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LOOKING BEYOND NUMBERS, SYMMETRY AND INVARIANTS IN THE LIGHT OF NATURE OF SCIENCE TEACHING. AN INTERDISCIPLINARY LEARNING PATH FROM PRIMARY TO SECONDARY SCHOOL.

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To my beloved parents Giorgio and Irene,

and to my dearest Danièle and Remo.

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Keywords

Artefacts, Big Ideas, Interdisciplinary Teaching, Invariance, Symmetry, Teaching Learning Path, Symmetry Operations, Reflections, Rotations, Nature of Science.

Abstract

This PhD study examines the effectiveness of an interdisciplinary instructional approach to symmetry and invariance in teaching mathematics to fifth graders as they transition from elementary to secondary school. Symmetry and invariance are potential Big Ideas that benefit interdisciplinary instruction because they are principles of invariance used to classify and categorize phenomena, data, and information into coherent knowledge. Introducing the unitary nature of symmetry and invariance early in the classroom allows pupils to develop generalization, synthesis methods, and cognitive processes that span many domains.

The study employed action research with three cycles of reflection: designing and implementing a teaching-learning sequence for elementary students (N=96, aged 10), conducting pre- and post-tests with a control group (N=41), following up with the same group at the secondary level and providing an online course for teachers (N=26) who participated in the research project. The 15 hours of lessons focused on enhancing teachers' classroom practices regarding the pedagogical and curriculum content associated with symmetry and invariance. Teachers investigated the project's tools and materials for feasibility and transferability on the didactic framework.

Results suggest that introducing interdisciplinary symmetry concepts to fifth graders improves their spatial skills and motivates them to see mathematics as more connected to the real world and science. The study highlights the potential of manipulative games and artefacts to promote cognitive processes relevant to mathematical modeling and the generalization of symmetry concepts. The didactic path to symmetry should focus on finding invariants through various symmetry operations, especially rotations. Research suggests that a paradigm shift in symmetry teaching can positively impact student understanding and skill development and that teachers can harness the potential of symmetry and invariance as powerful tools for interdisciplinary knowledge.

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List of Abbreviations

Agreement Marker .... (AM)
Application Programming Interface.. (API)
Biserial Point Correlation Coefficient...(Rp-b)
Classical Test Theory ....(TCT)
Confidence Level .... (CL)
Cronbach's Alpha Index .... (α)
Difficulty Index .... (QI)
Discrimination Index.. (DI)
Experiment Model Transition ....(EMT)
Facility Index ....(PI)
Gunning-Fog Index ....(FMG)
Item Response Theory... (IRT)
Kaiser-Meyer-Olkin .... (KMO)
Kuder-Richardson Index ....(RK20)
Learning Trajectories ....(LT)
Learning Path or Learning Progression..(LP)
Learning Teaching Progression... (LTP)
Learning Teaching Sequence .... (LTS)
Metacognitive Explanation Based...(MEB)
Ministry of Education, University and Research .... (MIUR)
Natural Language Processing...(NLP)
National Research Council.. (NRC)
Nature of Science ....(NOS)
Negative Agreement.. (TN)
Negative Disagreement .... (FN)
Organisation for Economic Co-operation and Development ....(OECD)
Positive Agreement ....(TP)
Positive Disagreement.. (FP)
Programme for International Student Assessment.................................(PISA)
Relative Significance Index .... (ISR)
Sentiment Analysis.. (SA)
Significance Index..(IS)
Standard Deviation .... (SD)
Trends in Mathematics and Science Study ....(TIMSS)

Statement of Original Authorship

The work contained in this thesis has not been previously submitted to meet requirements for an award at this or any other higher education institution. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made.

The intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged. Parts of this work have been published in the form of articles or book chapters (see the Publications section).

Signature:

Date: 11/03/2023

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As I reflect on my journey through the fascinating world of symmetries and its vast realms, I feel compelled to express my deep gratitude to all those who have supported me on this path of exploration. Now that we have arrived - at least for the moment - on the shores of the mainland, it is appropriate to pause for a moment and reflect.

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Chapter 1: Introduction

Even though mathematical proficiency is highly relevant in and to the research fields, there is solid evidence showing that mathematics education is failing to support students in developing a positive science identity with meaningful connections to physics and other scientific disciplines. In fact, sometimes from students' perspective, mathematics and physics, generally speaking science education, typically appear as separate subjects with few interconnections (Redish & Kuo, 2015).

Although many studies (Boniolo et al., 2010; Fuson et al., 2005) have been devoted to the problem of teaching mathematical and science contents together, the problem of how to use math in teaching science remains unsolved. In science education, mathematics is seen as a mere tool to describe and calculate, whereas physics is usually viewed as a possible context for the application of mathematical concepts. This compartmentalization might generate many theoretical misunderstandings in the process of learning certain important concepts (Meltzer, 2002; Buick, 2007). In this sense, a key aspect of educational innovation is to promote creative and flexible frameworks for integrating productive ideas across disciplines.

Symmetry represents a possible epistemic core for its special power to capture a universal value between disciplines. According to Darvas (1997, p. 328), symmetry has an important role in mathematics education as a vehicle of a general scientific method applicable throughout the sciences as a set of principles. Therefore, teaching methods that place symmetry as a cross-disciplines big idea provide a basis for interdisciplinary sequence of learning teaching activities.

The thesis aims at developing an interdisciplinary approach based on symmetry and invariance across mathematics and physics to foster the teaching of the Nature of Science (NOS). NOS refers to principles and ideas which provide a description of science as a way of knowing, as well as characteristics of scientific knowledge (Svendsen, 2021). The unitary nature of symmetry across mathematics and science is definite throughout historical and scientific development. However, symmetry has a more far-reaching role in the educational context, even if its role in the teaching/learning process is a story seldom told (Dreyfuss & Esienberg, 1990). Indeed, symmetry intervenes in different disciplines, and it is often presented as a collection of disconnected concepts.

Many phenomena represent symmetry as reflections or bilateral symmetry, rotations, translation, glide reflections, similitude, affine projection, topological symmetries. Nevertheless, the list is far from complete. Thus, the unifying approach to the modern concept of symmetry constitutes a fundamental notion in education as a big idea because it equips learners with the ability to explain a broad range of phenomena within and between it (Shin et al., 2009). The modern meaning of symmetry refers to invariance properties of physical or abstract objects under certain operations (i.e., transformations such as rotation, reflection, inversion, or other abstract and complex processes). Although the relationship between symmetry and invariance is relevant in mathematics and science, it does not receive the attention it deserves in education (Libeskind et al., 2018; Dreyfus & Eisenberg, 1990; Schuster, 1971).

This PhD thesis aims at putting symmetry and invariance in the heart of teaching-learning process at the primary toward secondary school level. Besides the consolidated pedagogical-educational practices, we introduce an interdisciplinary approach based on the connection between symmetry and invariance. This study presents an educational trajectory based on the modern concept of symmetry as a big idea within our project "Mathematics beyond numbers, symmetry and the search of invariants," initiated at the primary school of Montegranaro in the school year 2018/2019.

The introductory chapter provides a comprehensive overview of the background and purpose of the study. Section 1.1 gives a brief theoretical overview, while Section 1.2 identifies the research problem. Section 1.3 outlines the proposal and objectives of the study, along with its research problem. Section 1.4 explains the significance and scope of the study, as well as the definitions of the research questions. Section 1.5 provides an overview of the research methodology that will be used to answer the research questions. Finally, Section 1.6 outlines the remaining chapters of the dissertation to provide the reader with a roadmap. Overall, this introductory chapter serves as a comprehensive guide for the study and lays the foundation for the study and subsequent research.

1.1 BACKGROUND

International assessments in mathematics and science are open-source data impacting educational methods and policies in Italy since 2003. The Programme for International Student Assessment (PISA) reports show that Italian 15-year-old pupils' knowledge in mathematics and science is below the Organisation for Economic Co-operation and Development (OECD) average. As illustrated in Figure 1.1, the trends are positive for mathematics but very negative in science from 2012 scores. In short, the scientific expertise is declining.

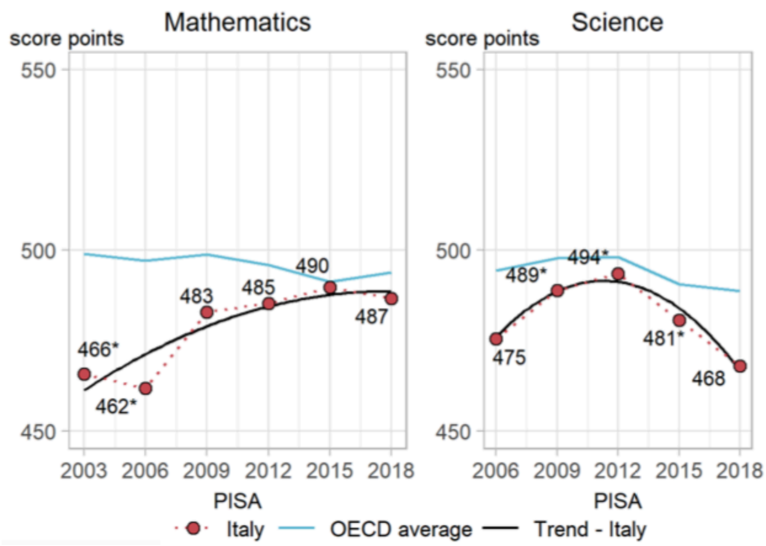


Figure 1.1. Trends in performance in mathematics and science.
Source: OECD, Database PISA 2018

These disappointing tendencies are in agreement with the most recent research on Trends in Mathematics and Science Study (TIMSS). Like PISA, TIMMS is a large-scale international assessment of learning outcomes in mathematics and science at primary (grade 4) and lower secondary (grade 8) levels. TIMMS evaluates student achievement over time, focusing on math and science concepts with social background and learning environment variables. TIMMS 2019 shows a reduction in student interest in science as they transition from primary to secondary school. The situation is more severe in mathematics as the number of students who dislike mathematics goes from 20% to 49%.

In addition, students who feel confident in mathematics have higher average achievements than those who do not. Liking learning mathematics and feeling confident in mathematics are strongly associated with higher achievements (Mullis et al., 2020). There is a need to improve and innovate teaching to provide students with long-term active and constructive mathematics learning as an indispensable backbone for acquiring scientific skills.

The National Guidelines for the curriculum (MIUR, 2012) set as a fundamental objective the acquisition of mathematical tools for the understanding of scientific disciplines and being able to operate in the field of applied sciences. The European Recommendations on key competences for lifelong learning (European Parliament, & Council of Europe, 2006; Council of the European Union, 2018) also consider mathematical language to be a fundamental communication tool, together with fundamental language, to be acquired at the end of compulsory education as a key citizenship competence.

Galileo Galilei had already identified mathematics in his famous writing "*The Assayer*" as a language for describing and understanding the world and with which the sciences developed. In 1960, Eugene Wigner observed in his article "*The Unreasonable Effectiveness of Mathematics in the Natural Sciences*" that mathematical language is a wonderful and incomprehensible gift for accurately describing the laws of physics and every expectation of the various branches of knowledge. Many researchers and philosophers have tried to explain the unreasonable efficacy of mathematics (Bangu, 2016; Harvey, 2011; Longo, 2005), but what is more crucial than its explanation is its decisive contribution to teaching. The learning-teaching process of Mathematics and Physics cannot fail to take into account this inseparable complexity.

1.2 PROBLEM IDENTIFICATION

Mathematics education does not provide students with a solid mathematical foundation, enough awareness and skills for physics and scientific disciplines. One of the key concerns in science education, at whichever level, is the lack of students' familiarity with mathematics. It is rather common that students need to learn more mathematics to succeed in science. As it has already been sufficiently established, the absence of some basic mathematical abilities is a significant reason for students' failure in physics courses (Rossdy et al., 2019; Awodun & Ojo, 2013). However, it is equally accepted that mastering these skills does not guarantee success in physics (Vinitsky & Galili, 2014; Hudson & Rottmann, 1981). Students display a largely nominal rather than substantial understanding of physical and mathematical ideas. In other words, students focus on basic comprehension and memorization of concepts and formulas rather than on critical thinking.

Many causes have been attributed to the lack of pupils' mathematical proficiency and literacy. For a clear picture of the complexity of the problem, it needs to ascertain all educational and cognitive factors besides social aspects (Cai et al., 2020; Nilsen & Gustafsson, 2016). Among various interrelated causes restraining pupils' mathematical skills, we can mention the lack of interdisciplinary settings with poor connections between concepts and real-life, which inhibits conceptual understanding. Conceptual understanding is one of the primary components of scholars' mathematical proficiency (Henningsen & Stein, 1997). Disharmony in the pedagogical curriculum and lack of inquiry-based learning also influence students' mathematical achievement. Furthermore, enhancing reasoning ability and abstraction process from primary school has a key role in developing pupils' mathematical skills.

1.3 RESEARCH PROBLEM AND RESEARCH PROPOSAL

The above discussion raises important problems about the connections and interplay between mathematics and physics in an educational context. The evidence particularly shows barriers between mathematics and scientific disciplines. On this basis, the research is guided by the following driving questions:

What might be done to avoid this lack of harmony in the pedagogical curriculum? How can we make physics and mathematics accessible, motivating and more integrated in a natural and organic way?

This study uses Michelsen's (2015) didactic framework to answer these questions by merging mathematics and science and creating a coherent learning progression that seamlessly transitions from elementary to higher mathematics. Of the various frameworks available, Michelsen's method is ideal for integrating mathematics and science concepts around a central *big idea*.

This approach views learning as a continuous process of spiralling construction and reconstruction of important mathematical concepts and functions essential for analysing physical models and making interdisciplinary connections. The concept of "spiralling trajectories" recognizes that learning is not a linear but a cyclical process in which learners repeat and refine their understanding of mathematical concepts over time to gain a more comprehensive and advanced understanding of the subject. In addition, the instructional design is conducive to action research because it allows teachers to participate in multiple cycles of creating, implementing, and assessing instructional sequences. Section 2.2 explains how the learning pathway is constructed based on Michelsen's framework for modelling interdisciplinary activities between mathematics and science.

The designed didactic teaching-learning sequence puts abstraction at the heart of teaching. Abstracting is a process beginning with reality and using some tool to pare away the excess to reveal a critical essence. From a curriculum design perspective, the abstracting process offers common grounds for theoretical frameworks. The development of pattern-based thinking is a powerful tool for doing mathematics and it leads to hierarchical mathematical platforms for integrating and developing scientific knowledge.

In this integrated learning progression, mathematics becomes a metalanguage that allows descriptions and understandings of the physical world. The elements of mathematics suit a set of fundamental ideas that will build the architecture of complete scientific knowledge, typically of NOS. Specifically, we focus on grasping the challenges of an interdisciplinary approach study at the transition from primary to secondary school. The key aspect of educational innovation in NOS Interdisciplinarity Teaching is to assume some organizing concepts as possible core-ideas for linking teaching in mathematics and physics and, more generally, science.

Our assumption of symmetry and invariance as central curriculum units (i.e., *big idea*) establishes a paradigm change for an efficacy interdisciplinary teaching approach at school, facilitating in-depth learning and conceptual understanding across science disciplines.

1.4 RESEARCH QUESTIONS

Several notable studies point out that the concepts and the principles of the application of the modern concept of symmetry can be taught and understood at the high school and early college level (Bertozzi et al., 2014). Our experimental project aims at answering the following General Research Question:

GRQ. How can we use symmetry and the search of invariants as bridging concepts in science education for fifth grade pupils?

This leads to the following Specific Research Questions:

SRQ1. What is the effect of this change of paradigm on the students' understanding and skill development?

SRQ2. What is the effect of this change of paradigm on the didactical framework of linking teaching in Mathematics and Physics?

1.5 SIGNIFICANCE AND SCOPE OF THE STUDY

The study focuses on three years of action research in mathematics education around the transition from primary to secondary school in Montegranaro (Italy). The investigations evaluate how the specific teaching action makes the modern concept of symmetry appropriate for primary school students in three domains: cognitive, affective, and psychomotor dimensions.

We expect that this study will provide valuable insight into the possibility of applying the modern concept of symmetry in primary school and the possibly arriving challenges. Also, it is hoped to help students enhance interdisciplinary knowledge and develop mathematical and scientific skills. We hope the study's findings would furnish the scholar community with rich stimuli and deep insights as a habit of mind on the theme of symmetry to link mathematics science topics. In addition, the findings would bring important suggestions and contents of reflection that teachers can consider for exploiting the potential interdisciplinary learning path on symmetry and invariance for teaching methods and pedagogical curriculum.

1.6 THESIS OUTLINE

The following chapters explore the potential of symmetry as a pedagogical tool in mathematics and science education and present three research studies related to the focus areas described in Chapter 1.

The literature review in Chapter 2 addresses various aspects of interdisciplinarity, learning pathways, and theoretical frameworks that provide the foundation for using symmetry in science education. In particular, the roots of the language of symmetry, the interplay of mathematics and physics through symmetry, and the simplification of concepts through abstraction are explored. It also explores how learning symmetry can be facilitated through educational artefacts.

Chapter 3 discusses the methodology and research design used in the first reflection cycle of the experiment. Specifically, it presents the design and application of the didactic EMT model for the transition from elementary to secondary school that informed the two cycles of reflection in the action research. Chapter 4 presents the results of the first cycle, primarily through the analysis of students' responses to pre-post questionnaires assessing learning changes in their knowledge of symmetry.

Chapter 5 outlines the design of the second cycle of the experiment and the partial implementation and results for the breakout of covid-19. Chapter 6 focuses on the design and evaluation of a teacher training course that integrates symmetry pedagogy as a third experiment to fill the gap left by the breakout of covid-19 explosion in the second cycle. The thesis concludes in Chapter 7 with a summary of the main findings, limitations, and implications of the study.

Chapter 2: Literature Review

This chapter provides a comprehensive overview and background information regarding symmetry in mathematics and physics supporting justification for the study. It covers various topics, such as interdisciplinary teaching and the didactic model of teaching symmetry. Emphasis is placed on promoting interdisciplinary understanding by connecting and organizing knowledge from different disciplines in the Section 2.1.

The Section 2.2 explores in detail the learning pathway by examining the different phases of the didactic model that students go through when learning about symmetry. The chapter also discusses in Section 2.3 the theoretical framework, including the roots of symmetry language (Section 2.3.1), group theories (Section 2.3.2), and the interaction between mathematics and physics through symmetry (Section 2.3.3). It also discusses symmetry phenomena (Section 2.3.4), the modern concept of symmetry (Section 2.3.5), simplification through abstraction (Section 2.3.6), and learning symmetry through artifacts (Section 2.3.7).

Finally, the chapter summarizes the main points and their implications and provides an insightful and thorough literature review on symmetry in mathematics and physics.

2.1 INTERDISCIPLINARITY

The theme of interdisciplinarity education has been acknowledged in literature since the 1920s (Vars, 1991), and it is yet increasingly topical and widely discussed. The current debate involves organizing knowledge for the challenges of 21st Century skills and a global economy (Dowden, 2007). Indeed, interdisciplinary is identified as one of the four types of knowledge by the OECD Future of Education and Skills 2030 project (OECD, 2018). 21st Century education demands interdisciplinary skills and knowledge to tackle complex issues that require overcoming the boundaries of disciplines.

Menken & Keestra (2016, p. 34) emphasize that complexity is the main driving force behind interdisciplinarity. According to J. T. Klein (1990, p. 55, 196), the defining characteristic of interdisciplinary is integrating disciplinary knowledge into solving problems and answering questions. He considers multidisciplinary as a simple additive process, while interdisciplinarity integrates disciplinary knowledge and complex cognitive skills.

Although the integrative side of interdisciplinarity is essential, as is its innovative character (Davies et al., 2010, p. 12), defining it is less obvious than it may seem. In H. H. Jacobs's (1989) view interdisciplinarity is a "knowledge view and curriculum approach that consciously applies methodology and language from more than one discipline to examine a central theme, issue, problem, topic, or experience" (p. 14). Repko et al. (2014) define, furthermore, interdisciplinarity in terms of the cognitive process of individuals or groups: "interdisciplinary [studies] is a cognitive process by which individuals or groups draw on disciplinary perspectives and integrate their insights and modes of thinking to advance their understanding of a complex problem with the goal of applying the understanding to a real-world problem" (p. 100).

The previous different definitions underscore a lack of conceptual clarity concerning the nature of interdisciplinarity. In this regard, Jacobs & Frickel (2009, p. 52) refer to the relative absence of epistemic clarity and warn about possible fragmentation of disciplines. Bennington (1999, p. 104) alleges that ambiguity arises from the "slippery" nature of the interdisciplinary. The *inter*-disciplinary prefix is often exchanged with other prefixes such as *multi*-, *cross*-, *pluri*- and *trans*-. Nevertheless, the plethora of hierarchies of interdisciplinary nuances is not terminated (J. A. Jacobs, 2014). As a result, the terms are used interchangeably, causing further confusion. Regarding the interdisciplinary definition, J. T. Klein (2005, p. 55) argues that it furthers differing purposes of education and research, the role of disciplines and critique. Therefore, interdisciplinarity is to be understood as various ways of bridging and comparing the different disciplinary approaches. In this view, the different approaches can be classified by the depth of integration of disciplines.

From *intra*-disciplinarity (i.e., the knowledge and skills within a single discipline) to *multi*-, *cross*-, *pluri*-, *inter-,* and *trans*-disciplinarity, the integration between disciplines increases gradually as a "continuum" (Fogarty, 1991; Lattuca, 2003). At one of the extremities of the *continuum* of connection between disciplines is the notion of *multi*-disciplinarity. It refers to a juxtaposition of several disciplines focused on one problem without a direct attempt to integrate (Piaget, 1972). The slightest level of disciplinary interaction is *cross*-disciplinarity. The boundaries of disciplines are crossed at a general and descriptive level. Meeth (1978) describes *cross*-disciplinarity as viewing one discipline from the perspective of another. With *pluri*-disciplinarity, the juxtaposed disciplines are linked together by short-term cooperation on a particular issue (Piaget, 1972; Halloun, 2020). Increasing the extent of disciplinary integration leads to *inter*-disciplinarity. The disciplines are yet identifiable but produce a real synthesis with a social, scientific and personal added value (Wilthagen et al., 2018, p. 30). Finally, in *trans*-disciplinarity, interactions are extended in stronger forms. Jantsch (1972) considers *trans*-disciplinarity indeed as the highest degree of interdisciplinarity. In particular, Johnston (2008) emphasizes that *trans*-disciplinarity is at a profound level of connection and connectedness beyond the disciplinary perspectives. In practice, interdisciplinarity has developed into a broad range of terms and definitions.

Regardless of the variety of expressions in the literature, the plurality of interdisciplinary approaches shares the same objective of overcoming the traditional separation between disciplines to build complex and interdisciplinary knowledge. Boix Mansilla et al. (2000) name the capacity to integrate knowledge and modes of thinking in two or more disciplines to produce a cognitive advancement in social and natural/physical sciences as *interdisciplinary understanding* (p. 219).

In teaching-learning terms, *interdisciplinarity understanding* arises as a necessity to build students' learning knowledge and scientific experience in a full and cross-sector sense. Undoubtedly, *interdisciplinary understanding* is best fostered when the interdisciplinary knowledge is meaningfully organised and related to robust and accurate awareness of the disciplines. Students should learn to integrate and synthesize the disciplinary contents, methods and languages. Under the interdisciplinary approach, these elements provide the foundation for building new interdisciplinary knowledge and way of thinking. According to the National Research Council (NRC, 2014), *interdisciplinary understanding* occurs through a convergence approach that combines "knowledge, tools, and ways of thinking" throughout the application of interdisciplinary nodes "to tackle scientific and societal challenges that exist at the interfaces of multiple fields" (p. 1).

The interdisciplinary convergence approach can be seen through the metaphor of Leonardo's octagonal mirror chamber (Da Vinci, Manuscript B, folio 28r, c.1488). If we look at the problem as an object located in the room's centre, every single mirror wall reflects the object as a single discipline; at the same time, it reflects the object's reflections in infinite sequence of reflections from all sides, not just from one. In this way, interdisciplinarity comes from the multiple and resonant points of view on the complex problem to find the convergence of the methods and the synthesis of the answers to the problem.

Leonardo's Mirror room sheds light on the relationship between complexity and the interdisciplinary act of knowing. Indeed, Newell (2001) sees complexity as a prerequisite for interdisciplinary studies. Rather, Morin (1990, 2007) considers interdisciplinarity to be part of "complex thought". To that extent, he hopes for a reform of thought through the interdisciplinary approach to the complexity of reality: "Pour qu'il y ait véritable interdisciplinarité, il faut des disciplines articulées et ouvertes sur les phénomènes complexes, et, bien entendu, une méthodologie ad hoc. Il faut aussi une théorie - une pensée - transdisciplinaire qui s'efforce d'embrasser l'objet, l'unique objet, à la fois continu et discontinu, de la science : la physis[1]" (Morin, 1973, p. 229). Indeed, he advocated a new science since *complex thought* reform is closely linked to education.

Therefore, starting from the earliest stages, a reform of teaching in the school entails a mental paradigm revolution for organizing and connecting the fields of knowledge confined within the disciplines. Such paradigm revolution toward the complexity of the world demands *interdisciplinary teaching*. Unlike traditional disciplinary teaching, which generates isles of knowledge, *interdisciplinary teaching* grants a way to form a complex web of knowledge, opening the mind to *interdisciplinary understanding*. Using Aristotle's terms (Ross, 1924), the interdisciplinary approach in education makes the whole learning process not a mere heap of parts (i.e., the disciplines) but something besides them.

The wide need for interdisciplinarity concerns the complex cooperation of the two cerebral hemispheres in acquiring and building knowledge. According to Darvas (2007, p. 365), the differences between the two hemispheres play a role in sensation, perception, understanding and gaining knowledge, and consequently in learning. For example, students with the dominant right hemisphere learn intuitively and creatively through manipulation activities and spatial operations. In contrast, abstract thinking

[1] The nature of knowledge is multi-faceted and requires a transdisciplinary approach and a new scientific paradigm to grasp the complexity of physis, which encompasses everything that exists, including the various forms of life on Earth, and recognizes the crucial role of the thinking subject in the cognitive process.

is better conferred by pupils whose left hemisphere is prevailing. Although the brain's function division is deeply individual and complex, the corpus callosum allows a great two-way exchange influencing personal learning (Darvas, 2015, p. 80).

The corpus callosum is a band of nerve fibres that connects the two hemispheres of the brain and allows communication between the two sides. Studies have shown that a larger corpus callosum (i.e., fibre density and cross-sectional area) is associated with better mathematical abilities because it allows for a more efficient transfer of information between the hemispheres (O'Boyle, 2008; Collins et al., 2021). Arnold V.I. emphasizes the importance of both hemispheres of the brain for mathematics because the combination of spatial intuition and algebraic manipulation can be challenging but also fascinating: "Our brain has two halves: one is responsible for multiplying polynomials and languages, the other for orienting figures in space and all the things that are important in real life. Mathematics is geometry when you have to use both halves" (Lui, 1997). This difficulty in making intuition precise makes geometry challenging but also fascinating, as is the intuitive concept of the difference between straight lines and curves (Brooks, 2003; Geiges, 2008). Brain plasticity, the ability of the brain to reorganize itself in response to new experiences and learning, is another important factor in the development of mathematical skills.

Early education and encouragement can maximize brain plasticity and optimize mathematical skills, as plasticity decreases with age. In addition to the brain structure, *interdisciplinary teaching* can improve mathematical skills by exposing students to real-world problems requiring mathematical thinking and promoting cognitive flexibility. From this point of view, *interdisciplinary teaching* offers many possible methodological deviations to organizing the complexity of cognitive processes with elegance, originality and creativity (Berthoz, 2009). *Interdisciplinary teaching* focuses on creating a network of links between pupils' ideas to develop a peculiar, harmonised and normative understanding to interpret new situations (Linn & Eylon, 2011). Accordingly, *interdisciplinary teaching* employs a wide range of models and methods to teach wholeness across different curricular disciplines creating a new body of knowledge and generating the motivation for the learning process (Petere, 2003).

The American Association for the Advancement of Science (AAAS, 2011) recognised the importance of interdisciplinarity in science education by including the ability to tap into the interdisciplinary nature of science (NOS) as one of six core skills (p. 15). Students best master this core competence if it explicitly addresses disciplinary core ideas and crosscutting concepts within science learning (Lederman & Lederman, 2014). Interdisciplinary connections are achieved through unifying concepts, namely *big ideas*, which promote better *interdisciplinary understanding* by setting up "contingent moments that are the essence of powerful [interdisciplinary] teaching" (Hurst, 2017, p. 117).

Interdisciplinary teaching towards and through *big ideas* is an essential shift and a pedagogical challenge for teachers in science disciplines (Bravo González & Reiss, 2021; Harlen, 2015) and mathematics (Gadanidis & Hughes, 2011; Loh & Choy, 2021). By *big ideas*, McTighe et al. (2004) mean relationships between core concepts, principles, theories, and processes that should serve as the focal point of curricula, instruction, and assessment in a field of study. According to Mitchell et al. (2016), "*big ideas* are meaningful patterns that enable one to connect the dots of otherwise fragmented knowledge" (p. 3). Tomlinson & McTighe (2006) extend the fields of application of *big ideas* as central to students' understanding, learning, and work, providing the conceptual pillars that anchor the various disciplines (p. 33). Indeed, knowledge from various science fields is interrelated into a coherent and scientifically based worldview (Chalmers et al., 2017). In short, *big ideas* are key ideas that link numerous discipline understandings into coherent wholes (Charles, 2005; Harlen, 2010).

The shift for the crosscutting educational concept as unifying themes across scientific topics and various domains of science was first outlined in the framework edited by the National Research Council (NRC) in the U.S. The framework supports interdisciplinary learning and teaching, presenting mathematical proficiency as a key and fundamental skill "such as modelling, developing explanations, and engaging in critique and evaluation" (NRC, 2012, p. 3) for the future science education of K–12 students. Interdisciplinary connections are visible to students if they gain enough mathematical literacy and proficiency. As students mature mathematical skills, the connections become more explicit through *big ideas*.

In order to see and build bridges between mathematics and science in education, interdisciplinary teaching-learning should focus on big ideas and thinking development in mathematics (Li et al., 2019), commencing from primary science classrooms (Geiger, 2019). Consequently, scientific literacy advances focus on better understanding the relationships between science and mathematics teaching. Maass et al. (2019) highlight the need to understand mathematics around big ideas from the beginning of education to connect such key ideas underlying science. These concepts have, by the way, already been expressed in the AAAS publication (Science for All Americans). Indeed, Rutherford & Ahlgren (1990) consider science and mathematics part of the same endeavour because they try to discover general patterns and relationships (p. 34). Thanks to mathematical language and scientific theories, the world and its structures are thus possible to understand, explain, and depict.

The rationalization of the world is the fundamental starting point of the mathematical-logical episteme (Lähdesmäki & Fenyvesi, 2017, p. 4). Looking at the evolution of mathematics in ancient Greece we find that Greek gods assured the existence of deep links between mystic thought and nature, between our human sphere and external reality; the unique way and the fundamental principle to fill the gaps between these realms could be expressed not ambiguously by mathematical languages. In Plato's view, mathematics had a divine origin: it was given to humanity by the Greek sky god Uranus, who was identified with the cosmos[2] and expressed the laws of numbers and relationships in the movement of the stars (Zellini, 2016, p. 34). Such a gift was the most powerful tool to assist and train the mind in thinking and trying to reach the truth. According to him, the physical world was made by substantial objects in material reality, but the expression of intelligible forms was revealed only through arithmetic and geometry.

So, the study of nature leads to establishing relations between the "sensible" world and the intelligible world by mathematical methods. The nature of the visible is such that it requires some branch of mathematics to be understood. Thereby the qualities of symmetry and orderliness found in nature acted as a guide to conceive a growing variety of subjects in mathematics such as arithmetic, geometry, algebra, trigonometry and calculus. The formulas of mathematics, inspired by nature, have the power to dynamically design the forms of reasoning, describing and knowing cognitive thoughts by their autonomy.

[2] "For if one enters on the right theory about it, whether one be pleased to call it *World-order* or *Olympus* or *Heaven* - let one call it this or that, but follow where, in bespangling itself and turning the stars that it contains, it produces all their courses and the seasons and food for all. And thence, accordingly, we have understanding in general, we may say, and therewith all number, and all other good things: but the greatest of these is when, after receiving its gift of numbers, one has covered the whole circuit." (Plato, Epinomis, 977b)

In the opinion of Darvas (2007), reason, logic, and the objectivity of perception are determinants, which are often related to science (p. 375). The special relation between science and mathematics is shifted from the ontological field to the linguistic sphere. Mathematics does not grab the "essence of the world" but provides the language to shape and craft reality in its quantitative aspects.

Nowadays, mathematics has increased its power and is seen in a new light as the science of all patterns. It consists of patterns and relationships between them that ultimately fit all phenomena in the real world through deep connections between different areas of science that seem far apart. These links enable us to find order in apparent chaos and glimpses of hidden structures underlying reality. Metaphorically, mathematics is like solving a huge jigsaw puzzle. We do not know what the final picture is going to look like, and we try to find a way to make all pieces of the jigsaw puzzle fit together; at the beginning, we create little parts of this jigsaw puzzle that represent active contents as units of different disciplines, and then we try to see how to connect them. There is a secret universe interconnected out in nature, invisible to our eyes.

Mathematics *big ideas* provide a conceptual lens (mathematical glasses) to reveal patterns in the world around us and build bridges between different science continents (Tomlinson & McTighe, 2006, p. 27). A scientist is similar to a poet because he has the power to see deeper, looking beyond numbers and exploring the unknown. Scientific ideas are articulated and shaped by the language of mathematics, which creates a powerful and coherent structure, helping us think and make sense of a wide range of natural phenomena. Gielis (2017, p. 19) stated that all scientific progress relies on the availability and development of new pairs of mathematical glasses as mental instruments, quoting a famous mathematician of twentieth-century René Thom (1991, p. 96). Indeed, mathematical glasses are mostly new ways of looking at the natural world to find regularities and connections of things in such a way as to conceive and develop scientific ideas. The minds of scientific giants such as Kepler, Newton, Leibniz, Euler Gauss, and Einstein provided surprising insights and creative approaches to solving difficult and complex problems by supplying new mathematical glasses as mental devices for looking at natural phenomena. Such mental perspectives have revealed previously hidden connections and infinite possibilities to integrate knowledge across different domains of science, showing their effective applicability to various contexts of knowledge.

According to McClintock, as cited by Keller (1983), seeing things in various forms is an important instrument for all students and scientists: "Basically everything is one. There is no way in which you draw a line between things. What we normally do is to make subdivisions, but they are not real. Our educational system is full of subdivisions that are artificial, that should not be there" (Keller, 1983, p. 204).

The school education should indeed encourage students to go beyond their way of seeing by wearing new mathematical glasses and searching for unity. Students can develop a deeper understanding of the elegance and beauty of mathematics by emphasizing the connections between different mathematical concepts and domains. Ramanujan and Boole are excellent examples of mathematicians who recognized the importance of unity in their work. Boole (1854) was particularly interested in the unity of thought in his day: "It may be that the progress of natural knowledge amounts to the recognition of a central unity in nature" (p. 417). Boole's insights into the unity of thought and the interconnectedness of different areas of natural knowledge helped lay the foundation for modern logic and computer science. Ramanujan, for his part, expressed this unity through his ability to connect seemingly disparate mathematical concepts and derive new insights. Indeed, Ramanujan's work contains an identity that precisely connects three pillars of number theory: the golden ratio ϕ with a conversion to terms of e and π (Olsen, 2017). Encouraging learners to view the world through mathematical glasses is a key to fostering learning mathematical skills and developing a deeper understanding of mathematics and a greater appreciation of its beauty and power.

Several studies (Xu, 2017; Innabi & Emanuelsson, 2020) in the learning theory examine the pupils' variation in seeing and understanding mathematics for making sense of the science subjects in the teaching-learning sequence. The results of educational research show the potential of students' reflections and descriptions of their mathematical thinking and seeing process during lessons. The shifting in the way of seeing mathematics concepts is critical in understanding teaching/learning to the interdisciplinary educational outcome. Svensson (2016) argues that learners' awareness of mathematics is linked to and revealed through the different use of language in expressing understanding of the subject matter. Further, Tytler et al. (2021) argue that mastering the way of mathematical seeing and thinking promotes pupils' additional learning, generating interdisciplinary knowledge. Based on the constructivist perspective, learners build interdisciplinary knowledge by reflecting on their educative experiences and transforming the language and the mathematical way of seeing the world (Veine et al., 2020).

Mathematics can have a richer function along with the potential of mathematics in building bridges within and between disciplines. Indeed, mathematics can play the cultural role of corpus callosum connecting the two hemispheres because of its humanistic content, since it describes and creates possible worlds, and its scientific methods since it uses logic (Odifreddi, 2000, p. 8). Thus, mathematics is cognitively functional and foundational to complex mathematical thinking and scientific reasoning (Clements & Sarama, 2011).

2.2 LEARNING PATH

Mathematical thinking and scientific reasoning can be achieved in education by the design and application of *learning trajectories* or *learning progression* (LTP). The original notion of *learning trajectories* applied to mathematics belongs to Martin Simon (1995). He was the first to explicitly use the term *learning trajectory* in the Journal for Research in Mathematics Education as a segment of a learning path along which students achieve mathematical ideas or concepts (p. 133). According to Clements & Sarama (2004, p. 83), the *learning trajectory* is a “conjectured route through a set of instructional tasks designed to engender those mental processes or actions to move children through a developmental progression of levels of thinking”.

The sequence of learning-teaching activities comprises the important components of learning trajectories such as learning goals, teaching content and methods. The learning goal is often related to the pupils’ understanding of big ideas to build a framework of body of knowledge and high-level mathematical reasoning which allow them to interpret new and interdisciplinary situations. The sophistication of students’ thinking about big ideas over time is common with the *learning progression* used mostly in science education (Mohan & Plummer, 2012). Besides the centrality of big ideas, the learning trajectory and progression share approaches and methods. However, there is no clear distinction in the use of these terms, which often get specified according to the field of application: mathematics versus science education (Lobato & Walters, 2017, p. 75).

The development and validation processes of the learning sequence in the LTP are not well specified either (Krajcik, 2012, p. 28), depending on the specific learning goals of the teaching activities (Stevens et al., 2010). There exist various strategies to validate students' progression. Among them, to mention just a few, we find multiple-choice questionnaires (Neumann et al., 2013; Confrey et al., 2020), interviews and video recordings (Wilson et al., 2013), written tasks or drawings (Demosthenous et al., 2019; Villarroel et al., 2019), and instructional activities (Seah & Horne, 2019b; Stevens et al., 2010). Beyond the LTP's validation, the focus on developing pupils' thinking and reasoning is a hallmark of a *learning path* in education. In this perspective, we refer to the *learning path* as a general learning progression or trajectory. The educational path is individual and social simultaneously. Indeed, a *learning path* can differ for every pupil depending on his/her learning styles and rhythm of acquiring knowledge.

The *learning path* runs through different phases. From first making sense of the mathematical idea, the pupils pass through an in-depth reflection on acquired information, integrating the knowledge nodes into their mental models. The teaching-learning sequence aids in guiding learners to make correct arrangements of the knowledge and to organize them in a meaningful way into the schematic and cognitive structure. In a meaningful way means that the acquisition of connection between information is characterised by "learning with understanding which proceeds by organization appropriate to the inherent structure of the material" (G. Katona, 1940, p. 238). According to Gómez-Veiga et al. (2018), meaningful learning is a complex and sequential task that solicits students to activate various levels of understanding, from intuitive to sophisticated and abstract reasoning. Therefore, the learning efficacy is determined by providing significant and well-prepared learning material to crystallize the logical relations inherent and resonant with the pupils' cognitive skills.

As Gravemeijer et al. (2003, p. 52) emphasised, the importance of the *learning path* is to create artefacts and tools for the classroom practice and build a sequence of progressive activities along which each pupil's thinking progresses toward the big idea. Moreover, the complexity of the teaching-learning process proposes a circular relationship between the three moments of the educational process. The planning phase is inextricably intertwined with the execution and empirical validation in progressive implementation and reflection and tuning cycles. The specific characteristic of an action research experiment is the reflection and tuning cycle in iterative design, focusing on the students' understanding of the big idea while the experiment is in progress: "The iterative nature of the action inquiry process is perhaps its single most distinguishing characteristic" (Tripp, 2005, p. 452).

An action research design refers to studies involving iterative reflective cycles that provide didactic and pedagogical feedback before moving on with the activities. According to O'Connor and Diggins (2002), action research is a paradigm of research and daily practice that uses reflection into action in real classroom settings to provide a sense of teaching and learning issues and make positive, conscious pedagogical changes to improve future decisions and actions. As a result, it includes the data collection and analysis processes where the analysis informs the next activities implementation and data collection cycle. A spiral path of different phases generally represents the reflection cycle. Lewin (1946) and Kemmis & McTaggart (1988) first designed the ongoing learning spiral cycle in four fundamental phases: planning, observing, acting, and reflecting. Several researchers adopted and adapted the basic structural framework for their action research in the field of education (Hopkins, 1993, p. 44-45; Elliot, 1991, p.71-76; Ferrance, 2000, p. 9; Stringer et al., 2010, p. 23; Lovemore et al., 2021, p. 6) including more steps as moments of conducting, exploring, and evaluating the possibilities and limits.

In the chapter presenting the research methodologies, we will describe our version of the six-step iterative spiral process based on the didactical model of Michelsen (2015). The latter consists of an educational framework (Figure 2.1) for modeling interdisciplinary activities between mathematics and science.

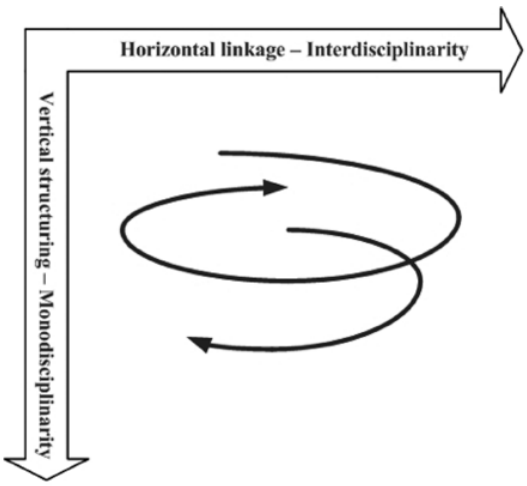


Figure 2.1. Michelsen's model: the spiral shape shows the repetitive movements between horizontal linkage and vertical structuring.

Specifically, this model presents a cyclic process of designing interdisciplinary and non-fragmentary learning path, providing longitudinal development of mathematics contents around a big idea or core concept. Interdisciplinary coordination and interaction are operative in horizontal linkage and vertical structuring. In the horizontal phase, thematic integration is established to connect the concept and process skills of mathematics and science. The horizontal linkage is characterised by the process of modelling activities in an interdisciplinary context.

Michelsen (2015) points out that "modelling in an interdisciplinary context is a specific problem-solving strategy with scientific and pragmatic purposes beyond the traditional and historical boundaries determined between subjects" (p. 491). In other words, the interdisciplinary competence of modelling is the glue that connects interacting subjects. In vertical structuring, new concepts are anchored with the development of the mathematical core. Along the vertical direction, pupils should move up through cognitive and language development logically and analytically linked to the main idea within mathematics and science. This process is thought to create a firmer and generalised mathematical structure.

In accordance with this model, we have designed the teaching-learning progression, developing the activities around the big idea of symmetry for the vertical increment and linking them to invariance for the horizontal dimension. Weaving together symmetry and invariance allows pupils to construct a new conceptual image of symmetry. By *conceptual image* we mean all cognitive structures in the mind of an individual, associated with a given concept, including all mental images, properties and processes (Tall & Vinner, 1981, p. 152). The pupils' ability to construct mental structures involves a qualitative change in their perception of symmetry linked to invariance. Such transformation is a cognitive shift in perceiving symmetry as a process (or a principle) of invariance for organizing and categorizing many phenomena, data, and information into coherent knowledge. The challenge is to develop the learning path with activities and materials that initially involve students in non-routine problematic situations. The teaching-learning sequence evokes the construction of meaningful concepts that are successively expanded and explored in other problematic situations and anchored in successive iterations. Thus, the vertical and horizontal progression occurs from iterative improvements in which pupils develop the habit of using different articulations of ideas to understand the behaviour of and changes in materials. Once the mathematical concepts and skills in search of invariants are conceptually anchored in pupils at primary school for the respective subjects, they can evolve in a new interdisciplinary context as part of a horizontal linkage. In this way, we expect to identify pupils' conceptions and the persistence of these conceptions and focus on difficulties children might have in mastering the concept of symmetry and translating it across other contexts. The analysis of learners' conceptions is critical because it provides the learning requirements for the student-centered model and is a valid tool to evaluate the effectiveness of the teaching-learning experiment. A common way of evaluating the effectiveness of a TLS is to compare the students' cognitive final state with their cognitive' initial state. Some other methodological approaches illuminate learners' cognitive pathways through the teaching-learning process (Méheut & Psillos, 2004).

Our study investigates the pupils' cognitive path about symmetry during the sequence through a mixed-methods approach. Johnson & Onwuegbuzie (2004, p. 17) define mixed methods methodology as "the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study". Mixed methods use intentionally different data (e.g., qualitative and quantitative) to address different analysis dimensions within a system. The qualitative and quantitative data for empirical investigation aim at broadening the dimension level of study understanding and reducing errors of interpretation.

We addressed the analysis in three domains devoted to cognitive, affective, and psychomotor dimensions. It involves collecting quantitative data on learning changes in pupils' knowledge of symmetry concepts and qualitative data about motivation, interests and learning requirements. To grasp the whole spectrum of dimensions, a comparative experimental study was applied by analyzing pre- and post-sequence questionnaires with closed and open-ended items. Open questions prompting students to reflect on the insight they gained and sensed during the teaching-learning units can shed more light on their thought processes. There is much evidence that open questions are useful for developing students' cognitive skills. Open-ended questions engage pupils in cognitively challenging conversations (Ogu & Schmidt, 2009), provide rationales for their thoughts (Lee et al., 2012) and likely promote higher-order thinking (Roth, 1996; Fuson et al., 2005). In addition, open-ended questions positively influence the development of pupils' vocabulary and language skills. For example, Kwon et al. (2006) found advantages in fluency, flexibility, and originality in learners' language of primary school when subjected to open-ended questions and problems in mathematics.

Within this context, exploring the pupils' reflections and observations allows examining the extent to which the students have elaborated and mastered the contents and how conscious they were of the learning activities. According to Duval (1998, p. 38), learning activities in geometry involves three cognitive processes affecting practical functions: visualization, construction, and reasoning. Although Duval differentiates visual and reasoning processes, geometry proficiency involves a complex interaction of three cognitive processes. The reasoning is the ability to think logically and coherently, integrating several ideas into a more coherent whole (Brodie, 2010). The reasoning is a process that provides depth to the existing body of knowledge (Duval, 1998).

The reasoning about symmetry is not simply about recognizing symmetrical shapes but refers to the in-depth knowledge and regulation of the cognitive process of symmetry (Seah & Horne, 2019b). The cognitive processes of symmetry are fundamentally based on geometric transformations, as symmetry is determined by invariance under geometric transformation. As specified by Sarama & Clements (2009, p. 225), working with shape recognition and transformations at early stages is important to children's mathematical development. Polyakov (2019) points out that geometric transformations are essential for bridging cognitive and spatial skills, integrating information from sensory modalities, and compacting the representation of acquired skills. The primary task in symmetry learning is to identify invariant properties and processes that generalize and abstract symmetries across different contexts.

Learning occurs when there is a change in cognitive schemes (Piaget, 1964). Piaget's theory of genetic epistemology (1970) considers that learning occurs through assimilation and accommodation of new notions on the existing mental structure. The cognitive structures are the basic building blocks of intelligent behavior, called schemas. The collection of schemas organizes knowledge around a concept, forming its *conceptual image*. The process involves intuition and reflective abstraction (Dubinsky, 1991) activated by the set of activities of increased challenge. Bartlett (1932) first introduced the shift in cognitive schemas, built on Head's (1920) schema for human memory.

Drawing on this conceptual background, Skemp (1986, p. 43) employed cognitive schema in mathematical education, stating that "to understand something [mathematical ideas] is to assimilate it into an appropriate schema". Learning should be considered as a dynamic process of modifying and restructuring representative schemes inherent in a given phase and potential development from concrete action to an internalised and abstract idea (concept). In addition, learning requires students to explain to themselves and others what they see, what they discover, and what they think and conclude (Hershkowitz, 1998, p. 30). Problems and investigations using manipulative tasks and artefacts can help learners improve their mathematical thinking and understanding.

Several studies (Boggan et al., 2010; Cope, 2015; Laski et al., 2015; Ojose & Sexton, 2009; White, 2012; Holmes, 2013) show that manipulatives enable students to understand abstract concepts through concrete experiences and provide long-term consistency in mathematical skills. Relying on the Theory of Conceptual Fields (Vergnaud, 1998, p. 84), the learning activities are the primitive and prototypical references for the scheme concept. Mathematical understanding depends on the degree of connectedness among learners' schemas. Students develop connected schemas progressively as they find similarities and differences between mental networks and operational activities, which leads to abstract models. Such abstract models are the *big ideas* of mathematics as critical and central organizing ideas (Schifter & Fosnot, 1993), without which the development of the structure of children's reasoning would be impossible or complicated.

Generally, the conceptualization process is favored by both linguistic and conceptual evolution. Moerk (1974) demonstrated close parallels between cognitive development theory and independent language development. According to Sfard (2008), learning occurs when a definite change in vocabulary, visual mediators, routines, and narratives operates cognitively. The above-mentioned information constitutes the foundation for investigating the pupils' narratives about symmetry in terms of keywords and signs showing interaction with the teaching-learning path's visual mediators as artefacts, actions, and dynamics in the classroom social setting.

2.3 THEORETICAL FRAMEWORK

As discussed in the previous sections, big ideas have the potential to bring order and simplicity to the full disciplinary body of knowledge, connecting concepts and structures towards a coherent whole with a high transfer value across disciplines and contexts. In this perspective, symmetry can be regarded as a big idea, considering its marriage with invariance.

Symmetry is indeed one of the most fundamental and fruitful concepts in human thought, with which the mind has tried to comprehend and create order, beauty, and perfection (Schroeder, 1998, p. 3; Weyl, 1952, p. 5). We consider symmetry a principle of invariance for organizing and categorizing many phenomena, data, and information into coherent knowledge. Symmetry carries a structural common denominator among a broad range of entities, conferring a bridging function in art as an aesthetic value and an interdisciplinary role in science (Darvas et al., 1995).

Symmetry, therefore, plays the leitmotiv of order canon spanning from nature, culture, arts, and sciences. Symmetry is a unifying concept due to its ability to connect a variety of domains (Dreyfus & Eisenberg, 1990, p. 53). Symmetries also simplify, bringing about the economy of description. How symmetry really works and controls our world is fascinating. The more complex the structure, the more useful the symmetry is (Iachello, 2011, p. 13). It seems that natural laws have chosen to be expressed in the language of symmetries. As expressed by the words of Steven Weinberg (1986) in his Dirac Memorial lecture: “At the deepest level, all we find are symmetries and responses to symmetries”. The cohesive nature of symmetry across mathematics and science is apparent throughout historical and scientific development. The relationship between symmetry and invariance in mathematics and science has been symbiotic, each contributing to the other’s development (P. Klein, 1990; Weyl, 1952, p. 135).

“Symmetry is one of the thematic melodies of 20th-century Theoretical Physics”. In these very colourful terms, the Nobel Prize-winning physicist C. N. Yang (2003) emphasizes the invasion of the ideas of symmetry into modern physics, which played an essential guiding role from primordial concepts in the cognitive history of humankind, and it continues to play this role with the same power in all branches of science. Nevertheless, symmetry can play a far deeper role in the educational context. P. Klein (1990, p. 85) relates symmetry in school as not just another topic but a vital and multifunctional constituent [*big idea*] of cognitive formation and mental development. Based on historical, epistemological analysis of symmetry and its mathematical language, we pose its unitary structure at the heart of learning progression in the early stage of education. Below we report the theoretical ground that underpins our use of the unifying concept of modern symmetry in education.

2.3.1 THE ROOTS OF THE SYMMETRY LANGUAGE

Symmetry has different meanings, some more intuitive than others. The same term can take on different definitions over time (Hon & Goldstein, 2008, p. 203). If we go back to the origins of this term, the understanding of symmetry had completely different connotations from that of present-day (Hubert, 2020, p. 210). From ancient times, symmetry has been conceptually close to proportionality, the right balance, beauty, equality, and harmony. As a matter of fact, in Greek, the term "*σύμμετρος"* (symmetry) is a word composed of *"σύμ"* (with or together) and *"μετρον"* (measure). Together they describe something with equal measure, well-ordered, and well-proportioned; in Euclid's Elements (Book X), it is equivalent to commensurable. In Latin translations, symmetry is adapted literally in "*commensuration*" (in measure with or sharing a common measure) (Weyl, 1952, p. 75). These senses of harmony made symmetry important in arts, referring to the aesthetic proportion of part-whole relationships, *"commodulatio"*, particularly in the human body (Pelkey, 2022, p. 4). Hon & Goldstein (2016, p. 48) specify the usage of symmetry in two trajectories corresponding to its two different interpretations: "a sense of relation which constitutes the mathematical path, and the sense of property which marks the aesthetic path". Since symmetry's inception, however, it has presented a fascinating intertwining of aesthetic and rational elements. This fusion is visible in the works of the sculptors Polykleitos first (5th century BC) and Vitruvius after (1st century BC), who link the canon of art to science since symmetry is established on two purely mathematical concepts: number and measure.

According to Gielis (2017, p. 10), the deliberate act of measuring by comparison means to symmetrize, thus forming the real basis for mathematics and geometry. In this regard, Vitruvius suggests a "mathematization" and a sort of "geometrization" of the figure, assumed many centuries later by Leonardo da Vinci

(15th century BC). His most famous drawing, the Vitruvian Man (Figure 2.2), symbolizes Renaissance culture bridging art, math, and science.

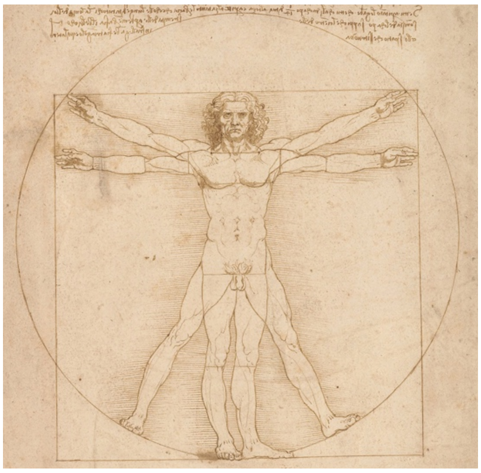


Figure 2.2. "The Vitruvian Man" (c. 1487) has his natural center in the navel, and his arms and legs are inscribed in a circle and a square centered on the navel. (Wikipedia Commons).

Da Vinci presents the same man in two different overlapping positions, inside a circle and a square, to show the different ratios. Vitruvius coined the term "*ratio symmetriarum*" as a symmetry principle of design (Rowland et al., 1999, p. 78). It requires a module, a fundamental unit, on the basis of which fixed proportions can be established (Hon & Goldstein, 2016, p. 56). In short, the aesthetic sense of symmetry is described by the language of mathematics. Size, scale, numbers, ratio, and proportion are central to such a concept of symmetry.

In modern times, symmetry has become a mathematical property, more abstract, with no clear references to aesthetic components, grounded not on proportion but on equality relation instead (Brading & Castellani, 2003, p. 2). This new mathematical sense of symmetry first refers to equality between opposed parts as a bilateral (left-right) mirror image. It was first C. Perrault (1673, p. 10) to move away from the ancient symmetry, explicitly introducing the notion of reflection symmetry. The breakthrough came with Legendre (1794), who presented his revolutionary concept of symmetry as a relation of equality between two distinct objects or entities in a scientific context in solid geometry (Hon & Goldstein, 2008, p. 51). Legendre noted that similar and equal non-superposable polyhedra might be symmetrical to one another. Such relation was called "*equality by symmetry*". From that time, the notion of symmetries assumed an increasingly general and modern characteristic because it referred to the use of mathematical operations such as symmetry transformations (i.e., reflections, rotations, and translations) and the idea of invariance. The mathematical formalization of modern symmetry occurred only at the beginning of the 19th century when the mathematical language apt to describe it was at its disposal: the group theory.

2.3.2 TOWARDS GROUP THEORIES

Unlike what one might think, the roots of the abstract concept of the group do not lie in the study of geometric symmetries but of algebraic symmetries in equation theory, a very abstract domain apparently unrelated to geometry. The concept of the group appeared for the first time, albeit implicitly, in work on the theory of algebraic equations carried out by J. L. Lagrange (1770-71).

After 60 years, the notion of a group was explicitly defined by Galois shortly before his early death in a duel[3]. Indeed, just on the eve of that duel he wrote down the first technical definition of a group, called the *Galois group*, in some last-minute additions to his manuscript (P.M. Neumann, 2011, p. 115). His great achievement was far beyond solving algebraic problems of determining the solvability of polynomial equations by radicals, which was a huge success. Galois' key insight was to analyse the characteristics of operation rules (i.e., the structure of the equation roots, rather than calculate their solutions). Studying their symmetries determines whether radicals can express the roots of the polynomial. The emphasis is on the operations that reveal the symmetry of an object rather than on the object. In Galois' theory, the object is the set of roots of the polynomial, and the operation is the permutation of the roots. The collection of symmetries is a group. When group theory appeared, its relationship with geometry was immediately evident, offering new perspectives.

The development of the group concept in geometry was due to A. Cayley (1854). He first defined *abstract finite groups* as a set of symbols regardless of their nature, be it permutations, transformations, numbers, matrixes, or any other type of elements verifying a series of axioms. F. Klein and S. Lie highlighted a closer connection between *group theory* and *geometry*. F. Klein (1872) published a manuscript *Vergleichende Betrachtungen über neure geometrische Forschungen* (A Comparative Review of Recent Research in Geometry), known as the very famous

[3] Galois died on May 31, 1832, at the age of twenty, from a gunshot wound received in a duel the day before.

Erlanger Programme (Erlanger Program). The *Erlanger Programme* was the climax and synthesis of many previous mathematical studies, influenced mostly by S. Lie. The project's manifesto addresses geometry as a study of invariants under a certain transitive group action. According to Yaglom (1988, p. 115), "Geometry is the science which studies the properties of figure preserved under the transformation of a certain group of transformations, or, as one also says, the science which studies the invariants of a group of transformations".

Up to F. Klein's work, mathematicians understood geometry in terms of objects such as points, lines, triangles and circles, the relationships between them as distances, angles, and so on. F. Klein's point of view was different: it suggested that the geometry itself was not characterised by objects but by the group of transformations that allowed certain properties to remain unchanged. Geometry became expressed and unified as a study of invariants under a group of transformations. In other words, the geometry type is determined by its symmetry group. For example, *Euclidean geometry* was defined as studying the properties of "figures" invariant under isometries (i.e., translations, rotations, reflections, and glide-reflections that leave the lengths, the amplitudes of the angles and the areas unchanged). Spherical and hyperbolic geometry can be defined similarly. F. Klein focused on discrete groups of symmetries with a finite number of elements (e.g., rotations of an n-sided regular polygon contain n rotations $\{r_\theta,\ \theta = 2\pi k/n\,, k = 0, \dots, n-1\}$), while Lie studied infinite continuous groups, depending on continuous parameters (e.g., rotations of a circle where the rotation angle can be changed continuously).

From that moment, group theory began appearing in practically all branches of mathematics. Symmetry also played an important role in crystallography and a wide range of physical phenomena. Therefore, symmetry and invariance to a group of transformations became almost synonymous. In a broader sense, symmetry has acquired the meaning of *immunity to possible changes* (Livio, 2006; Rosen, 1995, 2008) or *sameness within change* (van der Veen, 2013; Leikin et al., 1998), even when the elements to which it is applied have no concrete representation. Indeed, the study of symmetries of the physical objects shifted to that of symmetries of physical laws. The great mathematician H. Weyl (1952, p. 77) characterised this fact in his elegant, marvellous, and multidisciplinary booklet *Symmetry*, and defined a swan song himself in this way: “We still share his [Kepler’s] faith in the mathematical harmony of the universe. It has withstood the test of ever widening experience. But we no longer seek this harmony in static forms like regular solids, but in dynamic laws”.

Thus, opened what we can call the modern era of the application of symmetry to physics. Lie’s contributions were to extend the *Erlanger Programme* to the study of physical systems and to apply group theory to differential equations, as Galois had applied it to algebraic equations. Since physical laws are expressed precisely through differential equations, which describe the spatial and temporal variations of the systems, the step from Lie’s theory to physics was short. F. Engel (1916), a German mathematician who worked with S. Lie on *Theorie der Transformationsgruppen* (publ. 1888–1893; tr., “Theory of transformation groups”), showed in 1916 mutual connections between the invariance group of classical mechanics and conservation laws of momentum, angular momentum, and velocity of gravity centre. Two years later, E. Noether generalised this result to all finite and infinite continuous groups. Indeed, she established in her paper *Invariante Variationsprobleme* (tr., “Invariant Variant Problem”; Noether, 1971) that in correspondence to each continuous symmetry of physical laws, there is a corresponding conservation law and a corresponding preserved quantity, i.e., a measurable physical quantity unchanged, whatever the process considered (e.g., the total energy of a physical system). Even the opposite is, almost always, valid. Whenever we find some quantity preserved in physical processes in nature, we know there must be a corresponding continuous symmetry. More precisely, Noether’s first theorem states that if the dynamic laws of a system are invariant under transformations of an n-parameter Lie group, whose elements are organised continuously and smoothly (Thyssen & Ceulemans, 2017, p. 58), then the system has n motion constants. The constants got named “*Noether charges*” in modern physics during the 1980s (Takeda, 1985, p. 196). Whenever a system has some symmetry, the corresponding continuity equation depicts the flux of the conserved quantity through the system. Energy conservation comes from translation symmetry in time, and translation symmetry in space accounts for momentum conservation. Likewise, from rotation symmetry emerges the conservation of angular momentum.

Property of the system	Symmetry	Noether charges conserved
Homogeneity of time	Translation symmetry in time	Energy
Homogeneity of space	Translation symmetry in space	Momentum
Isotropy of space	Rotation symmetry	Angular Momentum

Table 2.1. The correspondence between symmetries and conservation laws

Even though Noether's theorem was poorly understood during the first half of the last century, it has been later called "certainly one of the most important mathematical theorems ever proved in guiding the development of modern physics, possibly on a par with the Pythagorean theorem" (Hill & Lederman, 2004, p. 73). Why has her work been considered of great impact as part of the cornerstone on which modern science is built? How come that the concept of symmetry invaded physics?

2.3.3 MATHEMATICS AND PHYSICS INTERPLAY VIA SYMMETRY

Although Mathematics and Physics independently influenced the development of the study of symmetry (Darvas, 1997, p. 322.), only the Noetherian *pure formulas*[4] put the gain in mathematicians' insight of symmetry at physicists' disposal in understanding nature. The most important enlightened and penetrating idea was to promote the study of symmetries from the aesthetic-mathematical scope to the physical one. Indeed, the extremely wide-ranging aspect of the theorem led mathematics to have a completely new heuristic scope: it became possible to derive the existence of well-determined physical entities *a priori* in a completely amazing way. From a description of what is present in physical laws, symmetry became the architect of reality with applications in numerous fields of physics, from electrodynamics to relativity, from quantum mechanics to the physics of elementary particles.

The presence of a certain group of symmetry and invariants linked to every conservation principle changed not only the methodology of physicists but also their point of view: instead of a set of deterministic laws that predict what can happen, there were laws of conservation to determine what cannot happen.

[4] Einstein placed the theorem of Noether in the special category *"spiritual formulas"*. The quotation is from Einstein's tribute to her, published in the New York Times. (Einstein, A., Professor Einstein Writes in Appreciation of a Fellow-Mathematician, The New York Times, 5 May 1935). "One seeks the most general ideas of operation which will bring together in simple, logical and unified form the largest possible circle of formal relationships. In this effort toward logical beauty *spiritual formulas* are discovered necessary for the deeper penetration into the laws of nature."

In M. Henneaux's (2020, p. 11) words, "the importance of symmetry, on account of the variance it expresses, imposes constraints, which can be very strong, up to determining the form of physics laws uniquely". Symmetry transformation and the search for invariants became an abstract principle from particularities to universality; as Dutch physicist V. Icke (1995, p. 103) stated: "Symmetry prohibits. Forbidding imposes order, but many things that possess a certain order may derive from the same symmetry (e.g., all planets possess approximate rotational symmetry but have very different masses and compositions). That is why physicists believe that the underlying symmetry, which forbids whole classes of occurrences at one stroke, is, in a sense, more fundamental than the individual occurrences themselves and is worth discovering".

While physical laws express relationships that physical events must satisfy, the principles of symmetry represent conditions imposed on the laws themselves, conditions that often allow drawing observable consequences, even if the explicit form of laws is unknown to us. There are certainly many uncanny connections left to discover via the power of invariance symmetry unified by its mathematical language, namely group theory, as we see in the above paragraph.

Symmetry imposes its rule to catch the universe's structure, as in the case of the great discoveries of *gauge symmetries*[5], which characterize the 20th century. All this has only increased the mystery about the nature of the creative power of mathematical language, apparently a simple concatenation of symbols, yet so close to the nature of things. The most striking and disturbing feature is how nature displays the reflections of mathematical developments and abstract concepts dealing with symmetry. Even greater is the secret of how symmetry organizes our thinking when exploring the unknown universe. Once again, it proves what the extraordinary Nobel Prize-winning physicist E. P. Wigner (1960) named the "*unreasonable effectiveness*" of mathematics in science. Mathematicians initially considered group theory behind symmetry too abstract to find applications in physics.

[5] Hermann Weyl conceived in 1918 an ambitious project of building a unified theory of electromagnetism and gravity, based on a new symmetry under two simultaneous transformations: a scaling of space-time and a redefinition of electromagnetic potential. He called this symmetry Eichinvarianz (invariance "of measure", or "of calibre") translated into English with gauge symmetry. And from then on, the name is in use although the transformations in current gauge theories are no longer a change of "measure". For further information on gauge theory, see Rosen (2008), Redhead, (2003), Bangu (2013).

Remarkable studies (Darvas, 2015; Bangu, 2016; Zee, 1990) have been devoted to the concerns raised by Wigner more than a half-century ago. They argue, among other things, that symmetry is becoming more pervasive throughout many disciplines, not only scientific (e.g., non-physical science, including genetic biology and humanities), and that its powerful effectiveness is growing in sciences based on the language of geometric symmetry. Going back to C. N. Yang's lecture at UNESCO, "in the primordial concept of the beauty of geometrical forms, there have been profound revolutions in our understanding, revolutions that had brought forth a more beautiful, more subtle, more precise and more unified description of nature".

2.3.4 SYMMETRY PHENOMENA

Symmetry is unavoidably everywhere. Whatever you do, wherever you go, symmetry is there. Whether knowingly or not, symmetry permeates our whole life and relationship with the natural world. As Avital (1996, p. 30) notes, "from the moment one becomes aware of symmetry, it is impossible not to see it wherever we look, for symmetry is not only one of the building blocks of nature but also the foundation of civilization".

From the first years of life, we are exposed to symmetrical figures and relationships (Dreyfus, & Eisenberg, 1998, p. 189). We experience the symmetry of plants, animals, objects, music, drawings, and architecture. The human being is particularly skilled in recognizing symmetrical visual models, and s/he is strongly attached to them. The origin of this ability very likely lies in the most evident symmetry: bilateral symmetry in humans and many other organisms. However, symmetry runs beyond the embodiment of its bilateral character. Indeed, reflection (or mirror) symmetry is one simple type of a huge variety of possible geometrical symmetries. Picking only from living nature, the possible types of symmetry are multiple: reflection (or mirror) symmetries in butterflies, rotational (or radial) symmetries in flowers, translational symmetries in honeycomb patterns, helical symmetries in seashells, spiral (or scale) symmetries in a nautilus shell, glide-reflection symmetries in footprints, cubic symmetries in a salt crystal, spherical symmetries in molecular structures of certain viruses. The above examples of symmetry represent numerous phenomena, yet the list is far from complete.

As a result, the plethora of symmetry forms constitutes a hierarchy of symmetry definitions that students must learn in their studies. In addition, teachers often distinguish symmetry across various disciplines at school. Even within the very mathematics and geometry, the teaching approaches to symmetry are often different, neglecting to point out the underlying common feature of all symmetries (Leikin et al., 1998). Thus, students see symmetry as a collection of disconnected concepts. Evidence shows that students' knowledge is fragmented and confused with other mathematics ideas. Even though learners can visualize some basic symmetries, their reasoning skills and critical thinking are poor.

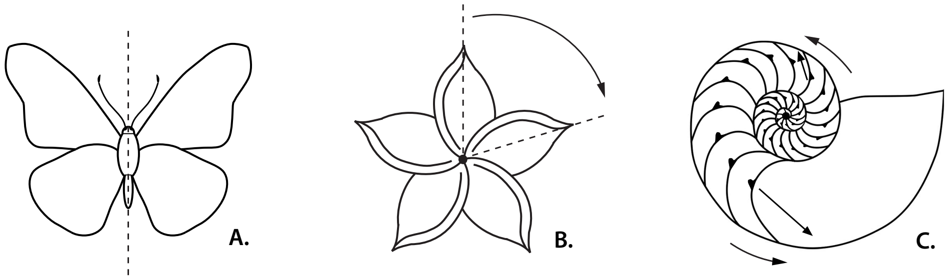


Figure 2.3. Diagrams showing three types of symmetry in the living organism: A. Bilateral symmetry in a butterfly (*Morpho didius*). B. Rotational symmetry in a flower (*Plumeria rubra*). C. Spiral (a combination rotation, translation, and dilations) symmetry in a nautilus shell (*Nautilus pompilius*).

The issue is exposed by Seah & Horne (2019a, p. 634), Ng & Sinclair (2015), and Knuchel (2004) for elementary school. The continuity and ruptures identified in applying the symmetry concept are analysed by Chesnais & Munier (2013) for the French primary to secondary school transition. Leikin et al. (2000a) illustrate the difficulties and misconceptions for seven-eighth graders students associated with symmetry transformations. In summary, we refer to symmetry with several ideas and terms depending on different situations and domains in which it is used. A possible rationale is that symmetry is not precisely defined (Lowrey, 1989, p. 485). Fortunately, the concept underlying most symmetry definitions is the same (Petitjean, 2008).

2.3.5 SYMMETRY IN MODERN SENSE

The valuable connection of all this body of knowledge that underpins the study of symmetries in the modern sense is obtained by focusing on the invariance of a general class of transformations. Under this perspective, all symmetry instances mentioned above show one or more geometrical properties unchanged after performing some geometrical operation (i.e., symmetry transformation). This characteristic proved to be invariant under any given transformation. In Schuster's (1971, p. 83) words, "once the transformation is available, the symmetry concept is apparent". According to Darvas (2007, p. 20), the generalization of symmetry toward its modern concept involves three conditions:

(i) under any kind of *transformation* (operation),

(ii) at least one *property*

(iii) of the affected *object* remains invariant (identical, indistinguishable).

There are several crucial points embedded in the modern symmetry definition.

First, the term "symmetry" has no connotation apart from symmetry operations (Gould, 2004). The emphasis is indeed placed on the *active operation*, and not on a thing perceived passively. Symmetry is a process, a method, and an exploration (Thyssen & Ceulemans, 2017, p. 11). It is necessary to transform an object and look for invariants that preserve its structure (Stewart, 2007, p. 120).

Stewart (2007, p. 12) refers to symmetries as special transformations, a way to move an object that makes it stay the same; such transformations are called symmetry operations for the object. From this operational perspective, symmetry and the search for invariance are appropriate in the educational field since learning has to be active; students construct their knowledge of symmetry through dynamic explorations and operations (Leikin, 2007). P. Klein (1990, p. 86) agrees that learning with symmetries is based on *action* and gives a good basis for understanding complicated problems. The degree of operating transformational geometry with the search for geometric regularities is related to the children's intellectual development (Swoboda, 2016); they are also good indicators for pupils' achievements at school in mathematics and geometry (Marchini & Vighi, 2007). Thus, the concept of geometrical transformation is indispensable to learning geometry, including symmetries. Students have a vast opportunity to experiment with the various symmetry operations.

In geometry, transformations are broad depending on different invariant properties. Consequently, a specific class of symmetry has its proper invariants. Figure 2.4 summarises some symmetry operations, illustrating images of the white letter R under various transformations.

Rigid motions such as sliding, turns, and flips of the white letter R preserve areas, distances, and angles, thus conserving parallelism. Such transformations are called *isometries*; they fundamentally comprise *translations* (moving the object from one place to another in some direction by a specific distance), *reflections* (mapping each point to its mirror image on some fixed line), *rotations* (rotating the object through some angle direction about a fixed point) and some combinations of them as glide reflection.

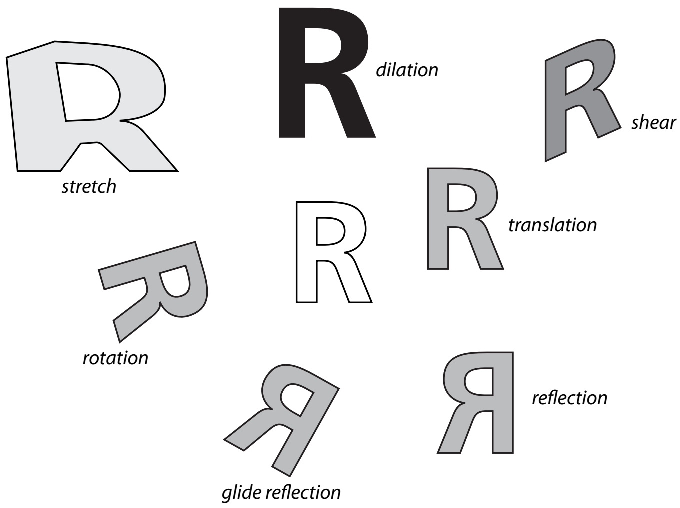


Figure 2.4. Symmetry transformations of the white letter R.

The symmetry language provides a modern way of studying congruence for such a class of transformations (Ellenberg, 2021, p. 129). We say that each of the four letters R in grey is congruent with the white one, if we apply a rigid motion to one, it coincides with the other. Properties that were invariants before for the above letters, like area, and distances, are no longer invariant for the black R. The notion of sameness is not valid under that transformation of dilatation. Other things, however, remain invariant.

Non-rigid motions such as *dilations* and *shear* of the white letter R preserve parallelism, but neither distances nor areas. Particularly, *dilation*, on the one hand, is a similar *transformation* (or *similitude*) that maintains the ratios of corresponding side lengths, angles, and parallel and perpendicular lines. On the other hand, the *shearing transformation* is an *affine projection* where straight lines are transformed into straight lines, and area and parallelism are conserved but not distance or angles, and thus perpendicularity. Another transformation of the letter R is the *stretch* or *squeeze*. The shape is deformed, and its measures change but preserve only the continuity of points, namely the topological relations between points; if this last characteristic is invariant, the *topological symmetry* manifests. Poincaré founded this extravagantly forgiving kind of geometry, where a coffee cup and a donut are considered the same thing, to count the holes in a straw (Ellenberg, 2021, p. 142). Although topology has been regarded as abstract mathematics without applications, it is fundamental to understanding many real-world phenomena. According to Hasan & Kane (2010), symmetry and topology are two conceptual pillars that underlie the understanding of physics (e.g., nano systems, electronic phases of matter).

Second, within such a definition of symmetry, the *object* is *arbitrary* and not *necessarily geometrical*, as well as its *property* and *transformations*. This arbitrary character of applicability confers an *interdisciplinary* character to the modern notion of symmetry as it deals with formal conditions for the understanding of functionality regarding all objects of experience (P. Klein, 1990, p. 86). As a result, transformation and invariance are considered rudimentary notions that pervade all mathematics and science.

The application of symmetry is appropriate to both animate and inanimate material objects, as well as to products of our minds (Katona, 2021, p. 464). The possibility of widening the fields of symmetry application, of choosing the "rules" (in the sense of regularity) to be considered from time to time offers the starting point for reflections on the relationships of equality and, more generally, equivalence; it also encourages reflections on the relativity of the concept of equality and further on the form as a physical system that is preserved (Rosen & Copié, 1982; Weyl, 1928).

The processes of change or modelling of a rule are subject to a principle of legality[6] that guarantees compatibility and consistency with those already in existence: in physics, this set of binding conditions is represented by symmetry principles. The symmetry perceived with the meaning of the invariance of a changing form becomes a powerful tool of *interdisciplinary knowledge*, including applied sciences both in history (Pisano, 2011), in society (Marchis, 2009), and in education (van der Veen, 2012; Hill & Lederman, 2000). Symmetry as a mathematical tool is

[6] The principle of legality or regularity registered at the level of nature, physical constants, or specific natural properties points toward the existence of "forms" in nature. The transformations in the material world refer not only to an "efficient causality" but also, necessarily, to a "formal causality". It is explicitly demonstrated that some physics laws reflect such order and can be derived directly from it (Knuth, 2016).

also advanced in a general scientific method (Darvas, 1997, p. 328, 2015). Thus, it can establish principles generally applicable throughout the sciences.

The essential idea is to extend symmetry starting from line symmetry and aesthetic qualities with the search for regularity to more general and interdisciplinary aspects of dynamic principles of transformation (Leikin et al., 2000b). The figures with characteristics common to our eye from different points of view somehow refer to invariance for which various formulas have been coined (see Section 2.3.2). The search for invariants is inherent to the description of reality, introducing dynamic principles of transformation that give meaning to the modern concept of symmetry. In other words, a system is said to possess symmetry if one can make a change (a transformation) in the system so that, after the change, the thing appears the same (is invariant) as before (Hill & Lederman, 2000). As an empirical definition, we need to make assumptions about the systems' properties and entities' shape until the action has been accomplished; thus, symmetry is a guideline that helps to get at those invariances and differences without decreeing what they are in advance (Olsen & Witmore, 2015).

Third, another important task is to shift the attention from the relationships between geometric shapes to the transformations of geometric objects (van der Veen, 2013, p. 483; Hill & Lederman, 2000, p. 3). The search for symmetry in the object results in the search for transformations that send it into itself and, more precisely, in the determination of the symmetry group of the object. Besides describing a square with four identical sides and angles, we can say that it does not change under four rotations around the centre of 90 degrees. In Section 2.3.7., we will detail the symmetry group of a square, also known as a dihedral group of order 8 and denoted D_4, where eight is the number of elements in the group (i.e., identity, three rotations, two mirror images and two diagonal flips).

The identification and description of the invariants, starting from the basic school, allow for simplifying the problems and grasping their essential aspect. The search for conserved quantities is a method that characterizes scientific research and allows the formulation of unifying theories. The features linked to transformations can be generalised in a manner more obvious and easier to understand. The former are symmetry, correspondence, and relation properties that allow the acquisition of a transferable investigation method. The search for the invariant quantities leads, to a greater economy of thought (Jourdain, P. 1914), from concrete objects to the abstract concept of *formless invariants*. What is preserved in certain systems is, in fact, the *form* as a broad concept of shape: phrases, laws, covariant mathematical expressions (that remain unchanged under certain transformations), and properties of artistic objects or entities like colour, tone, shading, and weight. In short, anything can be a *form*.

The following is an overview of special and unusual invariant forms with a potential for education that may let scholars reflect, explore, and invent symmetry operations. The phrases "Never odd or even" and "So many dynamos" or numbers like "909" and even many films[7] are symmetric for the back and forth reading (Tsvetkova, 2020). Terms with this symmetry are called palindromes from the Greek *palin* (back) and *dromos* (direction). It is a type of sameness about mirror symmetry. Among many examples of symmetry in art and mathematics, as well as in poetry and music, Gould (2004) uses *palindromic symmetry* (i.e., palindrome) in its course *Seeing Through Symmetry - An Interdisciplinary Multimedia Course*.

Letters such as S, Z, and the number "609", or special calligraphic designs, namely ambigrams coined by D. R. Hofstadter (Polster, 2000), are unchanged when rotated 180 degrees. It is a type of sameness displaying rotational symmetry. Other ambigrams exploit more symmetry operations depending on how the characters are written. Mishra & Bhatnagar (2014) emphasize the role of ambigrams as a beautiful idea for introducing symmetry and invariance in mathematics. Its interdisciplinary use at school is meaningful as ambigrams combine "the mathematics of symmetry, the elegance of typography and the psychology of visual perception to create surprising, artistic designs" (Mishra & Bhatnagar, 2013, p. 28).

Consider a white square (a), which we colour as the square (b) in black and white in Figure 2.5. Analysing the symmetry, the colour action changes the original white square's symmetries. Furthermore, if we exchange the colour that characterizes each point, the white part becomes black and vice versa (c).

[7] *Symmetry. A palindromic film* (2013), written, directed, and edited by Yann Pineill, is a mirrored narrative that progresses whether watched from the beginning, from the middle or reversed from the end; *Tenet* (2020) by the director Nolan represents artfully the concept of reflectional symmetry from front to back or back to front.

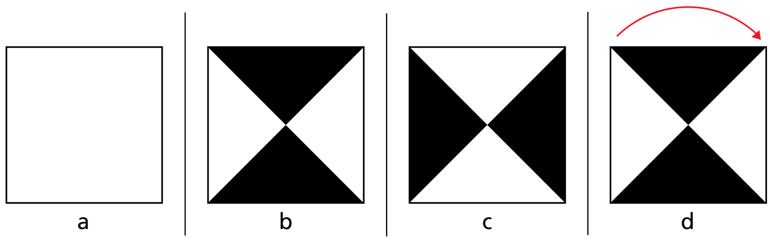


Figure 2.5. Combination of colour and rotation transformation.

This colour transformation does not transform the square into itself. As stated by Rosen (1975), “no system can be invariant in colour transformation, but there can be symmetry under the combination of colour transformation with some other kind of transformation, where neither is a symmetry transformation by itself” (p. 74). Hence, if we subsequently perform a 90 degrees rotation around its centre, the figure coincides with the original one (d). In this case, the notion of the square’s *sameness within change* does not apply to 90 degrees rotations alone nor to the exchange of white with black, but to the combination of the two transformations, regardless of the order. As far as invariance is concerned, more general transformations, and/or complex symmetries, composed of bulk of simple symmetries, not just geometric ones, must be evaluated. (Darvas, 2021, p. 6).

“This is a radical departure from the traditional approach which attempts to incarcerate the perceived shape of objects in the ideal, formal description of classical geometry” (Shaw et al., 1974, p. 295). Hofstadter (1990, p. 108) considers the concept of symmetry extremely subtle as it contains in it “the (seemingly exact) concept of *sameness*, but that concept implicitly means *sameness in certain respects*, which is tantamount to the concept of *similarity*, which after all is not at all exact at all, but highly dependent on human perception and categorization”. It is important to note that many invariant properties of figures are not seen except when they see themselves transformed, as we noted above. For instance, from a generic parallelogram to a rectangle, there is the preservation of parallelism and congruence of opposite sides, of congruence or constant sum in the corners.

Therefore, the analysis of the properties of the form through the study of transformations and invariants facilitates learners’ acquisition of cognitive procedures and processes that can be extended to many areas, including modelling and generalization processes (i.e., synthesis and abstraction). As Dreyfus (2002, p. 36) put it, “the process of abstraction is intimately linked to generalization. Another incentive is the achievement of synthesis." Symmetries and invariance show students that it is possible to describe with the language of group theories a lot of situations, which otherwise would be separate and independent. Abstractions, thereby, make numerous connections among disciplines and topics within disciplines.

2.3.6 SIMPLIFICATION THROUGH ABSTRACTION

Simplification through abstraction is a powerful and paradoxical idea in mathematics (Riehl, 2021). The powerful aspect of simplification through abstraction "is that many different situations become the same when you forget some details" (Cheng, 2018, p. 35). The art of Picasso (Figure 2.6) can illustrate how simplification through abstraction works, allowing educational innovations. Picasso stated that "a picture used to be a sum of additions. In [his] case, a picture is a sum of deconstructions." Picasso's Bull is a series of eleven lithographs of bull images showing how Picasso gradually depicted the animal in various stages of abstraction through basic forms until he reached a minimalistic, linear outline of its shape. Step by step, Picasso dissected the animal along its muscles and skeleton lines and simplified its anatomy, erasing the parts and reducing the form. In the end, he reached the level of a simple outline, which perfectly captures the essence of the animal. In this case, the essence is equal to the *formless invariant*.

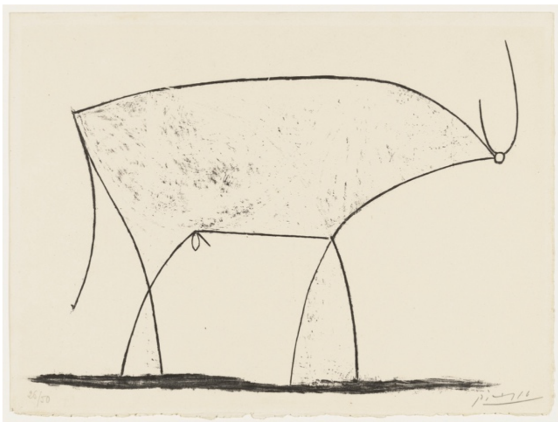


Figure 2.6. The essence of the bull, Le Taureau, P. Picasso, 1946 © 2021MoMa New York/Scala.

Abstracting begins with reality and uses some tool to pare away the excess to reveal critical essence. The essence is achieved by ignoring as many details as possible and making crystalline the concepts simplified out of things that we notice in the world around us. In agreement with this idea, we can mathematically describe an object in terms of its properties and explore it abstractly, regardless of the scaffolding of examples or constructions. Furthermore, we can unify various things under the same *formless invariant* by studying their common relationships, in Poincare's mathematics vision as "the art of giving the same name to different things" (Verhulst, 2012, p. 157). Once we find the relationships between different entities, we understand them from different points of view; in addition, we can study many structures simultaneously in the same unifying act, resulting in much more efficiency. Stroup (2005, p. 193) proposes mathematics as the systematic study of forms of 'sameness' or invariance: "Mathematical invariance is what allows us to see a commonality in many situations that, on the surface, may appear to be very different". According to Riehl (2021, p. 37), "the cascading levels of abstraction, from concrete mathematical structures to axiomatic systems, present a new challenge: it is no longer very clear what it means to say that one thing is the same as another thing".

Symmetry is a possible terrain and even an intuitive "language", among others, to help students to take the right road toward simplification through abstraction. Shaw et al. (1974, p. 298) consider symmetry a powerful tool of knowledge detecting formless invariants over time. Properties that remain invariant under transformations, such as the phenomena of depth and shape, are detectable by humans depending on their age and definable by appropriate symmetry operations. From an early age, children have experiences with shapes, sizes, and movements in space. They structure space based on their bodies, emotional and affective experiences, and motor actions. Children can visually distinguish shapes at preschool age but cannot represent them per Euclidean or projective relationships (Figure 2.9). They draw what they know, and the invariants are the topological ones (Piaget & Inhelder, 1967; Hiigli, 2013).

Figure 2.7. Topological Shapes: a large shape with a small circle
in various arrangements, (Piaget & Inhelder, 1948).

Although Piaget's topology-primate hypothesis is controversial (Clements et al., 1999, p. 193; Chien et al., 2012), children are more likely to use the topological concepts of inclusion, separation, and closure to recognize only interior, exterior, and boundary points, but not angles or lines, and to recognize rectilinear shapes as indistinguishable from curved ones (Martin, 1976, p. 27). Only at age 7-8 do they begin to learn the Euclidean concepts of distance and its measurement, perpendicularity and parallelism, and vertical and horizontal coordinates. However, they do not yet have correct geometric vision (e.g., they have difficulty recognizing a triangle resting on one of its vertices and confuse the square resting on one of its vertices with the rhombus).

Regarding symmetry perception, general and global symmetry is probably found at the primitive level (Hu & Zhang, 2019). There follows the developmental sequence of symmetry from a process of differentiation. The concept of symmetry is abstracted, first perhaps subconsciously, later in more explicit forms (Yang, 1996). General symmetry refers to a kind of balance in arrangement structures to express a sense of unity and order through an almost exact repetition of one or more elements. Nevertheless, symmetry is not just something that we perceive statically by looking at what surrounds us, and that we try to reproduce with a mixture of pleasure and astonishment behind the feeling of harmony.

Furthermore, according to Weyl's ideas[8], invariance is primarily a *measure* among geometric transformations and algebraic laws. It captures an incredible relational value of comparative invariant measurements acquired in different systems. The measure is the degree of symmetries, i.e., the number of different invariant transformations. We can intuitively give some examples by limiting ourselves to the field of geometry. Let us consider the shapes in Figure 2.8.

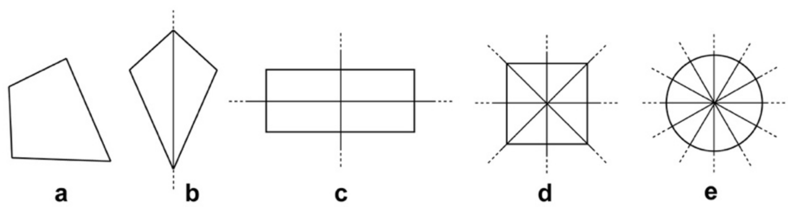


Figure 2.8. Different shapes and their symmetries.

The initial four shapes represent some quadrilaterals. The first (a) is asymmetrical; no transformations are sending it into itself, except identity. The symmetries grow from a kite (b) to a square (d) and the circle is the most symmetric

[8] During a talk at the end of the 1940s, Hermann Weyl demonstrated the impossibility of formalizing mathematical theories. A formal theory cannot be an ideal structure without physical use. This epistemological passage is linked to the deep unity between mathematics and physics (Weyl 1939).

two-dimensional figure. The more symmetries a figure has, the more symmetric it is. Presumably, for this reason, symmetric figures are common, beautiful, and familiar. To Weyl (1952), "Beauty is linked to symmetry", so human beings instinctively prefer regularity. The circle provides us with an important task. It intuitively furnishes one example of continuous symmetry, unlike the other discrete symmetries, in which the students somehow come across the concept of infinity and infinitesimal transformations, an example referring to Lie's group, discussed in Section 2.3.2. Yet, these shapes show how symmetries impose constraints, expressing a property of a variance. Figures can, therefore, be defined in terms of this property (Henneaux, 2020). As a result, a symmetry principle in geometry imposes constraints on *n*-sided polygons[9].

A clue to the depth of these observations is provided precisely by what has been seen concerning all the operations that retain the shape or the arrangement of objects. The number of operations that can be carried out leaving a figure unchanged provides a first assessment of how "symmetrical" it is and how it will behave under transformations. Moreover, two shapes can have the same *measure* of symmetries and not be equivalent. It is necessary to consider how the transformations are carried out, i.e., the effect shown when they are performed one after the other and their parity. In other words, the symmetry is measured by the algebraic structure determined by the composition of transformations, that is, the structure of the group or symmetry group.

As stated by Darvas (2007, p. 28), "the concept of the group has proved to be the most useful tool for the mathematical description of symmetries". A group is an algebraic structure $\mathcal{G}(G,*)$ consisting of a (non-empty) set of elements G with a binary operation $\circ : G \times G$ that combines any of two elements to yield a third element satisfying the following properties, namely group axioms:

1. Closure: *for any element* $f, g \in G$ *we have* $f \circ g, g \circ f \in G$; (i.e., the operation cannot bring a result outside the group and the set G is said to be *closed* under the law of composition).
2. Associativity: *for any element* $f, g, h \in G$ *we have* $(f \circ g) \circ h = f \circ (g \circ h)$; (i.e., operations can be grouped regardless of the order).

[9] We can prove the *n*-sided polygon's regularity by comparing the number of sides with rotational symmetries.

3. Identity: *there exists an element* $e \in G$ *such that for any* $f \in G$ *we have* $e \circ f = f \circ e = f$; (i.e., the existence of an identity element or neutral element for the group $\mathcal{G}$; the element e is unique because of the assumptions).

4. Invertibility: *for any element* $f \in G$ *there exists* $f^{-1} \in G$ *such that* $f \circ f^{-1} = f^{-1} \circ f = e$; (i.e., the existence of an inverse element for the group $\mathcal{G}$; as with the identity element, the element f^{-1} is unique because of the assumptions).

The group's structure is completely independent of the object on which the transformations act, permitting the study of symmetries without referring to any object from which we extracted them. Thus, many objects can correspond to the same symmetry or symmetry type. Such characteristic gives the group the power to study the general properties of symmetry transformations without inspecting each object on its own (Schwichtenberg, 2018, p. 29). As a result, the symmetry relationship helps to organize a jumbled mess of real objects and abstract entities into an orderly classification system.

Now, we show that symmetries of shapes or forms can be studied carefully using the tools of group theory to make them clear. A simple way is to link them to concrete actions upon a line, a plane or even a higher dimensional space. Connecting objects in a group to concrete geometric transformations allows students to grasp abstract reasoning because geometric transformations are easy to concretize visually and physically. Consider the example of rotating a circle or a square on a plane.

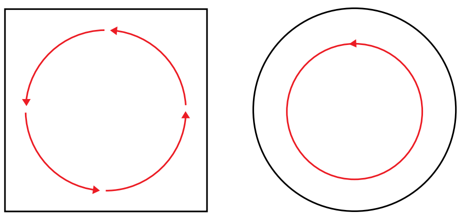


Figure 2.9. Rotational symmetries of a circle and square.

The circle is invariant under rotations by any angle around its centre, while the square looks the same from four angles (90, 180, 270 and 0/360 degrees). We are not referring to all symmetries, but a set of all rotational ones; moreover, these shapes prove that the same rotational symmetry has two subtypes, i.e., continuous, and discrete.

Let us prove that the set of rotations satisfies all criteria (four axioms) to form a group:

1. Closure: Performing a circle rotation of any angle followed by a second rotation of any angle is again a symmetry rotation. A rotation by 90 degrees of the square followed by 180 degrees is a rotation by 270 degrees, which is again a symmetry rotation, as it is (always) a rotation by a multiple of 90 degrees.
2. Associativity: A square rotation by 90 degrees followed by a rotation by 180 degrees, then by 90 degrees is the same as a square rotation by 270 degrees followed by a rotation by 90 degrees which is the same as a square rotation by 90 degrees followed by a rotation by 270 degrees. This property is valid for the circle by rotations of any three angles.
3. Identity: Rotating by 0 degrees, "in the intuitive sense, the non-action of doing nothing" (Rosen, 1975, p. 22), serves as the identity element, leaving the square and circle unchanged.
4. Invertibility: For a rotation through any angle α degrees, there is an inverse of a rotation through the angle $(360 - \alpha)$ degrees, namely its explementary angle. A rotation through any angle followed by one of $(360 - \alpha)$ degrees yields a combined rotation of $\alpha + (360 - \alpha)$ degrees. However, rotating by 360 degrees (full circle) returns to the initial position as a rotation of 0 degrees, i.e., the identity element.

Another simplest and most visual way to verify whether symmetry transformations form a group is by using the multiplication table as the composition of the transformations (Rosen, 1975; Richardson & Kallen, 2015). We analyse the multiplication tables of square and triangle shapes in Section 2.3.4. Based on Figure 2.10, it is easily determined that rotations form a group, proving that the four axioms are satisfied again:

1. Closure: There are no empty cells in the table, and thus the set is closed.
2. The composition of the elements is associative.
3. There is an identity element.
4. Every element has a unique inverse specified due to the exactly one identity element in each row and column.

Thus, the set of rotations has been promoted as a group. Such a procedure can be extended to all geometric shapes and transformations and beyond abstract entities and operations.

After introducing the basic issue and concepts of group theory, we take up the challenge raised by Riehl, primarily at the educational level. From the perspective of curriculum design, the abstracting process provides common grounds for the theoretical frameworks relating the development of student thinking to the structure of mathematical concepts. Methodologically speaking, abstraction has an operational perspective and transdisciplinary character[10]. The abstraction of patterns is the fundamental block of innovation. Innovation arises from the fact that through the use of patterns the mind moves toward the rules, designs, and laws that connect collections of movements and thoughts from one area of learning to another.

Developing such connections using the mathematical language of symmetries is central to education. The challenge in education, then, is to use symmetry as an organizing idea that holds learners' minds together, far from a compartmentalised vision. By encouraging children to focus on relational invariants from which they can abstract a general rule, the link between symmetry and invariance is operationalised, and the paradigm shift toward the modern sense of symmetry is promoted.

This connection can be applied as a generative and consistent method, but it requires several steps. First, it is necessary to introduce symmetries and transformations since they are often taught as distinct concepts. Then, one goes beyond the most immediate geometric symmetry, i.e., line symmetry, and examines all symmetry transformations. Our systematic approach includes rigid transformations other than reflections, e.g., rotations and translations, and begins in the earliest grades. According to Mizsaz et al. (2019), the foundations for good mathematical progress are indeed laid in the early years, and a lack of adequate instruction leads students to academic failure.

[10] "Transdisciplinary" involves the development of an overarching paradigm encompassing several scholarly disciplines. The theory of symmetrisation and a-symmetrisation as a theory of form, formation, and form-variation/mutation ("evolutionism", "neo-evolutionism") illustrates a transdisciplinary framework. The components of this theory (modern painting, art theory, evolutionary biology, natural philosophy, psychology, geometry, physics, chemistry, cosmology, music, etc.) are not only linked internally and closely interwoven through integration by the theme "symmetry"; in addition, the disciplines (monodisciplinary fields) are subsumed in the transdisciplinary concept "symmetry" under a supradisciplinary paradigm" (Hahn, 1998).

Although teachers believe that such topics are premature for elementary school and reserved for high school students (Montone et al., 2017) and a source of stress or strain on already crowded curricula, these concerns are not justified. The inherent logic and internal dynamics of symmetries with the proposed materials and methods generate a kind of self-organization in students.

Thus, although the concept requires hard motor and intellectual work, it should bring simplicity, unity, joy, and relief (P. Klein, 1990). The goal is to get pupils to grasp the operational side of symmetry in addition to the basic notion of bilateral balance and to investigate dynamic symmetries. We design a series of activities where students independently construct the scientific notion of symmetry and improve their spatial skills using 2D and 3D artefacts[11]. Actions such as flipping, rotating, imagining, cutting, mirroring, drawing, and discussing shapes train spatial creativity and foster insights and intuitions about symmetry concepts.

2.3.7 SYMMETRY LEARNING VIA ARTEFACTS

The learning progression we designed aims at engaging students in a sequence of didactic cycles that use multimodal artefacts. These activities allow students to expand their knowledge through a series of progressive experiences to understand the general concept of symmetry. Such activities provide a rich context for experiencing the mathematical process and the power of symmetries. In this framework, the design and modelling activities have the character of games that follow certain rules but have a serious basis and background of symmetry operations.

In the first cycle of our action research, we introduce planar transformations using cardboard boxes as a stimulating and concrete situation. The connection between concrete operations, observational aspects, explanations, and intuitions suitable for the transition from elementary to complex symmetries is established by solving the game of closing special cardboard boxes.

We construct models of boxes of different shapes, as shown in Figure 2.10. The manipulative work involves folding and taping these cardboard models to obtain boxes with their lids. The lids are designed in a such way that they can only be rotated.

[11] An artefact is any device conceived and realised by human beings for any purpose; besides, mathematical meanings may be related to the artefact and its use (Rabardel, 1995; Bartolini et al., 2008).

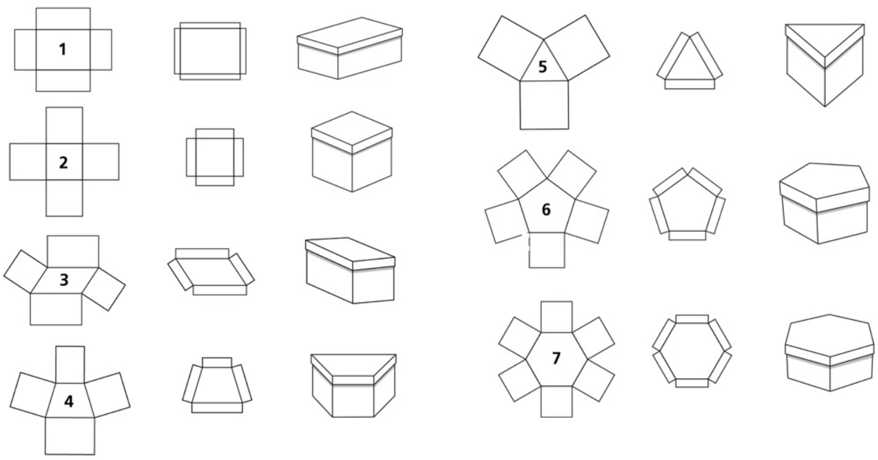


Figure 2.10. Models of cardboard boxes and lids of 7 shapes.

The game task is to figure out how many ways one can put the lid on each box. This game refers to the rotational symmetries of the differently shaped lids. For example, we take the triangular lid. The symmetries of the equilateral triangle can be thought of as the transformations (i.e., rotational actions) that make it possible to cover each box.

In this way, students can playfully and intuitively deal with triangle invariants under rotations (Figure 2.11). In elementary school, the approach to symmetry must be a joyful and concrete situation. Activities are designed to stimulate discussion and build the mathematical concept behind the game. At this stage, the concepts of symmetry and invariance can be explored through a non-technical approach. However, these elements of symmetry represent fundamental ideas that will provide the building of more comprehensive scientific knowledge. The symmetries of the square (Figure 2.12) are more numerous than those of the equilateral triangle.

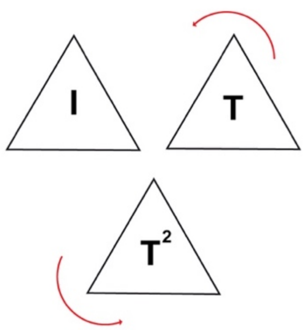


Figure 2.11. The cyclic 3-group represents the rotation symmetries of an equilateral triangle, where I is the identity, T is rotation of 120 degrees, and T^2 240 degrees.

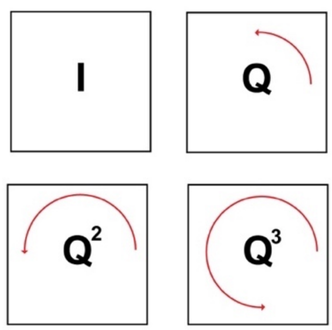


Figure 2.12. The cyclic 4-group represents the rotation symmetry of a square, where I is identity, Q is rotation of 90 degrees, Q^2 180 degrees, Q^3 270 degrees.

We can compare the composition of two different rotations in the equilateral triangle (Table 2.2) and square (Table 2.3). The procedure is repeated for all shapes.

a·b (rows: a, columns: b)	I	T	T^2
I	I	T	T^2
T	T	T^2	I
T^2	T^2	I	T

Table 2.2. Composition of two rotations (a•b) in sequence of an equilateral triangle[12].

a·b (rows: a, columns: b)	I	Q	Q^2	Q^3
I	I	Q	Q^2	Q^3
Q	Q	Q^2	Q^3	I
Q^2	Q^2	Q^3	I	Q
Q^3	Q^3	I	Q	Q^2

Table 2.3. Composition of two rotations (a•b) in sequence of a square.

In the later stage of the game design, a new version of lids is used. The lids can be rotated and flipped. Figure 2.13 and Figure 2.14 show the different design and movements associated with the new lids.

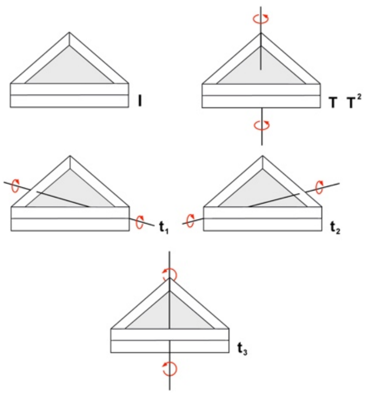


Figure 2.13. The *dihedral 3-group* represents the rotation symmetries I, T, T^2 and reflexion symmetries t_1, t_2, t_3 of an equilateral triangle.

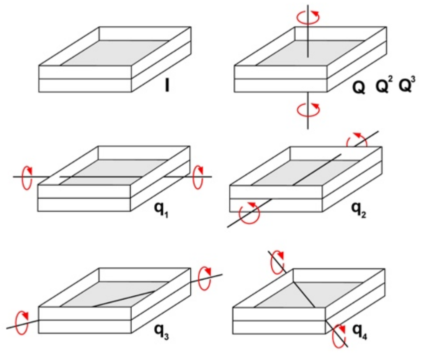


Figure 2.14. The *dihedral 4-group* represents the rotation symmetry I, Q, Q^2, Q^3 and reflexion symmetries q_1, q_2, q_3, q_4 of a square.

The game of covering boxes becomes more difficult and complex but stays rich in stimuli and feasible for the pupils. The complexity of symmetry lies in the fact that the rules for the transformations related to the two types of cardboard box lids differ. In rotative manipulations, the lid returns to its original position if and when the rotations are repeated three or four times, depending on the shape of the lid.

[12] The notation (a•b) is read "*a* composed with *b*" or "*a* after *b*". It means that action *a* follows action *b* in the column. For example, the composition $T \bullet T^2$ indicates a rotation of 240 degrees followed by one of 120 degrees. The element $T \bullet T^2$ associated in the table is the intersection of the second column T^2 and the first row T. The product of two symmetry rotations in the cyclic group is commutative. The order in which we perform the operations does not matter (i.e., $T \bullet T^2 = T^2 \bullet T = I$). The cyclic table is indeed symmetric across the main diagonal, running from top left to bottom right.

The *cyclic n-group* describes the symmetries of a regular n-sided polygon rotating in a plane. With the new covers, the actions are not in the plane but in three-dimensional space. A new dimension is needed for flipping the lids around. Different moves, rotations, and reflections must be combined to achieve all the elements without applying the same rule repeatedly.

The *dihedral n-group* is the group of regular n-sides polygon symmetries, that includes both rotations and reflections. These symmetry groups are not commutative. Therefore, the order of actions is important. A rotation followed by a reflection is not always the same as a flip followed by a rotation. By assigning a different colour to each action in the table group, we can colour each cell based on the action occurring in it. The different properties between the cyclic and dihedral groups become apparent when we look at the coloured pattern in the tables.

Table 2.4 and Table 2.5 show the results of all possible combinations between all symmetry elements connected with differently shaped lids: 36 and 64 entries in a coloured spectrum in Table 2.4 and Table 2.5, respectively. Each row or column has all 6 equal colours for the symmetries of the triangle and all 8 equal colours for the symmetries of the square.

b

a	**I**	**T**	**T^2**	**t_1**	**t_2**	**t_3**
I	I	T	T^2	t_1	t_2	t_3
T	T	T^2	I	t_2	t_3	t_1
T^2	T^2	I	T	t_3	t_1	t_1
t_1	t_1	t_3	t_2	I	T^2	T
t_2	t_2	t_1	t_3	T	I	T^2
t_3	t_3	t_2	t_1	T^2	T	I

Table 2.4. Composition of any two operations (a•b) in sequence of an equilateral triangle.

b

a	**I**	**Q**	**Q^2**	**Q^3**	**q_1**	**q_2**	**q_3**	**q_4**
I	I	Q	Q^2	Q^3	q_1	q_2	q_3	q_4
Q	Q	Q^2	Q^3	I	q_2	q_3	q_4	q_1
Q^2	Q^2	Q^3	I	Q	q_3	q_4	q_1	q_2
Q^3	Q^3	I	Q	Q^2	q_4	q_1	q_2	q_3
q_1	q_1	q_4	q_3	q_2	I	Q^3	Q^2	Q
q_2	q_2	q_1	q_4	q_3	Q	I	Q^3	Q^2
q_3	q_3	q_2	q_1	q_4	Q^2	Q	I	Q^3
q_4	q_4	q_3	q_2	q_1	Q^3	Q^2	Q	I

Table 2.5. Composition of any two operations (a•b) in sequence of a square.

All combinations of rotations and flipping actions can be replaced by a single symmetry operation (i.e., 6 symmetries of the triangle or 8 symmetries for the square). The colour pattern reveals the structure of a smaller group within these tables. The tables are divided into four quadrants. The upper left quadrant has the same colours as the lower right, and the lower left quadrant has the same colours as the upper right. It means that the upper left and the lower right quadrants consist only of rotations, while the other two quadrants are pure rotations.

We can replace the dihedral groups with a multiplication table for the group of order two, as shown in Table 2.6. A sequence of two rotations or flips always results in one rotation, while one rotation and one flip result in one flip. Just like geometric symmetries, the dynamic ones occur when transformations leave unchanged the equations of the system.

	Rot.	**Flip**
Rot.	Rot.	Flip
Flip	Flip	Rot.

Table 2.6. The resulting table group C_2 from the partition of D_n.

2.4 SUMMARY AND IMPLICATIONS

The chapter provides a comprehensive overview of the multifaceted concept of symmetry and its many applications in various fields, with particular emphasis on science and mathematics education. The discussion emphasizes the importance of understanding symmetry beyond its geometric aspects and highlights its functional properties and characteristics of phenomena. In addition, the chapter traces the historical development of symmetry, addresses pedagogical issues related to effective teaching, and explores the concept of linking symmetry and invariance with particular attention to interdisciplinary connections and bridges between the fields. The final section of the chapter introduces cardboard boxes as teaching tools for exploring symmetry operations and their connections to group theory.

The implications for our educational research are numerous. It highlights the need to promote interdisciplinary learning and teaching by integrating symmetry-related concepts across disciplines and exploring effective teaching methods that connect concrete examples with abstract mathematical concepts and link symmetry to group theory by using innovative teaching tools and techniques, such as cardboard boxes. In addition, it is important to evaluate the effectiveness of such an approach to determine how best to teach symmetry to students of different ages. Ultimately, this chapter offers valuable insights for educational researchers who wish to deepen their understanding of symmetry and its many applications.

Chapter 3: Experiment Model Transition

This chapter describes the study design to achieve the objectives stated in Section 1.4 of Chapter 1:. Section 3.1 explains the methodology used in the study, the research design, and the phases in which the methodology was implemented. Chapter 3: then addresses the first reflection cycle of the experiment (Section 3.2), which includes several reflection phases such as diagnosis, action planning, pre-test, action implementation, post-test, and evaluation.

Section 3.2.1 on diagnosis discusses the premise and the participants involved in the experiment, while Section 3.2.2 on action planning focuses on the process, lists all the tools used in the experiment and justifies their use. Section 3.2.3 on the pre-test outlines the rationale for the action, and Section 3.2.4 on the action taking describes the process used to conclude and reflect on the action taken during the experiment. Section 3.2.5 on the post-test addresses the reflections following the experiment, and Section 3.2.6 on the evaluation discusses the results of the experiment. Section 3.2.7 focuses specifically on the analysis of the data resulting from the pupils' responses to the open-ended question Q3 in the questionnaire.

The chapter also includes Section 3.3 on ethics, which discusses the ethical considerations associated with the experiment. Overall, the chapter provides a detailed look at the experimental model and the different phases of the experiment.

3.1 METHODOLOGY AND RESEARCH DESIGN

The methodology used in the development, construction, and implementation of the teaching learning sequence (TLS) is a reflection cycle of ongoing action research to provide investigative, didactic, and pedagogical feedback before proceeding with the sequence.

We adopted the spiral educational model proposed by Michelsen (Section 2.2) and adapted it for our action research experiment, calling it Experiment Model Transition (EMT). The EMT is a process model used in the transition from primary to secondary school to plan, evaluate, and revise the teaching and learning of ideas, materials, and strategies related to the concept of symmetry in cycles.

Each reflection cycle in our action research experiment consists of six steps, which are specified in Table 3.1.

Steps	Reflection Process	Objectives
1. Diagnosis	Reflection on premise	Identify materials, artefacts, and classroom context
2. Action Planning	Reflection on process	Develop tasks and actions for transition in learning pathways
3. Pre-Test	Reflection for action	Perform the pre-test assessment with the control group
4. Action Taking	Reflection in action	Implement the actions in the teaching-learning sequence
5. Post-Test	Reflection on action	Perform the post-test assessment with the target group
6. Evaluation	Reflection on findings	Evaluate how students understand the concept of symmetry

Table 3.1. The main features of each step of the reflection cycle.

The six steps are consistent with our research and correspond to a pre-test and post-test design with a control group. Each student in the control group is tested twice: before the control group receives the educational intervention and once after. The TLS is used between the two assessments. As highlighted in Table 3.1, reflection is not limited to the end of research activities but is practiced throughout the process, with specific objectives for each phase of the action research cycle.

Indeed, in reflection cycles, reflection at each stage is critical to integrating action and research on symmetry at multiple levels: to enhance student learning and understanding, to promote deep reflection on mathematical and pedagogical content for teachers, and to create an opportunity for meaningful change in consolidated instructional practice with the interdisciplinary approach based on the connection between symmetry and invariance.

According to Silva & Colombo Jr. (2017), the cycle of reflection is a methodological tool for educational research that explores ways of innovative teaching based on the joint elaboration of the teaching-learning sequence by teachers, students, and researchers. The process of spiral education focuses on the progress of the teaching-learning sequence. Such progress in EMT builds on refinements made in incremental vertical ways through articulations of simple and intuitive to complex mathematical ideas of symmetry and transversal pathways through the student's ability to use acquired understanding to interpret different materials and contexts with an interdisciplinary mindset (Michelsen, 2015; Acher & Arcà, 2014).

The use of concrete artefacts with associated coherent and progressive activities is central to this type of transition for fifth graders to increase their learning potential in terms of symmetry and invariance.

Following the logic of the study considerations and the design aspects described above, we set up the educational research project in two experiments with reflection cycles, as shown in Figure 3.1.

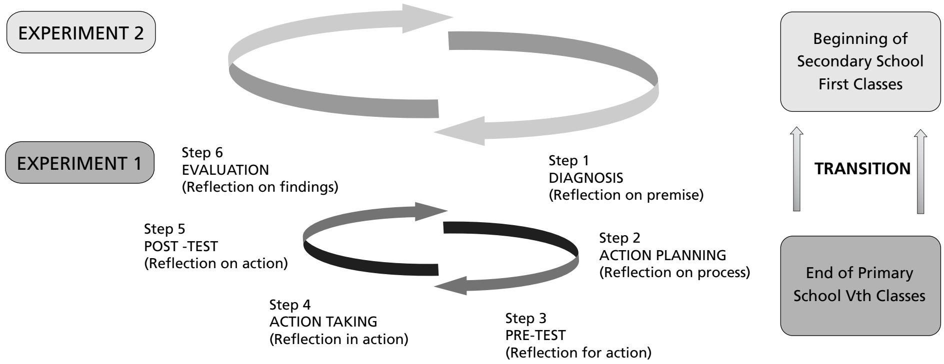


Figure 3.1. Representation of first reflection cycle in Experiment Model Transition.

The first cycle experiment led to new insights into teaching the big idea of symmetry and invariance, including further and more refined material and questions for the second cycle. The results were channeled into a second experiment as part of the action research process. This process may extend over several iterations of the action research spiral, the distinctive feature of which is that the learning never ends. The following sections detail the steps of the first experimental cycle of the action research project by explaining the context, the materials presented to the students and their rationale, the data sources and analysis methods, and the results.

3.2 EXPERIMENT 1 - FIRST CYCLE

The action research started in November of the school year 2018/2019: "Mathematics beyond numbers, symmetry and the search for invariants" at the primary school of the Comprehensive Institute in Montegranaro (Italy). The first cycle of experiments was completed in June 2009 and was aimed at fifth graders. Table 3.2. contains the detailed schedule for the first experiment as follows.

Steps	Nov. 2018	Dec. 2018	Jan 2019	Feb. 2019	March 2019	April 2019	May 2019	June 2019
Diagnosis	4 h.							
Action Planning		3 h.	2 h.					
Pre-Test				3 h.				
Action Taking					18 h.	12 h.		
Post-Test						6 h.		
Evaluation							4 h.	6 h.

Table 3.2. The schedule of the first experimental cycle in hours (h.) for each phase.

At that time (i.e., June 2019), the second cycle of reflection was already in the evaluation phase to renew and refine the teaching and learning materials for a successful transition to the secondary school. The nature of the reflections in the first experiment will be discussed in the next sections using the experimental structure illustrated in Figure 3.1.

3.2.1 DIAGNOSIS: REFLECTION ON PREMISE AND PARTICIPANTS

After the initial contact with the school administration to officially approve the implementation of the action research, the initial phase of the first cycle began with two meetings to explain to all mathematics teachers the body of knowledge around the concept of symmetry, which is central to the teaching-learning process. The sharing of ideas and research goals, as well as the identification of strengths and weaknesses of the material, make teachers reflect on their teaching methods. These moments are critical in selecting volunteer teachers to participate in the first cycle.

Steps have been taken to create a shared vision of the goals and focus of the research and to give a more concrete form to the organization, especially to the working group. In this regard, we selected the working group of students and teachers to consider the research hypothesis and objectives, work dynamics, and availability to prepare and carry out the research proposals. The voluntary appointment of two tutors, each in his or her subject area (i.e., primary, and lower secondary mathematics), as mediators in researcher-teacher communication led to better practices in the work context. Prior to the formation of the working group, a diagnostic was conducted through a discussion at meetings and a semi-structured questionnaire with closed, open, and Likert scale questions (see Appendix A) that allowed teachers to specify and contextualize their knowledge and experience in teaching symmetry.

In this step, teachers reflect on their thoughts, feelings, or actions in the context of personal symmetry background to gain further insight into classroom experiences and materials that better guide their actions in action research. Kreber (2004, p. 44) refers to reflection on premises as an engagement with knowledge content and, more importantly, instrumental, communicative, and emancipatory learning processes that lead to the confirmation or rejection of research hypotheses. Reflecting on the practices and questioning the body of knowledge or actions of teachers in relation to the concept of symmetry influenced the focus on the research questions. Based on these premises, the research group decided to investigate the use of symmetry and the search for invariants as bridging concepts in teaching. Therefore, the sample group chosen for the research action consisted of students at the transition from primary to secondary school. The sample group was then divided into an experimental and a control group. Neither group was randomly selected, nor were participants randomly assigned to groups, as the setting could not lead to the group being artificially created for the experiment, but rather existing intact fifth graders were used. It is common in education to use intact school classes; indeed, quasi-experimental designs are often used (Creswell, 2015, p. 311).

In summary, the first experiment included a sample group of 96 students, i.e., all fifth grades (10 years old) of the primary school participated in the study. Five classes and four teachers experienced the first year of the 3-year research project; children participated again in the second year, along with their new secondary school classroom teachers. Classes VF and VG, a total of 41 students, formed the control group, which had to answer the same questionnaire at the beginning (pre-test) and the end of the learning and teaching sequence (post-test). Table 3.1 shows in detail the composition of the sample group of students.

Classes	N° Pupils	Gender	Pre-Test	TLS	Post-Test
V A	16	6 F – 10 M	No	Yes	Yes
V C	19	7 F – 12 M	No	Yes	Yes
V D	20	8 F – 12 M	No	Yes	Yes
V F	20	11 F – 9 M	Yes	Yes	Yes
V G	21	10 F – 11 M	Yes	Yes	Yes

Table 3.3. Scheme of sample group.

3.2.2 ACTION PLANNING: REFLECTION ON PROCESS AND MATERIALS

The action planning step is crucial for action research. It benefits from teachers, as active members of the school environment, providing feedback on materials prepared by researchers, which are deemed important to the teaching-learning sequence. Preparing activities, tests, and artefacts are based on positive negotiation between teachers and researchers. The group of teachers reflected on the process by discussing the implicit curriculum and materials highlighted by the researchers regarding the vertical and horizontal directions of the EMT didactic model. The goal of the planning is to generate expertise and practical knowledge useful for implementation in teaching-learning classroom. According to De Jong et al. (2017), action research is about understanding practice through a combination of intervention and explanation in the light of individual, collective, and community learning and expanding collective intelligence.

Special attention was given to the study of artefacts, which are considered by researchers to be conducive material in terms of symmetry learning potential. The choice of cardboard boxes (Section 2.3.7) as the most important concrete artefact was based on their great potential to redefine and conceptualize the concept of symmetry in conjunction with invariance through manipulative play.

Furthermore, the use of cardboard boxes embodies the aspect of scientific practice, namely learning and reflection through action, and the learning goals of Nature of Science (NOS) in teaching learning sequences for increasingly challenging and complex non-routine activities. This planning phase is also about realizing physical prototypes in many materials and refining them with appropriate activities for a gradually developing understanding of symmetry. We chose to develop activities where students could build cardboard boxes to engage them directly through manipulative games and problems.

Besides cardboard boxes, the investigations for a coherent set material were extended to mirrors and kaleidoscopes, origami and paper folding, shading tasks with symmetric shapes, magic squares, shortest path problems, and simple invariance experiments. Deepening teachers' knowledge of various resources that can be included in the classroom experiment opens the possibility of addressing symmetry and invariance coherently and reinforcing their teaching-learning process. From this point of view, the aim of using concrete manipulative activities and games with different and related objects is to promote relational understanding and reasoning in relation to the big idea of symmetry as sameness within change (see Section 2.3.2).

Action planning also included important considerations of pedagogical strategies to change students' experiences and perceptions of the concept of symmetry. As a result, metaphors were incorporated into the teaching-learning process. According to Martinez et al. (2001), metaphor exerts a strong influence on the processes of analysis and planning in education and influences teachers' reflecting about teaching and learning. Teachers' interest, motivation and positive thoughts can be triggered using metaphors that create an atmosphere full of joy and deep meaning content in the teaching-learning process (Maulana et al., 2019).

In summary, reflecting on the process was characterised by a dialog about prospective teaching practice, in which the participants acknowledged the researcher's mastery and the other teachers' experience. As an effect of this collective and participatory process, a teaching-learning progression is built. The teaching and learning structure, shown below in Table 3.4, consists of five activities divided into three lessons of two hours each. The total duration is six hours, during which the researcher proposes the various activities in each classroom, and the teachers participate in the lessons as observers.

We decided to use the first-person perspective in implementing the lessons in the action step of the first cycle, in which the researcher (i.e., I) puts the activities into action. This decision was negotiated in the working group to avoid bias, inconsistency, and contradiction in practice and to promote strong reflection in action among teachers. Coghlan & Shani (2021, p. 472) suggest that the critical role of the first person in action research throughout the inquiry process is to "facilitate the process of sense-making in the second person, as this is likely to bring some coherence to the next phase of the action and inquiry process". Therefore, the active participation of teachers in the implementation of the teaching actions is reserved for the second cycle of the experiment.

Activity	Type	Title	Duration
A1	Brainstorming - Writing - Drawing	If I say the word symmetry, what are you thinking about?	2 h
A2	Solving Problems	Boxes and lids	1 h
A3	Listening - Watching	Lesson	1 h
A4	Task Game in group	Strange boxes and lids	2 h
A5	Homework task	Hunting for symmetries	/

Table 3.4. Series of Activity in TLS.

3.2.3 PRE-TEST: REFLECTIONS FOR ACTION

A pre-/post-test research design is a valuable diagnostic tool for measuring learning growth in an educational experiment because it measures student progress in a particular subject. Namely, a pre-test provides information about content or trait that you assess in students in a pre-treatment experiment, while a post-test measures the same content or trait after treatment. On the pre-test at the beginning of class, students are not expected to know the answers to all questions; however, they should be expected to use their prior knowledge to predict rational answers. When answering the same questionnaire at the end of the lesson (i.e., the post-test), students should answer more questions correctly because their knowledge and understanding have improved. According to Andrade et al. (2015), using the same questionnaire is a better way to relate student learning and the impact of the teaching methods and materials used in the classroom.

Our first-cycle action research used a pre-test/post-test study design to evaluate the effectiveness of the TLS by analysing learning changes in students' knowledge of symmetry that resulted from the teaching-learning session. The questionnaire (see Appendix B) consists of four questions of increasing difficulty that relate to activities that will take place in the next action phase. The logical sequence of questions is designed to encourage students to reflect on their knowledge of symmetry, question it, and renegotiate its meaning through the concept of invariance. As the Italian National curriculum guidelines provide, we assess initial skills by asking pupils to justify their answers. The test is divided into three sections related to the cognitive, affective, and psychomotor dimensions. The time allotted for answering the questions is about 45 minutes and must not exceed one hour. Table 3.5 shows the structure of the pupils' questionnaire used for the test before and after class.

Question	Type	Aim
Q1	Dichotomous Choice Open Justification	To verify how the concept of symmetry is perceived in the assumed didactical frame.
Q2	8 Items Test	To assess and measure the level of knowledge and skills related to the tasks and/or specific areas.
Q3	Narrative Text	D.1: Emotional and affective dimension. D.2: Cognitive dimension of learning. D.3: Possible presence of a cognitive conflict
Q4	A Task Open Justification	To check the extent to which students have mastered the skills in a (task) exercise with a higher level of cognitive difficulty.

Table 3.5. Scheme of pupils pre-/post-test questionnaire.

It contains a dichotomous choice, an item test, a narrative text, and an open-ended thinking task, each of which relates to a specific objective and learning item. In the questionnaire before the teaching sequence, students do not have to answer the third open question, Q3. This question has no significance in the pre-test because it stimulates and shapes the pupils' reflection on their actions and personal experiences during the TLS (reflection on action). For a detailed discussion of the individual questions, see Section 3.2.5 of the post-test.

In the phase of pre-test administration, reflection pro action takes place. It refers to thinking about future actions to improve or change the practice associated with the learning object. Olteanu (2016) points out that reflection for action is related to the object of learning and considers three aspects of a given process. First and foremost, reflection for action refers to the content that students learn, and that is covered in class (i.e., the intended learning object), then to what appears in class (i.e., the implemented learning object), and finally to what students experience in a learning environment (i.e., the lived learning object). Therefore, reflection pro action has a proactive nature. According to Gencel & Saracaloglu (2018, p. 9), reflection for action takes place when individuals consider unexpected experiences from action as a new path for new situations. In our context, teachers and students were preparing for the future by using and reflecting on the knowledge gained from the pre-test. When the students were given the questionnaire, they were told that the questions were related to future teaching practice. In addition, no special study or prerequisite was required before taking the test. We encouraged students to do their best on the tests by answering according to their knowledge and intuition and thinking about the questions without cheating. Teachers were responsible for carefully reading the questions to all students and assisting only those with special needs. In general, pre-assessment may have potential advantages and disadvantages. For successful implementation, the potential advantages must be exploited, and the potential disadvantages avoided, keeping in mind the central purpose of using pre-testing (Guskey & McTighe, 2016). For this reason, careful consideration of the implications for action in pre-testing is necessary.

On the one hand, taking the pre-test before the lecture probably increased attention, curiosity, and enthusiasm to listen to the lecture. Some studies (Hartley, 1973; Berry, 2008) confirm that the pre-test can be a motivating and orienting tool that leads to better classroom performance. On the other hand, the pre-test could influence the experimental treatment because students anticipated the post-test questions based on their experiences with the pre-test. Coover & Angell (1907) noted early in experimental psychology that pre-tests lead to performance gains by practicing responding to test content, even when there is no treatment effect. In contrast, neither a statistically significant positive nor negative effect of the pre-test was found in the Hartley (1973) experiment. To ensure that the pre-test was useful and valid, we considered these background factors and caution in designing the pre-test. The structure of our study sample presented in Section 2.3.2 is suitable for investigating the influence of the pre-test and the didactic intervention on the post-test outcome. An overview of the quasi-experimental study for the first cycle experiment, i.e., the one-group pre-post research design, is provided in the following table.

Measure	**Treatment Cycle 1**	**Measure**
Pre-test	*TLS Experimental*	*Post-test*
O1 (VF –VG)	T1	O2 (VF –VG)
	T1	O2 (VA-VC-VD)

Table 3.6. Diagram of one-group pre-/post research design.

All students participate in the post-test, labelled O2 in the scheme, and only two classes (VF-VG) participated in the pre-test, labelled O1. The effect of intervention T1, administered to all participants, is measured by comparing the results of the post-test O2 and the pre-test O1 for the control group (VF-VG). This result can be statistically correlated with the post-test result for the whole sample. This quantitative analysis can determine if the pre-test affects the results. The analysis used to evaluate the empirical research and validate the questionnaire design is explained in detail in Chapter 4.

3.2.4 ACTION TAKING: REFLECTION IN ACTION

Reflection in action is the practice of reflection explicitly linked to the performance of activities. Thus, it is used in the context of action to learn more about one's understanding, to improve one's actions, and to build new knowledge (Schön, 1984, p. 77; Helmke, 2012, p.116). Reflection in teaching action is characterised by the ability to observe and reflect on one's actions, both those of teachers and learners, in teaching-learning practice to adapt them to prevailing conditions based on reflection findings. According to Schön (1984, p. 424), "reflection in action is both a consequence and cause of surprise".

For learners, the surprise factor is a means of reflection because it stimulates their attention and curiosity about the subject matter and enhances the learning process (Adler, 2008). Based on Metacognitive Explanation Based (MEB) theory, Foster & Keane (2018) propose that the higher the degree of surprise about learners' prior knowledge, the more likely they are to retain information that elicits different explanatory strategies. Accordingly, learning can be triggered and reinforced by planned and random surprises. Planned vs. incidental surprises refer to instructional materials created by teachers and educators who use surprises to support pupils' learning tailored to their knowledge and prerequisites.

For teachers, the surprise factor is an unexpected experience or outcome when conducting classroom activities. Such unexpected events allow teachers to react and smoothly change the course of classroom activities when they occur. From a practical and scientific point of view, the designed teaching-learning sequence is constantly reshaped by the researcher based on the dynamics of the ongoing events in the classroom during the action research and the corresponding reflections in practice. Therefore, the schedule for the implementation of our teaching sequence has included a break of a few days between lessons to allow for such a possible surprise factor.

To consider the great importance of such a surprise factor for children, the entire first teaching-learning cycle was built with the metaphor of a journey into the world of symmetry. The protagonists of the adventure about symmetry were the students themselves. They made surprising encounters with characters such as the Little Prince and the Rose, Paolo and his strange boxes, and were asked to pass some tests about symmetry. As mentioned in Section 3.2.2, the first teaching-learning cycle consists of five activities. The first lesson is essentially an introduction to symmetry and is divided into two parts. In the first part, we introduced the topic by reading the Little Prince's dialogue (see Appendix C) with the Rose from the original version of *The Little Prince* (De Saint-Exupéry, 1945).

To solicit pupils' ideas and assess their prior knowledge about symmetry, we asked them the following questions, "You have already studied the topic of symmetry; what do you know about it?" and "What comes to mind when the Little Prince talks about symmetry?"

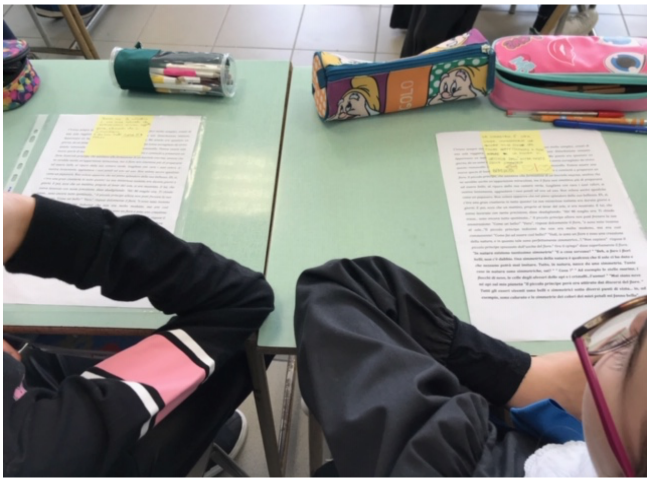


Figure 3.2. Children are ready to read their answers on post-it notes during activity A1.

After students were given a post-it note and a drawing sheet, they briefly wrote the answer to the questions on their post-it notes and drew everything symmetrical around them on the sheet, which the teacher then collected. Drawings provide pupils with an interesting way to reflect and express their ideas and core beliefs about symmetry (Villarroel et al., 2019) and the teaching and learning of mathematics, more so than questionnaires (Stiles et al., 2008). At the same time, it is a valid method to collect and interpret data about students' mathematics-related beliefs (Hatisaru, 2021).

Learners' drawings show their prototypical representations of symmetry by depicting living organisms, plants, and man-made objects, usually with bilateral symmetry. Many children explicitly draw symmetry axes in their symmetry representations, usually vertically, a few horizontally, and in some cases, both. Figure 3.3 shows how pupils VC12 and VG8 emphasize the vertical axes in all symmetrical objects. These pictorial motifs, common to many pupils in all classes, suggest that, for them, symmetry is a property of objects that relates almost exclusively to bilateral or reflexive symmetry.

Reflection in action confirmed this general view in the second phase of the lesson, in which each child read aloud his or her post-it note response and stuck it on a previously prepared poster with the word symmetry; we then rearranged all the slips of paper and discussed their conceptual content with the children. This phase served to test the premises for the following activities and to show again that the prevailing notion of symmetry at the end of the first cycle of elementary education is to mirror a figure on a line (e.g., the axis of symmetry), which can be horizontal, vertical, oblique, internal, or external. This definition is in their elementary school textbook (Carai et al., 2016, p. 324), but it can be found in all Italian elementary school textbooks. In Section 4.1, we will discuss in more detail the analysis of the discrepancy between the conceptions of symmetry before and after the teaching-learning sessions.

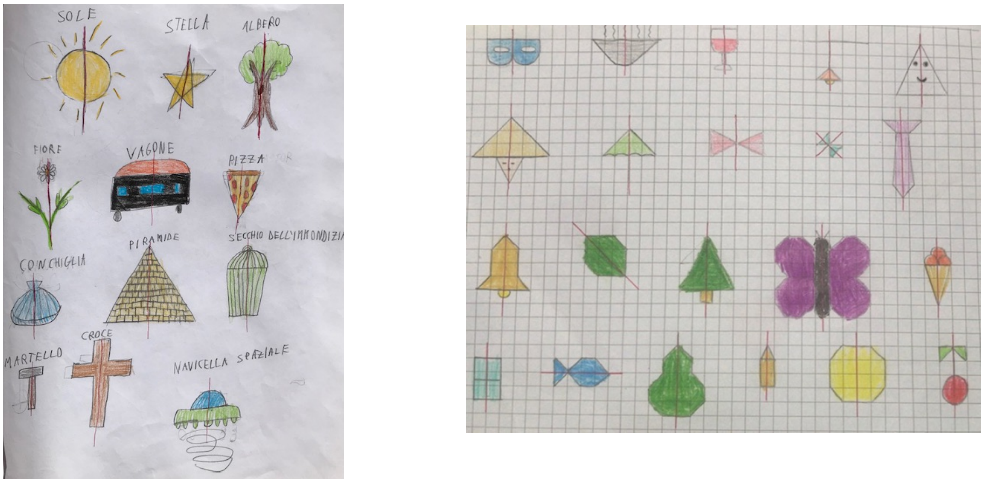


Figure 3.3. The drawings of the pupils VC12 (left) and VG8 (right) show elements of bilateral symmetry with emphasis on the vertical axes.

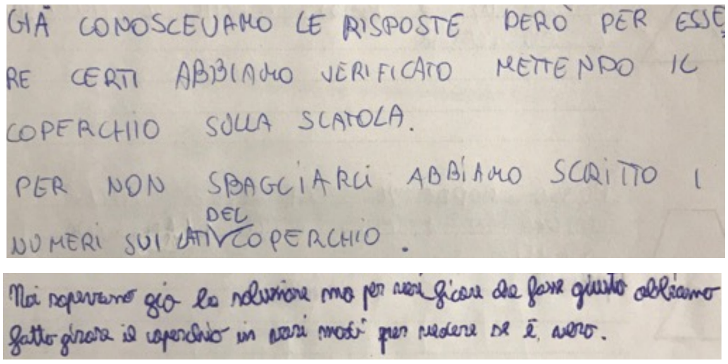
In the second lesson, symmetry is introduced through an operational approach using curious and unusual problems rather than expressions such as "the axis of symmetry" or "being symmetrical". Students were asked to solve the following problem of boxes with lids from the book by Caronni et al. (2007, p. 32). They are asked to help Paolo, who finds that there are not one but two ways to put the lid on a box. Then he finds a strange box with a square bottom: how many ways can he put the lid on the box? Are you sure? How can you find out?

Each pupil works individually by writing down the situation, drawing it, and answering the question. Although most pupils solved the task, some initially had difficulty drawing the box in three dimensions and imagining a box with a square lid. After we showed and concretised the box, the entire class was able to help Paolo. In the second phase of the lesson, we had the students examine the symmetries of an apple as a function of how it was cut. They were excited to see that the same fruit had bilateral or radial symmetry when cut vertically or horizontally. Several students noted that the apple could be turned over about five times because of the fivefold internal arrangement of the seed carpels. The exploration did not end with the apple. We also had them play with palindromes of words or numbers and ambigrams and watch the film *Symmetry, a palindromic film (2013)* (see Section 2.3.5.) and each time they saw symmetries in the short film, they wrote them down on paper.

The main activity A.4 *Strange boxes and lids* was carried out during the next lesson to reinforce the theme of symmetry introduced in the previous activities and actions. It consisted of a task game with cardboard boxes described in Section 2.3.7. The teacher divided the students into small homogeneous groups of 4/5 learners and made them follow the instructions on the cards. The cardboard boxes and paper were provided for each group. The groups had to build the boxes and answer the question on the paper (see Appendix D). Throughout the activity, the children showed great interest, active participation, and a desire to engage with each other to find a common solution in each group. Collaboration was encouraged from the beginning of the cooperative activity through the choice of the group captain and the group name, which had to be original and refer to symmetry, and through the construction of the strange boxes and lids.

Figure 3.4. The pupils, divided into small groups of 4/5 pupils, perform manipulative work by following instructions and completing the task game questionnaire.

Reflection during the activity was encouraged through thought-provoking questions such as “Are you sure?” and “How do you know?” as well as additional space to fill in with further reflection and free thought. The children used different strategies and techniques depending on the problem: some groups chose the graphical solution, while others worked concretely by turning the lids on the boxes and giving the answers.


GIÀ CONOSCEVAMO LE RISPOSTE PERÒ PER ESSE-
RE CERTI ABBIAMO VERIFICATO METTENDO IL
COPERCHIO SULLA SCATOLA.
PER NON SBAGLIARCI ABBIAMO SCRITTO I
NUMERI SUI ~~LATI~~ DEL COPERCHIO.

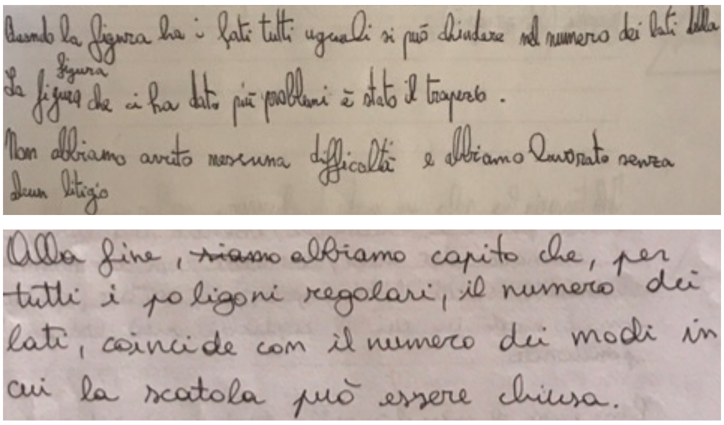
Noi sapevamo già la soluzione ma per verificare che fosse giusto abbiamo
fatto girare il coperchio in vari modi per vedere se è vero.

Figure 3.5. Free thoughts from two groups in classes VG and VB, respectively.

VG above: *We already knew the answers, but to be sure, we checked them by putting the lid on the box. We wrote the numbers on the lid to ensure we did not make a mistake.*

VB below: *We already knew the solution, but to check if it was correct, we turned the lid in different ways to see if it is [was] true.*

All 24 groups gave correct answers to the questions about closing the box, and almost all groups attempted to abstract a relationship between the number of ways to close the box and the sides, axes of symmetries, or angles, especially for regular polygons. Some groups did very well and gave correct reasoning, while many statements needed to be corrected. The correct relationship between the number of sides and the rotational symmetries within the regular polygons was conjectured by some groups, as can be seen in Figure 3.6, a theorem described in note 7 in Section 2.3.6.


Quando la figura ha i lati tutti uguali si può chiudere nel numero dei lati della
figura
La figura che ci ha dato più problemi è stato il trapezio.
Non abbiamo avuto nessuna difficoltà e abbiamo lavorato senza
alcun litigio

Alla fine, ~~riasso~~ abbiamo capito che, per
tutti i poligoni regolari, il numero dei
lati, coincide con il numero dei modi in
cui la scatola può essere chiusa.

Figure 3.6. Correct conjectures from two groups in classes VG and VF, respectively.

VG above: *If the figure has all the same sides, it can be closed with the number of sides. The figure that gave us the most problems was the trapezoid. We had no difficulties and worked without any disputes.*

VF below: *Finally, we found that for all regular polygons, the number of sides equals the number of ways to close the square.*

It is interesting to observe how the children approach the closure of the round jar, where the first encounter with the concept of infinity occurs. All groups answered that there are infinite possibilities for putting the lid on the round jar. Even more interesting are their justifications for this answer because most of them refer to the infinite possibilities as the infinite point of the circumference, the infinite rays, or the infinite axis of symmetry.

The function of the teacher in this phase was essentially that of a facilitator and mediator, especially for lexical questions or children who showed difficulty, and that observer of the relational dynamics in the groups from the outside without directly intervening in the methods and strategies of the children themselves.

After the first hour of problem-solving, the results and the strategies used by the different groups were discussed and shared. This was an important moment to reflect on the theme of symmetry since the exchange allowed the verbalization of the activity and provided the opportunity to share and modify their solutions, whether correct or not. All the concepts and feelings that emerged from the activities were then organised and formalised in the following brainstorming session to arrive at a tentative definition of *symmetry* and *invariant in rotation* that was formulated by the learners and reviewed at the end of the process.

To make the teaching-learning sequence on symmetry and invariance meaningful and applicable inside and outside the classroom, we designed the last activity as a homework and reality task by linking mathematics to real-world problems: *Hunting for Symmetries*, which also served as a self-assessment of what had been learned. The task was to identify at home the elements, such as the boxes, of the symmetries and invariants for rotations and record them with notes, drawings, and photos.

3.2.5 POST-TEST: REFLECTION ON ACTION

After completing the teaching and learning sequence, we administered the post-test survey to all pupils using the same methods and environment as the pre-test. Each class participated in the post-test for one hour on April 15-17, 2019. The pupils knew the teaching and learning sequence would end with the post-test survey. They were also aware of the importance of the final assessment to complete the educational journey on symmetry that had begun with the pre-test. The post-test provided students with a platform of questions to demonstrate their knowledge and ability to apply it in a context relevant to the symmetry learning outcome.

Let us discuss the post-test questions in detail. As mentioned in Section 3.2.3, the post-test differs from the pre-test survey in that it contains an open-ended question. However, the questionnaire is consistent with the development of the teaching-learning progression. Namely, the first question, Q1, refers to the reading of the novel *The Little Prince* in class at the beginning of the lesson to introduce students to the concept of symmetry.

Q1) If you were the "Rose" trying to explain symmetry to the "Little Prince", which of the following sentences would you use?

a) There is symmetry when you can see that, if a line divides a figure in two parts, these parts reflect each other like in a mirror.

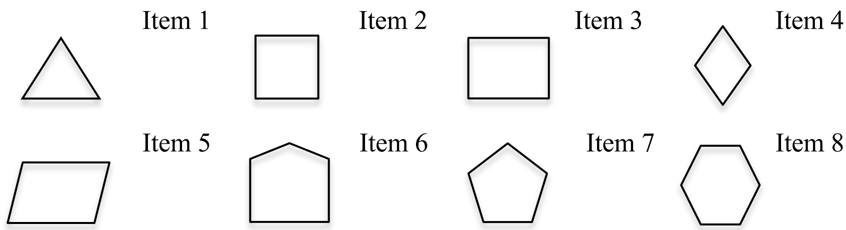
b) Among the many symmetries that exist there are also the movements that transform a figure so that the resulting figure coincides with the original.

Write the reason of your choice.

It is a semi-open question designed to test how the concept of symmetry is perceived in the didactic framework chosen. The question offers a dichotomous choice between two options (a or b) and asks for an open-ended rationale.

The second question, Q2, refers to classroom activities with cardboard boxes. It aims at assessing and measuring theoretical and practical knowledge (i.e., skills) related to cardboard box closing tasks and the specific geometric symmetries of the main polygons. Specifically, the question proposes eight polygons corresponding to the lids' 2D geometric shapes of the cardboard boxes: four regular polygons from triangle to regular hexagon and four irregular polygons, including a rectangle, a parallelogram, a rhombus, and an irregular pentagon.

Q2) After helping Paolo to close various boxes, note the number of ways the lid can be placed on each box and draw the lines of symmetry (axes), if any.



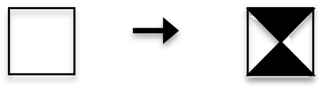
The third question, Q3, is open-ended and has special value compared to the other questions because it leaves much room for reflection on past actions, expression, and inventiveness. It also gives the survey new qualitative elements of stimulation and evaluation that are not explored with the purely quantitative instrument.

Through written verbalization, students undertake to describe their own experiences synthetically and develop them in three main dimensions: D1) the emotional and affective dimension in the description of what they liked most about the course; D2) the cognitive dimension of learning in the description of the acquired knowledge about what they learned; and D3) the possible presence of a cognitive conflict in the processing of information about the difficulties encountered.

> Q3) Tell "The Little Prince" about the classroom activities, what you enjoyed most, what you learned about symmetries, and if you had any difficulties.

The fourth question, Q4, aims at testing the learners' mastery of skills in an exercise of higher cognitive difficulty.

> Q4) The "Rose" explains to "The Little Prince" that there are also colour symmetries. By decorating a figure with colours, you can change its symmetry. Do you agree with this? Explain what happens to the symmetry of the square if you change it from all white to coloured in the following way.



The problematic situation they face in this question is different from the specific demands made in each activity of the learning sequence. Therefore, the question tests the learners' ability to articulate the different components of the acquired knowledge and their ability to analyse and abstract in a new context.

Reflection on action at this stage is the most important part of the reflective cycle for students, teachers, and researchers alike. Student reflection on assessment is a process that provides a rich source of information for self-evaluation and allows them to recall their personal experiences and cognitive thoughts during the teaching-learning sequence. For teachers, reflection on assessment reveals strengths and weaknesses that help them improve their reflection on teaching. Reflecting on classroom teaching, in turn, allows teachers to renew their practice and understand the impact of their teaching (M. Jacobs et al., 2011, p. 60). Such reflection is also critical to action research as part of the final assessment, which incorporates all findings from the reflection cycle into the results. We collected student responses to questionnaires before and after educational treatment, as well as reflections and observations from teacher logbooks. The resulting data collection was analysed using a mixed method and according to the objectives of the research questions. The first objective of the analysis refers to the evaluation of the effectiveness of the questionnaires. The next section, dealing with the evaluation step of the reflection cycle, examines in detail the reliability and validity of the questions used to evaluate the effectiveness of the teaching tool.

3.2.6 EVALUATION: REFLECTION ON FINDINGS

Evaluation is at the heart of action research for the final step and continuously at every stage. Action research must ensure a standard of quality that is directly related to effectiveness for intended outcomes and usefulness of search results to target audiences. This level of quality in action research can be referred to as rigor. According to Mertler (2017, p. 25), rigor typically refers to validity and reliability in quantitative studies and the accuracy of instruments, data, and research findings, as well as accuracy, credibility, and reliability in qualitative studies.

In general, rigor in action research is achieved through qualitative and quantitative data triangulation using a mixed methods approach. The quantitative and qualitative results are compared in order to see if they lead to a convergence of findings from different sources. Moreover, the spiral nature of action research contributes to its rigor, as each phase informs the next steps in a refinement perspective. Our evaluation aims at continuously refining the pedagogical tools and the implementation of the learning pathways in the light of the understanding developed in each previous phase to increase their reliability, validity, and effectiveness for the next step and the entire experimental cycle. Quantitative and qualitative data from all primary sources, such as pupils' written responses to the pre-test and post-test, and secondary sources, such as classroom observations, teacher logbooks, and teacher questionnaires, are equally important for critical evaluation at the end of the first cycle.

The development of pupils' knowledge and skills in terms of symmetry between pre- and post-surveys is analysed and discussed in more detail in the next chapter. Here we consider quantitative data using appropriate statistical psychometric models and qualitative data using text analysis statistics. The multidimensional approach of the empirical study aims at increasing the understanding of the study dimensions and reducing interpretation errors as much as possible, as mentioned already in Section 2.2. Linking the data increases the trustworthiness of the results.

In analysing the quantitative data from the learner survey, we evaluate its main aspects, such as validity and reliability. By validity, we mean the ability of the question to reveal a certain study factor, while by reliability, we mean the ability to measure it correctly. Validity and reliability are not two different concepts but have an important relationship with each other (Mertler & Charles, 2011). In some forms of action research, reliability and validity are achievable when applied with no less rigor (Dick, 2014). In our analysis, reliability refers to the internal consistency or coherence of the test and is derived from some indices such as Cronbach’s alpha or the Kuder-Richardson index KR20. Validity refers to the effectiveness of the items and is derived from the difficulty index, the discrimination index, and the item-total correlation of the set.

We developed a marking scheme and criteria model in Table 3.7 for scoring the second question Q2. Items from one to eight are correlated with geometric shapes, where R1 and R2/R2* are the variables associated with each item. The R1 variable corresponds to the number of ways to close the box; the R2 variable identifies the lines of symmetry and those in R2* to draw them. The score for each item is 1 if the answer is correct for the R1 and R2 variables; for the R2* variable, the score is 0.5 if the drawing is inaccurate.

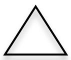
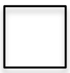
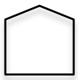
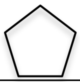
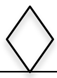
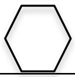
Geometric Shape	**Item**	**R1**	**R2/R2***	**Geometric Shape**	**Item**	**R1**	**R2/R2***
	Item 1	3	3		Item 5	2	0
	Item 2	4	4		Item 6	1	1
	Item 3	2	2		Item 7	5	5
	Item 4	2	2		Item 8	6	6

Table 3.7 Marking scheme and criteria model for Q2 analysis.

We applied the techniques of classical test theory (TCT) and item response theory (IRT) to assess the validity and reliability of the assessment model developed. In classical test theory, Cronbach’s alpha index (α) and the Kuder-Richardson index (RK20) are used to calculate the internal consistency of items with their global variability. Here are the formulas for these indices:

$$\alpha = \frac{n}{n-1}\left(1 - \frac{\sum_i \sigma_i^2}{\sigma_{tot}^2}\right) \quad [1] \quad RK20 = \frac{n}{n-1}\left(1 - \frac{\sum_i x_i(1-x_i)}{\sigma_{tot}^2}\right) \quad [2]$$

where *n* is the number of items, σ_i^2 is the variance of the items, σ_{tot}^2 is the variance of the total score, x_i is the number of correct item responses. The indices can be contained between 0 and 1. The results of the second question of the post-test (see Table 3.8) show good internal consistency for second question, considering that a

Cronbach’s alpha greater than 0.90 for pupil questionnaires might indicate that some items are redundant (Allwood et al., 2018; Streiner, 2003).

Index	R1	R2	R2*		Reference Range	
Cronbach’s Alpha (α)	0.70	0.69	0.73		<0.00 – 0.49>	Unacceptable
					<0.50 – 0.59>	Poor
					<0.60 – 0.69>	Acceptable
Kuder Richardson 20 (RK20)	0.70	0.70	0.73		<0.70 – 0.89>	Good
					<0.90 – 1.00>	Excellent

Table 3.8 RK20, α scores and reference range for R1, R2 and R2* variables.

The effectiveness of the second question is evaluated using the following indices: Facility Index (PI), Difficulty Index (QI), Discrimination Index (DI), and biserial point correlation coefficient (Rp-b). The main assumption is that the question must contain items that provide different levels of increasing difficulty and allow to distinguish the students who answer mostly well from those who are less good so that the different levels become clear.

The P.I. is the ratio between the number of students who gave a correct answer to the item ($\sum_i x_i$) and the number of the group (N=96):

$$PI = \frac{\sum_i x_i}{N} = \frac{\sum_i x_i}{96} \quad [3]$$

The greater the number of correct answers, the easier the item. The acceptance threshold is between 0.25 and 0.75, but it is important to analyse such an index in terms of the purpose of the questionnaire and its minimum objectives. To obtain the DI, we compared the number of correct responses to the item in the best group (the x_{es} pupils with the highest score) with the number of correct responses in the worst group (the x_{ei} pupils with the lowest score). This difference is normalised by the number of pupils in the top and bottom extreme groups (*n*=24 pupils or 25%).

$$DI = \frac{x_{es} - x_{ei}}{n} \quad [4]$$

The point-biserial correlation coefficient (Rp-b) is the difference between the average of the test scores of the students in the best group ($\bar{x}_e$) and the average of the test scores of the entire statistical sample ($\bar{x}_t$), normalised by the standard deviation of all students (σ) and the square root of the ratio of the indices of difficulty and facility:

$$R_{p-b} = \frac{\bar{x}_e - \bar{x}_t}{\sigma}\sqrt{\frac{P.I.}{1-P.I.}} \quad [5]$$

These indices range from -1 to 1. The maximum value of 1 is reached when all the best students (those with the highest total score) answer the question correctly, and there is no answer for the worst students (those with the lowest score). The value -1 is reached in exactly the opposite situation. A value of 0 means that the item is not discriminatory. A positive index greater than 0.20 means that the item is discriminatory enough to distinguish the two groups and measure their competence on the variable in question. Biserial correlation reflects discriminative power independent of difficulty (Crocker & Algina, 1986, p. 319). The following tables are based on the estimation of efficiency parameters for variables R1, R2, and R2*, which are listed in Appendix E.

Index	Avg. R1	SD R1	Avg. R2	SD R2	Avg. R2*.	SD R2*
P.I.	0.74	0.18	0.65	0.25	0.59	0.25
Q.I.	0.26	0.18	0.35	0.25	0.41	0.25
D.I.	0.54	0.28	0.61	0.20	0.64	0.23
$R_{p\text{-}b}$	0.58	0.08	0.55	0.11	0.58	0.15

Table 3.9 Average values and standard deviation of efficiency parameters for variables R1, R2 and R2*.

Reference Range P.I.		Reference Range D.I.		Reference Range $R_{p\text{-}b}$	
<0 – 0.25>	Hard	<0 – 0.19>	Inadequate	<0 – 0.19>	Inadequate
<0.26−0.50>	Medium-Hard	<0.20−0.29>	Poor	<0.20−0.29>	Poor
<0.51−0.75>	Medium-Easy	<0.30−0.39>	Good	<0.30−0.39>	Good
<0.76 – 1>	Easy	<0.40 – 1>	Very good	<0.40 – 1>	Very good

Table 3.10 Reference range of efficiency parameters for variables R1, R2 and R2*.

Based on Dichoso & Joy's (2020) item analysis, question Q2 appears to be easy on average and to have a good discriminant value for all variables studied, considering the age of the pupils and the short duration of the teaching experiment (6 hours). The items, ordered as in Table 3.7, show an increasing index for the variables, which becomes more evident in R1 (see Appendix E). The last two items are indeed moderately difficult (pentagon and hexagon). It is possible to compare the respective difficulty of the questions: R2 turns out to be more difficult than R1, and in R2*, some values are affected by the lack of accuracy of the drawing.

Both qualitative and quantitative data were collected via the third open-ended question Q3 of post-test, which allowed participants to describe their experience with the concept of symmetry during the teaching learning sequence. The study combines text analysis and Natural Language Processing (NLP) statistics to identify and extract information from pupils' responses. The use of text analysis technologies from NLP reconciles the qualitative character of semantic meaning with the quantitative aspect of its relevance and pertinence (Wiedemann, 2016).

Text analysis requires transforming qualitative data into statistical data, making the document a "bag of words" (Chakrabarti & Frye, 2017, p. 1358). According to Mills (2019, p. 37), quantifying the word bag at its simplest level involves translating qualitative textual data into frequencies and creating visualizations of word and topic patterns. This class of analysis methods is called content analysis, a generic term for the statistical analysis of qualitative data. Quantitative and qualitative approaches can be used in content analysis. Different techniques can be used to derive different types of conclusions or results from the content.

Based on pragmatic analysis strategies of narratives through open-ended questions (Rohrer et al., 2017), we used a combination of qualitative and quantitative approaches according to the diagram in Figure 3.7 to obtain the most accurate results and to ensure that all findings collected on the study dimensions were relevant to the research questions. According to Wiedemann (2016, p. 251), combining the extraction of qualitative knowledge from texts to underpin an understanding of social reality with the quantification of the extracted knowledge structures to infer their relevance is inherently a mixed method research design. The next section is devoted to detailed explanations of process evaluation in analysing student narratives about the concept of symmetry in the social context of classroom activities using a mixed-method approach.

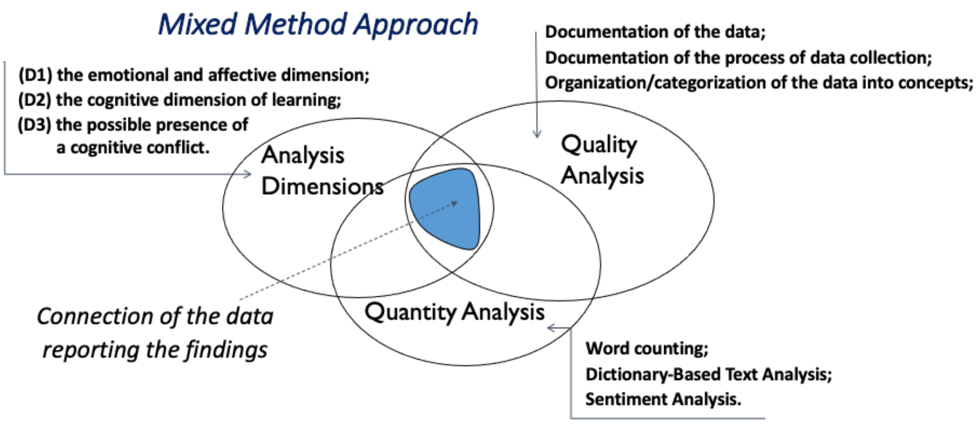


Figure 3.7. Integration of different methods in the work plan for the analysis of Q3 data.

3.2.7 ASSESSMENT PROCESS FOR THE OPEN-ENDED QUESTION Q3.

We used the text analysis tool *Textalyser* (Gregori-Signes & Clavel-Arroitia, 2015) to identify the most relevant lexicographic information in our corpus, which consists of all the pupils' narratives.

Total word count (Tokens, N)	4.452	Total number of characters	26 523
Distinct lexical items (Types, V)	660	Total number of characters w/o spaces	19 967
Lexical Density Factor (LD, V/N) [<20%]	15%	Average number of syllables per word	2.08
Lexical Richness (V_1/V) [<50%]	48%	Sentence count (S)	175
Readability (Gunning-Fog) [6 easy - 20 hard]	13.5	Number of words per sentence (WPS)	27.1
Average frequency (FMG, N/V)	6.7	Max sentence length (words) (MW)	87
Guiraud's Index (V/√N)	10	Min sentence length (words) (LW)	4

Table 3.11 Full report on the lexicometry measurements of the corpus (elab. Textalyser).

The basic lexicometry measures allow us to evaluate the statistical adequacy of the corpus. Based on these results, we listed all the distinct words (i.e., types) in the corpus with their frequency (i.e., the number of occurrences for each word; see Appendix G). We then examined the vocabulary for each frequency and created a table showing the frequency and corresponding position or rank (see Appendix F). The set of individual words (i.e., lexemes or word tokens) determines the lexical size of the vocabulary (V). We label the number of lexemes occurring *i* times as vocabulary V_i. The vocabulary V is the sum of the individual vocabularies V_i, starting with the words that occur only once in the text V_1 (i.e., hapax) and ending with the vocabulary with the maximum number of occurrences V_{fmax}, where *fmax* indicates the maximum frequency:

$$V = V_1 + V_2 + V_3 + \ldots + V_{fmax} = \sum_{i=1}^{fmax} V_i \quad [6]$$

The corpus contains a total number (N) of 4452 words (i.e., types) and 175 sentences with 660 unique forms (V) (i.e., tokens). Lexical density (LD) is the ratio between the width of the vocabulary and the size of the corpus (V/N). It expresses the lexical expansion of the text or the diversity of the vocabulary. The LD index is one of the basic parameters for evaluating the suitability of the corpus, since a language that has a corpus with a large lexical expansion (i.e., a rich vocabulary) should have an equally large number of occurrences. To extrapolate useful information from text analysis, the acceptance threshold of LD should be less than or equal to 20%. In our case, this is a good value since the basic vocabulary represents 15% of the whole corpus, and the average number of words per sentence (AWPS) is

27.1. Another important parameter for evaluating the statistical comprehensibility of texts is the percentage of hapax in the vocabulary volume (V_1/V). This index measures lexical richness, indicating how many graphic forms occur once compared to individual words. The number of hapaxes should not exceed half of the vocabulary. Our corpus meets this requirement, as V_1 contains 320 hapaxes, and the proportion of hapaxes is 48%. Thus, our corpus is sufficiently large for quantitative analysis. Consequently, the lexical volume of the text is considered representative of the language of primary school pupils. In contrast, a vocabulary that is too large compared to the size of the corpus would not make the text a characteristic example of the language (Corral et al., 2015). The parameter readability (Gunning-Fog Index) and average total frequency (FMG, N/V) provide additional corpus information. Readability refers to how easy the text is to read and understand. Since complex words are assumed to contain three or more syllables, readability is based on sentence length and word length in syllables. The average total frequency, in its turn, is the average number with which each graphic form occurs in the vocabulary. This last parameter corresponds to the characteristic values of a corpus created by elementary school students and is acceptable: each lexeme occurs on average between 6 and 7 times in the vocabulary. The readability index has a mean value of 13.5, which means that the corpus is quite difficult to read for the age of the students. Consequently, the corpus does not have very large vocabulary (V) within a relatively large corpus (N), making the text quite repetitive and well-suited for statistical analysis without affecting the reading too much.

Our analysis of the graphic forms highlights the dominant terms. The number of times each word occurs in a text is counted, and the words are then arranged in a table, with the first word being the most frequent, the second word the second most frequent, and so on. Briefly, the word list (see Appendix F) is arranged in decreasing vocabulary order. The rank of a graphic form is the number of its position in the ranking list. In the first place, there is only one graphic form with a repetition of 168 times ($V_{168} = 1$: "*che*" [*that*]) with a relative number of 168 words (tokens) in the corpus, while in the 20th position, two lexemes with a repetition of 45 times ($V_{45} = 2$; "*avuto*" [*had*] and "*anche*" [*also*]) appear with a relative number of 90 (tokens) words in the corpus. The Types Cumulative and Tokens Cumulative columns (see Appendix G) give, for each row, the total number of unique graphic forms and the total number of words in the corpus that occur in all preceding rank classes, including the value itself. In the last position, corresponding to the 55th rank, 320 lexemes (types) appear since the number of tokens is equal to 320. Thus, they occur only once (V_1 = 320) and form the hapax. With this last class, the corpus includes 4542 words (tokens) and 660 unique forms (types), as indicated in Table 3.11.

Figure 3.8 shows the full range of vocabulary that you get when you plot the data as a function of lexical frequencies (i.e., the occurrence of the words). The word distribution is not homogeneous because many words with low frequency correspond to the highest rank positions, and only a few terms with a lower rank persist. On the right side of the diagram, the first positions are occupied by high-frequency words that are not very expressive, i.e., empty terms such as articles, conjunctions, simple prepositions, and so on. They frequently occur in speech because they form the morphosyntactic scaffolding of sentences, but they have nothing to do with content analysis.

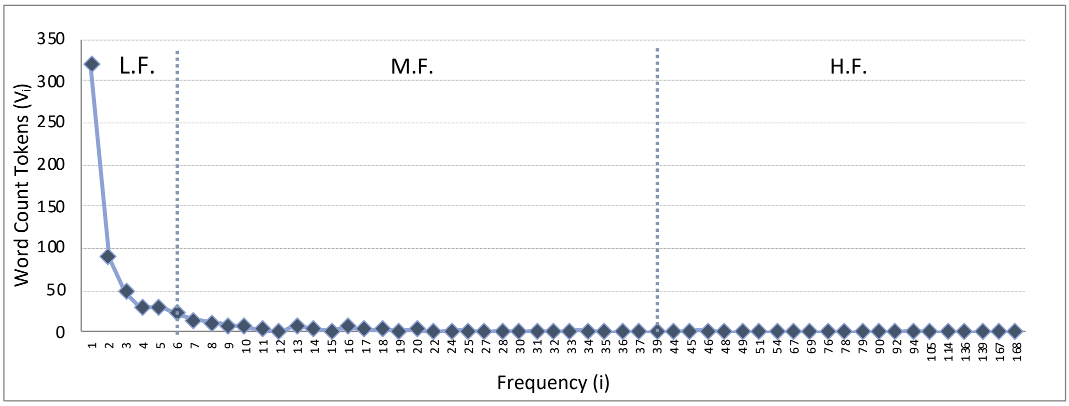


Figure 3.8. The rank-frequency curve is divided into three frequency classes: low frequencies (LF), medium frequencies (MF), and high frequencies (HF).

Immediately after that, in the middle part of the diagram, we find the concentration of essential words related to the topic of the text. On the left side of the spectrum is a scattering of lesser-known terms that have to do with characterizing the text and that constitute a measure of vocabulary richness. The most diffuse different terms form the core of the keywords of the corpus. Identifying the keywords is important to understand the meaning of the content of the pupils' answers. They can highlight the latent structure of the text by reducing the representational space of the linguistic variables. To obtain the list of thematic keywords of the corpus, we determined the frequency threshold below which it is possible to eliminate words without causing a significant loss of corpus information. Thus, we divided the terms into three frequency classes: high frequency (HF), medium frequency (MF), and low frequency (LF). The first pair of types in the rank with the same number of occurrences determines the boundary from the high-frequency band to the middle frequencies (i.e., the upper cut-off point). The first gap in the occurrence represents the boundary from the low to the middle frequencies (i.e., the lower cut-off point). In our case, there are six types in the high-frequency class since two graphical forms with the same number of occurrences (94) occupy the seventh rank. The low-frequency class groups all the graphic forms with a frequency of 20 or less corresponding to a rank of 36 or more (i.e., starting with the terms "*parole*" [words], "*vedere*" [see], and "*palindromo*" [palindrome]), because there is a gap at rank 35 with the occurrence of 21.

Freq. Class	Types (Vi)	Token (Vi)	Types C.	Token C.	% Types	% Token
HF	6	829	660	4542	0.9%	42.7%
MF	38	1775	654	3713	5.8%	39.1%
BF	616	1938	616	1938	93.3%	18.2%

Table 3.12 Summary of types and tokens in the frequency bands (HF, MF, LF).

As shown in Table 3.12, the first six graphic forms in the high-frequency class have a cumulative frequency of 42.7%. In short, they alone account for this percentage of occurrences (tokens) in the corpus, reflecting the trend of the Zpif-Guiraud law that a few words cover half of the occurrences of terms in the corpus (Corral et al., 2015). When we add the medium and low-frequency classes, we get 57.3% of the occurrences. This corresponds to 99% of the vocabulary.

Figure 3.9. Keyword cloud from the MF-LF sub-corpus (elab. Wordle and Textalyser).

Consequently, we consider the union of the two groups, MF and LF, as a sub-corpus to extract the set of keywords. Figure 3.9 above shows the most frequently occurring terms of such a sub-corpus in a word cloud displaying text data in a font directly proportional to their frequency.

We have further refined the list of keywords by grouping the lexical forms corresponding to the different inflections of the same lemma. In addition, we converted all verbs to the infinitive form and each noun to the singular form by using a special POS tagging program (i.e., TreeTagger) developed by the Institute of Computational Linguistics at the College of Stuttgart. The pos-tagging operation consists in assigning to each text term the morphosyntactic category to which it belongs and its specific lemma. See Appendix H for an example of applying pos-tagging to the text of a student VA2 response.

We then reduced the graphic forms based on similarity criteria at the semantic and syntactic levels (i.e., "*film*," "*video*," "*filmato*" [film]; "*professore*," "*maestro*," "*insegnante*" [teacher]). The resulting keyword core, shown in the word cloud in Figure 3.10 below, was extracted from a reduced vocabulary of 616 lexemes (V) and a sub-corpus of 3124 words (N) with an LD index of 19.7% (< 20%), and an FMG of 5.

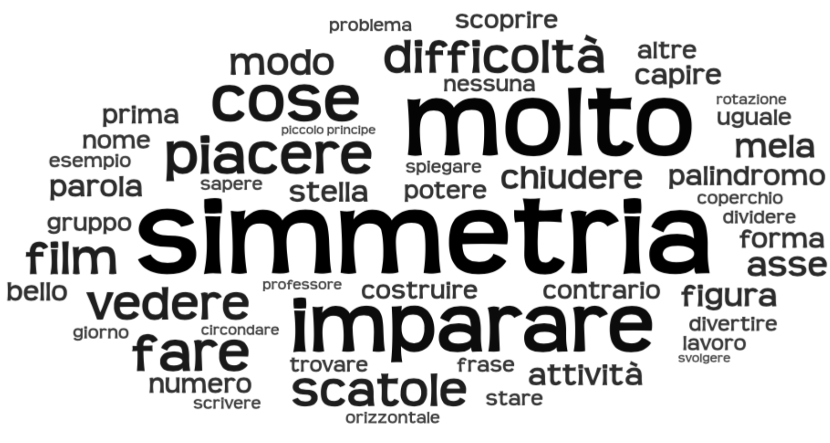


Figure 3.10. Keyword cloud from the reduced sub-corpus MF-LF (elab. Wordle/Textalyser).

Such a selection of the most frequently occurring terms should provide a synthetic representation of their respective semantic references, even if they constitute less than 10% of the vocabulary. These terms form a descriptive structure, not as a summary of the content but as important descriptive indices for the three dimensions of analysis described in Section 3.2.4.

The term "*simmetria*" [symmetry] is the theme word par excellence (122 occ.), followed by "*molto*" [a lot] (76 occ.), "*imparare*" [to learn] (74 occ.), "*piacere*" [pleasure] (73 occ.), "*cosa*" [thing] (60 occ.), "*difficoltà*" [difficulty] (54 occ.), "*scatole*" [boxes] (51 occ.). If the most frequently used words are necessarily a consequence of answering the three dimensions of question Q3, the high presence of the term "*molto*" [very], which conveys a positive attitude, is less predictable. In contrast, the word "*cosa*" [thing] is not relevant, being a common word misused as a substitute for proper terms, especially by elementary school students.

To examine the specifics related to the central theme of symmetry, we highlight the presence of other critical keywords, such as "*film*" (33 occ.), "*palindromo*" [palindrome] (32 occ.), "*parola*" [word] (27 occ.), "*figura*" [figure] (25 occ.), "*mela*" [apple] (17 occ.), "*stella*" [star] (14 occ.), "*numero*" [number] (13 occ.). These graphic forms represent essential traces of the activities carried out in the classroom, especially in relation to the main activity A2 "*Boxes and Lids*". They define the wide range of semantic features revolving around the symmetry argument, even better framed by the most frequent verbs "*vedere*" [to see] (49 occ.), "*fare*" [to do] (47 occ.), "*potere*" [can] (29 occ.), "*chiudere*" [to close] (25 occ.), "*capire*" [to understand] (24 occ.), "*scoprire*" [to discover] (23 occ.), "*costruire*" [to build] (21 occ.), "*leggere*" [to read] (15 occ.), "*divertire*" [to enjoy] (16 occ.), "*tagliare*" [to cut] (13 occ.). To gain a semantic perspective on the preceding keywords considering the different levels of interaction with the learning pathway context, we compared them to teacher observation reports and logbooks.

As noted earlier at the beginning of the section, triangulating methods and data can offset potential biases resulting from using a single method or data source in a study and increase the validity and reliability of action research (Bryman, 2004, p. 73). According to Dawadi et al. (2021), results from one method may be based on or evolve from another. Thus, the validity of the analysis can be assessed by comparing the results of different methods and testing for convergence. See Appendix I for an excerpt from the classroom teacher's protocol VC, which includes at least ten terms (*symmetry, numbers, words, phrases, palindromes, discover,* etc.) that correspond to the keywords identified in the word cloud. The relationship between the keywords and the context turns out to be the relevant meaning markers perceived by the whole community of participants in the didactic experiment on symmetry.

The study of word associations has the potential to specify the relationships between keywords and semantic structures. We have examined the proximity of two or more graphic forms occurring in similar contexts (i.e., co-occurrence) to quantify semantic relations (Lafon & Salem, 1983, p. 162; Rozeva & Zerkova, 2017). Some approaches use the co-occurrence of words to identify and generate semantic themes of the text, confirming further that it is useful for numerous applications in text analysis (Bullinaria & Levy, 2007; Bourgeois et al., 2015; Wang et al., 2017). The co-occurrence analysis is intended to capture the most descriptive features of text meaning and focus the study on reduced dimensions. The result is a group of related, co-occurring terms that suggest an overall meaning. Furthermore, significant co-occurrence of topics in segmented subcollections can be visualised as a network diagram to show global thematic structures (Wiedemann, 2016, p. 123).

We used the TreeCloud application, which displays additional information compared to "classical" word clouds. Classical word clouds only consider the frequency of words to reflect their importance in the text by varying the font size or colour, whereas TreeCloud arranges the most frequent words in a tree that reflects their proximity in the text and shows their semantic proximity to the text (Gambette & Véronis, 2010; Amstuz & Gambette, 2010). TreeCloud uses a neighbourhood linkage algorithm (Saitou & Nei, 1987) that creates a matrix based on the proximity measures between the most frequent keywords and is visualised by SplitsTree (Huson & Bryant, 2006). The tree diagram in Figure 3.11 helps to identify the main themes of the corpus and consequently to track the dimensions developed in the pupils' responses.

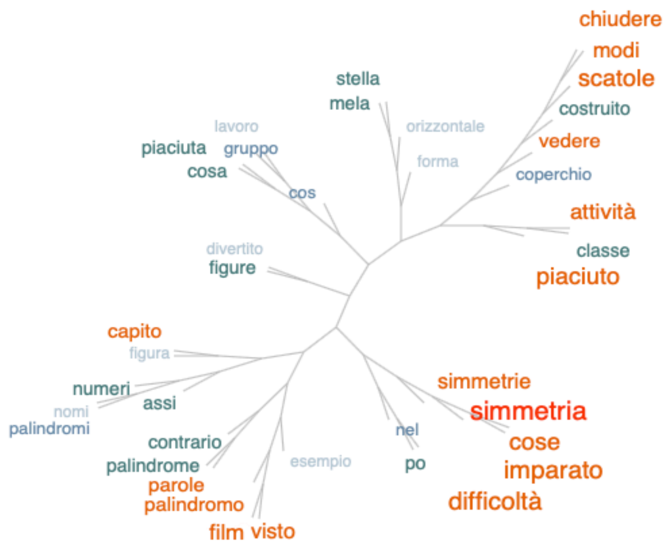


Figure 3.11. Reduced frequency tree cloud MF-LF sub-corpus (elab. TreeCloud).

We can observe how the tree structure has expanded to four significant branches as words gather in specific cores. Thus, they form classifications of similarities with the specifics of the educational path and clarify its structure. We distinguish the word group for the main activity (top right) from the core for the secondary activities (bottom left). The identification of word groups allows better understanding of the information contained in the corpus.

The lexical analysis techniques become even more effective when they reveal other textual regularities, such as repeated groups of words with a precise semantic meaning (i.e., repeated segments). Repeated segments are statistical units with more precise meanings than individual words. Thus, they are particularly useful for highlighting syntagmatic connections between graphic forms, as they have clearer meanings than the same words taken individually. Repeated text segments with three graphic forms (i.e., 3-grams or trigrams) are very connotative for content extraction. The cloud in Figure 3.12 shows the full list of 48 segments extracted from the corpus using the Textalyser application with a font directly proportional to their frequency.

Figure 3.12. The trigram cloud extracted from the corpus (elab. Wordle and Textalyser).

We introduced a textual significance index (IS) for trigrams to determine their effectiveness. For each segment, the IS index is calculated based on the ratio between the occurrences of the segment ($f_{seg.}$) and the frequencies (f_i) of each graphic form q in the segment (i.e., the degree of absorption of the segment to the words that compose it) and multiplied by the number of significant words (P) according to the following formula:

$$IS = \sum_{i=1}^{q} \frac{f_{seg}}{f_i} \times P \quad [7]$$

Then we get the relative significance index ISR comparing IS index to its maximum value q^2. Thus, the relative significance index ISR ranges from 0 to 1.

$$0 \leq ISR = \frac{IS}{q^2} \leq 1 \quad [8]$$

The most frequent segment is "*mi è piaciuto*" [is liked by me], with 59 occurrences composed of "*mi*" [by me] (94 occ.), "è" [is] (148 occ.) and "*piaciuto*" [liked] (73 occ.); hence the significance indices:

$$IS = \left[\frac{59}{94} + \frac{59}{148} + \frac{59}{73}\right] \times 2 = 3.7 \ ; \ ISR = \frac{3{,}7}{4} = 0.9 \quad [9]$$

In this case, the ISR index is 0.9 because the maximum value q^2 is 4. The ISR index for the "*non ho avuto*" [I did not have] segment with 28 occurrences is 0.6. From calculating the degree of absorption of every graphic form in the segment, it is possible to obtain an index of the repeated segment's relevance in the corpus. The stronger the degree of absorption of the component words is, the more relevant the segment is. When the segment is less relevant, ISR tends to zero. If the words occur only within the segment and not on other occasions in the text, then the segment becomes very relevant, and the ISR indicator tends to be one. Thus, the IS index calculation identifies which trigrams can be used to construct the generative processes of latent meaning within the text (see Appendix J).

The trigram study was combined with a concordance analysis to determine the different uses and meanings of the words in the segments. Comparing such results and combining them into a more comprehensive picture is beneficial. Full words and meaningful repeated segments were classified into thematic categories.

The Word Tree application allows the selected word to be placed as a pivot in a dynamic graphical format of phrases in context (i.e., words that follow or precede it). All sequences are navigable to highlight their semantic context. The example in Figure 3.13 below shows the frames of words or segments associated with the keyword "*film*". Once the concept or significant expression is identified, we code all fragments in which that expression occurs. We then analyse the contexts throughout the corpus by adding special characters at the end of the graphic forms. This way, we obtain a complete list of similar graphic forms denoting the same topic or theme. Such a topic-based study provides a model for validating the analysis of repetitive segments and for quantifying the distribution of semantic cores in the corpus.

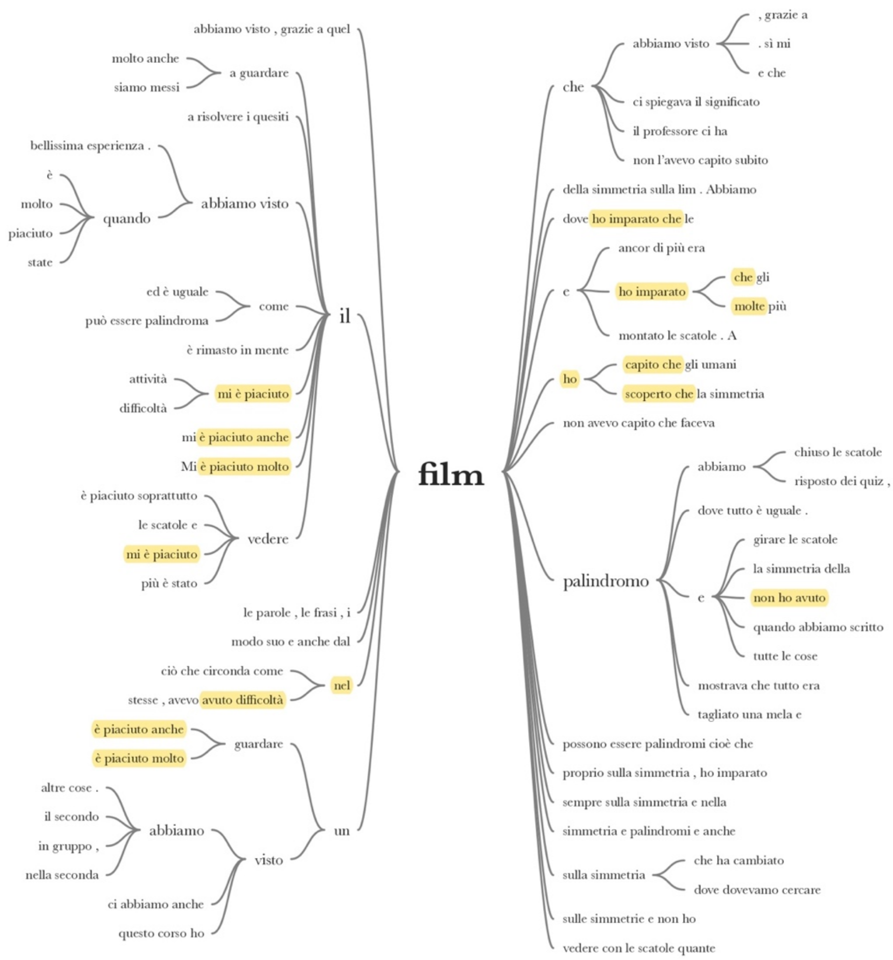


Figure 3.13. Concordance analysis of the pivotal word "*film*" (elab. Word Tree).

The common underlying assumption is that the text has a latent semantic structure that can be inferred from the distribution of words. The topic-based approach uses basic regularity and pattern extraction tools to automatically or semi-automatically organise, search, and analyse word data collections. Text similarity methods, in their turn, extract and create knowledge directly from the text. We have identified the predominant trigrams that provide precise clues to the dimensions of the analysis (D1, D2, D3).

(D1) Emotional D.	**ISR**	**(D2) Cognitive D.**	**ISR**	**(D3) Conflict D.**	**ISR**
mi è piaciuto [I liked it]	0.92	ho imparato che [I learned that]	0.62	non ho avuto [I did not have]	0.70
è piaciuto molto [I loved it]	0.36	in quanti modi [in how many ways]	0.48	ho avuto difficoltà [I had difficulty]	0.83
mi sono divertito [I had fun]	0.68	imparato molte cose [learned things]	0.48	po' di difficoltà [little bit of difficulty]	0.31
mi sono piaciute [I liked them]	0.59	abbiamo scoperto che [we found that]	0.50	avuto nessuna difficoltà [had no difficulty]	0.30
piaciuta di più [I liked most]	0.36	ho capito che [I understood that]	0.47	ho avuto nessuna [had not any]	0.26

Table 3.13 Summary of the prevailing trigrams for dimensions D1, D2, D3.

To this end, we extracted a list of 15 segments in Table 3.13 from the text corpus using the Textalyser application. We then identified the three corresponding dimensions in the text corpus by measuring the absolute and relative frequencies of the segments with verbal labels. Figure 3.14 shows the analysis results of dimensions D1, D2, and D3 diffusion on students' responses to open-ended question Q2. The emotional component D1 is 77% (SD 16%). These indices show that the classes had positive but not equal emotional and affective attitudes. The most positively involved class is VG (100%), and the least involved are VA and VF (63%). The cognitive dimension of D2 learning is correct and relatively constant across classes, with an average percent index of 71% (SD 6%). The cognitive conflict dimension is 55% (SD 14%). The distribution of cognitive conflict shows that 10% of the students had difficulty, 22% had some difficulty, 23% had no difficulty, and 45% had no mention of difficulty.

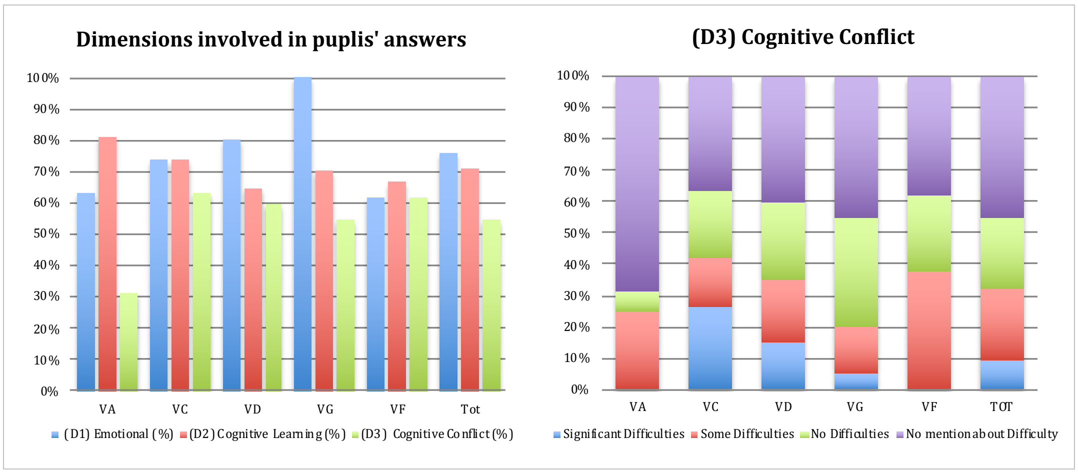


Figure 3.14. Bar charts of the distribution of dimensions D1, D2, D3 (left)
with the detail of the distribution of cognitive conflicts (right).

The narratives were then analysed using natural language processing (NLP) techniques to assess the emotional and affective dimensions further. NLP is a field of artificial intelligence, linguistics and computer science that encompasses a set of technologies that enable computers to process text written in natural language adequately, i.e., the language used by humans daily (White & Cambria, 2014). The statistical text analysis used is the sentiment analysis (SA), which consists of a set of NLP techniques developed to extract information from texts about the attitudes (i.e., sentiment) associated with them. There are different attitudes: judgment or evaluation of the topics and the pupils' emotional response. The basic task of SA is to classify the overall polarity of the answer (i.e., positive, negative, or neutral value related to the topic in question). This analysis was performed using application programming interface (API) library functions provided by free software Meaning Cloud, which use machine learning to explore the structure and content of individual responses. Words are converted into sequences of structured data (i.e., segments) that are assigned emotional response values ranging from -1 (extremely negative, [N+]) to +1(extremely positive, [P+]). The neutrality range is from - 0.5 to + 0.5 ([N]), and the value None means no emotional response. Table 3.14 shows the sentiment analysis of three sentences with their polarity score, confidence level (CL), and agreement marker (AM). The CL index indicates the confidence associated with the sentiment analysis of the text, ranging from 0 to 100. The AM index indicates the agreement between the sentiments recognised in the sentence or segments. The two possible values indicate the agreement or disagreement between the different polarities of the elements.

Level	Text	Score	AM/CL
Sentence	*"Questa attività mi è piaciuto moltissimo perché ho scoperto e imparato cose che non avrei mai immaginato."* [I enjoyed this activity very much because I discovered and learned things I never imagined]	P	Agreement/ 100%
Segment	"*Questa attività mi è piaciuto moltissimo*" [I really enjoyed this activity]	P	Agreement/ 100%
Segment	*"ho scoperto"* [I discovered]	NONE	Agreement/ 100%
Segment	"*imparato cose che non avrei mai immaginato.*" [learned things I never imagined]	P	Agreement/ 100%
Sentence	*"Sulla matematica ci sono molte cose meravigliose e sorprendenti."* [There are many wonderful and surprising things about mathematics.]	P+	Agreement/ 98%
Segment	"*Sulla matematica ci sono molte cose meravigliose e sorprendenti*" [There are many wonderful and surprising things about mathematics.]	P+	Agreement/ 98%
Sentence	"*Io ho passato un momento difficile cioè le simmetrie del rettangolo."* [I went through difficult time, that is the symmetries of the rectangle.]	N	Agreement/ 100%
Segment	"*Io ho passato un momento difficile*" [I went through difficult time]	N	Agreement/ 100%
Segment	"*le simmetrie del rettangolo*" [the symmetries of the rectangle]	NONE	Agreement/ 100%

Table 3.14 Sentiment Analysis Excerpt from VA2's response (elab. Meaning Cloud).

Although sentiment analysis now uses more precise methods with word embedding to represent objectivity and subjectivity, procedures for evaluating the results are needed to ensure their accuracy or at least give us a measure of their objectivity (Liu, 2010; Feldman, 2013; Lee et al., 2021). The best way to validate the results would be to compare them with the results of different methods, e.g., semi-automated techniques and supervised models (Sokhin & Butakov, 2018).

The human factor is allowed in the evaluation of sentiment analyses, which leads to a key component of the validation process. Our mixed-method study allows us to direct question Q3 toward objectivity and reduce interpretation errors.

It is possible to observe a coherence of judgments between the two assessments obtained with different methods. Indeed, it has been shown that the polarity of the attitude is positive with 60% (SD 5%) (tab. 3.15), comparable to the D.1 index in Figure 3.14.

Sentiment	Count	Percentage	Agreement	Disagreement	Sentiment Performance Metrics	
Positive	112	60%	103 (T.P.)	9 (F.P.)	Accuracy	89%
Neutral	15	8%	4	11	Precision	92%
Negative	17	9%	14 (T.N.)	3 (F.N.)	Recall	97%
No Sentiment	42	23%	42	0	F-measure	94%
Total	186	100%	163	23	Accuracy	89%

Table 3.15 Summary of sentiment analysis and performance metrics (elab. Meaning Cloud).

The performance metrics used to evaluate the classification results are accuracy, precision, recognition, and F-measure. The metrics are calculated using positive agreement (TP), positive disagreement (FP), negative agreement (TN), and negative disagreement (FN) scores (Altrabsheh et al., 2014).

Accuracy is a measure that refers to the number of agreements identified relative to the amount of data.

$$Accuracy = \frac{Tot.Agreement}{Tot.Count} \quad [10]$$

Precision is the number of positive agreements out of all positive elements, while Recall is the number of positive agreements out of the actual positive elements.

$$Precision = \frac{TP}{TP + FP} \quad [11] \quad ; \quad Recall = \frac{TP}{TP + FN} \quad [12]$$

F-measure is a weighted method of Precision and Recall, namely their harmonic mean. F-measure values range from 0 to 1. The closer it is to 1, the better the results.

$$F - measure = \frac{2 \times Precision \times Recall}{Precision + Recall} = \frac{2}{\left(\frac{1}{Precision} + \frac{1}{Recall}\right)} \quad [13]$$

The values obtained in processing the data set are relatively good and show that the model provides a fairly accurate characterization of the sentiment in the corpus of student responses.

The bubble plot in Figure 3.15 below shows the distribution of sentiment polarity for each pupil class and the aggregate.

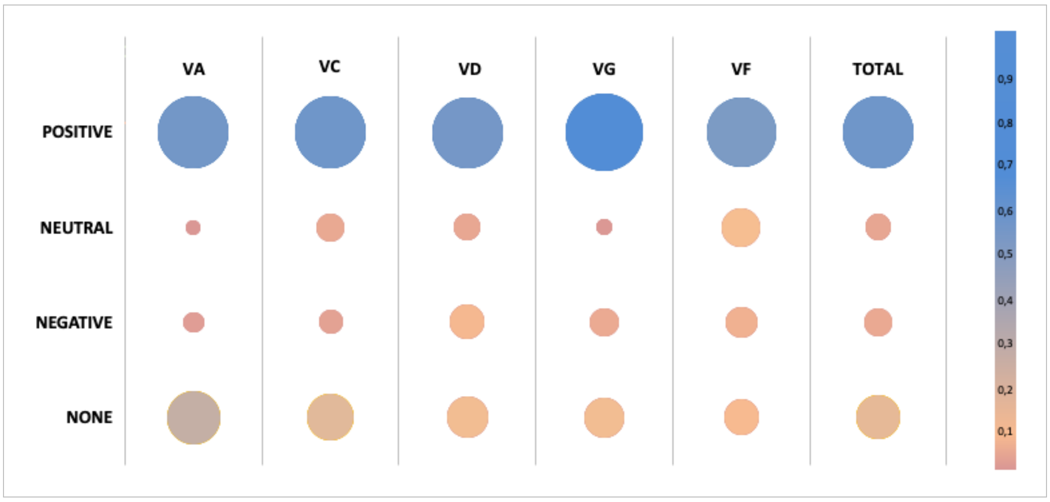


Figure 3.15. Bubble chart showing sentiment polarity distribution (elab. Meaning Cloud).

Each detection of a sentiment polarity is represented by a bubble colour, with the relative frequency in each student class indicated by the bubble size. The large circles in shades of blue represent a stronger presence of polarity, while the small circles in shades of orange indicate limited polarity detection. It can be noted that the prevalence of positive polarity is essentially homogeneous between classes (60%; SD 5%), with the highest value found in the class VG.

This finding is consistent with the results in Figure 3.12, which show that the class VG is the most emotionally involved. There is a correlation between the two estimates obtained by different methods.

3.3 ETHICS

Ethics are paramount when conducting action research as part of an educational experiment with teachers and students. We have ensured that the anonymity of all participants is maintained, and their confidentiality is protected. It is also important to respect the rights and dignity of all individuals involved in the study and to ensure they are not harmed.

We obtained informed consent from all participants (for undergraduates, parents signed the informed consent) and ensured that they fully understood the nature and purpose of the study. In addition, we made the motives and methods transparent and avoided any form of deception or coercion.

Adherence to these ethical guidelines is essential to ensure that educational action research is conducted respectfully and responsibly. Finally, scientific supervision and peer reviews are critical to ensure research is conducted responsibly and respectfully. By following these ethical guidelines, we can ensure that the research findings are valid and trustworthy and that the well-being of all participants is protected.

3.4 SUMMARY

This chapter provides a comprehensive summary of the first cycle experiment developed to teach symmetry and invariance to fifth-grade elementary students. The reflection cycle method used in the experiment is described in detail. It includes an incremental vertical and transversal teaching-learning progression and the use of manipulative games to enhance student motivation and learning.

In addition, the chapter discusses the analysis conducted to ensure the consistency and validity of the questionnaire. It highlights the importance of open-ended questions in questionnaires and recommends the use of text analysis and natural language processing in mixed-methods analysis to explore student responses.

Ethical guidelines are given to ensure the validity and trustworthiness of the experimental design. The following chapter focuses on the results of the questionnaire presented in this chapter.

Chapter 4: Results of First Cycle

This chapter describes the results of the first action research cycle, which focused on the changes in students' understanding of symmetry that resulted from the teaching-learning progression. As mentioned in the previous chapter, the research methodology included the use of questionnaires at the beginning and end of the educational experiment. Sections 4.1 through 4.4 provide an in-depth analysis of the four questions (Q1 through Q4) in the questionnaires before and after the experiment. The chapter concludes with a discussion of the results obtained using a mixed methods approach, and Section 4.5 provides the overall conclusions of the study. In addition, Section 4.6 identifies the limitations of the first cycle of the study.

4.1 FINDINGS CONCERNING THE FIRST QUESTION Q1

To answer the specific research question SRQ1, the data analysis summarised in Figure 4.1 for question Q1 shows that the vast majority (68%) (SD 5%) of pupils in the sample chose the modern concept of symmetry (i.e., answer b). Pupils in the control group also preferred the modern concept of symmetry in the final questionnaire (78%) (SD 5%), whereas they had chosen line symmetry (i.e., answer a) in the first test (83%) (SD 5%).

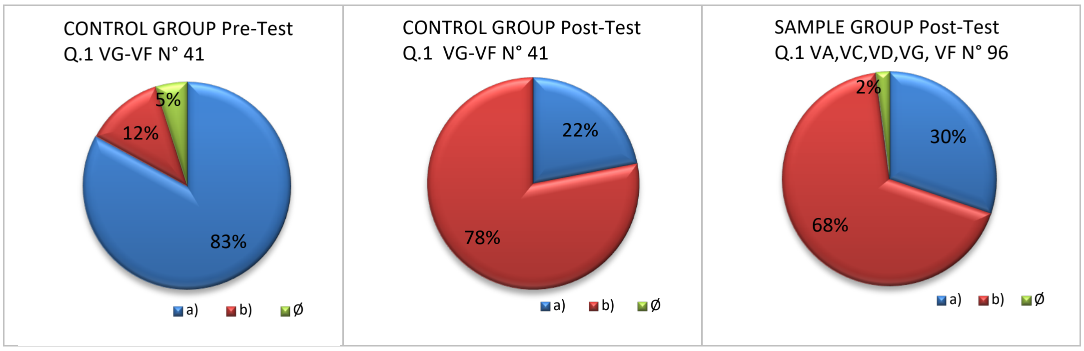


Figure 4.1. Comparison of answers to the question Q1 between pre- and post-test.

Thus, the experiment positively affects changing the notion of symmetry associated with the search for invariance in geometric transformations. However, these results are only meaningful when combined with asking how the students interpreted, experienced, and assimilated the new concept during the teaching-learning unit. To make a qualitative comparison, we provide some relevant examples of pupil responses in VF-VG classes at the beginning and end of the teaching cycle.

Pupil VG17 answered question Q1 on the pre-test:

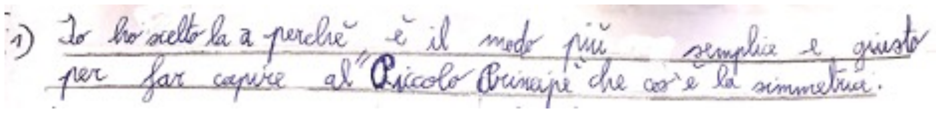

1) Io ho scelto la a perchè è il modo più semplice e giusto per far capire al Piccolo Principe che cos'è la simmetria.

Figure 4.2. "*I chose option (a) because it is the simplest and most correct way to explain symmetry to the Little Prince.*".

In the questionnaire that follows the teaching-learning sequence, he/she justifies his/her choice, which at this stage is (b), as follows:

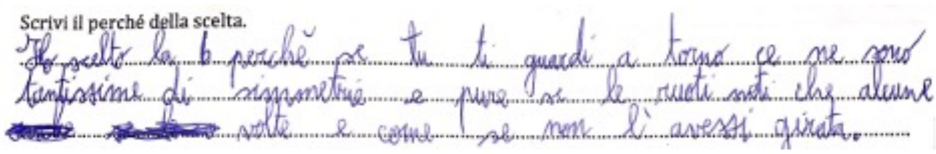

Scrivi il perché della scelta.
Ho scelto la b perchè se tu ti guardi a torno ce ne sono tantissime di simmetrie e pure se le ruoti noti che alcune volte è come se non l'avessi girata.

Figure 4.3. *"I chose option (b) because if you observe your surroundings, you find that there are many symmetries, and even if you turn them over, it sometimes happens that it is as if you had not turned them over."*

Pupil VG19 responds in the pre-test as follows:

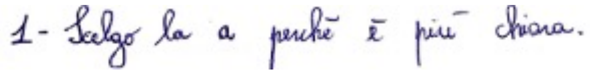

1- Scelgo la a perchè è più chiara.

Figure 4.4. "*I choose (a) because it is clearer than (b*)."

In the post-test, he/she answers:

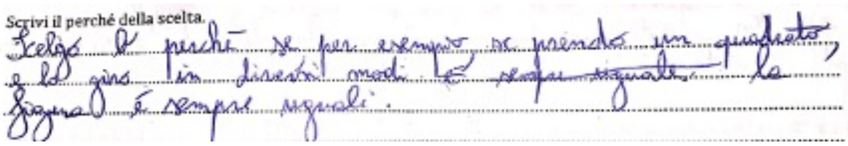

Scrivi il perché della scelta.
Scelgo b perchè se per esempio, se prendo un quadrato, e lo giro in diversi modi ~~le resta uguale~~ la figura è sempre uguale.

Figure 4.5. "*I choose (b) because, for example, if I take a square and rotate it in different ways, the figure always remains the same."*

Pupil VF16 on pre-test:

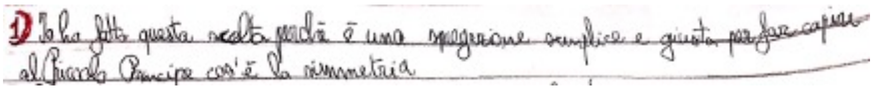

1) Io ho fatto questa scelta perchè è una spiegazione semplice e giusta per far capire al Piccolo Principe cos'è la simmetria

Figure 4.6. *"I chose (a) because it provides a simple and correct explanation that lets the Little Prince understand what symmetry is."*

On the post-test:

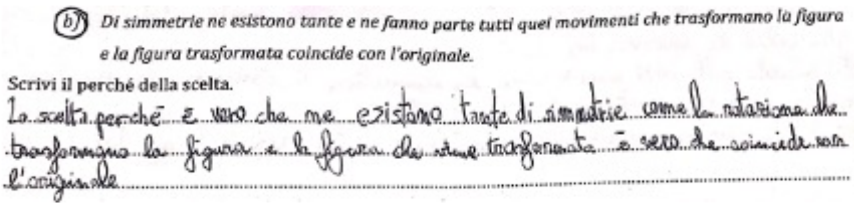

b) *Di simmetrie ne esistono tante e ne fanno parte tutti quei movimenti che trasformano la figura e la figura trasformata coincide con l'originale.*
Scrivi il perché della scelta.
Lo scelto perchè è vero che ne esistono tante di simmetrie come la rotazione che trasformano la figura e la figura che viene trasformata è vero che coincide con l'originale

Figure 4.7. *"I chose (b) because it is true that there are so many symmetries like a rotation that transform a figure, and this transformed figure is really congruent with the original figure."*

To answer the specific research question SRQ2, we have picked out the most frequent keywords, the recurring and most important arguments that appear in the justifications, to divide them into thematic groups according to the objectives of the teaching learning sequence. The results are shown in the following graphs.

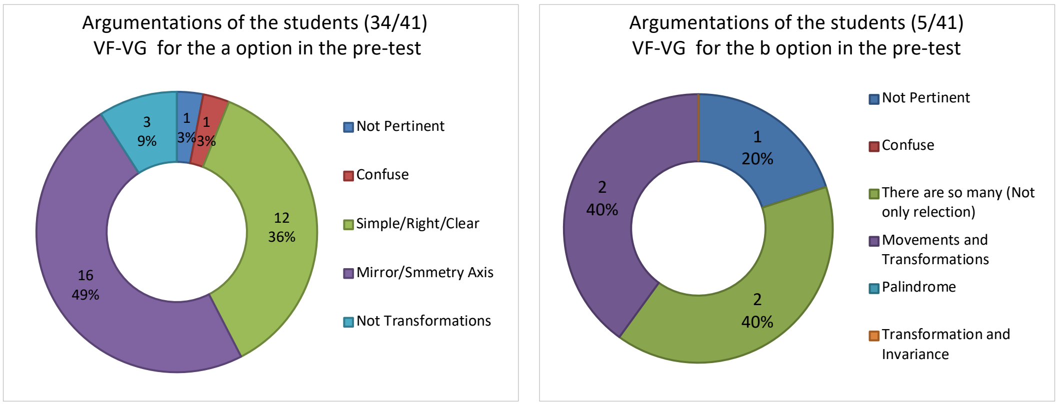


Figure 4.8. Logical reasoning of students in the control group VF-VG in the pre-test.

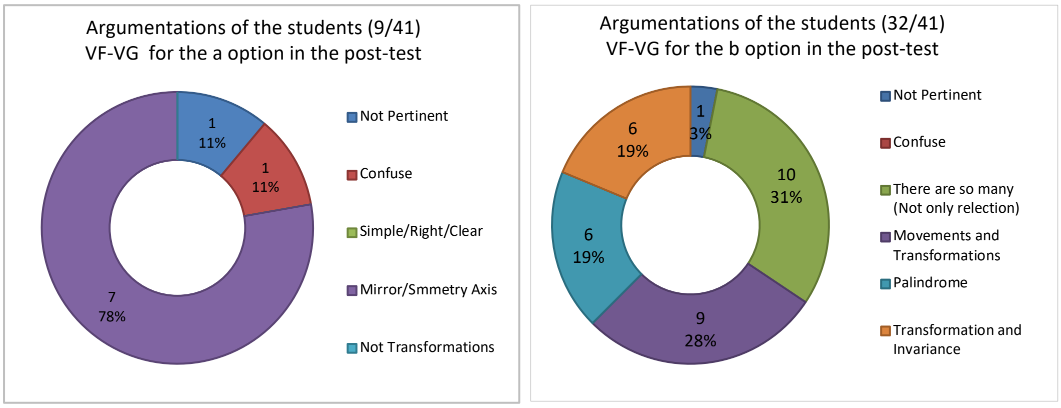


Figure 4.9. Logical reasoning of students in the control group VF-VG in the post-test.

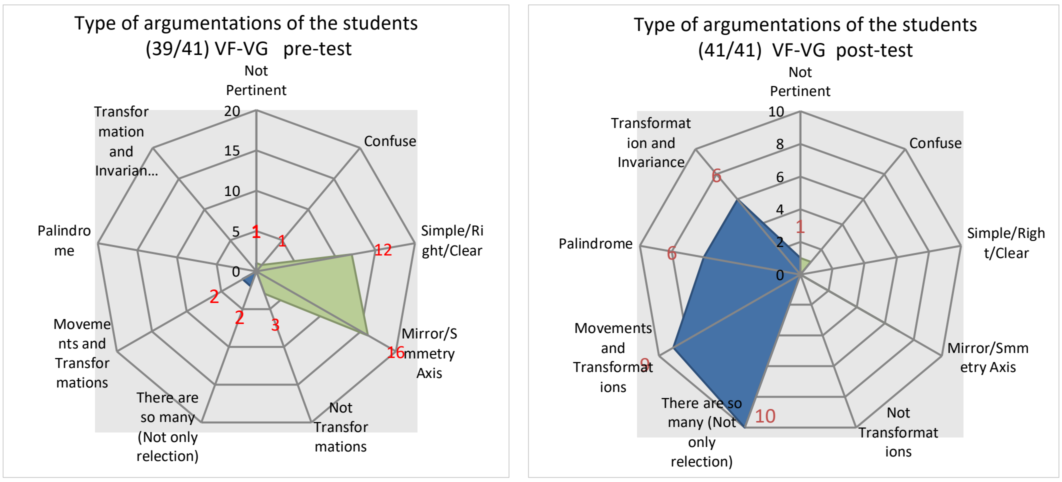


Figure 4.10. The difference in the use of language terms by pupils in
the control group VF-VG between the pre-test and post-test.

In particular, we should note the low percentage of incorrect, confusing, or non-existent responses in the pre-test (12%), as shown in Figure 4.8, and the post-test (7%), as shown in Figure 4.9. The radar plots in Figure 4.10 clearly show the difference between the arguments pupils used when answering question Q1 in the pre-test and the post-test. The expressions "*many symmetries*", "*movements*", "*transformations*", "*palindrome*", and "*coincidence with the original*", which appear in the post-test answers, are almost absent in the pre-test answers. This result suggests that the teaching-learning sequence helped students acquire a new lexicon and that they could cognitively process the new concepts, at least to some degree, and review the concepts they had already acquired. The pupils' ability to reflect on and interpret the knowledge they experienced in the lesson shows that a promising path was taken toward a connection between symmetry and invariance.

4.2 FINDINGS CONCERNING THE SECOND QUESTION Q2

To answer the specific research question SRQ1, we aggregated the data sets of variables R1, R2, and R2* (see Section 3.2.6.) for question Q2 to compare the distributions of the data between the pre-test and post-test control group and to contextualise the results in comparison to the entire sample. We used descriptive statistics, specifically boxplots, to visually represent the data and summarise the characteristics of each variable. Mathematician Tuckey (1977) introduced boxplots to provide a picture of a data set using summary statistics. Boxplots are a powerful way to summarise both continuous and discrete data distributions and allow visual comparisons of centres and dispersions by summarising five indices (i.e., minimum (Min.), lower quartile (q.1), median, upper quartile (q.3), maximum (Max.)).

R1 (N° Pupils)	Tot.	Aver.	Mode	Min.	q.1	Median	q.3	Max.	SD	CV
VF-VG Pre-test (41)	185	4,5	6	0	2	5	6	8	2.49	0.55
VF-VG Pos-test (41)	242	6.2	8	0	5.5	6	8	8	2.28	0.37
Whole Sample (96)	567	5.9	8	0	5	6	7	8	1.85	0.31
R2 (N° Pupils)	**Tot.**	**Aver.**	**Mode**	**Min.**	**q.1**	**Median**	**q.3**	**Max.**	**SD**	**CV**
VF-VG Pre-test (41)	184	4.49	5	1	4	5	5	8	1.75	0.39
VF-VG Pos-test (41)	223	5.44	8	2	3	6	7	8	2.06	0.38
Whole Sample (96)	499	5.20	8	1	4	5	7	8	1.88	0.36
R2* (N° Pupils)	**Tot.**	**Aver.**	**Mode**	**Min.**	**q.1**	**Median**	**q.3**	**Max.**	**SD**	**CV**
VF-VG Pre-test (41)	173.5	4.23	5	1	3	4.5	5	8	1.76	0.42
VF-VG Pos-test (41)	214.5	5.23	8	2	3	6	7	8	2.12	0.41
Whole Sample (96)	475	4.94	5	0.5	3.5	5	6.25	8	1.94	0.39

Table 4.1 Descriptive statistics of aggregate data sets for variables R1, R2, and R2*.

In Table 4.1 above, all these indices are presented with the standard deviation (SD) and coefficient of variation (CV) along the columns. The coefficient of variation is a measure of the dispersion of data and describes the spread of the standard deviation in relation to the mean value:

$$CV = \frac{SD}{Mean\ Value} \quad [13]$$

For R1, the boxplot diagram in Figure 4.11 shows that the average score for the control group increased by 40% between the pre-test and post-test. Variability also decreased somewhat (although maximum and minimum scores remained the same) as the distance between the 25th and 75th percentiles decreased from 4 to 2.5. For the whole sample, the distribution is as for the control group: 50% of the students scored between 5 and 7, with an average score of 5.9. For the R2 and R2* variables, the mean score for the control group increased by 20%, and variability increased as the gap between the 25th and 75th percentiles raised from 2 to 4. For the whole sample, the post-activity situation is similar to the control group: 50% of students scored between 4 and 7 on R2, with a mean of 5.2, and between 3.5 and 6.25 on R2*, with a mean of 4.94.

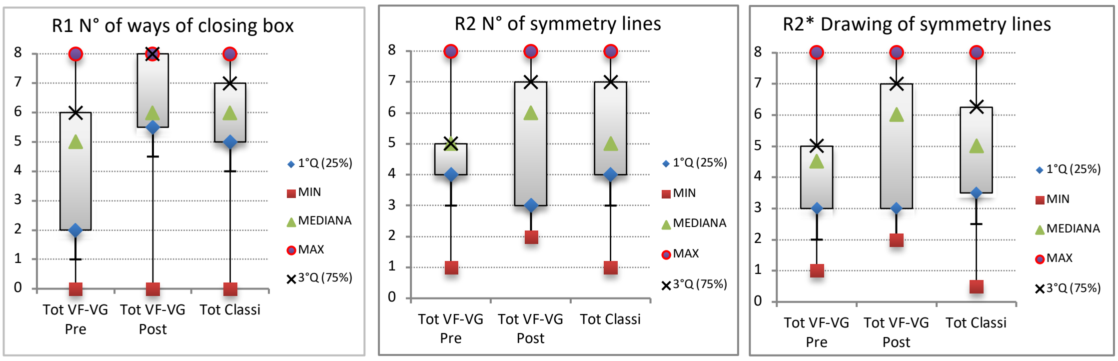
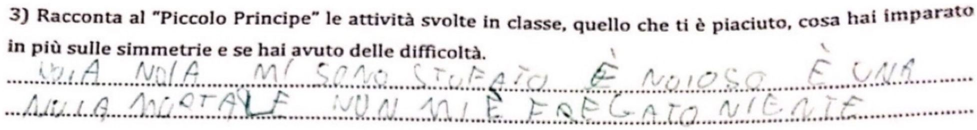


Figure 4.11. Comparison of boxplot data sets for R1, R2 and R2* variables between the pre-test and post-test of the control groups and the entire sample.

4.3 FINDINGS CONCERNING THE THIRD QUESTION Q3

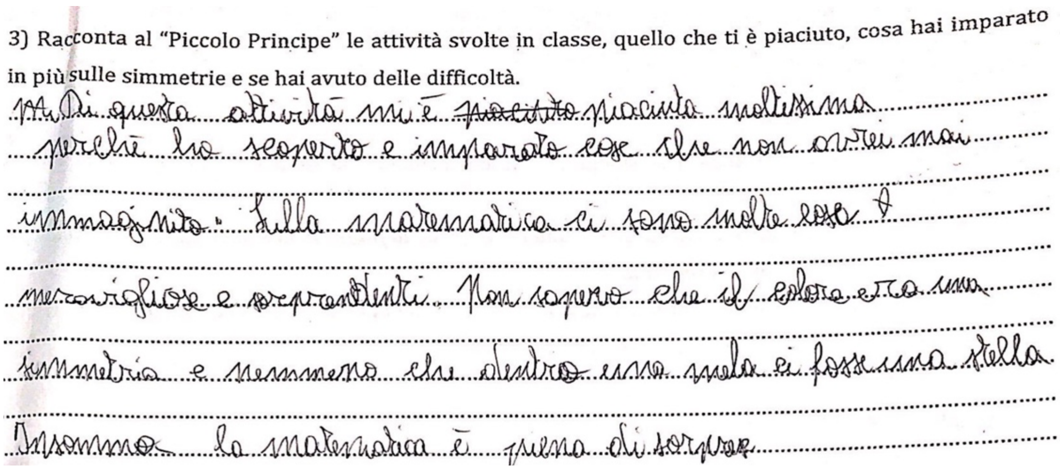
The findings of the detailed analysis in Section 3.2.7 show that the open-ended questions effectively explore pupils' responses to the questionnaire. The study offers some insights into the validity and potential of using text analysis and natural language processing in mixed-method analysis. Qualitative information supplementing the structured analysis responses provides clues and important insights into students' cognitive pathways. Their positive feedback on the teaching learning sequence suggests that the activities associated with the manipulative games motivate and enhance learning about symmetry and mathematics in general. We give some revealing examples of the answers proposed by the students.


3) Racconta al "Piccolo Principe" le attività svolte in classe, quello che ti è piaciuto, cosa hai imparato in più sulle simmetrie e se hai avuto delle difficoltà.

UNA NOIA MI SONO STUFATO È NOIOSO È UNA NOIA MORTALE NON MI È FREGATO NIENTE

Figure 4.12. VD14's answer: "*A bore, I am bored, it is boring, it is a deadly bore, I did not care.*"

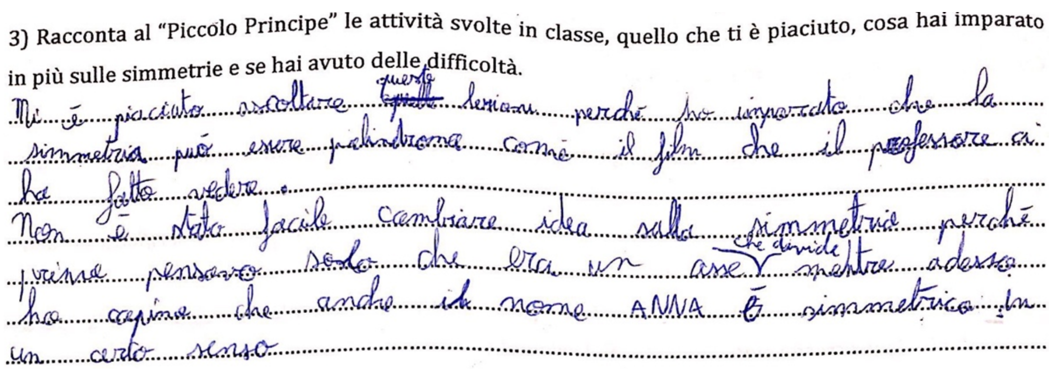
The only case of pupil VD14 in Figure 4.12, who expresses a very negative emotional reaction to the activities, contrasts with many other positive responses, such as that of pupil VA2 (Figure 4.13), who uses simple and sometimes incorrect words to express all his amazement and wonder at the discovery of some of the contents of the activities.


3) Racconta al "Piccolo Principe" le attività svolte in classe, quello che ti è piaciuto, cosa hai imparato in più sulle simmetrie e se hai avuto delle difficoltà.

Di questa attività mi è ~~piaciuto~~ piaciuta moltissimo perché ho scoperto e imparato cose che non avrei mai immaginato. Nella matematica ci sono molte cose meravigliose e sorprendenti. Non sapevo che il colore era una simmetria e nemmeno che dentro una mela ci fosse una stella. Insomma la matematica è piena di sorprese.

Figure 4.13. VA2's answer: "*I really enjoyed this activity because **I discovered and learned things that I never imagined**. There are wonderful and surprising things in mathematics. I did not know **that colour could be symmetry** or **that a star was inside an apple**. In short, **mathematics is full of surprise**.*"

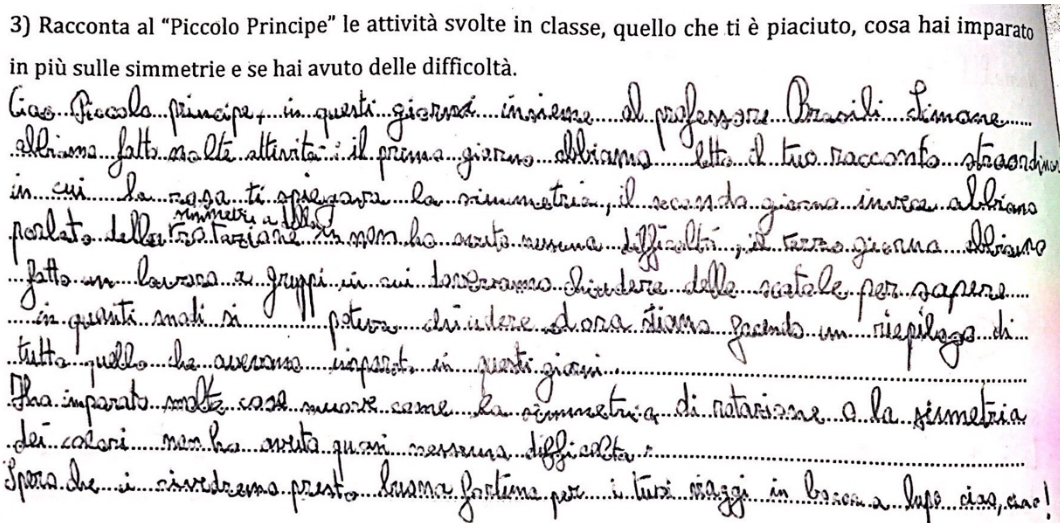
VG11's response in Figure 4.14 is significant because it illustrates the paradigm shift in symmetry.


3) Racconta al "Piccolo Principe" le attività svolte in classe, quello che ti è piaciuto, cosa hai imparato in più sulle simmetrie e se hai avuto delle difficoltà.

Mi è piaciuto ascoltare queste lezioni perché ho imparato che la simmetria può essere palindroma come il film che il professore ci ha fatto vedere.
Non è stato facile cambiare idea sulla simmetria perché prima pensavo solo che era un asse che divide mentre adesso ho capito che anche il nome ANNA è simmetrico in un certo senso.

Figure 4.14. VG11's answer: "*I enjoyed listening to these lessons because I learned that symmetry can also be palindromic, like the movie the professor showed us. **It was not easy to change my mind about symmetry**. Before, I just thought it was an axis that divides, while now **I have realised that even the name Anna is symmetrical** in a way.*"

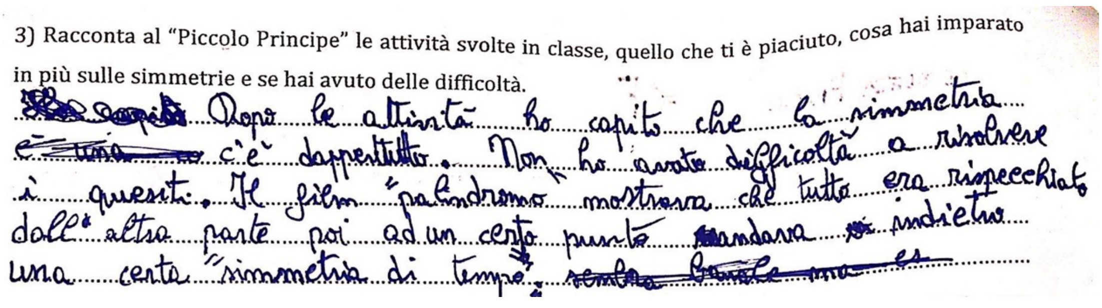
He/she states that he/she can recognize and regulate his/her emotions and listens more attentively. He/she admits that through personal reflection, he/she has deepened his/her knowledge of some aspects of symmetry that he/she had not previously considered and has arrived at a new concept of symmetry. Moreover, VG19's narrative shows that he/she has in mind the entire didactic path of symmetry and invariance covered in class. The response in Figure 4.15 shows personal cognitive work in focusing on symmetry issues and reorganizing and synthesizing the content.


3) Racconta al "Piccolo Principe" le attività svolte in classe, quello che ti è piaciuto, cosa hai imparato in più sulle simmetrie e se hai avuto delle difficoltà.

Ciao Piccolo principe, in questi giorni insieme al professore Brasili Simone abbiamo fatto molte attività: il primo giorno abbiamo letto il tuo racconto straordinario in cui la rosa ti spiegava la simmetria, il secondo giorno invece abbiamo parlato della rotazione non ho avuto nessuna difficoltà, il terzo giorno abbiamo fatto un lavoro a gruppi in cui dovevamo chiudere delle scatole per sapere in quanti modi si poteva chiudere. Adesso stiamo facendo un riepilogo di tutto quello che abbiamo imparato in questi giorni.
Ho imparato molte cose nuove come la simmetria di rotazione e la simmetria dei colori non ho avuto quasi nessuna difficoltà.
Spero che ci rivedremo presto buona fortuna per i tuoi viaggi in bocca al lupo ciao, ciao!

Figure 4.15. VG19's answer: "*Hello little prince, we have done many activities with Professor Brasili Simone these days. On* ***the first day****, we read your extraordinary story in which the* ***rose explained symmetry to you****; on* ***the second day****, we did group work in which we had to* ***close the boxes*** *to find out how many ways there are to close them, and now we are doing a* ***summary of everything we have learned in these days****. I learned a lot of new things, such as* ***rotational symmetries and colour symmetries****, and I had no difficulties. I hope to see you again soon. Good luck on your journey. Good luck! Bye-bye!*"

The ability of some students to structure experiences can sometimes lead to original and unexpected explanations. VF4's response in Figure 4.16 demonstrates the ability to extend the meaning of symmetry as he/she interprets the palindromic feature of the short film "*Palindromic Film*" (see footnote 5 in Section 2.3.5) by mentally connecting it to the argument of a certain symmetry of time.


3) Racconta al "Piccolo Principe" le attività svolte in classe, quello che ti è piaciuto, cosa hai imparato in più sulle simmetrie e se hai avuto delle difficoltà.

Dopo le attività ho capito che la simmetria ~~è una~~ c'è dappertutto. Non ho avuto difficoltà a risolvere i quesiti. Il film "palindromo" mostrava che tutto era rispecchiato dall'altra parte poi ad un certo punto andava indietro una certa "simmetria di tempo".

Figure 4.16. VF4's answer: "*After the activities, I realised that* ***symmetry is everywhere****. I had no difficulty in solving the questions. The movie "Palindrome" showed that* ***everything is mirrored on the other side*** *and then at some point runs backwards,* ***as a certain symmetry of time****.*"

VF4's reasoning in developing the spontaneous concept of symmetry supports the research hypothesis about symmetry and invariance. The teaching-learning sequence aims at understanding the connection between practical operations, the observational aspect, explanation, and intuitions suitable for describing complex situations. When we learn something and that something causes a structural change, we are ready to learn something more complicated in the same domain.

Based on the triangulation of the information obtained from the different evaluation methods, the third question provides data and evidence to answer the specific research question SRQ2. The effectiveness of the pedagogical activities is particularly evident in the positive impact on students' knowledge and skills and the positive and authentic attitude towards a pedagogical process of integration and innovation of the operational and abstract meaning of symmetry in the whole pedagogical context.

This result is consistent with the reflections of the teachers. In the excerpt from the teacher logbook of the class VF-VG (see Appendix K), the teacher states that "*thanks to the explanations and reflections made during the activities with the teacher, she feels more confident and trained*" in terms of content and teaching methods. The project provided broad space for reflection and renegotiation of consolidated pedagogical-didactic practices on the concept of symmetry (e.g., folding papers and mirror images) and new impulses for future pedagogical experiments.

The teacher also reported that "*the project proved to be an excellent, incisive training for her personal experiences.*" These results make it clear that the teaching-learning sequence (TLS) is a promising step toward applying the modern concept of symmetry.

4.4 FINDINGS CONCERNING THE FOURTH QUESTION Q4

The problematic situation that students face in question Q4 is different from the questions asked in the specific activities of the lesson. As mentioned in Section 3.2.5, the question aims at testing the students' ability to articulate the different components of the knowledge they have acquired and their ability to analyse and abstract. The students' answers show that they have better control of their skills. The data in Table 4.2, illustrated in Figure 4.17, show that 55% of the students in the sample (53/96) gave a correct and relevant answer to the question. Of the students in the control group, 27% (11/41) answered the initial questionnaire correctly, while

51% (21/41) answered the final questionnaire correctly, a significant increase of 25%.

Q4 (N° Pupils)	Pertinent	Not Pertinent	Confused	Not Answered
VF-VG Pre-test (41)	11 (27%)	28 (68%)	0 (0%)	2 (5%)
VF-VG Pos-test (41)	21 (51%)	20 (49%)	0 (0%)	0 (0%)
Whole Sample (96)	53 (55%)	36 (38%)	4 (4%)	3 (3%)

Table 4.2. Data from responses to the fourth question Q4.

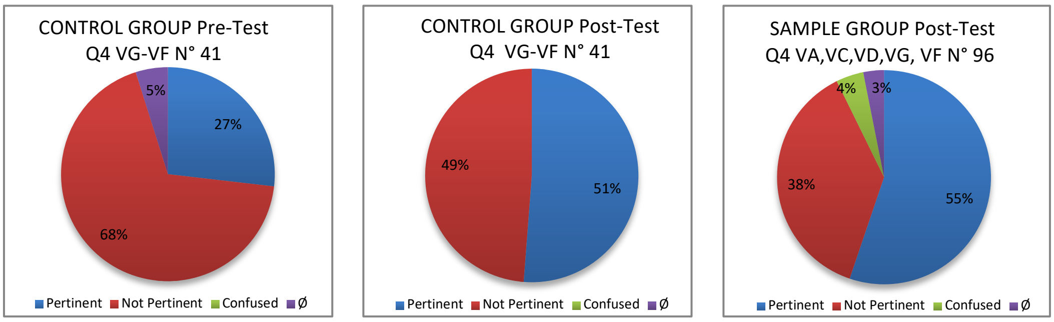


Figure 4.17. The difference in the use of pertinent or not pertinent language by pupils in the control group VF-VG between the pre-test and post-test and all sample groups.

The qualitative analysis shows a different choice of vocabulary in the students' answers. The correct answers in the control group VF-VG, the answers can be divided into three categories in terms of justifications, as can be seen in Figure 4.18: reduction of the number of symmetry axes, reduction of the order of rotation, and other types of correct comments.

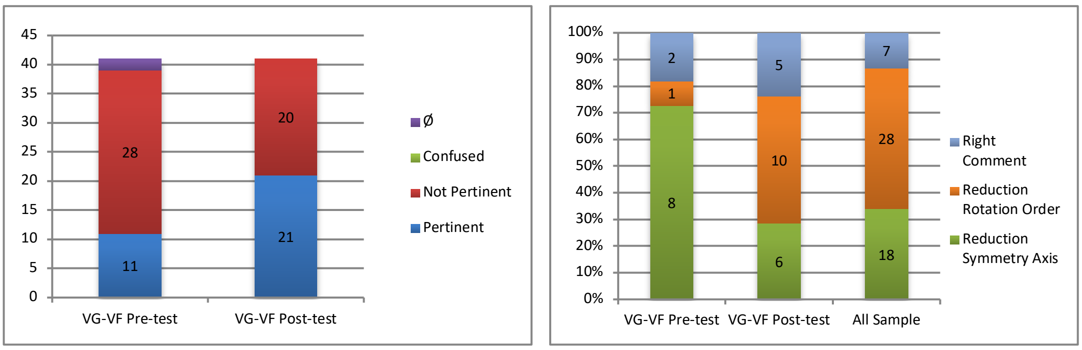
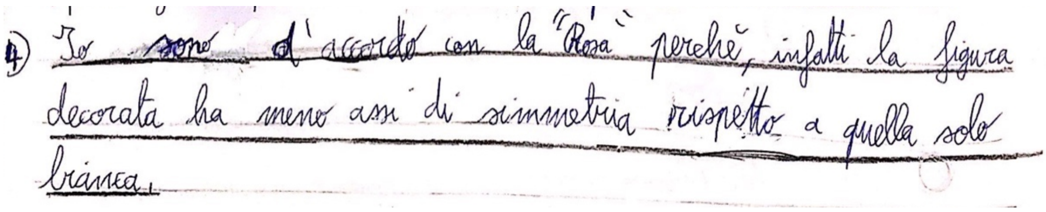


Figure 4.18. The difference in the use of specific justifications by pupils in the control group VF-VG between the pre-test and post-test.

The main rationale for the pre-test is a 70% reduction in the number of symmetry axes and a 48% reduction in the order of rotation for the post-test. For the full sample, the justifications are a 34% (18) reduction in axes of symmetry, a 53% (28) reduction in the order of rotation, and 13% (7) other correct responses.

The following examples show the difference between students' argumentative responses in the post-test and pre-test.

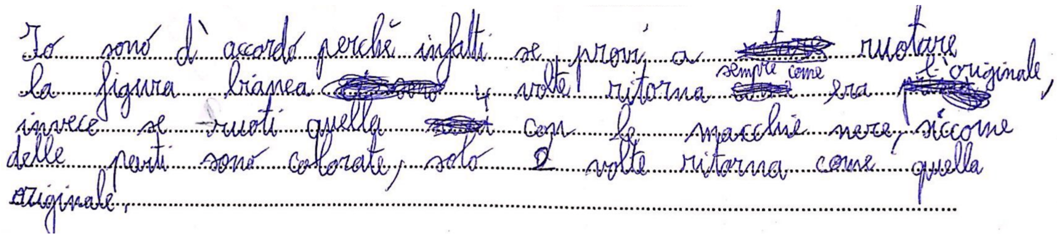
Pupil VG1 gave the following response in the pre-test (Figure 4.19): "*I agree with the "Rose" because it is a fact that the decorated figure has fewer symmetry axes than the one that is just white.*"



4) Io sono d'accordo con la "Rosa" perchè, infatti la figura decorata ha meno assi di simmetria rispetto a quella solo bianca.

Figure 4.19. VG1's pre-test.

In the final questionnaire (Figure 4.20), the same VG1 student justifies his/her answer, "*I agree because it is a fact that if you rotate the white figure four times, it remains the same as before. On the other hand, if you rotate the figure with the black stains, which has some coloured parts, (only) twice, it looks the same as before.*"



Io sono d'accordo perchè infatti se provi a ruotare la figura bianca 4 volte ritorna sempre come era l'originale, invece se ruoti quella con le macchie nere, siccome delle parti sono colorate, solo 2 volte ritorna come quella originale.

Figure 4.20. VG1's post-test.

The case of VG1 shows how this student was able to develop the exercise correctly in both the pre-test and the post-test and to consistently revise the information he/she had when he/she handed in the questionnaire.

Pupil VF2 initially gave a negative answer (Figure 4.21): "*No, the symmetry of the square does not change when the square is coloured. The symmetries are always the same, only the colour changes.*"

No la simmetria del quadrato non cambia se viene colorato le sue simmetrie sono sempre le stesse cambia solo il colore del quadrato.

Figure 4.21. VF2's pre-test.

However, in the post-test, he/she gives a positive answer (Figure 4.22): "*Yes, I agree with Rose. Colour can change the symmetry of figures. If the square is coloured, the symmetry is different. If I turn the square over, it looks different than before because where it was black, it is now white, and vice versa. However, if I turn* *it another time, it returns to the starting point*."

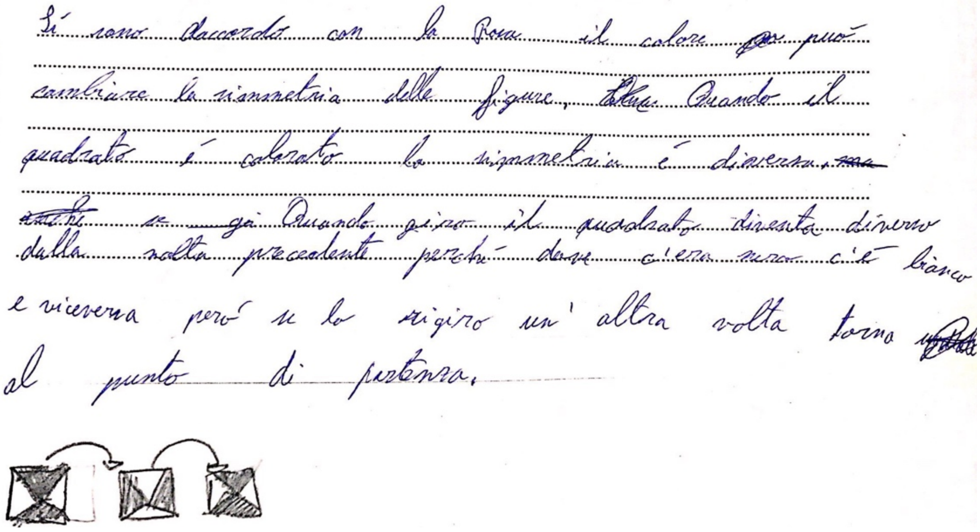

Sì sono daccordo con la Rosa il colore ~~po~~ può cambiare la simmetria delle figure, ~~tutte~~ Quando il quadrato è colorato la simmetria è diversa, ~~ma anche~~ se gi Quando giro il quadrato diventa diverso dalla volta precedente perché dove c'era nero c'è bianco e viceversa però se lo rigiro un' altra volta torno al punto di partenza.

Figure 4.22. VF2's post-test.

Pupil VF4 responded to the pre-test (Figure 4.23): "*Yes, I agree with the "Rose" because you can divide the square into two parts, horizontally and vertically, but not diagonally because the two black triangles face each other. In contrast, the white square can be divided horizontally, vertically, and diagonally.*"

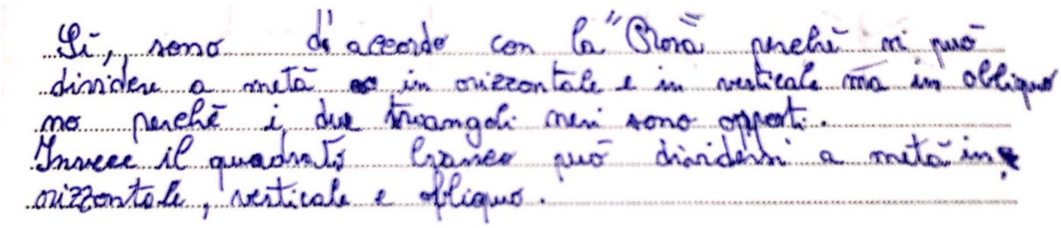

Sì, sono d'accordo con la "Rosa" perché si può dividere a metà in orizzontale e in verticale ma in obliquo no perché i due triangoli neri sono opposti. Invece il quadrato bianco può dividersi a metà in orizzontale, verticale e obliquo.

Figure 4.23. VF4's pre-test.

However, in the post-test (Figure 4.24), he/she answered, "*Yes, I agree with the "Rose" because the square has four axes of symmetry, while there are no diagonal axes of symmetry for the coloured triangles. When I rotate the square, I can see that it resembles the original figure only after two rotations.*"

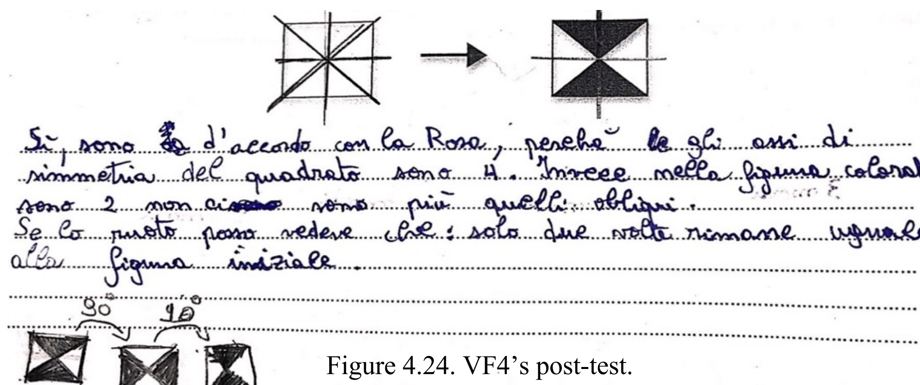

Sì, sono d'accordo con la Rosa, perché ~~le~~ gli assi di simmetria del quadrato sono 4. Invece nella figura colorata sono 2 non ci sono più quelli obliqui. Se lo ruoto posso vedere che: solo due volte rimane uguale alla figura iniziale.

Figure 4.24. VF4's post-test.

In summary, question Q4 provides convergent evidence that the children have grasped the notion of general symmetry in the context of the concept of invariance because they have learned a new method for answering the question. They try to perform an operation on the squares and see if they change. Thus, the concept of modern symmetry, described in Section 2.3.5 as invariance in any transformation, has properties that are intuitively easy for children to learn through the activities of the teaching-learning sequence.

4.5 DISCUSSION ON THE FINDINGS AND CONCLUSIONS

The first action research cycle aims at implementing a teaching-learning sequence focused on symmetry and invariance as a paradigm shift for an effective interdisciplinary teaching approach in schools. Specifically, we investigate how such a paradigm change affects fifth graders' understanding and skill development (SRQ1) and the didactic framework that links mathematics and physics (SRQ2).

Two hypotheses were formulated in relation to the first specific research question (SRQ1) The first states that the paradigm change is familiar and within the acquaintance of fifth-grade students, while the second states that the paradigm shift enhances the pupils' understanding and skill development. Data analysis shows that the paradigm shift of symmetry can be implemented in elementary school for fifth graders, as students were able to grasp the changed concept of symmetry in the context of the search for invariance. The results also provide a good foundation in terms of knowledge and skills appropriate to the age and prior knowledge of the pupils and promote the process of abstraction. It also conveys the basic ideas that symmetry and invariance concepts enable the construction of a more comprehensive scientific architecture.

The challenge to explain phenomena in different descriptions can lead pupils to a more comprehensive learning of scientific and mathematical concepts related to symmetry. Thus, the study of transformations and invariants facilitates pupils' acquisition of cognitive procedures and processes that can be extended to many domains. The experience of manipulative games has been shown to trigger intense spatial thinking activity in addition to the perception of line symmetry associated with traditional materials.

As for the second specific research question (SRQ2), our hypothesis is that there is a significant difference in the didactic framework due to the impact of the paradigm shift. Unlike the first two hypotheses, the third hypothesis is more complex and requires a mix of quantitative and descriptive analysis methods. The overall pedagogical context shows a positive and authentic attitude towards the pedagogical process of integration and innovation of the operational and abstract meaning of symmetry. These results open new space for research in the pedagogical framework of symmetry and geometric transformations.

According to Swoboda (2012), pedagogical paths to transformations in the plane should be based on rotations and not on mirror symmetry, as in many proposals. Our results are consistent with this statement, as rotations led children to perform the most intuitive movements in learning the concept of symmetry. Manipulation with rotations was independent of visual recognition of axis symmetry. Such an approach may strongly influence the intuitive knowledge used in further learning about geometric transformations. As a result, the dynamic concept of rotation may be closer to familiarity than other rigid motions in the plane, giving more weight to the concept of symmetry associated with the search for invariance.

Based on this consideration, it would be interesting to implement combined symmetries of different designs and movements using cardboard boxes with new lids and other teaching materials. For this reason, we plan to continue the second cycle of action research to deepen the results of the present work.

4.6 LIMITATIONS ON THE FIRST CYCLE

The first action research cycle has some limitations that will be further considered in subsequent cycles and future research. The first important limitation relates to the quasi-experimental research design, which does not include random assignment. Using non-random samples may limit the generalizability of the results and compromise the findings. Thus, we included a control group with a before-and-after assessment of outcomes to increase the rigor of the research design. In any case, the first cycle aims at testing the existence of a causality relationship between the teaching experiment and the outcomes.

Following the research diagram in Table 3.6 (Section 3.2.3), we evaluated the effectiveness of the teaching-learning sequence by comparing the results between the post-test and the pre-test with a non-randomly selected control group and then extrapolating the results to the entire sample group. As described in the previous sections, the comparison showed a significant improvement in the results, indicating an advance in attitudes toward the modern concept of symmetry and a difference in students' knowledge and practice in finding invariants. Teacher observations and triangulation of mixed method data analysis provided sufficient evidence that generalization to the entire sample was possible. However, the generalizability of the results still needs to be improved.

Therefore, the second action research cycle aims at establishing a stronger causal link between the classroom experiment and the results, which may positively impact the generalizability potential of the findings.

Chapter 5: Second Cycle Experiment

Based on the findings, considerations, and limitations of the first cycle, we planned and conducted the second action research cycle using the research design of the EMT model for the transition from primary to secondary school. Figure 5.1. below shows the diagram of the research design of the EMT model with the two cycles of experiments. The second cycle of the teaching experiment combined the longitudinal study with a cross-sectional approach to provide more evidence of causality between the T1 teaching-learning sequence and its outcomes.

Measure	**Treatment Cycle 1**	**Measure**
Pre-test	*TLS Experimental*	*Post-test*
O1 (VF-VG)	T1	O2 (VF-VG) [C.G.1]
	T1	O2 (VA-VC-VD)

TRANSITION FROM PRIMARYTO SECONDARY SCHOOL

Measure	**Treatment Cycle 2**	**Measure**
Pre-test	*TLS Experimental*	*Post-test*
O3 (1A,1B,1C,1D,1E)	T2	O4 (1A,1B,1C,1D,1E)
O3 (2B-2E) [C.G.2]	T2	O4 (2B-2E) [C.G.2]

Figure 5.1. Diagram of the overall experimental model transition EMT research design, where O is an observation process and T is an exposure of the group to the treatment.

In this research design, all students in the full first-cycle sample entering first year of secondary school were compared to a new control group (CG2) of students in secondary grades 2B-2E who were one year older and had not participated in the earlier symmetry classes. The effectiveness of T1 was demonstrated by assessing differences between the sample and the control groups on pre-test O3. The post-test O4 should have shown learning gains for the two groups of students to confirm the hypothesis of a positive effect of T1 and T2.

The second action research cycle is generally designed to refine the focus of investigations incorporating further information and data, leading to increasingly effective answers to the research questions that form the core of the research project. The following sections describe the phases of the second experimental action research cycle, which aim at evaluating, reformulating, and revising the materials and actions from the first experiment.

The chapter is organised into several sections, each dealing with a different aspect of the study. These sections include diagnosis (Section 5.1.1), action planning (Section 5.1.2), preparation and follow-up (Section 5.1.3), reflection on actions taken during the Covid-19 pandemic (Section 5.1.4), and evaluation (Section 5.1.5).

In addition, this chapter presents the results of the survey, including a breakdown of responses to each of the four questions (Q1 through Q4) on the questionnaire, which are discussed in Sections 5.1.6 through 5.1.8. The results are analysed and discussed in Section 5.2, and the limitations of the second cycle are presented in Section 5.3.

Overall, the chapter provides a thorough overview of the study, from the initial diagnosis and planning stages to the analysis of the results. The breakdown of the survey results and subsequent discussion of the findings provide an understanding of the impact of the study, while the second cycle limitations help identify opportunities for improvement for future iterations of the research.

5.1 EXPERIMENT 2 – SECOND CYCLE

The second action research cycle consisted of six steps, like the first teaching experiment, but the scope of the cycle was larger. As can be seen in Figure 5.2, following Michelsen's model in Section 2.2, the second reflection cycle was more extensive than the first in terms of the number of lessons, the research sample, and the complexity of the activities.

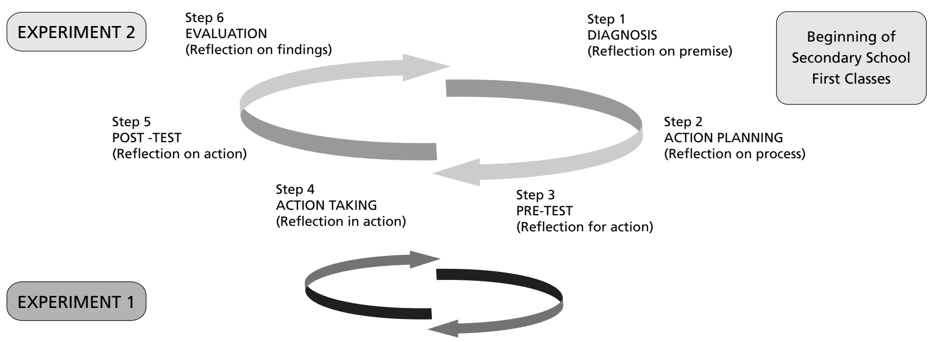


Figure 5.2. Representation of second reflection cycle in Experiment Model Transition.

The group of students was larger than that of the first cycle, as a total of 143 students were included in the sample group. Table 5.1 provides a detailed list of all classes participating in the experiment.

Classes	N° Pupils	Gender	Cycle 1	Pre- Test	Cycle 2	Post-Test
I A	18	8 F – 10 M	Yes	Yes	Yes	Yes
I B	18	8 F – 10 M	Yes	Yes	Yes	Yes
I C	19	6 F – 13 M	Yes	Yes	Yes	Yes
I D	19	8 F – 11 M	Yes	Yes	Yes	Yes
I E	19	6 F – 13 M	Yes	Yes	Yes	Yes
II B	21	7 F – 14 M	No	Yes	Yes	Yes
II E	20	7 F – 13 M	No	Yes	Yes	Yes

Table 5.1. Details of the students and classes in the sample group.

The teacher work group of the second action research cycle was composed of 5 secondary school teachers in Montegranaro as each of them teaches a single course from A to E, and of the same cohort of primary school teachers of the previous experiment, only for the theoretical action research phases, without the classroom implementation. The composition of the research group aimed at ensuring continuity in the transition from primary to secondary school and influencing mutual participation in order to share and increase responsibility for the process. In addition, changes in group composition kept group members attentive and led to shared reflection on the common goal within the action research framework.

The second action research cycle started in November 2019 and should have been completed in June 2019, according to the planning of the 6 phases listed in the table below.

Steps	Nov. 2019	Dec. 2019	Jan 2020	Feb. 2020	March 2020	April 2020	May 2020	June 2020
Diagnosis	4 h.							
Action Planning		3 h.	2 h.					
Pre-Test				7 h.				
Action Taking				20 h	30 h.	20 h.		
Post-Test						7 h.		
Evaluation							4 h.	3 h.

Table 5.2. The schedule of the second experimental cycle in hours (h.) for each phase.

Unfortunately, the ongoing experiment was interrupted by the Covid-19 outbreak, as of March 5, 2020, classes for the 2019/2020 school year had been suspended in early childhood education institutions and schools throughout Italy at all levels, as well as activities for the 2019/2020 academic year in universities. We conducted at least the first three steps, including pre-test O3 and part of the fourth phase, with only three hours per class, as shown in Figure 5.3. Therefore, we were able to consider the results of the pre-test O3 according to the diagram in Figure 5.1

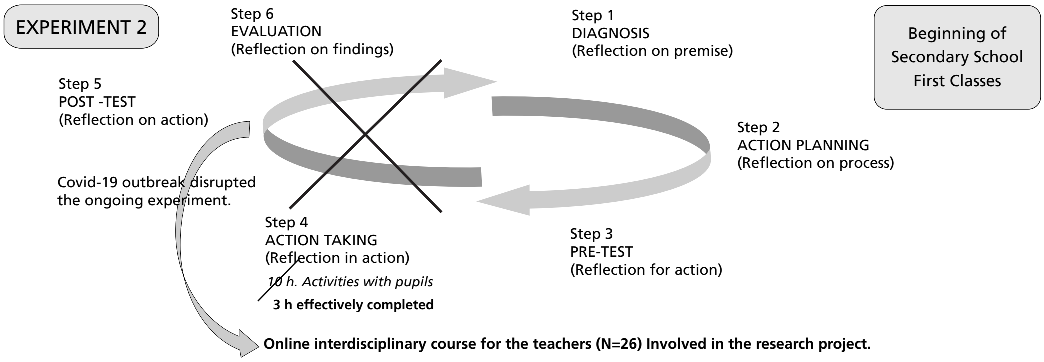


Figure 5.3. Illustration of the second reflection cycle interruption in the Experiment Model Transition due to the Covid-19 outbreak and replacement with an online course for teachers.

Since it was not certain that students could carry out the previously designed activities because the school could be closed for the whole school year, the last steps were transformed into an online course for all teachers involved in the research project. We discarded the possibility of conducting all remaining steps online with the students, as this would have been a different experiment with different conditions and settings that would not have allowed comparison with the results already available. The online course for teachers was considered to be the third experiment in action research that focused on bringing pedagogical and curricular content related to symmetry and invariance into classroom practice. Teachers examined the project's tools and materials for their feasibility and transferability to the didactic scene. Triangulation methods were used to compare the results of the third experiment with those of the previous experiments.

Sections immediately below describe the phases of the second experiment, conducted before the Covid-19 epidemic, and the analysis of the results, while the following chapter presents the third experiment, entirely devoted to the online course for teachers.

5.1.1 DIAGNOSIS: REFLECTION ON PREMISE AND PARTICIPANTS

The first step of the second cycle was the reflection on the premise. It was different and more active than in the first cycle because the reflection on the premise in the second cycle was built on the previous experiences, observations, findings, and reflections of the first experiment. Each step was different because the relative starting points were higher than in the previous steps.

As a result, the research group found several prior pedagogical premises to use as empirical evidence and as a reference source for improving and refining materials, tools, and the sequential teaching-learning progression in the second experiment. As mentioned in the previous section, the study group of the second experiment consisted of 5 secondary school teachers, 134 students, 41 of whom served as the control group, and 4 elementary school teachers exclusively in the first two phases of the second cycle. The other 12 elementary school teachers should have participated in the last phase, but because of the Covid-19 epidemic, they did not participate.

At the beginning of the first session, secondary teachers were presented with the semi-structured questionnaire with closed and open-ended questions and Likert scale questions from the first cycle (see Appendix A). The test aimed at comparing their knowledge and experience in teaching symmetry with that of the primary teachers before and after the teaching intervention in the first experiment. The discussion took place in two two-hour sessions in which the secondary teachers were informed by those who had supported the previous teaching. The focus was on the pedagogical and methodological dynamics of teaching the different tasks and artefacts on symmetry and invariance in the classroom.

Often students' abilities are underestimated, and what they learned in the previous school is assumed, breaking the continuity of the curriculum, and making the transition more difficult. Such a gap in the transition from primary to secondary school could be reduced if there were a transfer of plans so that secondary school teachers valued and used information from previous teachers. It should be noted that a shared understanding of progress in symmetry means that teachers within the school, preferably across schools, should agree on what implementing symmetry is about. Teachers need also to be aware of the progression of challenges and difficulties in symmetry for students so that teachers' ideas about the progression of teaching and learning are consistent as pupils move between grade levels and schools without thinking that there is one right pathway of progression for all pupils.

5.1.2 ACTION PLANNING: REFLECTION ON PROCESS AND MATERIALS

In order to reinforce continuity in the teaching and learning of symmetry as pupils move from primary to secondary school, we explored the possibility of refining the nature of learning about the relationship between symmetry and invariance through transformations. We identified further applications of the materials and the possibility of adding cardboard boxes with lids that allow reflections and rotations (see Section 2.3.7) in the classroom. The repetitions of the activities with the cardboard boxes and the new lids represent a progression of symmetry on the vertical path through the complex articulation of the symmetry transformation associated with invariance. However, we can also make a progression along the horizontal axis through the ability to use the articulations of symmetry and invariance by using other materials in other contexts, according to our EMT model.

Mirrors were the first educational tool explored for transversal progression in the model. A mirror reflects everything in front of it, including another mirror. Symmetries can be used ingeniously with mirrors that allow students to visualize, classify, and empirically measure symmetries. The kaleidoscope is a special device that can create amazingly beautiful reflections of reflections of real objects. The beauty of symmetry patterns depends on complex mirror arrangements. An object produces three reflections when placed equidistantly in front of two mirrors that form a 90-degree angle. The smaller the angle becomes, the more reflections are generated, up to an infinite number of reflections when the object is placed at the same distance between two parallel mirrors. Rotational symmetry is easy to understand by putting two mirrors together to make a book. Studying the mirror arrangements of kaleidoscopes makes a connection between symmetry and regular polygons. A regular polygon is formed when a line is reflected in two mirrors open to a given point, as shown in Figure 5.4.

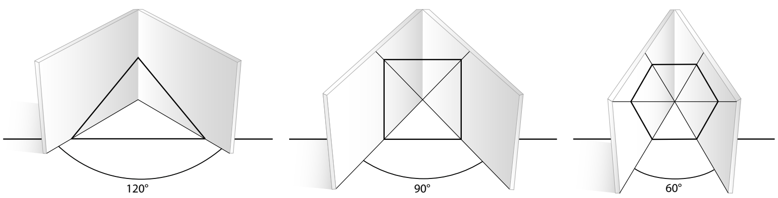


Figure 5.4. Regular polygons are formed by the kaleidoscope depending on the angle between the mirrors.

The construction is such that the mirror and the line form an isosceles triangle in the plane. The smaller the angle, the greater the number of sides of the polygon, and we can conclude that the polygon approaches a circle as the angle approaches zero. We have called the device with hinged mirrors Archimedes' machine because of his famous method of exhaustion. The basic idea is to approach the area of a circle from above and below by circumscribing and drawing in regular polygons with a larger number of sides. We can indeed consider two polygons: a regular polygon inscribed in a circle whose vertices lie on the circle and the circumscribing polygon consisting of the tangents to the circle at a point P. This point P divides both the arc and the circle. This point P divides the arc connecting two vertices of the inscribed polygon and the edges or sides of the circumscribing polygon into two equal parts. By iterating the procedure, Archimedes determined the value of π. When the number of sides of the inscribed and circumscribed polygons approaches infinity, the vertices have points that form the circle as a limiting case. The sides of the inscribed and circumscribed polygons then coincide. The Archimedes machine was made by cutting sheets of reflective Plexiglas into rectangles of 18 cm x 26 cm with a thickness of 1 cm, into which an incision was made on the back in the middle of the base so that they could be folded like a book. This decision was made by the teacher group keeping safety in mind. Therefore, we checked that all corners were round and sharp edges were removed. Then, we have created a worksheet for the activity with the kaleidoscope to ensure its usability and safety (see Appendix L).

The transformation of an artefact into a pedagogical tool continues beyond its theoretical conception or its physical construction. It takes place at important stages of development, considering the potential changes the tool can bring about in the teaching and learning processes of teachers and students. It involves social interaction and intersubjectivity between teachers and learners both of which modify the existing schema of the learning environment. The community constantly refines the artefact in everyday school life in terms of its compatibility with the intended goals. Therefore, the artefact must be tested in practice, as it often involves different interrelated actions and activities.

In this context, secondary teachers chose to participate in the classroom as observers rather than as instructors proposing activities during the action steps of the cycle, at least in the initial phase. They assumed the same role as the elementary teachers in the first experimental cycle. Besides other artefacts to be used, we discussed the potential of using and developing complex games by proposing tasks with multiple simultaneous symmetry operations.

To this end, I have proposed two activities that take advantage of the symmetries of the square. The first is inspired by the “Symmetry Challenge” task that you can find on the University of Cambridge’s NRICH Projects website. It aims at creating symmetrical designs that correspond to the rotation and reflection operations by colouring whole squares of a 3 x 3 grid. In the figure below, shading a single square results in 3 symmetrical patterns. The patterns are considered equivalent if one is an image of the other by rotation. The last two graph solutions correspond to 8 graphs rotated 90 degrees four times, while the first pattern is not repeated. There are various applications for this task, from simple to complex.

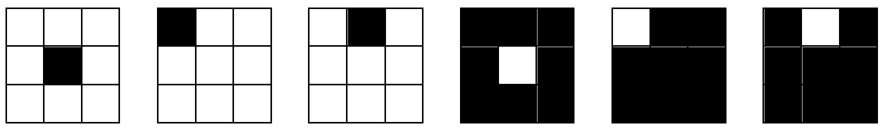


Figure 5.5. Three symmetric patterns from the “Symmetry Challenge” task by shading one square correspond to the same number of diagrams shading the other 8 squares (© 1997-2022 the University of Cambridge, with permission).

The goal is to explore all 64 patterns created by colouring 1 to 9 squares. The following table shows all the solutions corresponding to the coloured squares and the number of symmetry axes in each diagram. In Appendix M, each solution pattern is plotted, where the number of patterns when shading 1,2,3,4 squares is the same as when shading the corresponding 8,7,6,5 cells.

		N° Symmetry Axes				
N° Squares	N° Patterns	0	1	2	3	4
0	1	/	/	/	/	1
1	3	/	2	/	/	1
2	6	/	5	1	/	/
3	10	/	8	2	/	/
4	12	/	10	/	/	2
5	12	/	10	/	/	2
6	10	/	8	2	/	/
7	6	/	5	1	/	/
8	3	/	2	/	/	1
9	1	/	/	/	/	1
All	64	/	50	6	/	8

Table 5.3. The number of solutions to the task of colouring squares symmetrically and the number of corresponding symmetry axes.

The task is closely related to magic squares since the functional scheme is identical: recognize equivalent configurations leading to the identity property or invariance to geometric transformations. For third-order magic squares, a 3x3 matrix with 3^2 cells filled with various positive integers 1, 2, ...3^2, the invariances are of two natures: the arithmetic invariance is the sum of the numbers (i.e., 15) in each row, column, and diagonal, and the geometric invariance is the equivalence of the eight solutions of the magic square. The symmetry operations of the square lead to 8 different magic squares. The operations are rotations and reflections of the squares forming the closed group D_4 (i.e., the dihedral group of order 4, as seen in Section 2.3.7). A closer look at all the magic series up to 15 reveals arithmetic structures and geometric symmetries in 3×3 magic squares.

1+5+9	**2**+4+9	**3**+4+8	**4**+2+9	**5**+1+9	**6**+1+8	**7**+2+6	**8**+1+6	**9**+1+5
1+6+8	**2**+5+8	**3**+5+7	**4**+3+8	**5**+2+8	**6**+2+7	**7**+3+5	**8**+2+5	**9**+2+4
	2+6+7		**4**+5+6	**5**+3+7	**6**+4+5		**8**+3+4	
				5+4+6				

Table 5.4. List of magic series in 3x3 magic squares.

There are eight distinct series, underlined in the table above, as the eight-magic series that form a magic square of order 3. Consequently, there is only one magic square (basic form) with the first nine natural numbers. Figure 5.6 shows all eight different and equivalent magic squares formed by the symmetry operations of the square.

6	7	2		8	1	6		4	3	8		2	9	4
1	5	9		3	5	7		9	5	1		7	5	3
8	3	4		4	9	2		2	7	6		6	1	8

2	7	6		8	3	4		6	1	8		4	9	2
9	5	1		1	5	9		7	5	3		3	5	7
4	3	8		6	7	2		2	9	4		8	1	6

Figure 5.6. All 8 equivalent magic squares of order 3, related by symmetry.

Symmetries often remain hidden from learners when they look only at the numbers in the grids. We have taken a more intuitive approach to discovering symmetries. Figure 5.7 shows how pupils can detect symmetries through drawing magic lines or using colourful visualisations. The magic line consists of a straight line from 1 to 9 connecting each whole number in a sequence of numbers.

Considering the age of the students, we preferred that they use colours to represent the 8 corresponding squares. Therefore, we allowed the children to reproduce their magic squares with colours on tracing paper, as shown in the following figure.

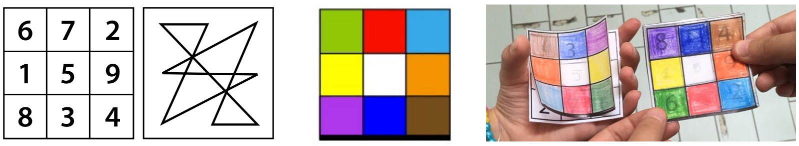


Figure 5.7. Creative visualisation of order 3 magic squares.

As a result, the coloured visualisation transforms the eight solutions of the third order magic square into the following coloured squares.

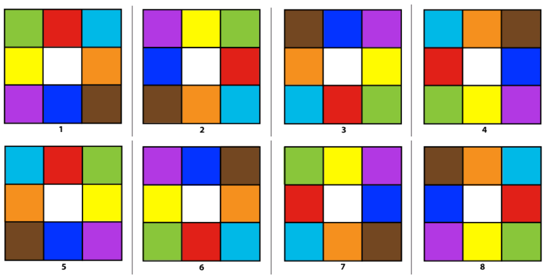


Figure 5.8. Creative visualisation of order 3 magic squares.

We then developed a manipulative game related to the type of visualisation, in which each student can perform symmetry operations such as rotation and flipping to match their magic square to the eight solutions. By exploring all the solutions of the magic square, the manipulative games allow students to become familiar with the symmetry groups of the square and to internalise the language of transformations with the concept of invariance through a joyful and concrete situation.

In close connection with the above material on geometric symmetries and invariants, we have briefly discussed how invariance extends horizontally across different subject areas and vertically across different levels by giving examples of invariance in different geometric, mathematical, and scientific areas.

Accordingly, the tasks were designed to be suitable for primary and secondary school students. For example, we modified a problem-solving task presented by Libeskind et al. (2018, p. 114) about the invariance. Specifically, the task involves cutting a triangle from a square piece of paper (see Figure 5.9) and determining the percentage of wasted paper. Then, cutting out 4 triangles or 9 triangles and so on (up to n^2) from the same square piece of paper maintains the same percentage of wasted paper.

Figure 5.9. A geometrical example of invariant properties: no matter how many n^2 triangles are cut out of the square, the percentage of wasted paper is always half of the square.

The problem can be solved in many ways, algebraically or geometrically. At the primary level, it is best to concretely recreate the situation and use scissors and glue to show that the n^2 triangles correspond to the first triangle and that the latter corresponds to half of the square. Therefore, the percentage of wasted paper is the same no matter how many squares are cut out of the square.

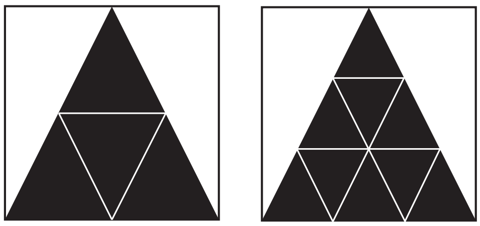


Figure 5.10. Decomposition of the first large triangle into 4 and then into 9 small triangles, proving the equivalence of the shapes in the squares of the previous figure.

The search for invariance can lead to surprising results, which stimulate further investigation and bring interesting mathematical results. The constant area of the wasted paper is an invariant of the problem, even independent of the shape of the paper cut out (e.g., the constant percentage of wasted paper for the circles is slightly more than 20%, i.e., equal to ($1-\pi/4$) %). In this problem-solving task, pupils play with shapes and experience activities related to symmetry and invariance. Working with symmetry operations helps students determine the area of shapes even without using the terminology of symmetry (Preeti, 2021). Therefore, the connection between symmetry and invariance and the concept of area, especially in the context of the given problem, can be built by exposing pupils to well-designed problems early in their schooling.

The problems described above from different areas of mathematics form the basis for developing activities for learning the conceptual and procedural knowledge of symmetries and invariants in different topics. In this context, we decided to design further activities that encourage students to explore symmetry and invariance through various interdisciplinary examples that connect them to science and daily life.

According to Knuchel (2004, p. 5), allowing students to explore and develop their ideas and concepts of geometry, including symmetry and invariances, leads to higher levels of thinking.

The floating ping-pong ball experiment is an interesting, quick, and simple experiment that can be done in the classroom for elementary students to illustrate the importance of invariance in science, specifically for fluid dynamic phenomena. It is easy to perform because all you need is a hairdryer and ping-pong balls. If you hold a ping-pong ball in the preferably cold air stream of the hair dryer, which is pointed at the ceiling, the ball will be pushed up and will be in a stable equilibrium position. Even if you move the ball a little to the right or left, it will return to its original position. It looks as if some mysterious force does not allow the ball to deviate from the air jet (Merkulov, 2012, p. 97). The same result is obtained if the ping-pong ball floats on a jet of water coming, for example, from a small drinking fountain.

Without going into the details of Bernoulli's theorem or the pressure and velocity of the air, the experiment is based on a dynamic equilibrium due to the symmetry of the system. The airflow of the hair dryer follows all contact surfaces it encounters, and when it encounters a symmetrical surface, such as a ping-pong ball, a boundary region is created. Figure 5.11 shows that the air stream flowing on the right side of the ball (and curving to the left) pushes the ball to the right, while the air stream flowing on the left side (and curving to the right) pushes the ball to the left, creating a dynamic equilibrium, as mentioned earlier.

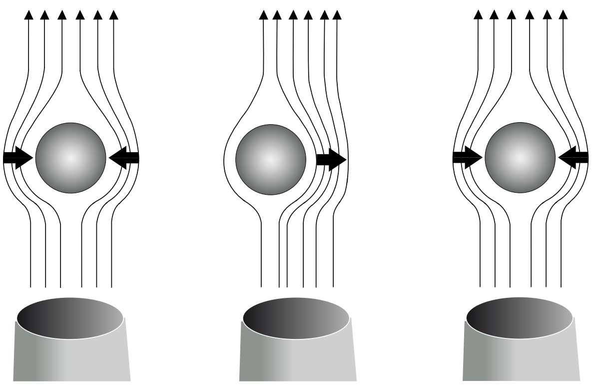


Figure 5.11. The ping-pong ball in the air stream[13].

[13] The ball is in dynamic equilibrium in the centre of the air stream (left drawing) because the forces of the air flowing around it from both sides cancel each other out. When the ball moves out of the centre of the air stream (e.g., to the left in the middle drawing), an asymmetry of pressures occurs, indicated by the flow lines, and as a result, the ball is pushed back to the centre of the air stream (right drawing).

Pupils can experience the importance of symmetry in the previous experiment by taking an asymmetrical ping-pong ball (e.g., by simply pushing it to one side), which will cause the magical effect of the suspension to disappear. The lack of spherical symmetry of the ball disrupts the equilibrium dynamics of the system and, thus, the floating effect of the ball in the air. As explained in Section 2.3.2, the symmetry of a system is associated with a physical dimension that is preserved and vice versa. Symmetry breaking is a process in which one or more physical quantities that define the properties of the dynamical system undergo a series of transformations in which they lose their corresponding invariance states, resulting in an asymmetric state (Dhiraj & Gehan, 2022, p. 3-1). In the previous experiment, the asymmetric state is precisely represented by the loss of dynamic equilibrium, causing the ball to fall to the ground. The activity can be deepened with different fun materials that change the air propulsion (e.g., a straw bent upward) or the ping-pong ball (e.g., an inflated balloon). There are also various ways to explore and extend the work, such as using real-life scenarios in which students look for related everyday phenomena.

Table 5.5 summarises the programmed teaching-learning activities based on the materials discussed above with their type of methodology and the time spent on each, the total time for the action step being 10 hours.

Activity	Type	Title	Duration
A1	Task Game in group	Strange boxes and new lids	2 h
A2	Solving Problems	Symmetry challenge	2 h
A3	Task Game in group	Archimedes' machine	2 h
A4	Solving Problems	Symmetries of magic squares	2 h
A5	Discovery and inquiry	Search for symmetries and invariances	2 h

Table 5.5. Summary of activities in the second cycle TLS.

5.1.3 PRE- AND POST-TEST: REFLECTIONS FOR ACTION

According to the research design, the pre- and post-test of the second reflection cycle differed from those of the first experiment due to the change in the control group and in the specific target. Consequently, the pre-test was designed to assess the knowledge of students who completed the first experiment the previous year and compare it to that of a new control group of students (one year older) from secondary grades 2B-2E who had learned symmetry in traditional classes. Indeed, the questionnaire's design was based on the previous assessment questions from the first cycle. Namely, we used the same metaphor of the dialogue between the Little Prince and the Rose for the first question to provide continuity for students with prior knowledge from the first cycle and to include students without basic knowledge in the control group.

> Q1) If you were the "Rose" trying to explain symmetry to the "Little Prince", which of the following sentences would you use?
>
> a) There is symmetry when you can see that, if a line divides a figure in two parts, these parts reflect each other like in a mirror.
>
> b) Among the many symmetries that exist there are also the movements that transform a figure so that the resulting figure coincides with the original.
>
> Write the reason of your choice.

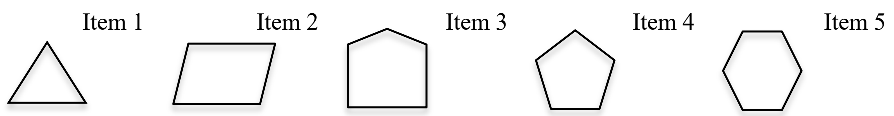
The second and third questions asked pupils to write down words or school subjects that were directly related to the concept of symmetry.

> Q2) Write five/six meaningful words that come to mind about symmetry.
>
> Q3) In which school subject (or subjects) is symmetry studied?

These questions aimed at identifying gaps in the breadth and depth of the sample's knowledge of symmetry by examining their word choices. As discussed in Section 3.2.6, the analysis of word associations with the word *symmetry* can capture a qualitative scope of symmetry learning. However, the responses to these questions needed to be combined with the results of the fourth questions.

The fourth and last question was related to the first reflection cycle task of closing the box for the analysis of the geometric symmetries of the main polygons.

> Q4) After helping Paolo to close various boxes, note the number of ways the lid can be placed on each box and draw the lines of symmetry (axes), if any.



This question included 5 figures to be analysed in the pre-test. Specifically, the question suggests five polygons that correspond to the 2D geometric shapes of the lids of the cardboard boxes. The number of figures in the post-test was expanded to 16 to include real objects such as flags, flowers, and traffic signs. The decision to increase the number of figures in the post-test was taken to assess student progress on symmetry and the ability to see its applicability in real-world situations.

Question	Type	Aim
Q1	Dichotomous Choice Open Justification	To verify how the concept of symmetry is perceived in the assumed didactical frame.
Q2/Q3	Word Choice	To assess the keywords related to symmetry.
Q4	Items Test	To assess and measure the level of knowledge and skills related to the tasks and/or specific areas.

Table 5.6. Schema of the pre-test/post-test questionnaire for students in the second TLS cycle.

The table above provides an overview of the four questions with the typology and objectives for each question. The actual pre-test and post-test can be found respectively in Appendix N and Appendix O.

5.1.4 REFLECTION IN ACTION DURING THE COVID-19 PANDEMIC

The Covid-19 outbreak in early March 2020 led to an unprecedented disruption of human activity worldwide, including our ongoing second-cycle study. In the Marche region of Italy, schools were closed from March 4th until the end of the school year (i.e., June 2020). The national law of April 8th also introduced distance education officially and voluntarily.

As a result, there were different approaches to school-based learning, ranging from daily online classes to no distance learning. Empirical researchers also chose different methods for proceeding with their experiments. Some completely redesigned their research, others postponed data collection until the indefinite future, yet a third option was to adapt to the changing environment using online methods (Kobakhidze et al., 2021).

As explained in Section 5.1, we did not conduct the remaining steps of the action research with students online. We felt that Covid-19 had shocked everyone, especially the students. Therefore, their participation in continuing the instructional sequence would not have been guaranteed due to psychological and technological issues and possible parental interference. We also considered that the actual hours taught were too few to reach a conclusion (i.e., 30% of the planned 10 hours) and that it would have been a different experiment because the conditions and environment were different, which would have prevented comparison with the results already available from the pre-test and the reflection of the first three hours. Since we could not wait for the schools to open due to time constraints, we decided to conduct a different experiment with an online course on symmetry and invariance for all math teachers in the school.

However, for the second experiment, we conducted the pre-test and the first activity (A1) during the action period, while the second activity (A2) was started in all classes and completed in some. Although the second action research was not completed, the steps already taken had produced promising results. One sign of the promising results was that the teachers were amazed at the students' ability to solve challenging tasks. Indeed, the problem associated with the Symmetry Challenge task proved accessible to most students.

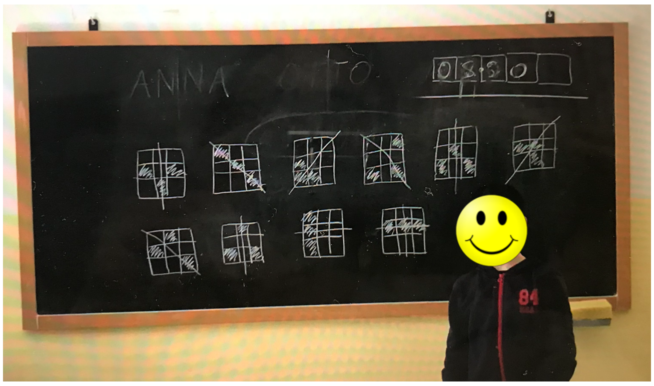


Figure 5.12. A student solves the task of shading 3 (or 6) squares in the symmetry challenge by identifying the 10 symmetrical patterns and drawing the symmetry axes.

Figure 5.10 shows a student who solved the Symmetry Challenge of shading 3 (or 6) squares obtaining all 10 symmetric patterns. During these activities, teachers had the opportunity to reflect on their approach to teaching symmetry and to realise that challenging tasks were appropriate for students learning symmetry in terms of teaching-learning progression. Such reflection in action can be a wedge that gradually opens the door to change the mindset of teaching symmetry.

One element of rethinking symmetry teaching has been the use of challenging and progressive tasks that differ from those teachers typically use to promote student engagement and learning. Developing activities that ranged from simple to complex and were accompanied by informative comments and supportive tasks was important to engage students and help them if they were initially stuck while solving and learning the task. According to Sullivan et al. (2015), the approach described above should change teachers' beliefs about teaching and learning processes and the opportunities and constraints they face when teaching mathematical topics such as symmetry and invariance.

5.1.5 EVALUATION: REFLECTION ON FINDINGS

Data were collected by analysing primary sources, such as students' pre-test questions, and secondary sources, such as classroom observations and teachers' questionnaires before and after the teaching-learning sequence in the first and second experiments. In the next sections, we present the results of the data analysis based on the research questions of the second action research cycle. First, we assess the difference in students' knowledge of symmetry between the sample and control groups based on the results of the pre-test questions.

5.1.6 FINDINGS CONCERNING THE FIRST QUESTION Q1

Analysis of the data for question Q1 in Figure 5.13 shows that the vast majority (70%) (SD 5%) of students in the sample chose answer (b), while almost all students in the control group preferred option (a) (98%) (SD 5%).

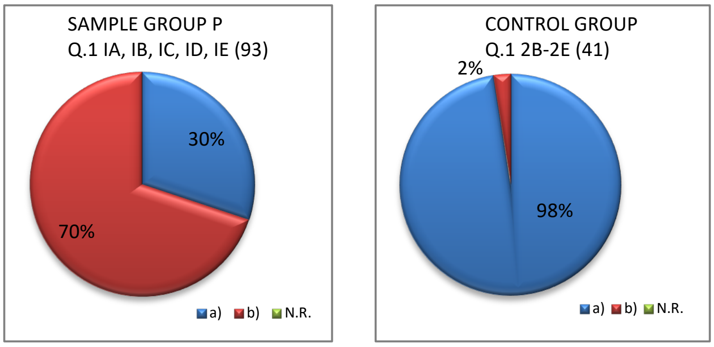


Figure 5.13. Comparison of answers to question Q1 between pupils in the sample and control group.

The results confirm the effectiveness of the treatment of T1 according to the research design in Figure 5.1. Only one student in the control group chose the option b and gave a good explanation, "If I draw a fir tree and copy it in another place, it will look the same as the original." This statement is exactly the concept of sameness in change. I questioned the pupil to get more information and found that he/she was stimulated to think by the question and wrote down the simplest case in his/her head.

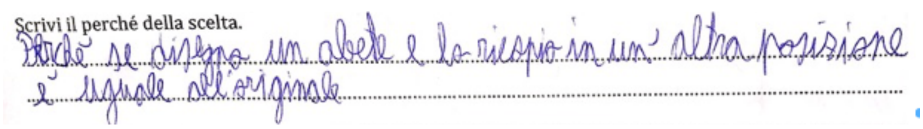


Figure 5.14. Explanation of choice (b) by a class IIB pupil in the control group.

IIB above: *I choose (b) because if I draw a fir-tree and copy it in another position, it is the same as the original.*

This fact proves that children can easily grasp the definition of the modern concept of symmetry. To answer the specific research question SRQ1. the above results can be combined with the analysis of the fourth question, Q4. The analysis of the answers to Q4 aims at refining the results and the efficiency of the teaching-learning sequence T1 of the first cycle.

5.1.7 FINDINGS CONCERNING THE QUESTIONS Q2 AND Q3

The second and third questions of the questionnaire aimed at providing data and evidence to answering the specific research question SRQ2. Questions Q2 and Q3 asked students to indicate which words and disciplines they thought were most related to symmetry. For each question, students could associate up to six terms with the word *symmetry* without being given any suggestions.

We analysed the responses by splitting the data from the students in the sample and control groups to determine the differences. The average number of words written by students was almost the same in the sample and control groups. Students wrote an average of 4.3 words (SD 1.6) in the sample group and an average of 4.8 (SD 1.8) in the control group.

As discussed earlier in the Section 3.2.7 and highlighted in many studies, there is a correlation and impact of students' language use in terms of word frequency, vocabulary effect, and semantic variety on learning and mathematical ability. Therefore, we analysed the word frequency and reduced the list of words by eliminating those with a frequency lower than two. We then represented the most frequent words in word clouds that display text data in a font directly proportional to their frequency. The resulting word clouds for the sample, which you can see in Figure 5.15 below, were extracted from a reduced sub-corpus of 247 words (tokens), representing 60% of the original corpus and 40 words (types).

Figure 5.15. Word cloud from the sample sub-corpus (elab. Wordle).

For the control group, on the other hand, Figure 5.16 shows the reduced sub-corpus of 122 words (tokens), which corresponds to 62% and 22 words (types).

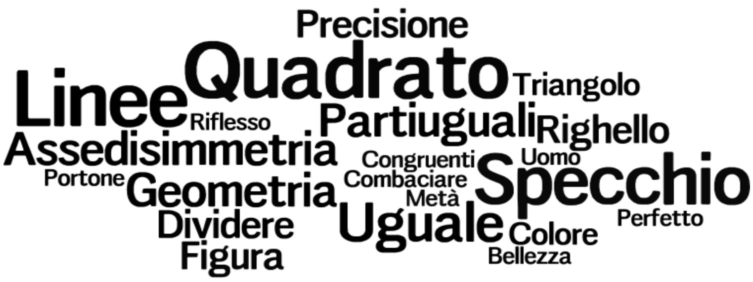


Figure 5.16. Word cloud from the control group sub-corpus (elab. Wordle).

In both word clouds, some identical terms appear, such as "*specchio*" [mirror], "*due parti uguali*" [two equal parts], "*precisione*" [precision], "*parti uguali*" [equal parts], "*geometria*" [geometry], "*metà*" [half], "*quadrato*" [square], "*riflesso*" [reflection], "*dividere*" [divide], "*figura*" [figure], "*linee*" [lines], "*uguaglianza*" [equality], "*asse di simmetria*" [axis of symmetry]. This vocabulary refers exclusively to line symmetry, reflection, and the perfection of this type of symmetry. From this, we can conclude that the two groups have the same base of experience with reflective symmetry at school and in life.

Among these words, terms like "*rotazione*" [turns], "*mela*" [apple], "*scatole*" [boxes], and "*palindromo*" [palindrome] occur only in the sample and not in the control group. These terms come from the activities of the first experiment and refer to rotational symmetry, to which only the students of the control group were exposed. According to Lipka et al. (2019), symmetry remains a simplified and underrepresented idea in today's school mathematics.

Working with other types of symmetry and different manipulations expands learners' experiences with symmetry. Students can easily expand their understanding of symmetry by participating in various invariance activities. Symmetry should be a foundational framework for school mathematics beyond the line of symmetry because it promotes spatial reasoning, visualisation, and mental rotation that fosters scientific and mathematical thinking.

The fact that symmetry is an interdisciplinary topic is clear to all students in both groups, as indicated by their responses to Question Q3. From the histograms of responses in Figure 5.17, pupils view symmetry as a concept that is important in geometry, science, technology, and in art. Symmetry is also found in mathematics and music, although there, as in all other subjects, to a lesser extent.

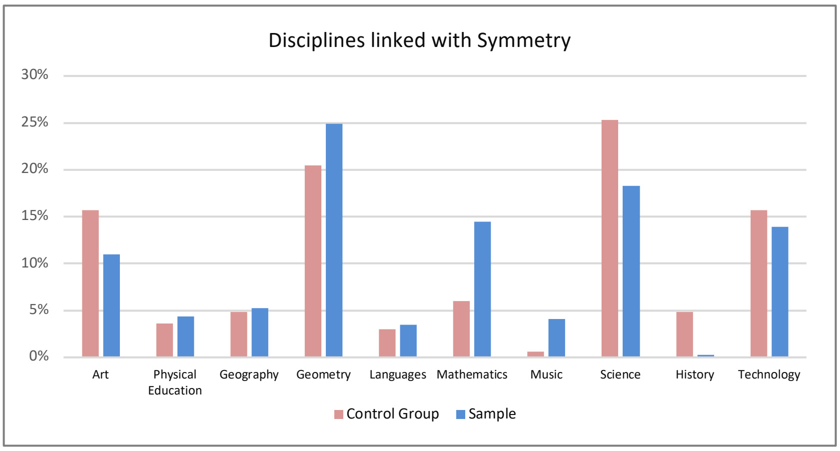


Figure 5.17 Comparison of answers to question Q3 between pupils in the sample and control group.

5.1.8 FINDINGS CONCERNING THE FOURTH QUESTION Q4

We used the same analysis methods as described in Section 4.2, since the question has the same structure as in the pre-test questionnaire of the previous cycle.

In Figure 5.18, the data for question Q4 show a 50% difference in correct answers between the sample and control groups for both rotations and reflections. This difference is more pronounced for symmetry rotations (R1) than symmetry reflections (R2). One explanation could be that the students in the sample group had already gained experience with rotations by working with the lids in the first experiment, which could only be rotated over the boxes. However, the sample group also performed better on the symmetry reflections, as their scores for R2 were much higher than those of the control group.

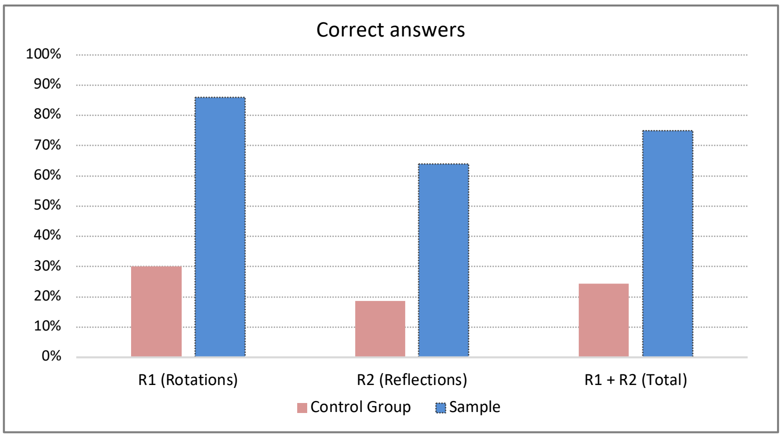


Figure 5.18. Comparison of answers to question Q4 between pupils in the sample and control group. There is a 50% difference in correct answers for both rotations and reflections (R1+R2).

Specifically, for R1 the boxplot in Figure 5.19 shows that the mean score for the sample is much higher than for the control group (i.e., the means differ by 2.8 from 4.3 to 1.5, with the same variability as 2), as do the median scores from 1 to 5. For the sample, 50% of the students scored between 3 and 5, whereas the control group scored lower than 2. For R2, the difference in mean between the control group and the sample is 2.3 (from 0.9 to 3.2), with increased variability from 1 to 4, as the median scores from 1 to 3. The difference is evident because 50% of the students in the control group scored between 0 and 1, and the students in the sample scored between 2.5 and 4.

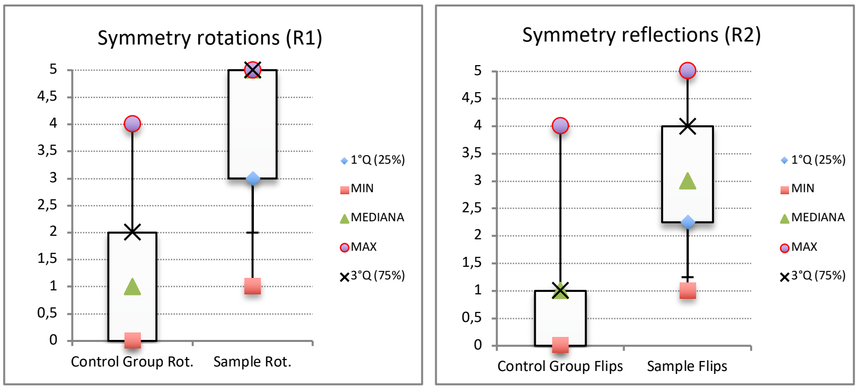


Figure 5.19. Comparison of boxplot data sets for variables R1, R2 between pre-test question Q4 of pupils in the sample and the control group.

The above results show that the learners who participated in the TLS of the first experiment (T1) performed significantly better on the task than the students in the control group.

5.2 DISCUSSION ON THE FINDINGS AND CONCLUSIONS

The second action research cycle aimed at deepening the results of the first experiment and at gaining more knowledge and insight into the specific research questions. Although the second experiment was interrupted due to the spread of Covid 19 and the closure of the school, we were able to draw some conclusions by comparing the pre-test results between the sample and the control group. The results of questions Q1 and Q4 show a discrepancy between the groups representing different levels of symmetry understanding and skills. This discrepancy confirms that the teaching-learning sequence T1 is positively effective in learning symmetry in terms of invariance in the fifth grade and results in even better performance than the control group in the transition to the first grade of secondary school. Consequently, the teaching-learning progression T1 can be considered an effective approach for changing the symmetry paradigm in elementary school for fifth-grade students.

The results in terms of the specific research question SRQ1 can be interpreted as follows: students move from the notion of line symmetry to the recognition of invariants under transformations, mainly in geometric domains, which represents a paradigm shift in understanding symmetry. This way, pupils make rapid and significant conceptual progress in their structural awareness of symmetry. This paradigm shift is triggered by manipulative work with artefacts and the dynamic nature of progressive geometric tasks in the search for invariants related to transformations of reflections and rotations, ultimately leading to a generalisation of symmetry properties. During the first and part of the second experiment, students reinforce some successful problem-solving skills based on their understanding of symmetry as a dynamic method for identifying the invariants.

Our study has demonstrated that the children acquire new words, gestures, and knowledge about symmetry through intensive spatial work in the dynamic geometry environment. The results for questions Q2 in the pre-test, presented in the previous section, highlight linguistic differences that are apparent in the in-depth analysis of the open-ended narratives in the first cycle (see Section 3.2.7). Given these differences, this study highlights the value of a qualitative analysis that takes into account students' language as well as individual and social gestures to gain insight into how children learn symmetry. From a qualitative perspective, the observation of pupils' responses and postures during the activities combined with quantitative analysis of test results and teacher reflections shows encouraging progress toward the specific research question SRQ2.

Results for question Q3 show a general awareness of the interdisciplinary value of symmetry, with no significant differences between groups. For pupils, symmetry is a concept that is strongly present in science, technology, and art; they also perceive symmetry as an object that belongs to geometry and mathematics. The effectiveness of early teaching and learning of symmetry in pedagogical contexts depends on providing more examples, tasks, and artefacts for a coherent model of activities that reinforce the awareness of interdisciplinarity. Our study highlights the interdisciplinarity and generalizability of the concept of symmetry through its connection to invariance. We have provided many teaching examples of how to extend and advance the pedagogical framework that incorporates teaching and learning of isolated skills through the fundamental approach of the modern symmetry concept into a more integrated mathematics and science classroom. In this regard, our research suggests that this process may facilitate young children's early abstraction and generalization of symmetry ideas in ways that promote mathematical thinking and build the architecture of complete scientific knowledge, typically NOS. The second cycle of action research thus provides evidence for a more solid causal relationship between the class experiment and the results, as well as an important basis for the generalization potential of the results.

5.3 LIMITATIONS ON THE SECOND CYCLE

As mentioned earlier, the second cycle of reflection suffered from the limitation of the planned development to reflect more explicit aspects of reasoning and interdisciplinary exploration using the symmetry paradigm. The post-test O4 and its analysis data, which should have shown learning gains after the educational treatment T2, were basically not conducted. For this reason, we do not have complete and robust data from the second cycle of reflection to answer the second research question, SRQ2, which refers to the impact of the paradigm shift of symmetry on the didactic framework.

In an effort to fill this gap, we decided to continue our research with a third experiment involving all teachers who directly and indirectly participated in the two action research cycles in an online course. This decision was based on the results of the teachers' questionnaires at the beginning and the end of the teaching experiments (see Appendixes Q and R) in the first and second cycles. Namely, the results of the surveys showed the following:

- Most teachers believe that the learning unit broadened students' view of symmetry and invariances.
- Most teachers believe that their knowledge of symmetry and invariance improved during the teaching-learning process.
- Some teachers agree that their knowledge has increased but feel uncomfortable implementing what they have learned in the classroom.

- Most teachers feel that in-service training on symmetry and invariance would be useful.

Therefore, the third experiment aimed at consolidating and developing the knowledge, materials, and methods of the action research project. It also aimed at improving the teachers' pedagogical approach in synergy with our EMT teaching model, both vertically and horizontally, which may have a positive impact on the generalizability and interdisciplinary potential of the previous research results.

Chapter 6: Third Experiment

The improvement of the didactic framework related to the paradigm shift of symmetry implies a significant change in the related teaching-learning process. This important step takes place at different levels and through a variety of actors involved in the framework, but mainly through the teachers who have to enforce the innovation. One way to accomplish this, therefore, is to provide teachers with a solid scientific and interdisciplinary knowledge of symmetry and invariance upon which they can design and develop practices to experiment with in their classrooms. Building on the findings, reflections, and limitations of the first and second cycles, we designed and implemented the third experiment, which focused entirely on training service teachers through synchronous online instruction, in response to the Covid 19 epidemic. Online teaching presents several challenges and issues. As we transition from the classroom to virtual learning, we face the challenge of transforming offline materials and artefacts into online materials. Using information technology allows us to effectively organize and design activities and lessons in an online environment.

In this chapter, we describe the structure of the teacher training program and the steps taken to develop and implement the lessons. Section 6.1 outlines the content and structure of the program, while Section 6.2 provides background information on the participating teachers. Section 6.3 explains how the lessons were created and what materials were used. Section 6.4 also discusses the tools used to evaluate the training, including questionnaires for teachers before and after the training and concept maps. This section also explains the data collection procedures and analysis methods used in the study. Sections 6.5 through 6.7 present the results of the mixed-methods analysis used to answer the research questions. Section 6.8 analyses the data and draws conclusions about the effectiveness of the program, including a discussion of the results and conclusions. Finally, in Section 6.9, we will identify limitations or areas for improvement in the study. Overall, this chapter provides a comprehensive overview of the teacher training program and the evaluation methods used to measure its impact.

6.1 DESIGN OF THE TEACHER TRAINING COURSE

The design of teacher training on symmetry and invariance in our ongoing action research study aims at establishing learning objectives and methods in the form of learning activities and related pedagogical tools to achieve the overall outcomes of the action research. As explained in the previous chapter, the learning objectives refer to the consolidation and development of knowledge, materials, and action methods, considering the needs and requirements of teachers. Therefore, all the efforts of our educational program aim at deepening knowledge and crystallizing teaching methods based on symmetry and invariance, specifically (a) improving the teachers' pedagogical approach in synergy with pedagogical continuity of our EMT model both vertically and horizontally, and (b) their ability to independently plan interdisciplinary didactic pathways.

Whether it is possible to convert goals into outcomes depends, among several factors, primarily on the ability to engage individual teachers in online instruction. Indeed, there is a risk that the course will become ineffective due to limited interactivity among participants in the online format. In this regard, there are ways to facilitate interactions and discussions between teachers during the course. According to Yilmaz et al. (2017), there is a direct link between the high efficiency of the teaching-learning process in education and the motivation of those involved, both teachers and learners, because teaching and learning only take place when there is constant motivation.

One way to awaken and strengthen the motivation of teachers is to include training in the SOFIA (Sistema Operativo per la Formazione e Aggiornamento) platform of the Italian Ministry of Education, University and Research. In order to obtain the certificate of attendance through SOFIA, teachers must register for the training on the platform and be present for at least 75% of the total teaching time (according to Article 1, paragraph 5 of Directive No. 170/2016 of the Italian Ministry of Education, University and Research).

Another way to motivate teachers is to invite different experts, such as my PhD supervisor, Prof. Piergallini, and Maestro Veneri, to give interdisciplinary lectures. Multiple lecturers provide more dynamism and deeper insights into the relationship between symmetry and the world and disciplines from different angles. Palmer (2006) argues that instructors can better motivate learners if they foster a deeper understanding of content, including pedagogy. The first lesson relates to symmetry in art, while the second relates to music, both emphasising the importance of symmetry. In particular, the lessons illustrate the interdisciplinary aspect of symmetry through various historical, artistic, and scientific examples, including applicable activities such as mosaic work or music listening.

An important factor in maintaining teacher engagement in online courses is also to allow sufficient time in each class session for concrete activities and adequate breaks between sessions so that students have enough time to engage with the material. According to Garet et al. (2001), active learning rather than passive knowledge absorption during instruction is a fundamental characteristic of effective teacher education. In designing lessons, we pay attention to these aspects and believe that individual engagement is more effective in small groups. Therefore, in each lesson, after explaining the initial topic, there is a workshop in which teachers are divided into groups of five, each participating in a small session on Google Meet with an internal mentor chosen by the same teachers.

Many virtual platforms (e.g., Zoom, Edmodo, Google Meet, Teams, Webex Meet) allow classes to be divided into simultaneous small groups during the online session. We deployed Google Meet video conferencing as a G Suite Premium for Education application, free to schools until July 2020, to address the Covid 19 shortage. To support teachers during virtual classes, we used Google Classroom to manage communication, material assignments, and deadlines with teachers and Google Drive to create, store and edit documents directly online in collaborative mode through the school's account. Hutajulu (2022) has shown that Google Meet in online learning improves students' mathematical communication skills as well as their learning outcomes. In this regard, Google Meet proves to be a suitable interactive learning medium for online learning.

In general, each session consists of two parts. In the first part of the lesson, instructors impart knowledge about pedagogical theories and didactic methods through participatory frontal teaching. In the second part of the lesson, teachers, as learners, cooperatively put these theories and methods into practice through small group workshops. Classes are taught in synchronous mode, but asynchronous activities, such as learning assessment, review of materials, and viewing recorded lecture videos, are also provided. Such a didactic and technological design ensures that each teacher contributes to the class activity and promotes understanding and reflection on symmetry.

6.2 PARTICIPANTS

Twenty-six teachers, most from the school of Montegranaro (FM) and 3 from S. Elpidio (FM) in Italy, participated in the in-service training course and were divided into five groups, with a professor of secondary education as mentor in each group. Table 6.1 summarises the characteristics of the sample, including the number of the group to which they belong, the gender of the teachers, the school where they teach, whether they have participated in previous experimental cycles, and their experience as mathematics teachers.

Num.	Code	Group	Gender	Type of S.	School	Part. P. Cycle	T. Exp. (years)
1	TA1	1	F	Secondary	I.C. M.	yes	38
2	TA2	1	F	Primary	I.C. M.	no	5
3	TA3	1	F	Primary	I.C. M.	no	21
4	TA4	1	M	Secondary	I.C. M.	no	Music Teacher
5	TA5	1	F	Primary	I.C. M.	no	20
6	TB6	2	F	Secondary	I.C. M.	no	13
7	TB7	2	F	Primary	I.C. M.	yes	20
8	TB8	2	F	Secondary	I.C. M.	no	17
9	TB9	2	F	Primary	I.C. M.	no	1
10	TB10	2	F	Primary	I.C. M.	no	1
11	TC11	3	M	Secondary	I.C. M.	yes	14
12	TC12	3	F	Primary	I.C. M.	no	5
13	TC13	3	F	Primary	I.C. M.	yes	2
14	TC14	3	F	Secondary	I.C. S. E.	yes	12
15	TC26	3	F	Primaria	I.C. M.	no	10
16	TD15	4	M	Secondary	I.C. M.	yes	3
17	TD16	4	F	Primary	I.C. M.	no	5
18	TD17	4	F	Primary	I.C. M.	no	30
19	TD18	4	F	Primary	I.C. M.	yes	15
20	TD19	4	F	Primary	I.C. M.	no	1
21	TE20	5	F	Secondary	I.C. M.	yes	14
22	TE21	5	F	Secondary	I.C. S. E.	no	13
23	TE22	5	F	Primary	I.C. M.	no	1
24	TE23	5	F	Primary	I.C. M.	yes	18
25	TE24	5	F	Primary	I.C. M.	no	10
26	TE25	5	F	Secondary	I.C. S. E.	no	2

Table 6.1. Description of the 26 teachers divided into 5 groups participating in the training.

They have up to 38 years of experience, with 40% up to ten years, 40% between 11 and 20 years, and 20% between 21 and 38 years. However, the gender ratio is not uniform. Gender balance is not achieved, as 90% of the teachers are female, which was to be expected given the preponderance of female mathematics teachers in the school.

6.3 LESSONS AND MATERIALS

The structure of the in-service training course, shown in the following table, consists of 6 consecutive 2-hour sessions. Three hours were devoted to completing questionnaires, tests, and concept maps, making the course of 15 hours total. Each lesson was a learning unit that addressed a particular side of the interdisciplinary concept of symmetry. Thus, the course created an organic learning process directly linked to materials development and guidance on the modern concept of symmetry.

Lessons	Title	Speaker	Duration
L1 (9/11/2021)	The path from ancient to the modern symmetry.	Brasili S.	2 h
L2 (13/11/2021)	Symmetries in art: tessellations.	Piergallini R.	2 h
L3 (16/11/2021)	From symmetry to the search for invariants.	Brasili S.	2 h
L4 (20/11/2021)	Symmetries and harmonies in music.	Veneri C.	2 h
L5 (26/11/2021)	The language of symmetries.	Brasili S.	2 h
L6 (30/11/2021)	Conclusion of the teaching path.	Brasili S.	2 h

Table 6.2. Outline of the sessions of the in-service teacher training course.

The first session was dedicated to the pedagogical and historical path from the ancient to the modern concept of symmetry. In the second session, Prof. Piergallini talked about symmetry in art and showed the connections between mathematics, art, and science through various examples of mosaics in art, craft, and architecture. The third session was dedicated to the potential of the connection between symmetry and invariance in education, especially in geometry and science, and the real world. In the fourth session with Maestro Veneri, teachers explored many examples of symmetry, such as reflection, rotation, and glide-reflection symmetry, found in music by listening to some excerpts from sonatas by Bach and Chopin. Maestro Veneri also introduced the relationship between music and mathematics by examining harmonies expressed as special integer ratios of frequencies. This connection to mathematics was deepened in the fifth lesson by explaining some elements of geometry and mathematics that form the basis for a language in which symmetries are explored. The sixth lesson concluded the course by reviewing all the topics covered in class and developing the final discussion and reflection among teachers.

As a result of the design, each lesson had its specific content and learning objectives so that the teacher could measure the effectiveness of his or her learning and the materials used in the lesson by the attainability of the above objectives. In each lesson, the materials were developed with an eye toward (re)building the concept of symmetry and pedagogical content knowledge. The focus was on developing, integrating, and refining materials analysed in previous cycles to put knowledge about symmetry and invariance into classroom practise and didactic framework contexts. Teachers' reflections on existing materials heled them generate ideas and determine which materials needed to be developed and introduced into classroom practice. During the virtual workshop in each session, teachers discussed and compared their ideas with each other and selected the best ideas to put into material form and use in the development of curriculum materials. A key element for the effectiveness of the training was the implementation of the developed materials in the classroom, as this is the best way to implement pedagogical changes in teaching practice.

However, adoption of the new curriculum materials is not a foregone conclusion, and often implementation is partial and not in line with reform ideas. Pedagogical change in classroom practice is a complex process that rarely happens (Fullan, 2007). We were aware of this difficulty and complexity, so we tried to address all the details and provide teachers with all the tools and resources they needed to collaborate effectively and learn the best from their experiences for a possible paradigm shift in the concept of symmetry in the pedagogical framework. These tools included collaborative lesson design and delivery, guidelines for collaborative reflection, online learning materials such as power point slides, educational videos, documents, and tangible offline materials. By offline materials, we mean all kinds of manipulatives and physical tools that can support and promote understanding of symmetry and invariance concepts in the classroom. The workshop sessions therefore focused on exploring the potential for designing interactive activities based on such mathematical and geometric manipulatives. Most of the proposed tools and activities have been presented in the previous educational experiments listed below, such as mirrors, kaleidoscopic mirrors (Archimedes' machine), boxes and lids, symmetrical shading shapes and simple science experiments on invariance. Others are new, such as origami activities, magic squares, Platonic solids, tessellations, geometric invariance, and shortest path problems. The list of sample artefacts is extensive and provides teachers with possible examples of artefacts that demonstrate best practices for accessing and understanding symmetry and invariance.

Some materials were handed out to teachers in secure, disinfected plastic bags. Figure 6.1 shows the contents: a ping pong ball for home science experiments with a hair dryer and sticks for building geometric polyhedral models. The cardboard boxes and lids used for the previous experiment were transformed into paper printouts (see Appendices U) that each teacher could download via Google Driver and print at home. Figure 6.2 shows miniature models of the boxes with lids that the teachers made with paper and tape. We have selected ready-to-use teaching materials and compiled them into toolboxes tailored to the specific pedagogical and instructional needs of online teachers. The materials cover all instructional topics and encourage teachers to engage in new experiences that have the potential to transform their ideas about the nature of symmetry and their teaching.

Figure 6.1. The disinfected plastic bag with some materials.

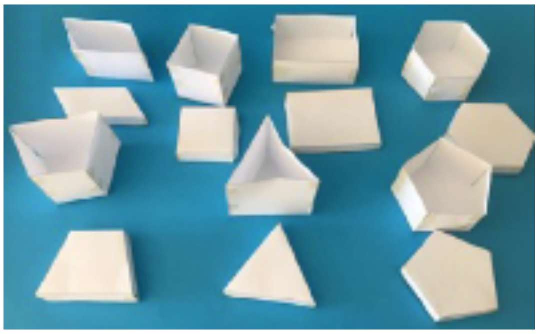


Figure 6.2. Miniature models of the boxes with lids.

With the help of the facilitator/teacher responsible for creating a supportive learning environment, teachers explored, discussed, and evaluated the materials in each group. Teachers were also invited to extend and refine pedagogical tools to support cross-curricular instruction on this big idea that had been the foundation for our entire research project. Communication among teachers is an important, albeit subtle, sign of progress, as pedagogical frameworks often change over time. In the final lesson, reflections were gathered and shared to draw on the collective expertise of the different groups and to support each other in overcoming challenges in the classroom.

6.4 TRAINING EVALUATION INSTRUMENTS

An evaluation of training refers to collecting, analysing, and summarising the information needed to determine the effectiveness of the training (Desimone, 2009). The appropriate design and use of qualitative and quantitative instruments aim at measuring all factors that contribute to determining effectiveness. The effectiveness of teacher training is a multidimensional phenomenon, including phenomena related to the achievement of learning objectives, the degree of congruence of aspects of the educational process and the results of the training with personal expectations and needs, and the requirements of teachers. The training course is also evaluated, considering the established goals of the overall action research, and as a final result, the didactic framework is provided with changes in the teaching-learning process in terms of knowledge and skills for symmetry. Thus, an effective training process measures the development and implementation of the training program, the improvement of individual performance, and the environmental impact (Mohammed Saad & Mat, 2013). The following instruments were developed to collect evaluation data to assess the specific objectives.

We used a pre- and post-test and concept maps before and after the training as evaluation tools to test teachers' knowledge and consolidation of activated knowledge about symmetry and invariance. In addition, survey data on a Likert scale were used to collect information to evaluate the effectiveness of the training from the perspective of the quality of the educational programs, the quality of the teaching staff, the quality of the materials and teaching methods, and the virtual environment of the educational system. Short semi-structured interviews were planned with some teachers to discuss the issues that emerged from the findings in more detail. In this way, there was the possibility of discovering theoretical horizons and validating interpretive hypotheses that have more to do with the concrete reality of developing activities related to teaching about symmetry and invariance.

Teachers' skills and knowledge acquired during in-service training on symmetry and invariance were also assessed by analysing their ability to plan, implement, and evaluate a didactic progression on these topics. Thus, teachers' knowledge of symmetry could be viewed as an integration of knowledge, skills, and attitudes about classroom action. Assessment is not only about what the teacher knows, but also about what the teacher can do in a particular teaching and learning context. The final learning objective was for teachers to design interdisciplinary lessons on course topics that could be used in the classroom at the end of the school year or next year. The following table shows teacher turnaround times for completing documents posted to Google Classroom.

31 November - 6 December	7 December – 13 December	14 December – 23 December
Post- Test		
	Survey	
		Interviews
	Concept Maps	

Table 6.3. Timeline for teacher submission of assessment documents.

6.5 PRE- AND POST-TRAINING TEACHER QUESTIONNAIRES

Pre- and post-training questionnaires were used to measure the knowledge gains of a group of learners by analysing the difference between their test scores at the beginning and the end of a teaching sequence. We have discussed the potential of such instruments in Section 3.2.3 and used them in the previous experiments for students to determine the extent to which they have learned and mastered the concept of symmetry. In the third experiment, comparing the teachers' post-test results with the pre-test results provided information about whether their symmetry knowledge activated by the training was successfully improved.

The questionnaire (see Appendix S) consists of four questions related to specific objectives, and each question is composed of different test items. Specifically, Table 6.4 shows the structure of the teacher questionnaire used for the pre- and post-training tests. It contains eight test items to identify the symmetry of shapes, two test items to investigate teachers' guesses about the symmetry of simple shapes, one open-ended thinking task to measure the symmetry of shapes, and four test items to select the specific symmetry of letters. Each item is assigned an ordinal score based on how well the teacher found the solution. The scale ranges from 0 to 3 depending on the scoring criteria[14]. We use a colour system to visualize the differences between the pre-test and post-test scores according to the scoring criteria. Appendix T shows the scoring of the colour gradient to highlight the difference between the pre-test and post-test for A1-A8 items.

[14] Scoring criteria: 0 for missing or wrong answer, 1 for correct answer with wrong explanation, 1 for partial answer without explanation, 1 for partial answer with wrong explanation, 2 for partial answer with correct explanation, 2 for correct answer without explanation, 2 for correct explanation only and 3 for correct answer.

Question	Type	Aim
A1-A8	8 Items Test	To evaluate and measure the level of teachers' knowledge of symmetry.
B1-B2	2 Items Test	To explore the validity of teachers' conjectures regarding symmetries of simple shapes.
C1	A task Open Justification	To evaluate the symmetry and rotation skills of teachers.
D1-D6	6 Items Test	To verify to what extent the teachers are controlling the skills of classifying symmetries of letters.

Table 6.4. Scheme of teachers' pre-/post-training test questionnaire.

The study differs from previous experiments in that questionnaires were administered to all teachers (n=26) before and after training, with no control group or random assignment, referred to as a pre-experimental single-group study (McMillan & Schumacher, 2001). As shown in Table 6.5, such a research design is only suitable for drawing conclusions about the effectiveness of symmetry training if the results are followed up and mixed with other instruments and further studies. While it is possible to demonstrate that a change in test scores occurred (i.e., O2 - O1), it is impossible to verify whether this change would have occurred without the treatment (i.e., ST). Therefore, we collected data using both quantitative and qualitative methods and with different instruments that were mixed and triangulated to compensate for such research limitations and answer the research questions.

Measure	Treatment	Measure
Pre-test O1 (26 Teachers)	*Symmetry Training* ST	*Post-test* O2 (26 Teachers)

Table 6.5. Diagram of one-group pre-/post research design.

The descriptive statistics for all questions in the pre-test and post-test can be found in Table 6.6, which shows that the mean score of all teachers in the sample group is 16.77 in the pre-test and 29.96 after conducting the post-test, which is an average improvement of 13.19 between the two scores.

Variable	Tot.	Aver.	Mode	Min.	q.1	Median	q.3	Max.	SD	CV
Score Post-test (26)	779	29.96	36	6.00	25.25	30.50	36	43.00	9.05	0.30
Score Pre-test (26)	436	16.77	22	0.00	10.25	16.50	22	38.00	8.76	0.52
Gain (post – pre)	343	13.19	14	6.00	15	14	14	5.00	0.29	-0.22

Table 6.6. Descriptive statistics of pre-/post-test aggregated scores of each teacher.

The boxplot diagram in Figure 6.3 shows that the mean score increased by 79% from the pre-test to the post-test, as did the median score of 84%, indicating better teacher performance on the post-test after training.

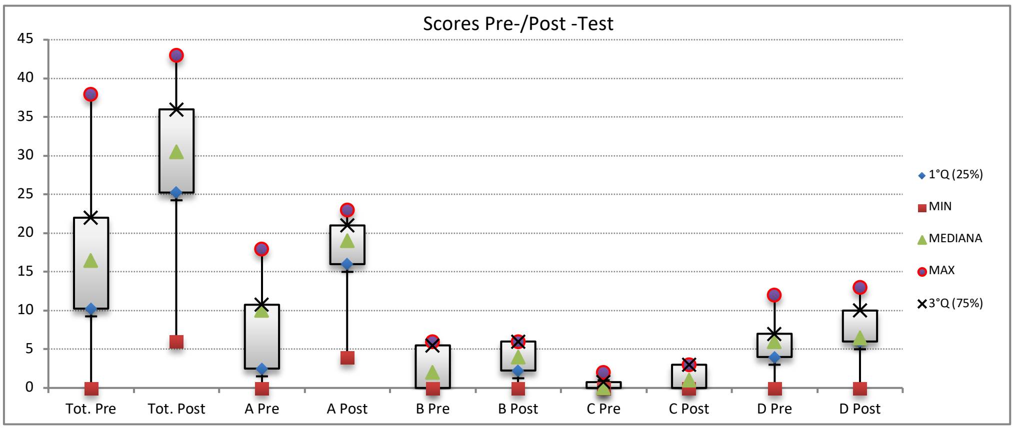


Figure 6.3. Comparison of boxplot data sets between the pre- and post-test.

A paired-sample t-test was conducted to compare the mean scores of teachers in the samples before and after the training. The null hypothesis (i.e., H_0) was tested at the significance level alpha equal to 0.05 and formulated as follows: There is no significant difference between the pre-test and post-test results. The statistical significance of the t-test in Figure 6.4 shows us that we can reject the null hypothesis. The critical value of t for a two-sided test for DF = 50 at $\alpha = 0.05$ is 2.009. Since the observed t-value of 5.340 is greater than the absolute value of the critical t-value, the two means are unequal, and their difference is significant.

Difference of scores	13,192
t (Observed value)	5,340
|t| (Critical value)	2,009
DF (Degree of Freedom)	50
p-value (Two-tailed)	<0,0001
Alpha	0,050
Cronbach's Alpha (Pre-test)	0,846
Cronbach's Alpha (Post-test)	0,795

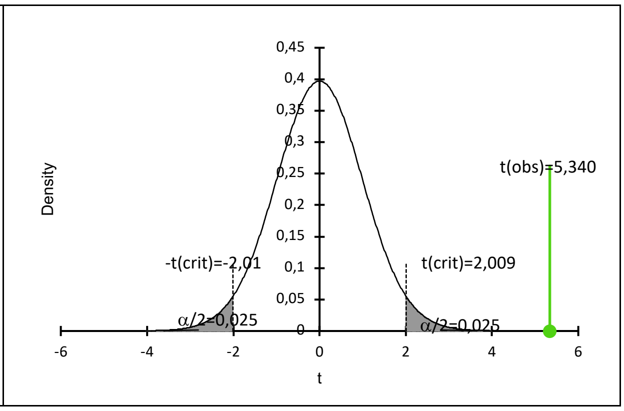


Figure 6.4. Two-tailed t-test with 95% confidence interval on the difference between the means: [8.230; 18.155] (elab. XLSTAT).

As in Section 3.2.6, the reliability of the questionnaire was evaluated using Cronbach's alpha. The pre-test instrument has a coefficient alpha of 0.846; for the post-test, Cronbach's alpha is 0.795 for n=17 (number of items in the questionnaire). Both reliability measures show a good level of internal consistency according to the reference range in Table 3.8. A pre-/post-training questionnaire can be considered a viable, effective, and reliable instrument for measuring the established learning objectives related to teachers' symmetry knowledge.

6.6 PRE- AND POST-TRAINING CONCEPT MAPS

In addition to the above-mentioned tools, concept mapping is a method for determining the level of knowledge achieved in a training course. They are widely used in schools, including math and science classes and educational research. Concept maps are graphical tools for organizing knowledge related to concepts, developed by Novak in 1972 (Novak & Cañas, 2006). As discussed in Section 1.2, conceptual knowledge is more than memorizing facts and is one of the most important components of students' mathematical competence. Concept maps aim at providing learners with meaningful learning that can change their cognitive structures related to the concept, as opposed to memorization, based on the rigid acquisition of memory content.

Novak (2010, p. 19) distinguishes between memorization, in which learned knowledge is recalled in exactly the form in which it was stored, and meaningful learning, in which experiences are transformed, organized, and linked to concepts; moreover, the two learning processes are completely different in terms of the neurobiology of brain function. Therefore, concept maps offer great potential for helping learners at different educational levels and in different disciplines to formulate and understand new concepts. They reflect a learner's cognitive framework and provide insight into their understanding as they graphically link topics and depict complex relationships between concepts.

According to Ausubel et al. (1978), changing a learner's cognitive structure requires using prerequisites or existing knowledge and integrating or changing relationships (i.e., cross-connections, links between concepts in different segments or domains). In this way, students must think critically to connect the newly learned concepts to their prior knowledge by developing the relationships between concepts in a concept map. Consequently, concept maps allow students to develop conceptual understanding and promote critical thinking. Teachers, in turn, can use concept maps to learn how students have related different concepts and to compare their prior knowledge with their final knowledge.

By observing the structure of a concept map, a teacher can interpret students' framework of knowledge about the concept (i.e., how they perceive, connect, and understand concepts). Concept maps thus serve as both a learning and assessment tool to examine and evaluate learners' content and structural knowledge about a particular concept. Concept mapping tool is particularly useful in assessing learners' knowledge organization at different points in their education to track short- or long-term changes in their cognitive structures. The concept maps technique has been used in educational research to assess acquired knowledge and conceptual changes in training courses by comparing concept maps created by participants at the beginning and end of the course.

We examined whether the concept maps reflect differences and expected changes in teachers' conceptual structures. Teachers in our research sample were asked to create concept maps about symmetry and invariance concepts at two-time points: before the beginning (i.e., pre-training concept map) and at the end of the teaching program (i.e., post-training concept map).

The web-based tool Mindomo allowed teachers to link topics and graphically represent relationships between concepts. Mindomo (www.mindomo.com) is an online concept map maker that is widely used in both K12 and higher education (Raman et al., 2015), as well as in teacher education and training (Sabourin, 2019) due to its high pedagogical functionality and integration with other platforms such as Google Classroom. In this way, concept maps are particularly useful for assessing interdisciplinary knowledge integration.

The increased visual complexity of concept maps before and after the course indicates an expected increase in knowledge integration through interdisciplinary instruction. Quantitative analysis allows for the assignment of a number that represents the quantitative assessment (i.e., score) of the concept maps.

The score is determined by the traditional metrics of concept maps that consider the number of concepts (NC), the hierarchy level (HL), the highest hierarchy level (HH) in the map, and the number of cross-links (NCL). Traditional scoring (Besterfield-Sacre et al., 2004) requires quantifying the number of the above components in each concept map to determine the sub-scores for knowledge breadth, depth, and interconnectedness summarised in the following Table 6.7.

Knowledge Breadth	Knowledge Depth	Knowledge Connectedness
The number of concepts (NC) in the concept map is counted. No consideration given to quality or correctness of concepts. In the formula cross linked concepts are not double counted. $KB = (NC - NCL) \times 1$	The number of hierarchies (HH), defined by propositions that include the concept map topic, in the concept map is counted. The highest level of hierarchy (HH) as number of concepts in the longest path down a hierarchy is recorded. $KD = (HH) \times 5$	The number of cross-links (NCL), which create propositions using concepts from different hierarchies, is counted. No consideration generally given to quality or correctness of cross-links. $KC = (NCL) \times 10$
$\text{Total Score} = KB + KD + KC = (NC - NCL) \times 1 + (HH) \times 5 + (NCL) \times 10$		

Table 6.7. Rubric for Traditional Concept Map Scoring (Watson et al., 2016; Novak & Gowin, 1984).

The overall metric is derived by combining the sub-scores using specific weights. Following the metric of Novak & Gowin (1984), each statement or concept is assigned 1 point, each level of the hierarchy is assigned 5 points, and each cross-connection is assigned 10 points (Table 6.7).

The total score for the concept map (i.e., structural complexity, Pruett & Weigel, 2020) is calculated from the sum of the weighted sub-scores. Such a scoring system favours learners with deeper and more interconnected knowledge over those with basic knowledge (Ruiz-Primo, 2000). Moreover, this assessment method is considered the most objective and appropriate for measuring changes in concept understanding (Erdimez et al., 2017) and for assessing progress in interdisciplinary integration of concept (Borrego et al., 2009).

We quantitatively analysed teachers' concept maps consistent with the above assessment measures. Although the task was given to all teachers participating in the study, only 13 teachers provided the final and initial concept maps because they needed to be adequately prepared for this teaching tool and needed more time to acquire appropriate knowledge. The data obtained were sufficient for the analysis.

For an example of assessment, see the TB9 Teacher Concept Map in Figure 6.5. We have highlighted and coded the main features of the concept maps with letters and numbers in each branch line with a different colour (i.e., a concept with C#, hierarchy level with H#, and cross-links with CL# in light blue). Teacher TB9's concept map shows 17 concepts, 2 of which are at the highest hierarchy level, and 6 cross-links, giving a score of 81 (see Figure 6.5).

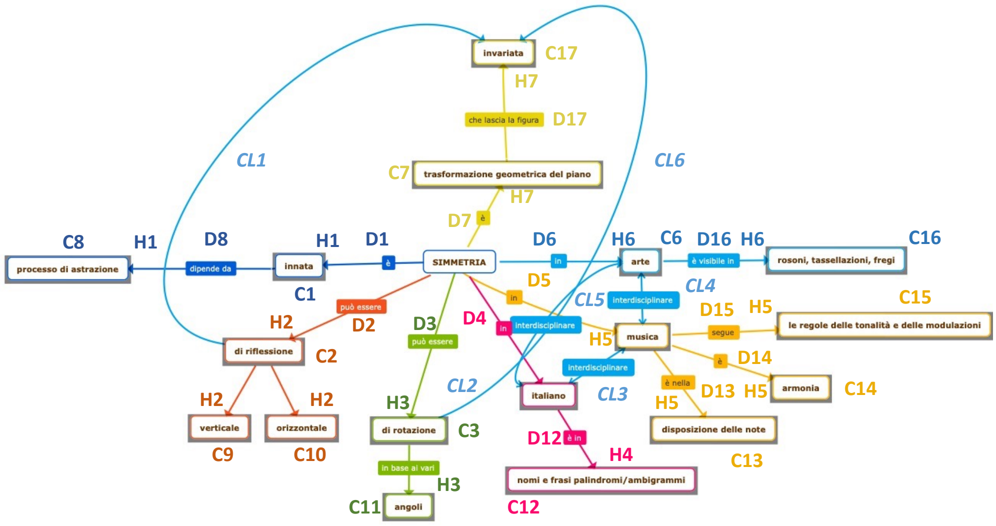


Figure 6.5. Teacher TB9 concept map after training with N.C.= 17, H.H.= 2, N.C.L.= 6, Total Score $= (17-6)+2\times5+6\times10 = 11+10+60 = 81$.

A comparison between the concept maps at the beginning and end of the training and the relative learning gains of some teachers can be found in Appendix V. The concept maps before and after training show differences in structure when viewed visually. The descriptive analysis in Table 6.8 confirms this visual difference. Specifically, the average knowledge breadth increases from 133 to 388, knowledge depth increases by 120 from 85 to 205, and the knowledge linkage jumps from 0 to 560. From the pre-training concept map to the post-training concept map, the increase in the average total score is 935.

The boxplot diagram in Figure 6.5 shows that the average score increased by 81% from the pre-map to the post-map, indicating that teachers performed better after training. Consequently, the concept maps after training result in a higher structural complexity score because the number of concepts and the highest hierarchy have significantly increased.

A paired-samples t-test was conducted to compare the mean scores of the teachers in the samples before and after the training. The null hypothesis (i.e., H_0) was tested at the significance level alpha equal to 0.05. According to Labovitz (2006), the significance level of 0.05 is supported by the moderate sample size (n=13) of the teachers, as small sample sizes would require a significance level greater than 0.05 to detect meaningful differences.

The statistical significance of the t-test in Figure 6.4 tells us that we can reject the null hypothesis (i.e., there is no difference between the mean scores of the concept map before and after the training). The critical value of t for a two-sided test for DF = 24 at α = 0.05 is 2.009. Since the observed t-value of 71.923 is greater than the absolute value of the critical t-value, the two means are unequal, and their difference is significant.

Variable (13)	Tot.	Aver.	Mode	Min.	q.1	Median	q.3	Max.	SD	CV
K.B. Pre-map	133	10,23	10	5	8	10	12	18	3,27	0,32
K.B. Post-map	388	29,85	24	11	24	33	36	38	8,60	0,29
Gain K.B. (post-pre)	255	19,62	14	6	16	23	24	20	5,33	-0,03
K.D. Pre-map	85	6,54	5	5	5	5	5	20	4,27	0,65
K.D. Post-map	205	15,77	15	10	10	15	15	35	6,72	0,43
Gain K.D. (post-pre)	120	9,23	10	5	5	10	10	15	2,45	-0,23
K.C. Pre-map	0	0,00	0	0	0	0	0	0	0,00	0,00
K.C. Post-map	560	43,08	40	20	40	40	50	60	9,47	0,22
Gain K.C. (post-pre)	560	43,08	40	20	40	40	50	60	9,47	0,22
Score Pre-map	218	16,77	15	10	13	15	17	32	6,31	0,33
Score Post-map	1153	88,69	93	76	85	91	93	94	5,56	0,07
Gain Sc. (post – pre)	935	71,92	78	66	72	76	76	62	-0,75	-0,26

Table 6.8. Descriptive statistics on the results of the concept maps before and after training, and the gains in Knowledge Breadth, Knowledge Depth, and Knowledge Connectedness.

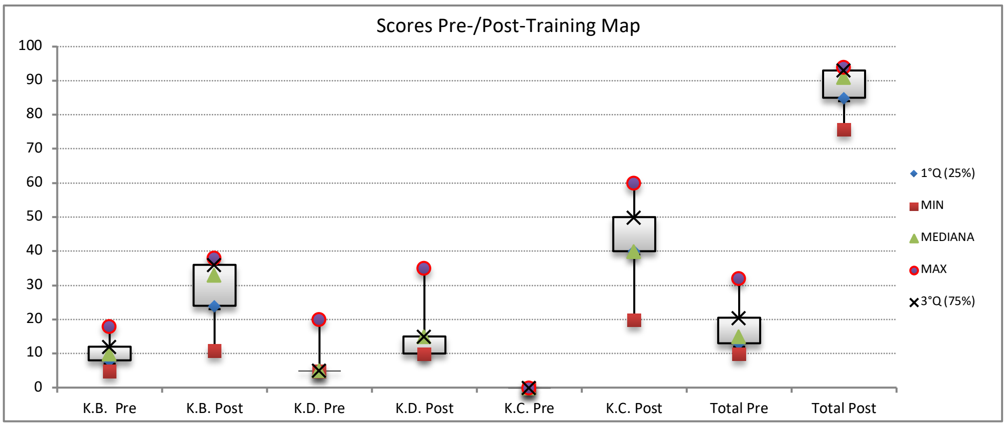


Figure 6.6. Comparison of boxplot data sets between the pre- and post-training concept maps.

The reliability of the concept maps is determined using Cronbach's alpha statistic. The results are 0.692 and 0.679 at the beginning and end of the online training, respectively, showing an acceptable level of internal consistency according to the reference range in Table 3.8 in Section 3.2.6.

The effect size (r) is calculated to measure the strength of the significant increment according to the following formula $z/\sqrt{N}$ (Field, 2013), where Z score is associated with t students and N is the number of observations. In our case, the effect size is 0.39 for Z=1.960 and N=26, because each of the 13 students drew a concept map before and after the course, and it is classified as in the middle range according to Cohen (1992).

Difference of scores	71,923
t (Observed value)	30,829
|t| (Critical value)	2,064
DF (Degree of Freedom)	24
p-value (Two-tailed)	<0,0001
Alpha	0,050
Cronbach's Alpha (Pre-test)	0,692
Cronbach's Alpha (Post-test)	0,679

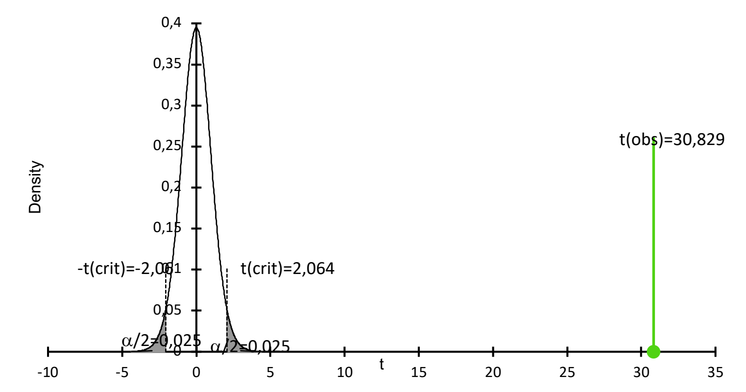


Figure 6.7. Two-tailed t-test with 95% confidence interval on the difference between the means: [67.108; 76.738] (elab. XLSTAT).

The results suggest that the concept map is an appropriate assessment tool to provide valuable information about the overall results of our educational research. Indeed, the concept map results suggest that half of the teachers have made significant progress in reorganizing their knowledge structure in terms of symmetry and invariance as a result of the training experience.

The most significant improvement relates to the contexts of the concept of symmetry, which the teachers lacked at the beginning of the training (see the feature KC in the pre-concept map in Table 3.8). Given the limitations of the study, primarily due to its small size, the training likely improved learning in the interdisciplinarity of symmetry and invariance, showing one side of the effectiveness of the course. This result needs to be further investigated and compared to the overall results of the other instruments used in the study.

6.7 TEACHERS SURVEY QUESTIONNAIRES

Among the tools we used to draw valid conclusions about the effectiveness of the course is the survey we asked student teachers at the end of the course (see Appendix W). Questionnaires are commonly used as research instruments in educational and social research (Tuzlukova et al., 2022). They typically consist of closed-ended questions (i.e., multiple-choice, scaled, and ranked questions) mixed with open-ended questions to allow for in-depth analysis when there are difficulties in interpreting the data with closed-ended questions.

Our questionnaire contained 34 Likert scale questions to collect information on various aspects of course effectiveness. As in questionnaires used in previous experiments, we used the 5-point Likert scale, which allowed respondents to indicate the extent to which they agreed or disagreed with each statement on a scale of 1 to 5. A score of 1 indicated strong disagreement, while a score of 5 indicated strong agreement. One of the advantages of using a Likert scale questionnaire is its ease of use and interpretation. The questions are simple and easy to understand, and the responses are generally easy to evaluate and analyse.

The use of a standardised Likert scale also allows for easier comparisons between different groups of teachers. Consequently, the Likert scale questionnaire is an important tool for evaluating the effectiveness of the training course. The use of this questionnaire allows for the collection of quantitative data that can be analysed to identify areas for improvement in the design and delivery of the course.

As shown in Figure 6.8, the questionnaire was divided into 5 parts with different colours and letters. The basic idea was to obtain aggregate information about certain general aspects of the course (i.e., A - background information about the course, B - teachers' knowledge of symmetry and invariance, C - materials and resources, D - teaching methods, and E - evaluation of teaching). The questionnaire included questions about the clarity of course objectives, the usefulness of course materials, and the level of interaction with teachers. In particular, questions about teachers' knowledge assessed their understanding of symmetry and invariance and how well they were able to apply that knowledge in the classroom. The questionnaire also asked about the effectiveness of teaching methods, such as lectures, discussions, hands-on activities, and instructor evaluations.

However, using a Likert scale questionnaire also has its limitations. Responses are limited to a predetermined number of options that may not fully capture the complexity of teachers' experiences. For this reason, student teachers were also asked to share their thoughts and experiences through open-ended questions in the survey and in further interviews as needed.

Q1	**A - General Aspects of the Course**
Q1.1	The training course met my expectations.
Q1.2	The concept of symmetry was dealt with in-depth.
Q1.3	The concept of invariance has been fully illustrated.
Q1.4	The integration between symmetry and invariance has been developed exhaustively.
Q1.5	The duration of the course was adapted to the topics covered.
Q1.6	The discussions and debates had adequate space within the course.
Q1.7	Considering that the course took place online, the time dedicated to group work and guided exercises was sufficient.
Q2	**B - Personal Body of Knowledge**
Q2.8	I have expanded my knowledge of the concept of symmetry thanks to the course.
Q2.9	I have expanded my knowledge on the concept of invariance thanks to the course.
Q2.10	The course has developed my ability to analyze interdisciplinary integrations about symmetry and invariance.
Q2.11	The course created new needs for personal study.
Q2.12	The course motivated me to follow the lessons and apply myself in the different activities.
Q3	**C – Materials and Resources**
Q3.13	The course materials were satisfactory and met my expectations.
Q3.14	The lessons of symmetry in music and art have been useful and satisfying.
Q4	**The following tools introduced to the course are consistent, suitable and useful for learning/teaching and applying the modern concept of symmetry and invariance:**
Q4.15	Cardboard boxes with normal and flip lids of different shapes.
Q4.16	Mira, mirrors, kaleidoscopic mirrors (Archimedes' machine).
Q4.17	Activities with letters, numbers, palindromes, viewing of the film "Palindromic Film".
Q4.18	Origami activity, symmetrical shape coloring, magic squares.
Q4.19	Activities with platonic solids, tessellations.
Q4.20	Geometrical invariance and shortest path problems and simple scientific experiments on invariance.
Q5	**D - Teaching Method**
Q5.21	The course allowed me to critically reflect on the teaching of symmetry and mathematics in general.
Q5.22	The course allowed me to understand the effectiveness and importance of symmetry in vertical development between primary and lower secondary schools and intercepted continuity with higher-level schools.
Q5.23	The course allowed me to understand the effectiveness and importance of invariance in vertical development between primary and lower secondary schools and intercepted continuity with higher-level schools.
Q5.24	Introducing the modern concept of symmetry (invariance in transformation) is inclusive for all pupils and more participatory in educational dialogue.
Q5.25	Introducing the modern concept of symmetry (invariance in transformation) facilitates the links between mathematics topics.
Q5.26	Introducing the modern concept of symmetry (invariance in transformation) facilitates links between mathematics and scientific disciplines.
Q5.27	The training course will influence my teaching practice of symmetry and mathematics in general.
Q5.28	I introduce the modern concept of symmetry in the classroom.
Q5.29	I can put into practice the knowledge and teaching/learning methods that I acquired during the course.
Q5.30	I am giving continuity to the action research on symmetry and invariance.
Q6	**E - Teaching Evaluation**
Q6.31	The speakers of the course proved to be competent for the topics addressed.
Q6.32	The intervention of the course speakers was effective in teaching.
Q6.33	The course speakers were available and ready to respond to the needs of the participants.
Q6.34	The speakers used coherent and appropriate language for the level of the course.

Figure 6.8. The questionnaires for the student teachers contain 34 questions divided into 5 categories.

Open- and closed-ended questions in the teacher surveys with additional interviews and online classroom observations complement each other and increase the reliability of the results. By using multiple methods, we can triangulate the data and cross-check the results, leading to a more comprehensive and accurate understanding and assessment of the impact of the training course on teachers' knowledge and instructional practices related to symmetry and invariance. In addition, this approach allows for a deeper exploration of teachers' views and experiences, resulting in more complex and reliable data. As a result, the data are more comprehensive, nuanced, reliable, and valid.

Despite several studies on the use of Likert scale questionnaires to evaluate the effectiveness of courses and some investigations of their psychometric properties, very few comprehensive studies have analysed the psychometric properties of these questionnaires to evaluate the effectiveness of courses using online and blended learning methods (Matosas-López & Cuevas-Molano, 2022). In this field, it is common to study the suitability of instruments to evaluate the effectiveness of courses. The two most important criteria to determine if an instrument is psychometrically suitable remain its validity and reliability to also measure the effectiveness of an online course (Kember & Leung, 2008; Fernández-Cruz et al., 2018; Abd ElHafeez et al., 2022). Validity and reliability are key factors that must be considered when developing a questionnaire to ensure that it provides accurate and reliable results. Several statistical approaches are used to ensure the reliability and validity with the 5-point Likert scale questionnaire.

Results for each item were analysed using descriptive statistics, including mean, median, minimum, maximum, and standard deviation. In addition, the Cronbach's alpha coefficient and item-total correlation (ITC), specifically Pearson correlation analysis for the Likert scale, were evaluated to measure the internal consistency of the instrument. These coefficients were discussed in detail in Section 3.2.6. All items of the questionnaire have high internal consistency with a Cronbach's alpha value of 0.96, indicating good reliability of the scale (Tapsir et al., 2018). The correlation values of the items range from 0.38 to 0.81 and have a median value of 0.69. Consequently, all items with a value of 0.30 and higher show good individual discrimination, and all items should remain in the questionnaire (George & Mallery, 2003, p. 231).

Most items are worth keeping because Cronbach's alpha is almost the same if you omit items from the constructs and dimensions. Table 6.9 below summarizes all the above psychometric factors.

Item No.	Range	Min	Max	Mean	Median	St. Dev.	Scale Means if Item deleted	ITC	KMO	Cronbach's Alpha if Item deleted
Q1.1	5	3	5	4,00	4,00	0,63	109,03	0,78	0,795	0,955
Q1.2	5	3	5	4,65	5,00	0,63	108,52	0,76	0,796	0,955
Q1.3	5	3	5	4,54	5,00	0,71	108,61	0,64	0,852	0,955
Q1.4	5	2	5	4,23	4,50	0,91	108,85	0,75	0,749	0,954
Q1.5	5	2	5	3,96	4,00	1,11	109,06	0,61	0,636	0,956
Q1.6	5	3	5	4,12	4,00	0,71	108,94	0,38	0,508	0,957
Q1.7	5	1	5	2,69	3,00	1,16	110,06	0,50	0,625	0,957
Q2.8	5	3	5	4,46	5,00	0,71	108,67	0,69	0,767	0,955
Q2.9	5	3	5	4,15	4,00	0,73	108,91	0,75	0,821	0,955
Q2.10	5	3	5	4,08	4,00	0,80	108,97	0,71	0,760	0,955
Q2.11	5	2	5	3,77	4,00	0,86	109,21	0,66	0,730	0,955
Q2.12	5	2	5	4,04	4,00	0,77	109,00	0,77	0,848	0,954
Q3.13	5	2	5	4,12	4,00	0,82	108,94	0,48	0,708	0,956
Q3.14	5	2	5	4,19	4,00	0,94	108,88	0,57	0,701	0,956
Q4.15	5	2	5	4,54	5,00	0,86	108,61	0,56	0,639	0,956
Q4.16	5	3	5	4,54	5,00	0,71	108,61	0,66	0,770	0,955
Q4.17	5	3	5	4,62	5,00	0,64	108,55	0,55	0,700	0,956
Q4.18	5	4	5	4,69	5,00	0,47	108,48	0,73	0,775	0,955
Q4.19	5	3	5	4,27	4,00	0,72	108,82	0,72	0,867	0,955
Q4.20	5	3	5	4,04	4,00	0,77	109,00	0,69	0,813	0,955
Q5.21	5	3	5	4,19	4,00	0,69	108,88	0,82	0,872	0,954
Q5.22	5	2	5	4,19	4,00	0,85	108,88	0,81	0,805	0,954
Q5.23	5	2	5	4,19	4,00	0,85	108,88	0,89	0,848	0,953
Q5.24	5	3	5	4,35	4,00	0,63	108,76	0,74	0,827	0,955
Q5.25	5	3	5	4,31	4,00	0,74	108,79	0,74	0,757	0,955
Q5.26	5	3	5	4,31	4,00	0,74	108,79	0,76	0,803	0,955
Q5.27	5	1	5	4,12	4,00	0,91	108,94	0,73	0,802	0,955
Q5.28	5	1	5	3,81	4,00	1,06	109,18	0,63	0,716	0,956
Q5.29	5	2	5	3,73	4,00	0,92	109,24	0,47	0,674	0,957
Q5.30	5	2	5	3,62	3,50	0,98	109,33	0,74	0,752	0,955
Q6.31	5	4	5	4,81	5,00	0,40	108,39	0,42	0,568	0,957
Q6.32	5	2	5	4,27	4,00	0,78	108,82	0,63	0,703	0,955
Q6.33	5	3	5	4,69	5,00	0,55	108,48	0,51	0,657	0,956
Q6.34	5	2	5	4,12	4,00	0,91	108,94	0,52	0,751	0,956

Table 6.9. Descriptive statistics of Item scales, Item-Total Correlation (ITC), Kaiser-Meyer-Olkin (KMO) and Cronbach's Alpha (elab. Xlstat).

In addition, the Kaiser-Meyer-Olkin (KMO) statistic was performed to assess the sampling adequacy and validity of the questionnaire for factor analysis. The resulting KMO value of 0.76 indicates an average suitability of the questionnaire for factor analysis. A KMO statistic of 0.6 or higher means that the variables might be suitable for structure recognition (Kaiser, 1974; Lloret et al., 2017). In our case, the results of the confirmatory factor analysis show that all items load significantly on a single factor. Consequently, the questionnaire is reliable and valid because all items measure the same construct and contribute to a single factor. This result is important because it indicates that the measurement is consistent and accurate in assessing the construct of interest. The single-factor structure simplifies interpretation of the data and facilitates their use in subsequent analyses or decisions. Overall, these statistical measures provide good evidence of the questionnaire's reliability and validity on the 5-point Likert scale for measuring the effectiveness of the online training course in terms of symmetry and invariance.

Figure 6.9 shows the full survey results in a coordinated series of divergent bar graphs, with blue representing agreement, red representing disagreement, and grey representing neutrality to the statements of the 34 items. Most teacher responses were positive, only 5% were mostly negative, and 1% disagreed totally negative. The analysis of the negative responses, highlighted in the figure above, shows that the negative comments are mainly related to the duration of the course, the time for group work and the implementation of the acquired theoretical knowledge and methods in teaching practise. Specifically, the relative comments are that there should be more hours for conducting the laboratory activities and that the topics could be easier for elementary school students.

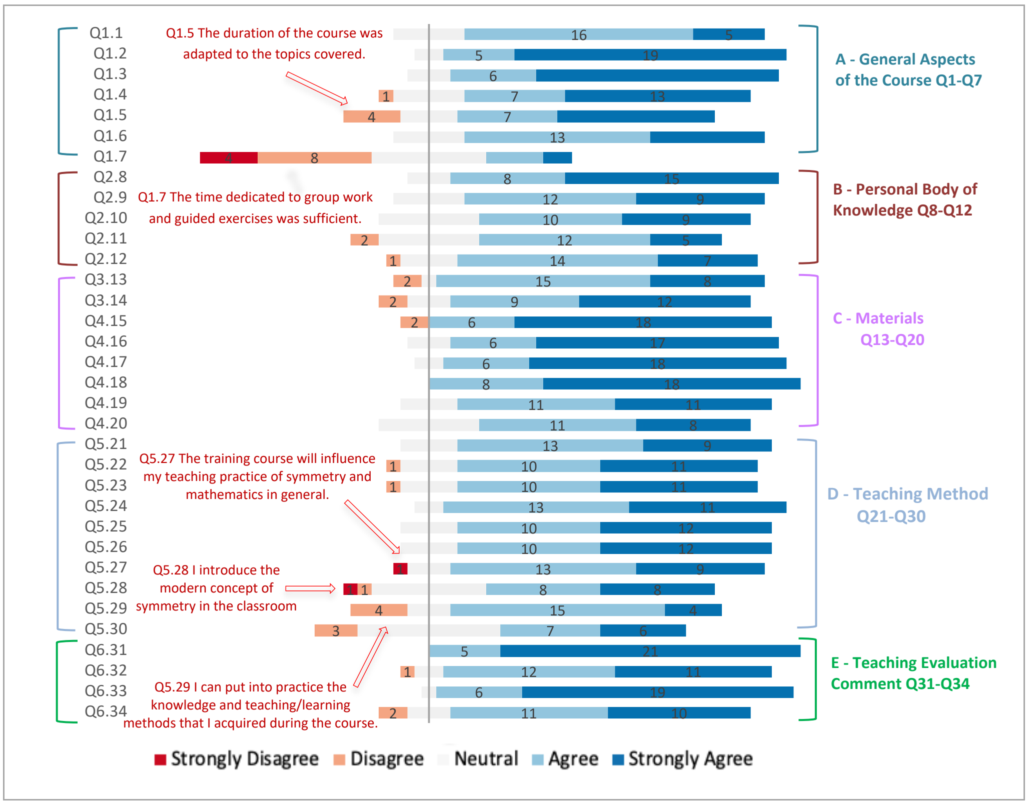


Figure 6.9. Divergent stacked bar graphs showing the distribution of responses for each question on the 5-point Likert scale, divided into 5 sections and with highlighting for the most negative responses.

Figure 6.10 shows the results in percent of the five-point Likert scale in sections A - E of the questionnaire. The sections are all positive, and the highest scores (about 90%) are obtained for the dimensions related to the materials and the didactic evaluation comment. For each section, I have singled out a few interesting comments from teachers to highlight the relative strengths in their perceptions.

On the first dimension A, teacher TD15 writes: “*The course stimulates my curiosity to ask me questions about topics I thought I fully understood*.”

Teacher TB8 writes on dimension B: “*The course is particularly effective, even for a distance learning course that usually increases the distance between teacher and learner. The ideas and incentives are numerous. In addition, the teaching is gradually becoming more engaging, and we have the opportunity to expand our contacts with teachers from other schools and even the university*.”

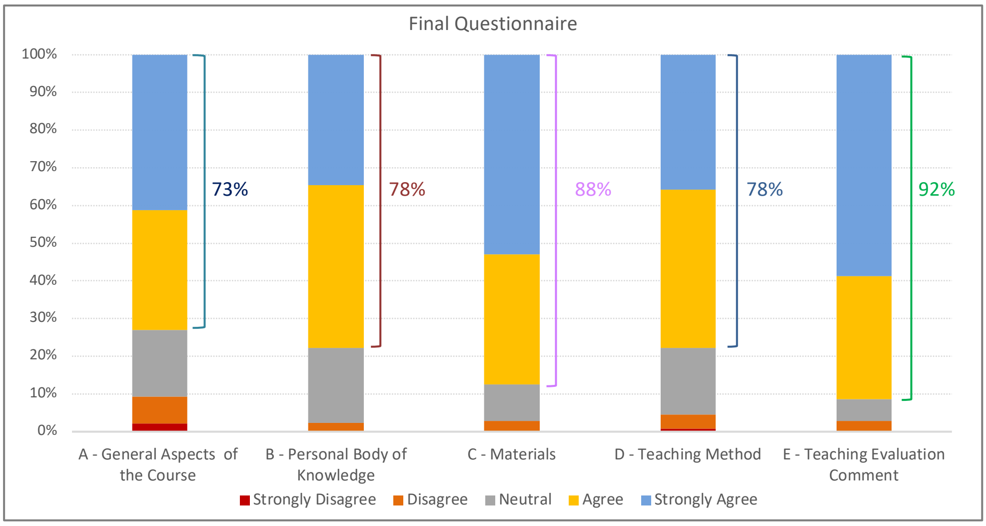


Figure 6.10. Histograms representing the distribution of teachers’ responses for each section A-E of the questions on the 5-point Likert scale.

Teacher TB6 expressed his interest in the materials as follows: “*I would like to introduce the concept of symmetry with the materials provided, adapting them to the needs of each class, especially boxes and mirrors, and linking the lessons to moments of laboratory teaching where students experiment concretely with the concept of symmetry*.”

TD18 is interested in magic squares: “*I will introduce the modern concept of symmetry in the second grade of elementary school by proposing lessons on magic squares in which students can have a first access to the modern concept of symmetry; this modality allows me to develop a topic of cardinal interest in the following years without putting it aside and losing the opportunity to build other knowledge on stable and consolidated learning pillars*.”

Teacher TE23’s feelings about the teaching method (i.e., dimension D) at the end of the training are positive and he/she is ready to implement activities in the classroom, as he/she personally states, "*I feel ready to carry out activities in the classroom that involve and connect different disciplines: math, science, music, drawing, art. By observing nature (fruits, animals, snowflakes, minerals...) and reading texts, simple songs, simple drawings, mosaics, projection of film clips, manipulative activities such as building origami, polyhedra, boxes, polygons, I will teach the children the importance of similarity and invariance, transformation, and conservation. The activities will generate many ideas and we will work toward understanding that symmetry reflects an image, but so is invariance; this can start in elementary school.*"

Regarding dimension E of teaching evaluation, teacher TB9 testifies his/her gratitude to the teaching professor: "*This course has allowed me to grow professionally, so I am grateful to all the professors*."

TD19 expresses positivity about the course and the instructors and disappointment about distance learning, "*I do not have any suggestions because the course is interesting, and the instructors are very good. It is a pity that it had to take place online, because it would certainly have been even more interesting and appealing in a face-to-face meeting.*"

Then, the results of the questionnaire presented to teachers show a positive response to the proposed online training program. Although some teachers expressed uncertainty about teaching symmetry differently than they were used to, most teachers agreed that proposing activities with new materials in the classroom was important. They indicated that they had deepened their knowledge of symmetry by recognizing the interdisciplinary nature of symmetry, especially when it was associated with invariance. They also reported that they had mastered the use of the new learning objects for teaching and would welcome more time to reflect on them.

Despite these positive results, the questionnaire also revealed some areas for improvement. For example, some teachers reported needing more support in implementing the new instructional strategies, and others requested additional resources to support their teaching. However, most teachers showed flexibility and creativity in implementing the workshop in their classrooms, indicating a willingness to adapt to the new instructional strategies.

In addition, the use of a questionnaire with a 5-point Likert scale proved to be one of the useful tools to gain more insight into the impact of the course on teachers' knowledge and teaching practices related to symmetry and invariance. The positive feedback from teachers underscored the potential benefits of online training programs for professional development in education and suggested that such programs could effectively promote interdisciplinary learning and new teaching strategies.

6.8 DISCUSSION ON THE FINDINGS AND CONCLUSIONS

Due to the Covid 19 pandemic, schools and research institutions worldwide have shifted to online education. Our research project adapted to this shift and provided online training for teachers to continue our earlier research cycles. Thus, the third experiment of the action research project was primarily a learning opportunity for in-service teachers on symmetry and invariance. Through this teaching and learning experience, we gained additional knowledge to answer the research questions of the entire experiment, specifically SRQ2, which referred to the impact of the paradigm shift of symmetry on the didactic framework. The use of mixed methods approaches allowed for a more comprehensive and complete understanding of the factors that may promote or hinder teachers' adoption of symmetry and invariance in the classroom.

We collected quantitative and qualitative data using a variety of instruments, including pre- and post-training questionnaires, pre- and post-training concept maps, final surveys based on Likert scale questions, interviews, and observations. Each instrument used in the experiment produced specific outcomes that were triangulated, showing convergence between the qualitative and quantitative methods. Teachers reported increased knowledge about symmetry, and pre- and post-training analysis showed increased learning. The final survey and interview analysis confirmed teachers' growth in skills and knowledge about symmetry, with most expressing a willingness to incorporate it into their teaching. Since numerous teachers expressed interest in conducting teaching and learning sessions on symmetry and invariance in their classrooms, their feedback suggests that the third experiment was successful.

This experiment served as a catalyst for change in the way teachers approach symmetry-based instruction, and it has the potential to spur ongoing professional development. As teachers implement the lessons, they will internalize the key components of the pedagogical framework, including interdisciplinary instruction and the use of learning materials and appropriate tasks based on the paradigm shift of symmetry. In this case, the innovative tasks and the interdisciplinary approach are the greatest strengths of the didactic implementation process.

Classes	Teachers	School	N° Pupils	Materials
IA IC ID IF IG	TD17 TE22 TC12 TB10	Primary	81	Mira, Numbers, Letters
IIA IIC IID IIF IIG	TD18 TE23 TA5	Primary	100	Magic Squares, Mirrors
IIIA IIIC IIID IIF IIIG	TA2 TB7	Primary	83	Mira, Mirrors
IV A	TB7	Primary	14	Cardboard boxes 1
VC VD VF VG	TB9 TE24 TD17	Primary	106	Cardboard boxes 1
IA IB IC ID IE	TA1 TC11 TE20 TD15 TB6	Secondary	106	Cardboard boxes 2, Kaleidoscope

Table 6.10. Summary of teachers and classes involved in implementing the symmetry learning materials in the classroom.

Table 6.10 contains all the teaching and learning units based on the provided online materials that teachers will design and use in class. Almost all school classes will be involved, except for the second and third grades of secondary school, as they had just participated in the previous experiments. Such a result suggests that the interdisciplinary approach is widely used and implemented throughout the school, which can lead to a more comprehensive and integrated learning experience for students. Furthermore, participants who complete the tasks and design of teaching units on the course are awarded a certificate of attendance in Attachment X, which may serve as a form of recognition for their efforts and accomplishments.

6.9 LIMITATIONS OF THE THIRD EXPERIMENT

The third experiment examined the effects of a training program to improve the understanding of symmetry and invariance among teachers-in-training. Although limited by a small sample size, the challenges associated with online instruction, and the selection of participants from primarily one school, the study nevertheless suggests that the course successfully improved teaching and learning of symmetry topics, highlighting the potential benefits of interdisciplinary training focused on concepts related to symmetry and invariance. However, future studies should include larger and more diverse samples and present the results at conferences with more researchers to obtain more meaningful results. Despite the limitations of the study, the results shed light on the importance of professional development for teachers and the potential benefits of interdisciplinary training to improve the teaching and learning of symmetry and invariance.

Chapter 7: Conclusions and Perspectives

7.1 CONCLUDING REMARKS

In this Ph.D. study, we conducted action research with cycles of reflection to evaluate the effectiveness of a particular teaching and learning approach that introduces elementary students to the modern concept of symmetry and to assess its impact on the didactic framework. Over the course of three years, our experiments demonstrated the effectiveness of an interdisciplinary teaching approach to symmetry and invariance that begins in elementary school and includes cognitive, affective, and psychomotor dimensions.

Our results suggest that introducing generalized and interdisciplinary symmetry concepts to fifth graders can improve their spatial skills and motivate them to see mathematics and its major concepts as more connected to the real world and scientific disciplines. An important finding of our study is that fifth grade students can understand the invariants associated with symmetry and adapt their understanding and mastery of symmetry to the new concept.

We also found that manipulative games and artefacts have the potential to facilitate the acquisition of cognitive processes relevant to mathematical modelling and generalization of symmetry concepts. Geometry activities for young children provide experiences that set the stage for real change in instruction and motivate students to process the creation of more complex symmetries in later stages of mathematics instruction. The didactic pathway to symmetry should focus on finding invariants through various symmetry operations, especially rotations, as manipulation with rotations can significantly influence tacit knowledge in further learning about geometric transformations.

Our study suggests that a paradigm shift in teaching symmetry can have positive effects on students' understanding and skill development, and that teachers can harness the potential of symmetry and invariance as powerful tools for interdisciplinary knowledge and the results can inform future mathematics education research and practice.

7.2 RESTATEMENT OF LIMITATIONS

The limitations of this work are rooted in the research design, which focused on only one school and collected data from a teacher training program adapted due to the Covid-19 pandemic. Although the results of the study demonstrate promising transferability of findings and enough participating students, it is not appropriate to assume that the conclusions are generalizable beyond the sample. Therefore, it is important to acknowledge these limitations to avoid overgeneralization or unwarranted claims.

To address these limitations, future research should prioritize including the perspectives of teachers from other schools to broaden the database and improve the generalizability of findings. In addition, conducting a randomized controlled trial or other robust research design can reduce the risk of bias and increase the validity of conclusions. Using these approaches, we can provide a more comprehensive and accurate account of study findings and make an important contribution to the knowledge base on applying symmetry and invariance in schools.

7.3 IMPLICATIONS AND FUTURE RECOMMENDATIONS

The dissertation contains several implications and recommendations for future research.

First, the limitations of the study suggest that the sample needs to be increased by including teachers and students from different schools. This would allow for a more comprehensive understanding of the topic and improve the generalizability of the findings.

Second, the successful implementation of the lessons suggests the potential benefits of incorporating modern concepts into elementary school curricula. The dissemination of the symmetry concept and teachers' engagement in planning and implementing instructional materials are promising.

After completing the third experiment, we had the opportunity to design and conduct a lesson in the framework of a teacher training organised by the USR Marche and the University of Camerino (Unicam). Numerous teachers from all schools of the Marche region, divided into nine groups, participated in the 40-hour lesson on the modern concept of symmetry related to invariance.

In addition, the results of the second grade experiment were discussed as part of the final assignment of the online training on the introduction to symmetry and invariance through the use of magic squares at two conferences: the first was the XXXV National Conference “Mathematics Meetings” in Castel S. Pietro Terme (Bo) from 5 to 7 November 2022 and the second was the 16th International Conference “Building on the Past to Prepare for the Future” in King’s University, Cambridge (UK) from 8 to 13 August 2022.

Thirdly, the new research in an elementary school in Fermo demonstrates a commitment to follow up on the study’s findings and develop a more comprehensive learning pathway on symmetry. Focusing on all isometries, especially rotations, is a valuable contribution to this topic.

Overall, the recommendations underscore the importance of ongoing research for improving teaching and learning, particularly in elementary schools.

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Appendices

Appendix A

FIRST TEACHER QUESTIONNAIRE

Dear teachers,

there are still some solid and sure figures in pedagogical action (however difficult and complex it may be due to the vastness of the spectrum of variables in this field), among which that of the maestro/a; from the Latin "*Magister*" or the French "*Maîtres à penser*", his influence in the most intimate and fragile matter of children is so significant that it etches, shapes and sets on fire the hearts of future students. In the words of Plato, the voice of the master is still far more decisive than any book...(and any tablet!).

Thank you for the opportunity to share your activities and valuable research information with you. I have prepared questions for you to replace the questions in the presentation. They will be very useful both for better organization of activities and for feedback on the effectiveness of the course. In a cheerful atmosphere of shared goals, I wish you Good Work!

Q1) How long have you been teaching mathematics in elementary school?

Q2) What subjects do you particularly enjoy teaching?

Q3) What adjectives/words would you associate with mathematics?

Q4) How would you define or what mathematics means to you?

Q5) Do you agree with the following statements?

	Indicate the level of agreement with each statement by using the scale from 1 (strongly disagree) to 5 (strongly agree) next to each statement.					
	1 Strongly Disagree 😩 / **2 Disagree** 🙁 / **3 Neutral** 😐 / **4 Agree** 🙂 / **5 Strongly Agree** 😃					
Q5 1	Mathematics is more difficult than other subjects.	**1**	**2**	**3**	**4**	**5**
Q5 2	Mathematics creates anxiety in students.	**1**	**2**	**3**	**4**	**5**
Q5 3	Mathematics creates anxiety in teachers.	**1**	**2**	**3**	**4**	**5**
Q5 4	I enjoy teaching mathematics.	**1**	**2**	**3**	**4**	**5**

Q6) To what would you attribute the difficulties in mathematics in each case?

Q7) How would you rate your knowledge of the topics listed in the box?

Use the scale of 1 (inappropriate) to 5 (appropriate) given next to each statement.						
1 Inappropriate 😩 / **2 Slightly inappropriate** 🙁 / **3 Neutral** 😐 / **4 Slightly appropriate** 🙂 / **5 Appropriate** 😃						
Q7 1	Symmetry	**1**	**2**	**3**	**4**	**5**
Q7 2	Invariance	**1**	**2**	**3**	**4**	**5**
Q7 3	Multidisciplinary integration with symmetry	**1**	**2**	**3**	**4**	**5**

Q8) What key words can you relate to the concept of symmetry?

Q9) What key words can you associate with the concept of invariance?

Q10) What definition or connotation do you associate with symmetry?

Q11) What activities do you generally use when teaching symmetry?

Q12) How important do you think it is for students to know symmetry on the vertical path?

Use the scale from 1 (Not important) to 5 (Important) given next to each statement.

1	2	3	4	5
Not important at all 😩	**Not important** 😕	**Neutral** 😐	**Important** 🙂	**Very important** 😃

Q12 1	For the mathematics curriculum.	**1**	**2**	**3**	**4**	**5**
Q12 2	For the science curriculum.	**1**	**2**	**3**	**4**	**5**
Q12 3	For the civic competencies.	**1**	**2**	**3**	**4**	**5**

Q13) Justify the previous answer.

Q14) With what themes is the concept of symmetry directly related?

Q15) Justify the previous answer.

Q16) Do you think that knowing about the vertical development of the concept of symmetry could be important for your teaching practice?

Use the scale from 1 (Not important) to 5 (Important) given next to each statement.

1	2	3	4	5
Not important at all 😩	**Not important** 😕	**Neutral** 😐	**Important** 🙂	**Very important** 😃

Q16 1	In the teaching of mathematics.	**1**	**2**	**3**	**4**	**5**
Q16 2	In the teaching of science.	**1**	**2**	**3**	**4**	**5**
Q16 3	For civic competencies.	**1**	**2**	**3**	**4**	**5**

Q17) Justify the previous answer.

Place ________________________________ Date __________________________

Appendix B

PUPILS PRE-/POST-TEST QUESTIONNAIRE

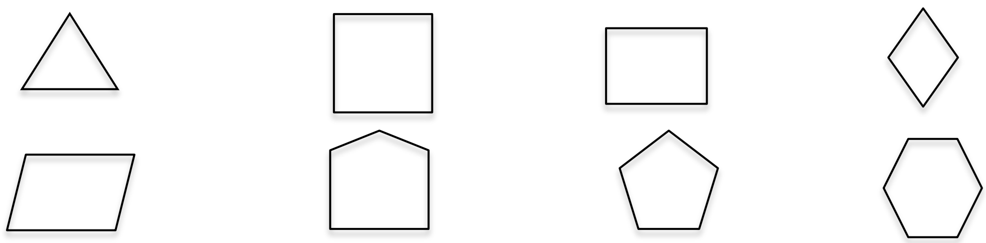
Q1) If you were the "Rose" trying to explain symmetry to the "Little Prince", which of the following sentences would you use?

a) There is symmetry when you can see that, if a line divides a figure in two parts, these parts reflect each other like in a mirror.

b) Among the many symmetries that exist there are also the movements that transform a figure so that the resulting figure coincides with the original.

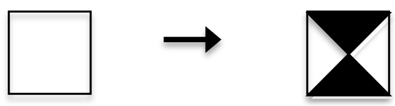
Write the reason of your choice.

Q2) After helping Paul close the various boxes, note the number of ways the lid can be placed on each box and draw the lines of symmetry (axes), if any.



Q3) Tell the "Little Prince" about the classroom activities, what you enjoyed most, what you learned about symmetries, and if you had any difficulties.

Q4) The "Rose" explains to the "Little Prince" that there are also colour symmetries. By decorating a figure with colours, you can change its symmetry. Do you agree with this? Explain what happens to the symmetry of the square if you change it from all white to coloured in the following way.



Appendix C

DIALOGUE BETWEEN ROSE AND THE LITTLE PRINCE

C'erano sempre stati sul pianeta del piccolo principe dei fiori molto semplici, ornati di una sola raggiera di petali, che non tenevano posto e non disturbavano nessuno. Apparivano un mattino nell'erba e si spegnevano la sera. Ma questo era spuntato un giorno, da un seme venuto chissà da dove, e il piccolo principe aveva sorvegliato da vicino questo ramoscello che non assomigliava a nessun altro ramoscello. Poteva essere una nuova specie di baobab. Ma l'arbusto cessò presto di crescere e cominciò a preparare un fiore. Il piccolo principe, che assisteva alla formazione di un bocciolo enorme, sentiva che ne sarebbe uscita un'apparizione miracolosa, ma il fiore non smetteva più di prepararsi ad essere bello, al riparo della sua camera verde. Sceglieva con cura i suoi colori, si vestiva lentamente, aggiustava i suoi petali ad uno ad uno. Non voleva uscire sgualcito come un papavero. Non voleva apparire che nel pieno splendore della sua bellezza.

Eh, si, c'era una gran civetteria in tutto questo! La sua misteriosa toeletta era durata giorni e giorni. E poi, ecco che un mattino, proprio al levar del sole, si era mostrato. E lui, che aveva lavorato con tanta precisione, disse sbadigliando: "Ah! Mi sveglio ora. Ti chiedo scusa... sono ancora tutto spettinato..."

Il piccolo principe allora non poté frenare la sua ammirazione: "Come sei bello!" "Vero", rispose dolcemente il fiore, "e sono nato insieme al sole..." Il piccolo principe indovinò che non era molto modesto, ma era così commovente! "Come fai ad essere così bello?" "Vedi, io sono un fiore e sono una creazione della natura, e in quanto tale sono perfettamente simmetrico..." "Non capisco" rispose il piccolo principe spiazzato dall'uscita del fiore."

Ora ti spiego" disse superbamente il fiore. "In natura esistono tantissime simmetrie" "E a cosa servono?""Beh, a fare i fiori belli, non c'è dubbio. Una simmetria della natura è qualcosa che il sole ci ha dato e che nessuno potrà mai imitare. Tutto, in natura, nasce da una simmetria. Tante cose in natura sono simmetriche, sai?" "Cosa?" "Ad esempio, le stelle marine, i fiocchi di neve, le celle degli alveari delle api e i cristalli...l'uomo!" "Mai stata neve né api sul mio pianeta". "Il piccolo principe però era attirato dai discorsi del fiore." "Tutti gli esseri viventi sono belli e simmetrici sotto diversi punti di vista... io, ad esempio, sono colorato e le simmetrie dei colori dei miei petali mi fanno bello".

Soon I found out more about this flower. Flowers on the planet of the Little Prince were always very plain. They had one layer of petals; they didn't occupy territories or hurt anyone. They broke out in the grass in the morning and faded peacefully in the evening. Once, a new flower grew up from an unknown seed, and the Little Prince watched very carefully a new sprout that was so unlike any other on the planet. You know, it could be a new sort of baobab trees. Soon, the bush stopped growing and prepared for a flower to appear. The Little Prince, who was present during the first breaking open of a huge bud, knew immediately it would be something marvellous. But the flower didn't want to stop beautifying itself in the shadows of the grass. The beauty was choosing colours very carefully. It was improving its petals one by one. It didn't want to arrive ragged as field poppies, it wanted to be beautiful.

Oh, yes! It was a teasing flower! And its mysterious improvement had been happening for days. Then one morning, at dawn, it showed itself at last. And after such a hard work it yawned and said. "Ah! I just woke up. Excuse me. My lapels are a mess..."

The little prince then could not restrain his admiration: "You look so beautiful!" "Do I?" the flower replied kindly. "I was born at the moment when the sun..." The Little Prince realised at once that it wasn't modest at all, but it was so touching and alluring! "How can you be so beautiful?" "Well, you see, I am a flower and am a creation of nature, and as such I am perfectly symmetrical..."

"I don't understand" replied the little prince, taken aback by what the flower had said.

"Now I'll explain it to you" said the flower haughtily. "In nature there are many symmetries" "And what is their purpose?"

"Well, to make flowers beautiful, there is no doubt. A symmetry of nature is something that the sun has given us and that no one will ever be able to imitate. Everything in nature, is born from a symmetry. Many things in nature are symmetric, did you know that?" "What?" "For example, starfish, snowflakes, the cells of the beehives and crystals... man!"

"Never been snow nor bees on my planet". The little prince, however, was attracted by what the flower was saying. "All living things are beautiful and symmetrical from different points of view... I, for example, am coloured and the symmetries of the colours of my petals make me beautiful".

Appendix D

HELP PAOLO TO SOLVE THE BOX PROBLEM!

CLASS: ……………. GROUP NAME: …………………… GROUP CAPTAIN: ……………………

GROUP MEMBERS: ………………………………………………………………………………

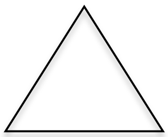
After Paolo sees that the boxes can be closed in different ways, he looks around the house for other boxes and jars with lids. He finds a jar of sugar based on a circle. How many ways can he put the lid on the jar? Why?

....

....

....

In how many ways can the lid of a box based on these shapes be placed? Are you sure about that? How can you know that?



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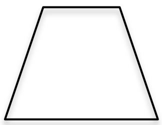

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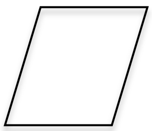

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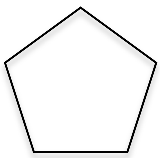

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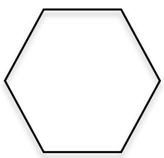

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FURTHER REFLECTION - FREE THOUGHT

Appendix E

ITEM EFFICIENCY PARAMETERS FOR VARIABLES R1, R2, R2* in Q2.

R1	*Item 1*	*Item 2*	*Item 3*	*Item 4*	*Item 5*	*Item 6*	*Item 7*	*Item 8*	*Avg.*	*S.D.*
P.I.	0,90	0,97	0,84	0,69	0,71	0,81	0,49	0,50	0,74	0,18
Q.I.	0,10	0,03	0,16	0,31	0,29	0,19	0,51	0,50	0,26	0,18
D.I	0,29	0,13	0,38	0,50	0,67	0,54	0,92	0,92	0,54	0,28
Rp-b	0,61	0,58	0,54	0,39	0,59	0,64	0,67	0,65	0,58	0,08

R2	*Item 1*	*Item 2*	*Item 3*	*Item 4*	*Item 5*	*Item 6*	*Item 7*	*Item 8*	*Avg.*	*S.D.*
P.I.	0,71	0,90	0,73	0,88	0,60	0,83	0,31	0,24	0,65	0,25
Q.I.	0,29	0,10	0,27	0,13	0,40	0,17	0,69	0,76	0,35	0,25
D.I	0,63	0,33	0,71	0,25	0,67	0,54	0,96	0,79	0,61	0,20
Rp-b	0,58	0,42	0,49	0,29	0,59	0,57	0,76	0,72	0,55	0,11

R2*	*Item 1*	*Item 2*	*Item 3*	*Item 4*	*Item 5*	*Item 6*	*Item 7*	*Item 8*	*Avg.*	*S.D.*
P.I.	(0,47*)	0,79	0,68	0,84	0,60	0,83	0,26	0,21	0,59	0,25
Q.I.	(0,53*)	0,21	0,32	0,16	0,40	0,17	0,74	0,79	0,41	0,25
D.I	0,73	0,42	0,75	0,29	0,67	0,54	0,92	0,77	0,64	0,23
Rp-b	0,64	0,44	0,55	0,41	0,60	0,57	0,74	0,70	0,58	0,15

Appendix F

RANK-FREQUENCY TABLE OF WORD TYPES.

Order	Unfiltered word count		Oc.
1.	che	that	168
2.	e	And	167
3.	è	is	139
4.	ho	I have	136
5.	abbiamo	we have	114
6.	in	in	105
7.	mi	me	94
8.	simmetria	symmetry	94
9.	di	from	92
10.	la	there	90
11.	non	Not	79
12.	le	the	78
13.	il	the	76
14.	imparato	learned	69
15.	sono	I am	67
16.	difficoltà	difficulty	54
17.	piaciuto	liked	51
18.	un	a	49
19.	una	a	48
20.	si	yes	46
21.	scatole	cans	46
22.	avuto	had	45
23.	anche	also	45
24.	cose	what's this	44
25.	più	more	39
26.	molto	Very	37
27.	attività	activities	36
28.	a	to	35
29.	molte	many	34
30.	fatto	done	34
31.	quando	when	33
32.	film	movie	33
33.	con	with	32
34.	i	the	31
35.	ci	there	31
36.	ma	but	30
37.	modi	ways	28
38.	sulla	on the	28
39.	simmetrie	symmetries	27
40.	visto	view	25
41.	delle	of the	25
42.	al	to the	24
43.	chiudere	to close	24
44.	per	for	22
45.	palindromo	palindrome	20
46.	parole	words	20
47.	vedere	see	20
48.	scoperto	discovery	19
49.	solo	alone	19
50.	capito	understood	18
51.	può	can	18
52.	possono	can	18
53.	classe	class	17
54.	mela	apple	17
55.	poi	then	17
56.	quanti	how many	17
57.	l	L	16
58.	io	I	16
59.	come	how	16
60.	cosa	What	16
61.	figure	figures	16
62.	tutto	everything	16
63.	queste	these	16
64.	se	self	15
65.	della	of the	15
66.	stella	star	14
67.	contrario	contrary	14
68.	cui	which	14
69.	ha	has	13

Order	Unfiltered word count		Oc.
70.	po	bit	13
71.	costruito	built	13
72.	palindrome	palindrome	13
73.	numeri	numbers	13
74.	piaciuta	liked	13
75.	era	was	13
76.	assi	axes	13
77.	nel	in the	12
78.	sulle	on	11
79.	cos	cos	11
80.	coperchio	cover	11
81.	perché	Why	10
82.	palindromi	palindromes	10
83.	gruppo	group	10
84.	alcune	someone	10
85.	dei	of the	10
86.	nella	in	10
87.	questo	this	10
88.	cioè	that is	9
89.	esempio	example	9
90.	figura	figure	9
91.	hanno	they have	9
92.	forma	form	9
93.	noi	we	9
94.	prima	Before	9
95.	divertito	amused	9
96.	fare	Do	9
97.	da	from	8
98.	lavoro	work	8
99.	principe	prince	8
100.	tante	many	8
101.	stata	was	8
102.	stato	state	8
103.	orizzontale	horizontal	8
104.	piccolo	small	8
105.	gli	the	8
106.	nomi	names	8
107.	altre	other	8
108.	nessuna	none	8
109.	o	or	7
110.	me	myself	7
111.	quella	that	7
112.	quello	that	7
113.	essere	to be	7
114.	parola	word	7
115.	professore	professor	7
116.	piaciute	liked	7
117.	all	annex	7
118.	del	of the	7
119.	tutte	all	7
120.	nelle	in	7
121.	questa	this	7
122.	circonda	surrounds	7
123.	diverse	different	6
124.	dentro	inside	6
125.	frasi	phrases	6
126.	state	been	6
127.	uguale	same	6
128.	rotazione	rotation	6
129.	nuove	new	6
130.	colore	color	6
131.	lezioni	lessons	6
132.	dopo	after	6
133.	dove	where is it	6
134.	seconda	second	6
135.	esistono	exist	6
136.	sapevo	I knew	6
137.	alcuni	some	6
138.	forme	forms	6

Order	Unfiltered word count		Oc.
139.	tutti	all	6
140.	mia	my	6
141.	natura	nature	6
142.	scatola	box	6
143.	divertita	amused	6
144.	asse	axis	6
145.	ad	to	5
146.	ed	and	5
147.	su	on	5
148.	video	video	5
149.	diversi	different	5
150.	scritto	written	5
151.	simmetrica	symmetrical	5
152.	simmetrici	symmetrical	5
153.	guardare	watch	5
154.	simmetriche	symmetrical	5
155.	problemi	problems	5
156.	capire	understand	5
157.	coperchi	lids	5
158.	costruire	to build	5
159.	puntata	bet	5
160.	volte	times	5
161.	colori	colors	5
162.	lezione	lesson	5
163.	dovevamo	we had to	5
164.	modo	way	5
165.	linea	line	5
166.	giorno	day	5
167.	ciò	that is	5
168.	lette	read	5
169.	esercizi	exercises	5
170.	tra	between	5
171.	anna	anna	5
172.	potevano	they could	5
173.	svolte	carried out	5
174.	svolto	carried out	5
175.	c	c	4
176.	spiegato	explained	4
177.	paolo	Paul	4
178.	ne	neither	4
179.	avevo	I had	4
180.	trova	find	4
181.	grazie	thank you	4
182.	problema	problem	4
183.	tanto	much	4
184.	porre	to pose	4
185.	orizzontalmente	horizontally	4
186.	bella	beautiful	4
187.	incontro	match	4
188.	maestro	master	4
189.	tagliato	cut	4
190.	divide	divides	4
191.	soprattutto	mostly	4
192.	vari	various	4
193.	inizio	Start	4
194.	secondo	according to	4
195.	comunque	anyway	4
196.	geometriche	geometric	4
197.	matematica	mathematics	4
198.	letto	read	4
199.	girare	spin	4
200.	nome	first name	4
201.	suo	his	4
202.	questi	these	4
203.	cartone	cardboard	4
204.	spiegava	he explained	3
205.	ce	there is	3
206.	bene	well	3
207.	cambia	changes	3

Order	Unfiltered word count		Oc.
208.	parte	part	3
209.	sempre	always	3
210.	mondo	world	3
211.	sinistra	left	3
212.	tagli	cuts	3
213.	terzo	third	3
214.	terza	third	3
215.	scoprire	to discover	3
216.	tipo	guy	3
217.	stesso	same	3
218.	stesse	same	3
219.	destra	right	3
220.	nemmeno	neither	3
221.	imparare	learn	3
222.	tantissime	many	3
223.	qualche	some	3
224.	sorpreso	surprised	3
225.	ciao	Hello	3
226.	difficile	hard	3
227.	filmato	movie	3
228.	certo	Certain	3
229.	esiste	exists	3
230.	uguali	the same	3
231.	poteva	he could	3
232.	dire	to say	3
233.	quasi	almost	3
234.	metà	half	3
235.	gruppi	groups	3
236.	sugli	on	3
237.	giorni	days	3
238.	rimasto	remained	3
239.	trovato	found	3
240.	rimangono	remain	3
241.	due	two	3
242.	niente	nothing	3
243.	fosse	is	3
244.	ruotare	rotate	3
245.	mettere	to put	3
246.	noia	boredom	3
247.	primo	first	3
248.	divertente	fun	3
249.	altra	other	3
250.	siamo	we are	3
251.	quante	how much it is	3
252.	ogni	every	2
253.	ai	to the	2
254.	is	is	2
255.	li	there	2
256.	no	no	2
257.	lì	there	2
258.	ti	you	2
259.	aveva	had	2
260.	persona	person	2
261.	parti	set off	2
262.	lavori	works	2
263.	scritta	written	2
264.	quelle	those	2
265.	rifare	redo	2
266.	leggere	read	2
267.	dividere	to divide	2
268.	leggiamo	let's read	2
269.	dovevo	I had to	2
270.	taglia	cut it	2
271.	risolvere	solve	2
272.	perfettamente	perfectly	2
273.	stessa	itself	2
274.	vorrei	I would like	2
275.	oggetti	objects	2
276.	oggetto	object	2
277.	provato	tried	2
278.	disegno	drawing	2
279.	quesiti	questions	2
280.	quesito	question	2
281.	caso	case	2
282.	facile	easy	2
283.	facili	easy	2
284.	degli	of the	2
285.	frase	phrase	2
286.	scrivere	to write	2
287.	tagliare	cut	2
288.	tagliata	cut	2
289.	cambiano	they change	2
290.	indietro	backwards	2
291.	precedente	previous one	2
292.	parlato	spoken	2
293.	proprio	own	2
294.	palindroma	palindroma	2
295.	volta	time	2
296.	miliardi	billions	2
297.	dalle	give her	2
298.	momento	moment	2
299.	spiegazioni	explanations	2
300.	quel	that	2
301.	vede	see	2
302.	montato	mounted	2
303.	foglio	sheet	2
304.	fortuna	fortune	2
305.	preferita	favorite	2
306.	trovare	to find	2
307.	solamente	only	2
308.	sapere	know	2
309.	svolgere	carry out	2
310.	riflettuto	reflected	2
311.	vuol	wants	2
312.	viaggio	voyage	2
313.	fotocopia	photocopy	2
314.	passato	past	2
315.	moltissimo	very very much	2
316.	moltissime	very many	2
317.	geometrica	geometric	2
318.	dovuto	due	2
319.	nell	in the	2
320.	significato	meaning	2
321.	invece	instead	2
322.	mai	never	2
323.	mie	my	2
324.	nei	in the	2
325.	ora	Now	2
326.	riuscito	succeeded	2
327.	allo	at the	2
328.	alla	at the	2
329.	sia	is	2
330.	sua	her	2
331.	sul	on	2
332.	quadrato	square	2
333.	tre	three	2
334.	altro	other	2
335.	interno		2
336.	aver	having	2
337.	brasili	brazil	2
338.	simone	simone	2
339.	semplice	simple	2
340.	rettangolo	rectangle	2
341.	5	5	1
342.	imparando	learning	1
343.	d	d	1
344.	h	h	1
345.	oggi	today	1
346.	stat	stat	1
347.	base	base	1
348.	spiegati	explain yourself	1
349.	rileggiamo	let's reread	1
350.	sula	sula	1
351.	suoi	his	1
352.	am	am	1
353.	frutta	fruit	1
354.	frutto	fruit	1
355.	gl	gl	1
356.	it	it	1
357.	ke	ke	1
358.	lo	the	1
359.	on	on	1
360.	pi	pi	1
361.	te	you	1
362.	sì	Yes	1
363.	amici	friends	1
364.	avere	to have	1
365.	diversa	different	1
366.	esistessero	existed	1
367.	professor	professor	1
368.	imperfezione	imperfection	1
369.	ricostruito	rebuilt	1
370.	durante	during	1
371.	inoltre	furthermore	1
372.	molti	lot of	1
373.	senza	without	1
374.	esempi	examples	1
375.	esperienza	experience	1
376.	senso	sense	1
377.	esistevano	existed	1
378.	subiscono	suffer	1
379.	tempo	time	1
380.	incisa	engraved	1
381.	osso	bone	1
382.	simmetrico	symmetrical	1
383.	serve	serves	1
384.	compreso	included	1
385.	impararne	learn about it	1
386.	approfondito	thorough	1
387.	straordinario	extraordinary	1
388.	guardava	watched	1
390.	insegnato	taught	1
391.	fuori	out	1
392.	ascoltare	to listen	1
393.	accorgiamo	we notice	1
394.	avrei	I would have	1
395.	intorno	environment	1
396.	veder	see	1
397.	testo	text	1
398.	testa	head	1
399.	facendolo	doing it	1
400.	pensavamo	we thought	1
401.	tipi	types	1
402.	riguardo	regard	1
403.	avuta	had	1
404.	avute	had	1
405.	questionario	survey	1
406.	breve	short	1
407.	dividono	divide	1
408.	infinite	infinite	1
409.	operazioni	operations	1
410.	disegni	drawings	1
411.	pensassi	I thought	1
412.	leggono	they read	1
413.	tanti	many	1
414.	tanta	a lot	1
415.	ancor	still	1
416.	faceva	did	1
417.	però	However	1
418.	insomma	in short	1
419.	fregato	screwed	1
420.	servivano	were needed	1
421.	posso	I can	1
422.	contatto	contact	1
423.	tuoi	yours	1
424.	capite	understand	1
425.	piacciono	like them	1
426.	parallelogramma	parallelogram	1
427.	descrivere	to describe	1
428.	tantissimo	very, very much	1
429.	bello	nice	1
430.	lati	sides	1
431.	poco	little	1
432.	poca	little	1
433.	sorprese	surprises	1
434.	costruite	built	1
435.	erano	were	1
436.	racconto	tale	1

Order	Unfiltered word count		Oc.
437.	perfetti	perfect	1
438.	perfetta	perfect	1
439.	post	post	1
440.	ritagliato	cropped	1
441.	maniera	manner	1
442.	osservato	observed	1
443.	potrebbero	they might	1
444.	varie	various	1
445.	importante	important	1
446.	prim	prim	1
447.	esercizio	exercise	1
448.	difficili	difficult	1
449.	percorso	path	1
450.	risultato	result	1
451.	quattro	four	1
452.	filmati	video clips	1
453.	così	Like this	1
454.	legger	read	1
455.	andava	it went	1
456.	andati	gone	1
457.	insicura	insecure	1
458.	andare	to go	1
459.	rileggeva	reread	1
460.	certa	certain	1
461.	certe	certain	1
462.	congruenti	congruent	1
463.	stati	States	1
464.	fiore	flower	1
465.	scoprendo	discovering	1
466.	cambiato	changed	1
467.	cambiare	change	1
468.	puntate	bets	1
469.	ricordata	remembered	1
470.	diviso	divided	1
471.	affrontare	face up to	1
472.	studiarla	study it	1
473.	formava	formed	1
474.	formato	format	1
475.	stufato	stew	1
476.	torna	come back	1
477.	stela	stela	1
478.	progetto	project	1
479.	verticalmente	vertically	1
480.	rotazioni	rotations	1
481.	dette	said	1
482.	detto	said	1
483.	restano	remain	1
484.	corrisponde	matches	1
485.	lupo	wolf	1
486.	differenza	difference	1
487.	stiamo	we are	1
488.	dappertutto	everywhere	1
489.	manifestare	manifest	1
490.	leggersi	read	1
491.	pensavo	I thought	1
492.	costruzione	construction	1
493.	viaggi	trips	1
494.	assomigliava	resembled	1
495.	dall	from	1
496.	ultima	last	1
497.	bocca	mouth	1
498.	arrotare	grind	1
499.	danno	damage	1
500.	spesso	often	1
501.	sentita	felt	1
502.	spiegarci	explain us	1
503.	mostrava	showed	1
504.	coperto	covered	1
505.	mentre	while	1
506.	piacevole	enjoyable	1
507.	schede	cards	1
508.	potevo	I could	1
509.	soltanto	only	1
510.	spero	I hope	1
511.	studiamo	we study	1

Order	Unfiltered word count		Oc.
512.	troppe	too many	1
513.	appreso	learned	1
514.	quale	which	1
515.	quant	quant	1
516.	lavorato	worked	1
517.	riepilogo	summary	1
518.	averlo	have it	1
519.	presto	soon	1
520.	quei	those	1
521.	quiz	quiz	1
522.	miei	my	1
523.	punto	point	1
524.	punte	spikes	1
525.	collegare	connect	1
526.	oppure	or	1
527.	farci	make us	1
528.	verticale	vertical	1
529.	osserviamo	we observe	1
530.	spezzata	broken	1
531.	immaginato	imagined	1
532.	scatoletta	little box	1
533.	immaginare	to imagine	1
534.	fatta	done	1
535.	colorate	colored	1
536.	scope	scope	1
537.	bicicletta	bike	1
538.	mole	mole	1
539.	specchiano	mirror	1
540.	trasformazione	transformation	1
541.	idea	idea	1
542.	movimento	movement	1
543.	insieme	together	1
544.	adesso	now	1
545.	corso	course	1
546.	stavamo	we were	1
547.	rispecchiato	mirrored	1
548.	linee	lines	1
549.	piaciuti	liked	1
550.	cerchio	circle	1
551.	pallini	shot	1
552.	ruota	wheel	1
553.	piccole	small	1
554.	piccoli	little ones	1
555.	ruotando	rotating	1
556.	indeciso	undecided	1
557.	preferite	favorite	1
558.	identiche	identical	1
559.	sorprendenti	surprising	1
560.	spostandole	moving them	1
561.	lettera	letter	1
562.	lettere	letters	1
563.	leggi	laws	1
564.	inni	hymns	1
565.	avevamo	we had	1
566.	noioso	boring	1
567.	compagnia	company	1
568.	alcuna	any	1
569.	vengono	they come	1
570.	cercare	to seek	1
571.	symmetry	symmetry	1
572.	piano	floor	1
573.	colpito	hit	1
574.	piana	flat	1
575.	diceva	he said	1
576.	chiuso	Closed	1
577.	tagliando	cutting	1
578.	coi	with	1
579.	rosa	rose	1
580.	dal	from the	1
581.	costruendo	building	1
582.	aiutare	to help	1
583.	aiutato	helped	1
584.	miglioramento	improvement	1
585.	significa	it means	1
586.	forse	perhaps	1

Order	Unfiltered word count		Oc.
587.	ero	I was	1
588.	neppure	not even	1
589.	incuriosito	intrigued	1
590.	piena	full	1
591.	elle	elle	1
592.	rivedremo	we will review	1
593.	emme	emme	1
594.	mente	mind	1
595.	diffcoltà	difficulty	1
596.	meravigliose	wonderful	1
597.	letti	beds	1
598.	stupito	amased	1
599.	naturali	natural	1
600.	lettura	reading	1
601.	identica	identical	1
602.	mortale	mortal	1
603.	acqua	water	1
604.	vedendo	seeing	1
605.	preso	taken	1
606.	lim	lim	1
607.	nello	in	1
608.	curiosità	curiosity	1
609.	esce	goes out	1
610.	mio	my	1
611.	costruirle	build them	1
612.	definirsi	define themselves	1
613.	esse	they	1
614.	agli	ai	1
615.	metri	meters	1
616.	mette	puts	1
617.	risposto	replied	1
618.	aperto	open	1
619.	girata	turn	1
620.	bellissima	beautiful	1
621.	bellissimi	beautiful	1
622.	bellissime	beautiful	1
623.	tranne	except for	1
624.	messi	put	1
625.	riuscivo	I could	1
626.	riuscita	successful	1
627.	scegliere	choose	1
628.	specchio	mirror	1
629.	calcoli	calculations	1
630.	girandola	pinwheel	1
631.	sim	sim	1
632.	sui	on	1
633.	tuo	your	1
634.	rivisto	revised	1
635.	umani	humans	1
636.	anzi	rather	1
637.	uno	one	1
638.	elezioni	elections	1
639.	finisce	ends	1
640.	conoscendo	knowing	1
641.	immaginaria	imaginary	1
642.	divertiti	Enjoy yourself	1
643.	ripetendole	repeating them	1
644.	buona	good	1
645.	attenzione	Attention	1
646.	combaciare	fit together	1
647.	uscire	go out	1
648.	uscita	exit	1
649.	facendo	doing	1
650.	quarto	fourth	1
651.	storia	history	1
652.	incontrato	encountered	1
653.	fine	end	1
654.	stravolgenti	upsetting	1
655.	spuntata	checked	1
656.	subito	immediately	1
657.	gironi	rounds	1
658.	pensiero	thought	1
659.	semplicissimo	very simple	1
660.	simpatica	nice	1
			1

Appendix G

WORD RANK-FREQUENCY DISTRIBUTIONS OF VOCABULARY.

Rank	Freq. (i)	Types (V_i)	Token	Types C.	Token C.	Freq. Cl.
1	168	1	168	1	168	HF
2	167	1	167	2	335	HF
3	139	1	139	3	474	HF
4	136	1	136	4	610	HF
5	114	1	114	5	724	HF
6	105	1	105	6	829	HF
7	94	2	188	8	1017	MF
8	92	1	92	9	1109	MF
9	90	1	90	10	1199	MF
10	79	1	79	11	1278	MF
11	78	1	78	12	1356	MF
12	76	1	76	13	1432	MF
13	69	1	69	14	1501	MF
14	67	1	67	15	1568	MF
15	54	1	54	16	1622	MF
16	51	1	51	17	1673	MF
17	49	1	49	18	1722	MF
18	48	1	48	19	1770	MF
19	46	2	92	21	1862	MF
20	45	2	90	23	1952	MF
21	44	1	44	24	1996	MF
22	39	1	39	25	2035	MF
23	37	1	37	26	2072	MF
24	36	1	36	27	2108	MF
25	35	1	35	28	2143	MF
26	34	2	68	30	2211	MF
27	33	2	66	32	2277	MF
28	32	1	32	33	2309	MF
29	31	2	62	35	2371	MF
30	30	1	30	36	2401	MF
31	28	2	56	38	2457	MF
32	27	1	27	39	2484	MF
33	25	2	50	41	2534	MF
34	24	2	48	43	2582	MF
35	22	1	22	44	2604	MF
36	20	3	60	47	2664	LF
37	19	2	38	49	2702	LF
38	18	3	54	52	2756	LF
39	17	4	68	56	2824	LF
40	16	7	112	63	2936	LF
41	15	2	30	65	2966	LF
42	14	3	42	68	3008	LF
43	13	8	104	76	3112	LF
44	12	1	12	77	3124	LF
45	11	3	33	80	3157	LF
46	10	7	70	87	3227	LF
47	9	9	81	96	3308	LF
48	8	12	96	108	3404	LF
49	7	14	98	122	3502	LF
50	6	22	132	144	3634	LF
51	5	30	150	174	3784	LF
52	4	29	116	203	3900	LF
53	3	48	144	251	4044	LF
54	2	89	178	340	4222	LF
55	1	320	320	660	4542	LF

Appendix H

EXTRACT FROM THE FILE OF POS TAGGED VA2 TEXT

WORD	POS	LEMMA	WORD	POS	LEMMA
Questa	DET:demo	questo	Non	NEG	non
attività	NOUN	attività	sapevo	VER:fin	sapere
mi	CLI	mi	che	CHE	che
è	VER:fin	essere	il	ART	il
piaciuto	VER:ppast	piacere	colore	NOUN	colore
moltissimo	ADV	molto	era	VER:fin	essere
perché	CON	perchè	una	ART	una
ho	AUX:fin	avere	simmetria	NOUN	simmetria
scoperto	VER:ppast	scoprire	e	CON	e
e	CON	e	nemmeno	ADV	nemmeno
imparato	VER:ppast	imparare	che	CHE	che
cose	NOUN	cosa	dentro	PRE	dentro
che	CHE	che	una	ART	una
non	NEG	non	mela	NOUN	mela
avrei	AUX:fin	avere	ci	CLI	ci
mai	ADV	mai	fosse	VER:fin	essere
immaginato	VER:ppast	immaginare	una	ART	una
.	SENT	.	stella	NOUN	stella
Sulla	ARTPRE	sulla	.	SENT	.
matematica	NOUN	matematica	Insomma	ADV	insomma
ci	CLI	ci	la	ART	la
sono	VER:fin	essere	matematica	NOUN	matematica
molte	DET:indef	molto	è	ADJ	essere
cose	NOUN	cosa	piena	ADJ	pieno
meravigliose	ADJ	meraviglioso	di	PRE	di
e	CON	e	sorprese	NOUN	sorpresa
sorprendenti	ADJ	sorprendente	.	SENT	.
.	SENT	.			

(elab. TreeTagger)

Appendix I

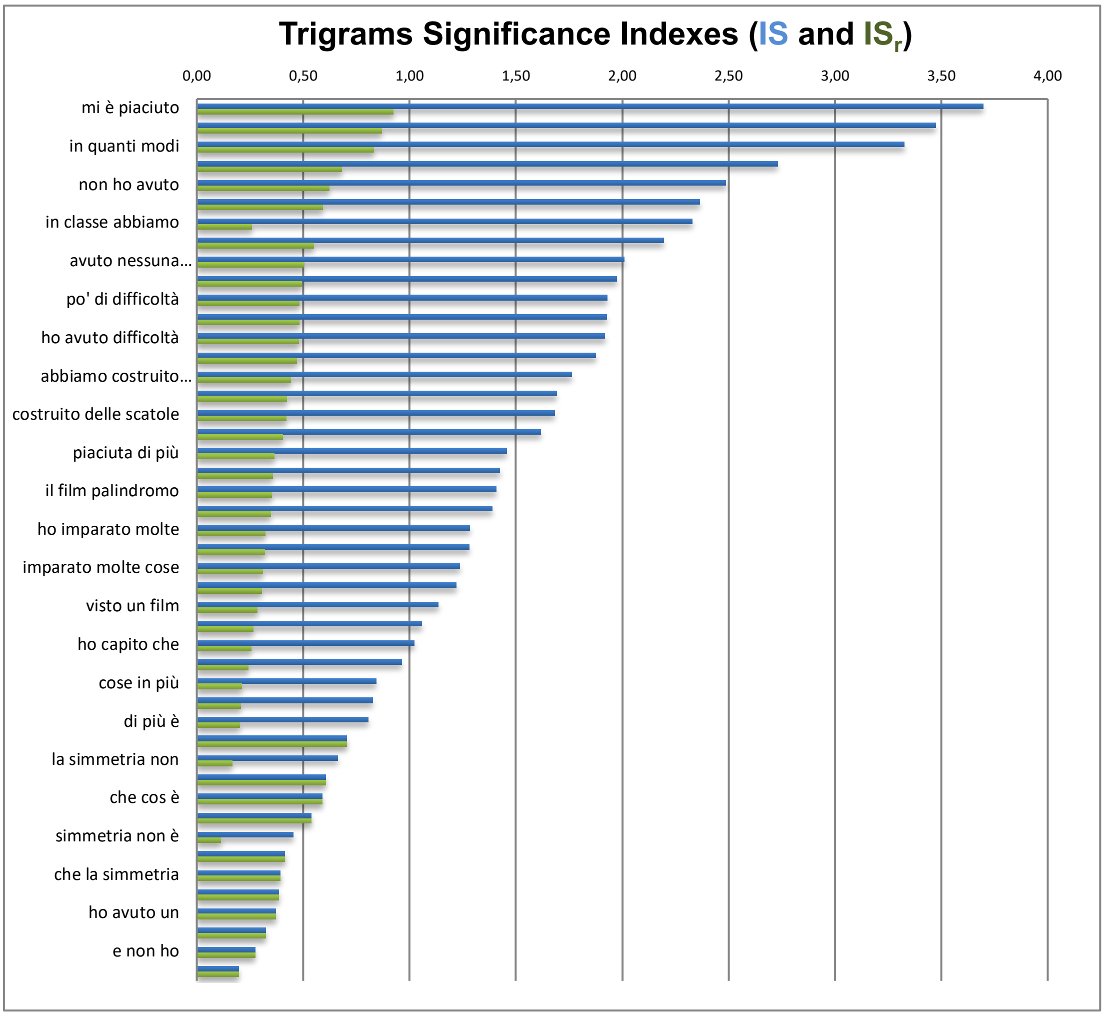
EXCERPT FROM VC TEACHER'S LOGBOOK

Dal resoconto dell'attività precedente emerge un buon contributo, ordinato e pertinente. Le riflessioni conducono al concetto di simmetria da cogliere, ad esempio, nei numeri (riferimento alla simmetria assiale, quella che i bambini hanno avuto modo di scoprire e conoscere). Apprezzabile la gradualità con cui i ragazzi vengono condotti ad addentrarsi nel concetto.
L'introduzione del concetto di numero-e poi parole o frasi- palindromo, incuriosisce e affascina i ragazzi: su alcuni volti si coglie meraviglia e stupore. Anche riguardo al video si avrà poi un feedback positivo: diversi alunni hanno colto l'istante preciso nel video in cui la storia si riflette per poi tornare alla situazione iniziale.
Altrettanto stupore per la mela con la sua simmetria assiale, per rotazione e interna. I bambini diranno poi che sia il film che l'osservazione del frutto, hanno permesso loro di cogliere come davvero ci siano molte simmetrie nella realtà.

"From the report on the previous activity, a good, ordered, and relevant contribution emerges. The reflections lead to grasping the concept of symmetry, for example, in numbers (via axial symmetry, which the students were able to discover and learn about). The gradual approach used to introduce the children to the concept is remarkable. The introduction of numbers and then palindromic words or phrases fascinates the children: you can see the amazement on some of their faces. Moreover, there is positive feedback on the video: several students captured the exact moment in the video when the story is reflected in the initial situation. The apple's axial, rotational, and internal symmetry is also amazing. The children then say that by watching the film and observing the structure of the fruit, they understood how many symmetries there are in reality."

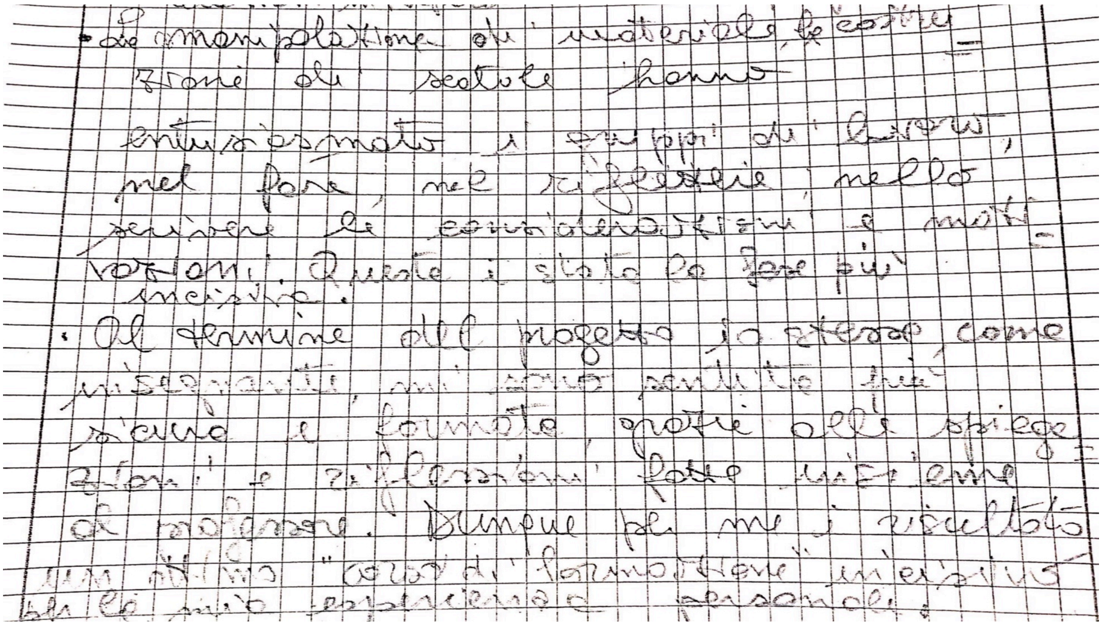
Appendix J

DISTRIBUTION OF TRIGRAMS IN DESCENDING ORDER OF SIGNIFICANCE INDEX IS.



Appendix K

EXCERPT FROM VF-VG TEACHER'S LOGBOOK



• La manipolazione di materiali, la costru-
zione di scatole hanno
entusiasmato i gruppi di lavoro,
nel fare, nel riflettere, nello
scrivere le considerazioni e moti-
vazioni. Questa è stata la fase più
incisiva.
• Al termine del progetto io stesso come
insegnante mi sono sentito più
sicuro e formato grazie alle spiega-
zioni e riflessioni fatte insieme
al professore. Dunque per me è risultato
un ottimo "corso di formazione" interattivo molto
utile per l'esperienza personale.

"Handling the materials and building the boxes encouraged the work groups to be active, reflect, and write down thoughts and motivations. At the end of the project, I felt more confident and trained, thanks to the explanations and reflections I had made with the teacher. It was excellent training that was very important for my professional experience."

Appendix L

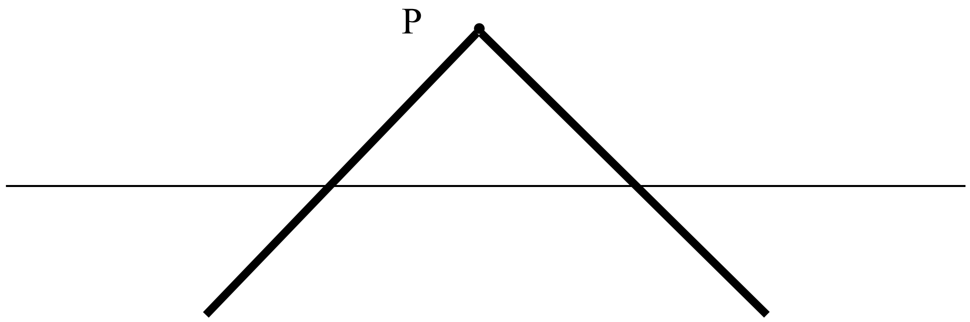
ARCHIMEDES MACHINE

Draw a line in the sheet with a point P, as in the figure below, and place the Archimedean machine on the sheet so that its hinge is at the point P.

Then unfold the two side mirrors so that part of the segment extends from one mirror to the other.

Move the two sides of the unfolded mirror until you see a regular polygon.

Move the two sides until you can see an equilateral triangle, a square, and a hexagon and then measure the angle formed by the two sides of mirrors.

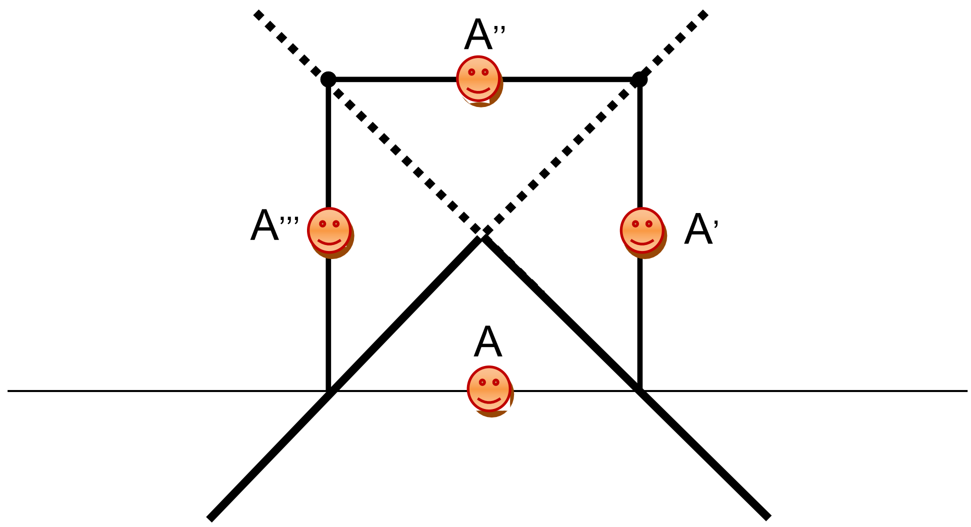


If the angle between two mirrors is a factor of 360 from 1 to 360, then the total number of reflections is finite and is calculated as 360 divided by the angle minus 1

For example, if the angle is 90 degrees, then the total number of reflections is 360/90 - 1 =4 - 1 = 3 so we have a square.

If the angle is 60 degrees, the total number of reflections is 360/60 - 1 = 6 - 1= 5 so we have a hexagon.

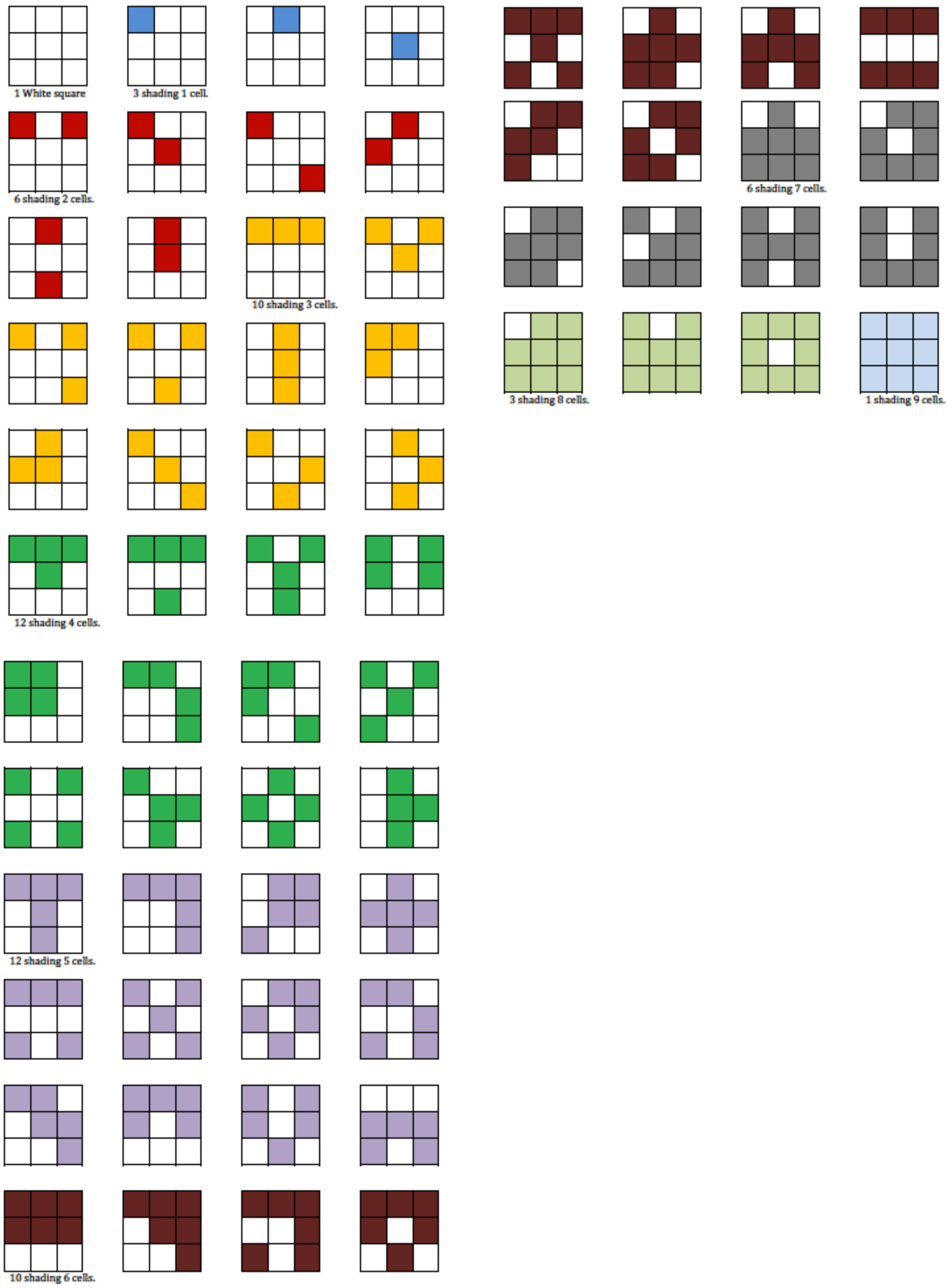
If the angle is not a factor of 360, there are infinitely many reflections.



Each point of the line is mirrored three times, just as the object A is mirrored in A', A'', A'''.

Appendix M

SOLUTIONS OF COLOURING SQUARES TASK SYMMETRICALLY



Appendix N

PRE-TEST QUESTIONNAIRE IN THE SECOND EXPERIMENT

Then one morning, at dawn, it showed itself at last. And after such a hard work it yawned and said, "Ah! I just woke up. Excuse me. My lapels are a mess..." The little prince then could not restrain his admiration: "You look so beautiful!" "Do I?" the flower replied kindly. "I was born at the moment when the sun..." The Little Prince realised at once that it wasn't modest at all, but it was so touching and alluring! "How can you be so beautiful?" "Well, you see, I am a flower and am a creation of nature, and as such I am perfectly symmetrical..." "I don't understand" replied the little prince, taken aback by what the flower had said.

*"Now I'll explain it to you" said the flower haughtily. "**In nature there are many symmetries" "And what is their purpose?" "Well, to make flowers beautiful, there is no doubt. A symmetry of nature is something that the sun has given us and that no one will ever be able to imitate. Everything in nature, is born from a symmetry. Many things in nature are symmetric, did you know that?" "What?" "For example, starfish, snowflakes, the cells of the beehives and crystals... man!"***

"Never been snow nor bees on my planet". The little prince, however, was attracted by what the flower was saying. "All living things are beautiful and symmetrical from different points of view... I, for example, am coloured and the symmetries of the colours of my petals make me beautiful".

1) If you were the "Rose" trying to explain symmetry to the "Little Prince", which of the following sentences would you use?

a) There is symmetry when you can see that, if a line divides a figure in two parts, these parts reflect each other like in a mirror.

b) Among the many symmetries that exist there are also the movements that transform a figure so that the resulting figure coincides with the original.

c) Or else:

……………………………………………………………………………………………

Write the reason of your choice.

…………………………………………………………………………………………………

2) Write five/six meaningful words that come to mind about symmetry:

a)………………………………… b)……………………………….. c) ……………………………….

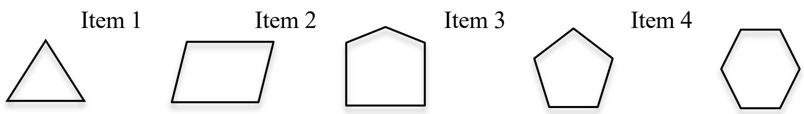
d)………………………………… e)……………………………….. f) ……………………………….

3) In which school subjects (or subjects) is symmetry studied:

a)………………………………… b)……………………………….. c) ……………………………….

d)………………………………… e)……………………………….. f) ……………………………….

4) For each figure, write the number of ways the lid can be placed on each box.



Appendix O

POST-TEST QUESTIONNAIRE IN THE SECOND EXPERIMENT

1) Write the definition of symmetry and justify your choice.

……

……

……

2) Write five/six meaningful words that come to mind about symmetry:

a)……………………………… b)…………………………….. c) ……………………………….

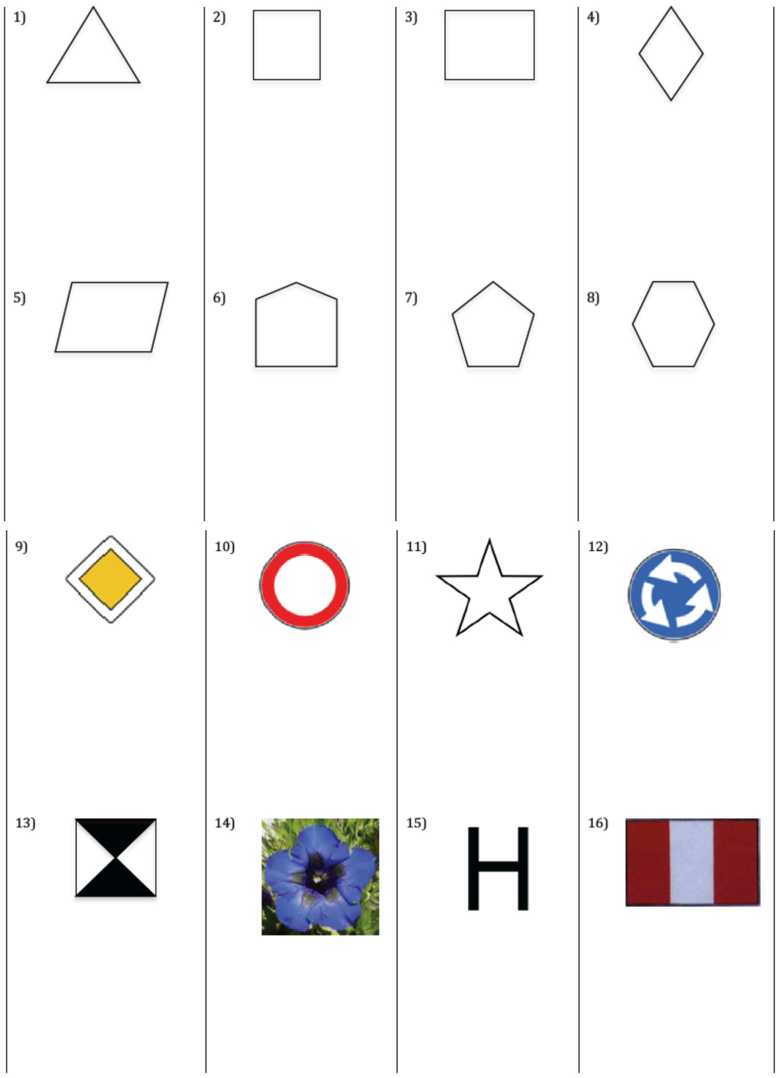
d)……………………………… e)…………………………….. f) ……………………………….

3) In which school subjects (or subjects) is symmetry studied:

a)……………………………… b)…………………………….. c) ……………………………….

d)……………………………… e)…………………………….. f) ……………………………….

4) For each figure, write whether it is symmetrical and why.



Appendix P

FINAL TEACHER SURVEY TO EVALUATE THE LEARNING UNIT

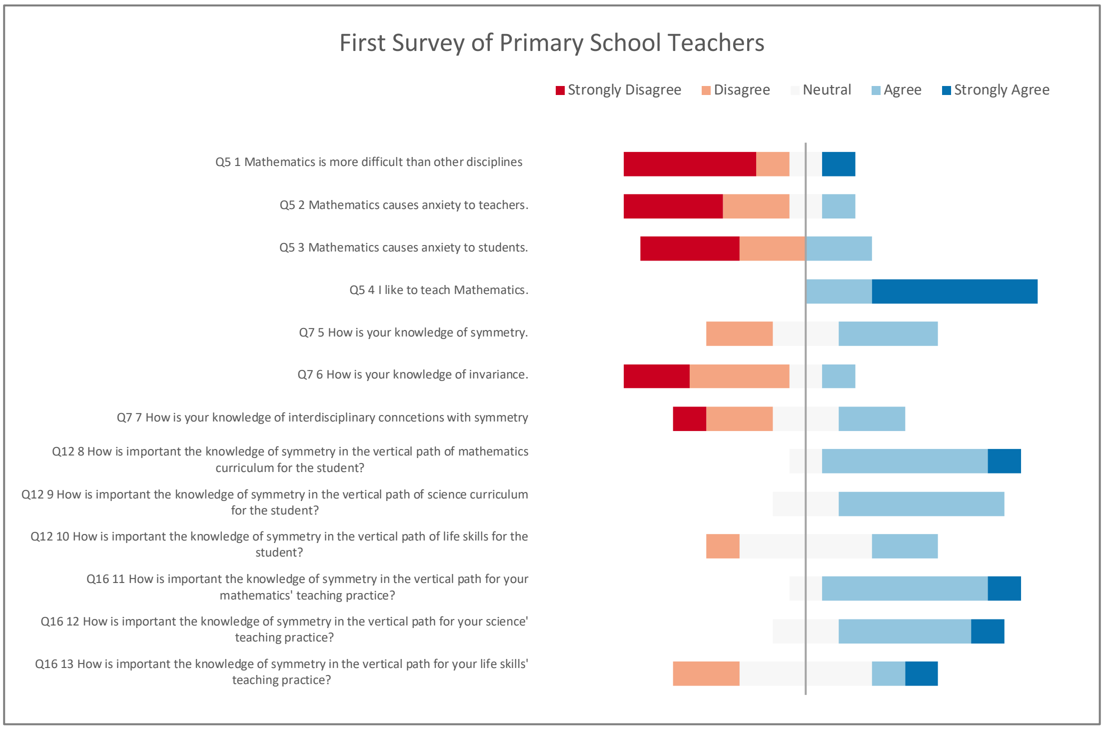
	Indicate the level of agreement with each statement by using the scale from 1 (strongly disagree) to 5 (strongly agree) next to each statement. 1 Strongly Disagree · 2 Disagree · 3 Neutral · 4 Agree · 5 Strongly Agree					
Q1 1	The learning unit has met my expectations.	**1**	**2**	**3**	**4**	**5**
Q1 2	My personal knowledge about the concept of symmetry has improved thanks to the didactic proposal.	**1**	**2**	**3**	**4**	**5**
Q1 3	My personal knowledge of the concept of invariance has improved thanks to the didactic proposal.	**1**	**2**	**3**	**4**	**5**
Q1 4	The learning unit has developed my ability to analyse interdisciplinary integrations on the topic of symmetry and invariance.	**1**	**2**	**3**	**4**	**5**
Q1 5	The learning unit has awakened in me new needs for further studies.	**1**	**2**	**3**	**4**	**5**
Q3 6	The learning unit has allowed me to think critically about the teaching of symmetry and mathematics in general.	**1**	**2**	**3**	**4**	**5**
Q3 7	I can put into practice the knowledge and teaching/learning methods I acquired in the didactic proposal.	**1**	**2**	**3**	**4**	**5**
Q5 8	The learning unit has shown me how important the concepts of symmetry and invariance are for students in vertical development and in continuity with higher schools.	**1**	**2**	**3**	**4**	**5**
Q5 9	The learning unit has broadened the students' view of symmetry.	**1**	**2**	**3**	**4**	**5**
Q5 10	The learning unit has broadened the students' view of invariance.	**1**	**2**	**3**	**4**	**5**
Q5 11	It would be useful to have a training session on symmetry and invariance.	**1**	**2**	**3**	**4**	**5**
Q5 12	It would be useful to develop the teaching proposal on symmetry and invariance and give it continuity.	**1**	**2**	**3**	**4**	**5**

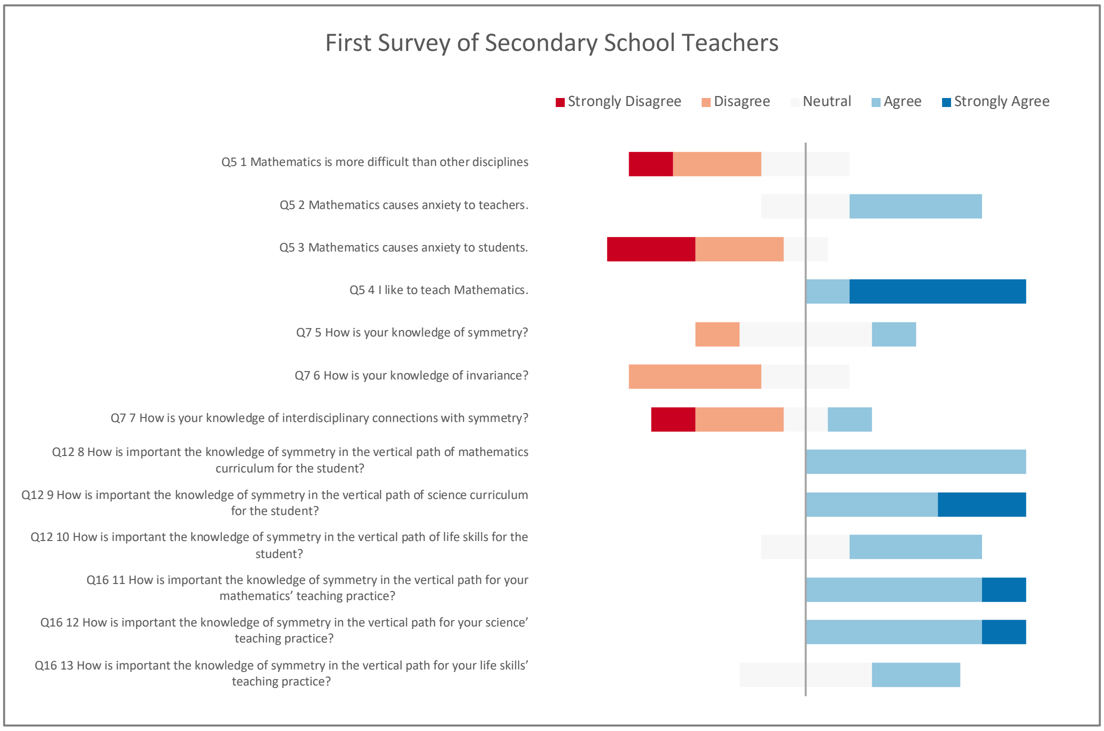
Now rate the lessons. Use the scale from 1 (Inadequate) to 5 (very adequate) given next to each statement. 1 Inadequate · 2 Slightly inadequate · 3 Neutral · 4 Quite adequate · 5 Adequate						
Q7 10	Topics addressed	**1**	**2**	**3**	**4**	**5**
Q7 11	Expository clarity	**1**	**2**	**3**	**4**	**5**
Q7 12	Teaching methodologies	**1**	**2**	**3**	**4**	**5**
Q7 13	Availability	**1**	**2**	**3**	**4**	**5**

Place ________________________________ Date __________________________

Appendix Q

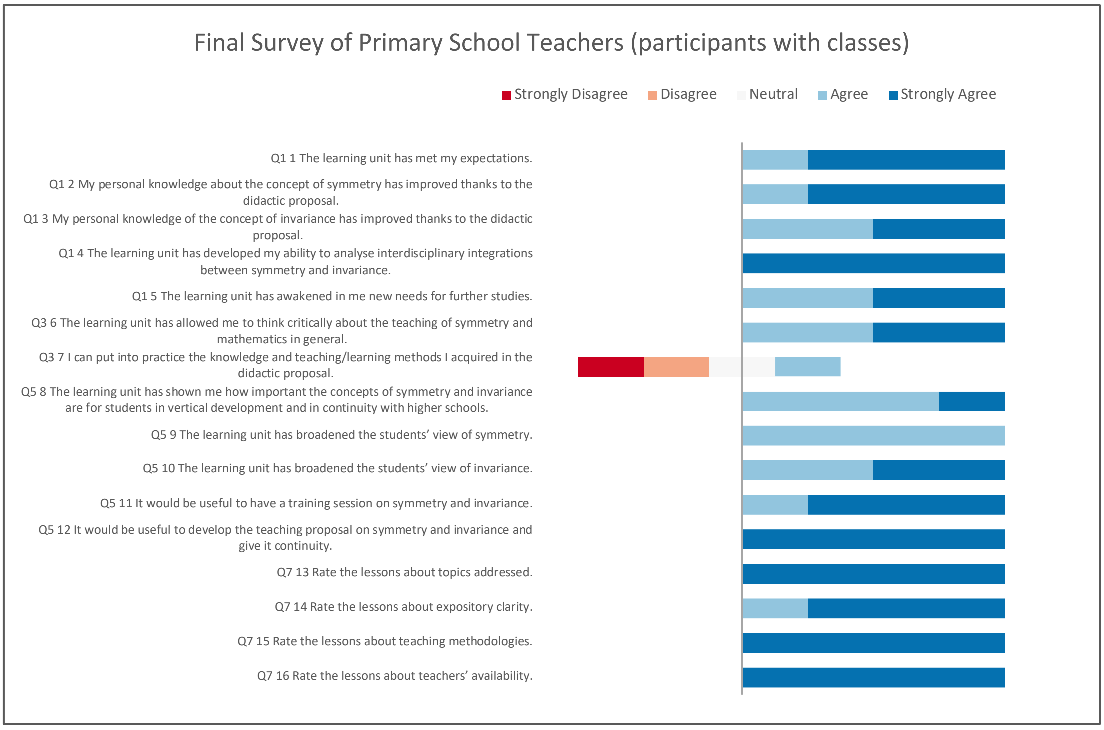
FINDINGS FIRST TEACHER SURVEY

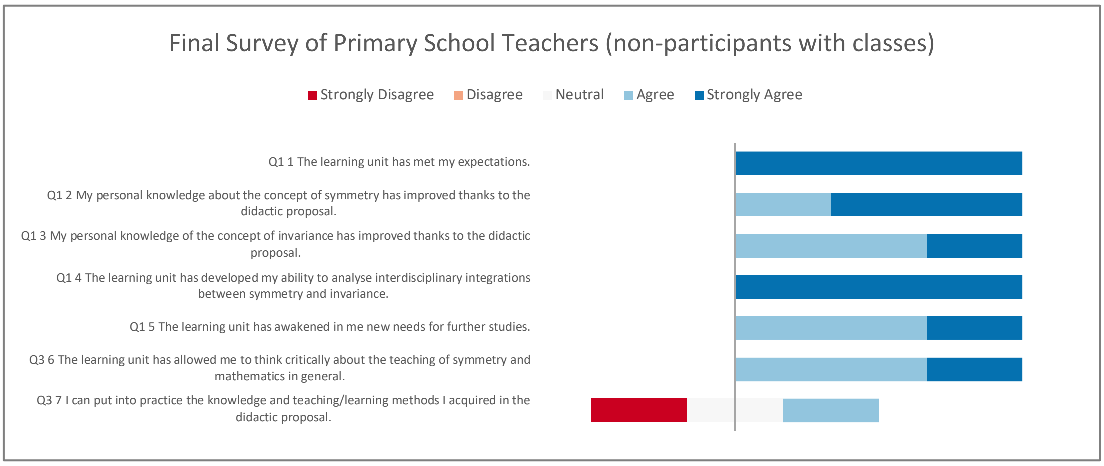


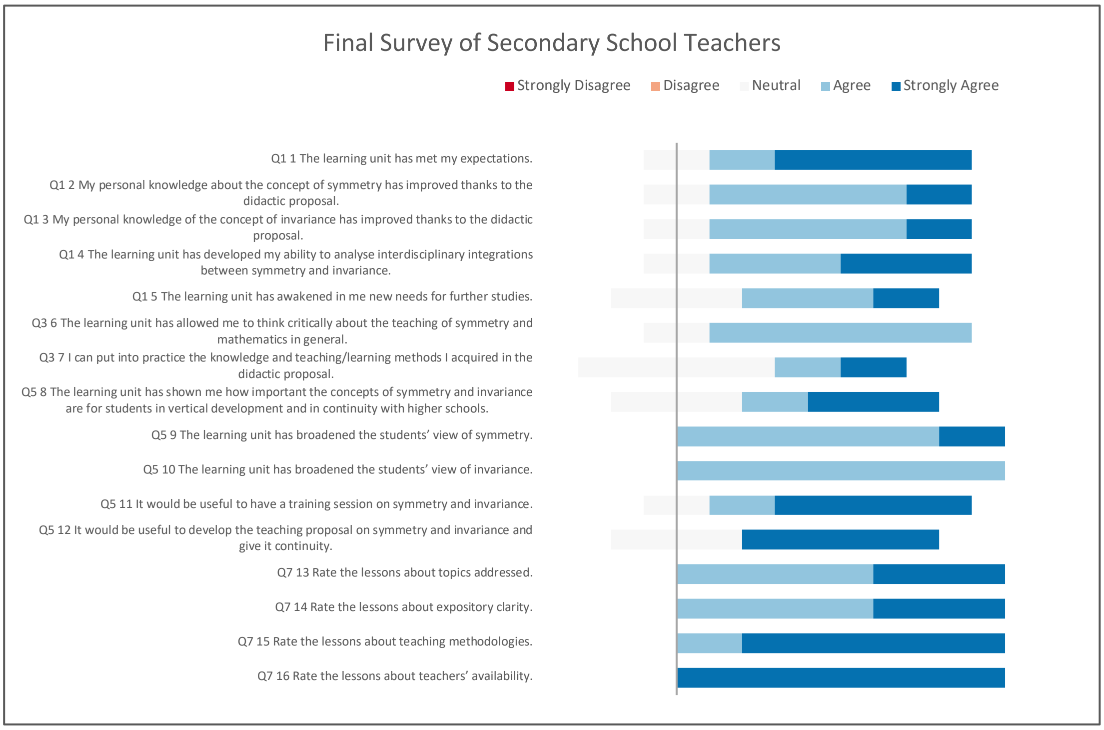


Appendix R

FINDINGS FINAL TEACHER SURVEY






Final Survey of Secondary School Teachers
Strongly Disagree
Disagree
Neutral
Agree
Strongly Agree
Q1 1 The learning unit has met my expectations.
Q1 2 My personal knowledge about the concept of symmetry has improved thanks to the didactic proposal.
Q1 3 My personal knowledge of the concept of invariance has improved thanks to the didactic proposal.
Q1 4 The learning unit has developed my ability to analyse interdisciplinary integrations between symmetry and invariance.
Q1 5 The learning unit has awakened in me new needs for further studies.
Q3 6 The learning unit has allowed me to think critically about the teaching of symmetry and mathematics in general.
Q3 7 I can put into practice the knowledge and teaching/learning methods I acquired in the didactic proposal.
Q5 8 The learning unit has shown me how important the concepts of symmetry and invariance are for students in vertical development and in continuity with higher schools.
Q5 9 The learning unit has broadened the students' view of symmetry.
Q5 10 The learning unit has broadened the students' view of invariance.
Q5 11 It would be useful to have a training session on symmetry and invariance.
Q5 12 It would be useful to develop the teaching proposal on symmetry and invariance and give it continuity.
Q7 13 Rate the lessons about topics addressed.
Q7 14 Rate the lessons about expository clarity.
Q7 15 Rate the lessons about teaching methodologies.
Q7 16 Rate the lessons about teachers' availability.

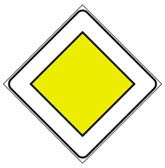
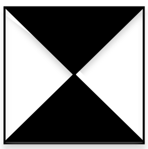
Appendix S

PRE- /POST-TEST TEACHER QUESTIONNAIRE

IN THE THIRD EXPERIMENT

COURSE IDENTIFIER CODE: ………… PARTICIPATION IN CLASS ACTIVITIES (Yes No): ………….

A) For each figure identify its symmetries:

Figure	Symmetries	Explanation
A1) 		
A2) 		
A3) 		
A4) 		
A5) 		
A6) 		

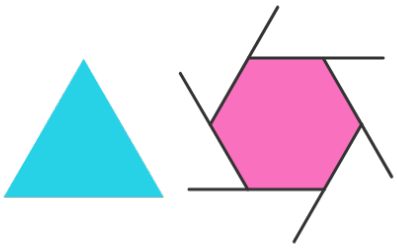
A7) 		
A8) 		

B) Judge whether these conjectures should be considered true or false and give a brief explanation.

B1) You can draw a quadrilateral that, when mirrored on one side, makes a pentagon.

B2) If half of a geometric shape has three sides, the mirrored shape must be a hexagon.

C) Which of the two shapes is more symmetrical?



Comment your choice (Required):

D) Divide the letters of the alphabet into groups according to their symmetry.

A B C D E F G H I
J K L M N O P Q R
S T U V W X Y Z

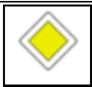
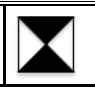
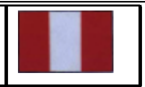
E) (Optional) Any comments on the questionnaire:

Appendix T

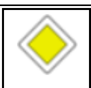
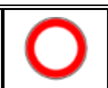
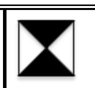
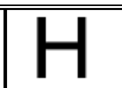
SCORING CRITERIA BY COLOR GRADIENT FOR A1-A.8 ITEMS

0	Missing Answer	1	Partial Answer without Explanation	1	Correct Answer and Wrong Explanation
0	Wrong Answer	1	Partial Answer with wrong Explanation	2	Correct Explanation Only
1	Correct Answer with Wrong Explanation	1	Partial Answer and right Explanation	2	Correct Answer

Pre- Test

							
A.1	A.2	A.3	A.4	A.5	A.6	A.7	A.8
2	2	0	0	2	0	2	2
0	3	2	0	2	0	2	2
2	2	0	0	2	0	2	2
0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0
0	2	0	0	0	3	2	2
0	0	0	0	0	0	2	2
2	3	2	0	2	0	2	2
2	2	0	0	2	0	2	2
0	2	0	0	0	0	0	0
2	3	2	2	2	3	2	2
0	2	0	0	2	0	2	2
2	2	2	2	2	0	0	0
2	2	2	2	2	0	2	2
0	0	0	0	2	0	2	2
2	2	0	0	0	0	0	2
2	2	2	0	2	3	2	2
0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0
0	0	0	0	0	0	2	0
2	2	0	2	0	0	2	2
2	2	2	0	2	2	2	2
2	2	2	0	0	0	2	2
2	2	0	0	2	0	2	2
2	0	0	0	0	0	0	0
2	2	2	0	2	2	2	2

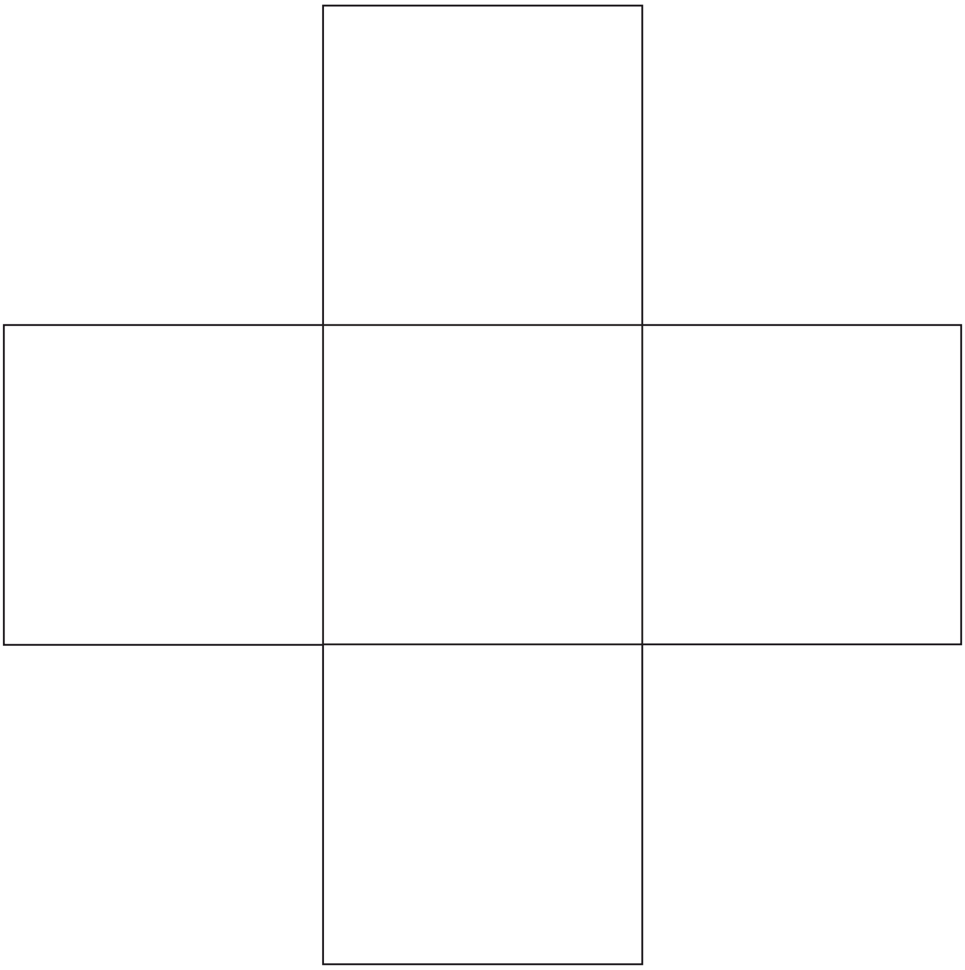
Post-Test

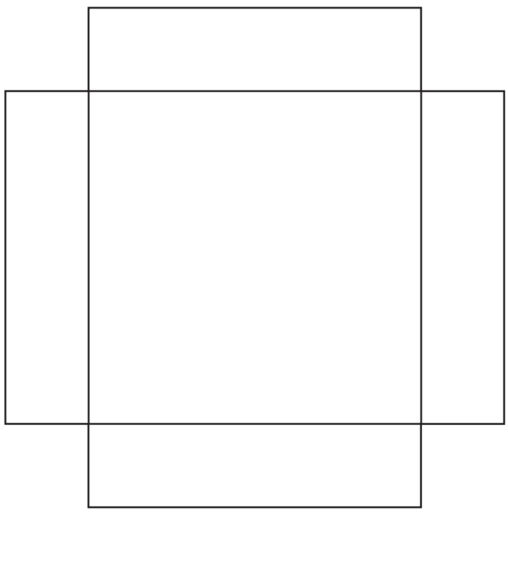
							
A.1	A.2	A.3	A.4	A.5	A.6	A.7	A.8
3	3	3	3	2	3	2	2
3	3	3	3	2	3	2	2
0	2	0	0	2	0	0	0
2	2	3	3	2	3	2	3
1	2	0	0	2	0	2	2
3	3	2	3	4	3	2	2
3	2	2	2	3	0	2	2
3	3	3	3	3	0	3	3
3	3	0	3	3	0	3	3
3	2	2	3	3	0	2	2
2	3	2	2	2	3	2	2
3	3	3	3	3	0	3	3
2	3	2	2	2	0	2	2
3	2	3	3	3	3	3	3
3	2	0	0	0	3	3	0
2	3	2	3	0	0	3	3
3	2	3	0	3	3	2	3
3	2	3	3	3	3	3	3
3	0	1	3	3	3	3	3
3	3	3	3	2	0	3	2
3	2	3	3	3	3	3	3
3	3	3	3	3	2	1	3
3	2	3	3	3	0	2	3
3	3	0	3	3	0	3	3
2	0	0	2	0	0	0	2
2	2	2	0	2	2	2	2

Appendix U

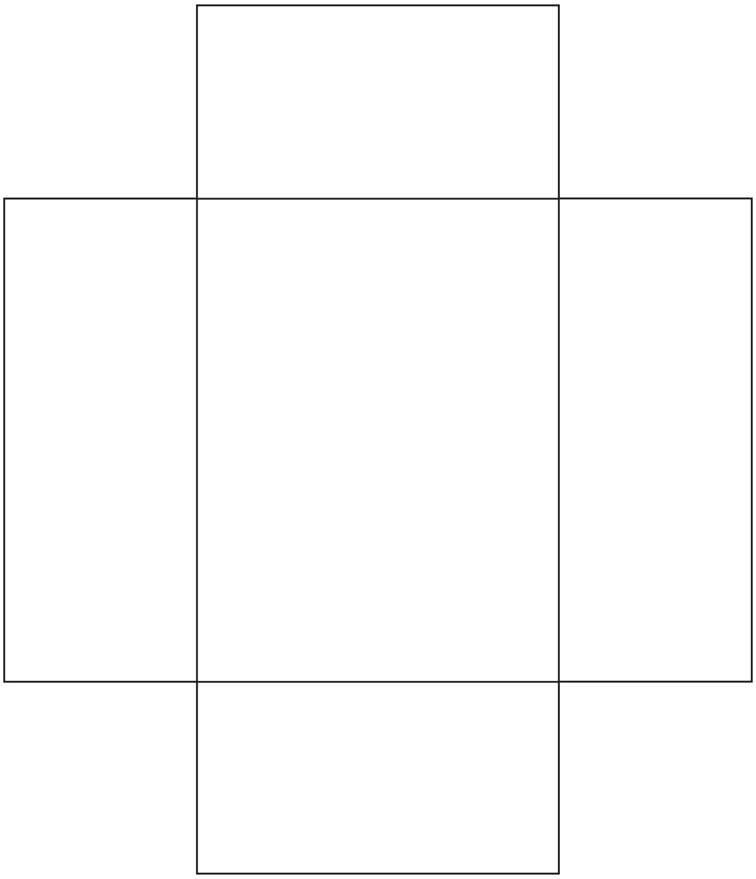
A4 TEMPLATES FOR POLYGONAL BOXES AND LIDS.

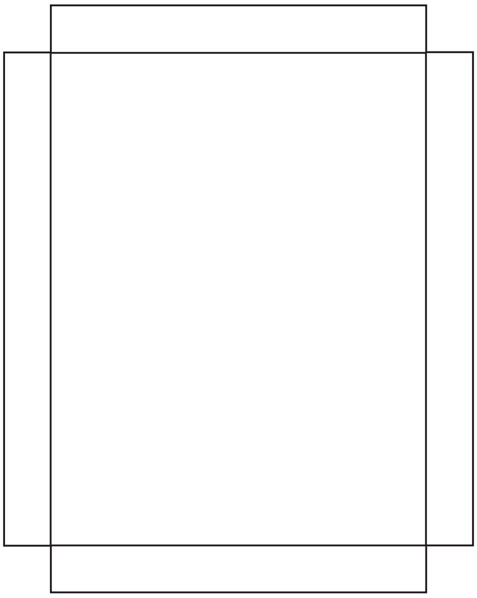
CUBE BOX



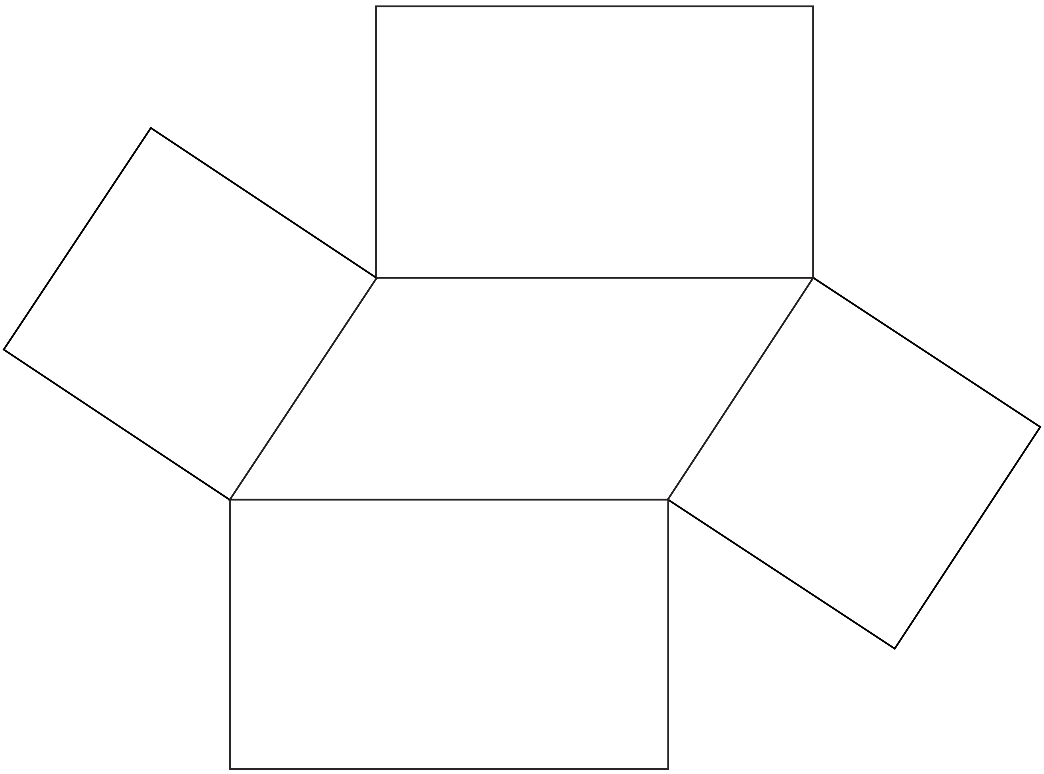


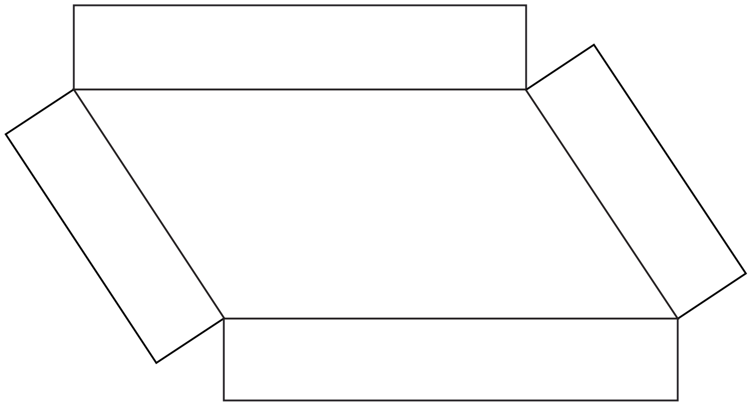
RECTANGULAR PARALLELEPIPED BOX



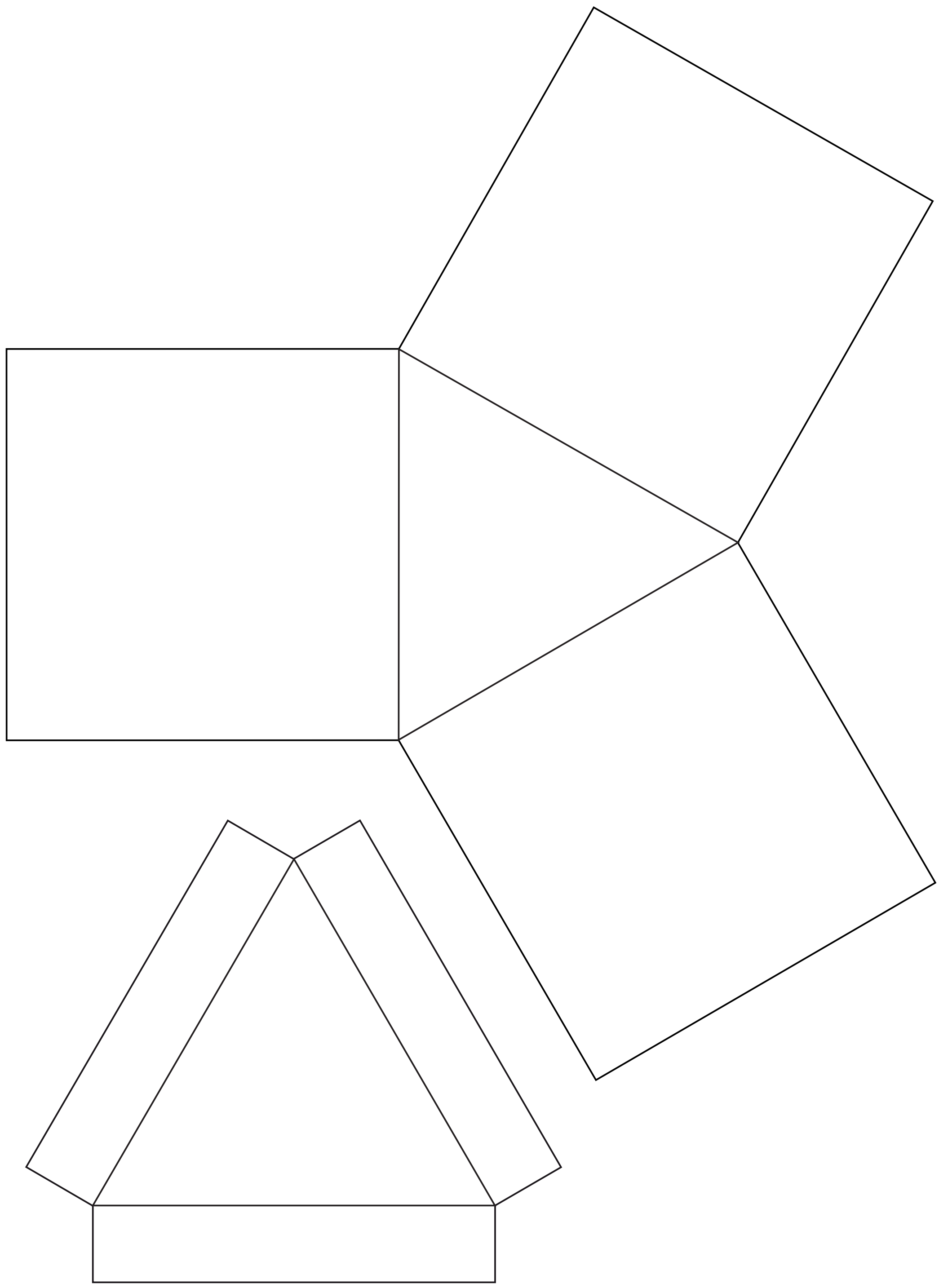


RIGHT PARALLELEPIPED BOX

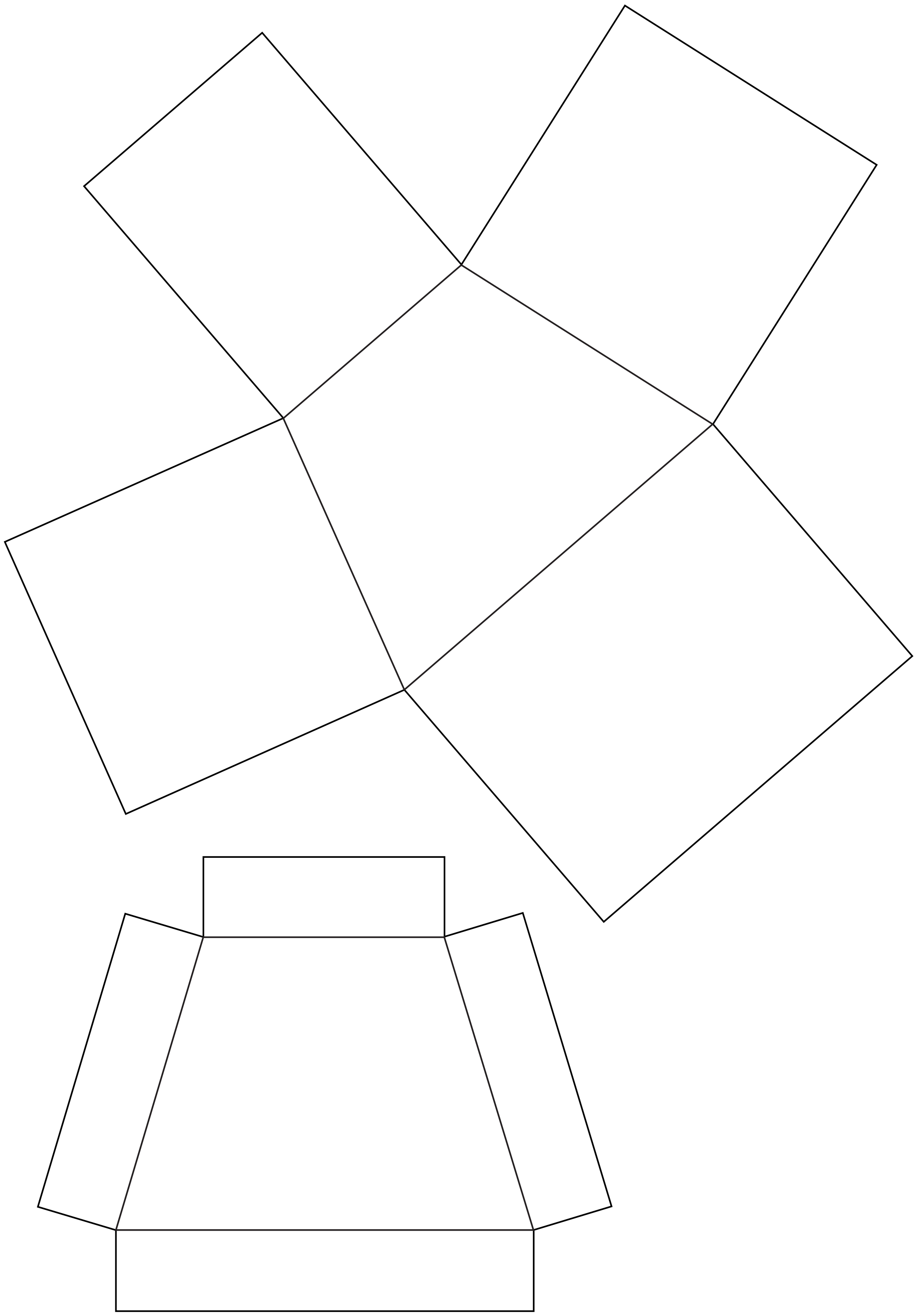




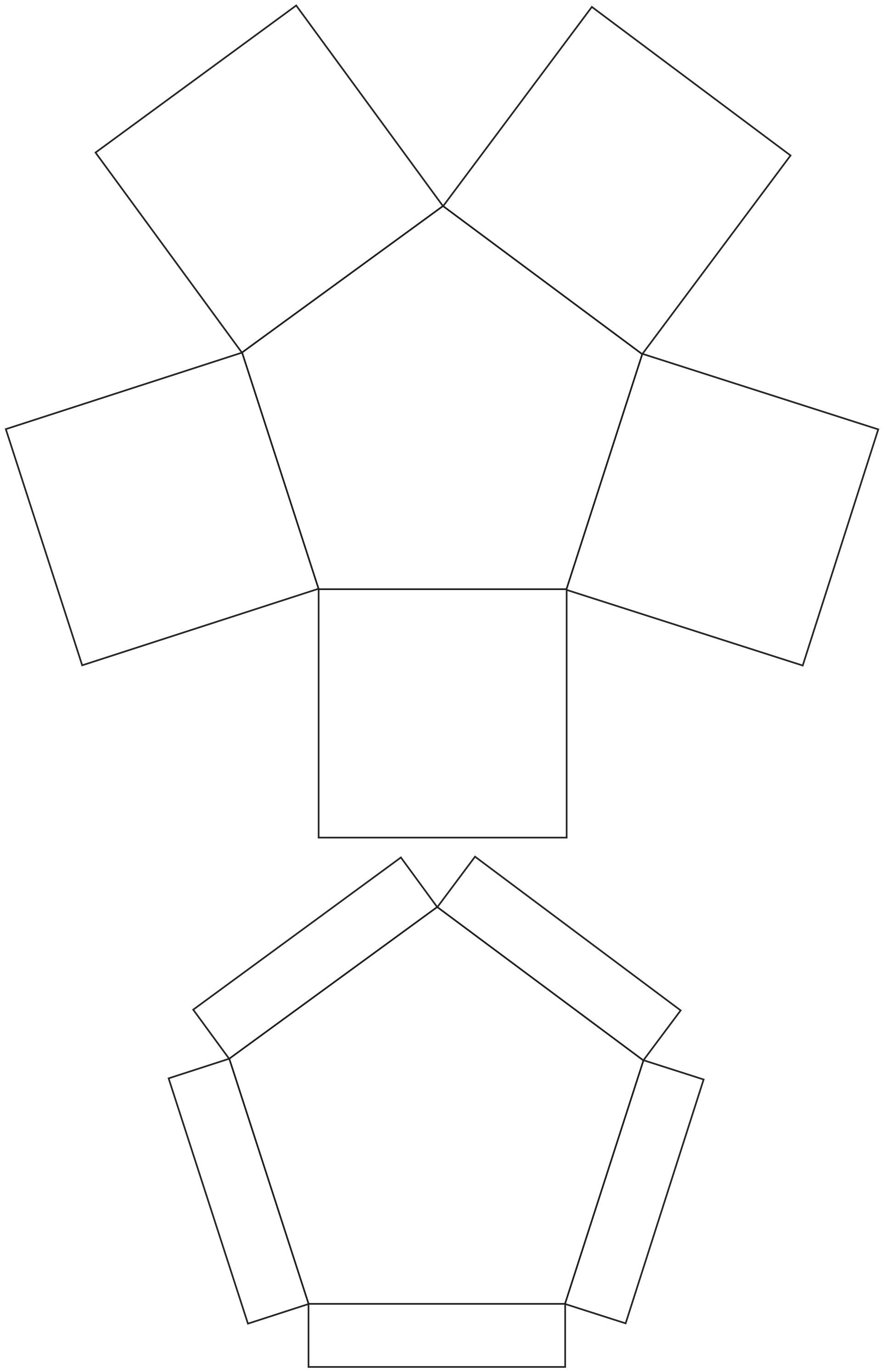
REGULAR RIGHT TRIANGULAR PRISM BOX



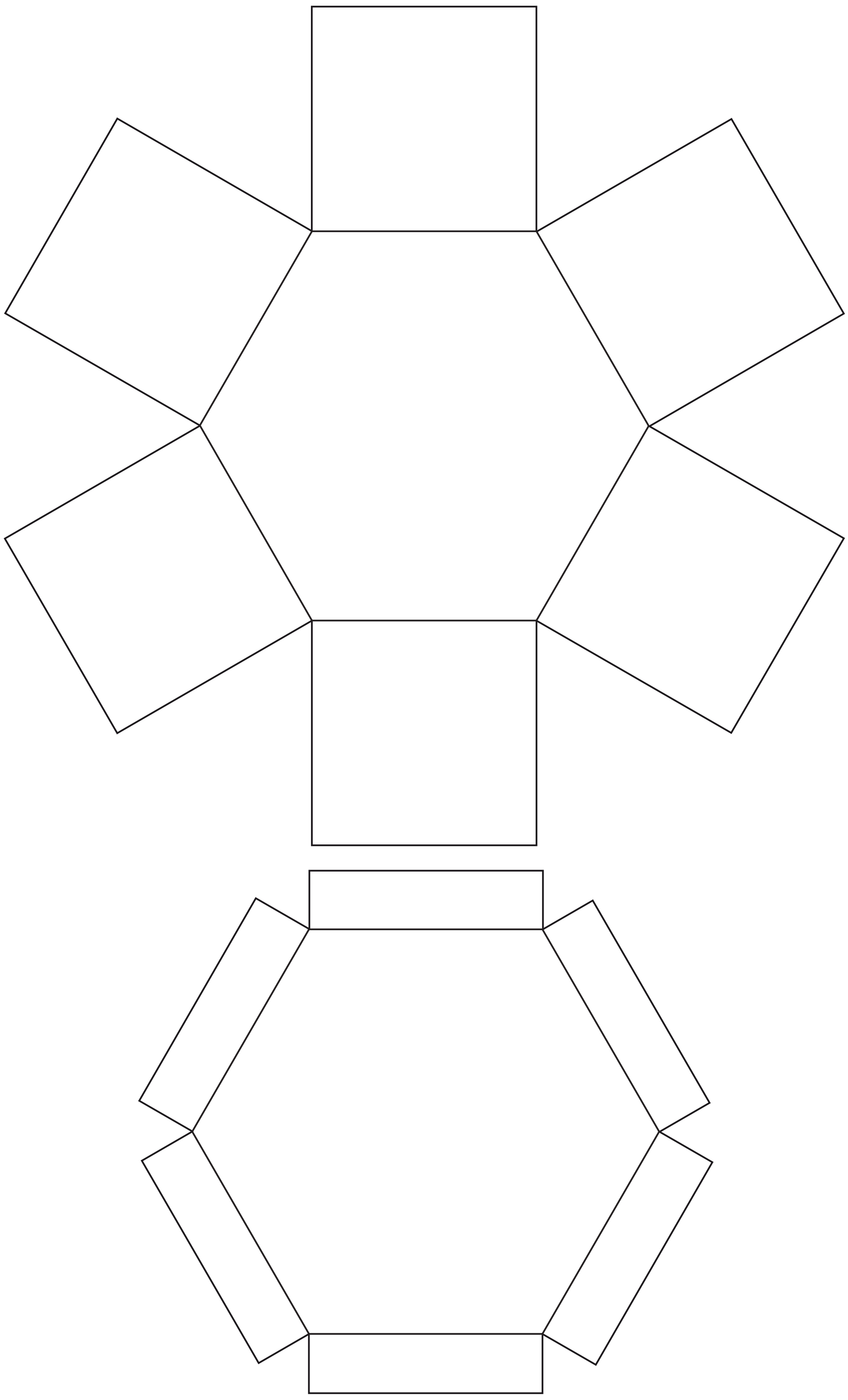
RIGHT TRAPEZOIDAL PRISM BOX



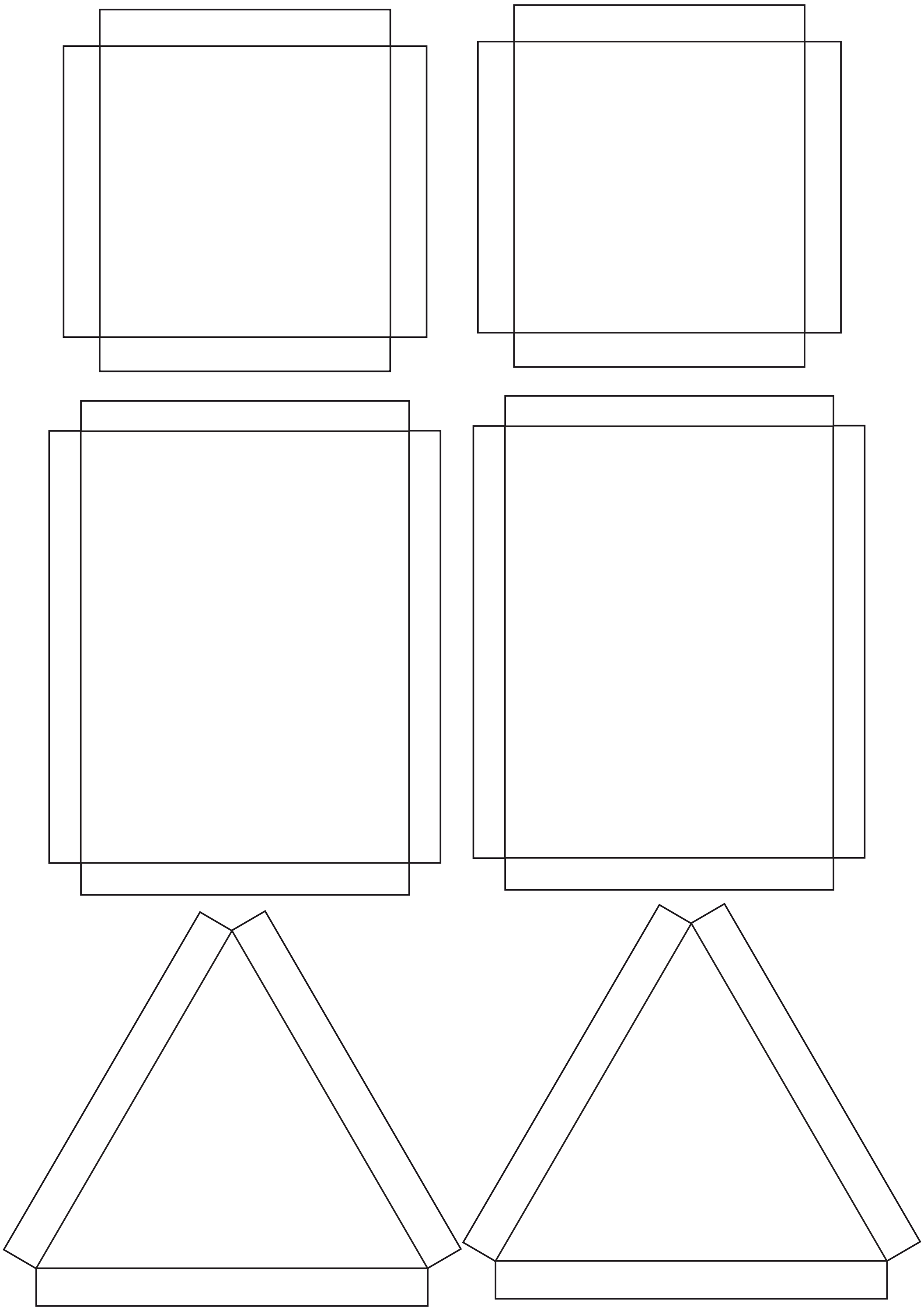
REGULAR RIGHT PENTAGONAL PRISM BOX

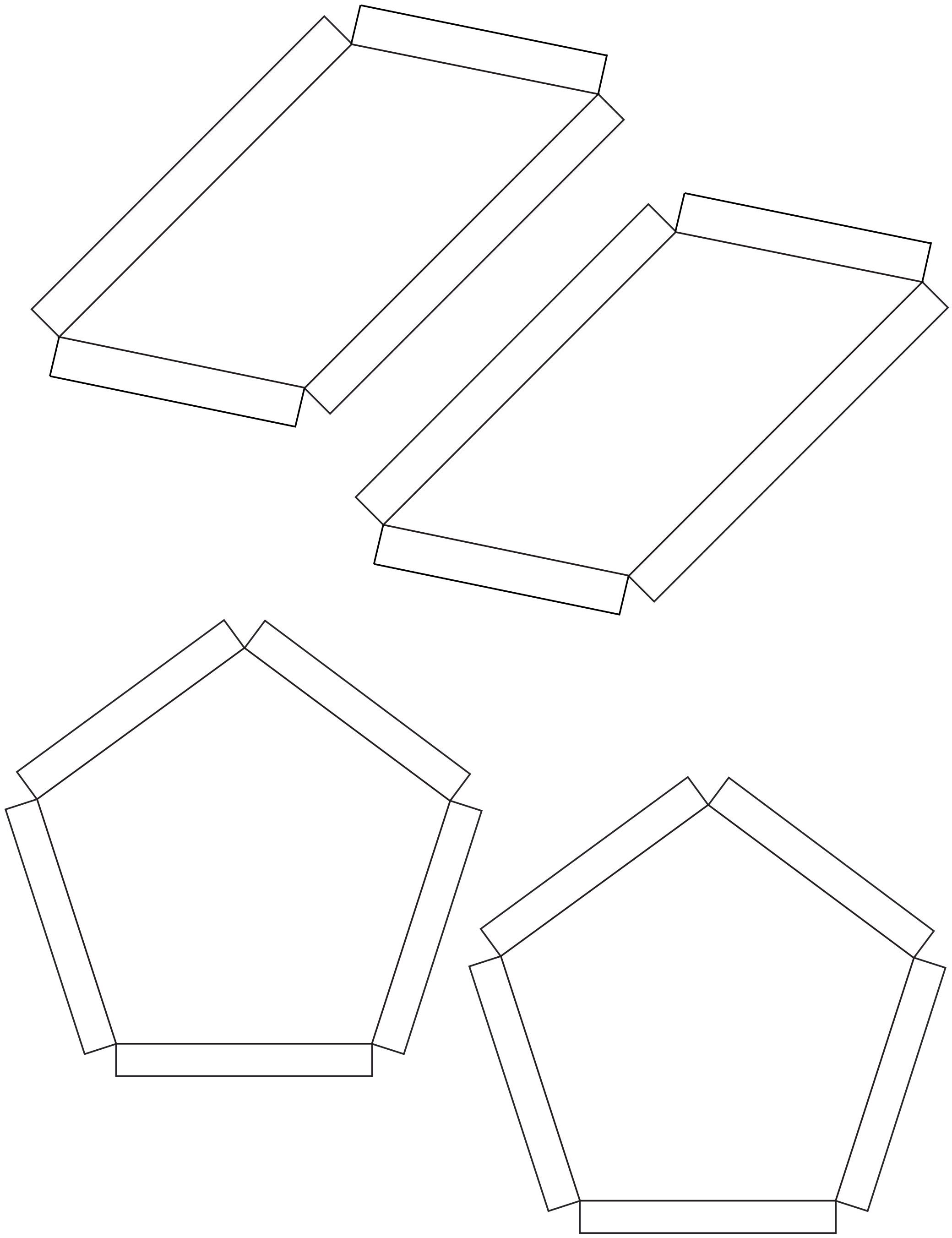


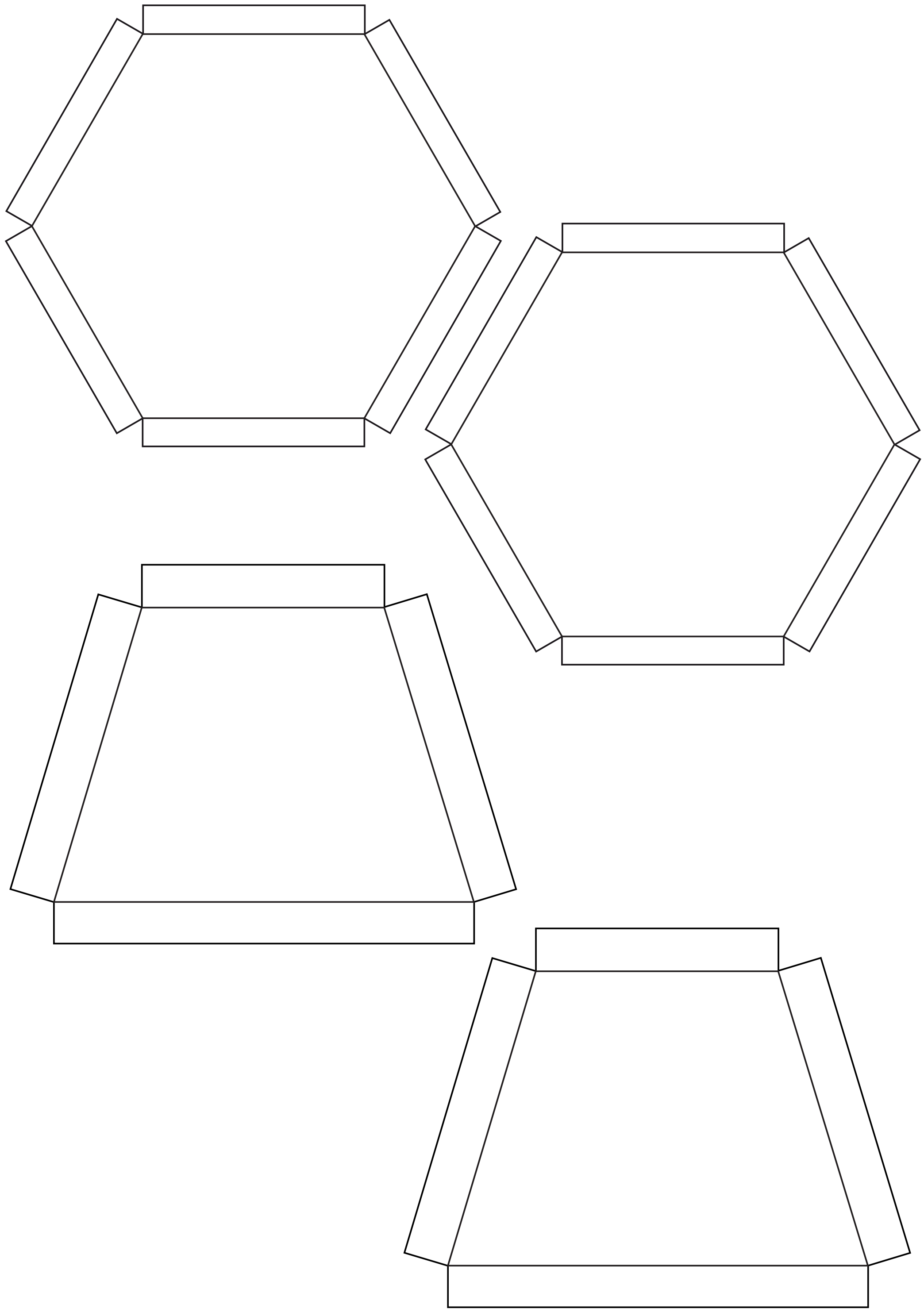
REGULAR RIGHT HEXAGONAL PRISM BOX



LIDS FOR ROTATIONS AND FLIPS

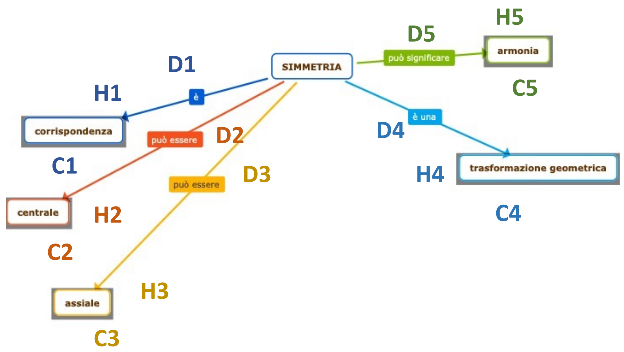






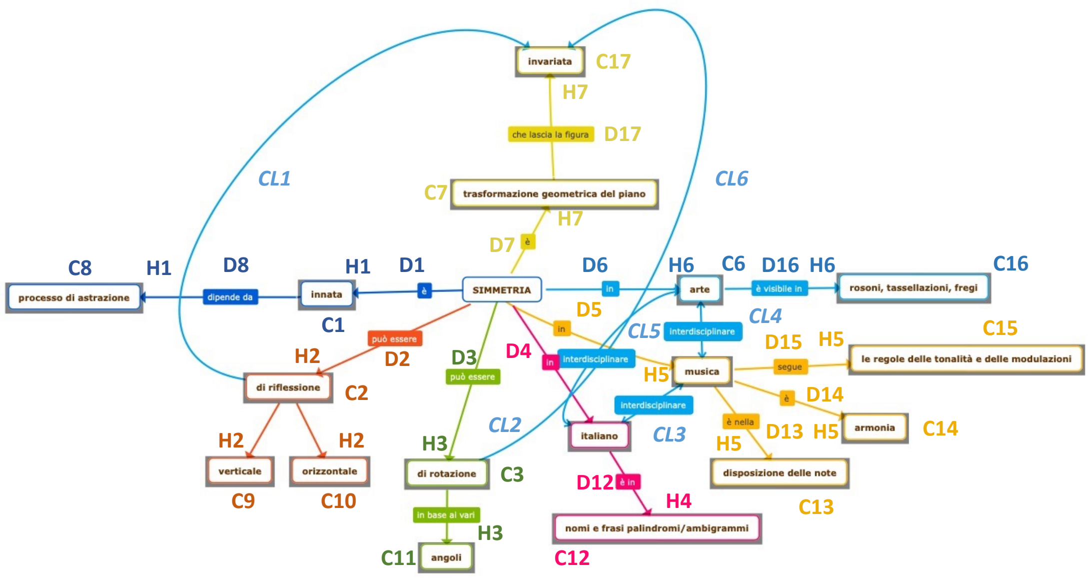
Appendix V

TB9 PRE-TRAINING CONCEPT MAPS



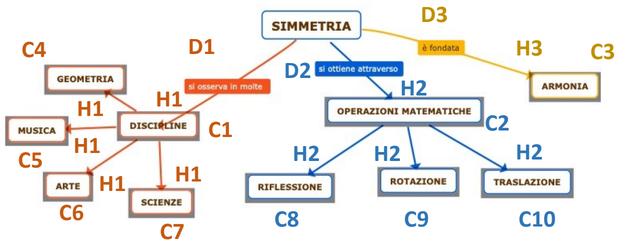
N.C.= 5, H.H.= 1, N.C.L.= 0, Score = (5-0) +1×5+0×10=5+5=10.

TB9 POST-TRAINING CONCEPT MAPS



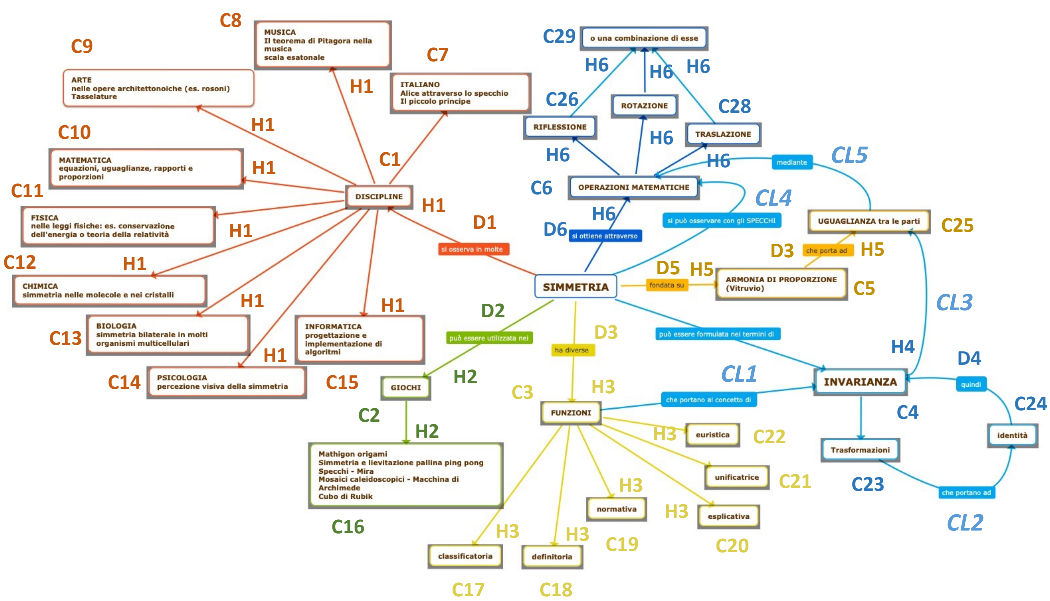
N.C.= 17, H.H.= 2, N.C.L.= 6, Score = (17-6) +2×5+6×10=11+10+60=81.

TE20 PRE-TRAINING CONCEPT MAPS



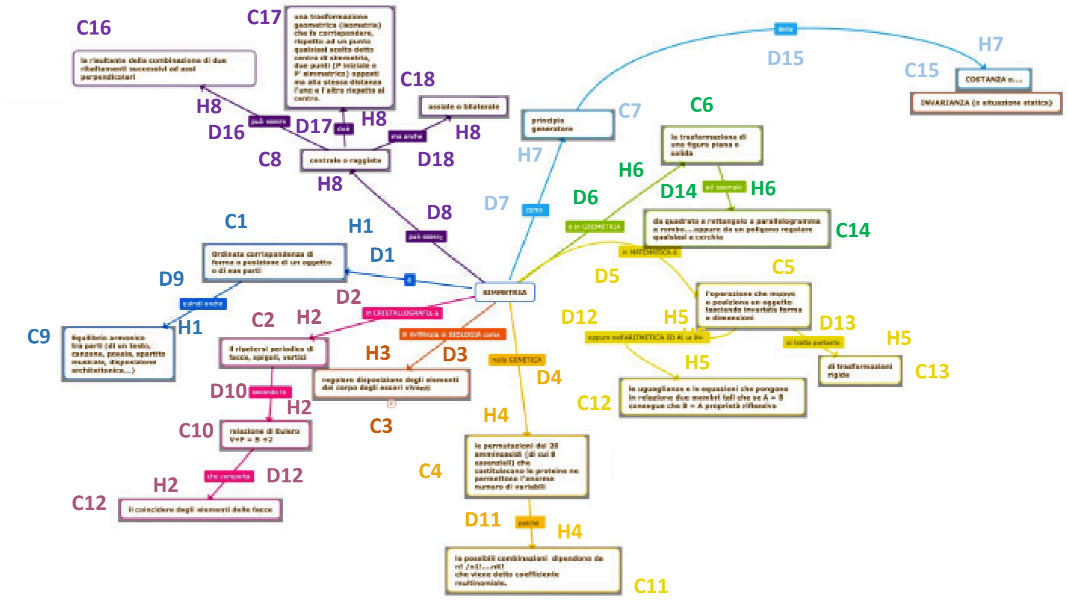
N.C.= 10, H.H.= 2, N.C.L.= 0, Score = (10-0) +2×5+0×10=10+10=20.

TE20 POST-TRAINING CONCEPT MAPS



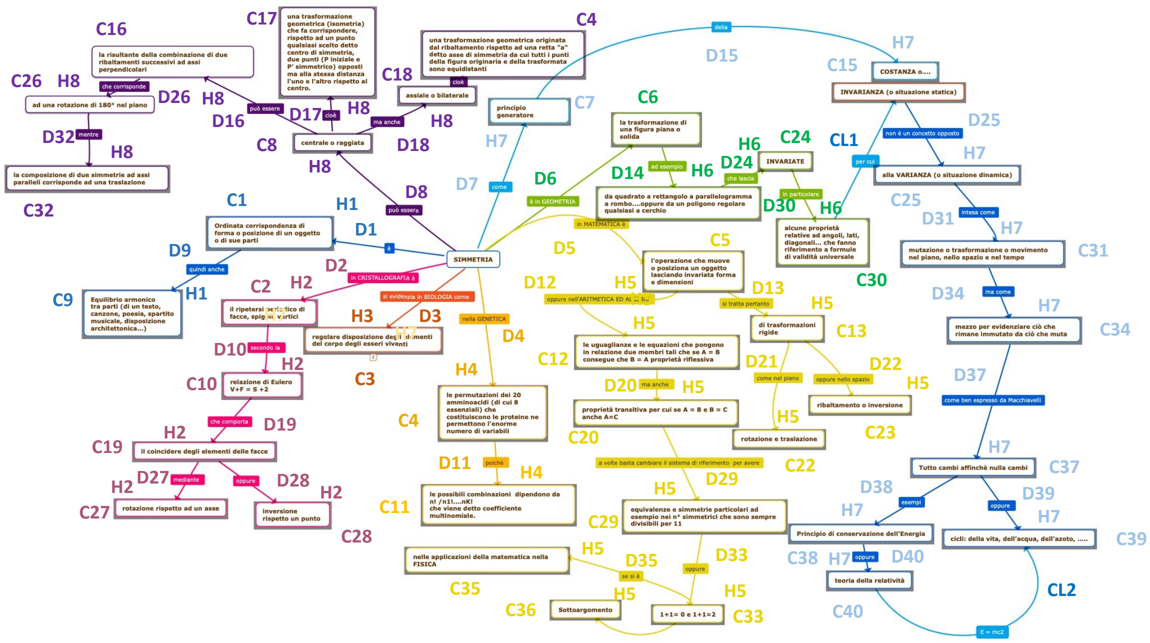
N.C.= 29, H.H.= 3, N.C.L.= 5, Score = (29-5) +3×5+5×10=24+15+50=89.

TA1 PRE-TRAINING CONCEPT MAPS



N.C.= 18, H.H.= 2, N.C.L.= 0, Score = (18-0) +2×5+0×10=18+10=28.

TA1 POST-TRAINING CONCEPT MAPS



N.C.= 40, H.H.= 7, N.C.L.= 2, Score = (40-2) +7×5+2×10=38+35+20=93.

Appendix W

TEACHER SURVEY QUESTIONNAIRE

COURSE IDENTIFIER CODE: …. PARTICIPATION IN CLASS ACTIVITIES IN PREVIOUS RESEARCH CYCLES (Yes/No): ….

Read the following statements carefully and rate the extent to which you agree with one (x) of each statement using the scale of 1 (Strongly disagree) to 5 (Strongly agree) given next to each statement.

1	2	3	4	5
Strongly disagree 😩	**Quite in disagreement** 🙁	**Nor disagree nor agree** 😐	**Enough agree** 🙂	**Strongly agree** 😃

A - GENERAL ASPECTS OF THE COURSE						
1.	The course met my expectations.	1	2	3	4	5
2.	The concept of symmetry was covered in depth.	1	2	3	4	5
3.	The concept of invariance was presented exhaustively.	1	2	3	4	5
4.	The integration between the concept of symmetry and that of invariance was covered in detail.	1	2	3	4	5
5.	The duration of the course was appropriate to the topics covered.	1	2	3	4	5
6.	Discussions and debates had sufficient space in the course	1	2	3	4	5
7.	Considering that the course was held online, the time allocated for group work and guided exercises was sufficient.	1	2	3	4	5

B - PERSONAL KNOWLEDGE						
8.	Thank you to the course, I have increased my knowledge of the concept of symmetry.	1	2	3	4	5
9.	I have expanded my knowledge of the concept of invariance thanks to the course.	1	2	3	4	5
10.	The course has developed my ability to analyse interdisciplinary integrations about symmetry and invariance.	1	2	3	4	5
11.	The course has awakened new needs for personal study.	1	2	3	4	5
12.	The course has motivated me to follow the class and participate in the various activities.	1	2	3	4	5

If you answered positively to questions 10, 11, 12 by choosing one of the alternatives 4 or 5, briefly explain the reasons, on which specific topics and purposes.

C - MATERIALS						
13.	The course material was sufficient and met my expectations.	1	2	3	4	5
14.	The lessons on symmetry in music and art were useful and satisfying.	1	2	3	4	5
Among the following tools introduced in the course I think are coherent, suitable and useful for learning/teaching and applying the modern concept of symmetry and invariance:						
15.	Cardboard boxes with regular and hinged lids in various shapes.	1	2	3	4	5
16.	Mira, mirrors, kaleidoscopic mirrors (Archimedes machine).	1	2	3	4	5
17.	Activities with letters, numbers, palindromes, watching the movie "Palindromic Film".	1	2	3	4	5
18.	Origami activities, symmetrical colouring of shapes, magic squares.	1	2	3	4	5
19.	Activities with platonic solids, tessellations.	1	2	3	4	5
20.	Geometric invariance and shortest path problems, and simple science experiments on invariance.	1	2	3	4	5

D – TEACHING METHODOLOGIES						
21	The course enabled me to think critically about the teaching of symmetry and mathematics in general.	**1**	**2**	**3**	**4**	**5**
22	The course enabled me to understand the effectiveness and importance of symmetry in the vertical progression between elementary and lower secondary schools, as well as the broken continuity with secondary schools.	**1**	**2**	**3**	**4**	**5**
23	The course enabled me to understand the effectiveness and importance of invariance in the vertical progression between elementary and lower secondary schools, as well as the broken continuity with secondary schools.	**1**	**2**	**3**	**4**	**5**
24	The introduction of the modern concept of symmetry (invariance in transformation) is inclusive for all students and more participatory in the educational dialog.	**1**	**2**	**3**	**4**	**5**
25	Introducing the modern concept of symmetry (invariance in transformation) facilitates connections between mathematical topics.	**1**	**2**	**3**	**4**	**5**
26	Introducing the modern concept of symmetry (invariance in transformation) facilitates connections between math and science disciplines.	**1**	**2**	**3**	**4**	**5**
27	The course will influence my teaching practice in relation to symmetry and mathematics in general.	**1**	**2**	**3**	**4**	**5**
28	I will introduce the modern concept of symmetry into the classroom.	**1**	**2**	**3**	**4**	**5**
29	I will be able to put into practice the knowledge and teaching/learning methods acquired in the course.	**1**	**2**	**3**	**4**	**5**
30	I will provide continuity to action research on symmetry and invariance.	**1**	**2**	**3**	**4**	**5**

If you answered positively to questions 28, 29, 30 by choosing one of the alternatives 4 or 5, briefly explain how you think you do it.

Evaluate teaching now. Indicate the degree to which you agree with (x) of each statement using scale from 1 (completely inadequate) to 5 (completely adequate).

1	2	3	4	5
Completely inadequate	**Enough appropriate**	**Nor adequate nor inadequate**	**Enough appropriate**	**Completely appropriate**
😩	🙁	😐	🙂	😃

E – CONSIDERATIONS ON TEACHING						
31.	The course speakers proved to be competent in the topics covered.	**1**	**2**	**3**	**4**	**5**
32.	Course speaker intervention was effective in the classroom.	**1**	**2**	**3**	**4**	**5**
33.	Course speakers were available and willing to respond to participants' needs.	**1**	**2**	**3**	**4**	**5**
34.	Speakers used language that was coherent and appropriate for the level of the course.	**1**	**2**	**3**	**4**	**5**

Other considerations and/or suggestions (strengths, weaknesses of the course, tools, and methodologies, …)

Appendix X

TRAINING CERTIFICATE

Scuola di Scienze e Tecnologie
Via Madonna delle Carceri – 62032 Camerino (Italy) – tel. +39 0737 402549 – fax +39 0737 632525

Matematica oltre i numeri, simmetria e ricerca invarianti

Percorso formativo svoltosi dal 09-11-2020 al 30-11-2020
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Nome Cognome

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e

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Camerino, 11 giugno 2021

prof. Riccardo Piergallini *dott. Simone Brasili*

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