Essays on Realized Covariance Estimation

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Preface

Allocation of assets in financial markets is a trade-off between expected gain and potential risks, which should be estimated before making an investment decision. Other financial activities, in as far as they are associated with investing, also involve risk estimation: risk-management, trading, hedging, option pricing, business valuation, etc.

A first approximation of portfolio risk is a variance of its return. Even if distribution of returns is not Normal, but is some fat-tailed distribution, variance of the portfolio may be considered as a scaling parameter of this distribution. Variance, and consequently, risk of the portfolio may be decreased through diversification by mixing different asset types, securities of the firms operating in different industries across different countries of the world. Volatility of the portfolio is a function of covariance matrix of assets, included in the portfolio. Correlation terms may completely change the estimation of volatility and benefits of the diversification.

Estimation of covariance matrix, historically performed on daily data of close prices on stock exchanges, became more precise with availability of high-frequency intraday data. Forecasting models, based on high-frequency data, such as HAR, provide better forecasts than models based on daily data.

However, high-frequency data is observable only when trading occurs. During the overnight periods, when stock exchanges are closed, for the most of the assets high-frequency data is not available. So, for each assets data is available for only about eight hours per each day, excluding holidays and weekends. At the same time, prices of assets change even during the nights and weekends, resulting in close-to-open overnight returns.

If all the assets are traded at one and the same stock exchange, there exist about 8-hours periods of high-frequency data to estimate the intraday realized covariance matrix, and only overnight covariance should be estimated with different methods. However, the largest stock exchanges are located throughout the world: in USA, UK, Japan and other countries. Due to the difference in time zones, these eight hours are different for each assets, leading to even smaller overlapping period, at which realized covariance of assets can be estimated.

The present dissertation develops methods of the estimation of the whole day covariance matrices based on available intraday high-frequency data.

In the first chapter, existing literature is reviewed. In empirically-oriented literature, it is argued that overnight returns appear to follow different price processes than returns within the trading day. In particular, overnight returns have lower volatility, higher Sharp ratios, higher tail risk, and higher correlations. Moreover, according to some studies, intraday and overnight volatilities mutually influence each other. Methodological literature proposes a number of models for univariate overnight realized volatility estimations, as well as correction procedures for covariance estimation in asynchronous markets based on daily data. Given the variety of existing approaches, it should be stated that there is an absence of high-frequency based methods for a whole day covariance estimation (except for very naive and noisy methods), both for synchronous and asynchronous case.

Chapter 2 and 3 of the present work fill this gap. In the second chapter a new concept of linear algebra is defined: proportionality of positive semi-definite symmetric matrices. It is based on existing concept of geometric means of the matrices. Direct proportionality function $\mathbf{Y}(\mathbf{X})$ is defined as:

$$\mathbf{Y}(\mathbf{X}) = \mathbf{SXS} \tag{1}$$

where $\mathbf{S}$ is some positive semi-definite symmetric matrix, called 'scaling' matrix.

The second chapter deals with the problem of estimation of the whole day covariance, based on intraday realized covariance matrix and a vector of overnight return. Assuming that overnight covariance is 'proportional' (in a sense of equation 1) to the intraday covariance, two conditionally unbiased estimators are obtained: one based only on intraday realized covariance, and the other one based only on overnight returns. They are weighted in a way, allowing to decrease the noise of the resulting estimator.

The third chapter is devoted to covariance in asynchronous markets. In this case, volatilities are estimated using regular methods, while the whole day correlation is assumed to be a function of realized correlation during overlapping period. A whole day correlation is obtained from 'rescaled' (in the sense of equation 1) covariance of overlapping period. A time series of whole day covariance matrices was used as input to the forecasting models. A bivariate extension of HAR model was proposed and compared with other multivariate HAR specification and EWMA model.

Both estimators as well as forecasting model were tested on real high-frequency data, and show a higher degree of precision then existing approaches.

The main scientific novelty of the dissertation is as follows:

- matrix proportionality function was defined, and its properties were examined,
- multivariate extension of Hansen & Lunde (2005) estimator was proposed,
- this extension was adapted for the case of asynchronous markets,
- new bivariate extension of HAR model was suggested, that takes into account both volatility spillovers and correlation-volatility dependence.

Methods proposed in the dissertation have high potential of practical application both by researchers and practitioners. In as far as the developed methods allow more precise estimation of the benefits of international diversification, they will contribute to increasing efficiency of risk-management, option pricing, asset allocation, and trading. Methods developed in this dissertation allow researchers to obtain time series of the whole realized covariance matrices: on the basis of these time series, they can test different empirical hypothesis of volatility structure, invent new forecasting models and back-test them.

CHAPTER 1

Overnight period in financial markets: survey of empirical researches and statistical methods

The present paper is a review of the literature devoted to statistical properties of overnight returns in financial markets. Particular attention was given to properties and estimators of variance and covariance overnight return. Additionally was studied the problem that arises due to the asynchronicity of overnight periods of stock exchanges located in different time zones.

Keywords:

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1.1 Introduction

Prices of financial assets are observed due to the trading in the stock exchanges. However, the true price process continues even when stock exchanges are closed: that is why open prices are usually different from the previous day close prices.

It it not necessary to assume that the price process is the same during the trading hours and overnight period: properties of the overnight returns are different from observed during the trading hours. This difference may be a consequence of both the fundamental differences in the latent true price processes, as well as the trading noise during the day, which is absent at overnight period. Consequently, the study of statistical properties of overnight returns will lead to better understanding of the asset pricing mechanism also during the intraday period.

The difference between intraday and overnight periods is especially important from the practical point of view: for many practical applications, such as risk-management, portfolio management, option pricing, it is necessary to estimate the volatility of an asset or variance-covariance matrix of a portfolio. There exist a lot of methods to compute realized variance or covariance matrix during trading hours (see McAleer & Medeiros (2008) for a review of realized volatility estimators). Absence of high-frequency data during non-trading period makes impossible estimation of overnight realized volatility, and consequently, creates handicaps for risk-management. Consequently, various special methods are used to estimate the overnight and whole day volatility. Especially notable situation arises when overnight periods on different stock exchanges do not concur. In this case even estimation of covariance based on daily data provides biased result.

This paper presents a review of the literature connected to the particular features of overnight returns. Special interest of the paper is an impact of these features on the volatility and covariance estimation.

In this paper following notations are used. Daily return r_t is a sum of overnight return $r_{on,t}$ and intraday return $r_{id,t}$. Their variances are σ_t^2, $\sigma_{on,t}^2$ and $\sigma_{id,t}^2$ respectively.

The remainder of the paper is organized as follows. Section 1.2 describes stylized facts on overnight return. Section 1.3 is devoted to the overnight variance and covariance estimation. Section 1.4 presents estimators of covariance in asynchronous markets. Section 1.5 is a conclusion.

1.2 Stylized facts about overnight return

1.2.1 Lower volatility during overnight

The fact that intraday and overnight prices follow different processes becomes clear by just comparing the variances of overnight and intraday returns. Not only overnight volatility is several times lower in absolute terms. Overnigth-to-intraday volatility ratio is even smaller in per hour terms, so, a simple "$\sqrt{t}$-rule" can never be applied to overnight volatility estimation. One of the early mention of low per hour overnight volatility was done in the work of Fama (1965), where he mentioned, that volatility of weekend return is only 22% higher than one of weekdays returns, while covers three times larger chronological time.

French & Roll (1986) proposed three possible explanations of this phenomenon: (1) arrival of public information is higher during the normal business hours, (2) private information arrives to market through trading only when it is open, (3) noisy traders induce market volatility. They used data of prices of all common stocks listed at NYSE and ASE (1963-1982). The first test was a volatility at the election days: at this day stock exchanges are closed, but the companies are open. Overnight volatility under the (1) hypothesis should not be affected at all, while under (2) hypothesis it should be postponed for the next trading day, at hypothesis (3) the decrease of the volatility does not move to the next days. It turned out that during election the variance ratio is slightly higher than during holiday (1.165 and 1.145 respectively), while weekly volatility ratio increases from 0.559 to 0.614. The fact that volatility is still smaller than during normal business day rejects public information hypothesis. While the difference between the volatility ratio supports the private hypothesis, it is not high hight enough to reject the noise hypothesis. The second test included autocorrelations: the first order positive autocorrelation supports the noise (mispricing) hypothesis. However, it does not totally explain the difference between overnight and intraday volatility. So, the authors suggested to interpret this difference as a cause of different factors.

The lower overnight volatility is combined with larger average overnight return. At the first approximation it means, that night holding provides higher return-to-risk ratio.

Cooper *et al.* (2008) used data for all the stocks included in S&P-500 index, 14 ETF, 44 stocks of AMEX (1993-2006). They have found out, that mean of overnight return is significantly positive, while intraday one is slightly negative. The return of the first trading hour is negative, while return at the last trading hour is positive, and these returns are significant. As this effect remains throughout all stock exchanges, it is not linked to the opening rule (see section 1.2.5).

This difference is not linked to information announcements, that happen mainly during overnight period: exclusion of earning announcements did not change the results. Neither it is explained by liquidity effects: for 2 out of 6 liquidity measures, difference between night and day return is larger for more liquid stocks. Finally, they regress the difference between overnight and intraday return by a set of explanatory variables: dummy for trading at Nasdaq, market capitalization, Amihud (2002) illiquidity measure, volume of transactions, average trade size, bid-ask spread, and standard deviation. Constant appeared significantly positive, showing that neither liquidity, nor market risk explain the overnight premium.

Similar research was performed by **Kelly & Clark (2011)**. They focused on ETF data (DIA, IWM, MDY, QQQQ, SPY, 1996-2006). To decrease the effect of opening price bias (see section 1.2.5), they used 5-min. VWAP prices. After correction of overnight returns for risk-free rate, Sharp ratio (mean-to-standard-deviation) was shown to be significantly higher during overnight than intraday period for each of the ETF. The result holds after fitting AR-GARCH(1,1) model with residuals follwing Student-t-skewed distribution, and calculation of the Sharp ratio dividing expected return on expected volatility. Overnight SR is significantly positive for all ETF, while intraday SR is negative for all, but significant only for one ETF.

However, it was shown, that applying a long overnight - short intraday trading strategy (correction for transaction costs, overnight interest rates and dividends was made), does not overperform buy-and-hold strategy for 4 out of 5 ETFs. Even with zero transaction costs two ETFs show better buy-and-hold return.

1.2.2 Higher tail risk during overnight

One of the possible explanation of high Sharp ratio may be in the wrong definition of risk. The fact of lower overnight volatility, does not yet mean that risk during overnight period is lower: kurtosis of overnight return is higher than kurtosis of intraday return. So, defining risk not as a volatility, but as a tail risk, may explain the overnight risk premium.

Riedel & Wagner (2015), used data for NASDAQ Composite, CAC 40, DAX 30 (1988-2010) and TOPIX (1990-2010) indices. They have confirmed that overnight volatility is smaller than intraday one, while kurtosis is higher. Fitting AR(2)-TGARCH(1,1,1) for overnight and intraday returns separately, they have shown, that overnight volatility especially depends upon the previous squared overnight return (the corresponding coefficient tends to be three times higher for overnight volatility than for intraday volatility) (see section 1.2.6).

They assumed, that return exceedances (upper and lower tails separately) have Generalized Pareto Distribution.

$$\bar{G}_{\xi,\beta}(y) = \begin{cases} (1 + \xi \frac{y}{\beta})^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\ e^{-\frac{y}{\beta}} & \text{if } \xi = 0 \end{cases} \tag{1.1}$$

Testing the equality of ξ to zero, they have shown that for intraday returns ξ is not significantly different from zero, so, it is possible to explain the exceedances by simple normal distribution with conditional volatility. At the same time for 7 out of 8 overnight returns this coefficient was significantly different from zero (the remaining is an upper tail of CAC). As a robustness check, they have performed test of equality of ξ coefficients for overnight and intraday returns, as well as overnight and whole day returns. It turnes out that for all assets overnight, intraday and whole day tail indices are significantly different. As a practical application they used VaR forecast. They have shown, that overnight tail risk is higher than intraday one starting from 2.5% VaR level in relative terms and for 0.1% even in absolute terms.

Consequently, there exists such a risk measure (in this case 0.1% VaR), for which overnight risk is higher than intraday one.

1.2.3 Higher correlation during overnight

Another difference between intraday and overnight is the increase of correlation between stock return during non-trading period. Increase of correlation decreases benefits of diversification, and consequently increases portfolio risk.

First study of this fact was done by **Pandey (2003)**. Using data of 30 stocks included in Sensex index of Mumbai Stock Exchange (1997-2001), he has shown, that correlation is higher during the overnight period for 425 out of 435 pairs of stocks.

In accordance with Boyer *et al.* (1997) theorem, conditional correlation should be lower when the volatility is lower, even if unconditional correlation remains the same. However, as the overnight volatility is lower, overnight increase in correlation can not be considered spurious.

He provided a possible explanation of this phenomenon: overnight information is more probably related to the common factors, while intraday trading is more driven by idiosyncratic (firm-specific) factors.

To test whether correlation increase during high-volatility period, the sample was sorted by Sensex squared daily return. It turned out that for all pairs of stocks (except one) both intraday and overnight correlations are higher during the period of higher volatility.

To prove that intraday and overnight volatilities are driven by different processes, GARCH(1,1) model with an overnight dummy was fitted.

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma I_{on} \tag{1.2}$$

After exclusion of 3 stocks with missing data, and consequent estimation of the equation, it turns out that γ coefficient is significantly negative for all the stocks. It confirms, that overnight volatility is lower than intraday one.

After fitting Bivariate GARCH-BEKK model for pair of stocks returns, for overnight and intraday returns separately, it was confirmed that average conditional correlation is higher for overnight period than for intraday one.

1.2.4 Opposite facts for ADR

These stylized facts, however, apply only for the original stocks. If take into consideration cross-listed depositary rights on this stocks, it turns out, that overnight and intraday periods are switching, as an intraday period for ADR is an overnight period for an original stock.

On a data of ADR prices on companies with highest capitalization in China, Japan, South Korea, Taiwan and India (2004-2014), **Leung & Kang (2016)** have shown, that non-overlapping ADR have higher overnight variance than intraday one. They have found that distribution of $q = \frac{r_{on}^2}{r_{on}^2 + r_{id}^2}$ is U-shaped for the most of the ADR. It was also shown, that overnight correlation of ADRs with SPDR S&P 500 ETF (SPY) is higher than intraday one. As a practical application they constructed a portfolio based on the fact that return spread between ADRs and SPY is mean-reverting.

This inverse of stylized facts may be considered as one of the counter-argument against noisy trading hypothesis of French & Roll (1986), and an argument for the public information hypothesis. Under the private information hypothesis, this fact shows, that there are more informed traders in the country of stock origin.

1.2.5 Overnight ends several minutes after opening

Opening price is not always a price, at which everyone can buy the stock, but frequently it has only an indicative sense. Tsiakas (2008) has compared different methods of determining opening price and have shown, that overnight-to-intraday volatility ratio depends upon the opening rule of the stock exchange. For this reason, for example, Ahoniemi & Lanne (2013) used either the price 5 minutes after opening, or a Special Opening Quote as an opening price.

A more detailed study of this effect was performed by **Heaton *et al.* (2011)**. They have shown that opening price in Australian market does not fully reflect overnight information. Due to asynchronous trading, returns of American indices should explain overnight return of Australian index, but not intraday one. They estimated the following equation:

$$lnS_{t,i} - lnS_{t,i-1} = \alpha_i + \beta_i' X_{t-1} + \epsilon_{t,i} \tag{1.3}$$

where the first term is a return of ASX for the period from time $i - 1$ to i at day t and X_{t-1} is a vector of returns of commodity indices and S&P-500. 'Full absorption time' i^* was defined as the first i for which β_i is non-significant. They used 15-minutes returns of ASX energy, industrial, materials and market-wide indices as dependent variables and corresponding commodities indices traded in UK and USA, as well as S&P-500 as explanatory variables. It was shown, that overnight information is absorbed at the first 15 minutes after the market is open. Consequently, they made possible estimation of the total overnight effect of international commodity prices.

1.2.6 Relation of overnight and intraday volatilities

Properties of overnight and intraday return of one and the same stocks are interconnected.

Wang *et al.* (2009) studied stylized facts on overnight returns using data of all 2215 stock traded at NYSE at the end of 2007 (1988-2007, 1000 to 5000 observations for each stock). As a contribution to the previous studies, they have shown how intraday and overnight statistical properties are linked across different stocks.

First, they have studied a tail distribution. Assuming power law, exponential and power law with exponential cut-off, they have performed goodness of fit test by Kolmogorov-Smirnov. For overnight return, exponential distribution was rejected for three times more stocks than for intraday returns. At the CDF plots overnight tails almost always decayed slower than intraday tails. They have also shown that intraday tail exponent (coefficient of the power law distribution) positively correlated with the whole day one, and this correlation is stronger than for an overnight tail exponent.

By fitting the fluctuation function of detrended returns $F(\epsilon) = \epsilon^{\alpha}$, they have shown absence of autocorrelation in the residuals ($\alpha \approx 0.5$), but a strong positive autocorrelation in short term absolute returns (average $\bar{\alpha} = 0.59$ for overnight and $\bar{\alpha} = 0.63$ for intraday) and long term absolute returns ($\bar{\alpha} = 0.71$ and $\bar{\alpha} = 0.75$ for overnight and intraday respectively). It shows the slightly higher persistence of intraday volatility. It turns out that autocorrelation coefficients of daily, overnight and intraday absolute returns (a measure of volatility persistence) are positively correlated (seeing each separate stock as an observation).

Finally, they have shown that overnight and intraday returns are negatively correlated, however, this difference is not significant. Correlation between intraday, overnight and the whole day return remained stable for the whole period of study.

So, the study confirmed stylized fact of overnight tail risk, found the positive correlation between intraday and overnight volatility persistence, and stability of cross correlation coefficients between overnight, intraday and the whole day returns.

Two periods GARCH models

This fact of mutual influence of intraday and overnight returns on each other is used in construction of two periods volatility forecasting model - separate forecast of intraday and overnight volatility.

Gallo (2001) estimated the influence of overnight return on the intraday volatility. Using the data on 20 large caps stocks from NYSE (1994-1998, 1235 observations) the following equation was estimated:

$$\frac{\zeta_t^2}{h_t} = \phi_0 + \phi_1\eta_t + \phi_2\eta_t I_{\eta_t<0} + \phi_3\eta_t^2 + u_t \tag{1.4}$$

where ζ_t are residuals of equation $r_{id,t} = a + br_{on,t} + \zeta_t$, η_t are residuals of $r_{on,t} = c + dr_{id,t-1} + \eta_t$ and h_t is a volatility of ζ_t estimated using simple GARCH(1,1) model. It turns out, that for 16 out of 20 stocks F-test shows that overnight information significantly influence the intraday variance.

So, a new version of GARCH was estimated, that takes into account overnight returns.

$$\sigma_{id,t}^2 = \omega + \alpha_1\zeta_{t-1}^2 + \beta_1\sigma_{t-1} + \gamma\zeta_{t-1}^2 I_{\zeta_{t-1}<0} + \phi\eta_t^2 \tag{1.5}$$

It turnes out, that for 14 stocks coefficient ϕ (accounting for overnight surprises) is significantly different from zero. Additional result was that negative overnight return do not have significant impact on intraday volatility.

An out-of-sample forecast (testing sample: 2.10.1998-7.1.1999) exercise has shown that Mean Absolute Error is lower in the GARCH specifications that includes η_t^2 for all 200 assets, while RMSE is lower for GARCH models without this term in 17 cases.

Tsiakas (2008) distinguished influence of four types of overnight return on the daily volatility: weeknight (stock exchange opens the next day after closure), holiday (one day after), weekend (two days after) and long weekend (three days after).

Using data for FTSE-100, CAC-40, DAX-30, DJIA, S&P-500, NASDAQ-100 indices (2000-2004), he has shown that overnight volatility ratios depend upon the opening procedure of the stock exchange and range from 3% (DJIA) to 41% (CAC-40) (see section 1.2.5).

He proposed the model of stochastic intraday volatility, that takes into account overnight return, type of preceding overnight period, as well as leverage effects.

$$\sigma^2_{id,t} = \mu + \gamma_D r_{id,t-1} + \gamma_D^a |r_{id,t-1}| + \gamma_N r_{on,t-1} + \phi(\sigma^2_{id,t-1} - \mu) + \epsilon_t \quad (1.6)$$

$$\begin{aligned} \gamma_N r_{on,t-1} = {} & (\gamma_{wn} I_{wn} + \gamma_{we} I_{we} + \gamma_{hol} I_{hol} + \gamma_{lw} I_{lw}) r_{on,t-1} + \\ & + (\gamma^a_{wn} I_{wn} + \gamma^a_{we} I_{we} + \gamma^a_{hol} I_{hol} + \gamma_{lw} I_{lw}) |r_{on,t-1}| \end{aligned} \quad (1.7)$$

where I_i is a dummy variable for the type of overnight period. The equation was estimated using Monte-Carlo-Markov-Chain algorithm.

It turned out, that overnight information increases the predictive performance of both intraday return and volatility models. Distinguishing between different types of overnight period leads to even more superior results, especially when taking into account leverage effect. Consequently the general model outperform more parsimonious specification both in-sample and at out-of-sample exercises at every stock exchange.

Additional finding was, that intraday returns have negative autocorrelation, so the observed positive daily autocorrelation is an effect of overnight positive autocorrelation.

Kang & Babbs (2012) presented a multivariate joint modelling of overnight and intraday returns. Mean of the returns of each asset follows autoregressive model:

$$r_{on,t} = \alpha_0 + \alpha_1 r_{id,t-1} + \alpha_2 r_{on,t-1} + \eta_{on,t} \quad (1.8)$$

$$r_{id,t} = \beta_0 + \beta_1 r_{on,t} + \beta_2 r_{id,t-1} + \eta_{id,t} \quad (1.9)$$

Overnight and intraday volatilities of the were modelled as GARCH process:

$$\sigma^2_{on,t} = \theta_0 + \theta_1 \eta^2_{id,t-1} + \theta_2 \eta^2_{on,t-1} + \theta_3 \sigma^2_{on,t-1} \quad (1.10)$$

$$\sigma^2_{id,t} = \delta_0 + \delta_1 \eta^2_{on,t} + \delta_2 \eta^2_{id,t-1} + \delta_3 \sigma^2_{id,t-1} \tag{1.11}$$

Residuals of the model are assumed to have standardized Student's t-distribution, linked with two Student's copulas: for overnight and intraday returns separately. Correlation matrices of both copulas follow DCC dynamics.

$$Q_{n,t} = \Pi_0 + \pi_1(\zeta_{cd,t-1}\zeta'_{cd,t-1}) + \pi_2(\zeta_{cn,t-1}\zeta'_{cn,t-1}) + \pi_3 Q_{d,t-1} + \pi_4 Q_{n,t-1} \tag{1.12}$$

$$Q_{d,t} = \Psi_0 + \psi_1(\zeta_{cn,t}\zeta'_{cn,t}) + \psi_2(\zeta_{cd,t-1}\zeta'_{cd,t-1}) + \psi_3 Q_{n,t} + \psi_4 Q_{d,t-1} \tag{1.13}$$

where ζ is a vector of realized quantiles of t-distribution. They have tested their model on the data of returns of 9 S&PDR ETF investing in different industries and as well as 6 currency and commodities funds. For most funds they have found that intraday returns have non-significant constant, but significant negative dependence of intraday return on overnight return. Overnight returns was shown to have lower degrees of freedom (supporting heavy-tails stylized fact, see section 1.2.2). Intraday volatility for 13 out of 15 funds is larger than overnight volatility. In constant conditional correlation model, intraday copulas have higher degrees-of-freedom parameter and lower correlations than overnight copulas. Dynamic conditional correlation have higher likelihood; it turns out that overnight correlation influence next day's correlation, but depends mostly upon previous overnight correlations. The model was applied to VaR, ES forecast as well as optimal portfolio allocation for a CRRA utility function.

Asymmetric influence of overnight and intraday volatility

In two periods GARCH models overnight and intraday volatility usually assumed to have symmetric influence on each other. However, it was shown not to be true.

Blanc *et al.* (2014) studied how intraday and overnight volatilities influence on each other. They estimated GARCH model, in which coefficients are the power-law functions of the lag number.

$$\begin{aligned}\sigma^2_{id,t} = s^2_d &+ \sum_{\tau=1}^{\infty} L_1(\tau) r_{id,t-\tau} + \sum_{\tau=1}^{\infty} K_1(\tau) r^2_{id,t-\tau} + 2\sum_{\tau=1}^{\infty} K_2(\tau) r_{id,t-\tau} r_{on,t-\tau} + \\ &+ \sum_{\tau=1}^{\infty} L_2(\tau) r_{on,t-\tau} + \sum_{\tau=1}^{\infty} K_3(\tau) r^2_{on,t-\tau} + 2\sum_{\tau=1}^{\infty} K_4(\tau+1) r_{id,t-\tau-1} r_{on,t-\tau}\end{aligned} \tag{1.14}$$

$$\begin{aligned}\sigma^2_{on,t} = s^2_n &+ \sum_{\tau=1}^{\infty} L_3(\tau) r_{on,t-\tau} + \sum_{\tau=1}^{\infty} K_5(\tau) r^2_{on,t-\tau} + 2\sum_{\tau=1}^{\infty} K_6(\tau) r_{id,t-\tau} r_{on,t-\tau} + \\ &+ \sum_{\tau=1}^{\infty} L_4(\tau) r_{id,t-\tau} + \sum_{\tau=1}^{\infty} K_7(\tau) r^2_{id,t-\tau} + 2\sum_{\tau=1}^{\infty} K_8(\tau) r_{id,t-\tau-1} r_{on,t-\tau}\end{aligned} \tag{1.15}$$

where $K_i(\tau)$ are exponentially truncated power-law kernel functions and $L_i(\tau)$ are simple exponential kernel functions.

The model was fitted to the data for normalized and deseasonalized returns of 280 stocks included in S&P-500 index (2000-2009, 2515 observations), assuming universal dynamic for all the stocks (the same coefficients of kernel functions for every asset).

It turns out, that overnight influence on the intraday volatility decays quicker than overnight to overnight and intraday to both intraday and overnight. Practically it means, that only the very last square overnight return influence the intraday volatility.

They also have shown that the proposed predictive model provides higher sum of pointwise log-likelihoods than plain ARCH, both on in-sample and out-of-sample tests.

It was also shown the presence of U-shaped weekly seasonality of overnight volatility: weekend and Thursday-to-Friday volatilities are higher than ones in the middle of the week. Also the fact that overnight returns have higher kurtosis (even after fitting the model) than intraday returns was confirmed.

1.3 Overnight volatility estimators

1.3.1 Estimators based on high-frequency intraday data

Adding squared overnight return

There exist many methods measuring intraday realized volatility. However, due to absence of high-frequency data during overnight period, it is not trivial to measure the whole day realized covariance.

The simplest way is to treat overnight return as just one regular observation and add a squared return to the intraday realized volatility.

$$\sigma_t^2 = \hat{\sigma}_{id,t}^2 + r_{on,t}^2 \tag{1.16}$$

where $\hat{\sigma}_{id,t}^2$ is a realized volatility of intraday period and $r_{on,t}$ is an overnight return.

The first time such estimator was introduced by **Blair *et al.* (2001)**. The target was to compare predictive abilities of ARCH model, implied volatility and realized volatility. They used data for daily returns, 5-min. returns and implied volatilities (VIX) of S&P-100 index. In-sample period was 1987-1992 (1519 observations), an out-of-sample period 1993-1999 (1768 observations).

Compared predictive models were: linear combination of the three estimators, and these estimators separately. At in-sample analysis it turned out, that implied volatility is the major explanatory variable, while realized volatility adds only a small increment. On out of sample forecast of 1, 5, 10 and 20 days horizon, it was shown, that even while realized volatility increases 1 day ahead prediction, at larger horizons VIX fully explain the variance.

Also, the same realized covariance estimator was used in the paper of **Becker *et al.* (2007)**, who studied the same question as Blair *et al.* (2001). They used data for returns and volatility indices of S&P-500 (1990-2003, 3481 observations). Unlike the previous study they have compared VIX with different specifications of GARCH, Stochastic volatility, ARMA-Realized volatlity models. Fitting linear regression of volatility forecasts as independent variables and VIX values as dependent variable, they received residuals (after exclusion of non-significant variables) - a part of VIX not explainable by volatility forecasting models. It turns out, that regression of residual VIX by 1, 5, 10, 15, 22-days ahead realized volatility is not significant neither by F-test nor by Hotelling test. It means that volatility index does not provide any more information than volatility forecasting models.

In the multivariate framework treating overnight return simply as regular observation will add an outer product of overnight return to the intraday realized covariance matrix.

$$V_t = \hat{V}_{id,t} + r_{on,t} r'_{on,t} \tag{1.17}$$

This estimator at first was used by **de Pooter *et al.* (2008)**. They discussed the optimal sampling frequency for the intraday realized covariance estimation at the example of 100 stocks included in S&P-100 index (1997-2004). They have shown that optimal decay parameter of EWMA model increases with the sampling frequency. Optimal sampling frequency for Min-Variance and Mean-Variance portfolio optimisation ranges from 30 to 130 minutes. They have also shown that using two-scales estimator and lead-lag bias correction decreases optimal frequency and increase performance of the portfolios.

The same methodology was used by Ubukata (2009) and Becker *et al.* (2015) for portfolio optimization.

Both in the univariate and multivariate estimators 1.16 and 1.17 overnight term is a very noisy estimator of overnight volatility/covariance, as it is obtained using only one observation.

Rescaling intraday realized volatility

Martens (2002) introduced another estimator of the whole day volatility - rescaling of intraday realized variance.

$$\sigma_t^2 = (1 + s)\hat{\sigma}_{id,t}^2 \tag{1.18}$$

Here, parameter s is a share of overnight volatility in the whole day volatility.

He compared three methods: rescaling of intraday volatility (equation 1.18), adding a squared overnight return (equation 1.16) and 30-minute sampling from overnight trading. The latest, unfortunately is available not for every asset.

On simulated data (data-generating process was GARCH 1,1) it was shown, that in the absence of overnight trading the most precise measure of the whole day volatility is a rescaled volatility (equation 1.18), while squared overnight returns is a very noisy measure of overnight volatility. It provides better forecasts for 1, 5 and 20 days ahead volatility. On a real data of S&P-500 index futures (1990-1998, overnight trading data was available only from 1994 year) the results were confirmed. He has also shown, that adding both previous day realized volatility measures increases the performance of GARCH(1,1) forecast.

Additionally he proposed a modelling of intraday and overnight volatility as separate GARCH processes.

$$\sigma^2_{on,t} = a_1 + a_2 r^2_{on,t-1} + a_3 \hat{\sigma}^2_{id,t-1} + a_4 \sigma^2_{on,t-1} \tag{1.19}$$

$$\sigma^2_{id,t} = b_1 + b_2 r^2_{on,t} + b_3 r^2_{id,t-1} + b_4 \sigma^2_{id,t-1} \tag{1.20}$$

At the same time volatility of 5-min. intraday returns also follows GARCH(1,1) process. For a one-day-ahead forecast, such specification provides increase in performance comparing with daily GARCH(1,1) model.

Optimal weighting

Hansen & Lunde (2005) proposed a procedure of weighting two volatility measures.

Assuming that whole day volatility is proportional to the intraday volatility, they have shown, that both rescaled realized volatility and squared overnight return are unbiased estimators of the whole day integrated volatility (IV):

$$IV_t = E((1 + \frac{1}{s}) r^2_{on,t}) = E((1+s)\hat{\sigma}_{id,t}) \tag{1.21}$$

They proposed the following realized volatility estimator for the whole day:

$$\tilde{\sigma}^2_t = (1-\omega)(1 + \frac{1}{s}) r^2_{on,t} + \omega(1+s)\hat{\sigma}^2_{id,t} \tag{1.22}$$

where weight ω is chosen in such a way to minimize the variance of the resulting estimator. They have shown, that under the assumption of unbiasness this weight also minimizes mean square error. At the example of 30 stocks of DJIA it was shown that assumption of proportionality holds. In other words, in this regression:

$$ln(\frac{r^2_{on,t}}{\hat{\sigma}^2_{id,t}}) = \alpha + \beta' Z_t + \epsilon_t \tag{1.23}$$

α is significant, while β is not. Z_t here is a list of instrumental variables. As instrumental variables were used: days of the week and realized volatility for the previous day. Only two out of 30 regressions provided significant β at 5% confidence level. To test assumption of unbiasness of the estimator, following regression was used:

$$ln(\frac{\tilde{\sigma}_t^2}{r_{on,t}^2 + \hat{\sigma}_{id,t}^2}) = \alpha + \beta' Z_t + u_t \tag{1.24}$$

in which, β is significant only for 3 out of 30 stocks.

Ahoniemi & Lanne (2013) tested the different methods of dealing with overnight return on a data for S&P-500 return index, as well as 30 stocks included in DJIA index (1994-2009, 3966 observations). Instead of using biased opening price of index, they use either Special Opening Quote or the value of the index 5 minutes after opening. Also they compare different ways of computing s in equation 1.18: in one case average overnight volatility was divided by average whole day volatility (computed on daily data), in the other it is divided by the sum of average overnight and average intraday realized volatility.

At an in-sample test, using MSE as a loss function, it was shown, that using both Model Confidence test and Diebold-Mariano test Hansen-Lunde weighting method provides statistically closer result for estimation of volatility of S&P-500 index. For single stocks, however, the best method was either not to include overnight volatility at all, or to use a scaling estimator. At an out-of-sample forecast exercise there were compared GARCH, GJR-GARCH (Glosten *et al.* (1993)) and APARCH (Ding *et al.* (1993)) models. It turned out that the choice of the model with the least MSE depends upon the underlying whole day volatility estimator. This difference have the maximum effect while forecasting S&P index, and decrease in single stock volatility forecast.

1.3.2 Estimators based on returns of other assets

Due to the fact, that volatilities of different assets are interconnected, it is possible to estimate an overnight volatility of one asset, using returns of other assets.

Triacca & Focker (2014) have proposed an estimator based on Dynamic Factor Model. Assuming that overnight log-variances of different assets follow generalized dynamic factor model:

$$\log \sigma_{o,it}^2 - E(\log \sigma_{o,it}^2) = \chi_{it} + \xi_{it} = \sum_{j=1}^{q} b_{ij}(L) u_{jt} + \xi_{it} \tag{1.25}$$

where χ_{it} is a common component, ξ_{it} is an idiosyncratic component, u_{jt} are common shocks, (L) is a lag operator, and b_{ij} is sensitivity of volatilities to factor shocks. Obtaining estimator of common component $\hat{\chi_{it}}$ through the method based on generalized dynamic factor model proposed in Forni *et al.* (2000), the resulting estimator for overnight volatility is:

$$\sigma^2_{o,it} = \exp(\frac{1}{T}\sum_{t=1}^{T}\log r^2_{o,it} + 1.27 + \hat{\chi_{it}}) \tag{1.26}$$

The proposed estimator was tested on simulated data, in which overnight returns have Student's t-distribution and volatilities follow autoregressive process as in Taylor (1989) but with leverage effect. It turns out, that the proposed estimator has lower RMSE than squared overnight return. As a practical example, they used data for stock prices of companies from S&P100 index to predict overnight volatility of American Express, Bank of New York, Intel Corporation and Merrill Lynch stocks. They have shown, that the proposed estimator has smaller variance in comparison with squared overnight returns.

If the dependence between an assets is known a priori, estimation becomes even simpler. For example, the value of stock market index depends upon the assets it includes.

Petroni & Serva (2016) proposed to estimate the whole day standard deviation of the index by averaging absolute returns of the stocks of which this index is composed. Assuming that market return is equal to $r_t = \sigma_t \epsilon_t$, where $\epsilon_t \sim U[-\sqrt{3}, \sqrt{3}]$, in the case of equiweighted index volatility is estimates as:

$$\hat{\sigma}_t = \frac{1}{\sqrt{3}N}\sum_{\alpha=1}^{N}|r_{\alpha,t}| \tag{1.27}$$

Using the data of 65 stocks of Dow Jones (1973 - 2014) they have shown, that estimated volatility is persistent (autocorrelation function of volatility decreases slowly), while the remainder ϵ_t is almost uncorrelated neither with volatility nor absolute return of the index. It also turns out that ϵ_t is uniformly distributed.

Through the idea of the paper is brilliant, there is a small concern about residuals test: the estimator was applied in-sample, and that seems to be a cause of uniform distribution of residuals, rather than properties of price process.

Overnight and whole-day volatility estimations are used as input in forecasting models, as well as back-testing benchmark for them. However, it is also possible to directly forecast the overnight and intraday volatilities as two separate processes as models of section 1.2.6.

1.4 Asynchronous markets

In the case, when overnight periods are different in different markets, covariance computed on daily close-to-close returns has downward bias, as its expectation is only covariance during overlapping period.

Taking asynchronicity into account is especially important to distinguish comovement and spillover effects.

1.4.1 Synchronous sampling

Sampling from overlapping trading hours

If there exists an overlapping period in stock exchanges, or at least an overlapping point, it is possible to measure synchronous prices even at asynchronous markets, as long as intraday high-frequency data is available.

In such a way **McAleer & Da Veiga (2008)** used synchronous prices observed at 16:00 GMT. They used European and USA data of S&P-500 (USA), FTSE-100 (UK), CAC-40 (France) and SMI (Switzerland) indices (1990-2004 years, 3720 observations). They proposed a Portfolio Spillover GARCH model in which volatilities of each asset depends not only upon its past realization, but also realization of volatilities other assets. They compared PS-GARCH model with VARMA-GARCH and CCC models in forecasting Value-at-Risk for equally weighted portfolio of indices. It turnes out, that even while spillover effect is significant, taking them into account does not increase the performance of 1% VaR forecasting.

However, synchronous observations are not always available. For example, Japan stock market opening hours do not overlap neither with American nor with European intraday periods.

Weekly subsampling

Weekly returns have larger share of overlapping period. So, the asynchronous bias of weekly observations may be neglected.

Berben & Jansen (2005) studied the increase of international correlations. They used weekly returns of market and industrial indices of US, UK, Germany and Japan (1980-2000). Volatilities were assumed to follow GARCH(1,1), while correlation coefficient changing between two regimes.

$$\rho_t = \rho_0(1 - \frac{1}{1 + exp(-\gamma(s_t - c))} + \rho_1 \frac{1}{1 + exp(-\gamma(s_t - c))} \qquad (1.28)$$

This model is called Smooth-Transition Correlation GARCH. The results are: correlations between Japanese and other markets didn't change. Correlation between UK and US market increases, as well as German-UK and German-US correlations. At industrial level UK-US correlation increased for eight industries out of ten, while German-UK and German-US correlations have changed at four industries out of ten.

Christiansen (2007) used weekly returns of governmental bonds to study volatility spillover at the international sovereign bond markets. The data was represented by JPMorgan government bond indices for US and Europe as explanatory variable, and Belgium, France, Germany, Italy, Netherlands, Spain, Denmark, Sweden and UK as dependent variables. Data was sampled each Wednesday (1988-2002 years, 777 observations). It turned out that there exist a significant volatility spillover effect from US and European index to the individual countries, but not the mean spillover from US to European countries. EURO introduction increased the volatility spillover effect.

In the same way **Abad *et al.* (2010)**, studied integration of European sovereign bond markets. They used Wednesday sampled returns of 10-year Governement bonds of 13 EU-countries (except for Luxembourg and Greece) (1990-2008 years). Testing the significance of integration coefficient (Bekaert & Harvey, 1995), they have shown, that EMU countries have higher degree of integration with German bond market, while non-EMU countries are more integrated with world market (represented by US data).

1.4.2 Sum of the lagged covariances

One of the solution to the problem is summation of day-to-day covariance with lagged covariances. It is easy to see, that equation 1.29 provides unbiased estimation of the true covariance $E(\hat{\sigma}_{ij}) = \sigma_{ij}$.

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T} r_{i,t}r_{j,t} + \frac{1}{T-2}\sum_{t=2}^{T} r_{i,t-1}r_{j,t} + \frac{1}{T-2}\sum_{t=2}^{T} r_{i,t}r_{j,t-1} \quad (1.29)$$

Disadvantage of such a solution, is that correlation coefficient $\hat{\rho}_{ij} = \frac{\hat{\sigma ij}}{\sqrt{\sigma_i^2\sigma_j^2}}$ is not bounded between -1 and 1. Consequently, the pair-by-pair estimation of covariance matrix does not guarantee positive semidefiniteness.

The problem of asynchronous trading was at first discussed by **Scholes & Williams (1977)**. They pointed, that asynchronous observations take place not only at asynchronous markets but also on illiquid market. The target of the paper was estimation of β coefficient - sensitivity of an asset return r_a to the market return r_M. They provided a following method to ensure the consistency and unbiasness of the estimator: sum *betas* for all possibly overlapping intervals. As long as there is at least one deal per day, it is enough to add *betas* for previous and last days. From efficient market hypothesis follows zero autocorrelation of market returns. However, in the presence of non-zero autocorrelation it is enough to divide calculated *beta* by one plus the double autocorrelation coefficient. The resulting estimator of *beta* coefficient is presented in equation 1.30.

$$\hat{\beta} = \frac{\beta^{-} + \beta_{unadj} + \beta^{+}}{1 + 2\gamma_M} \quad (1.30)$$

where:

$$\beta_{unadj} = \frac{cov(r_{a,t}, r_{M,t})}{\sigma_M^2} \tag{1.31}$$

$$\beta^- = \frac{cov(r_{a,t}, r_{M,t-1})}{\sigma_M^2} \tag{1.32}$$

$$\beta^+ = \frac{cov(r_{a,t}, r_{M,t+1})}{\sigma_M^2} \tag{1.33}$$

$$\gamma_M = \frac{cov(r_{M,t}, r_{M,t-1})}{\sigma_M^2} \tag{1.34}$$

The resulting estimator was tested on NYSE and ASE data (1963-1975). It was shown that trading volumes at each day explain the difference between β calculated on raw data and β_{true} calculated using equation 1.30.

The paper of Scholes & Williams (1977) provided a huge impact on the literature. Their method becomes very useful for estimation of covariances, correlations and betas in the asynchronous stock markets. Moreover, as asynchronous trading is observed also during the trading day, similar idea to sum the products of all overlapping returns was used by Hayashi & Yoshida (2005) for the intraday realized covariance estimation.

At **RiskMetrics (1996)** the problem of asynchronous markets was separated from the general case of nonsynchronous trading. It was shown that correlation between close-to-close sampling 10-years Australian and US government bonds is very underestimated in comparison with simultaneous sampling (at 00:00 GMT). Following Scholes & Williams (1977) they suggested to sum the covariance between market returns with the covariances between return of lagged market returns, as in equation 1.29. Consequently the adjusted correlation estimator is $\hat{\rho}_{ij} = \frac{\hat{\sigma ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}$. After adjustment correlation coefficient between USD and AUD bond increase from 0.305 to 0.560.

In the multivariate framework, however, the proposed method doesn't guarantee positive semi-definiteness of the covariance matrix. RiskMetrics provides solution to this problem: to use linear combination of adjusted and unadjusted covariance, setting to the latest as the largest possible weight such that resulting estimator is positive semi-definite.

$$V_{psd} = (1 - \omega)V_{unadj} + (\omega) \times \hat{V} \tag{1.35}$$

where V_{psd} is a positive semidefinite covariance matrix, V_{unadj} is a close-to-close covariance matrix and $\hat{V}$ is an unbiased covariance estimator, with elements as in equation 1.29. The resulting covariance matrix is supposed to follow Exponentially Weighted Moving Average Process (EWMA).

At the paper of **Bergomi (2010)** correlation in asynchronous markets is discussed in respect of option trading. It was shown, that options, written on assets in different markets, should be priced with a synchronized correlation that is a sum of current correlation between two assets and correlation between lagged return of first asset and current return of the second, given that first exchange is closed later. Unlike RiskMetrics or Scholes & Williams (1977) only the overlapping returns are used. If market j is closed after the market i, their unbiased correlation estimator is as in equation 1.36.

$$\hat{\rho} = corr(r_{i,t}, r_{j,t}) + corr(r_{i,t}, r_{j,t-1}) = \rho_s + \rho_a \tag{1.36}$$

He calls ρ_s a 'synchronous' correlation and ρ_a - an 'asynchronous' correlation.

Through the example of Stoxx50, S&P500 and Nikkei indices it was shown, that asynchronous term ρ_a becomes larger as the difference in trading hours increases. There was performed a comparison between estimator 1.36 and usage of weekly sampling. Difference between correlation computed using n-days sample and synchronized one is expected to be $-\frac{\rho_a}{n}$. A problem of positive semi-definiteness of the covariance matrix was mentioned: in case of serial correlation it can happen that $\hat{\rho} > 1$, but usage of $\rho = 1$ (a theoretical boundary) leads to underpricing of correlation swaps.

1.4.3 Return synchronisation

Unlike the case of intraday asynchronous trading, observations at asynchronous market at least have regular timestamps. Consequently, it is possible to make an adjustment for asynchronous correlation and obtain 'synchronized' prices and covariances. Practically Vector Moving Average and Vector Auto-Regressive models are used to fit the data.

This approach was at first introduced by **Burns *et al.* (1998)**. They provided a synchronization procedure for asynchronous returns, variances and correlations. They assume a moving average process for asynchronized returns:

$$r_t = \epsilon_t + M\epsilon_{t-1} \tag{1.37}$$

where, $\epsilon \sim N(0, \Omega)$. From equation 1.37 follows that synchronized return is described by:

$$\tilde{r}_t = \epsilon_t + M\epsilon_t \tag{1.38}$$

covariance matrices of asynchronized covariances are

$$E(r_t r_t') = \Omega + M\Omega M \tag{1.39}$$

$$E(r_t r_{t-1}') = M\Omega \tag{1.40}$$

and consequently the synchronized covariance matrix of assets is:

$$E(\tilde{r}_t \tilde{r}_t') = (I + M)\Omega(I + M)' \tag{1.41}$$

where Ω is a covariance matrix of ϵ. Matrix M is expected to be a triangular with zero diagonal elements (given that assets are ordered by the time of exchange closure), as returns for non overlapping periods should not correlate. Synchronized returns are fitted to the BEKK-GARCH model. Asynchronous GARCH model was tested on G-7 market indeces (CAC, DAX, FTSE, MIL, NIK, S&P, TTO). It was shown, that the residuals of the model have covariance matrix almost equal to Identity, almost all their autocorrelations are zeros, there are almost neither correlation in squared residuals nor leverage effect.

Synchronisation procedure of **Audrino & Bühlmann (2004)** is similar to Burns *et al.* (1998). Instead of MA(1) in this model asynchronized returns follow AR(1) process.

$$r_t = Ar_{t-1} + \epsilon_t \tag{1.42}$$

Consequently, synchronized returns are computed as follows:

$$\tilde{r}_t = r_t + A(r_t - r_{t-1}) \tag{1.43}$$

From matrix A they excluded non-significant coefficients. Having synchronized returns they fit CCC-GARCH model, in which volatilities follow GARCH and correlation coefficient assumed constant. Real-data example included DJIA, CAC40, DAX, BCI, CBS, FTAS, NIKKEI from 1990 to 1994 year. It was shown that usage of synchronized returns for CCC-GARCH provides better results in terms of out-of-sample likelihood difference test. Synchronized models also forecast 1%, 5% and 10% Value-at-Risk more precisely.

The same modification of Burns *et al.* (1998) as Audrino & Bühlmann (2004) was used by **Scherer (2013)**: he changed MA(1) assumption to AR(1). He has shown the practical importance of asynchronicity adjustment, usually neglected by practitioners. Using data for indices returns: S&P-500, DAX, NIKKEI (2007-2010), he compared three adjustment methods: VAR(1), VMA(1) and usage of weekly returns. Assuming that variance of the portfolio follows IGARCH, 5% weekly Value-at-Risk was forecasted. VAR(1) synchronisation method provides more close number of violations to confidence level than usage of weekly returns.

Butler & Okada (2007) instead of using first-order models, used VMA(2) model for asynchronous assets returns and EGARCH(2,2) for their volatilities with constant correlation. At the example of returns of Morgan Stanley Capital International stock market indices for Japan, US and UK (1996 - 2002) they show non-significance of second-order term in VMA(2) and significance of bivariate and second-order terms in EGARCH(2,2) regressions. However in out-of-sample test of forecasting conditional mean and two-days volatility second-order terms do not improve quality of prediction.

Comparison of RiskMetrics (1996) and Burns *et al.* (1998) models was done by **Martens & Poon (2001)**. They used close-to-close and synchronous prices (16:00 London time) of S&P-500, FTSE-100 and CAC40 indices (1990-1998 years, 1994 common trading days). Each model: exponencially weighted moving average (EWMA) of Morgan (1996) and assymetric dinamic covariance (ADC) was fitted using both synchronous prices and prices synchronized by corresponding method. It was shown, that correlation between correlation parameters, obtained by synchronous and synchronized data is only 0.247 for ADC and 0.451 for EWMA. On an out-of-sample VaR forecast exercise there was illustrated that EWMA Morgan (1996) model provides more volatile and more conservative 1% VaR forecasts than ADC Martens & Poon (2001). However, both model provide significantly more violations than the confidence level.

1.4.4 Four phase models

In the same way, as it is possible to model intraday and overnight volatilities separately (section 1.2.6), it is also possible to model the volatilities and covariances at each of the phases separately. A phase here means a time period at which no market opening or closure happens at any considered stock exchange.

This model was applied by **Golosnoy *et al.* (2015)** to estimate volatility spillovers across NIKKEI 225, DAX, Dow Jones indices at 1996-2009. Realized covariance/variance estimators were calculated separately for each of the four periods: overlapping DAX-DJ (here denoted as V_1), only DJ (σ_2), only NIKKEI (σ_3) and only DAX (σ_4). DAX-DJ covariance matrix is assumed to follow Conditional Autoregressive Wishart process (introduced in Golosnoy *et al.* (2012), a high-frequency adaptation of BEKK-GARCH Engle & Kroner (1995)). In this model covariance matrix of DJ-DAX has a Wishart distribution $V_{1t} \sim W(p_1, \hat{V_{1t}}/p_1)$, with average covariance matrix as in equation 1.44.

$$\hat{V_{1t}} = G_1 G_1' + \sum_{i=1}^{q} A_{i1} V_{t-1} A_{i1}' + \sum_{i=1}^{p} B_{i1} \hat{V}_{t-1} B_{i1}' + \sum_{i=1}^{z} D_{i1} \bar{V_t} D_{i1}' \tag{1.44}$$

where G_1 is a triangular 2×2 matrix, A_{i1} and B_{i1} are parameter 2×2 matrices, D_{i1} are 2×3 matrices and $\bar{V_t} = diag(\sigma_{2t-1}, \sigma_{3t-1}, \sigma_{4t-1})$ is a diagonal matrix of volatilities at other phases. Other volatilities are Gamma distributed $\sigma_{it} \sim G(p_i/2, 2\hat{\sigma}_{it}/p_i)$.

$$\hat{\sigma}_{it} = g_i + a_i \sigma_{i,t-1} + b_i \hat{\sigma}_{i,t-1} + c_i \sigma_{j,\tau} + d_i \sigma_{k,\tau} + \sum_{j=1}^{y} e_i V_{1\tau}' e_i \tag{1.45}$$

where $j, k = 2, 3, 4 \neq i$ denote the other phase, and $\tau = t, t-1$ such that only past information is included into equation.

They have added a crisis dummy (Aug.2007-Mar.2009) to the coefficients of equations. It turned out that crisis of 2007-2009 volatility spillover across markets increased, while the persistence of volatility decreased.

1.5 Conclusion

In this paper there were reviewed articles devoted to different facts and problems caused by overnight period. Several stylised facts about overnight returns were selected. Overnight return has lower volatility, but positive mean resulting in high Sharp ratio. This can partially be explained by its higher tail risk. A common question in the literature is whether the difference between overnight and intraday return is caused by trading or difference in arrived information. Some facts point to the latest: overnight correlations are significantly higher than intraday one, it means that overnight information is more referred to the whole marked rather than to specific companies. Additional argument for the private information hypothesis is different behaviour of ADRs: if differences were caused mostly by trading noise, ADRs would have shown the same behaviour as original stocks, however, overnight return of ADRs has higher volatility and lower correlations than their intraday returns.

Absence of trading during overnight period causes difficulties in the overnight volatility estimation. There are two approaches for overnight volatility estimation: either to use intraday high-frequency data or to use prices of other assets in the market. Alternative methodology is to directly forecast overnight and intraday volatilities as separate, but correlated processes.

If overnight periods at stock exchanges take place in different hours, the observed open and close prices are non-synchronous. Consequently, direct covariance estimation is biased toward zero. There are three solutions are proposed: subsampling with lower frequency (a week or a month), adding lagged cross-correlation terms to correlation estimator, and synchronisation of returns via vector versions of Moving Average or Auto-Regressive models. As in the case of overnight volatility, it is possible to divide the whole day into 4 periods and estimate volatility and covariance matrix during this periods separately.

There are many potential for the future research. Especially topical is an estimation of overnight covariance matrix, for which, up to now exist only a very naive approach - sum of intraday covariance matrix with an outer product of overnight return vector. This topic is especially relevant for the case of asynchronous markets, where the advantages of having high-frequency data is not yet fully implemented.

CHAPTER 2

Adding Overnight to the Daily Covariance

This paper proposes a new method of calculation the whole day covariance matrix that includes information from the overnight period. Proposed estimator is a multivariate extension of scaling-and-weighting estimator by Hansen & Lunde (2005), extended to the multivariate case. For the scaling, a new concept of matrix proportionality is proposed. Three ways of optimal weighting of matrices are introduced, each of them exploit the theorem from the paper Hansen & Lunde (2005). New method decreases the noise of the whole day covariance estimator in comparison with existing approach: a sum of intraday realized covariance and outer product of overnight returns. Performance of the two estimators is compared using both simulated and real data.

Keywords:

C58 Financial Econometrics
G17 Financial Forecasting and Simulation
G32 Financial Risk and Risk Management
Overnight Realized Covariance
Matrix Geometric Mean

2.1 Introduction

Trading day consists of the period when assets are traded, a trading day, and when they are not, an overnight period.This article is devoted to the calculation of covariance matrix for the whole day. There exist many methods of computing the realized covariance inside the day. However, the computation of the covariance matrix for the whole day is not trivial.

Some of the researchers (Becker *et al.* (2015), de Pooter *et al.* (2008), Ubukata (2009)) simply add the overnight return as another observation ('naive' estimator). This method has a disadvantage: as an overnight covariance is reconstructed only by one observation, the total estimator has too higher variance. Due to the high noise of such estimator some authors (Kyj *et al.* (2009)) do not include the overnight covariance at all. Kang & Babbs (2012) model overnight and intraday covariances as separate processes, connected through GARCH-copula model. However, their approach does not take into account intraday data.

For univariate case, similar problems were already discussed. Hansen & Lunde (2005) introduced the method of rescaling the intraday and overnight variance to get two unbiased estimators, which are weighted to minimize the variance of the resulting estimator. The same method is also used by Fleming & Kirby (2011), Fuertes & Olmo (2013), and Ahoniemi & Lanne (2013). Triacca & Focker (2014) proposed instead to estimate the overnight variance based on the dynamic factor model.

In this paper the method of Hansen & Lunde (2005) is extended to the multivariate case: overnight covariance is assumed to be directly proportional to the intraday one, as in the univariate case, but the concept of direct proportionality is extended to the multivariate case in such a way that preserves positive semi-definiteness and symmetry of the matrix. Positive semi-definiteness of the covariance matrix is important in a number of financial applications such as risk-management, option pricing, portfolio optimization. Intraday covariance matrix and an outer product of overnight returns are rescaled in order to obtain two conditionally unbiased estimators for the whole day covariance. The resulting estimator is a weighted average of those two estimators. Unlike simple addition of outer product to intraday realized covariance, the proposed estimator is not suffered by great noise.

Practical application of the methodology will lead to more precise estimation of the whole day covariance matrix, and consequently to more efficient risk-management, portfolio allocation, and option pricing.

The paper is organized in the following way. It consists of theoretical part, simulation study and empirical part. Theoretical part consists of introduction to the problem, extension of proportionality for the multivariate case, its application to the problem and propose of weighting procedure, used to obtain whole day covariance matrix. In simulation study it is shown, that proposed estimator is robust to misspecification of the model. Even if overnight covariance has dependence structure different than assumed, the proposed estimator is still closer to the true covariance then naive one. In the empirical part of the paper, the proposed methodology is applied to the real data. It is shown, that invented scaling-and-weighting estimator outperforms existing 'naive' one in a number of tests: normality of residuals, value-at-risk forecast and minimum-variance portfolio allocation.

2.2 Whole day covariance estimator

2.2.1 Notations

In this paper vectors of returns $\mathbf{r}_t$ are assumed to be independent and have multivariate normal distribution with zero means: $\mathbf{r}_t \sim N(0, \mathbf{V}_t)$, where $\mathbf{V}_t$ is a covariance matrix. The whole day is divided into two periods: intraday, in which assets are traded, and overnight, in which the stock exchange is closed. Let $\mathbf{r}_{id,t} \sim N(0; \mathbf{V}_{id,t})$ be a return for the intraday period and $\mathbf{r}_{on,t} \sim N(0; \mathbf{V}_{on,t})$ for the overnight.

The normality assumption can be weakened: prices can follow any process with finite covariance, for example a diffusion process with jumps, that make a distribution of returns fat-tailed. For the construction of the whole day covariance estimator it is only necessary, that realized covariance estimator $\hat{\mathbf{V}}id, t$ is an unbiased estimator of $\int_0^1 E(\mathbf{r}(t)\mathbf{r}(t)')dt$ - some model-independent true covariance. However, this assumption will be important during testing of the performance of the proposed estimator in section 2.4: true residuals are assumed to have Standard Normal distribution, Value-at-risk is also assumed to be a quantile of Normal distribution.

The problem is to estimate the covariance matrix for the whole day $\mathbf{V}_t = \mathbf{V}_{id,t} + \mathbf{V}_{on,t}$ using the intraday realized covariance matrix $\hat{\mathbf{V}}_{id,t}$ and an overnight realized covariance (calculated using the only overnight observation) $\hat{\mathbf{V}}_{on,t} = \mathbf{r}_{on,t}\mathbf{r}'_{on,t}$.

2.2.2 Intraday covariance estimators

For the target of the paper - obtaining covariance estimator positive semi-definite by construction - it is important that intraday realized covariance estimator $\hat{\mathbf{V}}_{id,t}$ is positive semi-definite by construction. There exist many positive semi-definite intraday realized covariance estimators: subsampling (Barndorff-Nielsen & Shephard (2004)), bootstrapping (Dovonon *et al.* (2013)), double subsampling (Zhang *et al.* (2005), Zhang (2011)).The latest method is commonly applied for realized covariance estimation in empirical papers. Many estimators are based on 'refresh-time' synchronisation procedure: this method is proposed in the paper of Barndorff-Nielsen *et al.* (2011). Some improvements of the method were proposed by Christensen *et al.* (2010), Boudt *et al.* (2014). Estimators, based on Bayesian statistics are also positive semi-definite by construction. Estimators of Tsay & Yeh (2004) and Peluso *et al.* (2015) have one basic idea - iteratively generate missed observations and compute the covariance matrix based on them.

There are also estimators that do not guarantee positive semi-definiteness covariance matrix by construction. Usually they are bivariate estimators.

The covariance matrix in this case consists of pair-by-pair estimations. Even if covariance matrix is not positive semi-definite, it is possible to find the closest positive semi-definite matrix (see Higham (2002)). One of the most frequently used non positive semi-definite estimator is a Hayashi & Yoshida (2005) covariance estimator: a sum of the product of all returns time intervals of which intersect. Hayashi-Yoshida estimator uses all the observations and is unbiased estimator. Application of the estimator to the interpolated data with rounded timestamps was proposed by Kanatani & Renó (2007) and Corsi & Audrino (2012). Other estimators, that are not positive definite by construction are estimators of Aït-Sahalia *et al.* (2010), Hansen *et al.* (2015).

In this paper intraday realized covariance will be estimated using two-scale Zhang (2011) approach.

2.2.3 Naive estimator

Obvious unbiased estimator for $\mathbf{V}_t$ is just the sum of the two realized covariances for the intraday and overnight periods - a naive estimator.

$$\hat{\mathbf{V}}_t^+ = \hat{\mathbf{V}}_{id,t} + \hat{\mathbf{V}}_{on,t} \tag{2.1}$$

However, the last term is a very noisy estimator of overnight covariance, as it is obtained using only one observation. The idea of decreasing the noisiness of the estimator is following: to create a two-step estimator of the whole day covariance matrix. At the first step, there would be constructed an estimator for the whole day covariance without considering overnight covariance at all. Also, at the first step there would be constructed an estimator for the whole day without considering the intraday realized covariance at all. At the second step there would be taken a weighted average of them.

So, the first target is to estimate the whole day covariance matrix without considering overnight return at all. It is natural in this case to think about overnight covariance as some function of intraday realized covariance matrix.

Assume that conditional expectation of unobserved overnight covariance is a function of intraday covariance.

$$E(\mathbf{V}_{on,t}|\mathbf{V}_{id,t}) = F(\mathbf{V}_{id,t}) \tag{2.2}$$

The function $F : Sym_n^+(\mathbb{R}) \to Sym_n^+(\mathbb{R})$ transforms $n \times n$ symmetric positive semi-definite matrix into symmetric positive semi-definite one. The function F should satisfy the following requirements:

- it should be possible to transform any covariance matrix into any other, as it is not a priori known whether the correlation structure remains the same inside the day and during overnight;

- the function should have the least possible number of parameters to estimate;
- in univariate case function should be equivalent to simple multiplication: $f(v_t) = sv_t$.

Shortly speaking, the function should rescale the matrix $\mathbf{V}_{day}$ by some scaling matrix $\mathbf{S}$ in such a way, to get a for a positive semi-definite symmetric estimation of $\mathbf{V}_{on}$ as a result.

2.2.4 Matrix proportionality

In the univariate case direct proportionality $y(x) = s^2 x$ may be defined as such function, that preserves one and the same geometric mean of y and x^{-1}. In other words $y(x) : s = g(y, x^{-1}) = const$. This definition of proportionality is not common, however, it naturally extend proportionality for the multivariate case. Geometric mean of positive semi-definite symmetric matrices was already introduced by Pusz & Woronowicz (1975). For matrices $\mathbf{A}$ and $\mathbf{B}$ it may be defined as an unique positive semi-definite symmetric solution of the equation $\mathbf{B} = \mathbf{X}\mathbf{A}^{-1}\mathbf{X}$ or in explicit form:

$$G(\mathbf{A}, \mathbf{B}) = \mathbf{X} = \mathbf{A}^{1/2}(\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2})^{1/2}\mathbf{A}^{1/2} \tag{2.3}$$

In the literature on linear algebra it is usually denoted as $\mathbf{A}\#\mathbf{B}$.

Consequently, matrix proportionality will be defined as such function, that preserves one and the same geometric mean of one matrix and inverse of another:

$$\mathbf{V}_{on,t} = F_S(\mathbf{V}_{id,t}) = \mathbf{S}\mathbf{V}_{id,t}\mathbf{S} \tag{2.4}$$

where $\mathbf{S}$ is a positive semi-definite symmetric matrix, called 'scaling' matrix. It can be estimated as in equation 2.5.

$$\hat{\mathbf{S}} = \bar{\mathbf{V}}_{id}^{-1/2}(\bar{\mathbf{V}}_{id}^{1/2}\bar{\mathbf{V}}_{on}\bar{\mathbf{V}}_{id}^{1/2})^{1/2}(\bar{\mathbf{V}}_{id})^{-1/2} \tag{2.5}$$

where $\mathbf{A}^{1/2}$ is a matrix square root, defined as unique positive semi-definite symmetric solution of equation $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$, $\bar{\mathbf{V}}_{on}$ is an average overnight covariance matrix and $\bar{\mathbf{V}}_{id}$ is an average intraday covariance matrix. The latest can be computed either as average intraday realized covariance matrix, or, using daily time series of intraday returns.

Geometric means have a list of useful properties listed by Ando *et al.* (2004) (ALM conditions). The function in equation 2.4 inherits these properties, which are especially valuable in the covariance estimation.

P1 *Consistency with scalars.* It is easy to see that in the univariate case function 2.4 becomes $v_{on,t} = |\bar{v}_{on}/\bar{v}_{id}|v_{id,t}$, that is equivalent to Hansen & Lunde (2005) assumption. Proportion of determinants of the matrices is also constant.

P2 *Joint homogeneity.* If $\mathbf{V}_{on,t}^{new} = \alpha \mathbf{V}_{on,t}$ and $\mathbf{V}_{id,t}^{new} = \beta \mathbf{V}_{id,t}$, then $\mathbf{S}^{new} = \sqrt{\alpha/\beta}\mathbf{S}$. Consequently, estimation is invariant to the scale at which covariance matrices are computed. For example, overnight covariance matrix may be computed in per-hour terms.

P3 *Permutation invariance.* Given that $\mathbf{S}$ is symmetric, it does not matter in which order assets are ranged, the resulting estimator will be the same (up to permutation of assets), just like in case of regular covariance matrix estimation.

P4 *Monotonicity.* If $\mathbf{V}_{id,q} \geq \mathbf{V}_{id,t}$ in Loewner ordering, then $\mathbf{V}_{on,q} \geq \mathbf{V}_{on,t}$. From this property, for example, follows, that increase of intraday volatility of any portfolio will cause increase of overnight volatility of this portfolio and vice versa (properties (b) and (c) of Stepniak (1985)).

P5 *Additivity.* $\mathbf{V}_{on,t} + \mathbf{V}_{on,t+1} = \mathbf{S}(\mathbf{V}_{id,t} + \mathbf{V}_{id,t+1})\mathbf{S}$. In the limit, this property allows to estimate scaling matrix $\hat{\mathbf{S}}$ using equation 2.5. This property correspond to *continuity* at Ando *et al.* (2004).

P6 *Self-duality.* $\mathbf{V}_{on} = F_S(\mathbf{V}_{id}) \Rightarrow \mathbf{V}_{id} = F_S^{-1}(\mathbf{V}_{on})$ This property means that inverse function of 2.4 preserves the same structure and properties. It also means that it is enough to estimate one matrix S to be able to calculate both $E(\mathbf{V}_{id,t}|\mathbf{V}_{on,t})$ and $E(\mathbf{V}_{on,t}|\mathbf{V}_{id,t})$.

These properties, except for P5, directly follow from corresponding Ando *et al.* (2004) properties by setting $G(\mathbf{A}, \mathbf{B}^{-1}) = const.$

Interpretation of scaling matrix

One of the possible interpretation is that model 2.4 is a special case of BEKK Engle & Kroner (1995) model. BEKK model is a multivariate generalization of GARCH model, in which:

$$\mathbf{V}_t = \mathbf{A} + \sum_k \mathbf{B}_k \mathbf{V}_{t-1} \mathbf{B}_k' + \sum_l \mathbf{C}_l \mathbf{r}_t \mathbf{r}_t' \mathbf{C}_l' \tag{2.6}$$

However, if we assume instead of equation 2.4 that:

$$\mathbf{V}_{on,t} = \sum_k \mathbf{S}_k \mathbf{V}_{on,t} \mathbf{S}_k' \tag{2.7}$$

some valuable properties will be lost. For $k > 1$ amount of parameters will not be the smallest possible, as it is in univariate case. If $\mathbf{S}$ is not symmetric, permutation invariance will not be hold, while it is natural to assume, that ordering of assets should not influence estimation of their covariance matrix. Otherwise, it would be necessary to assume some a priori ranking. Disadvantage of multivariate proportionality with respect to BEKK model is loss of generality. Still, in multivariate proportionality model, like in BEKK, any element of overnight covariance matrix $\sigma^2_{on,ij}$ is a linear combination of all elements $\sigma^2_{id,lk}$ of intraday covariance matrix, just with stronger restrictions on parameters.

Another advantage of using multivariate proportionality is easiness of interpretation of scaling matrix. If an overnight covariance matrix is proportional to the intraday one, the matrix $\mathbf{S}$ is equal to some scalar multiplied by identity matrix $\mathbf{S} = s\mathbf{I}$. However, each asset may have different ratio of overnight-to-intraday variance. In this case, if the correlation matrix remains the same, and only variances are changing, the matrix $\mathbf{S}$ keeps being diagonal, but not necessary with the same coefficients on the diagonal. Non-diagonal coefficients capture an approximate systematic difference in correlations between overnight and intraday period: the fact that overnight correlations are higher than during intraday.

2.2.5 Weighting

Once $\hat{\mathbf{S}}$ estimator of $\mathbf{S}$ is obtained (equation 2.5), it is possible to get two conditionally unbiased estimators for $\mathbf{V}_t$ from matrices $\hat{\mathbf{V}}_{id,t}$ - realized covariance matrix for intraday period and $\hat{\mathbf{V}}_{on,t} = \mathbf{r}_{on,t}\mathbf{r}'_{on,t}$ - product of overnight returns:

$$\hat{\mathbf{V}}_{t|id} = \hat{\mathbf{V}}_{id,t} + \hat{\mathbf{S}}\hat{\mathbf{V}}_{id,t}\hat{\mathbf{S}} \tag{2.8}$$

$$\hat{\mathbf{V}}_{t|on} = \hat{\mathbf{S}}^{-1}\hat{\mathbf{V}}_{on,t}\hat{\mathbf{S}}^{-1} + \hat{\mathbf{V}}_{on,t} \tag{2.9}$$

Estimator of the covariance for the whole day will also be unbiased:

$$\hat{\mathbf{V}}_t^{\sim} = \omega\hat{\mathbf{V}}_{t|id} + (1-\omega)\hat{\mathbf{V}}_{t|on} \tag{2.10}$$

where ω is a weighting coefficient that belongs to $[0;1]$.

Weighting coefficient should be chosen in order to minimize the expected mean square error $E(\|\hat{\mathbf{V}}_t - \mathbf{V}_t\|^2)$. However true matrix of realized covariance $\mathbf{V}_t$ is not observable.

In the paper of Hansen & Lunde (2005) it was proven, that in the univariate case solution of the minimization problem $E((\hat{V} - V)^2) \rightarrow min$ is equal to the solution of the problem $var(\hat{V}) \rightarrow min$.

In the multivariate case the way of defining the distance between matrices is not unique. Therefore there are many options of choosing ω.

- Minimize the squared Frobenius distance to the true matrix.
  Frobenius norm on the space of matrices is one of the most frequently used. It is defined as $\|\mathbf{A}\|_\Phi = \sqrt{\sum_{i,j} a_{ij}^2}$. It is possible to prove that under the condition of unbiasness of $\hat{\mathbf{V}}_t^{\sim}$ the minimization problem $E(\|\hat{\mathbf{V}}_t^{\sim} - \mathbf{V}_t\|_\Phi^2) \rightarrow min$ is equivalent to the sum of the elements' variances minimization problem $\sum_{i,j} var(a_{i,j}) \rightarrow min$, what is similar to the results of Hansen & Lunde (2005). In this case weighting coefficient

ω is equal to:

$$\omega_{\Phi}^{*} = \frac{\sum_{ij} var(\hat{v}_{ij}^{on}) - \sum_{ij} cov(\hat{v}_{ij}^{on}, \hat{v}_{ij}^{day})}{\sum_{ij} var(\hat{v}_{ij}^{day}) + \sum_{ij} var(\hat{v}_{ij}^{on}) - 2\sum_{ij} cov(\hat{v}_{ij}^{on}, \hat{v}_{ij}^{day})} \tag{2.11}$$

- Minimize the sum of squared variances error.
  This problem is similar to the previous one, but includes only elements on the main diagonal - variances and not includes covariances. It can be written as $E(\|diag(\hat{\mathbf{V}}_t - \mathbf{V}_t)\|_{\Phi}^2) \to min$. Overnight covariance matrix is built using only one observation and its rank is zero. Therefore it is presumed to be a very noisy estimator. However, the covariances may be less noisy than variances in this case: overnight correlation is always equal to one by module. So, there is a sense not to take into account variance of error of covariances.

$$\omega_{diag}^{*} = \frac{\sum_{i} var(\hat{v}_{ii}^{on}) - \sum_{i} cov(\hat{v}_{ii}^{on}, \hat{v}_{ii}^{day})}{\sum_{i} var(\hat{v}_{ii}^{day}) + \sum_{i} var(\hat{v}_{ii}^{on}) - 2\sum_{i} cov(\hat{v}_{ii}^{on}, \hat{v}_{ii}^{day})} \tag{2.12}$$

- Minimize the portfolio variance error.
  As the covariance matrix is typically used in portfolio optimization or in risk-management of the portfolio of assets, it may be reasonable to minimize the mean squared error of the portfolio variance: $E(\mathbf{w}'\hat{\mathbf{V}}\mathbf{w} - \mathbf{w}'\mathbf{V}\mathbf{w})^2 \to min$ where $\mathbf{w}$ is a vector of of weights of each asset in portfolio. Disadvantage of this method is that the noisiness of overnight covariance decreases greatly, therefore the weight of the $\hat{\mathbf{V}}_{t|on}$ may be overestimated.

$$\omega_{w}^{*} = \frac{var(\mathbf{w}'\hat{\mathbf{V}}_{t|on}\mathbf{w}) - cov(\mathbf{w}'\hat{\mathbf{V}}_{t|id}\mathbf{w}, \mathbf{w}'\hat{\mathbf{V}}_{t|on}\mathbf{w})}{var(\mathbf{w}'\hat{\mathbf{V}}_{t|on}\mathbf{w}) + var(\mathbf{w}'\hat{\mathbf{V}}_{t|id}\mathbf{w}) - 2cov(\mathbf{w}'\hat{\mathbf{V}}_{t|day}\mathbf{w}, \mathbf{w}'\hat{\mathbf{V}}_{t|on}\mathbf{w})} \tag{2.13}$$

2.2.6 Scaling-and-weighting estimator

To sum up, Scailng-and-weighing estimator of the whole day covariance matrix may be computed using the following algorithm:

- calculate matrix $\hat{\mathbf{S}}$ by equation 2.5 using average intraday and average overnight covariance matrices for the previous days
- calculate weight ω^* using one of the criteria 2.11 2.12 or 2.13
- estimate the covariance matrix $\hat{\mathbf{V}}_t^{day}$ for the current day using one of the commonly used estimator (see section 2.2.2)
- compute the scaled-and-weighted estimator of covariance for the whole day $\hat{\mathbf{V}}_t^{\sim} = \omega^*(\hat{\mathbf{V}}_{id,t} + \hat{\mathbf{S}}\hat{\mathbf{V}}_{id,t}\hat{\mathbf{S}}) + (1-\omega^*)(\hat{\mathbf{V}}_{on,t} + \hat{\mathbf{S}}^{-1}\hat{\mathbf{V}}_{on,t}\hat{\mathbf{S}}^{-1})$

2.3 Simulation study

The crucial assumption of the paper is that overnight covariance is a known function of intraday one as in equation 2.4. In this section is shown, that even if this assumption doesn't hold, the proposed scaling-and-weighting estimator is still a good approximation of the whole day covariance. So, this estimator is robust to misspecification. It is shown, that at least it outperforms the existing 'naive' estimator (equation 2.1).

2.3.1 Data-generating process

The data generating process for the intraday covariance matrix is a multifactoral model. In this model the prices of the stocks are determined by many factors, and the amount of factors exceeds the amount of stocks.

$$\mathbf{V}_{id} = \mathbf{AFA}' \tag{2.14}$$

$\mathbf{V}_{id}$ is a $n \times n$ covariance matrix of the stocks, F is a diagonal $m \times m$ matrix of factor variances. So, $\mathbf{A}$ is a $n \times m$ matrix that represents sensibility of prices to the dynamic of factors.

At each period of time matrix A remains constant, while the matrix of volatility of the factors changes. The dynamic of the volatility of each factor follows Ornstein-Uhlenbeck process.

$$df_{ii} = \theta(\mu - f_{ii})dt + \sigma dW_t \tag{2.15}$$

It is a mean-reverting process, so, even if variance of factor is stochastic, it is still a stationary process.

Overnight covariance is determined as a weighted sum of two models.

The first model assumes, that half of the factors remain constant during the night. So, their volatility is zero. So, the matrix $\mathbf{G}$ is the same as $\mathbf{F}$ up to some diagonal element $g_{\frac{n}{2},\frac{n}{2}}$ and all other element are equal to zero. In this case the overnight covariance is described by the following equation.

$$\mathbf{V}^1_{on} = \mathbf{AGA}' \tag{2.16}$$

If the 'turned off' elements of $\mathbf{F}$ represent factors that influence mostly only one of the stock prices, while factors that remain 'turning on' influence all the stocks equally, the correlation during the night will be higher than during the day. It is a realistic assumption, as the overnight news will more probably be referred to the whole world or a country, while the news important for the industry or one single firm will more probably appear during the day.

Another model is simpler, but more different to the initial assumption. It just represent the fact, that correlation during the night is higher during overnight period, without assumptions about reasons for it. In this model, variances of the stock remains the same as in equation 2.16. But the correlation matrix is a weighted average of intraday correlation matrix and a matrix of ones.

$$\mathbf{V}_{on}^{2} = \sqrt{diag(\mathbf{V}_{on}^{1})}((1-s)\mathbf{C}_{id} + s\mathbf{U})\sqrt{diag(\mathbf{V}_{on}^{1})} \tag{2.17}$$

Here $\mathbf{C}_{id}$ is an intraday correlation matrix, $\mathbf{U}$ is a matrix of ones and $s \in (0,1)$ is a weight than represents the increase of the correlation during overnight period.

The resulting true overnight covariance is an average of two models.

$$\mathbf{V}_{on} = w\mathbf{V}_{on}^{1} + (1-w)\mathbf{V}_{on}^{2} \tag{2.18}$$

2.3.2 Properties of the simulations

In this simulations dimensionality of matrix $\mathbf{V}_{id}$ is 10 and amount of factors is three times more. So, $n = 10, m = 30$.

Matrix $\mathbf{A}$ is composed by three $n \times n$ matrices, last of them is diagonal. Each element of the matrix is determined by a random number ϵ that has an Uniform distribution from zero to one.

$$\mathbf{A} = \begin{bmatrix} \epsilon_{1,1} & ... & \epsilon_{1,n} & \epsilon_{1,n+1} & ... & \epsilon_{1,2n} & h\epsilon_{1,2n+1} & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ \epsilon_{n,1} & ... & \epsilon_{n,n} & \epsilon_{n,n+1} & ... & \epsilon_{n,2n} & 0 & ... & h\epsilon_{n,3n} \end{bmatrix} \tag{2.19}$$

The diagonal matrix represents individual factors, that affect only one single stock. So, during the overnight period all the individual factors are turned off, as well as a quarter of common factors. At the same time even during the night amount of factors exceed amount of stocks.

As price of each stock depends upon 20 common factors and only one individual, the latest is multiplied by constant h. In the basic simulation this constant is set to be equal to 10. So, in average, two thirds of the stock volatility is explained by common factors and one third by individual one. During the overnight period, variance is explained only by common factors. This leads to the significant increase in correlation during overnight period.

Vectors of parameters θ, μ, σ for equation 2.15 are also initialized in a similar way.

$$\theta_i = \tilde{\theta}\epsilon_{i,\theta} \tag{2.20}$$

$$\mu_i = \tilde{\mu}\epsilon_{i,\mu} \tag{2.21}$$

$$\sigma_i = \tilde{\sigma}\epsilon_{i,\sigma} \tag{2.22}$$

In the basic scenario of simulations parameters are equal to: $\tilde{\theta} = 0.02$, $\tilde{\mu} = 2$, $\tilde{\sigma} = 1$.

Weight s is set to be 0.5, so, an overnight correlation lies just in the middle between intraday correlation and 1.

After generating the constant parameters the simulation is run for 1000 periods to get 2000 of 'true' covariance matrices: for each day there is a true intraday covariance matrix $\mathbf{V}_{id,t}$ and a true overnight covariance matrix $\mathbf{V}_{on,t}$.

Estimator of intraday covariance matrix (realized covariance) and a vector of overnight returns are assumed to be observed with noise. Overnight return is a realization of multivariate Normal distribution $\mathbf{r}_{on,t} \sim N(0, \mathbf{V}_{on,t})$. Realized covariance estimator is modelled as a rescaled Wishart distribution with l degrees of freedom, that represents the preciseness of the estimator. $\hat{\mathbf{V}}_{id,t} \sim W(\frac{1}{l}\mathbf{V}_{id,t}, l)$ In the standard scenario preciseness is set to 50, that is an equivalent of 10 minute sampling for 8 hours of intraday trading. In practice, realized covariance estimator are sampled at higher frequency. The frequently used Zhang two-scale estimator may have a short grid of 1 minute. However, the presence of noise and Epps effect decrease the preciseness of the estimator, so, 50 degrees of freedom seems to be a realistic assumption.

Based on the simulation of 1000 observed realized covariances and overnight returns two competing estimators are computed. For the scaling-and-weighting estimator ($\mathbf{V}^{\sim}$) the scaling matrix $\hat{\mathbf{S}}$ and optimal weights are computed for each day based on the previous 100 days observations.

The naive estimator ($\mathbf{V}^{+}$) is computed as a sum of realized covariance $\hat{\mathbf{V}}_{id,t}$ and outer product of vector $\mathbf{r}_{on,t}$.

2.3.3 Results of the simulations

Two estimators were compared using the Standardized Loss Function approach of Diebold & Mariano (1995).

The loss function of estimators is defined as a Frobenius distance between an estimator and a true covariance matrix.

$$LF_{sw} = ||V_t - V_t^{\sim}||_{\Phi} \tag{2.23}$$

$$LF_{nv} = ||V_t - V_t^{+}||_{\Phi} \tag{2.24}$$

There would be tested the hypothesis that in average $LF_{sw} = LF_{nv}$, so, there are no reason to prefer scaling estimator to the naive one. Consider the following statistics.

$$\tau = \frac{(\bar{LF}_{sw} - \bar{LF}_{nv})\sqrt{n}}{\sqrt{var(LF_{sw} - LF_{nv})}} \tag{2.25}$$

As shown in Diebold & Mariano (1995) the statistics 2.25 has a Standard Normal distribution. So, it is possible to test its equality to zero in order to check the significance of the difference between two estimators.

In the table 2.1 the results of the simulations for different scenarios are reported: value of τ statistics, and the percent of observations for which the loss function of naive estimator is less than loss function of scaling estimator. The critical 1% level of τ-statistics is equal to 2.33.

In the basic scenario the scaling-and-weighting estimator significantly outperforms the naive one.

Other scenarios were run it order to ensure the robustness of the estimator to different specifications of data-generating process. Increase of the mean-reverting coefficient μ ten times makes the variance process look more similar to white noise process, while increase of σ allows the process to walk far from its mean and on the finite sample look like Brownian motion. Increased variance of the factor volatility leads to the less precise estimation of scaling matrix, and consequently lower the performance of the scaling-and-weighting estimator. However, the difference is still highly significant, so the estimator is robust to the high volatility of the volatility.

Choosing only one model of overnight return ($w = 0, w = 1$) only increase the performance of the estimator. So, the scaling function fits the data well even if the true process of increasing overnight correlation is different from assumed.

The proposed estimator is robust also to the situation when the difference between overnight and intraday correlation is not so high. If we set $h = 1$ it would mean that only about 5% of the stock variance is explained by individual factor. It will decrease the difference in correlations to the values about 0.05-0.1. Setting $s = 0.1$ we get the overnight correlation is only 10% more close to one then intraday one. However, the estimator is robust to such changes even if they happen in both models. Even choosing only one model with a small difference in the correlations does not decrease the significance of the test.

The estimator is robust even to a decrease of preciseness of the realized covariance estimator. If we set degrees of freedom equal to $l = 15$ that is equivalent to 30 minutes sampling on one grid, the test statistics is still far from zero and significant at 1% level.

However, unprecise realized covariance estimation together with a small difference in correlation slightly decreases the performance of the estimator. 15 observations do not let to estimate the correlation coefficients with a desired preciseness, so in 5.3% of cases it a 'naive' estimator is closer to the true covariance matrix. In average, difference between loss functions is still significantly lower than zero.

So, using simulated data, it was shown, that the proposed estimator is robust to misspecification and to the increased variance of the market volatility. It was also shown that the estimator is robust to the non-preciseness of the realized covariance estimator and to the decrease of the difference between intraday and overnight correlations, but the both can decrease the performance of the proposed estimator: it is still necessary to have enough observation to estimate the difference between overnight and intraday correlation. In all the considered cases, the 'scaling-and-weighting' estimator significantly outperforms 'naive' summing estimator.

2.4 Empirical application

2.4.1 Data

Empirical validation was performed on the data from Russian stock market. There was chosen nine stocks: Gazprom, Lukoil, Sberbank, Magnit, Nornikel, Surgutneftegas, Novatek, VTB and Rosneft. These stocks have the highest free-float capitalization. Weights of each stock in the MICEX index exceed 4%. The sample covers companies from different industries: oil&gas, metals&mining, banking and retail. The sample period was from 2009 to 2015 years.

Tick data on intraday stock prices was downloaded from website of "Finam" (www.finam.ru) - the largest broker in Russia.

For each of the day in the sample there were calculated realized covariances using double subsampling estimator with 30-minute large scale and 1 minute small scale (see Zhang *et al.* (2005), de Pooter *et al.* (2008), Zhang (2011)).

Then, for each year there was computed a scaling matrix S and an optimal weight. To make an out-of-sample analysis, both the weight and scaling matrix $\hat{\mathbf{S}}$ were used from the previous year. Say, each estimator of covariance matrix for 2010 year was computed with a scaling matrix and weight of 2009 year and so on.

The problem with calculation of weights is that extreme observations enter the average squared. So, an optimal weight may be very sensitive to the one or two outlier. To decrease this effect, Hansen & Lunde (2005) delete 1% of outliers when computing weights.

In this paper from 0 to 10% of the extreme observations were excluded. In the Table 2.5 there are resulting weights.

Among three methods of weight computation, minimization of Frobenius distance with 5 observations excluded (out of 250) has the least variance from year to year. This weight was used in empirical testing.

Average scaling matrix is presented in table 2.4. The matrix is nearly diagonal. However, each non-diagonal element is positive, what could not happen if these elements were insignificant. Testing hypothesis about diagonal scaling matrix is equivalent to the test of equality of correlation matrices. Using Larntz & Perlman (1985) test of equality of correlation matrix it is possible to reject the hypothesis of equality of overnight and intraday correlation matrices. Maximum difference between overnight and intraday z-transformed correlations is equal to 0.69, while minimum difference is 0.29. Given 1500 observations it corresponds to 18.75 and 7.86 valued of the test statistics, what is far beyond the 1% critical 3.52 level. So, even if each non-diagonal element of the scaling matrix is relatively small, the hypothesis that scaling matrix is diagonal is rejected.

In order to show, that scaling estimator $\mathbf{V}^{\sim}$ (equation 2.10) is more precise then naive estimator $\mathbf{V}^{+}$ (equation 2.1), three tests will be performed: normality of residuals, Value-at-Risk forecast, Minimum Variance portfolio optimization. For all these tests a simple forecasting model will be assumed:

$$E(\mathbf{V}_{t+1}|\mathbf{V}_t) = \mathbf{V}_t \tag{2.26}$$

Such a parsimonious model is assumed in order to avoid model specific bias. The target of empirical part of the present paper is not to provide the better forecast of next day covariance matrix, but only to show that scaling estimator is more precise then naive one. Usage of equation 2.26 implicitly assumes, that more precise estimator of today covariance matrix is a better forecast of tomorrow covariance matrix.

2.4.2 Residual test

If we multiply the inverse of the square root of the covariance matrix by the vector of returns for this day, the resulting vector - vector of residuals - will have standard Gaussian distribution. $(\mathbf{V}_t)^{-1/2}\mathbf{r}_t \sim N(0, \mathbf{I})$ So, better estimator of the whole day covariance should provide residuals that are closer to Standard Gaussian. This test in some variations is used in recent papers, such Andersen *et al.* (2007), Peluso *et al.* (2015). It is noted, that even if in reality presence of jumps leads to more fat-tailed distribution than Normal, both estimators are equally suffered from this. So, in this section return of the next day divided by the square root of covariance matrix of today, following an out-of-sample model 2.26.

In the table 2.6 properties of the rescaled returns are shown. It is easy to see, that scaling estimator $\mathbf{V}^{\sim}$ provides residuals closer to Standard Gaussian then naive estimator $\mathbf{V}^{+}$. Its variance is closer to 1, excess kurtosis is closer to 0, consequently, Jarque-Bera test is lower. Autocorrelation of residuals is also slightly closer to 0. Frobenious distance of covariance matrix of residuals to identity $\mathbf{I}$ matrix is closer to 0, as well as one of correlation matrix.

So, the residuals of the scaling estimator are more close to the standard Gaussian distribution then the residuals of naive estimator. It means that the first estimator provides more accurate estimation of the whole day covariance matrix.

2.4.3 Portfolio risk-management

One of the practical problem for which it is necessary to calculate the covariance matrix is a risk-management of a portfolio of stocks. There was constructed a portfolio of each stock with the same weights 1/9. For each day, there was calculated a covariance matrix using two competing methods.

Then, each covariance matrix was assumed to be a prediction of the covariance for the next day. The variance of the portfolio is: $v_t = \mathbf{w}'\mathbf{V}_{t-1}\mathbf{w}$ where $\mathbf{w} = (1/9, \ldots, 1/9)$. Using this variance there was calculated Value-at-Risk of 1%, 5%, and 10% level, as a quantile of Normal Gaussian distribution.

Assumption that tomorrow covariance is equal to the today realized covariance is too strong. So, as one can see from the table 2.7 the amount of exceptions (days, at which return is lower then calculated Value-at-Risk) for both estimators is much higher then confidence level. For the 1% each Value-at-Risk estimator provides 3 to 14 exceptions per year that is higher then expected 2.5 exceptions. However, for each year, the number of exceptions of scaling estimator is not higher then the number of exceptions of 'naive' estimator.

With the increase of confidence level the number of exception is more closer to the expected one. For 5% and 10% Value-at-Risk the number of expected violations is 12.5 and to 25 (75 and 150 for the 6-year period). And number of violations is still smaller if use a 'scaling' estimator, rather then 'naive' one (except for one case of 5% VaR at 2013).

Unconditional Coverage Test by Kupiec (1995) shows, that number of violations for all the cases significantly exceed the expected level. As it is shown in the table 2.8, only for 10% Scaling VaR probability of exception is not significantly larger than confidence level.

Test of equality of probabilities of binomial distributions shows (table 2.9), that there are no significant differences in number of exceptions between Scaling and Naive estimators.

In order to show, that difference between two estimators is statistically significant, Diebold-Mariano statistics 2.25 of different loss functions is presented at table 2.10.

If we use the fact of exception ($I = I_{r_t<VaR}$) as a loss-function (following Lopez (1998)), then scaling-and-weighting estimator turns out to provide significantly better value-at-risk forecast than naive estimator. This difference is significant at 5% level for 1% and 5% VaR, while for 10% VaR it is significant even at 1% level. Significance of DM-test, while non-significant UCT means, that even if difference between number of violations is too small, violation of scaling VaR happens mostly in the same days when naive VaR also violates.

Other loss functions for Value-at-Risk forecasts can be divided into "regulator's" and "firm's" loss functions (see Abad *et al.* (2015)).

Regulator's loss functions penalize difference between VaR forecast and realized return only in the case of VaR violence. So, regulator loss functions look like $LF = I_{r_t<VaR} f(r_t, VaR)$, where $f(r_t, VaR)$ is some positive function, increasing with difference $VaR - r_t$. In table 2.10 there are results for the following loss functions. RQL is a Quadratic Lopez (1998) loss function with $f = 1 + (VaR - r_t)^2$. As for daily returns quadratic difference is much smaller then one, the DM-statistics for RQL is almost the same as statistic

for indicator loss function. RL is a Linear loss function $f = VaR - r_t$ that is equal to $f = |VaR - r_t|$ - a regulatory loss functions RC3 by Caporin (2008). RQ is a Quadratic loss function proposed by Sarma *et al.* (2003) equal to $f = (VaR - r_t)^2$. RC1 and RC2 are the Caporin (2008) loss functions, which show magnitude of violation divided by VaR. For RC1 $f = |1 - |\frac{r_t}{VaR}||$, and for RC2 $f = \frac{(|r_t| - |VaR|)^2}{|VaR|}$.

Firm's loss functions penalise not only the exceptions, but also regular difference, that represents opportunity cost for the firm. In other words firm's loss functions prevent firms from excess reserving. In general form these loss functions look like $LF = If(r_t, VaR) + (1 - I)g(r_t, VaR)$.

FS is a firm's Sarma *et al.* (2003) loss function, in which f is the same as in RQ, while $g = -\beta VaR_t$. FABL is a similar loss function by Abad *et al.* (2015), with only difference that $g = \beta(r_t - VaR)$. In both cases Abad *et al.* (2015) suggest to consider β as an interesting rate. In this paper this coefficient is set to be $\beta = 1.05^{\frac{1}{365}} - 1$, that is equivalent of 5% interest rate.

Caporin (2008) suggested to use symmetric loss $f = g$ for the Value-at-Risk forecasts that pass Unconditional Coverage Test. However, in this paper, only one out of six VaR forecasts have number of violations not significantly different form the confidence level. So, the minimum of the Caporin's loss function is not guaranteed to be at the true VaR, at least because they are independent from the confidence level. So, Caporin's loss functions were modified, and set $f = \frac{1-\alpha}{\alpha}g$, where $\frac{1-\alpha}{\alpha}$ is equal to 99 for 1% confidence level, 19 for 5% confidence level and 9 for 10% confidence level.

Note, that even after modification, none of the loss functions except FC3Vm are consistent loss functions Gneiting (2011a), in the sense, that their minimums are not the true quantile. However, the results of all functions are presented at the table 2.10 for illustrative purposes. Even if "regulator" loss functions are not consistent for quantile estimation, they provide an indication of magnitude of VaR violations. "Firm" loss functions provide an indication of cost to the firm to use one of the VaR prediction as a risk measure.

As one can see from the table 2.10 for the majority of loss functions the difference between two forecasts is significant. In all the cases scaling estimator, proposed in this paper, provides a better Value-at-Risk forecast than naive estimator, even in the cases, when this difference is not significant. The most important in this table are first and last rows. The first row - is a different way of measuring difference in number of exceptions, while the last - is the only consistent loss-function.

To be precise, for 1% VaR the difference is not significant for the loss functions that include quadratic term. It means that for both models 1% VaR violations have large magnitude.

In total, out of 33 test statistics, 10 are significant at 1% confidence level, 12 are significant at 5% level, 5 are significant at 10% level, and only 6 are insignificant at 10% level.

So, parsimonious forecasting model provides better Value-at-Risk predictions if scaling estimator is used rather then 'naive' one.

2.4.4 Portfolio minimum variance optimisation

Another test - is a comparison of performance of two portfolio managers that minimize the variance of the portfolio using computed covariance matrices as predictors for the next day covariance of the assets. In this framework it is also assumed that the portfolio is rebalanced every morning at the beginning of the trading day.

Unlike the previous case, the weights of the portfolio are calculated using equation:

$$\mathbf{w}_{t+1} = \frac{\mathbf{V}_t^{-1}\mathbf{i}}{\mathbf{i}'\mathbf{V}_t^{-1}\mathbf{i}} \tag{2.27}$$

where $\mathbf{i} = (1, \ldots, 1)$ is a unit vector. As estimator for $\mathbf{V}_t$ one portfolio manager uses $\hat{\mathbf{V}}_t^{\sim}$ as defined in the equation 2.10 and another manager uses $\hat{\mathbf{V}}_t^{+}$ from equation 2.1.

As you can see in the table 2.11 standard deviation of the portfolio returns at each year is lower for scaling estimator then for 'naive' estimator. On the one hand, the difference is not significant if we use F-test of equality of variances. But on the other hand, if these variances were completely equal, we could expect about 50% at each year 'naive' estimator outperforms a 'scaling' one. However, the scaling estimator provides better results for every year. The probability of such event is $0.5^6 = 1/64$. Moreover, if we use Diebold & Mariano (1995) statistics (equation 2.25) for the loss function $LF = r_t^2$ (simply test equality of the variances in different way) it will be equal to -2.4, that is significant at 1% level.

Both methods do not produce a good predictor of the Value-at-Risk as their violations rates are higher than a confidence interval. This is linked to the fact, that noise in covariance estimator leads to the double mistake: one when computing the weights of the portfolio, and another when computing the variance of such portfolio. That leads to underestimation of the variance of portfolio. However, Value-at-Risk calculated using the scaling estimator is also more reliable than one, computed by the old method - 'naive' estimator. It has much less violations for almost every year at 1%, 5% and 10% level. And this difference is even significant at 10% confidence level for 1% VaR and 5% VaR, as it is show at table 2.12.

Moreover, applying the same loss functions based test of equality of VaR forecasts as was applied for equiweighted portfolio, provides DM statistics significantly negative at 1% level, except for one, that is still significant at 5% level.

So, the proposed estimator outperforms the existing one also in this test.

Using scaling estimator rather than naive, investor will decrease the variance of his portfolio, and also get more precise Value-at-Risk forecast.

2.5 Conclusion

In this paper the function of direct proportionality was extended to the positive semi-definite symmetric matrices. This functions was used to rescale intraday and overnight covariance matrices and receive two component of the covariance estimation for the whole day. The resulting estimator is a weighted sum of them.

At the simulation studies the invented estimator shows robustness to the misspecification of the model. It is also robust to the situation when the difference between overnight and intraday correlation is very small, and robust to the noise of underlying realized covariance estimator, as long as it allows to capture difference between overnight and intraday correlations.

In the test of residuals, the proposed estimator shows higher performance than alternative typically used estimator - the sum of intraday and overnight covariance matrix. In practical applications such as forecasting the portfolio Value-at-Risk and minimization of portfolio variance, it also performs significantly better.

The proposed method still can be improved in different ways. For example it is possible to use shrinkage covariance estimation for the overnight covariance (see e.g. Ledoit & Wolf (2004), Bai & Shi (2011)), before using the method proposed in this paper. It may improve the overnight component of the estimator making it strictly positive definite and decrease its noise.

There is still a question unsolved in the paper, what the distribution of the scaling matrix $\hat{\mathbf{S}}$ is. Once the distribution will be known, it will be possible to construct confidence interval for elements of these matrix, test their equality to zero, and test some specific form of the matrix $\mathbf{S}$, such as diagonality or equality of all its diagonal terms.

2.6 Appendix

Table 2.1: Results of the simulations

Scenario	τ	$\tau < 0$
Standard	-89	0.0%
$\theta = 0.2$	-100	0.0%
$\sigma = 10$	-61	3.4%
$\theta = 0.2, \sigma = 10$	-107	0.0%
$w = 0$	-118	0.0%
$w = 1$	-108	0.0%
$s = 0.1$	-118	0.0%
$h = 1$	-84	0.5%
$h = 1, s = 0.1$	-91	0.0%
$w = 0, h = 1$	-81	0.0%
$w = 1, s = 0.1$	-104	0.0%
$l = 15$	-74	0.0%
$l = 15, h = 1, s = 0.1$	-45	5.3%
$l = 15, w = 0, h = 1$	-62	0.1%
$l = 15, w = 1, s = 0.1$	-52	2.3%
$\sigma = 10, l = 15, h = 1, s = 0.1$	-54	0.8%

Table 2.2: Average Intraday Correlation Matrix

	GAZP	LKOH	SBER	MGNT	GMKN	SNGS	NVTK	VTBR	ROSN
GAZP	100%	60%	62%	30%	49%	53%	47%	53%	61%
LKOH	60%	100%	50%	30%	47%	54%	47%	45%	59%
SBER	62%	50%	100%	28%	44%	48%	42%	56%	53%
MGNT	30%	30%	28%	100%	24%	29%	28%	24%	28%
GMKN	49%	47%	44%	24%	100%	44%	38%	40%	47%
SNGS	53%	54%	48%	29%	44%	100%	44%	43%	53%
NVTK	47%	47%	42%	28%	38%	44%	100%	36%	44%
VTBR	53%	45%	56%	24%	40%	43%	36%	100%	46%
ROSN	61%	59%	53%	28%	47%	53%	44%	46%	100%

Table 2.3: Average Overnight Correlation Matrix

	GAZP	LKOH	SBER	MGNT	GMKN	SNGS	NVTK	VTBR	ROSN
GAZP	100%	81%	87%	57%	72%	79%	73%	82%	82%
LKOH	81%	100%	77%	53%	68%	79%	71%	75%	79%
SBER	87%	77%	100%	57%	73%	77%	74%	87%	82%
MGNT	57%	53%	57%	100%	49%	55%	56%	57%	52%
GMKN	72%	68%	73%	49%	100%	69%	63%	72%	71%
SNGS	79%	79%	77%	55%	69%	100%	69%	76%	77%
NVTK	73%	71%	74%	56%	63%	69%	100%	72%	73%
VTBR	82%	75%	87%	57%	72%	76%	72%	100%	79%
ROSN	82%	79%	82%	52%	71%	77%	73%	79%	100%

Table 2.4: Average Scaling Matrix

35%	3%	5%	2%	2%	3%	2%	4%	2%
3%	34%	2%	1%	1%	3%	2%	2%	2%
5%	2%	36%	3%	4%	3%	4%	6%	4%
2%	1%	3%	33%	2%	2%	3%	3%	1%
2%	1%	4%	2%	40%	2%	2%	4%	3%
3%	3%	3%	2%	2%	32%	2%	4%	3%
2%	2%	4%	3%	2%	2%	35%	4%	3%
4%	2%	6%	3%	4%	4%	4%	33%	4%
2%	2%	4%	1%	3%	3%	3%	4%	33%

Table 2.5: Optimal Weights of Rescaled Intraday Covariance Matrix

0 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	88%	97%	82%	94%	96%	68%	99%	0.01227
Trace weight:	91%	95%	86%	93%	97%	70%	98%	0.00950
Portfolio weight:	76%	98%	71%	94%	93%	51%	103%	0.03466
5 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	90%	88%	92%	90%	92%	91%	93%	0.00025
Trace weight:	89%	82%	91%	89%	93%	92%	93%	0.00149
Portfolio weight:	84%	81%	93%	83%	78%	81%	84%	0.00221
10 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	87%	86%	89%	87%	88%	91%	89%	0.00029
Trace weight:	84%	78%	86%	85%	87%	92%	88%	0.00167
Portfolio weight:	81%	78%	86%	80%	78%	70%	80%	0.00205
15 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	87%	85%	88%	84%	86%	89%	87%	0.00037
Trace weight:	82%	76%	87%	78%	85%	90%	85%	0.00257
Portfolio weight:	83%	72%	83%	73%	76%	78%	76%	0.00180
20 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	87%	84%	87%	83%	84%	88%	85%	0.00044
Trace weight:	82%	79%	83%	76%	78%	83%	81%	0.00069
Portfolio weight:	86%	62%	80%	72%	74%	75%	73%	0.00536
25 observations excluded	2009	2010	2011	2012	2013	2014	2015	Variance
Frobenius weight:	88%	81%	86%	83%	82%	87%	84%	0.00069
Trace weight:	83%	73%	84%	78%	71%	85%	79%	0.00312
Portfolio weight:	85%	58%	73%	71%	77%	73%	68%	0.00699

Table 2.6: Properties of the Distribution of Rescaled Returns

	$V^{\sim}$	V^{+}
Variance	1.81	1.92
Excess Kurtosis	3.14	3.65
Jarque-Bera test	5565	7514
Autocorrelation	3.95%	4.04%
F-distance of covariance to $\mathbf{I}$	7.07	8.96
F-distance of correlation to $\mathbf{I}$	4.75%	4.88%

Table 2.7: Value-at-Risk Violations for Portfolio of Equal Weights

	Violations		Consequent Violations	
	Scaling Estimator	Naive Estimator	Scaling Estimator	Naive Estimator
VaR 1%, 2010	8	9	0	0
VaR 1%, 2011	12	14	0	0
VaR 1%, 2012	8	10	0	0
VaR 1%, 2013	6	6	0	0
VaR 1%, 2014	6	7	0	1
VaR 1%, 2015	3	3	0	0
VaR 1%, 2010-2015	43	49	0	1
VaR 5%, 2010	15	17	0	0
VaR 5%, 2011	21	25	1	1
VaR 5%, 2012	19	21	0	1
VaR 5%, 2013	18	17	0	0
VaR 5%, 2014	16	19	2	1
VaR 5%, 2015	18	18	0	0
VaR 5%, 2010-2015	107	117	3	3
VaR 10%, 2010	25	25	2	2
VaR 10%, 2011	30	37	1	2
VaR 10%, 2012	24	27	2	4
VaR 10%, 2013	31	32	1	1
VaR 10%, 2014	29	30	4	3
VaR 10%, 2015	28	30	2	2
VaR 10%, 2010-2015	167	181	12	14

Table 2.8: Unconditional Coverage Test for Equiweighted Portfolio VaR Forecasts

Value-at-Risk	UCT χ-value	prob.
$V^{\sim}$ 1% VaR	35.1	0.0%
V^{+} 1% VaR	48.8	0.0%
$V^{\sim}$ 5% VaR	12.8	0.0%
V^{+} 5% VaR	21.3	0.0%
$V^{\sim}$ 10% VaR	2.1	15.0%
V^{+} 10% VaR	6.7	0.9%

Table 2.9: Test of Equality of Equiweighted Portfolio VaR Forecasts

Value-at-Risk	z-stat.	prob.
1% VaR	-0.64	26.3%
5% VaR	-0.69	24.4%
10% VaR	-0.80	21.2%

Table 2.10: Diebold-Mariano Statistics for Equiweighted Portfolio Value-at-Risk Forecasts

Loss Function	1% VaR	5% VaR	10% VaR
I	-1.900	-1.828	-2.562
prob.	2.87%	3.38%	0.52%
RQL	-1.900	-1.828	-2.562
prob.	2.87%	3.38%	0.52%
RL	-2.336	-3.950	-3.512
prob.	0.97%	0.00%	0.02%
RQ	-0.605	-1.632	-2.005
prob.	27.27%	5.13%	2.25%
RC1	-1.862	-3.307	-3.961
prob.	3.13%	0.05%	0.00%
RC2	-0.407	-1.500	-2.028
prob.	34.22%	6.68%	2.13%
FABL (YR = 5%)	-0.314	-1.481	-1.895
prob.	37.67%	6.93%	2.90%
FS (YR = 5%)	-0.268	-1.434	-1.838
prob.	39.45%	7.58%	3.30%
FC1Vm	-1.743	-3.049	-3.738
prob.	4.07%	0.11%	0.01%
FC2Vm	-0.113	-1.159	-1.899
prob.	45.50%	12.31%	2.88%
FC3Vm	-1.885	-2.390	-1.399
prob.	2.97%	0.84%	8.09%

Table 2.11: Properties of the Minimum-Variance Portfolio

	$V^{scaling}$	V^{naive}
Std.dev. of portfolio returns 2010	1.54%	1.58%
Std.dev. of portfolio returns 2011	2.03%	2.10%
Std.dev. of portfolio returns 2012	1.34%	1.37%
Std.dev. of portfolio returns 2013	1.14%	1.15%
Std.dev. of portfolio returns 2014	1.64%	1.67%
Std.dev. of portfolio returns 2015	1.56%	1.57%
Std.dev. of portfolio returns 2010-2015	1.57%	1.60%
VaR 1%, Number of Violations 2010	17	17
VaR 1%, Number of Violations 2011	24	35
VaR 1%, Number of Violations 2012	25	26
VaR 1%, Number of Violations 2013	18	23
VaR 1%, Number of Violations 2014	16	18
VaR 1%, Number of Violations 2015	18	20
VaR 1%, Number of Violations 2010-2015	118	139
Var 5%, Number of Violations 2010	33	39
Var 5%, Number of Violations 2011	46	49
Var 5%, Number of Violations 2012	40	41
Var 5%, Number of Violations 2013	32	40
Var 5%, Number of Violations 2014	30	37
Var 5%, Number of Violations 2015	34	35
Var 5%, Number of Violations 2010-2015	215	241
Var 10%, Number of Violations 2010	51	56
Var 10%, Number of Violations 2011	58	59
Var 10%, Number of Violations 2012	51	54
Var 10%, Number of Violations 2013	57	60
Var 10%, Number of Violations 2014	44	50
Var 10%, Number of Violations 2015	51	55
Var 10%, Number of Violations 2010-2015	312	334

Table 2.12: Test of Equality of Minimum Variance Portfolio Value-at-Risk Forecasts

Value-at-Risk	z-stat.	prob.
1% VaR	-1.37	8.5%
5% VaR	-1.32	9.3%
10% VaR	-0.97	16.4%

Table 2.13: Diebold-Mariano Statistics for Minimum-Variance Portfolio Value-at-Risk Forecasts

Loss Function	1% VaR	5% VaR	10% VaR
I	-3.666	-3.933	-2.670
prob.	0.01%	0.00%	0.38%
RQL	-3.666	-3.934	-2.670
prob.	0.01%	0.00%	0.38%
RL	-5.206	-4.396	-4.823
prob.	0.00%	0.00%	0.00%
RQ	-4.899	-6.291	-7.447
prob.	0.00%	0.00%	0.00%
RC1	-5.876	-7.181	-7.702
prob.	0.00%	0.00%	0.00%
RC2	-4.428	-5.239	-5.658
prob.	0.00%	0.00%	0.00%
FABL (YR = 5%)	-2.979	-3.499	-1.888
prob.	0.14%	0.02%	2.95%
FS (YR = 5%)	-2.985	-3.499	-3.711
prob.	0.14%	0.02%	0.01%
FC1Vm	-5.85	-7.23	-8.03
prob.	0.00%	0.00%	0.00%
FC2Vm	-4.34	-5.21	-5.83
prob.	0.00%	0.00%	0.00%
FC3Vm	-6.178	-6.928	-6.845
prob.	0.00%	0.00%	0.00%

CHAPTER 3

Asynchronous Markets: Estimation and Forecast of Covariance Matrix

Estimation of the variance-covariance matrix is not a trivial exercise if markets are asynchronous. This paper presents different methods of estimation the whole day correlations using high-frequency data from the overlapping period. Once covariance matrix estimates for each day are obtained, forecasting models become available. Different HAR and EWMA models were compared on FTSE-100 and NASDAQ data.

Keywords:

C58 Financial Econometrics
G17 Financial Forecasting and Simulation
G32 Financial Risk and Risk Management
Asynchronous markets

3.1 Introduction

Efficient asset allocation demands risk and return estimates to be as precise as possible. In order to predict variance of the portfolio, Value-at-Risk, or other risk measures, it is necessary to estimate variance-covariance matrix of the asset returns. Consequently, covariance matrix is necessary in asset-management, for mean-variance or minimum variance portfolio optimization. Even estimation of fundamental value of the firm through discounted cash flow models uses *beta* coefficient obtained from covariance matrix. So, virtually any aspect of risk-return estimation in financial markets involves forecast of covariance matrix.

Investment portfolios of companies and individuals may consist not only of domestic assets, but also include assets on international financial market. Stock exchanges, at which these assets are traded, may be located at different time-zones and have different trading hours. This causes asynchronicity problem, making estimation of covariance matrix not trivial.

For daily data the question was already discussed in the literature. In order to avoid asynchronicity problem some authors (González (2016), Chalmers *et al.* (2001)), compare only synchronous markets. Other (Berben & Jansen (2005), Christiansen (2007), Abad *et al.* (2010)) use weekly returns to decrease bias. Some methods are based on summation of product of all overlapping returns (Scholes & Williams (1977), Morgan (1996), Bergomi (2010), Sotiropoulos (2016). However, these methods do not guarantee positive semidefiniteness of the covariance matrix. Another approach is synchronization of the data through VAR or VMA model (Burns *et al.* (1998), Audrino & Bühlmann (2004), Butler & Okada (2007), Scherer (2013)).

The target of the paper is to estimate and forecast the whole day covariance matrix for a case of asynchronous markets, based on the intraday high-frequency data.

The main part of the paper is divided into two parts: estimation and forecast. In the first part, four estimators of correlation matrix are introduced and tested on real data. In the second part, a new version of HAR model is introduced, that takes into account both volatility spillovers and increasing correlation during high volatility periods. The performances of different HAR and EWMA models are compared on real data of market indices from LSE and NASDAQ. Main parts are followed by conclusion, bibliography and appendix.

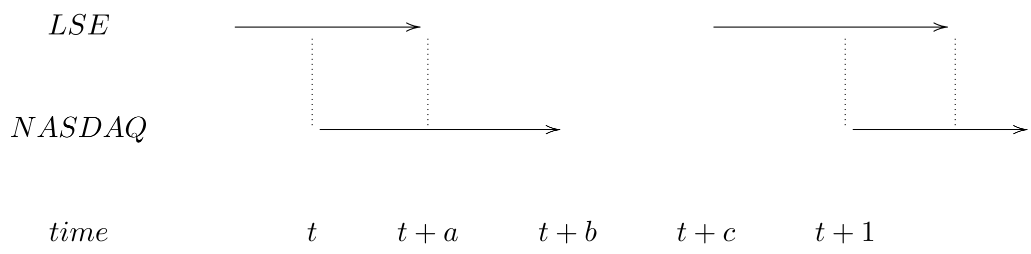
3.2 Estimation

3.2.1 Notations

Asynchronous markets are such markets, that have different trading hours. They are usually located at different time zones. Two stock exchanges can have a common trading period, like LSE and NYSE, or do not overlap at all, like NYSE and NIKKEI. The present paper is focused on the first case, when two stock exchanges have an overlapping period.

A picture 3.1 is a sketch illustrating two partially overlapping asynchronous markets: LSE and NASDAQ. At time t (14:30 UTC) NASDAQ opens and both stock exchanges are open. At time $t+a$ (16:30 UTC) trading day of LSE ends and only NASDAQ is open until $t+b$ (21:00 UTC). From $t+b$ to $t+c$ (next day 08:00 UTC) both stock exchanges are closed. Finally, from $t+c$ to $t+1$ only LSE is open.

Figure 3.1: Illustration of asynchronous stock exchanges



For definiteness, in the present paper we consider opening of NASDAQ as a beginning and ending point of the whole day.

Let us define $\mathbf{p}_t = (p_t^l, p_t^n)$ as a vector of logarithm of asset prices at time t, where the first asset is traded in London and second in New-York. So, logreturn for period $[t; t+i]$ is:

$$\mathbf{r}_{t:t+i} = \mathbf{p}_{t+i} - \mathbf{p}_t \tag{3.1}$$

Assume that logreturns are Normally distributed with zero mean and some covariance matrix.

$$\mathbf{r}_{t:t+i} \sim N(0, \mathbf{V}_{t:t+i}) \tag{3.2}$$

The target of the paper is to estimate the covariance matrix for the whole day $\hat{\mathbf{V}}_t = \hat{\mathbf{V}}_{t:t+1}$.

The assumption of normality can be weakened. In the present paper it is neither used in estimation, nor in forecast of covariance matrix, but only in the empirical tests: Value-at-Risk forecasts and Minimum-variance portfolio optimization.

3.2.2 Volatilities estimation

One approach is decomposition of covariance matrix into variances and correlations (see for example Engle (2002)).

$$\mathbf{V}_t = \begin{pmatrix} \sigma_{11,t}^2 & \rho_t \sigma_{11,t} \sigma_{22,t} \\ \rho_t \sigma_{11,t} \sigma_{22,t} & \sigma_{22,t}^2 \end{pmatrix} \tag{3.3}$$

Separate estimates of volatilities and correlation coefficient eliminate the asynchronicity problem for volatility estimation.

Estimation of the whole day volatility based on the intraday high-frequency data was already discussed in the literature (Hansen & Lunde (2005), Ahoniemi & Lanne (2013), Triacca & Focker (2014)). The present paper uses the method proposed in Hansen & Lunde (2005). Assuming that overnight volatility is proportional to intraday one, it is possible to define volatility estimator as a linear combination of intraday realized variance and overnight squared return.

$$\hat{\sigma}^2_{11,t} = \omega_1(1+s_1)(\hat{\sigma}^2_{11,t:t+a} + \hat{\sigma}^2_{11,t+c:t+1}) + (1-\omega_1)(\frac{1}{s_1}+1)(r^2_{l,t+a:t+c}), \quad (3.4)$$

$$\hat{\sigma}^2_{22,t} = \omega_2(1+s_2)(\hat{\sigma}^2_{22,t:t+b}) + (1-\omega_2)(\frac{1}{s_2}+1)(r^2_{n,t+b:t+1}), \quad (3.5)$$

where $\hat{\sigma}^2_{ii,T}$ is a realized volatility at the time interval T, s_i is a ratio between average overnight and intraday volatilities, and ω_i is chosen in such a way to minimize the variance of the whole day volatility estimator. Here, intraday realized volatility for LSE consists of two parts: one belongs to the next calendar day.

3.2.3 Correlation estimators

It is natural to use data from overlapping period for the whole day correlation estimation. However, whole day correlation appears to be higher then correlation during overlapping period. The present section proposes several methods of obtaining correlation coefficient, for simplicity called: $\mathbf{R}^S$ - scaling, ρ^L - linear, ρ^P - proportional and $\mathbf{R}^N$ - naive.

Scaling overlapping covariance matrix

Obtaining overnight covariance from the intraday realized covariance matrix is possible by multivariate rescaling defined in section 2.2.4. This method can be extended for the case of asynchronous markets. In comparison with section 2.2.4it is enough to take only correlation coefficient from it, neglecting the variances.

$$\mathbf{R}^S_t = diag(\mathbf{S}\hat{\mathbf{V}}_{op}\mathbf{S})^{-\frac{1}{2}}\mathbf{S}\hat{\mathbf{V}}_{op}\mathbf{S}\ diag(\mathbf{S}\hat{\mathbf{V}}_{op}\mathbf{S})^{-\frac{1}{2}} \quad (3.6)$$

where $\hat{\mathbf{V}}_{op} = \hat{\mathbf{V}}_{t:t+a}$ is a realized covariance matrix from overlapping period and scaling matrix $\mathbf{S} = GM(\bar{\mathbf{V}}, \bar{\mathbf{V}}_{op}^{-1})$ is a geometric mean (see Pusz & Woronowicz (1975), Ando *et al.* (2004), Iannazzo (2016)) of average covariance matrix $\bar{\mathbf{V}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t\mathbf{r}'_t$ and inverse of the average overlapping covariance matrix $\bar{\mathbf{V}}_{op} = \frac{1}{T}\sum_{t=1}^{T}\hat{\mathbf{V}}_{op,t}$.

This method guarantees that obtained correlation matrix will be positive semi-definite.

Linear regression of z-transformed correlations

The first approximation of functional dependence is a linear function. Assume, that whole day correlation ρ_t is a linear function of correlation $\rho_{op,t} =$

σ during overlapping period.

$$\rho_t = a + b\rho_{op,t} + \epsilon_t \tag{3.7}$$

However, linear dependence does not guarantee that correlation coefficient will belong to the interval $[-1; 1]$. A solution is to use Fisher's z-transformation.

$$z(\rho) = 0.5\log(\frac{1+\rho}{1-\rho}) \tag{3.8}$$

Z-transformed correlation belongs to interval $[-\infty; +\infty]$ and close to original correlation at low values of ρ. Assuming linear dependence between z-transformed whole day correlation and z-transformed overlapping correlation it is possible to fit the model:

$$z(\rho_t^L) = \alpha + \beta z(\rho_{op}) + \epsilon_t \tag{3.9}$$

As ρ_t is not directly observable, one of the solutions is to use $\bar{\rho}_t^{(m)}$ obtained from average covariance matrices for m days: $\bar{\mathbf{V}}_t^{(m)} = \frac{1}{m}\sum_{i=t-m}^{t}\mathbf{r}_i\mathbf{r}_i'$. In this paper optimal m was found by simulations (see Appendix, section 3.5).

Proportional z-transformed correlation

A modification to the previous approach is to use proportional function instead of linear at equation 3.9.

$$z(\rho_t^P) = \gamma z(\rho_{op}) + \epsilon_t \tag{3.10}$$

Both linear and proportional methods, however, guarantee positive semi-definiteness of the covariance matrix only for a case of two assets.

Naive estimator

The last approach is to treat a return out of overlapping period as one of the observation for realized covariance estimation.

$$\mathbf{V}_t^N = \hat{\mathbf{V}}_{op} + \mathbf{r}_{t+a:t+1}\mathbf{r}'_{t+a:t+1} \tag{3.11}$$

$$\mathbf{R}_t^N = diag(\mathbf{V}_t^N)^{-\frac{1}{2}}\mathbf{V}_t^N diag(\mathbf{V}_t^N)^{-\frac{1}{2}} \tag{3.12}$$

Disadvantage of this method is very high noise of correlation coefficient, as the major part of the covariance matrix 3.11 is constructed using only one observation.

3.2.4 Empirical validation

Data

Four methods were applied to the returns of FTSE-100 and NASDAQ indices. One minute sampled valued of indices were downloaded from www.finam.ru, one of the major Russian broker. To calculate intraday realized volatilities and ovelapping realized covariance matrices two scale estimator ((Zhang *et al.* (2005), Zhang (2011))) was used with 5-minute large grid and 1-minute small grid.

Data sample for 2012-2016 years consists of 1146 overlapping days. The first two years (452 observations) were considered as training sample and the last three years (694 observations) as control sample. All weights ω and scaling coefficients s for Hansen & Lunde (2005) volatilities were calculated using only training sample. The same is applied to calculation of parameters for correlation estimators: $\alpha, \beta, \gamma, \mathbf{S}$.

Average Frobenius Error and Root mean squared error

In order to choose the best estimator there was designed a simple out-of-sample test: forecast of the tomorrow covariance matrix. Frobenius error is defined as a Frobenius distance between covariance matrix forecast and an outer product of return vector $AFE = \frac{1}{T-1} \sum_{t=0}^{T-1} ||r_{t+1}r'_{t+1} - V_t||_\Phi$. A one-day-ahead return was chosen due to the fact, that the large part of 'Naive' estimator is an outer product of a large part of return vector, so, it will mistakenly provide the lowest error.

As one can see from the table 3.2, the lowest AFE is obtained while using 'Scaling' estimator. The difference between FE of estimators is very small, so Diebold & Mariano (1995) test was applied to ensure that the difference is significant (table 3.3). It turns out, that 'Scaling' estimator provides significantly lower error in comparison with alternatives. Consequently, in the following part, covariance matrices obtained using this method will be used in dynamic covariance modelling.

Another common measure of out-of-sample forecast preciseness is a Root Mean-Squared Error (Lopez & Walter (2000)). It is a root of average squared Frobenius distance. This measure is more sensitive to the outliers. The differences between RMSE, however, was not significant between any estimators.

3.3 Forecast

3.3.1 Covariance forecasting models

Estimated covariance matrices can be used as an input for the forecasting models. In this section there would be provided an empirical example of such a forecast.

Heterogeneous Autoregressive models were introduced by Corsi (2009). HAR represents volatility forecast as a linear combination of previous day, weekly and monthly volatilities. This model captures number of stylized facts, such as persistence of the volatility, fat tails of return distribution and self-similarity on different time horizons.

One of the multivariate approach is HAR-DRD model Oh & Patton (2016). In this model variance-covariance matrix is decomposed into variances and correlation matrix. Each variance follow univariate HAR process. In order to reduce influence of outlying observations and to guarantee positive variances, logarithm of variances is used. Correlation matrix in the same time, follows dynamic conditional correlation (DCC) model Engle (2002). In bivariate case this model is:

$$v_t^l = \alpha_l + \beta_{l1} v_{t-1}^l + \beta_{l2} v_w^l + \beta_{l3} v_m^l + \epsilon_{lt} \tag{3.13}$$

$$v_t^n = \alpha_n + \beta_{n1} v_{t-1}^n + \beta_{n2} v_w^n + \beta_{n3} v_m^n + \epsilon_{nt} \tag{3.14}$$

$$\rho_t = \alpha_\rho + \beta_{\rho 1} \rho_{t-1} + \beta_{\rho 2} \rho_w + \beta_{\rho 3} \rho_m + \epsilon_{\rho t} \tag{3.15}$$

where: $v_t = log(\sigma_t^2)$, $x_w = \frac{1}{4}\sum_{i=t-2}^{t-5} x_i$, $x_m = \frac{1}{15}\sum_{i=t-6}^{t-22} x_i$.

In the present paper, another extension of HAR is proposed, Bivariate Heterogeneous Autoregressive model (B-HAR), which includes both volatility spillovers and correlation-volatilities dependence.

$$\mathbf{v}_t = \mathbf{a} + \mathbf{B}_{HAR}\mathbf{v}_{HAR,t} + \epsilon_t \tag{3.16}$$

where, $\mathbf{v}_t = (v_t^l, v_t^n, z(\rho_t))$ is a vector of dependent variables, $\mathbf{a} = (\alpha_l, \alpha_n, \alpha_z)$ are intercepts, $\mathbf{v}_{HAR,t} = (v_{t-1}^l, v_w^l, v_m^l, v_{t-1}^n, v_w^n, v_m^n, z_{t-1}, z_w, z_m)$ is a vector of independent variables, and $\mathbf{B}_{HAR}$ is a matrix of coefficients:

$$\mathbf{B}_{HAR} = \begin{pmatrix} \beta_{lly} & \beta_{llw} & \beta_{llm} & \beta_{lny} & \beta_{lnw} & \beta_{lnm} & \beta_{lzy} & \beta_{lzw} & \beta_{lzm} \\ \beta_{nly} & \beta_{nlw} & \beta_{nlm} & \beta_{nny} & \beta_{nnw} & \beta_{nnm} & \beta_{nzy} & \beta_{nzw} & \beta_{nzm} \\ \beta_{zly} & \beta_{zlw} & \beta_{zlm} & \beta_{zny} & \beta_{znw} & \beta_{znm} & \beta_{zzy} & \beta_{zzw} & \beta_{zzm} \end{pmatrix} \tag{3.17}$$

Coefficients $\beta_{lly}, \beta_{llw}, \beta_{llm}, \beta_{nny}, \beta_{nnw}, \beta_{llm}, \beta_{zzy}, \beta_{zzw}, \beta_{zzm}$ represent simple multivariate HAR model, with only difference from HAR-DRD in z-transformation of correlations. Coefficients $\beta_{lny}, \beta_{lnw}, \beta_{lnm}, \beta_{nly}, \beta_{nlw}, \beta_{nlm}$ capture effect of volatility spillovers over the international markets. Other coefficients are used to include the relationship between correlation and volatilities.

In the present paper, the proposed model is called Bivariate HAR, as it guarantees positive semi-definiteness of the covariance matrix only in the case of two assets.

The effect of correlation increase at high-volatility times is not necessary direct. Longin & Solnik (2001) model tail dependence and Solnik &

Watewai (2016) use regime switching model with jumps. Consequently, it is possible to add other terms in BHAR model. However, there are already 30 parameters in the model (including constants), so, jumps and leverage effects were not included in the current model.

In order to decrease amount of parameters, another model is introduced with only 13 parameters: Bivariate restricted HAR (Br-HAR). In this model is assumed that all volatility spillovers take place only in short time intervals, such as one day, but not one week or one month. The same applies to correlations. Moreover, only volatilities are assumed to influence the correlation but not vice versa. The Br-HAR model is the same as in equation 3.16, but the matrix **B** is different, as in equation 3.18.

$$\mathbf{B}_{BrHAR} = \begin{pmatrix} \beta_{lly} & \beta_{llw} & \beta_{llm} & \beta_{lny} & 0 & 0 & 0 & 0 & 0 \\ \beta_{nly} & 0 & 0 & \beta_{nny} & \beta_{nnw} & \beta_{nnm} & 0 & 0 & 0 \\ \beta_{zly} & 0 & 0 & \beta_{zny} & 0 & 0 & \beta_{zzy} & \beta_{zzw} & \beta_{zzm} \end{pmatrix} \tag{3.18}$$

3.3.2 Full-sample analysis

In this section the significance of the coefficients will be studied on the full sample.

Following Corsi (2009) coefficients were estimated by standard OLS regression with Newey-West correction. The results of the estimation are provided in table 3.4.

All standard HAR coefficients are significant at 1% level and positive. It turns out, that volatility spillovers in fact is a short-term process: weekly spillover coefficients are not significant, while monthly are negative. The last fact may be interpreted either as overparametrization of autoregressive model, or that volatility spillover effect depends not upon increase in volatility per se, but upon increase of daily volatility above average monthly level.

Fengler & Gisler (2015) have shown, that covariances are relevant in the estimation of volatility spillovers. The present paper instead decompose the covariance into correlation and volatilities. It turns out, that increase in correlation does not provide a spillover effect to volatility on international markets, however, increase of volatility causes increase in correlations.

Note, that significant and interpretable coefficients are only those, that were included in Br-HAR model.

3.3.3 Out-of-sample forecast

In this section there would be compared a forecasting performance of B-HAR, Br-HAR, HAR-DRD, and a parsimonious Exponentially Weighted

Moving Averages model with standard decay rate $\lambda = 0.06$. So, there would be compared models with 30, 16, 12 and 0 fitted parameters.

As an input to the models 'Scaling' estimations of the whole day covariance matrix were used (see section 3.2.3). Data is described at section 3.2.4. All the regression coefficients, as well as scaling matrix for 'Scaling' estimator are estimated on training 2-years sample and tested on the following 3-years sample.

There were performed three tests: forecasting error, prediction of Value-at-Risk for equiweighted portfolio and Minimization of Variance of a portfolio.

Root mean squared error

As it is shown at tables 3.5 and 3.6, B-HAR model provides significantly lower Average Frobenius Error then any other compared models. Restricted version of the model, however, is not significantly better. On the other hand, Br-HAR provides significantly lower Root Mean Squared Error then all compared models, while RMSE of B-HAR is not significantly lower.

3.3.4 Value-at-Risk forecast

In this section compared models are used in Value-at-Risk forecast. Value-at-Risk is modelled a 1%, 5% or 10% quantile of normal distribution with zero mean and forecasted covariance of equiweighted portfolio.

For each computed forecast there was calculated number of exceptions, as well as a sum of loss functions for each day. There was a piecewise linear loss function used, defined as in equation 3.19

$$PWLF(\alpha, VaR_t, r_t) = (I_{VaR_t < r_t} - \alpha)(VaR_t - r_t) \tag{3.19}$$

where α is a VaR confidence level and I is an indicator function of VaR violation. It was shown by Gneiting (2011b), Gneiting (2011a) that the function in equation 3.19 is a consistent scoring function for Value-at-Risk forecast.

It turns out that ougmented B-HAR and Br-HAR models almost do not provide significantly superior VaR forecast in comparison with simpler HAR-DRD model. Only for 5% VaR forecast Br-HAR model provides significantly better prediction then HAR-DRD.

EWMA turns out to be a more conservative model then different specifications of HAR. Even while it provides lower VaR violation rate, it significantly underperforms at the 5% and 10% loss function test. It means that lower VaR violation rate is achieved not by more precise covariance estimation, but due to its overestimation. However, for 1% VaR average loss function is slightly lower, due to the fact, that real returns violate normality assumption.

Minimum variance portfolio

The next test is a prediction of minimal variance of the portfolio. Weights for such a portfolio are obtained with equation 3.20.

$$\mathbf{w}_t = \frac{\hat{\mathbf{V}}_t^{-1}\mathbf{i}}{\mathbf{i}'\hat{\mathbf{V}}_t^{-1}\mathbf{i}} \tag{3.20}$$

where $\hat{\mathbf{V}}_t$ is a covariance matrix forecast and $\mathbf{i}$ is a vector of ones.

The more precise is the covariance matrix estimation, the lower is the variance of the constructed minimum variance portfolio.

There were reported standard deviations of the portfolios, as well as DM-tests, in which squared min-variance portfolio return were used as a loss function.

In table 3.13 is shown that Bivariate HAR models provide lower variance of min-variance portfolio. However, Diebold-Mariano test (table 3.14) shows, that this difference is not statistically significant.

3.4 Conclusion

In this paper there was introduced a method of obtaining whole day realized covariance matrix estimation for asynchronously traded assets, based on intraday high-frequency data.

This method suggests to separately evaluate variances and correlation matrix. For estimation of the whole day variances there exist an estimator of Hansen & Lunde (2005). In the present paper was proposed to obtain the whole day correlation matrix from rescaled overlapping covariance matrix.

There were also proposed two alternatives for bivariate data: correlation coefficient for the whole day is obtained by linear or proportional function of z-transformed overlapping correlation.

These estimators were tested on real high-frequency data of FTSE-100 and NASDAQ indices using Frobenius distance to outer product of returns and RMSE. It turns out, that average Frobenius distance of covariance matrix estimator to the outer product of next day return is significantly lower if Scaling estimator is used.

Realized covariance estimators, obtained with the proposed method, may be successfully used as an input to the models of covariance forecast. Consequently, the proposed methodology have a potential impact in practical applications, such as risk-management, portfolio allocation and derivatives pricing. Potential scientific impact is also promising: estimator of the whole day covariance that is both based on intraday data and applicable for the asynchronous markets will be useful in studies of cross-border volatility spillovers, as well as changing of correlation structure between markets.

In order to illustrate the practical applicability of the estimator, there was used a forecasting exercise. Different specifications of multivariate HAR as well as EWMA model were used to forecast VaR and construct Minimum-Variance portfolio. Using out-of-sample exercise, it was shown, that even while more sophisticated models (with 30 and 16 parameters for 2 assets) provide significantly smaller Average Frobenious Distance or Root Mean Square Error then simple HAR-DRD (with 12 parameters), this difference becomes insignificant in practical exercise, such as VaR forecast and Min-Variance portfolio optimisation. All HAR models, however, significantly outperform EWMA covariance forecast.

Methodology, proposed in the present paper has a potential for improvement. Future researches are necessary to construct an estimator of realized covariance in the case, when there are no overlapping period at all - for example, NIKKEY trading hours don't intersect with European and American stock exchanges.

3.5 Appendix I

Optimal correlation averaging

In order to estimate α, β, γ in the equations 3.9 and 3.10 it is necessary to have a sample of two variables: correlations during overlapping period and correlations for the whole day. However, only the first variable can be directly assigned to each day. For this reason, the equation 3.9 will be estimated using sample of moving average correlations for several days.

Let m be number of days for which correlations are averaged. So, the first variable will be the correlation coefficient obtained from the sum of m covariance matrices, while the second will be obtained from the sum of m outer product of daily returns.

Using Monte-Carlo simulations, number m will be chosen in such a way, to minimize mean squared error of coefficient estimations.

Monte-Carlo simulations were organized in the following way. As a covariance matrix of overlapping periods were taken realized covariance estimations from real data (FTSE-100 and NASDAQ 2013-2014). Daily returns were generated from Multivariate Normal Distribution with artificially constructed covariance matrix: its diagonal elements (variances) were obtained from intraday realized volatilities, while correlation coefficient was calculated using the function 3.9.

In each scenario there was taken different parameters of α, β and variances of ϵ. The values of the parameters were the following: $\alpha = \{0, 0.1, 0, 5\}, \beta = \{0.5, 0.9, 1, 1.1, 1.5, 2\}, S.d.(\epsilon) = \{0, 0.05, 0.5\}$. Each of the 54 scenarios was run for 10 000 times. At each scenario estimation of values α and β were computed using average correlations for m periods, where $m =$

$\{2, 3, 4, 5, 10, 20, 50\}$. The result of estimation are presented in the table 3.1.

Table 3.1: Simulation results: mean squared error for different averaging period

m (averaging period)	2	3	4	5	10	20	50
Mean squared error	3.52	2.51	2.45	2.51	2.95	3.46	4.69

The least mean-squared error is achieved at $m = 4$. This value of m is used at the estimation of coefficients on real data.

3.6 Appendix II

Table 3.2: Average frobenius distance of covariance matrix estimator to the outer product of next day return

	V^S	V^L	V^P	V^N
AFE×10000	2.584	2.598	2.595	2.615
RMSE ×1000	1.198	1.198	1.198	1.199

Table 3.3: DM-test of equality of AFE for different covariance estimators

	V^S	V^L	V^P
V^L	***5.325		
V^P	***4.462	*-1.293	
V^N	*1.622	0.879	1.003

Table 3.4: Full-sample OLS estimation of BHAR parameters

	v^L	v^N	z
v^L_{t-1}	0.362	0.293	0.070
(prob.)	(0.0%)	(0.0%)	(0.0%)
v^L_w	0.413	-0.054	-0.038
	(0.0%)	(34.9%)	(11.4%)
v^L_m	0.151	-0.166	-0.024
	(0.2%)	(0.4%)	(35.6%)
v^N_{t-1}	0.120	0.423	0.043
	(0.0%)	(0.0%)	(0.1%)
v^N_w	-0.070	0.151	-0.035
	(19.3%)	(0.9%)	(12.4%)
v^N_m	-0.127	0.179	-0.015
	(3.1%)	(1.0%)	(57.8%)
z_{t-1}	0.027	0.016	0.138
	(73.6%)	(85.9%)	(0.0%)
z_w	-0.095	0.135	0.387
	(46.8%)	(31.6%)	(0.0%)
z_m	0.135	0.041	0.375
	(39.0%)	(80.2%)	(0.0%)
c	-1.516	-1.832	0.104
	(0.1%)	(0.1%)	(61.1%)

Table 3.5: AFE and RMSE of next day covariance matrix forecasts

	B-HAR	Br-HAR	HAR-DRD	EWMA
AFE×10000	2.309	2.323	2.326	2.618
RMSE×1000	1.168	1.167	1.169	1.174

Table 3.6: DM-test of equality of AFE for different covariance forecasts

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	***2.818		
HAR-DRD	***3.151	0.622	
EWMA	***10.679	***10.647	***10.724

Table 3.7: DM-test of equality of RMSE for different covariance forecasts

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	***-2.520		
HAR-DRD	0.994	*1.438	
EWMA	***11.856	***12.020	***11.955

Table 3.8: Percentage of VaR exceptions

	B-HAR	Br-HAR	HAR-DRD	EWMA
1% VaR	2.31%	2.17%	2.46%	2.17%
5% VaR	8.09%	7.66%	7.95%	5.64%
10% VaR	12.57%	12.28%	12.14%	10.40%

Table 3.9: VaR Average Loss Function

	B-HAR	Br-HAR	HAR-DRD	EWMA
1% VaR LF ($\times$1000)	0.4829	0.4784	0.4789	0.4740
5% VaR LF ($\times$1000)	1.1991	1.1908	1.1971	1.2388
10% VaRLF ($\times$1000)	1.8553	1.8551	1.8568	1.9159

Table 3.10: DM-test for 1% VaR LF equality

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	*-1.349		
HAR-DRD	-1.139	0.151	
EWMA	-0.466	-0.253	-0.296

Table 3.11: DM-test for 5% VaR LF equality

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	**-1.660		
HAR-DRD	-0.414	*1.297	
EWMA	**1.862	***2.428	**2.163

Table 3.12: DM-test for 10% VaR LF equality

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	-0.049		
HAR-DRD	0.309	0.387	
EWMA	***2.420	***2.662	***2.585

Table 3.13: Standard Deviation of MinVariance portfolios

	B-HAR	Br-HAR	HAR-DRD	EWMA
σ	0.993%	0.993%	1.005%	1.025%

Table 3.14: DM-test for equality of variances of MinVariance portfolios

	B-HAR	Br-HAR	HAR-DRD
Br-HAR	-0.073		
HAR-DRD	0.964	1.067	
EWMA	*1.542	*1.480	*1.229

Bibliography

Abad, Pilar, Chuliá, Helena, & Gómez-Puig, Marta. 2010. Emu and european government bond market integration. *Journal of banking & finance*, **34**(12), 2851–2860.

Abad, Pilar, Muela, Sonia Benito, & Martín, Carmen López. 2015. The role of the loss function in value-at-risk comparisons. *The journal of risk model validation*, **9**(1), 1.

Ahoniemi, Katja, & Lanne, Markku. 2013. Overnight stock returns and realized volatility. *International journal of forecasting*, **29**(4), 592–604.

Aït-Sahalia, Yacine, Fan, Jianqing, & Xiu, Dacheng. 2010. High-Frequency Covariance Estimates With Noisy and Asynchronous Financial Data. *Journal of the american statistical association*, **105**(492), 1504–1517.

Amihud, Yakov. 2002. Illiquidity and stock returns: cross-section and time-series effects. *Journal of financial markets*, **5**(1), 31 – 56.

Andersen, Torben G., Bollerslev, Tim, & Dobrev, Dobrislav. 2007. No-arbitrage semi-martingale restrictions for continuous-time volatility models subject to leverage effects, jumps and i.i.d. noise: Theory and testable distributional implications. *Journal of econometrics*, **138**(1), 125 – 180. 50th Anniversary Econometric Institute.

Ando, T., Li, Chi-Kwong, & Mathias, Roy. 2004. Geometric means. *Linear algebra and its applications*, **385**, 305 – 334. Special Issue in honor of Peter Lancaster.

Audrino, Francesco, & Bühlmann, Peter. 2004. Synchronizing multivariate financial time series. *Journal of risk*, **6**(2), 81–106.

Bai, Jushan, & Shi, Shuzhong. 2011. Estimating high dimensional covariance matrices and its applications. *Annals of economics and finance*, **12**(2), 199–215.

Barndorff-Nielsen, Ole E., & Shephard, Neil. 2004. Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. *Econometrica*, **72**(3), 885–925.

Barndorff-Nielsen, Ole E., Hansen, Peter, Lunde, Asger, & Shephard, Neil. 2011. Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading. *Journal of econometrics*, **162**(2), 149–169.

Becker, R., Clements, A.E., Doolan, M.B., & Hurn, A.S. 2015. Selecting volatility forecasting models for portfolio allocation purposes. *International journal of forecasting*, **31**(3), 849 – 861.

Becker, Ralf, Clements, Adam E., & White, Scott I. 2007. Does implied volatility provide any information beyond that captured in model-based volatility forecasts? *Journal of banking & finance*, **31**(8), 2535 – 2549.

Bekaert, Geert, & Harvey, Campbell R. 1995. Time-varying world market integration. *The journal of finance*, **50**(2), 403–444.

Berben, Robert-Paul, & Jansen, W Jos. 2005. Comovement in international equity markets: A sectoral view. *Journal of international money and finance*, **24**(5), 832–857.

Bergomi, Lorenzo. 2010. Correlations in asynchronous markets. *Risk*, **23**(11), 72.

Blair, Bevan J., Poon, Ser-Huang, & Taylor, Stephen J. 2001. Forecasting s&p 100 volatility: the incremental information content of implied volatilities and high-frequency index returns. *Journal of econometrics*, **105**(1), 5 – 26. Forecasting and empirical methods in finance and macroeconomics.

Blanc, Pierre, Chicheportiche, Rémy, & Bouchaud, Jean-Philippe. 2014. The fine structure of volatility feedback ii: Overnight and intra-day effects. *Physica a: Statistical mechanics and its applications*, **402**, 58–75.

Boudt, Kris, Laurent, Sébastien, Lunde, Asger, & Quaedvlieg, Rogier. 2014. Positive semidefinite integrated covariance estimation, factorizations and asynchronicity.

Boyer, Brian H, Gibson, Michael S, Loretan, Mico, *et al.* 1997. *Pitfalls in tests for changes in correlations.* Vol. 597. Board of Governors of the Federal Reserve System.

Burns, Patrick, Engle, Robert F, & Mezrich, Joseph J. 1998. Correlations and Volatilities of Asynchronous Data. *Journal of derivatives*, **5**, 7–18.

Butler, Kirt, & Okada, Katsushi. 2007. Bivariate and higher-order terms in models of international equity returns. *Applied financial economics*, **17**(9), 725–737.

Caporin, Massimiliano. 2008. Evaluating value-at-risk measures in the presence of long memory conditional volatility. *The journal of risk*, **10**(3), 79.

Chalmers, John MR, Edelen, Roger M, & Kadlec, Gregory B. 2001. On the perils of financial intermediaries setting security prices: the mutual fund wild card option. *The journal of finance*, **56**(6), 2209–2236.

Christensen, Kim, Kinnebrock, Silja, & Podolskij, Mark. 2010. Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data. *Journal of econometrics*, **159**(1), 116 – 133.

Christiansen, Charlotte. 2007. Volatility-spillover effects in european bond markets. *European financial management*, **13**(5), 923–948.

Cooper, Michael J, Cliff, Michael T, & Gulen, Huseyin. 2008. Return differences between trading and non-trading hours: Like night and day.

Corsi, Fulvio. 2009. A simple approximate long-memory model of realized volatility. *Journal of financial econometrics*, **7**(2), 174.

Corsi, Fulvio, & Audrino, Francesco. 2012. Realized Covariance Tick-by-Tick in Presence of Rounded Time Stamps and General Microstructure Effects. *Journal of financial econometrics*, **10**(4), 591–616.

de Pooter, Michiel, Martens, Martin, & van Dijk, Dick. 2008. Predicting the Daily Covariance Matrix for S&P 100 Stocks Using Intraday Data—But Which Frequency to Use? *Econometric reviews*, **27**, 199–229.

Diebold, Francis X., & Mariano, Roberto S. 1995. Comparing predictive Accuracy. *Journal of business and economic statistics*, **13**, 253–265.

Ding, Zhuanxin, Granger, Clive W.J., & Engle, Robert F. 1993. A long memory property of stock market returns and a new model. *Journal of empirical finance*, **1**(1), 83 – 106.

Dovonon, Prosper, Gonçalves, Sílvia, & Meddahi, Nour. 2013. Bootstrapping realized multivariate volatility measures. *Journal of econometrics*, **172**(1), 49 – 65.

Engle, Robert. 2002. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of business & economic statistics*, **20**(3), 339–350.

Engle, Robert F., & Kroner, Kenneth F. 1995. Multivariate simultaneous generalized arch. *Econometric theory*, **11**(1), 122–150.

Fama, Eugene F. 1965. The behavior of stock-market prices. *The journal of business*, **38**(1), 34–105.

Fengler, Matthias R., & Gisler, Katja I.M. 2015. A variance spillover analysis without covariances: What do we miss? *Journal of international money and finance*, **51**, 174 – 195.

Fleming, Jeff, & Kirby, Chris. 2011. Long memory in volatility and trading volume. *Journal of banking & finance*, **35**(7), 1714 – 1726.

Forni, Mario, Hallin, Marc, Lippi, Marco, & Reichlin, Lucrezia. 2000. The generalized dynamic-factor model: Identification and estimation. *Review of economics and statistics*, **82**(4), 540–554.

French, Kenneth R, & Roll, Richard. 1986. Stock return variances: The arrival of information and the reaction of traders. *Journal of financial economics*, **17**(1), 5–26.

Fuertes, Ana-Maria, & Olmo, Jose. 2013. Optimally harnessing interday and intra-day information for daily value-at-risk prediction. *International journal of forecasting*, **29**(1), 28 – 42.

Gallo, Giampiero M. 2001. Modelling the impact of overnight surprises on intra-daily volatility. *Australian economic papers*, **40**(4), 567–580.

Glosten, Lawrence R., Jagannathan, Ravi, & Runkle, David E. 1993. On the relation between the expected value and the volatility of the nominal excess return on stocks. *The journal of finance*, **48**(5), 1779–1801.

Gneiting, Tilmann. 2011a. Making and evaluating point forecasts. *Journal of the american statistical association*, **106**(494), 746–762.

Gneiting, Tilmann. 2011b. Quantiles as optimal point forecasts. *International journal of forecasting*, **27**(2), 197–207.

Golosnoy, Vasyl, Gribisch, Bastian, & Liesenfeld, Roman. 2012. The conditional autoregressive wishart model for multivariate stock market volatility. *Journal of econometrics*, **167**(1), 211 – 223.

Golosnoy, Vasyl, Gribisch, Bastian, & Liesenfeld, Roman. 2015. Intra-daily volatility spillovers in international stock markets. *Journal of international money and finance*, **53**, 95 – 114.

González, Mariano. 2016. Asymmetric causality in-mean and in-variance among equity markets indexes. *The north american journal of economics and finance*, **36**, 49–68.

Hansen, Peter, & Lunde, Asger. 2005. A realized variance for the whole day based on intermittent high-frequency data. *Journal of financial econometrics*, **3**(4), 525–554.

Hansen, Peter, Horel, Guillaume, Lunde, Asger, & Archakov, Ilya. 2015. *A markov chain estimator of multivariate volatility from high frequency data*. Creates research papers. School of Economics and Management, University of Aarhus.

Hayashi, Takaki, & Yoshida, Nakahiro. 2005. On covariance estimation of non-synchronously observed diffusion processes. *Bernoulli*, **11**(2), 359–379.

Heaton, Chris, Milunovich, George, & Passé-de Silva, Anthony. 2011. International commodity prices and the australian stock market*. *Economic record*, **87**(276), 37–44.

Higham, Nicholas J. 2002. Computing the nearest correlation matrix—a problem from finance. *Ima journal of numerical analysis*, **22**(3), 329–343.

Iannazzo, Bruno. 2016. The geometric mean of two matrices from a computational viewpoint. *Numerical linear algebra with applications*, **23**(2), 208–229.

Kanatani, Taro, & Renó, Roberto. 2007 (Apr.). *Unbiased covariance estimation with interpolated data*. Department of Economics University of Siena 502. Department of Economics, University of Siena.

Kang, Long, & Babbs, Simon H. 2012. Modeling overnight and daytime returns using a multivariate generalized autoregressive conditional heteroskedasticity copula model. *The journal of risk*, **14**(4), 35.

Kelly, Michael A, & Clark, Steven P. 2011. Returns in trading versus non-trading hours: The difference is day and night. *Journal of asset management*, **12**(2), 132–145.

Kupiec, Paul H. 1995. Techniques for verifying the accuracy of risk measurement models. *The j. of derivatives*, **3**(2).

Kyj, Lada, Ostdiek, Barbara, & Ensor, Katherine. 2009. Realized Covariance Estimation in Dynamic Portfolio Optimization.

Larntz, Kinley, & Perlman, Michael D. 1985. *A simple test for the equality of correlation matrices*. Tech. rept. Department of Statistics, University of Washington.

Ledoit, Olivier, & Wolf, Michael. 2004. A well-conditioned estimator for large-dimensional covariance matrices. *Journal of multivariate analysis*, **88**(2), 365 – 411.

Leung, Tim, & Kang, Jamie Juhee. 2016. Asynchronous adrs: Overnight vs intraday returns and trading strategies.

Longin, Francois, & Solnik, Bruno. 2001. Extreme correlation of international equity markets. *The journal of finance*, **56**(2), 649–676.

Lopez, Jose A. 1998. Methods for evaluating value-at-risk estimates. *Economic policy review*, **4**(3).

Lopez, Jose A, & Walter, Christian A. 2000. Evaluating covariance matrix forecasts in a value-at-risk framework. *Frb of san francisco working paper*.

Martens, Martin. 2002. Measuring and forecasting s&p 500 index-futures volatility using high-frequency data. *Journal of futures markets*, **22**(6), 497–518.

Martens, Martin, & Poon, Ser-Huang. 2001. Returns synchronization and daily correlation dynamics between international stock markets. *Journal of banking & finance*, **25**(10), 1805–1827.

McAleer, Michael, & Da Veiga, Bernardo. 2008. Forecasting value-at-risk with a parsimonious portfolio spillover garch (ps-garch) model. *Journal of forecasting*, **27**(1), 1–19.

McAleer, Michael, & Medeiros, Marcelo C. 2008. Realized volatility: A review. *Econometric reviews*, **27**(1-3), 10–45.

Morgan, JP. 1996. *Riskmetrics technical document*.

Oh, Dong Hwan, & Patton, Andrew J. 2016. High-dimensional copula-based distributions with mixed frequency data. *Journal of econometrics*, **193**(2), 349–366.

Pandey, Ajay. 2003. *Overnight stock returns and time-varying correlations.* Indian Institute of Management.

Peluso, Stefano, Corsi, Fulvio, & Mira, Antonietta. 2015. A Bayesian High-Frequency Estimator of the Multivariate Covariance of Noisy and Asynchronous Returns. *Journal of financial econometrics*, **13**(3), 665–697.

Petroni, Filippo, & Serva, Maurizio. 2016. Observability of market daily volatility. *Physica a: Statistical mechanics and its applications*, **444**, 838 – 842.

Pusz, W., & Woronowicz, S.L. 1975. Functional calculus for sesquilinear forms and the purification map. *Reports on mathematical physics*, **8**(2), 159 – 170.

Riedel, Christoph, & Wagner, Niklas. 2015. Is risk higher during non-trading periods? the risk trade-off for intraday versus overnight market returns. *Journal of international financial markets, institutions and money*, **39**, 53 – 64.

Sarma, Mandira, Thomas, Susan, & Shah, Ajay. 2003. Selection of value-at-risk models. *Journal of forecasting*, **22**(4), 337–358.

Scherer, Bernhard. 2013. Synchronize your data or get out of step with your risks. *The journal of derivatives*, **20**(3), 75–84.

Scholes, Myron, & Williams, Joseph. 1977. Estimating betas from nonsynchronous data. *Journal of financial economics*, **5**(3), 309 – 327.

Solnik, Bruno, & Watewai, Thaisiri. 2016. International correlation asymmetries: Frequent-but-small and infrequent-but-large equity returns. *Review of asset pricing studies*, raw005.

Sotiropoulos, Michael. 2016. Portfolio trading in asynchronous markets. *CFEM financial engineering seminar.*

Stepniak, Czeslaw. 1985. Ordering of nonnegative definite matrices with application to comparison of linear models. *Linear algebra and its applications*, **70**, 67 – 71.

Taylor, Stephen J. 1989. Simulating financial prices. *The journal of the operational research society*, **40**(6), 567–569.

Triacca, Umberto, & Focker, Fulvia. 2014. Estimating overnight volatility of asset returns by using the generalized dynamic factor model approach. *Decisions in economics and finance*, **37**(2), 235–254.

Tsay, Ruey S., & Yeh, Jin-Huei. 2004. Non-synchronous trading and high-frequency beta. *2004 ntu international conference on finance.*

Tsiakas, Ilias. 2008. Overnight information and stochastic volatility: A study of european and US stock exchanges. *Journal of banking & finance*, **32**(2), 251 – 268.

Ubukata, Masato. 2009. Large-scale portfolios using realized covariance matrix: evidence from the Japanese stock market. Sept.

Wang, Fengzhong, Shieh, Shwu-Jane, Havlin, Shlomo, & Stanley, H. Eugene. 2009. Statistical analysis of the overnight and daytime return. *Phys. rev. e*, **79**(May), 056109.

Zhang, Lan. 2011. Estimating covariation: Epps effect, microstructure noise. *Journal of econometrics*, **160**(1), 33–47.

Zhang, Lan, Mykland, Per A., & Ait-Sahalia, Yacine. 2005. A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Journal of the american statistical association*, **100**, 1394–1411.