (4.3.9) and (4.3.13) for the free case, and then we analyse the perturbed case. In both cases, we are going to show the relation between the two two parameters groups that appear in the integral expressions of $\Psi(T)$ and $\Phi(T)$.

After the change of variable (4.3.10), the integral equation (4.3.9) suggests us to consider the two parameters group $e^{i\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}$.

Moreover, as explained in the section above, we are looking for a suitable unitary operator $B(T)$ ables to simplify the group $e^{i\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}$ and the integral equation (4.3.9). Now we give two equivalent definitions to connect two two parameters groups.

Definition 4.4.1 (Similar groups)**.** *Let $0 \leq T, S \leq 1$. Let $U(T,S)$ and $U_0(T,S)$ be two two parameters groups with generators $-iA(T)$ and $-iA_0(T)$ having respectively dense domains $D(T)$ and $D_0(T)$ on the Hilbert space H. We say that $U(T,S)$ and $U_0(T,S)$ are similar if there exists a unitary operator*

$$B(T)\colon H \to H,$$

such that

$$U_0(T,S) = B(T)U(T,S)B^*(S), \tag{4.4.20}$$

and

$$D_0(T) = B(T)D(T).$$

Definition 4.4.2 (Splitting operators)**.** *Let $A(T)$ and $A_0(T)$ be self-adjoint operators with dense domains $D(T)$ and $D_0(T)$ on the Hilbert space H. We say that the couple $(A(T), A_0(T))$ is splitting if there exists a unitary operator*

$$B(T)\colon H \to H,$$

such that

$$B(T)\colon D(T) \to D_0(T),$$

and

$$B'(T) = i\left[B(T)A(T) - A_0(T)B(T)\right]. \tag{4.4.21}$$

Remark 4.4.3. We can verify that the two two parameters groups are similar if and only if the couple of the generators is splitting. The key is to compute $\partial_T U_0(T,S)$ using (4.4.20) and (4.4.21).

At first we consider the unperturbed Hamiltonian $\mathcal{H}_0$. In the following Lemma we prove that $\left(\frac{\mathcal{H}_0}{T^2}, \mathcal{H}_0\right)$ is a splitting couple.

Lemma 4.4.4. *Let $\mathcal{H}_0 = -\partial_x^2$. Then, one can find a unitary operator*

$$B(T)\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

so that for any $T, S \in (0,1]$ we have the relation

$$B(T)e^{\mathrm{i}\mathcal{H}_0\left(\frac{1}{T}-\frac{1}{S}\right)}B^*(S)g = e^{-\mathrm{i}\mathcal{H}_0(T-S)}g. \tag{4.4.22}$$

Proof. We consider the two-parameter groups

$$U_0(T,S) = e^{\mathrm{i}\mathcal{H}_0(1/T-1/S)}, \quad U_1(T,S) = e^{-\mathrm{i}(T-S)\mathcal{H}_0}$$

with generators

$$-\mathrm{i}\mathcal{H}_0(T) = -\mathrm{i}\frac{\mathcal{H}_0}{T^2}, \quad -\mathrm{i}\mathcal{H}_1(T) = -\mathrm{i}\mathcal{H}_0.$$

It is easy to show that the similarity condition for the two two parameters groups, (4.4.22) will follow from

$$B'(T) = \mathrm{i}\left[B(T)\frac{\mathcal{H}_0}{T^2} - \mathcal{H}_0 B(T)\right]. \tag{4.4.23}$$

We follow (4.1.20), (4.1.21) so we can choose

$$B(T) = M(T)\sigma_T, \tag{4.4.24}$$

where

$$M(T)f(x) = e^{\frac{\mathrm{i}x^2}{4}}f(x), \quad \sigma_T(f)(x) = \frac{1}{T^{1/2}}f\left(\frac{x}{T}\right). \tag{4.4.25}$$

We have to check the identity (4.4.23) with this choice of $B(T)$. For this we take $g(x) \in S(\mathbb{R})$ and use the relations

$$B'(T)g = -\frac{\mathrm{i}x^2}{4T^2}B(T)g(x) + M(T)\frac{d}{dT}\left[\frac{1}{T^{1/2}}g\left(\frac{x}{T}\right)\right] =$$

$$= -\frac{\mathrm{i}x^2}{4T^2}B(T)g(x) - \frac{1}{2T}B(T)g(x) - \frac{x}{T^2}M(T)T^{-1/2}g'\left(\frac{x}{T}\right)$$

so setting

$$L = x\partial_x, \tag{4.4.26}$$

we find

$$B'(T)g = -\frac{\mathrm{i}x^2}{4T^2}B(T)g(x) - \frac{1}{2T}B(T)g(x) - \frac{1}{T}B(T)L(g)(x).$$

In this way we arrive at the relation

$$B'(T) = \left(-\frac{\mathrm{i}x^2}{4T^2} - \frac{1}{2T}\right)B(T) - \frac{1}{T}B(T)L. \tag{4.4.27}$$

Further, we have

$$\begin{aligned}
\mathcal{H}_0 B(T)g(x) = -\partial_x^2\left(M(T)\frac{1}{T^{1/2}}g\left(\frac{x}{T}\right)\right) = \\
= -\frac{1}{T^{1/2}}\partial_x^2\left(e^{\frac{\mathrm{i}x^2}{4T}}g\left(\frac{x}{T}\right)\right) = \\
= -\frac{1}{T^{1/2}}g\left(\frac{x}{T}\right)\partial_x^2\left(e^{\frac{\mathrm{i}x^2}{4T}}\right) - 2\frac{1}{T^{1/2}}\left[\partial_x g\left(\frac{x}{T}\right)\right]\partial_x\left(e^{\frac{\mathrm{i}x^2}{4T}}\right) - \\
-\frac{1}{T^{1/2}}e^{\frac{\mathrm{i}x^2}{4T}}\partial_x^2\left[g\left(\frac{x}{T}\right)\right] = \\
= \frac{x^2}{4T^2T^{1/2}}e^{\frac{\mathrm{i}x^2}{4T}}g\left(\frac{x}{T}\right) - \frac{\mathrm{i}}{2T^{1+1/2}}e^{\frac{\mathrm{i}x^2}{4T}}g\left(\frac{x}{T}\right) - \\
-\frac{\mathrm{i}}{T^{1+1/2}}e^{\frac{\mathrm{i}x^2}{4T}}(Lg)\left(\frac{x}{T}\right) + \frac{1}{T^{1/2+2}}e^{\frac{\mathrm{i}x^2}{4T}}(-\partial_x^2 g)\left(\frac{x}{T}\right).
\end{aligned}$$

In this way we obtain

$$\frac{1}{T^2}B(T)\mathcal{H}_0 - \mathcal{H}_0 B(T) = \left[-\frac{x^2}{4T^2} + \frac{i}{2T}\right]B(T) + \frac{i}{T}B(T)L. \tag{4.4.28}$$

The relations (4.4.27) and (4.4.28) imply

$$B'(T) = i\left[\frac{1}{T^2}B(T)\mathcal{H}_0 - \mathcal{H}_0 B(T)\right], \tag{4.4.29}$$

so we have (4.4.23). This completes the proof of the Lemma. □

Now we proceed similarly for the perturbed case. We consider the perturbed Hamiltonian $\mathcal{H} = -\partial_x^2 + V(x)$. We can consider the two parameters group defined by

$$U(T,S) = B(T)e^{i\mathcal{H}/T}e^{-i\mathcal{H}/S}B^*(S), \tag{4.4.30}$$

where $B(T)$ and $B^*(T)$ is defined as in (4.4.24). We shall see that this two-parameter group has generator

$$i\Delta(T), \quad \Delta(T) = -\frac{1}{T^2}\sigma_T \mathcal{H}_V \sigma_T^*, \tag{4.4.31}$$

where

$$\Delta(T) = -\frac{1}{T^2}\sigma_T \left(-\partial_x^2 + V\right)\sigma_T^* = \Delta - \frac{1}{T^2}V\left(\frac{x}{T}\right).$$

Lemma 4.4.5. *Let $\mathcal{H} = -\partial_x^2 + V$, and let $B(T)$ be defined as in (4.4.24). Then, the genarator of the two-parameter group*

$$U(T,S) = B(T)e^{i\mathcal{H}/T}e^{-i\mathcal{H}/S}B^*(S) \tag{4.4.32}$$

is the operator $i\Delta(T)$ defined in (4.4.31), i.e. for any $g \in H^2(\mathbb{R})$ we have

$$\frac{d}{dT}U(T,S)g = i\Delta(T)U(T,S)g. \tag{4.4.33}$$

Proof. Again the relation (4.4.33) will follow from (4.4.32) and the splitting relation

$$B'(T) = i\left[B(T)\frac{\mathcal{H}}{T^2} + \Delta(T)B(T)\right]. \tag{4.4.34}$$

We know from Lemma 4.4.4 that

$$B'(T) = i\left[B(T)\frac{\mathcal{H}_0}{T^2} + \Delta B(T)\right].$$

Hence, the line above and the trivial identity

$$B(T)\frac{V}{T^2} - \frac{1}{T^2}V\left(\frac{\cdot}{T}\right)B(T) = 0, \tag{4.4.35}$$

imply (4.4.34). The thesis follows computing

$$\frac{d}{dT}U(T,S)g = \frac{d}{dT}\left[B(T)e^{i\mathcal{H}/T}\left(e^{i\mathcal{H}/S}B^*(S)g\right)\right],$$

with $g \in H^2(\mathbb{R})$ and using (4.4.34). $\square$

4.5 Equivalent Sobolev norms

Once established the relation (4.4.32), and keeping in mind the integral equation (4.4.11), it is useful to look for estimates for the Sobolev norms

$$\|U(T,S)f\|_{H^s(\mathbb{R})}\,, \tag{4.5.1}$$

where $0 < T < S \leq 1$ and $0 \leq s \leq 1$. In this section we want to establish an equivalence relation between the classical Sobolev norms defined by $\mathcal{H}_0$ and the modified ones defined by $\Delta(T)$. This result will turn to be crucial to get estimates for the norms (4.5.1) that we will study in the next section.

Fixed $s \in [0,1]$, we introduce the fractional powers $(-\Delta(T))^{s/2}$, of the generator $\Delta(T)$ of the two parameters group $U(T,S)$. We recall that these operators are defined by

$$(-\Delta(T))^{s/2} = \frac{1}{T^s}\sigma_T \mathcal{H}^{s/2}\sigma_T^*, \quad s \in [0,1]. \tag{4.5.2}$$

The following Lemma will give us an integral formulation of the operator $(-\Delta(T))^{s/2}U(T,S)$, with $s \in [0,1]$.

Lemma 4.5.1. *Let $0 < S \leq 1$ be fixed. If $f \in H^s(\mathbb{R})$, then for any $s \in [0,1]$ the function*

$$\Phi_s(T) = (-\Delta(T))^{s/2}U(T,S)f$$

satisfies the integral equation

$$\Phi_s(T) = U(T,S)(-\Delta(S))^{s/2}f - \int_T^S U(T,\tau)\sigma_\tau A(s)\sigma_\tau^* U(\tau,S)f\,\frac{d\tau}{\tau^{s+1}}, \tag{4.5.3}$$

where

$$A(s) = s\mathcal{H}^{s/2} + [L, \mathcal{H}^{s/2}], \tag{4.5.4}$$

$$A(s)\colon L^\infty(\mathbb{R}) \to L^1(\mathbb{R}),$$

is a bounded operator.

Proof. The function

$$\Phi(T) = U(T,S)f$$

is a solution to the equation

$$\mathrm{i}\partial_T \Phi(T) + \Delta(T)\Phi(T) = 0, \tag{4.5.5}$$

if f is sufficiently regular, namely $f \in H^2(\mathbb{R})$. If $s \in [0,1]$, then we can use a density argument and assuming $f \in H^{2+s}(\mathbb{R})$, we can assert that $\Phi(T) \in C([0,1]; H^{2+s}(\mathbb{R}))$ satisfies (4.5.5).

The proof of (4.5.3) can be reduce to the proof of the commutator relation

$$\Big[\partial_T, (-\Delta(T))^{s/2}\Big] = -\frac{s}{T}(-\Delta(T))^{s/2} - \frac{1}{T^{s+1}}\sigma_T[L, \mathcal{H}^{s/2}]\sigma_T^*, \tag{4.5.6}$$

where $L = x\partial_x$ is the delation operator. Hence we show that (4.5.6) is verified. The fractional powers

$$(-\Delta(T))^{s/2} = \frac{1}{T^s}\sigma_T \mathcal{H}^{s/2}\sigma_T^*,$$

involve the operators σ_T and $\mathcal{H}^{s/2}$ so we can use the following relations:

$$[\partial_T, \sigma_T] = -\frac{1}{T}\sigma_T L - \frac{1}{2T}\sigma_T, \quad [\partial_T, \sigma_T^*] = \frac{1}{T}L\sigma_T^* + \frac{1}{2T}\sigma_T^*. \tag{4.5.7}$$

To prove (4.5.6) we can use (4.5.7) combined with the definition of the commutator. In this way we find

$$\begin{aligned}\Big[\partial_T, (-\Delta(T))^{s/2}\Big] = \Big[\partial_T, \frac{1}{T^s}\sigma_T\mathcal{H}^{s/2}\sigma_T^*\Big] = -\frac{s}{T^{s+1}}\sigma_T\mathcal{H}^{s/2}\sigma_T^* - \\ -\frac{1}{T^{s+1}}\sigma_T L\mathcal{H}^{s/2}\sigma_T^* - \frac{1}{2}\frac{1}{T^{s+1}}\sigma_T\mathcal{H}^{s/2}\sigma_T^* + \\ +\frac{1}{T^{s+1}}\sigma_T\mathcal{H}^{s/2}L\sigma_T^* + \frac{1}{2}\frac{1}{T^{s+1}}\sigma_T\mathcal{H}^{s/2}\sigma_T^*.\end{aligned}$$

Hence we obtain

$$\Big[\partial_T, (-\Delta(T))^{s/2}\Big] = -\frac{s}{T}(-\Delta(T))^{s/2} - \frac{1}{T^{1+s}}\sigma_T[L, \mathcal{H}^{s/2}]\sigma_T^*.$$

To complete the proof we note that

$$\frac{1}{T^{s+1}}\sigma_T A(s)\sigma_T^* = \frac{s}{T}(-\Delta(T))^{s/2} + \frac{1}{T^{1+s}}\sigma_T[L, \mathcal{H}^{s/2}]\sigma_T^*, \tag{4.5.8}$$

where $A(s)$ is defined in (4.5.4). One can find a proof of boundness of $A(s)$ in [17]. Now, computing

$\partial_T\Phi_s(T)$, and using (4.5.6) and (4.5.8) we get

$$\mathrm{i}\partial_T\Phi_s(T) + \Delta(T)\Phi_s(T) = -\frac{\mathrm{i}}{T^{s+1}}\sigma_T A(s)\sigma_T^* U(T,S)f, \tag{4.5.9}$$

provided $f \in H^{2+s}(\mathbb{R})$. This equation and the definition of the two parameters group $U(T,S)$ give the integral equation (4.5.3). Now using density argument, we can assert that it is true for $f \in H^s(\mathbb{R})$. $\square$

In order to control the norm (4.5.1) thanks to the results of Lemma 4.5.1, we want to establish the following equivalence relation

$$\|\sqrt{\mathcal{H}_0}^s f\|_{L^2(\mathbb{R})} \approx \|\sqrt{-\Delta(T)}^s f\|_{L^2(\mathbb{R})},$$

with $s \in [0,1]$. Using the equivalence property of Lemma 4.2.3 as well as the fact that

$$\sqrt{-\Delta(T)} = \frac{1}{T}\sigma_T\sqrt{\mathcal{H}}\sigma_T^*$$

with

$$\sigma_T \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

an isometry, we deduce the following estimate

$$\|\sqrt{\mathcal{H}_0}f\|^2_{L^2(\mathbb{R})} \leq \left(1+\frac{1}{\delta_0}\right)\|\sqrt{-\Delta(T)}f\|^2_{L^2(\mathbb{R})} \leq 2\left(1+\frac{1}{\delta_0}\right)\|\sqrt{\mathcal{H}_0}f\|^2_{L^2(\mathbb{R})}, \tag{4.5.10}$$

for any $f \in H^1_{odd}(\mathbb{R})$. An interpolation argument implies the following result.

Lemma 4.5.2. *Suppose that the operator $\mathcal{H}$ satisfies the assumptions* (4.1.7), (4.1.8) *and* (4.1.9). *Then for any $s \in [0,1]$, there exists $C > 0$, so that*

$$C^{-1}\|\mathcal{H}_0^{s/2}f\|_{L^2(\mathbb{R})} \leq \|\left(-\Delta(T)\right)^{s/2}f\|_{L^2(\mathbb{R})} \leq C\|\mathcal{H}_0^{s/2}f\|_{L^2(\mathbb{R})}, \tag{4.5.11}$$

for any $f \in H^s_{odd}(\mathbb{R})$.

4.6 Estimates for the two parameters group $U(T,S)$

In this section we derive some a priori estimates for the group $U(T,S)$ and we prove the main technical lemmas essential to establish our main a priori estimates (4.1.27), (4.1.26).

4.6.1 Linear Strichartz estimates

Here we recall the classical Strichartz estimates for the Schrödinger group and then we get Strichartz type estimates for the two parameters group.

The Strichartz estimates for the free Hamiltonian in any time interval $(0,A)$ are

$$\left\| e^{-i\mathcal{H}_0 t} f \right\|_{L^p((0,A);L^q(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R})}, \tag{4.6.1}$$

where here and below the pair (p,q) verifies the admissibility relation

$$2 \leq q \leq \infty, \quad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}. \tag{4.6.2}$$

A standard $T - T^*$ argument implies the following estimate on the Duhamel term

$$\left\| \int_0^t e^{-i\mathcal{H}_0(t-s)} F(s) ds \right\|_{L^{p_1}((0,A);L^{q_1}(\mathbb{R}))} \leq C\|F\|_{L^{p_2}((0,A);L^{q_2}(\mathbb{R}))}, \tag{4.6.3}$$

where

$$\begin{aligned} 2 \leq q_1 \leq \infty, \quad & \frac{2}{p_1} + \frac{1}{q_1} = \frac{1}{2}, \\ 1 \leq q_2 \leq 2, \quad & \frac{2}{p_2} + \frac{1}{q_2} = \frac{5}{2}. \end{aligned} \tag{4.6.4}$$

For similar Strichartz estimate for the perturbed Hamiltonian $\mathcal{H}$ we can refer to [72] and Chapter 2 and 3. One can make a time shift and assume the initial data at $t = 1$. Then if we assume (4.1.7), (4.1.8) we shall have the estimates

$$\left\| e^{-i\mathcal{H}(t-1)} f \right\|_{L^p((1,A);L^q(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R})}, \quad \forall A > 1, \tag{4.6.5}$$

provided the pairs (p,q) satisfy (4.6.2) and

$$\left\| \int_1^t e^{-i\mathcal{H}(t-s)} F(s) ds \right\|_{L^{p_1}((1,A);L^{q_1}(\mathbb{R}))} \leq C \|F\|_{L^{p_2}((1,A);L^{q_2}(\mathbb{R}))}, \tag{4.6.6}$$

when (4.6.4) holds.

Now we turn to the two parameters group $U(T,S)$, introduced in (4.1.19), (4.1.20), (4.1.21). Our first step is to check the estimate

$$\|U(T,1)f\|_{L^p((a,1);L^q(\mathbb{R}))} \leq C\|f\|_{L^2(\mathbb{R})}, \tag{4.6.7}$$

where $0 < a < 1$ and (p,q) satisfy (4.6.2). Setting

$$g(t,x) = e^{i\mathcal{H}(t-1)} B^*(1) f(x),$$

we can see that

$$\|U(T,1)f\|_{L^p((a,1);L^q(\mathbb{R}))} = \left\| M(T)\sigma_T(g(T^{-1},\cdot)) \right\|_{L^p((a,1);L^q(\mathbb{R}))},$$

so, using the relation

$$\left\| M(T)\sigma_T(g(T^{-1},\cdot)) \right\|_{L^q(\mathbb{R})} = \left\| \sigma_T(g(T^{-1},\cdot)) \right\|_{L^q(\mathbb{R})} = T^{-1/2+1/q} \left\| g(T^{-1},\cdot) \right\|_{L^q(\mathbb{R})}$$

and (4.6.2), we obtain

$$\left\| \sigma_T(g(T^{-1},\cdot)) \right\|_{L^q(\mathbb{R})} = T^{-2/p} \left\| g(T^{-1},\cdot) \right\|_{L^q(\mathbb{R})}.$$

To this end we can set

$$h(t) = \|g(t,\cdot)\|_{L^q(\mathbb{R})},$$

so we use the relation

$$\left\| T^{-2/p} h(T^{-1}) \right\|_{L^p(a,1)} = \|h(t)\|_{L^p(1,A)}, \quad A = 1/a$$

and in this way we arrive at

$$\|U(T,1)f\|_{L^p((a,1);L^q(\mathbb{R}))} = \left\| e^{-i\mathcal{H}(t-1)} B^*(1) f \right\|_{L^p((1,A);L^q(\mathbb{R}))}$$

so applying the Strichartz estimate (4.6.5), since $B^*(T)$ is an isometry in $L^2(\mathbb{R})$, we find (4.6.7).

In a similar way, we can check the other Strichartz estimate

$$\left\| \int_T^1 U(T,S)F(S)dS \right\|_{L^{p_1}((a,1);L^{q_1}(\mathbb{R}))} \leq C\|F\|_{L^{p_2}((a,1);L^{q_2}(\mathbb{R}))} \tag{4.6.8}$$

for any $a \in (0,1)$ and for p_1, q_1, p_2, q_2, satisfying (4.6.4). Indeed, thanks to relation (4.6.7) we have that

$$\left\| \frac{1}{t^{1/2}t^2} F\left(\frac{1}{t}, \frac{x}{t}\right) \right\|_{L^{p_2}((1,A);L^{q_2}(\mathbb{R}))} = \|F\|_{L^{p_2}((a,1);L^{q_2}(\mathbb{R}))} .$$

Hence, operating the change of variable, $T \mapsto \frac{1}{t}$, in the right side of (4.6.8) and, using (4.6.3), we get the inequality (4.6.8).

4.6.2 Some a priori estimates

In the following we collect some important estimates for the two parameters group $U(T,S)$ that we are going to use for the proof of the Theorem 4.1.1.

The Strichartz estimates (4.6.7), (4.6.8) and the equivalence property of Lemma 4.5.2 imply the following result.

Corollary 4.6.1. *For any S, $0 < S \leq 1$ and for any $f \in H^s_{odd}(\mathbb{R})$, we have that:*

a) *If $s \in [0,1]$ and $0 < T < S$, then*

$$\|U(T,S)f\|_{H^s(\mathbb{R})} \leq C\left(\|f\|_{H^s(\mathbb{R})} + \left(\int_T^S \frac{d\tau}{\tau^{s\frac{4}{3}}} \right)^{3/4} \|U(\tau,S)f\|_{L^\infty((T,S);L^\infty(\mathbb{R}))} \right); \tag{4.6.9}$$

b) *If $s \in (3/4, 1]$, then*

$$\|U(T,S)f\|_{H^s(\mathbb{R})} \leq C\left(\|f\|_{H^s(\mathbb{R})} + \left(\frac{1}{T} \right)^{s-3/4} \|U(\tau,S)f\|_{L^\infty((T,S);L^\infty(\mathbb{R}))} \right); \tag{4.6.10}$$

c) *If $0 \leq s < 1/2$ or $1/2 < s < 3/4$ and $0 < T < S$, then*

$$\|U(T,S)f\|_{H^s(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}. \tag{4.6.11}$$

Proof. We consider $s \in [0,1]$. It is well known that

$$\|g\|_{H^s(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} + \|\mathcal{H}_0^{s/2} g\|_{L^2(\mathbb{R})}.$$

Moreover, Lemma 4.5.2 and equation (4.5.3) imply that

$$\|(-\Delta(T))^{s/2}U(T,S)f\|_{L^2(\mathbb{R})} \leq$$
$$\leq \|U(T,S)(-\Delta(S))^{s/2}f\|_{L^2(\mathbb{R})} + \left\| \int_T^S U(T,\tau)\sigma_\tau A(s)\sigma_\tau^* U(\tau,S) f \frac{d\tau}{\tau^{s+1}} \right\|_{L^2(\mathbb{R})}.$$

Using the Strichartz estimates (4.6.7), (4.6.8) we get the estimate (4.6.9). The estimate (4.6.10) is a trivial consequence of the estimate (4.6.9) in the case $s \in (3/4, 1]$. To complete the proof we note that, if $0 \leq s < 1/2$, we have

$$\begin{aligned}
\|U(T,S)f\|_{H^s(\mathbb{R})} &\leq \|U(T,S)f\|_{L^2(\mathbb{R})} + C\|(-\Delta(T))^{s/2}U(T,S)f\|_{L^2(\mathbb{R})} \\
&\leq \|f\|_{L^2(\mathbb{R})} + \|f\|_{\dot{H}^s(\mathbb{R})} + \left\| \frac{1}{\tau^{s+1}} \sigma_\tau A(s)\sigma_\tau^* U(\tau,S) f \right\|_{L^{4/3}((S,T);L^1(\mathbb{R}))} \\
&\leq \|f\|_{H^s(\mathbb{R})} + \left\| \frac{\tau}{\tau^{s+1}\tau^{1/2}} \tau^{1/2} U(\tau,S) f \right\|_{L^{4/3}((S,T);L^\infty(\mathbb{R}))} \\
&\leq \|f\|_{H^s(\mathbb{R})} + \left\|\tau^{-s}\right\|_{L^2(S,T)} \|U(\tau,S)f\|_{L^4((S,T);L^\infty(\mathbb{R})} .
\end{aligned}$$

The Strichartz estimates and the condition $s < 1/2$ imply the inequality (4.6.11). On the other side, if $1/2 < s < 3/4$, we use the interpolation Sobolev estimate

$$\|g\|_{L^\infty(\mathbb{R})} \leq C\|g\|^{1/2}_{H^{1/2-\delta}(\mathbb{R})} \|g\|^{1/2}_{H^{1/2+\delta}(\mathbb{R})},$$

for $g = U(\tau,S)f$ and $1/2 + \delta < s$. Hence, the inequality (4.6.9) and the interpolation Sobolev estimates above combined with Young inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}, \quad a,b > 0, \quad \epsilon > 0,$$

imply (4.6.11). This complete the proof of the Corollary. □

Now we want to get $H^\alpha(\mathbb{R})$ estimates, with $\alpha > 1/2$, for the operators involved in the expression of the remainder $\Phi_{rem}(T)$ (4.4.19).

We prove the fundamental lemmas to get the estimates (4.1.26) and (4.1.27).

Lemma 4.6.2. *Let $g \in H^\alpha_{odd}(\mathbb{R})$. The perturbed group $U(T,S)$ satisfies the estimates*

$$\|[U(T_1,T_2) - I]\, g\|_{H^\alpha(\mathbb{R})} \leq C|T_1 - T_2|^{\theta/8} \|g\|_{H^{\alpha+\theta}(\mathbb{R})} \tag{4.6.12}$$

provided

$$\alpha \in [0, 3/4),\ \ \theta \in [0, 1], \quad \frac{4\alpha}{3} + \theta < 1, \quad 0 \le T_2, T_1 \le 1. \tag{4.6.13}$$

Proof. We can consider the operator

$$B(\alpha, \theta) = |T_1 - T_2|^{-\theta/8}(1 - \Delta)^{\alpha/2}\,[U(T_1, T_2) - I]\,(1 - \Delta)^{-(\alpha+\theta)/2}$$

and show that (4.6.12) follows from the fact that it is $L^2 - L^2$ bounded operator. We have also the following simple observation: if

$$B(\alpha, \theta) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

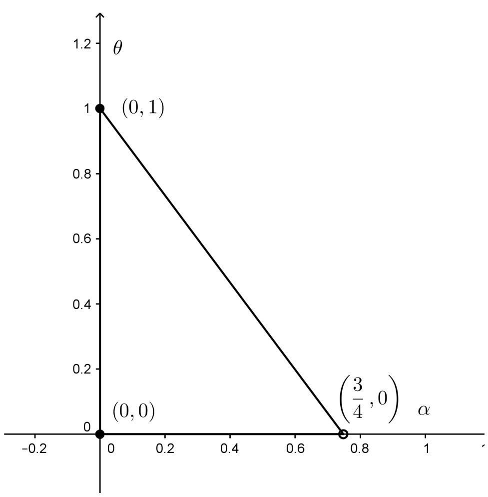
is bounded for some $\alpha, \theta \in \mathbb{R}$, then the operator

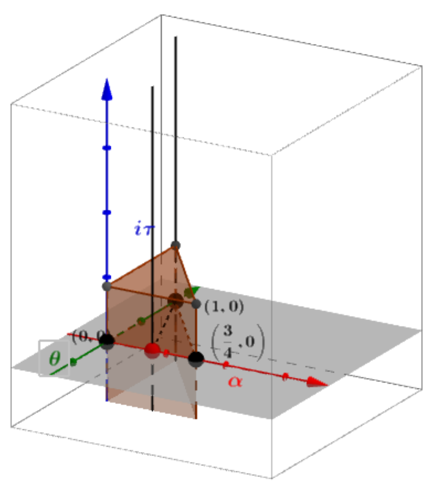
$$B(\alpha + i\tau, \theta + i\tau) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is bounded operator for any $\tau \in \mathbb{R}$. This observation and the Stein interpolation theorem guarantee that it is sufficient to show the L^2 boundedness of $B(\alpha, \theta)$ at three points

$$(\alpha, \theta) = (0, 0),\ \ (\alpha, \theta) = (3/4 - \varepsilon, 0), \quad (\alpha, \theta) = (0, 1),$$

with $\epsilon > 0$ arbitrarily small. Hence, we have to check (4.6.12) for the three points written above. The





estimate (4.6.12) is trivial in the point $(0, 0)$. Moreover, it is valid for the case $\theta = 0$, $\alpha \in [0, 3/4)$ due to the estimate (4.6.11) of Corollary 4.6.1. Therefore it only remains to prove the case $(\alpha,\theta)=(0,1)$, i.e.

$$\|[U(T_1,T_2)-I]\,g\|_{L^2(\mathbb{R})}\le C|T_1-T_2|^{1/8}\,\|g\|_{H^1(\mathbb{R})}\,,0<T_2<T_1\le 1. \tag{4.6.14}$$

To verify this estimate we note that

$$\|[U(T_1,T_2)-I]\,g\|^2_{L^2(\mathbb{R})}=\int_{T_2}^{T_1}\frac{d}{d\tau}\left(\|[U(\tau,T_2)-I]\,g\|^2_{L^2(\mathbb{R})}\right)d\tau. \tag{4.6.15}$$

We compute the derivative

$$\frac{d}{d\tau}\,\|[U(\tau,T_2)-I]\,g\|^2_{L^2(\mathbb{R})}=\lim_{\epsilon\to 0}\int_{\mathbb{R}}\frac{|(U(\tau+\epsilon,T_2)-I)\,g|^2-|(U(\tau,T_2)-I)\,g|^2}{\epsilon}\,dx,$$

and we get

$$\begin{aligned}
&\frac{d}{d\tau}\,\|[U(\tau,T_2)-I]\,g\|^2_{L^2(\mathbb{R})}=\\
&=2\Re\left(\int_{\mathbb{R}}i\Delta(\tau)U(\tau,T_2)g\cdot\overline{U(\tau,T_2)g}\,dx\right)+2\Im\left(\int_{\mathbb{R}}i\Delta(\tau)U(\tau,T_2)g\cdot\overline{g}\,dx\right)\\
&=2\Re\left(-i\int_{\mathbb{R}}\left|(-\Delta(\tau))^{1/2}U(\tau,T_2)g\right|^2\,dx\right)+2\Im\left(-i\int_{\mathbb{R}}(-\Delta(\tau))^{1/2}U(\tau,T_2)g\cdot\overline{(-\Delta(\tau))^{1/2}g}\,dx\right)\\
&=2\Im\left(-i\int_{\mathbb{R}}(-\Delta(\tau))^{1/2}U(\tau,T_2)g\cdot\overline{(-\Delta(\tau))^{1/2}g}\,dx\right).
\end{aligned}$$

Now, coming back to (4.6.15) and applying (4.6.10) we get

$$\begin{aligned}
\|[U(T_1,T_2)-I]\,g\|^2_{L^2(\mathbb{R})}&\le\int_{T_2}^{T_1}d\tau\left|\int_{\mathbb{R}}(-\Delta(\tau))^{1/2}U(\tau,T_2)g\cdot\overline{(-\Delta(\tau))^{1/2}g}\,dx\right|\\
&\le\|g\|_{H^1(\mathbb{R})}\int_{T_2}^{T_1}\left(1+\frac{1}{\tau^{1/4}}\right)\|g\|_{H^1(\mathbb{R})}\,d\tau\\
&\le\|g\|^2_{H^1(\mathbb{R})}\left((T_1-T_2)+\frac{4}{3}(T_1^{3/4}-T_2^{3/4})\right)\\
&\le C\,(T_1-T_2)^{1/4}\,\|g\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

Hence we deduce (4.6.14). The Stein interpolation theorem proves that the estimate (4.6.12) holds for any $\alpha\in[0,3/4)$, $\alpha\in[0,1]$ and $\theta<-\frac{3}{4}\alpha+1$. □

As consequence, we can justify the limits (4.4.14) and (4.4.15). More precisely, we have

Corollary 4.6.3. *Let $f\in H^{\alpha+\theta}_{odd}(\mathbb{R})$ and let $F(T)\in C\left((0,1];H^{\alpha+\theta}_{odd}(\mathbb{R})\right)$, with $\alpha+\theta<\frac{3}{4}$, $\alpha>\frac{1}{2}$ and*

$\theta \geq 0$, such that

$$\sup_{T\in(0,1]} T^{\theta/N} \|F(T)\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})} \leq C < \infty, \tag{4.6.16}$$

with $N > 16$. Then we have:

a) *Given any $S \in (0,1]$ the function*

$$T \in (0,1] \mapsto U(T,S)f \tag{4.6.17}$$

can be extended as a Hölder continuous function in $C^{0,\theta/8}\left([0,1]; H^{\alpha}_{odd}(\mathbb{R})\right)$;

b) *The function*

$$T \in (0,1] \mapsto \int_T^1 [U(T,S) - I]F(S)\frac{dS}{S} \tag{4.6.18}$$

can be extended as a continuous function in $C^{0,\theta/8-\theta/N}\left([0,1]; H^{\alpha}_{odd}(\mathbb{R})\right)$;

c) *The function*

$$T \in (0,1] \mapsto \int_0^T [U(0,S) - I]F(S)\frac{dS}{S} \tag{4.6.19}$$

can be extended as a continuous function in $C^{0,\theta/8-\theta/N}\left([0,1]; H^{\alpha}_{odd}(\mathbb{R})\right)$.

Proof. We shall prove the point a). We take $0 < T_1 < T_2 \leq 1$ and then we have the relation

$$U(T_1,S)f - U(T_2,S)f = [U(T_1,T_2) - I]\, U(T_2,S)f.$$

Taking

$$\alpha + \theta < \frac{3}{4},\ \alpha > \frac{1}{2},\ \theta \geq 0,$$

and applying Lemma 4.6.2 we find

$$\|U(T_1,S)f - U(T_2,S)f\|_{H^{\alpha}(\mathbb{R})} \leq C|T_1 - T_2|^{\theta/8} \|U(T_2,S)f\|_{H^{\alpha+\theta}(\mathbb{R})}.$$

So, the assumption

$$0 \leq \alpha + \theta < \frac{3}{4},$$

and the inequality (4.6.11) of the Corollary 4.6.1 below imply

$$\|U(T_1,S)f - U(T_2,S)f\|_{H^{\alpha}(\mathbb{R})} \leq C|T_1 - T_2|^{\theta/8} \|f\|_{H^{\alpha+\theta}(\mathbb{R})} \tag{4.6.20}$$

and this completes the proof of the point a). To prove b) we note that

$$\int_{T_1}^{1}[U(T_1,S)-I]F(S)\frac{dS}{S}-\int_{T_2}^{1}[U(T_2,S)-I]F(S)\frac{dS}{S}=$$
$$=\int_{T_1}^{T_2}[U(T_1,S)-I]F(S)\frac{dS}{S}+\int_{T_2}^{1}[U(T_1,S)-U(T_2,S)]F(S)\frac{dS}{S}.$$

Hence, using (4.6.12) and (4.6.20) we have that

$$\left\|\int_{T_1}^{1}[U(T_1,S)-I]F(S)\frac{dS}{S}-\int_{T_2}^{1}[U(T_2,S)-I]F(S)\frac{dS}{S}\right\|_{H^\alpha(\mathbb{R})}\leq$$
$$\leq C\int_{T_1}^{T_2}\frac{(S-T_1)^{\theta/8}}{S}\|F(S)\|_{H^{\alpha+\theta}(\mathbb{R})}+C\int_{T_2}^{1}(T_2-T_1)^{\theta/8}\frac{\|F(S)\|_{H^{\alpha+\theta}(\mathbb{R})}}{S}\,dS.$$

If we choose $N>16$, we have that

$$\left\|\int_{T_1}^{1}[U(T_1,S)-I]F(S)\tfrac{dS}{S}-\int_{T_2}^{1}[U(T_2,S)-I]F(S)\tfrac{dS}{S}\right\|_{H^\alpha(\mathbb{R})}\leq$$
$$\leq C|T_1-T_2|^{\theta/8-\theta/N}\sup_{T\in(0,1]}T^{\theta/N}\|F(T)\|_{H^{\alpha+\theta}(\mathbb{R}))}+$$
$$+C|T_1-T_2|^{\theta/16}\sup_{T\in(0,1]}T^{\theta/N}\|F(T)\|_{H^{\alpha+\theta}(\mathbb{R}))}.$$

Similarly, we prove c). Indeed, if $N>8$ we have that

$$\left\|\int_{0}^{T_1}[U(0,S)-I]F(S)\frac{dS}{S}-\int_{0}^{T_2}[U(0,S)-I]F(S)\frac{dS}{S}\right\|_{H^\alpha(\mathbb{R})}=$$
$$=\left\|\int_{T_1}^{T_2}[U(0,S)-I]F(S)\frac{dS}{S}\right\|_{H^\alpha(\mathbb{R})}\leq$$
$$\leq C|T_1-T_2|^{\theta/8-\theta/N}\sup_{T\in[0,1]}T^{\theta/N}\|F(T)\|_{H^{\alpha+\theta}(\mathbb{R})}.$$

□

4.7 Bound of the L^∞ - norm

In this section we prove the Theorem 4.1.1, i.e. Theorem 4.1.2. In particular we are going to prove the a priori estimates (4.1.27) and (4.1.26), when the initial datum $\phi_1(x)=e^{ix^2/4}\overline{\Psi}_1(x)$ verifies the hypothesis (4.1.13). Hence we are going to control the quantities on the left sides in (4.1.26) and (4.1.27). We recall the expressions in (4.4.19) and (4.4.17) that define respectively Φ_{rem} and Φ_{lead}.

Before starting with the proof, we need some additional estimates. Indeed, by the expression (4.4.19), we note that we need estimates of the Sobolev norms of products of the type

$$g(x) = e^{\mathrm{i}\Theta(x)} f(x),$$

for a real valued function $\Theta(x)$. For this purpose, we use the following equivalent norm of the Sobolev space $H^s(\mathbb{R})$ with $0 < s < 1$,

$$\|g\|_{H^s(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} + \left(\int_{\mathbb{R}} \int_{-1}^{1} \left| \frac{g(x+h) - g(x)}{h^s} \right|^2 \frac{dhdx}{|h|} \right)^{1/2}. \tag{4.7.1}$$

The definition and the equivalence properties of Sobolev and other interpolation spaces are discussed in details in [69], [59], [60].

Lemma 4.7.1. *If $\Theta(x)$ is a real valued function and*

$$\Theta \in H^s(\mathbb{R}), \quad f \in H^s_{odd}(\mathbb{R}) \cap L^\infty_{odd}(\mathbb{R}),$$

for some $s \in [0, 1)$, then

$$\left\| e^{\mathrm{i}\Theta} f \right\|_{H^s(\mathbb{R})} \leq C \|f\|_{H^s_{odd}(\mathbb{R})} + C \|\Theta\|_{H^s(\mathbb{R})} \|f\|_{L^\infty_{odd}(\mathbb{R})}. \tag{4.7.2}$$

By Sobolev embedding (for $s > \frac{1}{2}$, $\|f\|_{L^\infty(\mathbb{R})} \leq C_s \|f\|_{H^s(\mathbb{R})}$),

$$\left\| e^{\mathrm{i}\Theta} f \right\|_{H^s(\mathbb{R})} \leq C(1 + \|\Theta\|_{H^s(\mathbb{R})}) \|f\|_{H^s_{odd}(\mathbb{R})}. \tag{4.7.3}$$

Proof. We use the inequality

$$\left| e^{\mathrm{i}a_1} - e^{\mathrm{i}a_2} \right| \leq |a_1 - a_2|, \ \forall a_1, a_2 \in \mathbb{R},$$

and we have the inequalities

$$\left| e^{\mathrm{i}\Theta(x+h)} f(x+h) - e^{\mathrm{i}\Theta(x)} f(x) \right| \leq \left| e^{\mathrm{i}\Theta(x+h)} f(x+h) - e^{\mathrm{i}\Theta(x+h)} f(x) \right| +$$

$$+ \left| e^{\mathrm{i}\Theta(x+h)} f(x) - e^{\mathrm{i}\Theta(x)} f(x) \right| \leq |f(x+h) - f(x)| + |\Theta(x+h) - \Theta(x)| \, \|f\|_{L^\infty(\mathbb{R})}$$

so we can apply the equivalence property (4.7.1) to obtain (4.7.2) and complete the proof of the Lemma.

□

Finally, in order to prove the estimates (4.1.26), (4.1.27), we will also need the following Lemma.

Lemma 4.7.2. *If* $\Phi(T)$ *is a solution to the integral equation* (4.4.11), *i.e.*

$$\Phi(T) = U(T,1)(\phi_1) \pm i \int_T^1 U(T,S)\Phi(S)|\Phi(S)|^2 \frac{dS}{S},$$

then for any $s \in (1/2, 3/4)$ *we have*

$$\|\Phi(T)\|_{H^s(\mathbb{R})} \le C\|\phi_1\|_{H^s(\mathbb{R})} + C\int_T^1 \|\Phi(S)\|_{H^s(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\frac{dS}{S}. \tag{4.7.4}$$

Proof. The estimate (4.7.4) follows directly from the estimate (4.6.11) of the Corollary 4.6.1 and from the fractional Leibnitz rule

$$\|fg\|_{H^s(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + C\|g\|_{H^s(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}, \tag{4.7.5}$$

(more details on fractional Leibnitz rule can be found in [48], [50] or [35]). This completes the proof of the Lemma. □

Remark 4.7.3. A simple computation shows that

$$\int_T^1 \|\Phi(S)\|_{H^s(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\frac{dS}{S} \le C\frac{1}{T^{\theta/N}} \sup_{S\in[0,1]}\left(S^{\theta/N}\|\Phi(S)\|_{H^s(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\right).$$

Indeed, we have that

$$\begin{aligned}\int_T^1 \|\Phi(S)\|_{H^s(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\frac{dS}{S} \le \\ \le C\sup_{S\in(0,1)}\left(S^{\theta/N}\|\Phi(S)\|_{H^s(\mathbb{R})}\right)\|\Phi\|_{L^\infty((0,1);L^\infty(\mathbb{R}))}\int_1^T \frac{dS}{S^{\theta/N+1}} \\ \le C\frac{1}{T^{\theta/N}}\sup_{S\in(0,1)}\left(S^{\theta/N}\|\Phi(S)\|_{H^s(\mathbb{R})}\right)\|\Phi\|_{L^\infty((0,1);L^\infty(\mathbb{R}))}.\end{aligned}$$

Now we can start with the proof of the inequality (4.1.27).

First of all, we note that from Corollary 4.6.3, we have

$$\|U(T,S)f - U(0,S)f\|_{H^\alpha(\mathbb{R})} \leq CT^{\theta/8}\|f\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})}, \tag{4.7.6}$$

provided $f \in H^{\alpha+\theta}_{odd}(\mathbb{R})$ and $0 < T \leq S \leq 1$. Taking into account Corollary 4.6.3 b) and c), we get

$$\|L(f,F)(T)\|_{H^\alpha(\mathbb{R})} \leq CT^{\theta/8}\|f\|_{H^{\alpha+\theta}(\mathbb{R})} + CT^{\theta/8-\theta/N}\sup_{S\in(0,1]} S^{\theta/N}\|F(S)\|_{H^{\alpha+\theta}(\mathbb{R})}, \tag{4.7.7}$$

where $L(f,F)(T)$ is defined in (4.4.18). Replacing F by $|\Phi|^2\Phi$ and using the fractional Leibnitz rule (here we need $\alpha+\theta<1$)

$$\||\Phi|^2\Phi\|_{H^{\alpha+\theta}(\mathbb{R})} \leq C\|\Phi\|_{H^{\alpha+\theta}(\mathbb{R})}\|\Phi\|^2_{L^\infty(\mathbb{R})}, \tag{4.7.8}$$

we obtain

$$\begin{aligned}\|L(f,|\Phi|^2\Phi)(T)\|_{H^\alpha(\mathbb{R})} &\leq CT^{\theta/8}\|f\|_{H^{\alpha+\theta}(\mathbb{R})} + \\ + \quad & CT^{\theta/8-\theta/N}\sup_{S\in(0,1]}\left(S^{\theta/N}\|\Phi(S)\|_{H^{\alpha+\theta}(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\right).\end{aligned} \tag{4.7.9}$$

Our next step is to estimate

$$\int_T^1 G(S)\frac{|\Phi|^2(S)}{S}\exp\left(\pm i(\Theta(T)-\Theta(S))\right)\,dS,$$

where $\Theta(T)$ is the real-valued function defined in (4.1.25).By (4.7.3), we have

$$\begin{aligned}\|\int_T^1 e^{\pm\mathrm{i}[\Theta(T)-\Theta(S)]}L(f,|\Phi|^2\Phi)(S)\frac{|\Phi|^2(S)}{S}\,dS\|_{H^\alpha(\mathbb{R})} \leq \\ \int_T^1(1+\|\Theta(T)-\Theta(S)\|_{H^\alpha(\mathbb{R})})\|L(f,|\Phi|^2\Phi)(S)\|_{H^\alpha(\mathbb{R})}\|\Phi(S)\|_{L^\infty(\mathbb{R})}\|\Phi(S)\|_{H^\alpha(\mathbb{R})}\frac{dS}{S}.\end{aligned} \tag{4.7.10}$$

This necessitates estimating $\|\Theta(T)-\Theta(S)\|_{H^\alpha(\mathbb{R})}$. We have by the fractional Leibnitz rule

$$\begin{aligned}\|\Theta(T)-\Theta(S)\|_{H^\alpha(\mathbb{R})} \leq \int_T^S \|\Phi(z)\bar{\Phi}(z)\|_{H^\alpha(\mathbb{R})}\frac{dz}{|z|} \leq \\ \leq C|\ln(T)|\sup_{z\in[T,S]}(\|\Phi(z)\|_{H^\alpha(\mathbb{R})}\|\Phi(z)\|_{L^\infty(\mathbb{R})}),\end{aligned} \tag{4.7.11}$$

since $0 < T < S \leq 1$. To simplify the notation we put for a while

$$K_2 = K_2(\|\Phi\|_{H^\alpha(\mathbb{R})}) = \sup_{z\in[T,S]} (\|\Phi(z)\|_{H^\alpha(\mathbb{R})}\|\Phi(z)\|_{L^\infty(\mathbb{R})}),$$
$$K_3 = K_3(\|\Phi\|_{H^{\alpha+\theta}(\mathbb{R})}) = \sup_{S\in(0,1]} \left[S^{\theta/16}\|\Phi(S)\|_{H^{\alpha+\theta}(\mathbb{R})}\|\Phi(S)\|^2_{L^\infty(\mathbb{R})}\right].$$

Moreover we note that

$$K_2 \leq \frac{1}{S^{\theta/N}}\tilde{K}_2 = \frac{1}{S^{\theta/N}} \sup_{S\in[0,1]} \left(S^{\theta/N}\|\Phi(S)\|_{H^{\alpha+\theta}(\mathbb{R})}\|\Phi(S)\|_{L^\infty(\mathbb{R})}\right).$$

Hence, entering the estimate (4.7.11) together with (4.7.10) and (4.7.9) we get

$$\|\int_T^1 e^{\pm \mathrm{i}[\Theta(T)-\Theta(S)]} L(\phi_1, |\Phi|^2\Phi)(S)\frac{|\Phi|^2(S)}{S}\, dS\|_{H^\alpha(\mathbb{R})} \leq$$
$$\leq \int_T^1 \left(1+|\ln(T)|\frac{\tilde{K}_2}{S^{\theta/N}}\right)\frac{\tilde{K}_2}{S^{\theta/N}}\left(S^{\theta/8}\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} + S^{\theta/8-\theta/N}K_3\right)\frac{dS}{S}.$$

So, choosing N big enough the following estimate holds

$$\|\int_T^1 e^{\pm \mathrm{i}[\Theta(T)-\Theta(S)]} L(\phi_1, |\Phi|^2\Phi)(S)\frac{|\Phi|^2(S)}{S}\, dS\|_{H^\alpha(\mathbb{R})} \leq$$
$$\leq C\left(1+|\ln(T)|\tilde{K}_2\right)\tilde{K}_2\left(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} + K_3\right). \tag{4.7.12}$$

On the other side, we note that, the control of the L^∞ norm of the term above is easier. Indeed, we have that

$$\|\int_T^1 e^{\pm \mathrm{i}[\Theta(T)-\Theta(S)]} L(\phi_1, |\Phi|^2\Phi)(S)\frac{|\Phi|^2(S)}{S}\, dS\|_{L^\infty(\mathbb{R})} \leq$$
$$\leq C\|\Phi\|^2_{L^\infty([0,1]\times\mathbb{R})}\left(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} + K_3\right). \tag{4.7.13}$$

Thus, it becomes clear that we need an estimate of the term

$$\sup_{S\in(0,1]}\left(S^{\theta/N}\|\Phi(S)\|_{H^{\alpha+\theta}(\mathbb{R})}\right), \tag{4.7.14}$$

where N is chosen enough big. For the purpose we shall apply the result of Lemma 4.7.2. Indeed, turning back to the estimate (4.7.14), we find

$$\sup_{T\in(0,1]}\left(T^{\theta/N}\|\Phi(T)\|_{H^{\alpha+\theta}(\mathbb{R})}\right)\leq C\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+$$

$$+C\sup_{S\in(0,1]}\left(S^{\theta/N}\|\Phi(S)\|_{H^{\alpha+\theta}(\mathbb{R})}\right)\|\Phi\|^2_{L^\infty((0,1]\times\mathbb{R})}.$$

Setting

$$|||\Phi|||_{\alpha,\theta}=\sup_{T\in(0,1]}\left(T^{\theta/N}\|\Phi(T)\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})}\right)+\|\Phi\|_{L^\infty((0,1]\times\mathbb{R})}, \tag{4.7.15}$$

we can rewrite the last estimate as

$$\sup_{T\in(0,1]}\left(T^{\theta/N}\|\Phi(T)\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})}\right)\leq C\|\phi_1\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})}+C|||\Phi|||^3_{\alpha,\theta}. \tag{4.7.16}$$

Now, combining the estimates (4.7.9) and (4.7.12) and using the notation in (4.7.15), we deduce

$$\begin{aligned}\|[\Phi(T)-\Phi_{lead}(T)]\|_{H^{\alpha}_{odd}(\mathbb{R})}\leq& CT^{\theta/8}\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+CT^{\theta/8-\theta/N}|||\Phi|||^3_{\alpha,\theta}+\\&+C\left(1+|\ln(T)||||\Phi|||^2_{\alpha,\theta}\right)|||\Phi|||^2_{\alpha,\theta}\left(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+|||\Phi|||^3_{\alpha,\theta}\right).\end{aligned}$$

On the other side, using (4.7.13), we can easily estimate the L^∞ norm of the reminder as follows

$$\|\Phi(T)-\Phi_{lead}(T)\|_{L^\infty(\mathbb{R})}\leq C\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+C|||\Phi|||^3_{\alpha,\theta}+C|||\Phi|||^2_{\alpha,\theta}\left(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+|||\Phi|||^3_{\alpha,\theta}\right).$$

Since $T^{\theta/N}|\ln(T)|$ is bounded for $T\in[0,1]$, we have that

$$|||\Phi_{rem}|||_{\alpha,\theta}\leq C\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+C|||\Phi|||^3_{\alpha,\theta}+C(1+|||\Phi|||_{\alpha,\theta})|||\Phi|||^2_{\alpha,\theta}(\epsilon+|||\Phi|||^3_{\alpha,\theta}).$$

Now we want to estimate the leading term (4.1.24). Using the established estimates for the perturbed group $U(T,S)$, we get

$$\|\Phi_{lead}(T)\|_{L^\infty(\mathbb{R})}\leq C\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+C|||\Phi|||^3_{\alpha,\theta},$$

and

$$\|\Phi_{lead}(T)\|_{H^{\alpha}(\mathbb{R})}\leq C\left(1+|\ln(T)||||\Phi|||^2_{\alpha,\theta}\right)\left(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})}+|||\Phi|||^3_{\alpha,\theta}\right).$$

From the previous estimates we conclude that

$$|||\Phi|||_{\alpha,\theta} \leq C\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} + C|||\Phi|||^3_{\alpha,\theta} + C\left(1 + |||\Phi|||_{\alpha,\theta}\right)|||\Phi|||^2_{\alpha,\theta}(\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} + |||\Phi|||^3_{\alpha,\theta}).$$

Provided that the initial data $\phi_1(x) = e^{ix^2/4}\overline{\psi_1}(x)$ verify

$$\|\phi_1\|_{H^{\alpha+\theta}(\mathbb{R})} \leq C\epsilon,$$

with ϵ small enough, we have the uniform bound

$$|||\Phi|||_{\alpha,\theta} \leq C\epsilon,$$

for any $\alpha \in (1/2, 3/4)$ and $\theta \in [0,1]$ sufficiently small such that

$$3/4 > \alpha + \theta = s > 1/2.$$

Appendix A

Functional spaces and classical embeddings

One of the main point when we work with PDEs is to set the problem in the appropriate functional spaces. Since the initial data and the solutions of the evolution equations are functions that model waves, we need to set the problem in some Banach (or Hilbert) spaces able to quantify the size of these functions in terms of measure of their integrability and of their derivability.

We start enumerating the definitions of the functional spaces used in this work. In the following we use the letters s, k, α as regularity indices and the letters p, q to denote integrability indices. Let $f(x)\colon \mathbb{R}^n \to \mathbb{C}$ be a Borel-measurable functions and $n \in \mathbb{N}$, $n \geq 1$. We recall the following spaces.

- Lebesgue spaces: $L^p(\mathbb{R}^n) = \left\{ f \,|\, \|f\|_{L^p(\mathbb{R}^n)} < \infty \right\}$,

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p \, dx \right)^{1/p}, \ 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \mathrm{supess}_{\mathbb{R}^n} |f|;$$

- Schwartz spaces:

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \,|\, a(x)P(D)f(x) \in L^\infty(\mathbb{R}^n) \right\},$$

for all a, P polynomial functions. Here $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$;

- Hölder spaces: $C^{k,\alpha}(\mathbb{R}^n) = \{f \mid f\text{continuous}, \ \|f\|_{C^{k,\alpha}} < \infty\}$, $k \in \mathbb{N}$, $0 < \alpha < 1$,

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

$$\|f\|_{C^{k,\alpha}(\mathbb{R}^n)} = \sum_{|\beta|\leq k} \|D^\beta f\|_{C^{0,\alpha}(\mathbb{R}^n)};$$

- Sobolev spaces: $W^{k,p}(\mathbb{R}^n) = \left\{f | \|f\|_{W^{k,p}(\mathbb{R}^n)} < \infty\right\}$, $k \in \mathbb{N}$, $1 \leq p < \infty$,

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \left(\|f\|^p_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\mathbb{R}^n)} \right)^{1/p},$$

where D^α denotes the weak derivatives. Similarly we can define the spaces $W^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$ considering the fractional derivative D^s;

- Tempered distributions:

$$S'(\mathbb{R}^n) = \{L \colon S(\mathbb{R}^n) \to \mathbb{R} \mid L \text{ is a linear continuous functional in } \mathscr{S}(\mathbb{R}^n)\}.$$

Since we are basically dealing with waves function, it is helpful to introduce functional spaces that control the regularity of the functions in the frequency space. Thanks to the Fourier transform and to the classical localization procedure in frequency space (Paley-Littlewood theory) we can introduce the Sobolev and Besov norms (semi-norms) and their homogeneous counterparts. Then the relatives functional spaces are the set of the tempered distributions $f \in S'(\mathbb{R}^n)$ such that the connected norm is finite.

- Sobolev and homogeneous Sobolev norms:

$$\|f\|_{H^s(\mathbb{R}^n)} = \|\mathscr{F}^{-1}[\langle\xi\rangle^s \mathscr{F} f]\|_{L^2(\mathbb{R}^n)} = \|\langle\xi\rangle^s \mathscr{F} f\|_{L^2(\mathbb{R}^n)}, \quad s \in \mathbb{R},$$

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \|\mathscr{F}^{-1}[|\xi|^s \mathscr{F} f]\|_{L^2(\mathbb{R}^n)} = \||\xi|^s \mathscr{F} f\|_{L^2(\mathbb{R}^n)}, \quad s \in \mathbb{R},$$

where $\langle\xi\rangle = \sqrt{(1+|\xi|^2)}$, and $\mathscr{F}$ is the Fourier transform defined in (B.1.2),

$$\|f\|_{\dot{H}^s_p(\mathbb{R}^n)} = \left\| \left\| 2^{ks} \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f \right\|_{\ell^2_k} \right\|_{L^p(\mathbb{R}^n)}, \ s \geq 0, \ 1 \leq p \leq \infty;$$

- Homogeneous Besov norms:

$$\|f\|_{\dot{B}^s_p(\mathbb{R}^n)} = \left\| 2^{ks} \left\| \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^j}\right) f \right\|_{L^p(\mathbb{R}^n)} \right\|_{\ell^2_k}, \; s \geq 0, \; 1 \leq p \leq \infty.$$

Here $\varphi(\xi) \in C_0^\infty(\mathbb{R} \smallsetminus 0)$ is a non-negative even function, such that

$$\sum_{j \in \mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = 1 \;, \quad \forall \xi \in \mathbb{R} \setminus 0,$$

$\mathcal{H}_0 = -\partial_x^2$ and $\mathscr{F}\left(\varphi(2^{-j}\sqrt{\mathcal{H}_0})f\right)(\xi) = \varphi(2^{-j}\xi)\hat{f}(\xi)$.

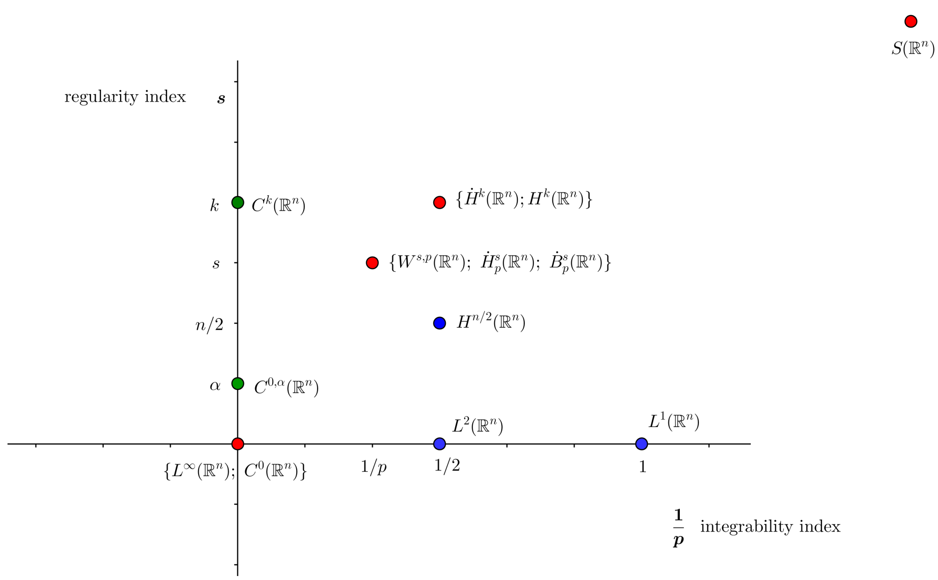


Figure A.0.1: Functional Spaces

We can also define more refined spaces if we introduce a second integrability index. Spaces of this kind are for instance the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$, or the two indices Sobolev and Besov spaces $\dot{H}^s_{p,q}(\mathbb{R}^n)$, $\dot{B}^s_{p,q}(\mathbb{R}^n)$. One can see [7] and [2] for a definition of these spaces. For a more complete diagram of the functional spaces one can see Terence Tao's web site, section "A type diagram for function spaces".

An important point is to understand the relations between the spaces. It is well known that there

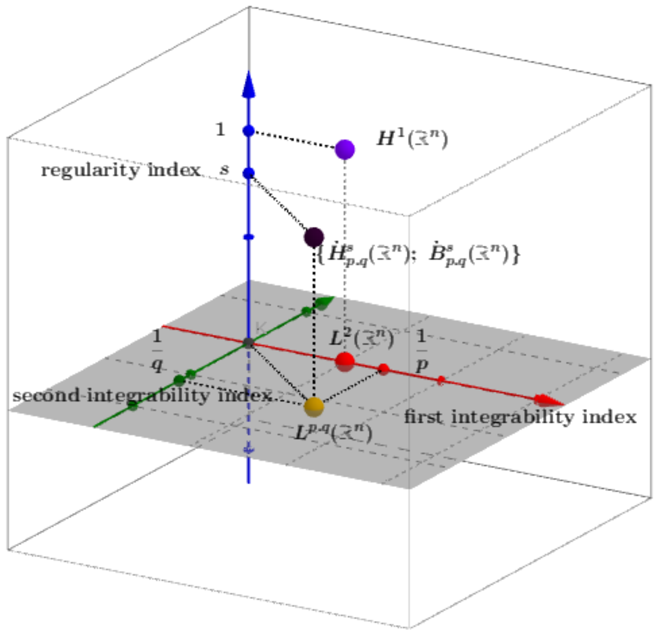


Figure A.0.2: Functional spaces

are no inclusions between spaces with different inegrability index, i.e. in general $L^p(\mathbb{R}^n) \nsubseteq L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n) \nsubseteq L^p(\mathbb{R}^n)$ if $1 \leq p \neq q \leq \infty$. Indeed, if we fix the integrability index p, the spaces with more regularity are contained in the spaces with less regularity, for instance, $H^2(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$.

Now we are interested to recall the most important embeddings between spaces with different integrability and regularity indices. One can find a brief exposition of the most relevant Sobolev inequalities in Section 1.3 and Section 1.4 in [13]. For a proof of these inequalities the classical books suggested are [21], [7] and references therein.

The critical Sobolev embeddings

$$H^{n/2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n),$$

holds for any $2 \leq p < \infty$. Representing this critical embedding in the plane $(1/p, s)$, we note that it is possible to recover all the numerology of the main Sobolev embeddings from this.

Indeed, fixed a Sobolev space $W^{s_1,p_1}(\mathbb{R}^n)$ (similarly for $\dot{H}^{s_1}_{p_1}$, $\dot{B}^{s_1}_{p_1}$), the numerology of the relative sharp embedding can be recovered intersecting the line $s - s_1 = n(1/p - 1/p_1)$ with the s-axis if $s_1 > n/p_1$ or intersecting the line $s - s_1 = n(1/p - 1/p_1)$ with the $1/p$-axis if $s_1 < n/p_1$. In this

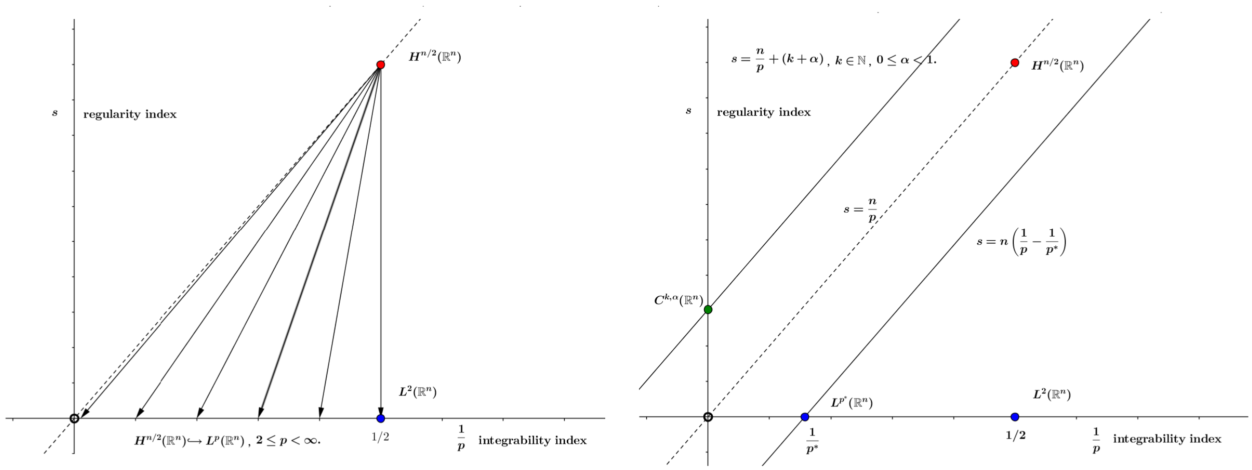


Figure A.0.3: Critical Sobolev embedding

way, we can easily remember the well known Sobolev embeddings with this graphic method:

$$s < n/p \qquad W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad p \le q \le p^*,\ p^* = \frac{np}{n-sp};$$

$$s = n/p \qquad W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad p \le q < \infty;$$

$$s > n/p \qquad W^{s,p}(\mathbb{R}^n) \hookrightarrow C^{k,\alpha}(\mathbb{R}^n), \qquad k = [s - n/p],\ \alpha = s - n/p - k.$$

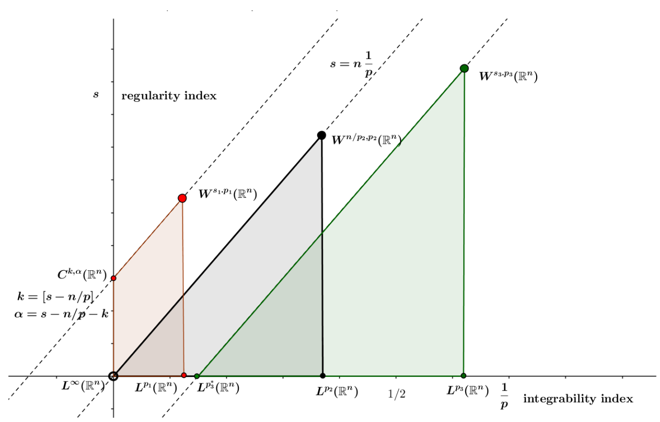


Figure A.0.4: Sobolev embeddings

We note that any pair $(1/p, s)$ does not identify a unique functional space but several of them. Here we just mention the relations between homogeneous Sobolev and homogeneous Besov spaces identified by the same pair $(1/p, s)$. Firstly it is well known that $\dot{H}^s_2(\mathbb{R}^n) \approx \dot{B}^s_2(\mathbb{R}^n) \approx \dot{H}^s(\mathbb{R}^n)$

(one can see Section 1.4 in [13]). In general, the following relations are satisfied:

$$\dot{H}^s_p(\mathbb{R}^n) \hookrightarrow \dot{B}^s_p(\mathbb{R}^n), \qquad 1 < p \leq 2,$$
$$\dot{B}^s_p(\mathbb{R}^n) \hookrightarrow \dot{H}^s_p(\mathbb{R}^n), \qquad 2 \leq p < \infty.$$

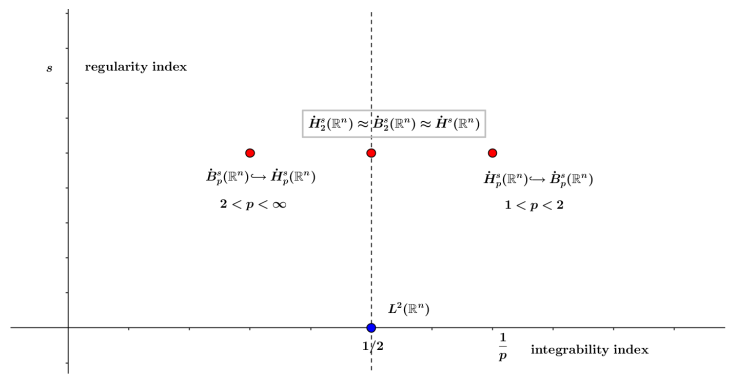


Figure A.0.5: Homogeneous Sobolev and Homogeneous Besov spaces

The spaces introduced up to now are spaces that take into account the spacial variable $x \in \mathbb{R}^n$. When we study the evolution equations, is appropriate to introduce the Bochner spaces that consider also the time variable. Let us denote with the generic notation $(Z(\mathbb{R}^n), \|\cdot\|_{Z(\mathbb{R}^n)})$ one of the Banach spaces introduced above. Let $0 < T \leq \infty$, $n \in \mathbb{N}, n \geq 1$ and let $g(t,x)\colon [0,T] \times \mathbb{R}^n \to \mathbb{C}$. The Bochner spaces are the spaces $X([0,T],\mathbb{R}^n) = \{g \mid \|g\|_{X([0,T],\mathbb{R}^n)} < \infty\}$, where $\|\cdot\|_{X([0,T],\mathbb{R}^n)}$ denotes one of the following norms:

$$\|g\|_{L^q([0,T],Z(\mathbb{R}^n))} = \left(\int_0^T \|g(t)\|^q_{Z(\mathbb{R}^n)}\,dt\right)^{1/q}, \ 1 \leq q < \infty;$$
$$\|g\|_{L^\infty([0,T],Z(\mathbb{R}^n))} = \text{supess}_{[0,T]}\|g(t)\|_{Z(\mathbb{R}^n)};$$
$$\|g\|_{C([0,T],Z(\mathbb{R}^n))} = \sup_{[0,T]} \|g(t,\cdot)\|_{Z(\mathbb{R}^n)}.$$

Appendix B

Classical latter for NLS

In this appendix we recall some general facts about the dispersive PDE. In particular we summarize the main ideas and the main tools to approach the linear and the nonlinear Schrödinger Cauchy problem to get local and global well posedness results. The reader can find an in-depth discussion of these arguments in [13], [67], [63].

B.1 A dispersive model: Linear Schrödinger equation

The linear Schrödinger equation

$$i\partial_t \Psi + \frac{1}{2}\partial_x^2 \Psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{B.1.1}$$

is the Quantum Mechanics model to describe the evolution of a free particle in a nonrelativistic regime and it is the classical model to explain the *dispersive* equations. According to this presentation, we will consider only the one dimensional case.

If we are on the whole space $\mathbb{R}$, we can solve the equation (B.1.1) by means of the Fourier transform. We define the Fourier transform of $f \in \mathscr{S}(\mathbb{R})$ and its inverse as follows:

$$\begin{aligned} \mathscr{F}[f](\xi) = \hat{f}(\xi) &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x)\, dx, \\ \mathscr{F}^{-1}[f](x) = \check{f}(x) &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\cdot\xi} f(\xi)\, d\xi, \end{aligned} \tag{B.1.2}$$

and then we can extend this operator on tempered distribution space $S'(\mathbb{R})$.

The space Fourier transform is a kind of change of variable able to simplify the problem.

Let $f \in S(\mathbb{R}^n)$ be the initial datum. It can be regarded as a wave packet

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

Then, thanks to Fourier transform, by ordinary differential equation theory, we can prove that the Cauchy problem

$$\begin{cases} i\partial_t \Psi + \frac{1}{2}\partial_x^2 \Psi = 0 \\ \Psi(0, \cdot) = f, \end{cases}$$

has a unique solution

$$\Psi(t) = S(t)f,$$

where

$$S(t) = e^{i\frac{\partial_x^2}{2}t} \colon S(\mathbb{R}^n) \to S(\mathbb{R}^n),$$

is defined via Fourier transform as follows:

$$S(t)f(x) = \mathscr{F}^{-1}(e^{-i\frac{|\xi|^2}{2}t}\hat{f}(\xi))(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{i(x\xi - \frac{\xi^2}{2}t)} \hat{f}(\xi)\, d\xi. \tag{B.1.3}$$

We can interpret the solution (B.1.3) as a superposition of plane waves, where ξ is the wave number (equivalently the momentum by De Broglie relation), $\omega = \xi^2/2$ is the angular frequency (or equivalently the energy), $\hat{f}(\xi)$ is the amplitude. We can see that the phase velocity, $v_p = \frac{\omega(\xi)}{\xi}$, depends on the wave number, so, waves with different wave number go to different velocities. The group velocity, $v_g = \partial_\xi \omega(\xi)$, is the velocity at which energy is conveyed along the wave. If the group velocity v_g depends on ξ, as in this case, the equation is dispersive and the relation $\omega = \omega(\xi)$ is called the *dispersive relation.*

We can think to the dispersive PDEs as to the equations which solutions are waves that spread out spatially as long as no boundary conditions are imposed as a result of different frequency components of the wave packet travelling at different velocities.

Always informally, we can also characterize the dispersive PDEs as follows. Let us consider the general linear evolution equation

$$\partial_t \Psi = iP(-i\partial_x)\Psi \tag{B.1.4}$$

where $P(-i\partial_x)$ is a linear differential operator of symbol $P(\xi)$, and $\mathscr{F}(P(-i\partial_x)\Psi)(\xi) = P(\xi)\hat{\Psi}(\xi)$. Then, using the space-time Fourier transform

$$\widetilde{\Psi}(\tau,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(t\tau+x\xi)}\Psi(t,x)\,dt\,dx,$$

we get

$$(i\tau - iP(\xi))\widetilde{\Psi}(\tau,\xi) = 0.$$

The space-time Fourier transform of the solution has to be supported on the surface

$$\Sigma = \{(\tau,\xi) \mid \tau = P(\xi)\}$$

that lives in the phase space $\mathbb{R}\times\mathbb{R}$ (energy - momentum). Hence, always informally, we can also say that the evolution equation (B.1.4) is dispersive if the surface Σ is curved (the energy is not a linear function of the momentum). The relation $\tau = \tau(\xi) = iP(i\xi)$ is called the dispersive relation.

To work with the nonlinear problem a preliminary step is to solve the non homogeneous Cauchy problem

$$\begin{cases} i\partial_t\Psi + \frac{1}{2}\partial_x^2\Psi = F(t,x), \ \ (t,x)\in\mathbb{R}_+\times\mathbb{R}, \\ \Psi(0,\cdot) = 0. \end{cases}$$

Interpreting the driving force F as a superposition of initial pulses in time we can produce the formula for the solution. Indeed, we rewrite the driving force as follows

$$F(t,x) = \int_0^\infty F(s,x)\delta(t-s)\,ds$$

and we look for a partial solution $\Psi(t,x;s)$ such that

$$\partial_t\Psi(t,x;s) = -iF(s,x)\delta(t-s) + i\frac{1}{2}\partial_x^2\Psi(t,x;s).$$

Suppose that the system is completely at rest in the distant past and that at some time s the driving force $F(t,x)$ starts to operate. Hence, we look for the partial solution $\Psi(t,x;s)$ such that

$$\begin{cases} \Psi(t,x;s) = 0 & t < s \\ \Psi(t,x;s) = -iF(s,x) & t = s \\ i\partial\Psi + \partial_x^2\Psi = 0 & t > s. \end{cases}$$

By the *Duhamel's principle* we have that the solution of the non homogeneous problem $\Psi(t,x)$, is obtained as a superposition of the partial solution $\Psi(t,x;s)$

$$\Psi(t,x) = \int_0^\infty \Psi(t,x;s)\,ds = -i\int_0^t S(t-s)F(s,x)\,ds.$$

This principle will turn to be crucial to introduce the definition of the solution of the Cauchy problem in the nonlinear contest. The ideas summarized in this section are broadly discussed in [21], [61], [23].

B.2 Abstract formalization of the nonlinear problem

In this section we recall the classical theory to prove the well-posedness of the nonlinear problem.

Let us consider the nonlinear Cauchy problem

$$\begin{cases} i\partial_t\Psi + \frac{1}{2}\partial_x^2\Psi = N(\Psi) \\ \Psi(0) = f, \end{cases} \tag{B.2.1}$$

where N is a nonlinear function with respect to Ψ. In particular, since we are considering N nonlinear only with respect to Ψ but linear with respect to all its derivatives, the PDE is said to be *semilinear*. The semilinear Schrödinger type equations are employed to model many quantum physical phenomena. Since it is a mathematical model, many physical laws have been simplified, hence it is necessary to verify the well-posedness of the problem before passing to any other questions.

First of all we have to define what we mean by a *solution* of the problem. So, we have to introduce suitable Hilbert space H in which the initial data live. Then, in order to have the well posedness of the problem, we want that the following features have to be satisfied:

- The model is coherent, i.e. the solution exists at least for short time $T > 0$ and its evolution in time lives in the same Hilbert space of the initial data, $\Psi(t,x) \in C([0,T],H)$;

- The mathematical model fits well in the numerical simulations, i.e. the uniqueness and the stability of the solution are satisfied.

Let H be an Hilbert space such that the initial datum $f \in H$ (keep in mind $H = H^s(\mathbb{R})$). Let X be a Banach space such that the *solution* $\Psi(t,x) \in X$ (keep in mind $X = C([0,T], H)$) for some positive time $T > 0$. Now we want to formalize the definition of the solution of the Cauchy problem (B.2.1). Let us introduce the operators T and T^*:

$$\begin{aligned} T\colon H &\to X \\ f &\mapsto S(\cdot)f, \\ T^*\colon X^* &\to H \\ F &\mapsto \int_{\mathbb{R}} S(-s)F(s)\,ds, \end{aligned}$$

and we define a restricted version of the operator TT^* as follows:

$$[(TT^*)_R F](t) = \int_0^t S(t-s)F(s)\,ds.$$

Definition B.2.1. *Let $f \in H$ and let $\mathscr{H}$ be the operator defined as follows*

$$\mathscr{H}\colon X \to X$$
$$\mathscr{H}(\Psi) = Tf - i(TT^*)_R N(\Psi).$$

We say that $\Psi \in X$ is a solution of the nonlinear Cauchy problem (B.2.1) *if Ψ is a fixed point of the map $\mathscr{H}$, i.e. $\mathscr{H}(\Psi) = \Psi$.*

Once we assume the initial data $f \in H$ and once we define the notion of solution we can deal with the following classical problems:

- *Local well possedness*: Check if there exists $0 < T < \infty$ such that the (B.2.1) admits a unique solution $\Psi \in C([0,T], H^s(\mathbb{R}))$. Check if the solution depends continuously on the initial data.

- *Global well posedness*: Check if the maximal existence time can be extended to $T = \infty$.

- *Behaviour near the maximal time*: Understand as the solution looks near the maximal time and if *scattering* or *blow up* phenomena can occur.

B.3 Tool box: Fixed point theorem - Dispersive estimates - Strichartz estimates

In this section we collect the main tools useful to prove well posedness theorems with the techniques introduced by Kato [47]. Moreover, we just give a sketch to underline the main keys and to show how these tools are used.

The main machinery used to prove well posedness is the fixed point theorem.

Theorem B.3.1 (Banach's fixed point theorem: Théorème 6, [4])**.** *Let (X, d) be a complete metric space and $\mathscr{H} : X \to X$. If there erists a constant $K < 1$ such that*

$$d(\mathscr{H}(\Psi_1), \mathscr{H}(\Psi_2)) < Kd(\Psi_1, \Psi_2)$$

for all $\Psi_1, \Psi_2 \in X$, then $\mathscr{H}$ has a unique fixed point $\Psi \in X$; i.e., there exists a unique $\Psi \in X$ such that $\mathscr{H} \in X$.

The key properties of this theorem used to prove local well posedness are the following:

- (X, d) is a complete metric space;
- $\mathscr{H}$ is a contraction;
- The proof of existence is obtained by successive approximations (*Picard iteration*).

Here we state the fixed time estimates (Dispersive estimates):

Theorem B.3.1. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, there exists a postive constant $C > 0$, such that the Schrödinger group $S(t) = T(\cdot)(t)$ satisfies the following estimates:

$$\|Tf(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\frac{1}{t^{n/2}}\|f\|_{L^1},$$
$$\|Tf(t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},$$
$$\|Tf(t)\|_{L^p(\mathbb{R}^n)} \leq C\frac{1}{t^{n/2-n/p}}\|f\|_{L^{p'}}.$$

These estimates give an a priori precise decay-rate on the linear solution and they have some strong consequences about the boundeness of some global norms (space-time) of nonlinear solution. On the other hand, these estimates is not quite handy for dealing with the nonlinearity.

Indeed, for the nonlinear case, the space-time integrability properties (Strichartz estimates) will turn to be crucial.

Definition B.3.2. *Let $n \geq 2$. We say that a couple of number (q,r) is Schrödinger admissible if the following relations hold:*

$$\frac{2}{q} = \frac{n}{2} - \frac{n}{r},\ q \geq 2,\ (q,r,n) \neq (2,\infty,2).$$

Let $n = 1$. We say that a couple of number (q,r) is Schrödinger admissible if the following relations hold:

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r},\ q \geq 4.$$

The pairs $(p,q) = (2, \frac{2n}{n-2})$ in dimension $n \geq 2$ are called endpoint pairs.

Theorem B.3.3 (Strichartz estimates)**.** *For any (q,r), $(\tilde{q},\tilde{r})$ Schrödinger admissible pairs one has the homogeneous Strichartz estimates*

$$(T) \qquad \|Tf(t)\|_{L^q(\mathbb{R},L^r(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)};$$

the dual homogeneous Strichartz estimates

$$(T^*) \qquad \|T^*F\|_{L^2(\mathbb{R}^n)} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^n))};$$

the inhomogeneous (retarded) Strichartz estimates

$$(TT^*) \qquad \|(TT^*)_R F\|_{L^q_t(\mathbb{R},L^r_x(\mathbb{R}^n))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(\mathbb{R}^n))}.$$

The proof of this theorem is due to different authors. In the original version, elaborated by Strichartz in [65], the statement is proved in the case $q = r = \frac{2(n+2)}{n}$ and $\tilde{q}' = \tilde{r}' = q' = r' = \frac{2(2+n)}{n+4}$. The non-endpoint case is proved in [32] and the end-point case is proved in [49]. For the nonlinear application is crucial the fact that the pairs (q,r) and $(\tilde{q},\tilde{r})$ are not related to each other in the (TT^*) estimate.

Remark B.3.4. We note that, the Schrödinger operator $T(\cdot)(t)$ commutes with the Fourier multipliers like $|D|^s = \mathscr{F}^{-1}(|\xi|^s\mathscr{F})$ or $\langle D\rangle^s = \mathscr{F}^{-1}(\langle\xi\rangle^s\mathscr{F})$. Hence, we easily get Strichartz estimates for Sobolev spaces substituting the spaces $L^2(\mathbb{R}^n)$, $L^r(\mathbb{R}^n)$, $L^{\tilde{r}'}(\mathbb{R}^n)$ with the Sobolev spaces $H^s(\mathbb{R}^n)$, $H^s_r(\mathbb{R}^n)$, $H^s_{\tilde{r}'}(\mathbb{R}^n)$ or their homogeneous counterparts $\dot{H}^s(\mathbb{R}^n)$, $\dot{H}^s_r(\mathbb{R}^n)$, $\dot{H}^s_{\tilde{r}'}(\mathbb{R}^n)$. Similarly, using the Minkowski's inequality

$$\left\| \|g_k(t)\|_{\ell^2(\mathbb{Z})} \right\|_{L^q(\mathbb{R})} \leq \left\| \|g_k(t)\|_{L^q(\mathbb{R})} \right\|_{\ell^2(\mathbb{Z})},$$

where $q \geq 2$, combined with the fact that the Schrödinger group commutes with the Fourier multipliers, we get the Strichartz estimates in Besov and homogeneous Besov spaces.

We note that the Strichartz estimates in homogeneous Besov spaces implies the Strichartz estimates in homogeneous Sobolev spaces since the following embeddings hold

$$\dot{B}^s_p(\mathbb{R}^n) \hookrightarrow \dot{H}^s_p(\mathbb{R}^n),\ 2 \leq p < \infty,$$
$$\dot{H}^s_p(\mathbb{R}^n) \hookrightarrow \dot{B}^s_p(\mathbb{R}^n),\ 1 < p \leq 2.$$

B.4 Semilinear Schrödinger equation and classical results

We recall some classical results about the model case of the pure power nonlinearity, well known as semilinear Schrödinger:

$$\begin{cases} i\partial_t \Psi + \frac{1}{2}\partial_x^2 \Psi = \alpha |\Psi|^{\gamma-1}\Psi, \\ \Psi(0) = f, \end{cases} \tag{B.4.1}$$

with $\gamma > 1$ and $\alpha = \pm 1$.

These equations are one of the universal model to describe the evolution of a wave packet in a weakly nonlinear and dispersive media. In particular, the case $\gamma = 3$ occurs to model different physical phenomena: the propagation of waves in optical fibers for $n = 1$, the focusing of laser beams for $n = 2$, the Bose-Einstain condensation phenomenon for $n = 3$, see [66], [46] and references therein.

As we can see in (B.4.1), the evolution is a competition between the linear part and the nonlinear one. So we can expect that the evolution can assume different behaviours: a *linearly dominated behavior* or a *nonlinearly dominated behavior* or also an *intermediate behavior*. In order to understand the behaviour, we shall classify the nonlinearity. Two basic features are crucial: the conservation laws and the natural scale-invariance of the equation. The structure of (B.4.1), in $H^1(\mathbb{R})$, implies the *mass conservation*

$$\|\Psi(t)\|_{L^2(\mathbb{R})} = \|\Psi(0)\|_{L^2(\mathbb{R})},$$

and the *energy conservation*

$$E(\Psi(t)) = \frac{1}{4}\|\nabla\Psi(t)\|^2_{L^2(\mathbb{R})} \pm \frac{1}{\gamma+1}\|\Psi(t)\|^{\gamma+1}_{L^{\gamma+1}(\mathbb{R})} = E(\Psi(0)).$$

Using the scale-invariance for (B.4.1)

$$\Psi_\lambda(t,x) = \lambda^{2/(1-\gamma)}\Psi(\frac{t}{\lambda^2},\frac{x}{\lambda}), \tag{B.4.2}$$

where $\lambda > 0$, we can classify the conservation laws as *subcritical*, *critical* (scale-invariant), or *supercritical*. In particular, in one dimension, using $L^2(\mathbb{R})$-conservation (similarly for $H^s(\mathbb{R})$ conservation), we have

$$\|\Psi_\lambda(t,\cdot)\|_{L^2(\mathbb{R})} = \lambda^{\frac{5-\gamma}{2(1-\gamma)}}\|\Psi(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{B.4.3}$$

We can give the following definition:

Definition B.4.1. Let $\gamma > 1$, we say that

- γ is L^2-subcritical if $1 < \gamma < 5$,
- γ is L^2-critical if $\gamma = 5$,
- γ is L^2-supercritical if $\gamma > 5$.

The adjective *critical* in this context concerns with the well posedness theory in L^2. Indeed, the scaling relation (B.4.3), could be interpreted as follows: in subcritical case, the norm of the initial data can be made small while the interval of time is made longer, in the critical case, the norm is invariant while the interval of time is made longer or shorter and finally in supercritical case, the norm grows as the time interval gets longer. In the last two cases we could find some problems for the well posedness.

Another important point is to distinguish if the equation is *focusing* ($\alpha = -1$) or *defocusing* ($\alpha = 1$). We can not make an exact theoretical distinction, but, in the defocusing case, since the nonlinearity has the same sign as the linear component, we guess that the dispersive effects of the linear equation are amplified. On the contrary, in the focusing case the dispersive effects can be attenuated, halted (stationary or travelling waves) or reversed (blow up of solution in finite time). For some literature on local existence results in the subcritical case, one can see [30], [47], [70] and [13]. For local existence in the critical case, one can see [14], [13]. Finally, for the global critical case one can see [31], [13].

Now we state a fundamental result based on the fixed point method and we give a sketch of the strategy of the proof underlining where the *critical* effects come out. As we will see, the problem of the *well-posedness* is closely intertwined with the quantitative estimates (*a priori estimates*).

Theorem B.4.2 (Tsutsumi [70]). Assume that $1 < \gamma < 1 + 4/n$. Then, for any $f \in L^2(\mathbb{R}^n)$ there exists a time $T = T(\gamma, n, \|f\|_{L^2(\mathbb{R}^n)}) > 0$ such that (B.4.1) has a unique local solution $\Psi(t)$:

(i) $\Psi \in C([0,T], L^2(\mathbb{R}^n)) \cap L^q([0,T], L^r(\mathbb{R}))$, for any (q,r) admissible pair;

(ii) $\Psi(t) = Tf(t) - i(TT^*)_R(|\Psi|^{\gamma-1}\Psi)(t)$;

(iii) Ψ depends continuosly on f in the following sense: Let $f_n, f \in L^2(\mathbb{R}^n)$ be such that $f_n \to f$, $(n \to \infty)$, in $L^2(\mathbb{R}^n)$. Let $\Psi \in C([0,T], L^2(\mathbb{R}^n))$ be the solution of (NLS) with initial data f. Then, the solutions $\Psi_n(t)$ with initial data f_n there exist in $[0,T]$ and $\Psi_n \to \Psi$, $(n \to \infty)$, in $C([0,T], L^2(\mathbb{R}^n))$;

(iv) $\|\Psi(t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$, for each $t \in [0,T]$.

We outline here the strategy of the proof since it is a classical argument.

Let $f \in L^2(\mathbb{R}^n)$ and let (q,r) be an admissible pair. We take T be a positive time (it will be determined later), and we consider the functional space $X = L^\infty([0,T], L^2(\mathbb{R}^n)) \cap L^q([0,T], L^r(\mathbb{R}^n))$. We define $\mathscr{H} \colon X \to X$

$$\mathscr{H}(\Psi) = Tf - i(TT^*)_R N(\Psi),$$

where $N(\Psi) = \alpha|\Psi|^{\gamma-1}\Psi$. The first step is to introduce the ball X_B

$$X_B = \left\{\Psi \in X \| \|\Psi\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} + \|\Psi\|_{L^q([0,T],L^r(\mathbb{R}^n))} \leq 2C\|f\|_{L^2(\mathbb{R}^n)}\right\},$$

with $C > 0$ and to prove that (X_B, d) is a complete metric space, where d is the distance defined as follows

$$d(\Psi_1, \Psi_2) = \|\Psi_1 - \Psi_2\|_{X_B} = \|\Psi_1 - \Psi_2\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} + \|\Psi_1 - \Psi_2\|_{L^q([0,T],L^r(\mathbb{R}^n))}.$$

Then we need to show that $\mathscr{H}\colon X_B \to X_B$ is a contraction, i.e. we need to verify the following properties

(1) $\|Tf\|_{X_B} \leq C\|f\|_{L^2(\mathbb{R}^n)}$;

(2) $\|N(\Psi)\|_{L^{\tilde{q}'}([0,T],L^{\tilde{r}'}(\mathbb{R}^n))} \leq CT^{\frac{4+n(1-\gamma)}{4}}\|\Psi\|^{\gamma}_{L^q([0,T],L^r(\mathbb{R}^n))}$;

(3) $\|(TT^*)_R N(\Psi)\|_{X_B} \leq C\|N(\Psi)\|_{L^{\tilde{q}'}([0,T],L^{\tilde{r}'}(\mathbb{R}^n))}$;

(4) $d(\mathscr{H}(\Psi_1), \mathscr{H}(\Psi_2)) \leq CT^{\frac{4+n(1-\gamma)}{4}}\left(\|\Psi_1\|^{\gamma-1}_{L^q([0,T],L^r(\mathbb{R}^n))} + \|\Psi_2\|^{\gamma-1}_{L^q([0,T],L^r(\mathbb{R}^n))}\right) d(\Psi_1.\Psi_2)$,

where $(\tilde{q}, \tilde{r})$ is an admissible pair with $\gamma\tilde{r}' = r$. Hence, we can choose $T = T(\gamma, n, \|f\|_{L^2(\mathbb{R}^n)})$ sufficiently small in order to apply the Theorem B.3.1. This implies that there exists a unique $\Psi \in X_B$ such that $\mathscr{H}(\Psi) = \Psi$. We note that this argument works in subcritical regime or in critical regime with small data. Now we can prove uniqueness for small time. Suppose that there are two solutions $\Psi_1, \Psi_2 \in C([0,T], L^2(\mathbb{R}^n)) \cap L^q([0,T], L^r(\mathbb{R}))$. The following estimate

$$\|\Psi_1 - \Psi_2\|_X = \|\mathscr{H}(\Psi_1) - \mathscr{H}(\Psi_2)\|_X \leq \|N(\Psi_1) - N(\Psi_2)\|_{L^{\tilde{q}'}([0,T],L^{\tilde{r}'}(\mathbb{R}^n))},$$

combined with estimate (2) gives the uniqueness for small time. A contradiction argument shows uniqueness in the interval $[0,T]$. Finally, to have the local well posedness we have to establish the continuous dependence of the solutions on the initial data. Let $f_1, f_2 \in L^2(\mathbb{R}^n)$ and let $\Psi_1(t), \Psi_2(t)$ be the corresponding solutions. For some time T depending on $\|f_1\|_{L^2(\mathbb{R}^n)}, \|f_2\|_{L^2(\mathbb{R}^n)}$ we have

$$d(\Psi_1, \Psi_2) \leq C\|f_1 - f_2\|_{L^2(\mathbb{R}^n)}.$$

The mass conservation is proved under more regularity condition. Indeed, we assume the solution $\Psi \in C([0,T], H^1(\mathbb{R}^n))$ and we multiply by $\bar{\Psi}$ the equation in (B.4.1) and we integrate by parts. The general statement is proved by density arguments combined with continuous dependence. One can see [70], [13] for a detailed proof.

We summarize the fundamental results of $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ theory in the following theorems:

Theorem B.4.1 ($L^2(\mathbb{R})$ well-posedness)**.** *(Cazenave [13], Section 4.6) Let $f \in L^2(\mathbb{R})$. The following statements hold:*

- *Let $1 < \gamma < 5$ and $\alpha = \pm 1$. Then there exists a unique global solution $\Psi \in C(\mathbb{R}, L^2) \cap L^q(\mathbb{R}, L^r(\mathbb{R}))$;*
- *Let $\gamma = 5$ and $\alpha = \pm 1$. Then there exists $\delta > 0$, quite small, such that, if $\|f\|_{L^2} \leq \delta$ we have a*

unique global solution $\Psi \in C(\mathbb{R}, L^2) \cap L^q(\mathbb{R}, L^r(\mathbb{R}))$.

In addition, the following conservation law holds:

$$M(t) = \|\Psi(t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},$$

for each $t \in \mathbb{R}$.

Theorem B.4.2 ($H^1(\mathbb{R})$ well-posedness)**.** *(Cazenave [13], Section 4.4) Let* $f \in H^1(\mathbb{R})$. *The following statements hold:*

- *Let* $1 < \gamma < 5$ *and* $\alpha = \pm 1$. *Then there exists a unique global solution* $\Psi \in C(\mathbb{R}, H^1) \cap L^q(\mathbb{R}, W^{1,r}(\mathbb{R}))$;
- *Let* $\gamma = 5$ *and* $\alpha = -1$. *Then there exists* $\delta > 0$, *quite small, such that, if* $\|f\|_{L^2} \leq \delta$ *we have a unique global solution* $\Psi \in C(\mathbb{R}, H^1) \cap L^q(\mathbb{R}, W^{1,r}(\mathbb{R}))$.

In addition, the following conservation laws hold:

$$\begin{aligned} M(t) &= \|\Psi(t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \\ E(t) &= \frac{1}{4} = \|\nabla\Psi(t)\|_{L^2(\mathbb{R}^n)} + \frac{\alpha}{\gamma - 1}\|\Psi(t)\|_{L^{\gamma+1}}^{\gamma+1} = E(0), \end{aligned}$$

for each $t \in \mathbb{R}$.

We underline here the fundamental steps to reach the results quoted above.

To pass from a local result to a global one the tools are the blow up alternative and the conservation laws.

Proposition B.4.3 (Blow-up alternative)**.** *Let* $T_{max} < \infty$ *(respectively , if* $T_{min} < \infty$ *), then, under the hypothesis of the Theorem B.4.2 we have*

$$\lim_{t \nearrow T_{max}} \|\Psi(t, \cdot)\|_{L^2(\mathbb{R})} = \infty \ \ (\lim_{t \searrow T_{min}} \|\Psi(t, \cdot)\|_{L^2(\mathbb{R})} = \infty).$$

In the critical case, as we remarked before, in the estimates 2. and 3. the time disappears, $T^{\frac{4+n(1-\gamma)}{4}} = T^0$, so we cannot play on the smallness of time. The absolute continuity of the Lebesgue integral helps us to establish local result. In the critical case the existence time $T = T(\gamma, n, f)$ depends on the shape of the initial data and so we cannot extend the local result to global one. The results in the space $H^1(\mathbb{R})$ are proved using Strichartz estimates in $H^1(\mathbb{R})$.

Appendix C

Gronwall's inequality on the real line

In this appendix we shall recall the classical Gronwall inequality and some modifications of its on $\mathbb{R}$.

We state the classical Gronwall's inequality in the integral form.

Lemma C.0.1. *Let $v\colon [x_0, x_1] \to \mathbb{R}^+$ be continuous and non-negative function and suppose that v satisfies the following inequality*

$$v(x) \leq a + \int_{x_0}^{x} b(s)v(s)\,ds$$

for all $x \in [x_0, x_1]$, where $a \geq 0$ and $b\colon [x_0, x_1] \to \mathbb{R}^+$ is continuous and non negative. Then, we have

$$v(x) \leq a \exp\left(\int_{x_0}^{x_1} b(s)\,ds\right)$$

for all $x \in [x_0, x_1]$.

This simple result is very useful in both the ordinary differential equations and the partial differential equations. Indeed, in the ODEs theory this result provide some control of the solution in terms of the initial data and of the linear perturbation. In the PDEs theory it is a very useful tool to get uniqueness results or to assure the absence of blow up in suitable norms.

The next Lemma is a slightly modified version of the previous one. It is a fundamental step to get the improved results on the modified Jost functions $m_\pm$ presented in Chapter 2, Lemma 2.1.2 and Lemma 2.1.3.

Lemma C.0.2. *If $v(x)$, $a(x)$, $b(x)$ are continuous non-negative functions on $\mathbb{R}$, and for any real r we* *have*

$$a(x),\, v(x) \in L^\infty((r,\infty)),\;\; b(x) \in L^1((r,\infty)) \tag{C.0.1}$$

that satisfy the inequality

$$v(x) \le a(x) + \int_x^\infty b(t)v(t)dt, \tag{C.0.2}$$

then we have

$$v(x) \le a(x) + \int_x^\infty a(t)b(t)\exp\left(\int_x^t b(s)ds\right)dt. \tag{C.0.3}$$

Proof. We shall sketch the proof for completeness. Set

$$\varphi(x) = \int_x^\infty b(t)v(t)dt.$$

The function is well-defined and $C^1(\mathbb{R})$ due to the assumption (C.0.1). Then

$$\varphi'(x) = -b(x)v(x) \ge -b(x)(\varphi(x) + a(x))$$

and

$$\left(e^{-B(x)}\varphi(x)\right)' \ge -e^{-B(x)}b(x)a(x)$$

with $B(x) = \int_x^\infty b(t)dt$. Integrating this inequality in the interval (x, R), we get

$$\varphi(x) \le e^{B(x)-B(R)}\varphi(R) + \int_x^R e^{B(x)-B(t)}a(t)b(t)dt.$$

Using again the assumption (C.0.1), we see that

$$\lim_{R\nearrow\infty} B(R) = 0,\;\; \lim_{R\nearrow\infty} \varphi(R) = 0$$

so we get

$$\varphi(x) \le \int_x^\infty e^{B(x)-B(t)}a(t)b(t)dt.$$

Then (C.0.2) implies $v(x) \le a(x) + \varphi(x)$ and we arrive at (C.0.3). This completes the proof. □

Since in the PDEs field the Lebesgue's spaces $L^p(\mathbb{R})$ play a fundamental rule, some generalization of the Gronwall's inequality for $L^p(\mathbb{R})$ functions are also very useful. One can see [73] and references therein for this kind of generalizations of the Gronwall's inequality and their discrete analogues. One can compare the Gronwall type estimate proposed below with Theorem 2 in [73] for example.

We shall assume $v(t) \in C([0,+\infty))$ and $a(t) \in L^\infty_{loc}((0,+\infty))$ are non-negative functions that satisfy the inequalities

$$0 < t \leq 1 \Longrightarrow \left(\int_0^t v(\tau)^p d\tau \right)^{q/p} \leq C_1, \tag{C.0.4}$$

$$t > 1 \Longrightarrow \left(\int_0^t v(\tau)^p d\tau \right)^{q/p} \leq C_1 + \int_1^t a(\tau) v(\tau)^q d\tau, \tag{C.0.5}$$

where $1 \leq q < p \leq \infty$ (with obvious modifications for $p = \infty$.)

Lemma C.0.3 ($L^p - L^q$ Gronwall's Lemma)**.** *Suppose that* $1 \leq q < p \leq \infty$ *and* $C_0 > 0$*. If* $v(t) \in C([0,+\infty))$ *and* $a(t) \in L^\infty_{loc}((0,+\infty))$ *are non -negative functions that satisfy the inequalities* (C.0.4), (C.0.5)*, then*

$$\left(\int_0^t v(\tau)^p d\tau \right)^{q/p} \leq \left(\frac{p}{p-q} \right)^{q/p} C_1 \ \exp \left(\left(\frac{p}{q} \right)^{q/(p-q)} \int_1^t a^{p/(p-q)}(\tau) d\tau \right). \tag{C.0.6}$$

Proof. First, we shall consider the case $1 < p < +\infty$, $q = 1$. Then the inequality (C.0.5) can be rewritten as

$$t > 1 \Longrightarrow \|v\|_{L^p(0,t)} \leq C_1 + \left(\int_1^t a(\tau) v(\tau) d\tau \right). \tag{C.0.7}$$

Set

$$\varphi(t) = C_1 + \int_1^t a(\tau) v(\tau) d\tau.$$

Then $\varphi(t)$ is increasing function and we have the relation

$$v(t) = \frac{\varphi'(t)}{a(t)},$$

so, we have

$$t > 1 \Longrightarrow \left\| \frac{\varphi'}{a} \right\|_{L^p(1,t)} \leq \varphi(t). \tag{C.0.8}$$

Further we can use the inequality (C.0.8) and we can derive the estimates

$$\begin{aligned} \varphi(t)^p =& \varphi(1)^p + p \int_1^t \varphi'(\tau) \varphi^{p-1}(\tau) d\tau = \\ =& C_1^p + p \int_1^t \varphi'(\tau) \varphi^{p-1}(\tau) d\tau \leq C_1^p + p \left\| \frac{\varphi'}{a} \right\|_{L^p(1,t)} \left\| a\varphi^{p-1} \right\|_{L^{p'}(1,t)} \leq \\ \leq& C_1^p + p\varphi(t) \left\| a\varphi^{p-1} \right\|_{L^{p'}(0,t)} . \end{aligned}$$

Hence we have

$$\varphi(t)^p \leq C_1^p + p\varphi(t)\left(\int_1^t a(\tau)^{p/(p-1)}\varphi^p(\tau)d\tau\right)^{(p-1)/p}. \tag{C.0.9}$$

We can use the Young inequality

$$ab < \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

so setting

$$A(t) = p\left(\int_1^t a(\tau)^{p/(p-1)}\varphi^p(\tau)d\tau\right)^{(p-1)/p},$$

we can write

$$\varphi(t)^p \leq C_1^p + \varphi(t)A \leq C_1^p + \frac{\varphi(t)^p}{p} + \frac{A^{p'}}{p'},$$

so we get

$$\begin{aligned}\varphi(t)^p &\leq p'C_1^p + A^{p'} = \\ &= p'C_1^p + p^{p'}\left(\int_1^t a(\tau)^{p/(p-1)}\varphi^p(\tau)d\tau\right),\end{aligned}$$

and we are in position to apply Gronwall's inequality and to derive the following estimate

$$\varphi^p(t) \leq p'C_1^p \exp\left(p^{p'}\int_1^t a^{p/(p-1)}(\tau)d\tau\right). \tag{C.0.10}$$

The line below

$$\varphi(t) \leq C_1\left(\frac{p}{p-1}\right)^{1/p} \exp\left(p^{1/(p-1)}\int_1^t a^{p/(p-1)}(\tau)d\tau\right), \tag{C.0.11}$$

and the inequality (C.0.7) completes the proof for the case $q = 1$. For the case $1 < q < p < \infty$ we can set

$$\varphi(t) = C_1 + \int_1^t a(\tau)v(\tau)^q d\tau$$

and repeating the above argument, we arrive at (C.0.6) for the case $1 < q < p < \infty$. Finally, the case $1 < q < p = \infty$ is well-known (see [73]) and can be reduced to the classical $L^\infty - L^1$ Gronwall estimate by the aid of the same transform $v^q(t) = u(t)$. This completes the proof of the Lemma. □

We show an interesting application of the $L^p - L^q$ Gronwall's Lemma that provide a control of the L^∞ norm of the solution of the Schrödinger equation.

Corollary C.0.4. *Let $\psi_1 = \psi(1,\cdot) \in L^2(\mathbb{R})$ and suppose V a potential continuing to support the* *assumptions made in Chapter 4. Then the unique solution of the integral equation*

$$\psi(t) = e^{-i(t-1)\mathcal{H}}\psi_1 \mp i \int_1^t e^{-i(t-s)\mathcal{H}}\psi(s)|\psi(s)|^2 ds, \ t > 1, \tag{C.0.12}$$

$\psi(t,x) \in C([1,\infty); L^2(\mathbb{R})) \cap L^4([1,\infty); L^\infty(\mathbb{R}))$ *satisfies the following control estimate*

$$\left(\int_1^t \|\psi(s)\|^4_{L^\infty(\mathbb{R})} ds\right)^{1/4} \leq C_1 \|\psi_1\|_{L^2(\mathbb{R})} \exp\left(C_2 \|\psi_1\|^3_{L^2(\mathbb{R})} t\right), \tag{C.0.13}$$

for some positive constants C_1, C_2.

Proof. The local and global existence in the Banach space $C([1,\infty); L^2(\mathbb{R})) \cap L^4([1,\infty); L^\infty(\mathbb{R}))$ can be derived using a fixed point argument in $L^\infty([1,\infty); L^2(\mathbb{R})) \cap L^4([1,\infty); L^\infty(\mathbb{R}))$ and Strichartz estimates (one can see Appendix B). Applying the Strichartz estimates to the integral equation (C.0.12) we get

$$\|\psi\|_{L^4((1,\tau);L^\infty(\mathbb{R}))} \leq C\|\psi_1\|_{L^2(\mathbb{R})} + C\|\psi_1\|^2_{L^2(\mathbb{R})} \|\psi\|_{L^{4/3}((1,\tau);L^\infty(\mathbb{R}))},$$

for any $\tau > 1$. Setting

$$\chi(t) = \|\psi(t)\|_{L^\infty(\mathbb{R})}$$

we rewrite the inequality above as follows

$$\left(\int_1^t \chi(s)^4 ds\right)^{\frac{1}{4}\cdot\frac{4}{3}} \leq C\|\psi_1\|^{4/3}_{L^2(\mathbb{R})} + \int_1^t (C\|\psi_1\|^2_{L^2(\mathbb{R})}\chi(s))^{4/3} ds.$$

Using the $L^p - L^q$ Gronwall estimate of Lemma C.0.3 we complete the proof of the Corollary. □

References

[1] G. Artbazar, K. Yajima, *The L^p-continuity of wave operators for one dimensional Schrödinger operators*, J. Math. Sci. Univ. Tokyo **7(2)** : 221 – 240, (2000).

[2] H. Bahouri, J. - Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften, 343, Springer, (2011).

[3] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math. **10** : 41 – 437 (1960).

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** : 133 – 181, (1922).

[5] J. E. Barab, *Nonexistence of asymptotic free solutions for a nonlinear equation*, J. Math. Phys. **25** : 3270 – 3273, (1984).

[6] F. A. Berezin, M. A. Shubin, *The Schrödinger Equation*, Dordrecht, Boston: Kluwer Academic Publishers, (1991).

[7] J. Bergh, J. Löfström, *Interpolation Spaces*, Springer, Berlin, (1976).

[8] H. Brezis, F. Browder, *Partial Dierential Equations in the 20th Century*, Advances in Mathematics, **135** : 76 – 144, (1998).

[9] F. Cajori, *The Early History of Partial Differential Equations and of Partial Differentiation and Integration*, The American Mathematical Monthly, **35(9)** : 459 – 467, (1928).

[10] R. Carles, *On the Cauchy problem in Sobolev spaces for nonlinear Schrödinger equations with potential*, Portugal. Math. (N. S.), **65** : 191 – 209, (2008).

[11] R. Carles, *Sharp weights in the Cauchy problem for nonlinear Schrödinger equations with potential*, Z. Angew. Math. Phys. **66(4)** : 2087 − 2094, (2015).

[12] R. Carles, C. Gallo, *Scattering for the nonlinear Schrödinger equation with a general one-dimensional confinement*, J. Math. Phys. **56(10)**, 101503, (2015).

[13] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**, American Mathematical Society, Providence, (2003).

[14] T. Cazenave, F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in* H^s, Nonlinear Anal. **14(10)** : 807 − 836, (1990).

[15] M. Christ, A. Kieselev, *One-dimensional Schrödinger operators with slowly decaying potentials: spectra and asymptotics (or: Baby Fourier analysis meets toy quantum mechanics),* Lecture notes for the University of Arkansas, Spring lecture series (2002).

[16] E. Cordero, D. Zucco, *Strichartz Estimates for the Schrdinger Equation*, CUBO A Mathematical Journal, **12(3)** : 213 − 239, (2010).

[17] S. Cuccagna, V. Georgiev, N. Visciglia, *Decay and scattering of small solutions of pure power NLS in* $\mathbb{R}$ *with* $p > 3$ *and with a potential*, Comm. Pure Appl. Math. **67**(**6**) : 957 − 981, (2014).

[18] P. D'Ancona, L. Fanelli, L^p*-Boundedness of the Wave Operator for the One Dimensional Schrödinger Operator*, Commun. Math. Phys. **268** : 415 − 438, (2006).

[19] P. Deift, E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** : 121 − 251, (1979).

[20] B. Erdogan, W. Schlag, *Dispersive estimates for Schrdinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three, II.* J. Anal. Math. **99**, 199 − 248, (2006).

[21] L. C. Evans, *Partial Differential Equations*, Second Edition, Graduate Studies in Mathematics, **19**, (2010).

[22] C. Fefferman, *Selected Theorems by Eli Stein. Essays on Fourier Analysis in Honor of Elias M. Stein* (PMS-42), 1 (2014).

[23] G. B. Folland, *Quantum Field Theory: A Touriste Guide for Mathematicians*, Mathematical surveys and monographs, AMS, **19**, (2008).

[24] V. Georgiev, A. R. Giammetta, *Sectorial Hamiltonians without zero resonance in one dimension,* Contemporary Mathematics, **666**, (2016).

[25] V.Georgiev, A. R. Giammetta, *On homogeneous Besov spaces for* 1*D Hamiltonians without zero resonance*, submitted, arXiv:1605.02581, (2016).

[26] V.Georgiev, A. R. Giammetta, *Hardy inequality and fractional Leibnitz rule for perturbed Hamiltonians on the line*, work in progress, (2016).

[27] V.Georgiev, A. R. Giammetta, A. Stefanov, *Scattering of the small solutions of the cubic NLS with short range potential*, work in progress, (2016).

[28] P. Germain, Z. Hani, S. Walsh, *Nonlinear resonances with a potential: Multilinear estimates and an application to NLS.* International Mathematics Research Notices, **18** : 8484 – 8544, (2015).

[29] V.Georgiev, N.Visciglia, *Decay estimates for the wave equation with potential*, Comm. Part. Diff. Eq. **28(7,8)** : 1325 – 1369, (2003).

[30] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations*, J. Funct. Anal., **32** : 1 – 71, (1979).

[31] J. Ginibre, G. Velo, *On the global Cauchy problem for some non linear Schrödinger equations*, Ann. Inst. H. Poincaré (C) Analyse non linéaire **1.4** : 309 – 323 (1984).

[32] J. Ginibre, J. Velo, *Generalized Strichartz Inequalities for the Wave Equation*, J. Funct. Anal., **133** : 50 – 68, (1995).

[33] M. Goldberg, W. Schlag, *Dispersive Estimates for Schrödinger Operators in Dimensions One and Three*, Commun. Math. Phys. **251** : 157 – 178, (2004).

[34] L. Grafakos, D. Maldonado, V. Naibo, *A remark on an endpoint Kato-Ponce inequality*, Differential Integral Equations, **27** : 415 – 424, (2014).

[35] L. Grafakos, S. Oh, *The Kato - Ponce inequality*, Comm. PDE., **39(6)** : 1128 – 1157, (2014).

[36] L. Grafakos, Z. Si, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angew. Math., **668** : 133 – 147, (2012).

[37] K. Gröchenig, A. Klotz, *Norm-controlled inversion in smooth Banach algebras*, I. J. Lond. Math. Soc. (**2**)**88** : 49 – 64, (2013).

[38] N. Hayashi, P. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. **120** : 369 − 389, (1998).

[39] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Series: Lecture Notes in Mathematics, **840**, (1981).

[40] P. D. Hislop, *Fundamentals of scattering theory and resonances in quantum mechanics*, **14 (3)** : 1 − 39, (2012).

[41] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, *vol. II Differential Operators with Constant Coefficients, Springer*, Berlin, (2005).

[42] A. Jensen, *Spectral properties of Schrodinger operators and time-decay of the wave functions*, Results un L2(R4) J. Math. Anal. Appl. **101** : 397 − 422, (1984).

[43] A. Jensen, T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. **46** : 583 − 611, (1979).

[44] J.-L. Journé, A. Soffer, C. D. Sogge, *Decay estimates for Schrödinger operators*, Comm. Pure Appl. Math. **44(5)** : 573 − 604, (1991).

[45] T. Kappeler, P. Perry, M. Shubin, P. Topalov, *The Miura transform on the line*, Int. Math. Res. Notices, **50** : 3091 − 3133, (2005).

[46] M. Karlsson, *Nonlinear propagation of optical pulses and beams*, Technical Report, Chalmers University of Technology, Göteborg, Sweden, **262**, (1994).

[47] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré, Phys. Théor. **46** : 113 − 129, (1987).

[48] T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure App. Math. **41** : 891 − 907, (1988).

[49] M. Keel, T. Tao, *Endpoint Strichartz Estimates.* Amer. J. Math. **120 (5)** : 955 − 980, (1998).

[50] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de-Vries equation via the contraction principle*, Comm. Pure App. Math. **46**(**4**), 527 − 620, (1993).

[51] K. Kirkpatrick, B. Schlein, G. Staffilani, *Derivation of the two dimensional nonlinear Schrödinger equation from many body quantum dynamics*, American Journal of Math., **133(1)** : 91 − 130, (2011).

[52] P. Lax, R. Philips, *Scattering theory*, Bull. Amer. Math. Soc. **70 (1)** : 130 – 142, (1964).

[53] H. Maassen, *Quantum Probability and Quantum Information Theory*, Springer Lecture Notes in Physics, **808**, (2010).

[54] H. McKean, J. Shatah, *The nonlinear Schrödinger equation and the nonlinear heat equation reduction to linear form.* Comm. Pure Appl. Math. **44** : 1067 – 1080, (1991).

[55] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. **139** : 479 – 493, (1991).

[56] J. Rauch, *Local decay of scattering solutions to Schrödinger equations*, Comm. Math. Phys. **61** : 149 – 168, (1978).

[57] J. Rauch, *Perturbation Theory for Eigenvalues and Resonances of Schrödinger Hamiltonians*, Journal Functional Analysis, **35** : 304 – 315, (1980).

[58] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol. I: Functional Analysis, Revised and Enlarged edition, (1980).

[59] T. Runst, *Mapping properties of non-linear operators in spaces of Triebel - Lizorkin and Besov type*, Analysis Mathematica , **12** : 313 – 346, (1986).

[60] T. Runst, W. Sinckel, *Sobolev Spaces of Fractional Order, Nemitskij Operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, (1996).

[61] S. Salsa, *Partial Differential Equations in Action: From Modelling to Theory*, Springer, (2008).

[62] W. Schlag, *Dispersive estimates for Schrödinger operators: A survey*, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, **163** : 255 – 285, (2007).

[63] G. Staffilani, *Dispersive equations: A survey*, MSRI, Women in mathematics: May 18 – 20, (2006).

[64] W. Strauss, *Nonlinear scattering theory*, Scattering Theory in Mathematical Physics, 53 – 78, (1974).

[65] R. S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations.* Duke Math. J., **44** : 705 – 774, (1977).

[66] C. Sulem, P. L. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, Springer-Verlag, New York, **139**, (1999).

[67] T. Tao, *Nonlinear Dispersive Equations - Local an Global Analysis*, CBMS, AMS, **106**, (2006).

[68] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrdinger Operators,* Second Edition, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, **99**, (2009).

[69] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel-Boston-Stuttgart, (1983).

[70] Y. Tsutsumi, L^2 *- solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcialaj Ekvacioj, **30** : $115-125$, (1987).

[71] R. Weder, *The* $W_{k;p}$ *- Continuity of the Schrödinger Wave Operators on the Line*, Commun. Math. Phys. **208** : $507-520$, (1999).

[72] R. Weder, $L^p - L^{p'}$ *Estimates for the Schrödinger Equation on the Line and Inverse Scattering for the Nonlinear Schrödinger Equation with a Potential*, Journal of Functional Analysis **170** : $37-68$, (2000).

[73] D. Willet, J. Wong, *On the discrete analogues of some generalisation of Gronwall's inequality*, Monat. sh. Math., **69** : $362-367$, (1965).

[74] K. Yajima, *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators,* J. Math. Soc. Japan., **47** : $551-581$, (1995).

[75] K. Yajima, *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators III, even dimensional cases* $m \geq 4$, J. Math. Sci. Univ. Tokyo **2** : $311-346$, (1995).

[76] K. Yajima, L^p*-boundedness of wave operators for two-dimensional Schrödinger operators*, Comm. Math. Phys. **208** : $125-152$, (1999).

[77] K. Yajima, *Dispersive estimate for Schrödinger equations with threshold resonance and eigenvalue*, Comm. Math. Phys. **259** : $475-509$ (2005).

[78] Q. S. Zhang, *Large Time Behavior of Schrödinger Heat Kernels and Applications* Commun. Math. Phys. **210** : $371-398$, (2000).

[79] A. Zheng, *Spectral multipliers for Schrödinger operators,* Illinois Journal of Mathematics, **54 (2)** : $621-647$, (2010).

[80] M. Zworski, *Resonances in physics and geometry*, Notices of Amer. Math. Soc. **46 (3)** : 319 – 328, (1999).