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Orthogonal decompositions in asset pricing

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Abstract

This thesis introduces several decompositions of financial processes in discrete and in continuous time. The overall purpose is the isolation of uncorrelated components classified according to the duration of their impact on aggregate processes, from temporary fluctuations to martingale terms (also called permanent components). The first article, *A persistence-based Wold-type decomposition for stationary time series* by F. Ortu, F. Severino, A. Tamoni and C. Tebaldi, deals with weakly stationary time series and constructs an orthogonal decomposition into components associated with different levels of persistence. This methodology is then applied to the analysis of realized volatility and provides evidence of the Heterogeneous Market Hypothesis. The second paper, *Weak time-derivatives and no arbitrage pricing* by M. Marinacci and F. Severino, introduces a differential tool that enables the identification of martingales within continuous-time stochastic processes. This instrument allows us to isolate the drift of a wide class of semimartingale no arbitrage price processes and to retrieve the decomposition of stochastic discount factors of Hansen and Scheinkman (2009). The third article, *Long-term risk with stochastic interest rates* by F. Severino, develops a conditional continuous-time asset pricing framework with stochastic interest rates. Differently from the constant rate case, in this setting no ex-ante measurable financial variable is known to capture both no arbitrage price dynamics and pricing kernel growth rates. Hence, we generalize our technique for martingale identification in order to characterize forward prices and illustrate the extent to which yields to maturity determine the evolution of prices and stochastic discount factors. Importantly, the construction is consistent with asymptotic results of long-term risk literature.

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Introduction

A challenge of asset pricing is modelling the driving forces that impinge on meaningful quantities for financial markets such as volatility, security prices and portfolio returns. The impact of each factor lasts for a proper amount of time. Therefore, investors require compensation for the exposure to risks with heterogeneous durations, especially when multiple horizons are considered. Moreover, the isolation of such factors is made difficult by the presence of simultaneous spurious correlation, that prevents drawing robust conclusions from statistical analyses.

In this thesis we illustrate several decompositions of financial processes in various contexts, both in discrete and in continuous time, allowing for both short- and long-term horizons. After decomposing a given process into several components, we are interested in classifying our factors according to their persistence, i.e. the relative duration of their impact on the process under consideration. On the one hand, we have martingale components - also called permanent components - that affect economic processes in the long term. Large part of the work collected here involves the development of tools for martingale identification applied to price processes in arbitrage-free markets. Specifically, we study financial markets with both deterministic and stochastic interest rates. On the other hand, when dealing with stationary processes, we provide a methodology to disentangle uncorrelated components with precise levels of persistence, according to their relative (finite) variance. This perspective is in line with the literature of heterogeneous agents that act on financial markets across different investment horizons and generate price movements in the aggregate.

This thesis consists of three chapters. Each of them is a self-contained paper, namely

- *A persistence-based Wold-type decomposition for stationary time series* by Fulvio Ortu, Federico Severino, Andrea Tamoni and Claudio Tebaldi (submitted);

- *Weak time-derivatives and no arbitrage pricing* by Massimo Marinacci and Federico Severino (submitted);

- *Long-term risk with stochastic interest rates* by Federico Severino.

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The first article, *A persistence-based Wold-type decomposition for stationary time series*, shows how to isolate uncorrelated persistent components from weakly stationary financial time series. This problem derives from the existence of heterogeneous driving factors underlying the observable outcome of economic processes. In particular, we provide an *Extended Wold Decomposition* based on an isometric *scaling operator* that exploits increasingly long averages of process innovations. Weakly stationary time series are decomposed into the sum, across time scales, of uncorrelated components associated with different degrees of persistence. Our representation of a time series allows to define *multiscale impulse response functions* and *multiscale* variance decomposition. Using these tools, we analyze the exchange rates realized volatility, and we quantify the role of investors with different time horizons, in agreement with the Heterogeneous Market Hypothesis.

The second paper, *Weak time-derivatives and no arbitrage pricing*, pertains to the no arbitrage valuation of random payoff in continuous-time markets. Despite the variety of payoffs, risk-neutral price dynamics are driven by the same interest rate process. We formalize this intuition by showing that risk-neutral prices constitute the weak solution of a differential equation, where interest rates have a prominent role. To reach this purpose, we introduce the notion of *weak time-derivative*, which permits to differentiate adapted stochastic processes. The weak time-derivative enables the identification of the drift term of semimartingale processes and, importantly, it is null for martingales. Finally, we reformulate the eigenvalue problem of Hansen and Scheinkman (2009) by employing weak time-derivatives.

The third article, *Long-term risk with stochastic interest rates*, is about conditional asset pricing in continuous-time arbitrage-free markets with stochastic interest rates. We study the impact on pricing of the intertemporal aggregation of exposures to short-term rates variability across arbitrarily large investment horizons. When interest rates are constant, the instantaneous rate drives both security prices (via the no arbitrage relation between instantaneous returns and risk-free rate) and the evolution of pricing kernels. However, no ex-ante measurable financial variable is known to play an analogous dual role when rates are time varying. Hence, without assuming any specific dynamics for interest rates, we show how pure discount bond yields address this issue when the forward measure is employed instead of the risk-neutral one. The weak time-derivative turns out to be a useful tool to achieve this goal. Our results properly generalize the constant rate case, with meaningful departures due to the variability of rates. Moreover, our analysis of prices and pricing kernels is consistent with long-term risk convergences when the horizon under consideration becomes arbitrarily large.

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1

A persistence-based Wold-type decomposition for stationary time series

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1.1 Introduction

Economic[1] and financial time series experience fluctuations whose frequencies range from monthly intervals – such as shifts due to non-farm payroll announcements – to several decades or longer – such as large-scale technological changes. For instance, the price-dividend ratio on the US stock market has crossed its mean value 29 times since 1872, with intervals between crossings ranging from one year to twenty years (see Campbell and Shiller (2001)).[2,3] Similarly, inflation and market volatility display often short-run, transitory movements occurring around a long-run, persistent trend, see e.g. Engle and Lee (1999) and Stock and Watson (2007).

This paper formalizes the idea that a time series can be seen as a superposition of (orthogonal) layers with heterogeneous levels of durations. Notably, our representation is obtained in the time-domain. Thus, although it shares many of the insights of frequency domain methods, our approach is well suited for predictive analysis where time adaptation, and localization in both time and frequency, are crucial. Also, by applying traditional time series formulations, our approach gives rise to a generalization of classical concepts such as impulse response functions (IRFs, see Frisch, 1933) and principal component analysis (PCA). The key novelty is that our IRFs and PCA speak about the role of both *time and frequency* in the evaluation of the impact of economic shocks, whereas classical time-series methods often have difficulty to disentangle the two dimensions.

Our main theorem (see Theorem 1.3) shows how to represent a stationary time series as a sum of uncorrelated components each one characterized by a specific level of persistence (or scale). Our novel representation produces an *Extended Wold Decomposition* in which each component at scale j is an infinite moving average driven by a sequence of uncorrelated innovations that are located on a time grid whose unit spacing is 2^j the original unit grid.

The features of our Extended Wold Decomposition give rise to generalizations of classical concepts such as PCA and IRFs, and provide a formal definition of persistence level. First, as a by-product of the orthogonality of components, our approach naturally provides a variance decomposition of the time series. Differently from the traditional PCA approach, each component in our analysis is associated with a specific scale, a fact that comes to aid when one has to provide an economic interpretation of the key driving factors. Indeed, different economic phenomena most

[1] Joint work with Fulvio Ortu, Andrea Tamoni and Claudio Tebaldi.

[2] The most pronounced "boom-bust" episodes are the one that began in the early 1920s and bust near the start of the Great Depression, and the one that began in the mid 1950s and bust in the 1970s.

[3] Analogously to the stock market, the aggregate economy experiences cyclical fluctuations of heterogeneous durations. For example, the Great Recession lasted 18 months, from December 2007 to June 2009. On the other hand, the recession in the early Nineties lasted only 8 months, from July 1990 to March 1991.

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likely operate at different frequencies: e.g., price movements can be related to announcements of macroeconomic conditions and corporate activity at the short-end of spectrum, to political cycles at the medium-end, to technological and demographic changes at the long-end, and to uncertainty regarding exhaustible energy resources and climate change at the very long-end.[4] Therefore, by being informative about the duration of the fluctuation most relevant for the time series variability, our approach sheds light on the potential economic mechanism driving the time series. For instance, in our empirical application, we show that a model for daily volatility based on daily, weekly, and monthly components (see Corsi (2009)) indeed proxies for phenomena occurring over longer time scales in between half-year and two years. If one views each volatility component as generated by the actions of specific types of market participants, then our analysis is informative about the horizons of such investors.[5]

Second, our approach yields (a set of) impulse response functions at specific levels of persistence (dubbed *multiscale impulse response functions*). These multiscale impulse response functions are linear transformation of the original Wold coefficients and, thus, easy to obtain in empirical applications. At the same time they are powerful tools to separate and zoom-in the different fluctuations which may co-exist in the original time series. For example, many U.S. macroeconomic time series have hump-shaped dynamics - in other words, they exhibit momentum in the short run and (partial) mean reversion in the long run (see, e.g., Fuster, Hebert, and Laibson (2012)). In a simulation exercise, we show that our multiscale impulse response function separates the momentum and reversal phases, and conveys information about their persistence (or anti-persistence), and hence about their build-up, and subsequent unravelling.

Third, our Extended Wold Decomposition induces a formalization of the notion of persistence for stationary processes. Indeed, our construction exploits the isometry of a *scaling operator* that smooths the original time series by making successive averages of the process innovations. Such procedure implicitly defines an index j that we dub *scale* or *degree of persistence*. In fact, a

[4] An extensive literature studies stock price reactions to corporate announcements including quarterly earnings and dividend policy (see MacKinlay (1997)), macroeconomic announcements (see Andersen, Bollerslev, Diebold, and Vega (2007)), political cycles (see Santa-Clara and Valkanov (2003)), technological innovation (see Jovanovic and Greenwood (1999) and Pastor and Veronesi (2005)), demographic changes (see Abel (2003) and Favero, Gozluklu, and Tamoni (2011)) and the prospect of rising global temperature (see Bansal and Ochoa (2011)).

[5] This idea stems from the so-called "Heterogeneous Market Hypothesis" presented by Müller, Dacorogna, Dave, Pictet, Olsen, and Ward (1993), which recognizes the presence of heterogeneity in the traders. An alternative view put forward by Andersen and Bollerslev (1997) is that the multi-component structure stems from the heterogeneous nature of information arrivals rather than from the heterogeneity of agents.

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shock at scale j has, by construction, a mean reversion time ranging between 2^{j-1} and 2^j, and its spectrum is localized in the corresponding band of frequencies.

Whereas our first result (see Theorem 1.3) shows how to decompose and analyze a time series in terms of its persistent components, our second result (see Theorem 1.5) shows how to build a stationary process starting from uncorrelated components with heterogeneous degrees of persistence. This second result is important as it offers a new family of (data generating) processes suitable to capture phenomena evolving over time scales of different length (or frequency). Interestingly, the work by Granger (1980) proposes contemporaneous aggregation of individual, *random* coefficient AR(1) processes as a mechanism that leads to long memory. Relying on Theorem 1.5, the simulations in Section 1.3.2 show that a *finite* (in fact as small as two) sum of autoregressive processes with *fixed* autoregressive coefficients is able to generate hyperbolically decaying autocorrelation functions as far as the processes evolve over different time scales (e.g., monthly, and annual). On the technical side, the proof of Theorem 1.5 highlights that the reconstructed time series is stationary because each scale-specific shock is assumed to be a weighted linear combination of underlying innovations. A suitable choice of the weighting coefficients yields once again the Extended Wold Decomposition. At the same time, this converse theorem opens up the possibility to deduce other Wold-type decompositions by choosing alternative suitable combinations of the underlying disturbances.

The rest of the paper is organized as follows. Section 1.2 employs our isometric scaling operator to derive the Extended Wold Decomposition. In that section we analyse the main properties of our decomposition and illustrate its empirical advantages by means of an example. In particular, Subsection 1.2.4 briefly discusses a generalization of the Extended Wold Decomposition to multivariate processes. We devote Section 1.3 to show a converse result that allows us to uncover the moving average representation of the weakly stationary time series $\mathbf{x}$ obtained by summing heterogeneous components at different scales. We show that, if a suitable energy (i.e. variance) conservation condition is imposed, one can uniquely pin down the coefficients that map the scale-specific shocks into fundamental innovations. In Section 1.4 we use spectral analysis to classify our theory as a low-pass filtering technique. Section 1.5 deals with the Extended Wold Decomposition of exchange rates Realized Volatility. Our orthogonal decomposition induces a persistence-based variance decomposition that allows us to detect the presence of heterogeneous market agents that trade according to different time horizons. Section 1.6 concludes. Proofs and additional results are collected in the Appendix.

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1.1.1 Related literature

Our multiscale impulse response functions are the multifrequency counterparts of the *shock elasticities* proposed by Borovička, Hansen, Hendricks, and Scheinkman (2011), Hansen (2012) and Borovicka and Hansen (2014). Both approaches generalize the impulse response functions familiar in the VAR literature. Whereas the shock elasticities suggested by Borovička, Hansen, Hendricks, and Scheinkman (2011) capture nonlinearities and state-dependent features of the dynamic model under study, our approach is useful to examine the reaction of dynamic economies hit by shocks of heterogeneous durations, following the intuition in Ortu, Tamoni, and Tebaldi (2013).

A common way to obtain impulse response functions is to estimate a Vector AutoRegression (VAR), and then invert it to get the moving-average model. Interestingly, in two recent contributions, Barnichon and Matthes (2017) and Plagborg-Moller (2017) have proposed empirical methods that side-step the VAR and instead directly estimate the vector moving average model. These methods could be applied to obtain the structural moving-average representation whose coefficients can later be used to compute our multiscale impulse response functions.

Our paper shows that a stationary time series can be described as the sum of uncorrelated components with different levels of persistence. A component at level of persistence j admits a classical Wold decomposition in terms of innovations that are moving averages of the original innovations and are localized on a time grid whose unit is 2^j units of the original observation grid. The coarsening of the grid where low-frequency shocks are located is closely related to the frequency-domain procedure adopted by Müller and Watson (2008) and Müller and Watson (2015), where long-run sample information is isolated using a small number of low-frequency trigonometric weighted averages which, in turn, can be used to conduct inference about long-run variability and covariability. Differently from them, our resulting representation uses information only up to time t and, thus, it does not suffer from the look-ahead bias typical of a two-sided frequency representation.

Our persistence-based Wold-type representation lays the foundation for processes that have been proposed in the literature to capture the long-range dependence exhibited by many financial time series, such the Heterogeneous Information Arrivals process suggested by Andersen and Bollerslev (1997), the Markov-Switching Multifractal work by Calvet and Fisher (2007), the Heterogeneous Autoregressive Model of Realized Volatility (HAR-RV) developed by Corsi (2009), the component-GARCH models of Engle and Lee (1999) and the Spline-Garch framework of Engle and Rangel (2008).[6]

[6]Specifically, Andersen and Bollerslev (1997) express the log volatility process as a sum of a large number of underlying independent AR(1) processes (dubbed information processes) and the autoregressive

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Our decomposition into components with different persistence levels is additive. This is in contrast with the MIDAS (Mi(xed) Da(ta) S(ampling)) regressions developed in Ghysels, Santa-Clara, and Valkanov (2004) and Ghysels, Santa-Clara, and Valkanov (2006), where the smoothing coefficients interact multiplicatively in a way that makes it difficult to isolate the effects of each components. As it is shown in Ortu, Tamoni, and Tebaldi (2013) the additivity of our decomposition is also particularly useful if one wants to separate out and price short-lived versus slower-decaying components.

Interestingly, a consensus has not yet been reached on how to measure persistence correctly. Campbell and Mankiw (1987) argue that persistence is well measured by the long-run impact of an innovation on the level of the series. Cochrane (1988) proposes an alternative measure based on the variance ratio of the long-run to one-period changes in the time series.[7] Cogley and Sargent (2005) characterize (inflation-gap) persistence in terms of the normalized spectrum at frequency zero. However, Cogley, Primiceri, and Sargent (2010) measure persistence in terms of short- to medium-term predictability since, as they argue, measures of short- and medium-term predictability are estimated more precisely than the spectrum at frequency zero and, thus, produce sharper results. As already discussed, the present approach offers a natural formalization of persistence since disturbances with different durations are associated with different scales. In addition, the uncorrelation of our persistent components allows one to employ relative variance as an indicator of persistence.

1.2 Extended Wold Decomposition

We consider, throughout, a zero-mean, weakly stationary process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$, where each x_t belongs to the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Through the whole paper, equalities between random variables are in the L^2-norm. Moreover, we assume that $\mathbf{x}$ is purely non-deterministic, i.e. there exists a

coefficients reflect the degree of persistence of each information arrival process; Calvet and Fisher (2007) specify the volatility series as the product of k distinct components, each of these components evolve independently as a two-state Markov chain and the transition probabilities control the persistence of the component; Corsi (2009) suggests to use the values of the time series realized over different interval sizes (i.e. sampled at different frequencies) to determine its future evolution.

[7] Campbell and Mankiw (1989) show that these two concepts of persistence are closely related and, in particular, the square root of Cochrane's measure of persistence is a lower bound on the Campbell-Mankiw one. Empirically the two measures have led to different perspectives on the importance of persistence shocks in output (see discussion in Durlauf (1989)).

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unit variance white noise $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ such that, for any t in $\mathbb{Z}$,

$$x_t = \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-h}.$$

In case $\mathbf{x}$ is not purely non-deterministic, the decomposition we describe is meant for the non-deterministic component of $\mathbf{x}$. The process $\boldsymbol{\varepsilon}$ is commonly called the *sequence of fundamental innovations* of $\mathbf{x}$. The coefficients α_h are square-summable, they do not depend on t and each α_h is the projection coefficient of x_t on the linear space generated by ε_{t-h}, namely $\alpha_h = \mathbb{E}[x_t\varepsilon_{t-h}]$. This is the Classical Wold Decomposition for time series (see Wold (1938)), recalled in Theorem 1.9 in Appendix 1.A.

Consider now the linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ spanned by the sequence of fundamental innovations $\{\varepsilon_{t-k}\}_{k\in\mathbb{N}_0}$, that is

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \left\{ \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} : \quad \sum_{k=0}^{+\infty} a_k^2 < +\infty \right\}.$$

In words, $\mathcal{H}_t(\boldsymbol{\varepsilon})$ is the space of infinite moving averages, commonly denoted by $MA(\infty)$, whose underlying white noise process is $\boldsymbol{\varepsilon}$. Next, we define the *scaling operator* $\mathbf{R} : \mathcal{H}_t(\boldsymbol{\varepsilon}) \longrightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ as follows

$$\mathbf{R}: \qquad \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \quad \longmapsto \quad \sum_{k=0}^{+\infty} \frac{a_k}{\sqrt{2}} \left(\varepsilon_{t-2k} + \varepsilon_{t-2k-1}\right) = \sum_{k=0}^{+\infty} \frac{a_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \varepsilon_{t-k},$$

where $\lfloor\cdot\rfloor$ associates any real number c with the integer $\lfloor c \rfloor = \max\{n \in \mathbb{Z} : n \leqslant c\}$. The scaling operator is well-defined, linear and isometric on $\mathcal{H}_t(\boldsymbol{\varepsilon})$, as stated by Proposition 1.10 in Appendix 1.B). Next, we show that an application of the Abstract Wold Theorem on the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, endowed with the operator $\mathbf{R}$, is suitable for determining a new Wold-type decomposition of x_t.

1.2.1 Orthogonal decomposition of the space

Given a Hilbert space $\mathcal{H}$ and an isometric operator $\mathbf{V} : \mathcal{H} \longrightarrow \mathcal{H}$, the Abstract Wold Theorem (i.e. Theorem 1.8 in Appendix 1.A) shows that $\mathcal{H}$ decomposes into an orthogonal sum $\mathcal{H} = \hat{\mathcal{H}} \oplus \tilde{\mathcal{H}}$, where

$$\hat{\mathcal{H}} = \bigcap_{j=0}^{+\infty} \mathbf{V}^j \mathcal{H}, \qquad \tilde{\mathcal{H}} = \bigoplus_{j=0}^{+\infty} \mathbf{V}^j \mathcal{L}^{\mathbf{V}}$$

and $\mathcal{L}^{\mathbf{V}}$, called the *wandering subspace for* $\mathbf{V}$, is defined as $\mathcal{L}^{\mathbf{V}} = \mathcal{H} \ominus \mathbf{V}\mathcal{H}$.

We apply the Abstract Wold Theorem to the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ jointly with the isometric operator $\mathbf{R}$. Hence, we consider the spaces

$$\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigcap_{j=0}^{+\infty} \mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon}), \qquad \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=0}^{+\infty} \mathbf{R}^j \mathcal{L}_t^{\mathbf{R}}$$

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and $\mathcal{L}_t^{\mathbf{R}} = \mathcal{H}_t(\boldsymbol{\varepsilon}) \ominus \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$. The theorem produces the orthogonal decomposition

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) \oplus \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigcap_{j=0}^{+\infty} \mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \bigoplus_{j=0}^{+\infty} \mathbf{R}^j \mathcal{L}_t^{\mathbf{R}}.$$

To fully characterize the spaces $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$ and $\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$, it is useful to define the *detail process at scale* 1, denoted by $\boldsymbol{\varepsilon^{(1)}} = \left\{\varepsilon_t^{(1)}\right\}_{t\in\mathbb{Z}}$, where

$$\varepsilon_t^{(1)} = \frac{\varepsilon_t - \varepsilon_{t-1}}{\sqrt{2}}, \qquad t \in \mathbb{Z}.$$

Each $\varepsilon_t^{(1)}$ has zero mean and unit variance by construction. Intuitively, the variables $\varepsilon_t^{(1)}$ mimic the differences typical of the Discrete Haar Transform (see, for instance, Addison (2002) as a reference). More generally, we provide the following definition of *detail process at scale* j.

Definition 1.1 Given a white noise process $\boldsymbol{\varepsilon}$, for any $j \in \mathbb{N}$, we call *detail process at scale* j the process $\boldsymbol{\varepsilon^{(j)}} = \left\{\varepsilon_t^{(j)}\right\}_{t\in\mathbb{Z}}$ where

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right). \tag{1.1}$$

For any given scale j, to avoid overlap among the variables $\varepsilon_t^{(j)}$ we consider the subseries of $\boldsymbol{\varepsilon^{(j)}}$ defined on the support $S_t^{(j)} = \{t - k2^j : \quad k \in \mathbb{Z}\}$.

It is interesting to observe that the variables $\varepsilon_{t-k2^j}^{(j)}$ exhibit a dual nature depending on the support on which they are considered. Indeed, the process $\boldsymbol{\varepsilon^{(j)}}$ is an $MA(2^j-1)$ with respect to the classical Wold innovations of $\mathbf{x}$, as stated in Definition 1.1. Therefore, some spurious correlation is present between variables $\varepsilon_{t-k2^j}^{(j)}$ and $\varepsilon_{\tau-k2^j}^{(j)}$ with $|t-\tau| \leqslant 2^j - 1$. Nevertheless, each subseries $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$ is a unit variance white noise on the support $S_t^{(j)}$, as shown by Proposition 1.11 in Appendix 1.B.

We focus now on the characterization of the subspace $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$. The first concern is to identify the image of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ through $\mathbf{R}$. Recalling the definition of the scaling operator, one readily sees that the subspace $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ consists of all linear combinations of the innovations ε_t with coefficients equal to each others 2-by-2, i.e.

$$\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) = \left\{ \sum_{k=0}^{+\infty} c_k^{(1)} \left(\varepsilon_{t-2k} + \varepsilon_{t-2k-1}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad c_k^{(1)} \in \mathbb{R} \right\}.$$

Iterating this logic with the operator $\mathbf{R}^j$, with $j \in \mathbb{N}$, we see that the subspaces $\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon})$ are made of linear combinations of the variables ε_t with coefficients equal to each others 2^j-by-2^j:

$$\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon}) = \left\{ \sum_{k=0}^{+\infty} c_k^{(j)} \left(\sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i} \right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad c_k^{(j)} \in \mathbb{R} \right\}.$$

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As a consequence, the intersection of all subspaces $\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon})$ contains only the zero element, that is $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$ is the null subspace: $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \{0\}$.

Now we turn to the subspace $\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$. The wandering subspace $\mathcal{L}_t^{\mathbf{R}}$ associated with the scaling operator is the orthogonal complement of $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ in $\mathcal{H}_t(\boldsymbol{\varepsilon})$. Since $\mathbf{R}$ is linear and bounded, the orthogonal complement of $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ coincides with the kernel of its adjoint operator, hence

$$\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(1)} \varepsilon_{t-2k}^{(1)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(1)} \in \mathbb{R} \right\}.$$

This result is justified by Lemma 1.7 and Proposition 1.12 in Appendix 1.A and 1.B respectively.

As a result, $\mathcal{L}_t^{\mathbf{R}}$ is the subspace of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ that contains all infinite moving averages driven by the detail process at scale 1 on the support $S_t^{(1)}$. More generally, for each $j \in \mathbb{N}$, the image of $\mathcal{L}_t^{\mathbf{R}}$ through the powers of the scaling operator $\mathbf{R}^{j-1}$ is the subspace

$$\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(j)} \in \mathbb{R} \right\},$$

as proved by Proposition 1.13 in Appendix 1.B. In sum, the subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$ consist of all infinite moving averages with innovations given by the detail process at scale j on the support $S_t^{(j)}$. The results of this section are summarized in the next theorem.

Theorem 1.1 *The Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ decomposes into the orthogonal sum*

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=1}^{+\infty} \mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}},$$

where

$$\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(j)} \in \mathbb{R} \right\}.$$

The proof of Theorem 1.1, as well as those of all following results, can be found in Appendix 1.B.

1.2.2 The Extended Wold Decomposition of x_t

Since $\mathbf{x}$ is a purely non-deterministic process, the Classical Wold Theorem guarantees that x_t belongs to $\mathcal{H}_t(\boldsymbol{\varepsilon})$. Therefore, the decomposition of the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ implies that there exists a sequence $\left\{g_t^{(j)}\right\}_{j\in\mathbb{N}}$ of random variables such that

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)}, \tag{1.2}$$

where each $g_t^{(j)}$ is the orthogonal projection of x_t on $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$.

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Definition 1.2 We call *persistent component at scale j* the orthogonal projection of x_t on the subspace $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$ of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ and we denote it by $g_t^{(j)}$.

By construction, given t, the persistent components $g_t^{(j)}$ are orthogonal to each others. Since each $g_t^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$,

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \tag{1.3}$$

for some square-summable sequence of real coefficients $\left\{\beta_k^{(j)}\right\}_k$. Each $\beta_k^{(j)}$ is the Fourier coefficient obtained by projecting x_t on the linear subspace generated by the detail $\varepsilon_{t-k2^j}^{(j)}$, i.e.

$$\beta_k^{(j)} = \mathbb{E}\left[x_t \varepsilon_{t-k2^j}^{(j)}\right].$$

Substituting the expression of $g_t^{(j)}$ into Eq. (1.2), we obtain the *Extended Wold Decomposition of* x_t.

Definition 1.3 We call *Extended Wold Decomposition of* x_t the decomposition

$$x_t = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)}.$$

Moreover, we call $\beta_k^{(j)}$ the *multiscale impulse response function* associated to the innovation at scale j and time translation $k2^j$.

It is important at this point to establish the relation between the coefficients $\beta_k^{(j)}$ of the Extended Wold Decomposition and the coefficients α_h of the Classical Wold Decomposition of x_t. Recall that the white noise $\boldsymbol{\varepsilon}$ is the process of classical Wold innovations of $\mathbf{x}$. Moreover, the Extended Wold Decomposition is based on the variables $\varepsilon_t^{(j)}$ that are finite linear combinations of the variables ε_t. Therefore, both decompositions actually exploit the same fundamental innovations. Since the Wold coefficients α_h are unique, given the process $\boldsymbol{\varepsilon}$, it follows that the coefficients $\beta_k^{(j)}$ are functions of the coefficients α_h. In addition, this connection ensures that the multiscale impulse response functions $\beta_k^{(j)}$ are independent of the time index t relative to the variable x_t that we decompose.

Proposition 1.2 *For any $j \in \mathbb{N}$, $k \in \mathbb{N}_0$,*

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right), \tag{1.4}$$

hence $\beta_k^{(j)}$ does not depend on t. In addition, $\lim_{k\to+\infty} \beta_k^{(j)} = 0$ for any $j \in \mathbb{N}$.

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Upon checking that the sequence $\left\{\beta_k^{(j)}\right\}_k$ is square-summable, the proof of Proposition 1.2 matches the coefficients of the Classical and the Extended Wold Decompositions of x_t, by exploiting the common underlying innovation process $\boldsymbol{\varepsilon}$. Since $\mathcal{H}_t(\boldsymbol{\varepsilon})$ decomposes into the direct sum $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \mathcal{L}_t^{\mathbf{R}}$, by iteratively applying the operator $\mathbf{R}$, it is readily shown that, at each scale $J \in \mathbb{N}$, the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ satisfies the orthogonal decomposition:

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}^J\mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \bigoplus_{j=1}^{J} \mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}.$$

We call *residual component* at scale j the orthogonal projection of x_t on the subspace $\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon})$ and we denote it by $\pi_t^{(j)}$. As shown in the proof of Proposition 1.2 in Appendix 1.B, this random variable has the following expression

$$\pi_t^{(j)} = \sum_{k=0}^{+\infty} \gamma_k^{(j)} \left(\frac{1}{\sqrt{2^j}} \sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i} \right),$$

where each coefficient $\gamma_k^{(j)}$ satisfies

$$\gamma_k^{(j)} = \frac{1}{\sqrt{2^j}} \sum_{i=0}^{2^j-1} \alpha_{k2^j+i}.$$

As a consequence, an Extended Wold Decomposition of x_t holds both in the finite case, i.e. when a maximum scale J is fixed, and in the infinite one:

$$x_t = \pi_t^{(J)} + \sum_{j=1}^{J} g_t^{(j)} \qquad \text{or} \qquad x_t = \sum_{j=1}^{+\infty} g_t^{(j)}.$$

The coefficients of $\pi_t^{(j)}$ and $g_t^{(j)}$ turn out to be the 2^j-by-2^j averages and differences that are generated by applying the Discrete Haar Transform to the sequence $\{\alpha_h\}_h$ of classical Wold coefficients of x_t. We summarize the results that we have discussed so far in the following theorem.

Theorem 1.3 Extended Wold Decomposition *Let* $\mathbf{x}$ *be a zero-mean, weakly stationary purely non-deterministic stochastic process. Then* x_t *decomposes as*

$$x_t = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

where the equality is in the L^2*-norm and*

i) for any fixed $j \in \mathbb{N}$*, the process* $\boldsymbol{\varepsilon^{(j)}} = \left\{\varepsilon_t^{(j)}\right\}_{t\in\mathbb{Z}}$ *is an* $MA(2^j-1)$ *with respect to the classical Wold innovations of* $\mathbf{x}$*:*

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right)$$

and $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$ *is a unit variance white noise;*

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ii) *for any* $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, *the coefficients* $\beta_k^{(j)}$ *are unique and they satisfy*

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right),$$

hence they do not depend on t *and* $\sum_{k=0}^{\infty} \left(\beta_k^{(j)}\right)^2 < +\infty$ *for any* $j \in \mathbb{N}$;

iii) *letting*

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

then, for any $j, l \in \mathbb{N}, p, q, t \in \mathbb{Z}$, $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right]$ *depends at most on* $j, l, p-q$. *Moreover,*

$$\mathbb{E}\left[g_{t-m2^j}^{(j)} g_{t-n2^l}^{(l)}\right] = 0 \qquad \forall j \neq l, \quad \forall m, n \in \mathbb{N}_0, \quad \forall t \in \mathbb{Z}.$$

Property *iii*) concerns the relation among the components $g_t^{(j)}$ at different layers of persistence. When t is fixed, the variables $g_t^{(j)}$ and $g_t^{(l)}$, with $j \neq l$, are orthogonal by construction. Besides, the orthogonality involves also shifted variables $g_{t-m2^j}^{(j)}$ and $g_{t-n2^l}^{(l)}$, for any $m, n \in \mathbb{Z}$, where the time translation is proportional to 2^j and 2^l respectively. In general, the covariance between $g_{t-p}^{(j)}$ and $g_{t-q}^{(l)}$ depends at most on the scales j, l and on the difference $p-q$.

The Extended Wold Decomposition allows us to decompose a zero-mean, purely non-deterministic process into the sum of uncorrelated components $g_t^{(j)}$ associated with the level of persistence j. The way in which persistence is captured by the scaling operator is described in Section 1.4 from a spectral point of view. Each variable $g_t^{(j)}$ has innovations on the grid $S_t^{(j)} = \left\{t - k2^j : \quad k \in \mathbb{Z}\right\}$, in which the time interval between two indices is a multiple of 2^j. For example, once a starting time t is set, $g_t^{(j)}$ may be the random variable that is affected by shocks that impinge an economy and have half-life in the interval $\left[2^{j-1}, 2^j\right)$. As j increases, the support $S_t^{(j)}$ becomes sparser and the degree of persistence of innovations rises. Thus, the multiscale impulse response function $\beta_k^{(j)}$ for large j quantify the propagation mechanism of low-frequency shocks.

Although the innovations of the components have support $S_t^{(j)}$, the variables $g_t^{(j)}$ are defined for every $t \in \mathbb{Z}$. In particular, if t and τ are any different time indices with $|t-\tau| \leqslant 2^j - 1$, the innovations of $g_t^{(j)}$ and $g_\tau^{(j)}$ belong to different grids, namely $S_t^{(j)}$ and $S_\tau^{(j)}$. Nevertheless, $g_t^{(j)}$ and $g_\tau^{(j)}$ share the same coefficients $\beta_k^{(j)}$, hence we are dealing with 2^j versions of the same process (in norm). Depending on the time index t we fix, we actually pick up one of these 2^j versions, namely the one with support $S_t^{(j)}$. Such structure is ultimately due to the weak stationarity of the process $\mathbf{x}$.

As a final remark, observe that it is possible to build an Extended Wold Decomposition based on time grids of N^j steps instead of 2^j (at scale j). This orthogonal decomposition exploits an

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isometric operator that generalizes $\mathbf{R}$ by averaging N subsequent innovations instead of just 2. Appendix 1.G provides a detailed derivation, which parallels the construction described in this section.

1.2.3 An example: the decomposition of an $AR(2)$ process

This section provides an example of how to compute the multiscale impulse response functions. The example shows that our multiscale analysis can potentially extract information about the dynamics of the process that would be otherwise difficult to detect using either the classical Wold coefficients or spectral analysis.

This example consists in deriving the Extended Wold Decomposition of the $AR(2)$ process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ defined by

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t,$$

where $\boldsymbol{\varepsilon} = \{\varepsilon\}_{t\in\mathbb{Z}}$ is a unit variance white noise and the parameters ϕ_1, ϕ_2 are chosen to guarantee the weak stationarity of $\mathbf{x}$. The moving average representation of x_t is (see Appendix 1.B)

$$x_t = \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-h} = \sum_{h=0}^{+\infty} \frac{\lambda_1^{h+1} - \lambda_2^{h+1}}{\lambda_1 - \lambda_2} \varepsilon_{t-h},$$

where

$$\lambda_1 = \frac{2\phi_2}{\sqrt{\phi_1^2 + 4\phi_2} - \phi_1}, \qquad \lambda_2 = \frac{-2\phi_2}{\sqrt{\phi_1^2 + 4\phi_2} + \phi_1}.$$

It follows from Proposition 1.2 that the coefficients of the Extended Wold Decomposition of x_t can be readily derived from the coefficients α_h. At scale $j \in \mathbb{N}$, for any $k \in \mathbb{N}_0$, the multiscale impulse response functions are then given by

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}(\lambda_1 - \lambda_2)} \left[\frac{\lambda_1^{k2^j+1}}{1-\lambda_1} \left(1 - \lambda_1^{2^{j-1}}\right)^2 - \frac{\lambda_2^{k2^j+1}}{1-\lambda_2} \left(1 - \lambda_2^{2^{j-1}}\right)^2 \right].$$

Figure 1.1 shows results for two alternative parametrization of the AR(2) process: $\phi_1 = 1.16$, $\phi_2 = -0.27$, and $\phi_1 = 1.3$, $\phi_2 = -0.41$. Both sets of parameters are such that the process is causal and with real roots λ_1 and λ_2. Panel 1.1(a) shows the classical impulse response function, Panel 1.1(b) shows the population spectral density, and Panel 1.1(c) and 1.1(d) report the multiscale impulse response functions for set 1 and set 2, respectively. Looking at Panel 1.1(a), it is obvious the two parameters' specifications generate processes that show momentum in the short run and some degree of mean reversion at longer horizons. The hump-shaped dynamics shown in Panel 1.1(a) is typical of many macroeconomic and financial time series as highlighted, e.g., in Fuster, Laibson, and Mendel (2010). The slower decaying autocorrelation for the first set of parameters

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suggests a higher persistence of the process. This intuition is confirmed in Panel 1.1(b) since the spectral density of the process with $\phi_1 = 1.16$, $\phi_2 = -0.27$ is more concentrated around frequency zero. To our eyes, this is all can be inferred by analyzing the classical Wold coefficients and the spectrum of these two processes. Our multiscale impulse response functions however provide new insights about their dynamics. In particular, looking at Panel 1.1(c) and 1.1(d), two facts emerge. First, the two processes display similar responses at scale 1 and 3 (and higher). Second and more important, the two processes display opposite behavior at scale 2: the impulse response functions in Panel 1.1(c) are positive, whereas the coefficients turn negative for several values of indices k in Panel 1.1(d).

To provide more intuition, suppose that x_t describes the dynamics of stock prices. Then our multiscale analysis suggests that although the stock price responses depicted in Figure 1.1(a) look on the surface similar, they are in fact generated by substantially different trading patterns. The first process is characterized by high-frequency (scale 1) "contrarian" trading strategy, and by market overreaction and momentum at scale 2 and 3, respectively. The second process, is characterized by "contrarian" trading strategy at scale 1 and 2, and by momentum at scale 3. Since it is known that contrarian trading strategies tend to provide liquidity to the marketplace (see e.g. Lo and Khandani, 2011), our analysis can be economically relevant because it characterizes the second process as supplying more liquidity than the first one.

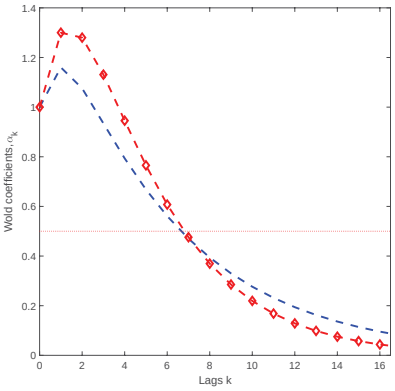
To conclude, our multiscale impulse response functions act as a microscope, one with many lenses and magnifications, that is suitable to examine the full range of financial-market activity and macroeconomic movements that are otherwise difficult to detect.

1.2.4 Multivariate Extended Wold Decomposition

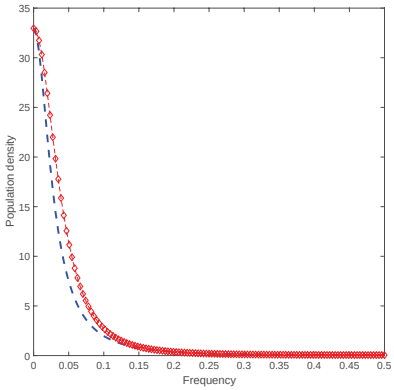
The theory discussed so far deals with weakly stationary univariate processes. However, the approach that exploits the Abstract Wold Theorem and a suitable isometric scaling operator to individuate persistent components of a given time series generalizes to multivariate processes. We address this issue in the paper by Cerreia-Vioglio, Ortu, Severino, and Tebaldi (2017b). We hereby summarize the main result of the work.

We consider an m-dimensional process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$, i.e. a vector of m time series, which is zero-mean and weakly stationary. According to the Multivariate Wold Decomposition (see Rozanov, 1967), any random vector x_t decomposes as $x_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k} + \nu_t$, where $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a unit variance m-dimensional white noise and ν_t is m-dimensional and predictable. The role of impulse response functions is played by the $m \times m$ matrices α_k such that $\alpha_k = \mathbb{E}\left[x_t \varepsilon'_{t-k}\right]$ and $\sum_{k=0}^{\infty} \mathrm{Tr}\left(\alpha_k \alpha'_k\right)$ is finite. Here $\mathrm{Tr}(M)$ denotes the trace of the $m \times m$ matrix M.

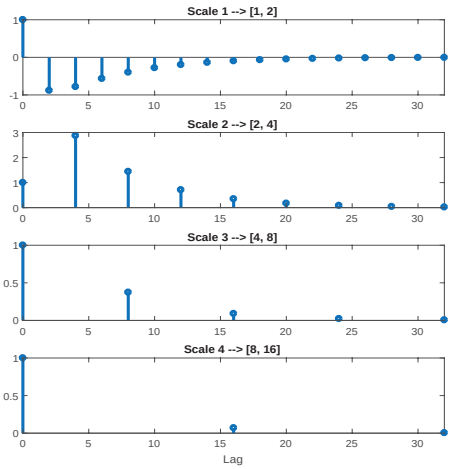
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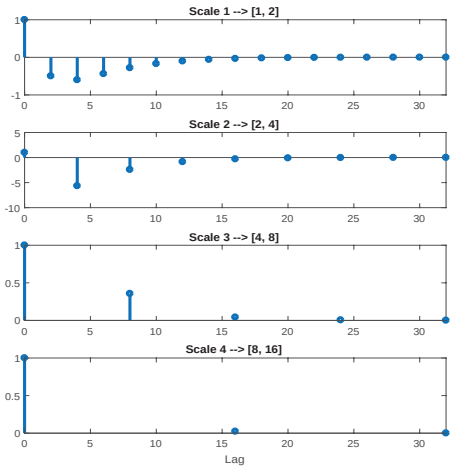
(a) Impulse response function



(b) Spectrum



(c) Multiscale impulse responses.



(d) Multiscale impulse responses.

Figure 1.1: Classical and multiscale impulse responses of two $AR(2)$ processes: parameters are $\phi_1 = 1.16$ and $\phi_2 = -0.27$ (blue dashed line in Panels A and B, and Panel C) and $\phi_1 = 1.3$ and $\phi_2 = -0.41$ (red line with diamonds in Panels A and B, and Panel D).

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We see that in multivariate modelling the field of scalars is replaced by the algebra A of squared matrices. Hence, we consider x_t as included in a Hilbert A-module instead of a Hilbert space (see, e.g., Cerreia-Vioglio, Maccheroni, and Marinacci (2017a) for the theory of Hilbert A-modules). In this setting two random vectors y and z are orthogonal when any entry y_i is uncorrelated with any entry z_j. The notion of isometry is defined, too. Moreover, we prove an Abstract Wold Theorem for Hilbert A-modules and, by defining a scaling operator analogous to $\mathbf{R}$, we obtain the following Multivariate Extended Wold Decomposition:

Theorem 1.4 *Let* $\mathbf{x}$ *be a zero-mean, weakly stationary purely non-deterministic* m*-dimensional process. Then* x_t *decomposes as*

$$x_t = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

where the equality is in norm and

i) for any fixed $j \in \mathbb{N}$*, the* m*-dimensional process* $\boldsymbol{\varepsilon^{(j)}} = \left\{\varepsilon_t^{(j)}\right\}_{t\in\mathbb{Z}}$ *is an* $MA(2^j-1)$ *with respect to the classical Wold innovations of* $\mathbf{x}$*:*

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right)$$

and $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$ *is a unit variance white noise;*

ii) for any $j \in \mathbb{N}$*,* $k \in \mathbb{N}_0$*, the* $m \times m$ *matrices* $\beta_k^{(j)}$ *are unique and they satisfy*

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right),$$

hence they do not depend on t *and* $\sum_{k=0}^{\infty} \operatorname{Tr}\left(\beta_k^{(j)} \beta_k^{(j)\prime}\right) < +\infty$ *for any* $j \in \mathbb{N}$*;*

iii) letting

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

then, for any $j, l \in \mathbb{N}, p, q, t \in \mathbb{Z}$*,* $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)\prime}\right]$ *depends at most on* $j, l, p-q$*. Moreover,*

$$\mathbb{E}\left[g_{t-m2^j}^{(j)} g_{t-n2^l}^{(l)\prime}\right] = \mathbf{0} \qquad \forall j \neq l, \quad \forall m, n \in \mathbb{N}_0, \quad \forall t \in \mathbb{Z}.$$

1.3 Reconstructing a time series from its components

This section investigates the Extended Wold Decomposition from the converse perspective: once the dynamics are given at various time scales, what can we say about the resulting process $\mathbf{x}$, built by summing up such components?

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1.3.1 The reconstruction theorems

We take for given the dynamics at different time scales $j \in \mathbb{N}$ and we want to recover the whole process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$. To make the sum of components feasible, we assume a common innovation process $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$. Indeed, MA processes may have multiple representations that involve different innovations. Considering MA processes driven by the same source of randomness has the advantage that the sum of such time series is still based on the same random shocks. Otherwise, we should question the existence of suitable innovations for the summed process, as observed in Ansley, Spivey, and Wrobleski (1977).

At each scale $j \in \mathbb{N}$, we define the detail process $\boldsymbol{\varepsilon^{(j)}} = \left\{\varepsilon_t^{(j)}\right\}_{t\in\mathbb{Z}}$ as an $MA(2^j-1)$ driven by the underlying innovations $\boldsymbol{\varepsilon}$. More precisely, we assume that

$$\varepsilon_t^{(j)} = \sum_{i=0}^{2^j-1} \delta_i^{(j)} \varepsilon_{t-i}$$

for some real coefficients $\delta_i^{(j)}$. Next, we consider the processes $\mathbf{g^{(j)}} = \left\{g_t^{(j)}\right\}_{t\in\mathbb{Z}}$ associated with level of persistence j. We assume that on the grids $S_t^{(j)}$ these processes are well-defined $MA(\infty)$ with respect to $\boldsymbol{\varepsilon^{(j)}}$. In particular, for any given j, we assume that there exists a sequence of real coefficients $\left\{\beta_k^{(j)}\right\}_k$ such that

1)
$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)};$$

2)
$$\sum_{j=1}^{+\infty}\sum_{h=0}^{+\infty}\left(\beta^{(j)}_{\left\lfloor \frac{h}{2^j}\right\rfloor}\delta^{(j)}_{h-2^j\left\lfloor \frac{h}{2^j}\right\rfloor}\right)^2 < +\infty;$$

3) for any $j,l \in \mathbb{N}, p,q,t \in \mathbb{Z}$, $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right]$ depends at most on $j, l, p-q$ and

$$\mathbb{E}\left[g_{t-m2^j}^{(j)} g_{t-n2^l}^{(l)}\right] = 0 \qquad \forall j \neq l, \quad \forall m,n \in \mathbb{N}_0, \quad \forall t \in \mathbb{Z}.$$

The first assumption guarantees that each $\mathbf{g^{(j)}}$ is a moving average with respect to the innovations $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$. While the variables $g_t^{(j)}$ are defined for every $t \in \mathbb{Z}$, given t the innovations of each $g_t^{(j)}$ have support $S_t^{(j)}$. Taken any different time indices t and τ with $|t-\tau| \leqslant 2^j - 1$, the innovations of $g_t^{(j)}$ and $g_\tau^{(j)}$ belong to the disjoint grids $S_t^{(j)}$ and $S_\tau^{(j)}$ respectively, although $g_t^{(j)}$ and $g_\tau^{(j)}$ share the same coefficients $\beta_k^{(j)}$. Assumption 2) is a square-summability requirement that ensures the well-definiteness of each component $\mathbf{g^{(j)}}$ and of the resulting process $\mathbf{x}$. Assumption 3) regards instead the interaction between components at different scales. For any fixed t, we require

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$g_t^{(j)}$ and $g_t^{(l)}$ to be orthogonal and we assume orthogonality also when such variables are shifted of $m2^j$ or $n2^l$ time units respectively. Moreover, the condition that $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right]$ depends just on the scales j, l and on the difference $p-q$ is crucial to ensure the weak stationarity of $\mathbf{x}$. Observe that Assumption 3) is satisfied in case $\delta_i^{(j)}$ are the Haar coefficients, that is

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \dots, 2^{j-1}-1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \dots, 2^j-1\}, \end{cases}$$

as a consequence of the orthogonal decomposition of the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ discussed in Section 1.2.

The following theorem states the properties of the process $\mathbf{x}$ obtained by summing the components $\mathbf{g^{(j)}}$ over all the scales $j \in \mathbb{N}$.

Theorem 1.5 *Let $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ be a unit variance white noise process. For any $j \in \mathbb{N}$, define the detail process $\boldsymbol{\varepsilon^{(j)}} = \left\{\varepsilon_t^{(j)}\right\}_{t\in\mathbb{Z}}$ as*

$$\varepsilon_t^{(j)} = \sum_{i=0}^{2^j-1} \delta_i^{(j)} \varepsilon_{t-i},$$

with $\delta_i^{(j)} \in \mathbb{R}$ for $i = 0, \dots, 2^j-1$, and consider a stochastic process $\mathbf{g^{(j)}} = \left\{g_t^{(j)}\right\}_{t\in\mathbb{Z}}$ such that there exists a sequence of real coefficients $\left\{\beta_k^{(j)}\right\}_k$ so that conditions 1), 2) and 3) are satisfied. Then, the process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ defined by

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)}$$

is zero-mean, weakly stationary purely non-deterministic and

$$x_t = \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-h},$$

where, for any $h \in \mathbb{N}_0$,

$$\alpha_h = \sum_{j=1}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)}.$$

The theorem provides the moving average representation, with respect to the innovations $\boldsymbol{\varepsilon}$, of the process $\mathbf{x}$ built via the aggregation of persistent components. In case the variables ε_t are the classical Wold innovations of $\mathbf{x}$, that is $\varepsilon_t = \left(x_t - \mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})} x_t\right) \left\| x_t - \mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})} x_t \right\|^{-1}$, Theorem 1.5 supplies the Classical Wold Decomposition of x_t, starting from its persistence-based representation. As a consequence, it constitutes a converse result of Theorem 1.3.

Collecting all processes $\mathbf{g^{(j)}}$ that satisfy the conditions of Theorem 1.5, we can define the set $\mathcal{C}_\delta$ of variables x_t that can be constructed by summing the components $g_t^{(j)}$:

$$\mathcal{C}_t^\delta = \left\{ x_t = \sum_{j=1}^{+\infty} g_t^{(j)} : \quad \mathbf{g^{(j)}} \quad \text{satisfies} \quad 1), 2), 3) \quad \forall j \in \mathbb{N} \right\}.$$

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$\mathcal{C}_t^\delta$ is a cone because, for any $x_t \in \mathcal{C}_t^\delta$ and any $\lambda \in \mathbb{R}$, the variable λx_t still belongs to $\mathcal{C}_t^\delta$. Since Theorem 1.5 guarantees that x_t is a linear combination of the innovations ε_t, it follows that $\mathcal{C}_t^\delta$ is included in the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$. It is interesting now to investigate the conditions under which the cone $\mathcal{C}_t^\delta$ coincides with the whole $\mathcal{H}_t(\boldsymbol{\varepsilon})$. This happens, for instance, when the coefficients $\delta_i^{(j)}$ in the moving average representation of the details $\boldsymbol{\varepsilon^{(j)}}$ are the Haar coefficients. Indeed, by Theorem 1.3, every variable $x_t \in \mathcal{H}_t(\boldsymbol{\varepsilon})$ has an Extended Wold Decomposition

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)} = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)}$$

with

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right)$$

and this implies that $x_t \in \mathcal{C}_t^\delta$. As a result, $\mathcal{C}_t^\delta = \mathcal{H}_t(\boldsymbol{\varepsilon})$ in this case.

On the converse, however, $\mathcal{C}_t^\delta = \mathcal{H}_t(\boldsymbol{\varepsilon})$ does not force the coefficients $\delta_i^{(j)}$ to coincide with the Haar ones. Indeed, the Haar coefficients embody several intrinsic features of the Extended Wold Decomposition. First, at any given scale j, they all have the same absolute value $1/\sqrt{2^j}$. Second, the sum of their squares at each scale is equal to 1. Finally, the coefficient $\delta_0^{(j)}$ is positive, while $\delta_{2^j-1}^{(j)}$ is negative. It is readily seen that one can modify any of these three properties and still satisfy conditions 1), 2) and 3), so to deduce other Wold-type decompositions. To obtain uniqueness in the Extended Wold Decomposition, in fact, one has to impose the following restrictions on the choice of the coefficients $\delta_i^{(j)}$:

a) $\left(\delta_0^{(1)}\right)^2 + \left(\delta_1^{(1)}\right)^2 = 1, \qquad \delta_1^{(1)} = -\delta_0^{(1)}, \qquad \delta_0^{(1)} > 0;$

b) for any $j \geqslant 2$ and any $k \in \{0, \ldots, 2^{j-1} - 1\}$,

$$\left(\delta_{2k}^{(j)}\right)^2 + \left(\delta_{2k+1}^{(j)}\right)^2 = \left(\delta_k^{(j-1)}\right)^2, \qquad \delta_0^{(j)} > 0, \qquad \delta_{2^j-1}^{(j)} < 0.$$

Conditions *a*) and *b*) can be interpreted as *mass* (or *energy*) *conservation* conditions that characterize the dichotomous structure of the coefficients $\delta_i^{(j)}$. These requirements ensure that the variance of each detail $\varepsilon_t^{(j)}$ is constantly equal to 1 at any scale. Moreover, weights are symmetric and there is no imbalance in the variance explained by the shifted details. Also, the choice of the signs at any scale makes the Haar coefficients arise naturally. We formalize this intuitive discussion in the form of

Theorem 1.6 *Under the assumptions of Theorem 1.5, the following statements are equivalent:*

i) $\mathcal{C}_t^\delta = \mathcal{H}_t(\boldsymbol{\varepsilon})$ *and the coefficients* $\delta_i^{(j)}$ *satisfy Conditions a*) *and b*)*;*

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ii) the coefficients $\delta_i^{(j)}$ *satisfy*

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \dots, 2^{j-1} - 1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \dots, 2^j - 1\}. \end{cases}$$

Under i) or ii),

$$\alpha_h = \sum_{j=1}^{+\infty} \frac{1}{\sqrt{2^j}} \beta^{(j)}_{\left\lfloor \frac{h}{2^j} \right\rfloor} \chi^{(j)}(h),$$

where

$$\chi^{(j)}(h) = \begin{cases} -1 & \text{if} \quad 2^j \left\lfloor \frac{h}{2^j} \right\rfloor \in \left\{ h - 2^j + 1, \dots, h - 2^{j-1} \right\}, \\ 1 & \text{if} \quad 2^j \left\lfloor \frac{h}{2^j} \right\rfloor \in \left\{ h - 2^{j-1} + 1, \dots, h \right\}. \end{cases}$$

1.3.2 An illustrative example of the reconstruction theorems

Relevant economic and financial quantities may often result from the aggregation of heterogeneous structural drivers evolving on different time scales. An example is provided by the heterogeneous autoregressive model of realized volatility of Corsi (2009). An interesting application of Theorem 1.5 consists in analyzing the properties of time series resulting from an aggregation of shocks with different frequency properties and evolving on different time grids. Assuming that the law of motion of each component is specified scale by scale, then the reconstruction theorem provides the scheme to determine the dynamics of the resulting aggregated stationary process as a function of the component dynamics. The simplest example of a multiscale construction of a stationary process is given by a two-scale process where a fast evolving autoregressive processes is modulated by injecting at a higher scale a second more persistent autoregressive process evolving on a coarser scale. We show by means of a numerical example that different choices of parameters have different impacts on the short and long lags of the ACF of the resulting process.

Given a common innovation process $\boldsymbol{\varepsilon} = \{\varepsilon\}_{t \in \mathbb{Z}}$, we let the coefficients $\delta_i^{(j)}$ in Theorem 1.5 be the Haar ones, i.e.

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \dots, 2^{j-1} - 1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \dots, 2^j - 1\}. \end{cases}$$

We start from two weakly stationary purely non-deterministic time series, $\mathbf{x} = \{x_t\}_{t \in \mathbb{Z}}$ and $\mathbf{y} = \{y_t\}_{t \in \mathbb{Z}}$, defined by two families of multiscale impulse responses: $\left\{\beta^{(j)}_{x,k}\right\}_{j,k}$ and $\left\{\beta^{(j)}_{y,k}\right\}_{j,k}$ with $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Theorem 1.6 ensures that the cone C_t^{δ} coincides with the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, hence $z_t = x_t + y_t$ belongs to $\mathcal{H}_t(\boldsymbol{\varepsilon})$.

We assume the multiscale impulse response functions of x_t to be

$$\beta^{(j)}_{x,k} = \frac{\rho_x^{k 2^j}}{\sqrt{2^j}} \frac{\left(1 - \rho_x^{2^{j-1}}\right)^2}{1 - \rho_x}, \qquad j \in \mathbb{N}, k \in \mathbb{N}_0,$$

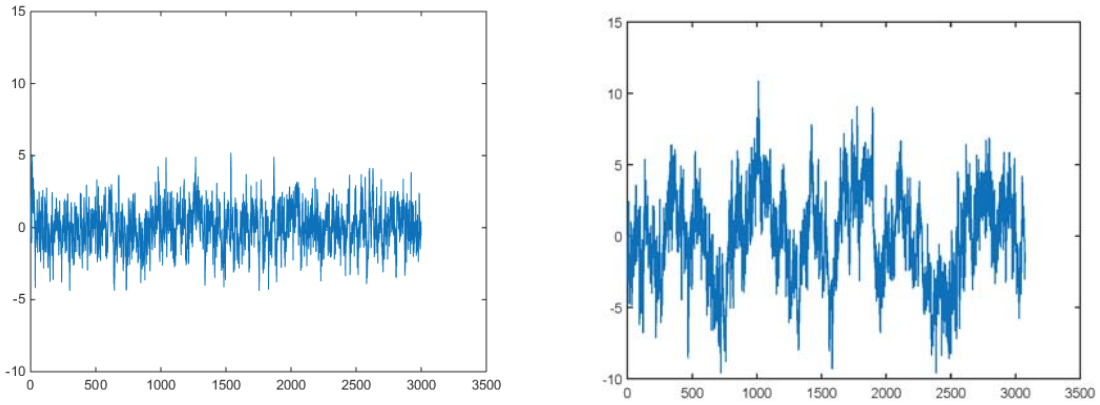
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i.e. the extended Wold coefficients of a weakly stationary $AR(1)$ process with parameter ρ_x, with $|\rho_x| < 1$.[8]

We now fix a scale J and we define the multiscale impulse responses of y_t by setting

$$\beta_{y,k}^{(j)} = \begin{cases} 0 & \text{if} \quad j \in \{0, \dots, J\}, \\ \frac{\rho_y^{k2^{j-J}}}{\sqrt{2^{j-J}}} \frac{\left(1-\rho_y^{2^{j-J-1}}\right)^2}{1-\rho_y} & \text{if} \quad j \geqslant J+1 \end{cases}$$

with $|\rho_y| < 1$. A simple comparison with $\beta_{x,k}^{(j)}$ shows that the coefficients $\beta_{y,k}^{(j)}$ identify an autoregressive process defined on the grid $S_t^{(J)}$.[9] In contrast with $\mathbf{x}$, which is a standard $AR(1)$ process, we call $\mathbf{y}$ an $AR(1)$ process *with horizon* 2^J. From the standpoint of interpretation we can think of $\mathbf{x}$ as a daily process, while $\mathbf{y}$ is a time series acting on longer lags (monthly, yearly...), depending on the choice of J.



(a) (b)

Figure 1.2: Simulated path for the process $\mathbf{x}$ with $\rho_x = 0.7$ and for the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = 0.9$, $J = 3$.

By construction, the multiscale impulse responses of $\mathbf{z} = \{z_t\}_{t\in\mathbb{Z}}$ are the sum of the extended Wold coefficients of $\mathbf{x}$ and $\mathbf{y}$, that is $\beta_{z,k}^{(j)} = \beta_{x,k}^{(j)} + \beta_{y,k}^{(j)}$ for any $j \in \mathbb{N}$, $k \in \mathbb{N}_0$. By Theorem 1.5 therefore, $\mathbf{z}$ is weakly stationary and purely non-deterministic.[10] In general, the sum of two

[8] See Appendix 1.D for the computation of the multiscale impulse responses of an $AR(1)$ process.

[9] In particular,

$$y_t = \sum_{k=0}^{+\infty} \rho_y^k \tilde{\varepsilon}_{t-k2^J}^{(J)}, \qquad \text{where} \qquad \tilde{\varepsilon}_t^{(J)} = \frac{1}{\sqrt{2^J}} \sum_{i=0}^{2^J-1} \varepsilon_{t-i}$$

denotes the unit variance white noise $\tilde{\boldsymbol{\varepsilon}}^{(\boldsymbol{J})} = \left\{\tilde{\varepsilon}_{t-k2^J}^{(J)}\right\}_{k\in\mathbb{Z}}$. Observe that the innovation process $\tilde{\boldsymbol{\varepsilon}}^{(\boldsymbol{J})}$ is different from the detail process $\boldsymbol{\varepsilon}^{(\boldsymbol{J})}$.

[10] All the assumptions of Theorem 1.5 are satisfied. Indeed, Assumption 1) holds by construction and

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autoregressive processes with different horizons is not stationary, but the structure of the shocks at different scales required by Theorem 3 ensures that this is the case. Hereafter we plot the ACF of $\mathbf{z}$ for some choices of the parameters ρ_x, ρ_y and of the scale J.

We start by setting $\rho_x = 0.7$, $\rho_y = 0.9$ and $J = 3$. Figure 1.2 compares a simulated path of the basic autoregressive process $\mathbf{x}$ with a path of the process $\mathbf{z}$ that is obtained after the addition of the persistent component. In the resulting process $\mathbf{z}$ the fast process corresponding to the process $\mathbf{x}$ mean reverts toward a trend which is itself a slowly mean reverting process, as determined by the low-frequency component. In Figure 1.3 we see that the ACF of $\mathbf{z}$ is piecewise approximated by the ACFs of $AR(1)$ processes with different parameters. Moreover, the persistence of $\mathbf{z}$ increases with the scale J, as we discuss in Appendix 1.E.

Alternative choices of ρ_y allow to modify the short lags of the ACF of $\mathbf{x}$, while keeping the long lags unchanged. For instance, this is possible by setting $\rho_y = \rho_x^{2^J}$, as we show in Appendix 1.E. Summing up, the scale-by-scale construction of $\mathbf{z}$ and the choices of the parameters ρ_x, ρ_y and of the scale J allow us to obtain ACFs with predetermined features. Further numerical examples are in Appendix 1.E.

1.4 Spectral analysis

Assume that the classical Wold coefficients of x_t satisfy $\sum_{h=0}^{\infty} |\alpha_h| < +\infty$, so that x_t has a well-defined spectral density function $f_x : (-\pi, \pi] \longrightarrow \mathbb{C}$, i.e.

$$f_x(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \gamma(n).$$

The scaling operator is an approximate low-pass filter and it is this feature that makes it valuable to capture persistence. Specifically, the spectral density functions of $\mathbf{R}x_t$ is

$$f_{\mathbf{R}}(\lambda) = 2\cos^2\left(\frac{\lambda}{2}\right) f_x(2\lambda),$$

as computed in Proposition 1.15 in Appendix 1.B. Hence, $f_{\mathbf{R}}$ associates the largest weight with frequencies around zero, while frequencies near to $|\pi|$ are negligible. An example of the impact of $\mathbf{R}$ on spectral density functions is described by Figure 1.4, where x_t is supposed to be a stationary $AR(2)$ process.

---

assumption 3) follows from the use of the Haar coefficients. As for 2), the square-summability of the $\beta_{z,k}^{(j)}$ is ensured by the square-summability of the $\beta_{x,k}^{(j)}$ and the $\beta_{y,k}^{(j)}$, that come from the well-defined Extended Wold Decomposition of the processes $\mathbf{x}$ and $\mathbf{y}$ respectively.

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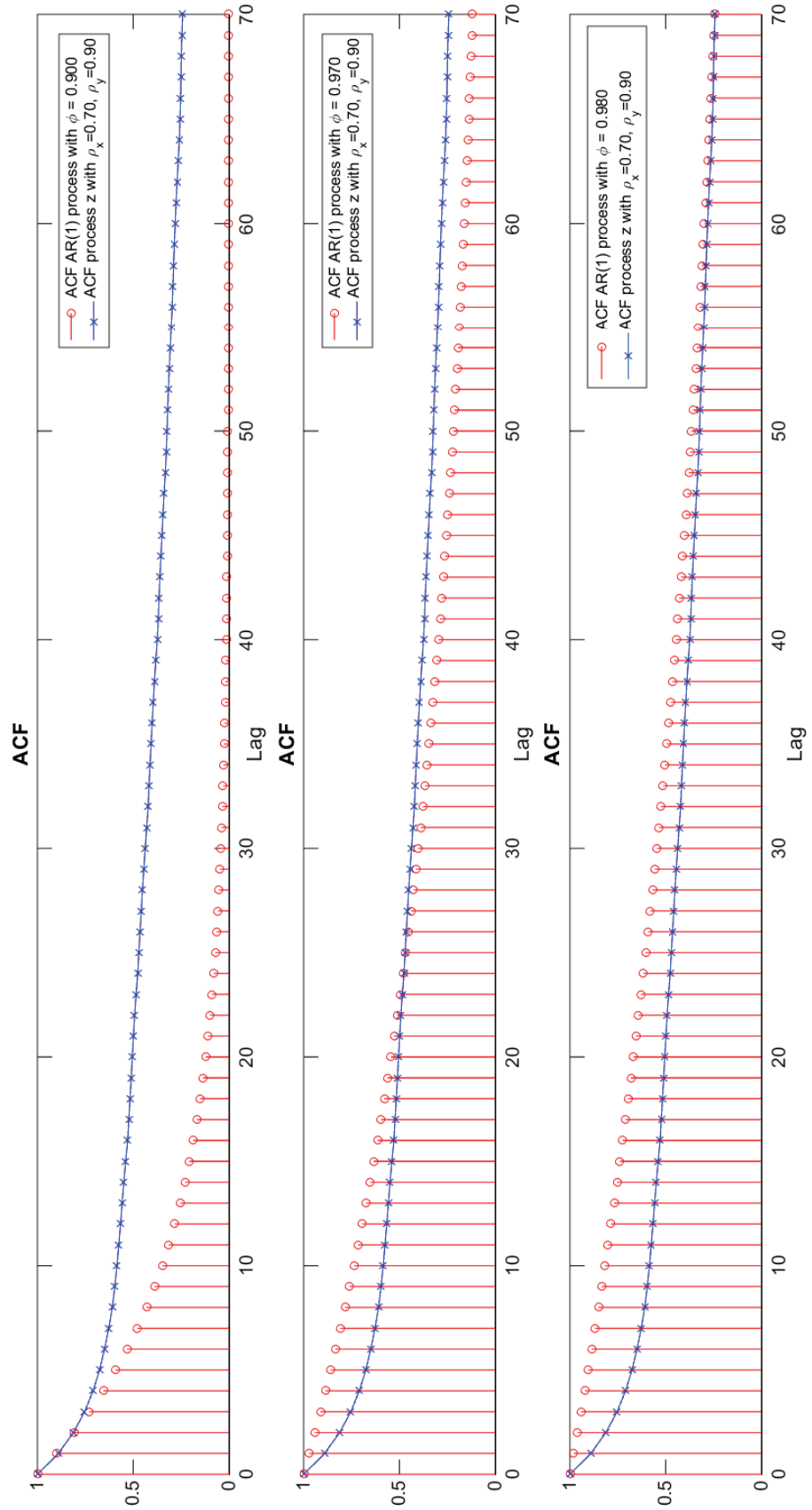


Figure 1.3: Comparison between the ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = 0.9$, $J = 3$ and the ACF of an $AR(1)$ process with parameter $\phi = 0.9, 0.97, 0.98$.

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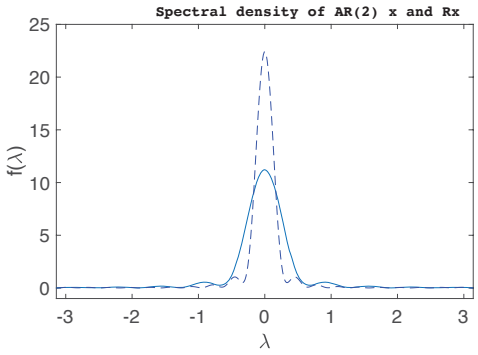


Figure 1.4: Comparison between the spectral density functions of x_t (solid line) and $\mathbf{R}x_t$ (dashed line) when x_t is an $AR(2)$ process with $\phi_1 = 1.16$ and $\phi_2 = -0.24$.

1.5 Empirical Analysis: Realized Volatility

A natural variable to be analyzed through the lens of our Extended Wold Decomposition is financial volatility. Perhaps the most intriguing feature revealed by empirical work on volatility is its long persistence. Among many alternative explanations for the source of such persistence, Andersen and Bollerslev (1997) argue that the long-memory features of volatility may arise naturally through the interaction of a large number of information processes, each one characterized by its own degree of persistence. Similarly, the Heterogeneous Market Hypothesis of Müller, Dacorogna, Davé, Olsen, Pictet, and von Weizsäcker (1997) suggests that the the long-memory feature of volatility can be the result of an additive cascade of different volatility components each of which generated by the actions of different (in terms of investment horizon) types of market participants. Inspired by the Heterogeneous Market Hypothesis, Corsi (2009) proposed a Heterogeneous Autoregressive model (HAR) of Realized Volatility. In this section we use the insight of our Extended Wold Decomposition to revisit this successful (for forecasting) empirical model and to shed new light on the role of persistent components in volatility of asset prices.

1.5.1 Econometric framework and data

Let's denote with $\mathbf{d} = \{d_t\}_t$ the daily Realized Volatility obtained by integrating high-frequency returns. For the specific application at hand, we consider the time series of daily USD/CHF exchange rate Realized Volatility built by Corsi (2009).[11,12] Starting from tick-by-tick series of

[11] We thank Fulvio Corsi for kindly providing us with his data.

[12] One may wonder whether the results that follow are induced by the conditional heteroskedasticity present in the series of Realized Volatility. Since the log Realized Volatility is approximately homoskedastic, in Appendix 1.F.2 we repeat the analysis replacing the Realized Volatility with its logarithmic trans-

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USD/CHF exchange rate from December 1989 to December 2003, d_t is constructed by computing spot logarithmic middle prices (averages of log bid and ask quotes) and the related returns as in Andersen, Bollerslev, Diebold, and Labys (2003):

$$d_t = \sqrt{\sum_{j=0}^{M-1} r_{t-j/M}^2}, \qquad r_{t-j/M} = p(t - j/M) - p(t - (j+1)/M),$$

where $M = 12$ refers to time intervals of two hours in a 24 hours trading day. Figures 1.F.1 and 1.F.2 in Appendix 1.F.1 display the realizations of d_t and its sample autocorrelation function (ACF), characterized by a slow decay.

Besides the daily volatility, the HAR model entertains weekly and monthly Realized Volatility, denoted by $\mathbf{w} = \{w_t\}_t$ and $\mathbf{m} = \{m_t\}_t$ respectively. Specifically, w_t and m_t are obtained by averaging subsequent past realizations of d_t:

$$w_t = \frac{1}{5}\sum_{h=0}^{4} d_{t-h}, \qquad m_t = \frac{1}{22}\sum_{h=0}^{21} d_{t-h}.$$

The HAR model reads as follows:

$$d_{t+1} = a_0 + a_d d_t + a_w w_t + a_m m_t + \nu_t. \tag{1.5}$$

Our empirical analysis proceed in three steps. First, we obtain the components of daily, weekly and monthly Realized Volatility. Second, we measure the relative importance of each persistence level in daily, weekly, and monthly Realized Volatility. This allows us to investigate whether w_t and m_t really capture weekly and monthly effects of d_t or, instead, they conceal long-term behaviors. Also, this analysis sheds light on the Heterogeneous Market Hypotheses and, in particular, on the most relevant investment horizons of different market participants.

Finally we introduce an econometric model that exploits three persistent components of d_t capturing long cycles between 128, 256 and 512 days and we show that this model performs as well as Eq. (1.5). Thus, our analysis shows that HAR derives its performance from exploiting the long-term dynamics of daily Realized Volatility rather then being truly based on short-term, namely weekly and monthly, components. This interpretation is consistent with the possibly infinite length of agents' time horizons. Throughout the analysis we devote particular attention to illustrate how one can practically implement our Extended Wold Decomposition.

1.5.2 Estimation methodology of components

We assume that $\mathbf{d}$ is a weakly stationary time series such that

$$d_t - \mu_d = \Phi(\mathbf{L})\varepsilon_t = \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-h}, \qquad \sum_{h=0}^{+\infty} \alpha_h^2 < +\infty,$$

formation, and we reach qualitatively and quantitatively similar conclusions.

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where μ_d is the first moment of d_t.

We assume that the operator $\Phi(\mathbf{L})$ is invertible and we estimate an autoregressive form for $\mathbf{d}$ of order N determined by the AIC or the BIC. The BIC delivers $N = 25$, which is in line with the $AR(22)$ specification of Eq. (1.5). Specifically, we implement the linear regression

$$d_t = \sum_{k=1}^{N} b_k d_{t-k} + \eta_t.$$

and we define the unit variance white noise $\varepsilon_t = \eta_t \mathbb{E}\left[\eta_t^2\right]^{-1/2}$. The estimates of b_k are provided by OLS coefficients, while we use the variance of the regression residuals to proxy for $\mathbb{E}\left[\eta_t^2\right]$. Since

$$d_t - \mu_d = \eta_t + \sum_{k=1}^{N} b_k \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-k-h} = \mathbb{E}\left[\eta_t^2\right]^{1/2} \varepsilon_t + \sum_{n=1}^{+\infty} \left(\sum_{h=\max\{n-N,0\}}^{n-1} \alpha_h b_{n-h} \right) \varepsilon_{t-n},$$

from the b_k-s we obtain the classical Wold coefficients α_h as

$$\alpha_0 = \mathbb{E}\left[\left(d_t - \sum_{k=1}^{N} b_k d_{t-k} \right)^2 \right]^{1/2}, \qquad \alpha_n = \sum_{h=\max\{n-N,0\}}^{n-1} \alpha_h b_{n-h} \quad \forall n \in \mathbb{N}.$$

Finally, we compute multiscale IRFs $\beta_k^{(j)}$ as prescribed by Eq. (1.4) and we use the residuals of the autoregressive regression to compute the innovations at each scale, as in Eq. (1.1). Finally, the component $d_t^{(j)}$ at level of persistence j is obtained from Eq. (1.3).

The very same procedure detailed for d_t is also applied to w_t and m_t to obtain their respective Extended Wold Decompositions.

1.5.3 Variance decomposition

To distinguish between the relative importance of, for instance, short-run from long-run components, we compute the variance explained by components $d_t^{(j)}$ at different scales j. Indeed, the orthogonality of persistent components is the key property that allows to decompose the total variance across persistence levels:

$$var(d_t) = \sum_{j=1}^{+\infty} var^{(j)}(d_t),$$

where we define the variance associated with scale j as

$$var^{(j)}(d_t) = var\left(d_t^{(j)}\right) = \sum_{k=0}^{+\infty} \left(\beta_k^{(j)}\right)^2.$$

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Thus, the Extended Wold Decomposition provides also a persistence-based variance decomposition of any weakly stationary process, thanks to the uncorrelation of components ensured by the Abstract Wold Theorem. This approach shares the same purposes of Dynamic Principal Components Analysis in (multivariate) factor models. A practical advantage of our construction is that it is entirely set in the time domain. Hence, it does not force us to work with frequencies in order to quantify persistence.

1.5.4 Results

Since the sample of daily Realized Volatility contains 3599 data, we estimate persistent components up to scale 10. Indeed, a shock on the eleventh scale involves 2048 past realizations, a fact that limits the estimation precision. However, this truncation is not restrictive because the first ten components of d_t explain roughly 94% of total sample variance. Figure 1.F.3 in Appendix 1.F.1 displays the first ten components of daily Realized Volatility estimated according to the procedure described in Subsection 1.5.2. Figure 1.5 shows instead the variance decomposition (across scales) of daily, weekly and monthly Realized Volatility.

The leftmost panel of Figure 1.5 provides evidence of the Heterogeneous Market Hypothesis and agrees with long memory of Realized Volatility. Indeed, most of the variance is explained by scales 7, 8 and 9, which involve shocks that last 128, 256 and 512 working days. This result is in line with the econometric literature and provides a quantitative insight into the relative importance of heterogeneous driving factors of volatility. While our analysis is consistent with the overlap of market actors with different horizons, our results show that most of the action is indeed in the long-run.

The importance of high scales is even more pronounced in the variance decomposition of w_t and m_t, see middle and rightmost panels of Figure 1.5. Compared to $var\,(d_t)$ in the first plot, we observe a shift in explained variance towards scales 7 and 8, while lower persistence levels are negligible. The sensitivity to shocks lasting 4 and 16 working days (captured by scales 2 and 4) is negligible with respect to semi-annual (128 days) or yearly (256 days) innovations (scales 7 and 8). This result is important since it shows that, differently from the HAR's intuition, w_t and m_t cannot be associated with weekly and monthly disturbances. Indeed, they are mostly driven by low-frequency factors.

The left panel in Figure 1.6 substantiates further this last statement by comparing w_t and m_t with the orthogonal components $d_t^{(2)}$ and $d_t^{(4)}$. These components approximately capture the reaction to weekly and monthly shocks, respectively. However the figure shows that the ACFs of w_t and m_t manifest long-term persistence and that this behavior is better captured by the components

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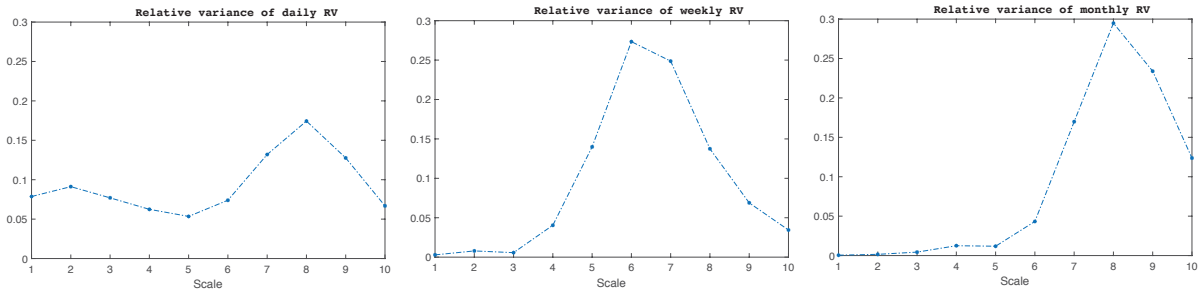


Figure 1.5: Variance ratio explained by each scale for daily, weekly and monthly Realized Volatility.

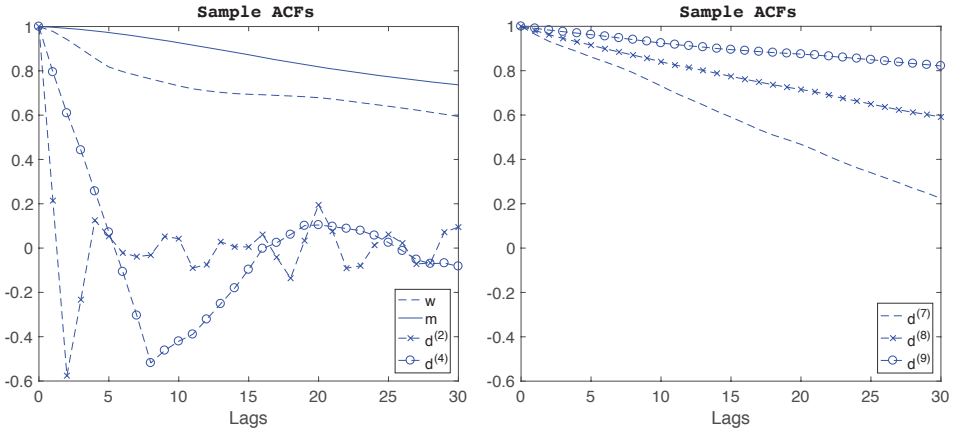


Figure 1.6: Left panel: comparison between sample ACFs of weekly and monthly Realized Volatility (dashed and solid line respectively) with ACFs of persistent components of daily Realized Volatility at scales 2 and 4 (lines marked with x and circles respectively). Right panel: sample ACFs of persistent components of daily Realized Volatility at scales 7 (dashed line), 8 (line marked with x) and 9 (line marked with circles).

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$d_t^{(7)}$, $d_t^{(8)}$ and $d_t^{(9)}$ (see rightmost panel). This result suggests once again that w_t and m_t, together with d_t, provide good forecasts of the volatility process through Eq. (1.5) because they properly capture low-frequency behavior of Realized Volatility rather than weekly and monthly dynamics.

Based on the analysis so far, we now establish whether Corsi's forecasting model is actually based on long-term features of the process. To this end we introduce a forecasting method based on the components of d_t that explain most of its unconditional variance: $d_t^{(7)}$, $d_t^{(8)}$ and $d_t^{(9)}$ (see again the first graph of Figure 1.5). We estimate the one-step ahead conditional expectation of these components as described in Appendix 1.C and we implement the linear regression:

$$d_{t+1} = a^{(0)} + a^{(7)}\mathbb{E}_t\left[d_{t+1}^{(7)}\right] + a^{(8)}\mathbb{E}_t\left[d_{t+1}^{(8)}\right] + a^{(9)}\mathbb{E}_t\left[d_{t+1}^{(9)}\right] + \xi_t. \tag{1.6}$$

Interestingly, whereas w_t and m_t display a certain amount of correlation, our Extended Wold Decomposition provides an orthogonalization of possibly correlated regressors, that generally improves statistical inference. In addition, as the number of components used as regressors grows large, the coefficients $a^{(j)}$ should equal 1 in order to retrieve the Extended Wold Decomposition of d_{t+1} inside the conditional expectation.

Figure 1.7 shows the relation between in-sample forecasting performance and the number of persistent components added according to their explained variance. After including the three scales $7, 8$ and 9, we do not see a meaningful improvement in forecasting ability.

Table 1.1 quantitatively compares performances of Corsi's HAR model and our persistence-based forecasting with components at scale $7, 8$ and 9. We also provide as a reference a model where we use the maximum number of available components. Not surprisingly, the forecasting equation with ten components as regressors outperforms HAR model. Our persistence-based forecasting model, instead, performs as well as the HAR model, while keeping the same parsimony of parameters. Persistent components at scales $7, 8$ and 9, which constitute the main drivers of daily Realized Volatility, provide the same forecasting power as daily, weekly and monthly averages. Hence, the forecasting ability of the HAR model, see Eq. (1.5), really depends on high scales; these persistent components are the fundamental drivers of the correlated variables w_t and m_t. As for out-of-sample forecasting, our three-component model outperforms the HAR specification just when the prediction horizon is high.

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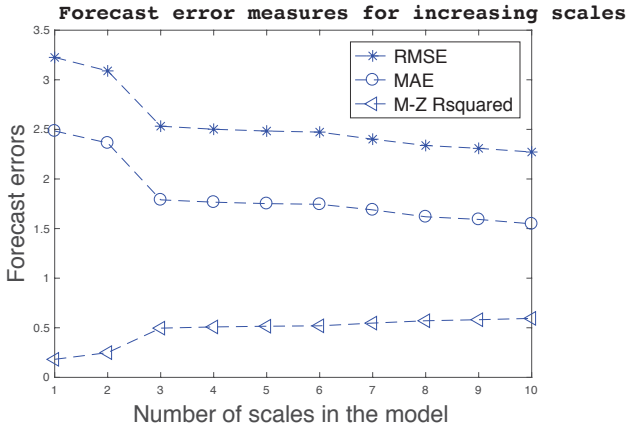


Figure 1.7: Forecasting performance of persistence-based models with increasing number of components as regressors. Persistent components are added according to their explained variance. RMSE, MAE and R^2 denote Root Mean Square Error, Mean Absolute Error and R^2 of Mincer-Zarnowitz regression.

Table 1.1: Forecasting performance of HAR model and persistence-based models with 3 and 10 component. RMSE, MAE and R^2 denote Root Mean Square Error, Mean Absolute Error and R^2 of Mincer-Zarnowitz regression.

	RMSE	MAE	R^2
HAR	2.607	1.757	0.565
Extended Wold (3)	2.532	1.790	0.496
Extended Wold (10)	2.272	1.547	0.597

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1.6 Conclusions

We have provided an orthogonal decomposition of x_t into uncorrelated components $g_t^{(j)}$ that are associated with different layers of persistence. This decomposition, which we dub *Extended Wold Decomposition*, results from the application of the Abstract Wold Theorem on the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, spanned by the classical Wold innovations of $\mathbf{x}$, where the scaling operator $\mathbf{R}$ is isometric. With our decomposition at hand, we introduce the concept of multiscale impulse response functions. We show that these impulse response functions are able to capture the effects of shocks with specific persistence. Finally, we show how to reconstruct a time series from its dynamics on various time scales.

From a technical perspective, our Wold-type decomposition of weakly stationary stochastic processes is the outcome of a multiresolution analysis of an abstract Hilbert space and it is connected to the work by Baggett, Larsen, Packer, Raeburn, and Ramsay (2010), where the authors study the wavelet decompositions of Hilbert spaces. Their construction and our approach rely on the isometry associated with the operator: this is key to permit the use of the Abstract Wold Theorem (see Nagy, Foias, Bercovici, and Kérchy (2010)).

Our work is also related to the literature on low-frequency econometrics. Our component $g_t^{(j)}$ at level of persistence j admits a classical Wold decomposition in terms of innovations that are localized on a time grid whose unit is 2^j units of the original observation grid. The coarsening of the grid where low frequency shocks are located is reminiscent of the frequency domain procedure adopted by Müller and Watson (2008) and Müller and Watson (2015) where long-run sample information is isolated using a small number of low-frequency trigonometric weighted averages which, in turn, can be used to conduct inference about long-run variability and covariability. Their approach to inference can be applied to the present decomposition by replacing the original trigonometric weighting scheme with an Haar basis weighting scheme. We view this as an interesting avenue for future research.

In all, we see our Extended Wold Decomposition as a useful tool for the analysis of time series that are commonly affected by shocks with different frequencies, such as macroeconomic and financial ones. Whereas traditional impulse response functions, based on the Classical Wold Theorem, fail to capture the sensitivity of macroeconomic variables to shocks with heterogeneous persistence, our multiscale impulse response functions can separate and zoom-in the different fluctuations which may co-exist in the original time series.

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Appendix

This Appendix includes five sections. Section 1.A provides a focused survey of the Hilbert space theory required for the decomposition of stationary time series. Section 1.B includes all the proofs. Section 1.C illustrates some formulas for forecasting based on the Extended Wold Decomposition. Section 1.D contains further simulations about ARMA processes. Section 1.E provides some complementary simulations related to the examples discussed in Subsection 1.3.2. Section 1.F contains additional figures about Realized Volatility analysis, both in level and in logs. Finally, Section 1.G describes the Extended Wold Decomposition in base N.

1.A Abstract Wold Theorem and Classical Wold Decomposition

Let $\mathcal{H}$ be a Hilbert space and $\mathbf{V} : \mathcal{H} \longrightarrow \mathcal{H}$ a linear operator.

Definition 1.4 Let $\mathbf{V}$ be bounded. The *adjoint*, or *transposed*, *of* $\mathbf{V}$ is the bounded linear operator $\mathbf{V}^* : \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies the relation

$$\langle \mathbf{V}x, y\rangle = \langle x, \mathbf{V}^* y\rangle \qquad \forall x, y \in \mathcal{H}.$$

In particular, $\|\mathbf{V}\| = \|\mathbf{V}^*\|$.

Lemma 1.7 *If* $\mathbf{V}$ *is bounded,* $\ker(\mathbf{V}^*) = \mathcal{H} \ominus \mathbf{V}\mathcal{H}$.

Proof See Luenberger (1968). ■

Definition 1.5 $\mathbf{V}$ is an *isometry* when

$$\langle \mathbf{V}x, \mathbf{V}y\rangle = \langle x, y\rangle \qquad \forall x, y \in \mathcal{H}$$

or, equivalently, $\mathbf{V}^*\mathbf{V} = I$, where I denotes the *identity map* on $\mathcal{H}$.

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If $\mathbf{V}$ is an isometric operator, its norm is equal to 1 and its powers $\mathbf{V}^j$, with $j \in \mathbb{N}$, are isometries, too. Isometries allow us to decompose orthogonally Hilbert spaces. The brick in the decomposition is the so-called *wandering subspace*, which is also termed *detail subspace* or *innovation subspace*.

Definition 1.6 Let $\mathbf{V}$ be an isometry. We call *wandering subspace for* $\mathbf{V}$ a subspace $\mathcal{L}^{\mathbf{V}}$ of $\mathcal{H}$ such that $\mathbf{V}^h\mathcal{L}^{\mathbf{V}}$ and $\mathbf{V}^k\mathcal{L}^{\mathbf{V}}$ are orthogonal for every $h, k \in \mathbb{N}_0$, with $h \neq k$. Since $\mathbf{V}$ is an isometry, we can just require that $\mathbf{V}^n\mathcal{L}^{\mathbf{V}}$ is orthogonal to $\mathcal{L}^{\mathbf{V}}$ for every $n \in \mathbb{N}$.

Given a wandering subspace $\mathcal{L}^{\mathbf{V}}$, it is possible to define the orthogonal sum $\bigoplus_{j=0}^{+\infty} \mathbf{V}^j\mathcal{L}^{\mathbf{V}}$, in which the convergence of the infinite direct sum is in the norm induced in the Hilbert space $\mathcal{H}$ by the inner product. It may happen that such an orthogonal sum covers the whole Hilbert space $\mathcal{H}$.

Definition 1.7 Let $\mathbf{V}$ be an isometry. We say that $\mathbf{V}$ is a *unilateral shift* when there exists a subspace $\mathcal{L}^{\mathbf{V}}$ of $\mathcal{H}$ which is wandering for $\mathbf{V}$ and such that

$$\bigoplus_{j=0}^{+\infty} \mathbf{V}^j\mathcal{L}^{\mathbf{V}} = \mathcal{H}.$$

Given a unilateral shift, the wandering subspace is uniquely determined by the relation $\mathcal{L}^{\mathbf{V}} = \mathcal{H} \ominus \mathbf{V}\mathcal{H}$. We name Abstract Wold Theorem the following decomposition of Hilbert spaces.

Theorem 1.8 Abstract Wold Decomposition *Let $\mathcal{H}$ be a Hilbert space and $\mathbf{V} : \mathcal{H} \longrightarrow \mathcal{H}$ be an isometry. Then, $\mathcal{H}$ decomposes uniquely into an orthogonal sum*

$$\mathcal{H} = \hat{\mathcal{H}} \oplus \tilde{\mathcal{H}},$$

such that $\mathbf{V}\hat{\mathcal{H}} = \hat{\mathcal{H}}$, $\mathbf{V}\tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}$ and the restriction of $\mathbf{V}$ on $\tilde{\mathcal{H}}$ is a unilateral shift. In particular,

$$\hat{\mathcal{H}} = \bigcap_{j=0}^{+\infty} \mathbf{V}^j\mathcal{H}, \qquad \tilde{\mathcal{H}} = \bigoplus_{j=0}^{+\infty} \mathbf{V}^j\mathcal{L}^{\mathbf{V}},$$

with $\mathcal{L}^{\mathbf{V}} = \mathcal{H} \ominus \mathbf{V}\mathcal{H}$.

Proof See Nagy, Foias, Bercovici, and Kérchy (2010). ■

We now turn to time series and we consider a stochastic process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ in which any variable x_t is measurable with respect to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that each x_t belongs to the vector space $L^2(\Omega, \mathcal{F}, \mathbb{P})$, in which two random variables are identified when they coincide almost surely.

Definition 1.8 The process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ is *weakly stationary* when, for any t,

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i) $\mathbb{E}\left[x_t^2\right]$ is finite,

ii) $\mathbb{E}\left[x_t\right]$ does not depend on t,

iii) the cross moment $\mathbb{E}\left[x_{t-h}x_{t-k}\right]$ depends at most on the difference $h-k$, for any $h,k\in\mathbb{Z}$.

With no loss of generality we assume that $\mathbf{x}$ has zero mean. Under stationarity, the *autocovariance function* $\gamma:\mathbb{Z}\longrightarrow\mathbb{R}$ is well-defined:

$$\gamma(n)=\mathrm{Cov}\left(x_t,x_{t-n}\right)=\mathbb{E}\left[x_tx_{t-n}\right],\qquad n\in\mathbb{Z}.$$

If $\gamma(n)=0$ for all $n\neq 0$ we are facing a *white noise* process.

$L^2(\Omega,\mathcal{F},\mathbb{P})$ is a Hilbert space when it is equipped with the inner product:

$$\langle A,B\rangle=\mathbb{E}\left[AB\right]\qquad\forall A,B\in L^2(\Omega,\mathcal{F},\mathbb{P}).$$

We are interested in the closed subspace $\mathcal{H}_t(\mathbf{x})$ of $L^2(\Omega,\mathcal{F},\mathbb{P})$ spanned by the sequence $\{x_{t-k}\}_{k\in\mathbb{N}_0}$, which is equivalently defined by

$$\mathcal{H}_t(\mathbf{x})=\mathrm{cl}\left\{\sum_{k=0}^{+\infty}a_kx_{t-k}:\quad\sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty}a_ka_h\gamma(k-h)<+\infty\right\},$$

where *cl* denotes the closure in the L^2-norm. $\mathcal{H}_t(\mathbf{x})$ is a Hilbert subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$ as discussed in Bierens (2012). We call *generators* of $\mathcal{H}_t(\mathbf{x})$ the random variables $\sum_{k=0}^{\infty}a_kx_{t-k}$ that satisfy the square-summability requirement

$$\left\|\sum_{k=0}^{+\infty}a_kx_{t-k}\right\|^2=\mathbb{E}\left[\left(\sum_{k=0}^{+\infty}a_kx_{t-k}\right)^2\right]=\sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty}a_ka_h\gamma(k-h)<+\infty.$$

Definition 1.9 Given a weakly stationary stochastic process $\mathbf{x}$, let $\mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})}x_t$ denote the orthogonal projection of the random variable x_t on $\mathcal{H}_{t-1}(\mathbf{x})$. We say that $\mathbf{x}$ is *regular* when $\left\|x_t-\mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})}x_t\right\|>0$ for all $t\in\mathbb{Z}$.

We now state the Classical Wold Decomposition Theorem for zero-mean, regular, weakly stationary time series.

Theorem 1.9 Classical Wold Decomposition *Let $\mathbf{x}=\{x_t\}_{t\in\mathbb{Z}}$ be a zero-mean, regular, weakly stationary stochastic process. Then, for any $t\in\mathbb{Z}$, x_t decomposes as*

$$x_t=\sum_{k=0}^{+\infty}\alpha_k\varepsilon_{t-k}+\nu_t,$$

where the equality is in the L^2-norm and

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i) $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ *is a unit variance white noise process;*

ii) *for any* $k \in \mathbb{N}_0$, *the coefficients* α_k *do not depend on* t,

$$\alpha_k = \mathbb{E}\left[x_t \varepsilon_{t-k}\right] \qquad \textit{and} \qquad \sum_{k=0}^{+\infty} \alpha_k^2 < +\infty;$$

iii) $\boldsymbol{\nu} = \{\nu_t\}_{t\in\mathbb{Z}}$ *is a zero-mean weakly stationary process,*

$$\nu_t \in \bigcap_{j=0}^{+\infty} \mathcal{H}_{t-j}(\mathbf{x}) \qquad \textit{and} \qquad \mathbb{E}\left[\nu_t \varepsilon_{t-k}\right] = 0 \quad \forall k \in \mathbb{N}_0;$$

iv)

$$\nu_t \in \operatorname{cl}\left\{ \sum_{h=1}^{+\infty} a_h \nu_{t-h} \quad \in \bigcap_{j=1}^{+\infty} \mathcal{H}_{t-j}(\mathbf{x}) : \quad a_h \in \mathbb{R} \right\}.$$

Proof A traditional proof is written in Brockwell and Davis (2009), while in Severino (2016) the result is proved by applying the Abstract Wold Theorem. ■

For any integer t, $\varepsilon_t = \left(x_t - \mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})} x_t\right) \left\| x_t - \mathcal{P}_{\mathcal{H}_{t-1}(\mathbf{x})} x_t \right\|^{-1}$ and $\boldsymbol{\varepsilon}$ is the process of *fundamental innovations* of $\mathbf{x}$. The term $\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k}$ constitutes the *non-deterministic component*, while $\boldsymbol{\nu}$ is the *deterministic component* of $\mathbf{x}$ and property *iv)* is known as *predictability*. We say that a weakly stationary time series is *purely non-deterministic* when the deterministic component is equal to zero. Similarly, we call *purely deterministic* those weakly stationary stochastic processes with null non-deterministic component.

1.B Proofs

Proposition 1.10 *Let* $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ *be a white noise process. The operator* $\mathbf{R} : \mathcal{H}_t(\boldsymbol{\varepsilon}) \longrightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ *is well-defined, linear and isometric.*

Proof Without loss of generality, assume that the variance of the white-noise process $\boldsymbol{\varepsilon}$ is equal to 1. In order to show that the scaling operator $\mathbf{R}$ is well-defined on $\mathcal{H}_t(\boldsymbol{\varepsilon})$, consider any element $X = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ belonging to $\mathcal{H}_t(\boldsymbol{\varepsilon})$, that is $\|X\|^2 = \sum_{p=0}^{\infty} a_p^2 < +\infty$. Our purpose is to prove that

$$\mathbf{R}X = \sum_{k=0}^{+\infty} \frac{a_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \varepsilon_{t-k}$$

belongs to $\mathcal{H}_t(\boldsymbol{\varepsilon})$ too, namely the series

$$\|\mathbf{R}X\|^2 = \frac{1}{2} \sum_{k=0}^{+\infty} a_{\lfloor \frac{k}{2} \rfloor}^2$$

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is convergent. Since index k is either even or odd, the sum actually is

$$\|\mathbf{R}X\|^2 = \frac{1}{2}\sum_{k=0}^{+\infty} a^2_{\lfloor \frac{k}{2} \rfloor} = \frac{1}{2}\sum_{p=0}^{+\infty} a^2_{\lfloor \frac{2p}{2} \rfloor} + \frac{1}{2}\sum_{p=0}^{+\infty} a^2_{\lfloor \frac{2p+1}{2} \rfloor} = \sum_{p=0}^{+\infty} a_p^2 = \|X\|^2$$

and this quantity is finite. As a result, $\mathbf{R}$ is well-defined. Note that $\mathbf{R}$ is also a bounded operator because $\|\mathbf{R}X\| = \|X\|$.

The linearity of the scaling operator is immediate.

We are left to prove that $\mathbf{R}$ is isometric on $\mathcal{H}_t(\boldsymbol{\varepsilon})$. Consider any two elements $X = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$, $Y = \sum_{h=0}^{\infty} b_h \varepsilon_{t-h}$ in $\mathcal{H}_t(\boldsymbol{\varepsilon})$. By exploiting the white-noise properties of $\boldsymbol{\varepsilon}$, we find that

$$\begin{aligned}\langle \mathbf{R}X, \mathbf{R}Y \rangle &= \sum_{k=0}^{+\infty} \frac{a_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \frac{b_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} = \frac{1}{2}\sum_{p=0}^{+\infty} a_{\lfloor \frac{2p}{2} \rfloor} b_{\lfloor \frac{2p}{2} \rfloor} + \frac{1}{2}\sum_{p=0}^{+\infty} a_{\lfloor \frac{2p+1}{2} \rfloor} b_{\lfloor \frac{2p+1}{2} \rfloor} \\ &= \sum_{p=0}^{+\infty} a_p b_p = \langle X, Y \rangle.\end{aligned}$$

Hence, $\mathbf{R}$ is an isometry. ■

Proposition 1.11 *For any fixed $j \in \mathbb{N}$, the process $\left\{\varepsilon^{(j)}_{t-k2^j}\right\}_{k\in\mathbb{Z}}$ is a unit variance white noise.*

Proof First of all, we show that $\left\{\varepsilon^{(j)}_{t-k2^j}\right\}_{k\in\mathbb{Z}}$ is weakly stationary.

i) Variables ε_t are the classical Wold innovations of the process $\mathbf{x}$, hence $\mathbb{E}\left[\varepsilon_{t-p}\varepsilon_{t-q}\right] = 0$ for all $p \neq q$ and $\mathbb{E}\left[\varepsilon_t^2\right] = 1$ for any time index t. Consequently, for any $k \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{E}\left[\left(\varepsilon^{(j)}_{t-k2^j}\right)^2\right] &= \frac{1}{2^j}\mathbb{E}\left[\left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-k2^j-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-k2^j-2^{j-1}-i}\right)^2\right] \\ &= \frac{1}{2^j}\sum_{i=0}^{2^j-1} \mathbb{E}\left[\varepsilon_t^2\right] = \frac{2^j}{2^j}\mathbb{E}\left[\varepsilon_t^2\right] = 1.\end{aligned}$$

Thus, $\mathbb{E}\left[\left(\varepsilon^{(j)}_{t-k2^j}\right)^2\right]$ is finite and it does not depend on k.

ii) Since $\mathbb{E}\left[\varepsilon_t\right] = 0$ for any t, we find that $\mathbb{E}\left[\varepsilon^{(j)}_{t-k2^j}\right] = 0$ for any $k \in \mathbb{Z}$ and so the expectation does not depend on k.

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iii) Let us analyse cross-moments in the support $S_t^{(j)}$. By taking $h \neq k$,

$$\mathbb{E}\left[\varepsilon_{t-h2^j}^{(j)}\varepsilon_{t-k2^j}^{(j)}\right] = \frac{1}{2^j}\mathbb{E}\left[\left(\sum_{i=0}^{2^{j-1}-1}\varepsilon_{t-h2^j-i} - \sum_{i=0}^{2^{j-1}-1}\varepsilon_{t-h2^j-2^{j-1}-i}\right)\right.$$
$$\left.\cdot\left(\sum_{l=0}^{2^{j-1}-1}\varepsilon_{t-k2^j-l} - \sum_{l=0}^{2^{j-1}-1}\varepsilon_{t-k2^j-2^{j-1}-l}\right)\right]$$
$$= \frac{1}{2^j}\left\{\sum_{i=0}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1}\mathbb{E}\left[\varepsilon_{t-h2^j-i}\varepsilon_{t-k2^j-l}\right] - \sum_{i=0}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1}\mathbb{E}\left[\varepsilon_{t-h2^j-i}\varepsilon_{t-k2^j-2^{j-1}-l}\right]\right.$$
$$\left. - \sum_{i=0}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1}\mathbb{E}\left[\varepsilon_{t-h2^j-2^{j-1}-i}\varepsilon_{t-k2^j-l}\right] + \sum_{i=0}^{2^{j-1}-1}\sum_{l=0}^{2^{j-1}-1}\mathbb{E}\left[\varepsilon_{t-h2^j-2^{j-1}-i}\varepsilon_{t-k2^j-2^{j-1}-l}\right]\right\}.$$

Since $h \neq k$, the sets of indices $\{h2^j, \ldots, h2^j + 2^j - 1\}$ and $\{k2^j, \ldots, k2^j + 2^j - 1\}$ are disjoint and so the last sums are null. As a result,

$$\mathbb{E}\left[\varepsilon_{t-h2^j}^{(j)}\varepsilon_{t-k2^j}^{(j)}\right] = 0 \quad \forall h \neq k.$$

To recap, $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$ turns out to be weakly stationary on its support $S_t^{(j)}$. In particular, it is a white noise process with variance equal to 1. ■

Proposition 1.12 *The adjoint of* $\mathbf{R}$ *is the operator* $\mathbf{R}^* : \mathcal{H}_t(\boldsymbol{\varepsilon}) \longrightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ *such that*

$$\mathbf{R}^* : \quad \sum_{k=0}^{+\infty} a_k\varepsilon_{t-k} \quad \longmapsto \quad \sum_{k=0}^{+\infty}\frac{a_{2k}+a_{2k+1}}{\sqrt{2}}\varepsilon_{t-k}.$$

In addition, the kernel of $\mathbf{R}^*$ *is*

$$\ker(\mathbf{R}^*) = \left\{\sum_{k=0}^{+\infty} b_k^{(1)}\varepsilon_{t-2k}^{(1)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(1)} \in \mathbb{R}\right\}.$$

Proof Firstly, note that $\mathbf{R}^*$ is well-defined. Indeed, taken any element $Y = \sum_{k=0}^{\infty} a_k\varepsilon_{t-k}$ in $\mathcal{H}_t(\boldsymbol{\varepsilon})$, that is $\|Y\|^2 = \sum_{h=0}^{\infty} a_h^2 < +\infty$, we have that

$$\|\mathbf{R}^*Y\|^2 = \sum_{k=0}^{+\infty}\frac{(a_{2k}+a_{2k+1})^2}{2} \leqslant \sum_{k=0}^{+\infty}\left(a_{2k}^2 + a_{2k+1}^2\right) = \sum_{h=0}^{+\infty} a_h^2 = \|Y\|^2 < +\infty.$$

Secondly, we check the validity of the relation $\langle \mathbf{R}X, Y\rangle = \langle X, \mathbf{R}^*Y\rangle$ for any variables belonging to $\mathcal{H}_t(\boldsymbol{\varepsilon})$ as, for instance, $X = \sum_{h=0}^{\infty} b_h\varepsilon_{t-h}$ and $Y = \sum_{k=0}^{\infty} a_k\varepsilon_{t-k}$. By using the white noise properties, we find that

$$\langle \mathbf{R}X, Y\rangle = \sum_{h=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{b_{\lfloor\frac{h}{2}\rfloor}}{\sqrt{2}}a_k\langle\varepsilon_{t-h},\varepsilon_{t-k}\rangle = \sum_{k=0}^{+\infty} b_{\lfloor\frac{k}{2}\rfloor}\frac{a_k}{\sqrt{2}} = \sum_{k=0}^{+\infty} b_k\frac{a_{2k}+a_{2k+1}}{\sqrt{2}}$$
$$= \sum_{h=0}^{+\infty}\sum_{k=0}^{+\infty} b_h\frac{a_{2k}+a_{2k+1}}{\sqrt{2}}\langle\varepsilon_{t-h},\varepsilon_{t-k}\rangle = \langle X, \mathbf{R}^*Y\rangle.$$

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Hence, $\mathbf{R}^*$ is the adjoint of the scaling operator.

As for the kernel of $\mathbf{R}^*$, we prove that

$$\ker(\mathbf{R}^*) = \left\{ \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad d_k^{(1)} \in \mathbb{R} \right\}.$$

Take any element of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ of the kind

$$X = \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right)$$

for some square-summable sequence of real numbers $\left\{d_k^{(1)}\right\}_k$. Such X can be rewritten as $X = \sum_{h=0}^{\infty} a_h \varepsilon_{t-h}$ with $a_{2k+1} = -a_{2k}$ for every $k \in \mathbb{N}_0$, that is $a_{2k} + a_{2k+1} = 0$. Hence, by the characterization of $\mathbf{R}^*$ we realize immediately that $\mathbf{R}^* X = 0$. Thus,

$$\left\{ \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad d_k^{(1)} \in \mathbb{R} \right\} \subset \ker(\mathbf{R}^*).$$

Conversely, consider an element $X = \sum_{h=0}^{\infty} a_h \varepsilon_{t-h}$ belonging to the kernel of $\mathbf{R}^*$. Since $\mathbf{R}^* X = 0$ in the L^2-norm, we have that

$$\sum_{k=0}^{+\infty} \frac{(a_{2k} + a_{2k+1})^2}{2} = 0.$$

Therefore, $a_{2k+1} = -a_{2k}$ for any $k \in \mathbb{N}_0$. This means that it is possible to write

$$X = \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right),$$

with $d_k^{(1)} = a_{2k}$. As a result,

$$\ker(\mathbf{R}^*) \subset \left\{ \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad d_k^{(1)} \in \mathbb{R} \right\}.$$

■

Proposition 1.13 *For any $j \in \mathbb{N}$,*

$$\mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(j)} \in \mathbb{R} \right\}.$$

Proof For any $j \in \mathbb{N}$, the operator $\mathbf{R}^j$ acts on the elements of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ as

$$\mathbf{R}^j : \quad \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \quad \longmapsto \quad \sum_{k=0}^{+\infty} \frac{a_k}{\sqrt{2^j}} \left(\sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i} \right).$$

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As the general case follows by induction, we prove that

$$\mathbf{R}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(2)} \varepsilon_{t-4k}^{(2)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(2)} \in \mathbb{R} \right\}.$$

In particular, we show that

$$\mathbf{R}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} d_k^{(2)} \left(\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad d_k^{(2)} \in \mathbb{R} \right\}.$$

Consider any element $Y \in \mathbf{R}\mathcal{L}_t^{\mathbf{R}}$. As Y is the image of some element $X \in \mathcal{L}_t^{\mathbf{R}}$, there exists a square-summable sequence of real numbers $\left\{d_k^{(1)}\right\}_k$ such that

$$X = \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right)$$

and

$$Y = \sum_{k=0}^{+\infty} \frac{d_k^{(1)}}{\sqrt{2}} \left(\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}\right).$$

As a result,

$$\mathbf{R}\mathcal{L}_t^{\mathbf{R}} \subset \left\{ \sum_{k=0}^{+\infty} d_k^{(2)} \left(\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}\right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad d_k^{(2)} \in \mathbb{R} \right\}.$$

Vice versa, consider any element Y of the kind

$$Y = \sum_{k=0}^{+\infty} d_k^{(2)} \left(\varepsilon_{t-4k} + \varepsilon_{t-4k-1} - \varepsilon_{t-4k-2} - \varepsilon_{t-4k-3}\right)$$

for some square-summable sequence of coefficients $\left\{d_k^{(2)}\right\}_k$. Then Y belongs to $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ too, because it is the image of the element

$$X = \sum_{k=0}^{+\infty} \sqrt{2} d_k^{(2)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right) \quad \in \mathcal{L}_t^{\mathbf{R}}.$$

■

Proof of Theorem 1.1

Proof On the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, the operator $\mathbf{R}$ is isometric by Proposition 1.10. Thus, we apply the Abstract Wold Decomposition, that is Theorem 1.8, which supplies the orthogonal decomposition $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) \oplus \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$, where

$$\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigcap_{j=0}^{+\infty} \mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon}),$$

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$$\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=0}^{+\infty} \mathbf{R}^j \mathcal{L}_t^{\mathbf{R}} \qquad \text{with} \qquad \mathcal{L}_t^{\mathbf{R}} = \mathcal{H}_t(\boldsymbol{\varepsilon}) \ominus \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}).$$

Firstly, we show that $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$ is the null subspace. Indeed, as remarked in Subsection 1.2.1, the subspaces $\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon})$ are made of linear combinations of innovations with coefficients equal to each others 2^j-by-2^j:

$$\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon}) = \left\{ \sum_{k=0}^{+\infty} c_k^{(j)} \left(\sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i} \right) \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad c_k^{(j)} \in \mathbb{R} \right\}.$$

As a result, $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon})$, being the intersection of all $\mathbf{R}^j\mathcal{H}_t(\boldsymbol{\varepsilon})$, can just include variables as $\sum_{h=0}^{\infty} c\varepsilon_{t-h}$ with $c \in \mathbb{R}$. Such elements must belong to $\mathcal{H}_t(\varepsilon)$, hence $\sum_{k=0}^{\infty} c^2$ is finite. Anyway, this is possible just in case $c = 0$. As a result, the one admissible element in $\hat{\mathcal{H}}_t(\varepsilon)$ is zero, that is $\hat{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \{0\}$. Hence, the orthogonal decomposition of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ simplifies to $\mathcal{H}_t(\boldsymbol{\varepsilon}) = \tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$.

Finally, let us focus on the subspace $\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon})$. By Theorem 1.8, we know that

$$\tilde{\mathcal{H}}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=0}^{+\infty} \mathbf{R}^j \mathcal{L}_t^{\mathbf{R}} = \bigoplus_{j=1}^{+\infty} \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}},$$

where $\mathcal{L}_t^{\mathbf{R}}$ is the wandering subspace. As the orthogonal complement of $\mathbf{R}\mathcal{H}_t(\mathbf{x})$ is the kernel of the adjoint operator $\mathbf{R}^*$ (see Lemma 1.7), by Proposition 1.12 the wandering subspace turns out to be

$$\mathcal{L}_t^{\mathbf{R}} = \mathcal{H}_t(\boldsymbol{\varepsilon}) \ominus \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) = \ker(\mathbf{R}^*) = \left\{ \sum_{k=0}^{+\infty} b_k^{(1)} \varepsilon_{t-2k}^{(1)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(1)} \in \mathbb{R} \right\}.$$

In addition, by Proposition 1.13, for any $j \in \mathbb{N}$,

$$\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(j)} \in \mathbb{R} \right\}$$

and so the whole decomposition is proved. ■

Proof of Proposition 1.2

Proof To begin with, we show that, for any fixed scale $j \in \mathbb{N}$,

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)}$$

belongs to $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$. Actually, we just have to prove that the series is convergent. By making variables $\varepsilon_{t-k2^j}^{(j)}$ explicit with respect to the classical Wold innovations of x_t, we derive that

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \left(\sum_{l=0}^{2^{j-1}-1} \frac{\beta_k^{(j)}}{\sqrt{2^j}} \varepsilon_{t-k2^j-l} - \sum_{l=0}^{2^{j-1}-1} \frac{\beta_k^{(j)}}{\sqrt{2^j}} \varepsilon_{t-k2^j-2^{j-1}-l} \right).$$

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For any $h \in \mathbb{N}_0$, we uniquely find $k \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, 2^j - 1\}$ such that $h = k2^j + l$. Consequently, we can express the component $g_t^{(j)}$ as

$$g_t^{(j)} = \sum_{h=0}^{+\infty} \eta_h^{(j)} \varepsilon_{t-h},$$

where coefficients $\eta_h^{(j)}$ are defined by

$$\eta_h^{(j)} = \begin{cases} \frac{\beta_k^{(j)}}{\sqrt{2^j}} & \text{for } k \in \mathbb{N}_0, \quad l \in \{0, \dots, 2^{j-1} - 1\} \\ -\frac{\beta_k^{(j)}}{\sqrt{2^j}} & \text{for } k \in \mathbb{N}_0, \quad l \in \{2^{j-1}, \dots, 2^j - 1\}. \end{cases}$$

Therefore, we check the convergence of the series $\sum_{h=0}^{\infty} \left(\eta_h^{(j)}\right)^2$. By using the inequality

$$\left(\sum_{i=0}^{n} a_i\right)^2 \leqslant (n+1) \sum_{i=0}^{n} a_i^2, \qquad n \in \mathbb{N},$$

it is easy to show that

$$\sum_{h=0}^{+\infty} \left(\eta_h^{(j)}\right)^2 = \sum_{k=0}^{+\infty} \left(\beta_k^{(j)}\right)^2 \leqslant \sum_{h=0}^{+\infty} a_h^2,$$

which is finite because x_t belongs to $\mathcal{H}_t(\boldsymbol{\varepsilon})$. As a result $g_t^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$. Furthermore, we actually showed that coefficients $\beta_k^{(j)}$ are square-summable. Inter alia, this ensures that, for any fixed scale $j \in \mathbb{N}$, $\lim_{k\to\infty} \beta_k^{(j)} = 0$.

In order to find the exact expression of coefficients $\beta_k^{(j)}$, we exploit the orthogonal decompositions of the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$ at different scales $J \in \mathbb{N}$:

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}^J \mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \bigoplus_{j=1}^{J} \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}}.$$

We call $\pi_t^{(j)}$ the orthogonal projection of x_t on the subspace $\mathbf{R}^j \mathcal{H}_t(\boldsymbol{\varepsilon})$ and we proceed inductively.

Let us start by the first decomposition $x_t = \pi_t^{(1)} + g_t^{(1)}$ coming from scale $J = 1$:

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \mathcal{L}_t^{\mathbf{R}}.$$

By using the definitions of elements belonging to subspaces $\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon})$ and $\mathcal{L}_t^{\mathbf{R}}$, as remarked in Subsection 1.2.1 and claimed in Proposition 1.12 respectively, we set

$$\pi_t^{(1)} = \sum_{k=0}^{+\infty} \gamma_k^{(1)} \frac{\varepsilon_{t-2k} + \varepsilon_{t-(2k+1)}}{\sqrt{2}} = \sum_{k=0}^{+\infty} c_k^{(1)} \left(\varepsilon_{t-2k} + \varepsilon_{t-(2k+1)}\right),$$

$$g_t^{(1)} = \sum_{k=0}^{+\infty} \beta_k^{(1)} \varepsilon_{t-2k}^{(1)} = \sum_{k=0}^{+\infty} d_k^{(1)} \left(\varepsilon_{t-2k} - \varepsilon_{t-2k-1}\right)$$

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for some sequences of coefficients $\left\{c_k^{(1)}\right\}_k$ and $\left\{d_k^{(1)}\right\}_k$, or equivalently $\left\{\gamma_k^{(1)}\right\}_k$ and $\left\{\beta_k^{(1)}\right\}_k$, to determine in order to have $x_t = \pi_t^{(1)} + g_t^{(1)}$, where we set $\sqrt{2}c_k^{(1)} = \gamma_k^{(1)}$ and $\sqrt{2}d_k^{(1)} = \beta_k^{(1)}$. The expressions above may be rewritten as

$$x_t = \sum_{k=0}^{+\infty} \left\{ \left(c_k^{(1)} + d_k^{(1)}\right) \varepsilon_{t-2k} + \left(c_k^{(1)} - d_k^{(1)}\right) \varepsilon_{t-2k-1} \right\}.$$

However, from the Classical Wold Decomposition of $\mathbf{x}$, we know that

$$x_t = \sum_{k=0}^{+\infty} \left\{ \alpha_{2k}\varepsilon_{t-2k} + \alpha_{2k+1}\varepsilon_{t-2k-1} \right\},$$

where we use the same fundamental innovations as before. By exploiting the uniqueness of writing deriving from the Classical Wold Decomposition, the two expressions for x_t must coincide. As a result, $c_k^{(1)}$ and $d_k^{(1)}$ are the solutions of the linear system

$$\begin{cases} c_k^{(1)} + d_k^{(1)} = \alpha_{2k} \\ c_k^{(1)} - d_k^{(1)} = \alpha_{2k+1}, \end{cases}$$

that is,

$$c_k^{(1)} = \frac{\alpha_{2k} + \alpha_{2k+1}}{2}, \qquad d_k^{(1)} = \frac{\alpha_{2k} - \alpha_{2k+1}}{2}.$$

In particular, we find that

$$\gamma_k^{(1)} = \frac{\alpha_{2k} + \alpha_{2k+1}}{\sqrt{2}}, \qquad \beta_k^{(1)} = \frac{\alpha_{2k} - \alpha_{2k+1}}{\sqrt{2}}.$$

Hence,

$$\pi_t^{(1)} = \sum_{k=0}^{+\infty} \frac{\alpha_{2k} + \alpha_{2k+1}}{\sqrt{2}} \frac{\varepsilon_{t-2k} + \varepsilon_{t-2k-1}}{\sqrt{2}}, \qquad g_t^{(1)} = \sum_{k=0}^{+\infty} \frac{\alpha_{2k} - \alpha_{2k+1}}{\sqrt{2}} \varepsilon_{t-2k}^{(1)}.$$

Now, focus on the scale $J = 2$. We exploit the decomposition of the space

$$\mathbf{R}\mathcal{H}_t(\boldsymbol{\varepsilon}) = \mathbf{R}^2\mathcal{H}_t(\boldsymbol{\varepsilon}) \oplus \mathbf{R}\mathcal{L}_t^{\mathbf{R}},$$

that implies the relation $\pi_t^{(1)} = \pi_t^{(2)} + g_t^{(2)}$. We follow the same track as in the previous case, by using the features of the coefficients of elements in $\mathbf{R}^2\mathcal{H}_t(\boldsymbol{\varepsilon})$ and in $\mathbf{R}\mathcal{L}_t^{\mathbf{R}}$ and, finally, by comparing the expression of $\pi_t^{(2)} + g_t^{(2)}$ with the (unique) writing of $\pi_t^{(1)}$ that we found before. By solving a simple linear system we discover that

$$c_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} + \alpha_{4k+2} + \alpha_{4k+3}}{4}, \quad d_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} - \alpha_{4k+2} - \alpha_{4k+3}}{4}$$

and, in particular,

$$\gamma_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} + \alpha_{4k+2} + \alpha_{4k+3}}{2}, \quad \beta_k^{(2)} = \frac{\alpha_{4k} + \alpha_{4k+1} - \alpha_{4k+2} - \alpha_{4k+3}}{2}.$$

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Consequently,

$$\pi_t^{(2)} = \sum_{k=0}^{+\infty} \frac{\alpha_{4k} + \alpha_{4k+1} + \alpha_{4k+2} + \alpha_{4k+3}}{2} \frac{\varepsilon_{t-4k} + \varepsilon_{t-(4k+1)} + \varepsilon_{t-(4k+2)} + \varepsilon_{t-(4k+3)}}{2},$$

$$g_t^{(2)} = \sum_{k=0}^{+\infty} \frac{\alpha_{4k} + \alpha_{4k+1} - \alpha_{4k+2} - \alpha_{4k+3}}{2} \varepsilon_{t-4k}^{(2)}.$$

As for the generic scale $J = j$, we claim that

$$\gamma_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^j-1} \alpha_{k2^j+i} \right), \quad \beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right).$$

In addition, it may be helpful to highlight the expressions of $\pi_t^{(j)}$ and $g_t^{(j)}$:

$$\pi_t^{(j)} = \sum_{k=0}^{+\infty} \left(\frac{1}{\sqrt{2^j}} \sum_{i=0}^{2^j-1} \alpha_{k2^j+i} \right) \frac{1}{\sqrt{2^j}} \sum_{i=0}^{2^j-1} \varepsilon_{t-k2^j-i},$$

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right) \varepsilon_{t-k2^j}^{(j)}.$$

■

Proof of Theorem 1.3

Proof The representation of x_t comes from the Wold decomposition of the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, claimed in Theorem 1.1. Indeed, by applying the Classical Wold Decomposition to the zero-mean, weakly stationary purely non-deterministic stochastic process $\mathbf{x}$, we find that x_t belongs to the Hilbert space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_t$ is the white noise process of classical Wold innovations of $\mathbf{x}$. Afterwards, by exploiting the orthogonal decomposition of $\mathcal{H}_t(\boldsymbol{\varepsilon})$ given by Theorem 1.1, justified by the fact that the scaling operator $\mathbf{R}$ is isometric on $\mathcal{H}_t(\boldsymbol{\varepsilon})$, we know that

$$\mathcal{H}_t(\boldsymbol{\varepsilon}) = \bigoplus_{j=1}^{+\infty} \mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}},$$

where

$$\mathbf{R}^{j-1} \mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} b_k^{(j)} \varepsilon_{t-k2^j}^{(j)} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad b_k^{(j)} \in \mathbb{R} \right\},$$

as stated in Proposition 1.13. Recall that random variables $\varepsilon_t^{(j)}$ are defined by

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right).$$

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Hence, by denoting $g_t^{(j)}$ the orthogonal projections of the variable x_t on the subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, we find that

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)},$$

where the equality is in the L^2-norm. Then, by using the characterizations of subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, for any scale $j \in \mathbb{N}$ we can find a square-summable sequence of real coefficients $\left\{\beta_k^{(j)}\right\}_k$ such that

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)}.$$

As a result, we are allowed to decompose the variable x_t as

$$x_t = \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

where the equality is in the L^2-norm.

i) As we can see in the definition of variables $\varepsilon_t^{(j)}$, the process $\boldsymbol{\varepsilon_t^{(j)}}$ is an $MA(2^j-1)$ with respect to the fundamental innovations $\boldsymbol{\varepsilon}$ of the process $\mathbf{x}$. In addition, as claimed in Proposition 1.11, the subprocess $\left\{\varepsilon_{t-k2^j}^{(j)}\right\}_{k\in\mathbb{Z}}$ is white noise with variance equal to 1.

ii) For any fixed scale $j \in \mathbb{N}$, once the detail process $\boldsymbol{\varepsilon^{(j)}}$ is defined, since the variables $\varepsilon_{t-k2^j}^{(j)}$ are orthonormal when k varies, the component $g_t^{(j)}$ has a unique representation of the kind

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)}.$$

Thus, the coefficients $\beta_k^{(j)}$ are uniquely defined.

By Proposition 1.2, the coefficients $\beta_k^{(j)}$ do not depend on t and they can be expressed in terms of the classical Wold coefficients α_h of x_t as

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right).$$

Moreover $\sum_k \left(\beta_k^{(j)}\right)^2 < +\infty$ for any $j \in \mathbb{N}$, as explicitly shown in the proof of Proposition 1.2. Indeed, it holds that

$$\sum_{k=0}^{+\infty} \left(\beta_k^{(j)}\right)^2 \leqslant \sum_{h=0}^{+\infty} \alpha_h^2 < +\infty.$$

iii) First of all, when t is fixed, $\mathbb{E}\left[g_t^{(j)} g_t^{(l)}\right] = 0$ for all $j \neq l$ because $g_t^{(j)}$ and $g_t^{(l)}$ are, respectively, the projections of x_t on the subspaces $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$ and $\mathbf{R}^{l-1}\mathcal{L}_t^{\mathbf{R}}$ which are orthogonal by construction. Now, consider any $g_{t-m2^j}^{(j)}$ with $m \in \mathbb{N}_0$. Clearly, $g_{t-m2^j}^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_{t-m2^j}^{\mathbf{R}}$

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but, by the definition of $g_t^{(j)}$, we can write

$$g_{t-m2^j}^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-(m+k)2^j}^{(j)} = \sum_{K=0}^{+\infty} \beta_K^{(j)} \varepsilon_{t-K2^j}^{(j)},$$

where

$$\beta_K^{(j)} = \begin{cases} 0 & \text{if } K \in \{0, \dots, m-1\}, \\ \beta_k^{(j)} & \text{if } K = m+k \quad \text{for some } k \in \mathbb{N}_0. \end{cases}$$

As a result, $g_{t-m2^j}^{(j)}$ belongs to $\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}}$, too. Similarly, at scale l, taken any $n \in \mathbb{N}_0$, it is easy to see that $g_{t-n2^l}^{(l)}$ belongs to $\mathbf{R}^{l-1}\mathcal{L}_t^{\mathbf{R}}$. Hence, the orthogonality of such subspaces guarantees that

$$\mathbb{E}\left[g_{t-m2^j}^{(j)} g_{t-n2^l}^{(l)}\right] = 0 \qquad \forall j \neq l, \quad \forall m, n \in \mathbb{N}_0.$$

As for the more general requirement concerning $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right]$ for any $j, l \in \mathbb{N}$ and $p, q, t \in \mathbb{Z}$, we have that

$$\begin{aligned} \mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right] &= \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty} \beta_k^{(j)}\beta_h^{(l)} \mathbb{E}\left[\varepsilon_{t-p-k2^j}^{(j)} \varepsilon_{t-q-h2^l}^{(l)}\right] \\ &= \frac{1}{\sqrt{2^{j+l}}} \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty} \beta_k^{(j)}\beta_h^{(l)} \sum_{u=0}^{2^{j-1}-1}\sum_{v=0}^{2^{l-1}-1} \Big\{ \mathbb{E}\left[\varepsilon_{t-p-k2^j-u}\varepsilon_{t-q-h2^l-v}\right] \\ &\quad - \mathbb{E}\left[\varepsilon_{t-p-k2^j-u}\varepsilon_{t-q-h2^l-2^{l-1}-v}\right] - \mathbb{E}\left[\varepsilon_{t-p-k2^j-2^{j-1}-u}\varepsilon_{t-q-h2^l-v}\right] \\ &\quad + \mathbb{E}\left[\varepsilon_{t-p-k2^j-2^{j-1}-u}\varepsilon_{t-q-h2^l-2^{l-1}-v}\right] \Big\} \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right] = \frac{1}{\sqrt{2^{j+l}}} \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty} \beta_k^{(j)}\beta_h^{(l)} \sum_{u=0}^{2^{j-1}-1}\sum_{v=0}^{2^{l-1}-1} \Big\{ &\gamma(p-q+k2^j+u-h2^l-v) \\ &- \gamma(p-q+k2^j+u-h2^l-2^{l-1}-v) \\ &- \gamma(p-q+k2^j+2^{j-1}+u-h2^l-v) \\ &+ \gamma(p-q+k2^j+2^{j-1}+u-h2^l-2^{l-1}-v) \Big\}, \end{aligned}$$

where coefficients $\beta_k^{(j)}$, $\beta_h^{(l)}$ do not depend on t. After the summations over u, v and k, h, the one remaining variables are $j, l, p-q$. In other words, $\mathbb{E}\left[g_{t-p}^{(j)} g_{t-q}^{(l)}\right]$ depends at most on $j, l, p-q$.

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Proposition 1.14 *Let* $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ *be a purely non-deterministic* $AR(2)$ *process, i.e.*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t,$$

and set

$$\lambda_1 = \frac{2\phi_2}{\sqrt{\phi_1^2 + 4\phi_2} - \phi_1}, \qquad \lambda_2 = \frac{-2\phi_2}{\sqrt{\phi_1^2 + 4\phi_2} + \phi_1}.$$

If $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$*, then* $\mathbf{x}$ *is weakly stationary and the moving average representation of* x_t *is*

$$x_t = \sum_{h=0}^{+\infty} \frac{\lambda_1^{h+1} - \lambda_2^{h+1}}{\lambda_1 - \lambda_2} \varepsilon_{t-h}.$$

Proof The proof is left to the reader. ■

Proof of Theorem 1.5

Proof First of all, we show that any process $\mathbf{g^{(j)}}$ is well-defined. Indeed, by using the moving average representation of each $g_t^{(j)}$ with respect to the innovations on the support $S_t^{(j)}$, mentioned in assumption 1), and the definition of detail processes $\boldsymbol{\varepsilon^{(j)}}$, we have

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)} = \sum_{k=0}^{+\infty} \sum_{i=0}^{2^j-1} \beta_k^{(j)} \delta_i^{(j)} \varepsilon_{t-k2^j-i} = \sum_{h=0}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j \left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \varepsilon_{t-h},$$

where we exploited the identity $h = k2^j + i$, with $k = \left\lfloor \frac{h}{2^j} \right\rfloor$ and $i = h - 2^j \left\lfloor \frac{h}{2^j} \right\rfloor$. Assumption 2) ensures that

$$\sum_{h=0}^{+\infty} \left(\beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j \left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \right)^2 < +\infty$$

and so each $\mathbf{g^{(j)}}$ is well-defined.

In addition, the process $\mathbf{x}$ is well-defined thanks to assumptions 2) and 3). According to the latter, the components $g_t^{(j)}$ are orthogonal to each others at different scales for fixed $t \in \mathbb{Z}$, that is $\mathbb{E}\left[g_t^{(j)} g_t^{(l)}\right] = 0$ for any $j \neq l$. Therefore,

$$\mathbb{E}\left[x_t^2\right] = \mathbb{E}\left[\left(\sum_{j=1}^{+\infty} g_t^{(j)}\right)^2\right] = \sum_{j=1}^{+\infty} \mathbb{E}\left[\left(g_t^{(j)}\right)^2\right] = \sum_{j=1}^{+\infty} \sum_{h=0}^{+\infty} \left(\beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j \left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \right)^2,$$

which is finite because of 2). In consequence, the process $\mathbf{x}$ is well-defined.

Now we show that $\mathbf{x}$ is weakly stationary, with zero mean.

i) We already observed that $\mathbb{E}\left[x_t^2\right]$ is finite and not dependent on t.

ii) Since the processes $\mathbf{g^{(j)}}$ have zero mean, $\mathbb{E}[x_t] = \sum_{j=1}^{\infty} \mathbb{E}\left[g_t^{(j)}\right] = 0$ for any $t \in \mathbb{Z}$.

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iii) Let $p \neq q$. Then,

$$\mathbb{E}\left[x_{t-p}x_{t-q}\right] = \mathbb{E}\left[\left(\sum_{j=1}^{+\infty} g_{t-p}^{(j)}\right)\left(\sum_{l=1}^{+\infty} g_{t-q}^{(l)}\right)\right] = \sum_{j=1}^{+\infty}\sum_{l=1}^{+\infty}\mathbb{E}\left[g_{t-p}^{(j)}g_{t-q}^{(l)}\right].$$

As $\mathbb{E}\left[g_{t-p}^{(j)}g_{t-q}^{(l)}\right]$ is assumed to be dependent at most on $j, l, p-q$, it follows that $\mathbb{E}\left[x_{t-p}x_{t-q}\right]$ depends at most on the difference $p-q$.

As a result, $\mathbf{x}$ is weakly stationary, with zero mean.

As we remarked before, assumption 1) and detail processes $\boldsymbol{\varepsilon^{(j)}}$ ensure that

$$g_t^{(j)} = \sum_{h=0}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \varepsilon_{t-h},$$

therefore each variable x_t decomposes as

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)} = \sum_{j=1}^{+\infty}\sum_{h=0}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \varepsilon_{t-h} = \sum_{h=0}^{+\infty}\left(\sum_{j=1}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)}\right)\varepsilon_{t-h}.$$

In consequence, the decomposition of x_t with respect to the process $\boldsymbol{\varepsilon}$ turns out to be

$$x_t = \sum_{h=0}^{+\infty} \alpha_h \varepsilon_{t-h} \qquad \text{with} \qquad \alpha_h = \sum_{j=1}^{+\infty} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \delta_{h-2^j\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \quad \forall h \in \mathbb{N}_0.$$

We finally observe that the process $\mathbf{x}$ has null purely deterministic part, hence it is purely non-deterministic. ■

Proof of Theorem 1.6

Proof *ii*) $\Longrightarrow$ *i*) Firstly, if $\delta_i^{(j)}$ are the Haar coefficients, it is easy to check that conditions *a*) and *b*) are satisfied.

Secondly, as $\mathcal{C}_t^{\delta}$ is a subset of $\mathcal{H}_t(\boldsymbol{\varepsilon})$, we have just to show that any variable x_t in $\mathcal{H}_t(\boldsymbol{\varepsilon})$ belongs to $\mathcal{C}_t^{\delta}$ too, when $\delta_i^{(j)}$ are the Haar coefficients. Indeed, Theorem 1.3 guarantees that x_t has an Extended Wold Decomposition

$$x_t = \sum_{j=1}^{+\infty} g_t^{(j)} = \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)},$$

where

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}}\left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i}\right).$$

The involved coefficients $\delta_i^{(j)}$ are exactly the Haar ones and so $x_t \in \mathcal{C}_t^{\delta}$. Hence $\mathcal{C}_t^{\delta} = \mathcal{H}_t(\boldsymbol{\varepsilon})$.

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i) $\Longrightarrow$ *ii)* Now we prove that assumptions *a)* and *b)*, together with $\mathcal{C}_t^{\delta} = \mathcal{H}_t(\boldsymbol{\varepsilon})$, force the coefficients $\delta_i^{(j)}$ to be the Haar ones, that is

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \dots, 2^{j-1}-1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \dots, 2^j-1\}. \end{cases}$$

We proceed by induction.

As for the basic step, we consider the scale $j = 1$. In this case, assumption *a)* implies that

$$\delta_0^{(1)} = \frac{1}{\sqrt{2}}, \qquad \delta_1^{(1)} = -\frac{1}{\sqrt{2}}.$$

About the inductive step, we suppose that

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \dots, 2^{j-1}-1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \dots, 2^j-1\} \end{cases}$$

for all $j \in \{1, \dots, J\}$ and we want to show that

$$\delta_i^{(J+1)} = \begin{cases} \frac{1}{\sqrt{2^{J+1}}} & \text{if} \quad i \in \{0, \dots, 2^J-1\}, \\ -\frac{1}{\sqrt{2^{J+1}}} & \text{if} \quad i \in \{2^J, \dots, 2^{J+1}-1\}. \end{cases}$$

Taken an arbitrary variable x_t in $\mathcal{C}_t^{\delta} = \mathcal{H}_t(\boldsymbol{\varepsilon})$, the related components $\mathbf{g^{(j)}}$ satisfy the orthogonality conditions mentioned in assumption 3). In particular, let us focus on the orthogonality between scales $J-p$ and $J+1$, where $p \in \{0, \dots, J-1\}$. By setting $z = k2^{p+1}+n$, with $n \in \{0, \dots, 2^{p+1}-1\}$, we have

$$\begin{aligned}
0 = \mathbb{E}\left[g_t^{(J-p)} g_t^{(J+1)}\right] &= \mathbb{E}\left[\left(\sum_{z=0}^{+\infty} \beta_z^{(J-p)} \varepsilon_{t-z2^{J-p}}^{(J-p)}\right)\left(\sum_{h=0}^{+\infty} \beta_h^{(J+1)} \varepsilon_{t-h2^{J+1}}^{(J+1)}\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{+\infty}\sum_{n=0}^{2^{p+1}-1} \beta_{k2^{p+1}+n}^{(J-p)} \varepsilon_{t-k2^{J+1}-n2^{J-p}}^{(J-p)}\right)\left(\sum_{h=0}^{+\infty} \beta_h^{(J+1)} \varepsilon_{t-h2^{J+1}}^{(J+1)}\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{+\infty}\sum_{n=0}^{2^{p+1}-1} \frac{\beta_{k2^{p+1}+n}^{(J-p)}}{\sqrt{2^{J-p}}}\left(\sum_{i=0}^{2^{J-p-1}-1} \varepsilon_{t-k2^{J+1}-n2^{J-p}-i} - \sum_{i=0}^{2^{J-p-1}-1} \varepsilon_{t-k2^{J+1}-n2^{J-p}-2^{J-p-1}-i}\right)\right)\right. \\
&\quad \left.\cdot\left(\sum_{h=0}^{+\infty} \beta_h^{(J+1)} \sum_{l=0}^{2^{J+1}-1} \delta_l^{(J+1)} \varepsilon_{t-h2^{J+1}-l}\right)\right] \\
&= \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty}\sum_{n=0}^{2^{p+1}-1} \frac{\beta_{k2^{p+1}+n}^{(J-p)}}{\sqrt{2^{J-p}}} \beta_h^{(J+1)} \\
&\quad \cdot \mathbb{E}\left[\left(\sum_{i=0}^{2^{J-p-1}-1} \varepsilon_{t-k2^{J+1}-n2^{J-p}-i} - \sum_{i=0}^{2^{J-p-1}-1} \varepsilon_{t-k2^{J+1}-n2^{J-p}-2^{J-p-1}-i}\right)\left(\sum_{l=0}^{2^{J+1}-1} \delta_l^{(J+1)} \varepsilon_{t-h2^{J+1}-l}\right)\right].
\end{aligned}$$

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In order to exploit the orthogonality of variables ε_t we match the indices in the summations. In particular, such variables are matched when $h = k$ and $l = n2^{J-p} + i$, or $l = n2^{J-p} + 2^{J-p-1} + i$, hence the expression above becomes

$$\sum_{k=0}^{+\infty} \sum_{n=0}^{2^{p+1}-1} \frac{\beta_{k2^{p+1}+n}^{(J-p)}}{\sqrt{2^{J-p}}} \beta_k^{(J+1)} \left(\sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+i}^{(J+1)} - \sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+2^{J-p-1}+i}^{(J+1)} \right) = 0,$$

namely

$$\sum_{n=0}^{2^{p+1}-1} \sum_{k=0}^{+\infty} \left(\sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+i}^{(J+1)} - \sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+2^{J-p-1}+i}^{(J+1)} \right) \beta_{k2^{p+1}+n}^{(J-p)} \beta_k^{(J+1)} = 0.$$

Observe that, actually, the term in brackets depends just on n. As coefficients $\delta_i^{(j)}$ come up in the representation of every variable in $\mathcal{H}_t(\boldsymbol{\varepsilon})$, while coefficients $\beta_k^{(j)}$ are arbitrary, it must hold that, for any $n \in \left\{0, \ldots, 2^{p+1} - 1\right\}$,

$$\sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+i}^{(J+1)} = \sum_{i=0}^{2^{J-p-1}-1} \delta_{n2^{J-p}+2^{J-p-1}+i}^{(J+1)}.$$

This is true for any $p \in \{0, \ldots, J-1\}$.

When $p = J - 1$, we find that, for any $n \in \left\{0, \ldots, 2^J - 1\right\}$,

$$\delta_{2n}^{(J+1)} = \delta_{2n+1}^{(J+1)},$$

meaning that coefficients $\delta_i^{(j)}$ are equal to each others in pairs.

When $p = J - 2$, we have that, for each $n \in \left\{0, \ldots, 2^{J-1} - 1\right\}$,

$$\delta_{4n}^{(J+1)} + \delta_{4n+1}^{(J+1)} = \delta_{4n+2}^{(J+1)} + \delta_{4n+3}^{(J+1)}.$$

But we already know that $\delta_{4n}^{(J+1)} = \delta_{4n+1}^{(J+1)}$ and $\delta_{4n+2}^{(J+1)} = \delta_{4n+3}^{(J+1)}$, thus

$$\delta_{4n}^{(J+1)} = \delta_{4n+1}^{(J+1)} = \delta_{4n+2}^{(J+1)} = \delta_{4n+3}^{(J+1)},$$

that is coefficients $\delta_i^{(j)}$ are equal to each others in groups of 4.

By induction we infer that, when $p = 0$, two patterns of 2^J equal coefficients $\delta_i^{(j)}$ arise:

$$\delta_0^{(J+1)} = \cdots = \delta_{2^J-1}^{(J+1)}, \qquad \delta_{2^J}^{(J+1)} = \cdots = \delta_{2^{J+1}-1}^{(J+1)}.$$

Now we employ assumption b) and the inductive hypothesis. Indeed, for any $k \in \left\{0, \ldots, 2^J - 1\right\}$,

$$\left(\delta_{2k}^{(J+1)}\right)^2 + \left(\delta_{2k+1}^{(J+1)}\right)^2 = \left(\delta_k^{(J)}\right)^2 = \frac{1}{2^J}.$$

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As $\delta_{2k}^{(J+1)}$ and $\delta_{2k+1}^{(J+1)}$ coincide, we find that

$$\left|\delta_{2k}^{(J+1)}\right| = \frac{1}{\sqrt{2^{J+1}}}.$$

Since $\delta_0^{(J+1)} > 0$ and $\delta_{2^{J+1}-1}^{(J+1)} < 0$ by assumption b), we deduce that

$$\delta_i^{(J+1)} = \begin{cases} \frac{1}{\sqrt{2^{J+1}}} & \text{if} \quad i \in \{0, \dots, 2^J - 1\}, \\ -\frac{1}{\sqrt{2^{J+1}}} & \text{if} \quad i \in \{2^J, \dots, 2^{J+1} - 1\} \end{cases}$$

and this completes the proof by induction.

Finally, a simple computation guarantees that coefficients α_h that represent a variable $x_t \in \mathcal{C}_t^\delta$ can be expressed as

$$\alpha_h = \sum_{j=1}^{+\infty} \frac{1}{\sqrt{2^j}} \beta_{\left\lfloor \frac{h}{2^j} \right\rfloor}^{(j)} \chi^{(j)}(h),$$

with

$$\chi^{(j)}(h) = \begin{cases} -1 & \text{if} \quad 2^j \left\lfloor \frac{h}{2^j} \right\rfloor \in \left\{h - 2^j + 1, \dots, h - 2^{j-1}\right\}, \\ 1 & \text{if} \quad 2^j \left\lfloor \frac{h}{2^j} \right\rfloor \in \left\{h - 2^{j-1} + 1, \dots, h\right\}. \end{cases}$$

■

Proposition 1.15 *The time series*

$$\mathbf{R}x_t = \sum_{k=0}^{+\infty} \frac{\alpha_{\lfloor \frac{k}{2} \rfloor}}{\sqrt{2}} \varepsilon_{t-k}$$

has spectral density function

$$f_{\mathbf{R}}(\lambda) = 2\cos^2\left(\frac{\lambda}{2}\right) f_x(2\lambda).$$

Proof Define the time-invariant linear filter

$$A(\mathbf{L}) = \sum_{h=0}^{+\infty} \alpha_h \mathbf{L}^h,$$

so that $x_t = A(\mathbf{L})\varepsilon_t$. Since $\sum_{h=0}^{\infty} |\alpha_h| < +\infty$ and the spectral density function of ε_t is $f_\varepsilon(\lambda) = \frac{1}{2\pi}$, the spectral density function of x_t is

$$\begin{aligned} f_x(\lambda) &= \left|A\left(e^{-i\lambda}\right)\right|^2 f_\varepsilon(\lambda) = \left|\sum_{h=0}^{+\infty} \alpha_h e^{-ih\lambda}\right|^2 \frac{1}{2\pi} \\ &= \frac{1}{2\pi}\left\{\left(\sum_{h=0}^{+\infty} \alpha_h \cos(h\lambda)\right)^2 + \left(\sum_{h=0}^{+\infty} \alpha_h \sin(h\lambda)\right)^2\right\} \\ &= \frac{1}{2\pi} \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} \alpha_h \alpha_k \cos(\lambda(k-h)). \end{aligned}$$

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Now consider $\mathbf{R}x_t$. As

$$\sum_{k=0}^{+\infty}\left|\frac{\alpha_{\lfloor\frac{k}{2}\rfloor}}{\sqrt{2}}\right| = \sqrt{2}\sum_{h=0}^{+\infty}|\alpha_h| < +\infty,$$

the spectral density function of $\mathbf{R}x_t$ is

$$\begin{aligned} f_{\mathbf{R}}(\lambda) &= \left|\sum_{k=0}^{+\infty}\frac{\alpha_{\lfloor\frac{k}{2}\rfloor}}{\sqrt{2}}e^{-ik\lambda}\right|^2\frac{1}{2\pi} = \frac{1}{2\pi}\sum_{h=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{\alpha_{\lfloor\frac{h}{2}\rfloor}\alpha_{\lfloor\frac{k}{2}\rfloor}}{2}\cos(\lambda(k-h)) \\ &= \frac{1}{2\pi}\sum_{h=0}^{+\infty}\sum_{k=0}^{+\infty}\alpha_h\alpha_k\left\{\cos(2\lambda(k-h)) + \frac{\cos(\lambda(2k-2h+1)) + \cos(\lambda(2k-2h-1))}{2}\right\} \\ &= \frac{1}{2\pi}\sum_{h=0}^{+\infty}\sum_{k=0}^{+\infty}\alpha_h\alpha_k\cos(2\lambda(k-h))\left\{1+\cos(\lambda)\right\} = 2\cos^2\left(\frac{\lambda}{2}\right)f_x(2\lambda). \end{aligned}$$

■

1.C Forecasting

In this section we provide some helpful formulas to determine conditional expectation and variance of a zero-mean weakly stationary purely non-deterministic time series $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ by exploiting its persistence-based decomposition. The computation of these quantities turns out to be quite straightforward. We assume that x_t has Classical and Extended Wold Decompositions respectively

$$x_t = \sum_{h=0}^{+\infty}\alpha_h\varepsilon_{t-h} \qquad \text{and} \qquad x_t = \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty}\beta_k^{(j)}\varepsilon_{t-k2^j}^{(j)}.$$

We consider the filtration generated by the white noise process of fundamental innovations $\boldsymbol{\varepsilon} = \{\varepsilon_t\}_{t\in\mathbb{Z}}$ and we assume that the variables ε_t are independent. Then, the conditional expectation at time t of the variable x_{t+1} is characterized by an off-set of the classical Wold coefficients:

$$\mathbb{E}_t\left[x_{t+1}\right] = \mathbb{E}_t\left[\sum_{h=0}^{+\infty}\alpha_h\varepsilon_{t+1-h}\right] = \sum_{h=0}^{+\infty}\alpha_{h+1}\varepsilon_{t-h} = \sum_{h=0}^{+\infty}\alpha_{h,1}\varepsilon_{t-h},$$

where $\alpha_{h,1} = \alpha_{h+1}$ for any $h \in \mathbb{N}_0$. Such offset is also present in the Extended Wold Decomposition of the variable $\mathbb{E}_t\left[x_{t+1}\right]$. Indeed,

$$\mathbb{E}_t\left[x_{t+1}\right] = \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty}\beta_{k,1}^{(j)}\varepsilon_{t-k2^j}^{(j)},$$

where, for any scale $j \in \mathbb{N}$ and any $k \in \mathbb{N}_0$, we set

$$\begin{aligned} \beta_{k,1}^{(j)} &= \frac{1}{\sqrt{2^j}}\left(\sum_{i=0}^{2^{j-1}-1}\alpha_{k2^j+i,1} - \sum_{i=0}^{2^{j-1}-1}\alpha_{k2^j+2^{j-1}+i,1}\right) \\ &= \frac{1}{\sqrt{2^j}}\left(\sum_{i=0}^{2^{j-1}-1}\alpha_{k2^j+i+1} - \sum_{i=0}^{2^{j-1}-1}\alpha_{k2^j+2^{j-1}+i+1}\right). \end{aligned}$$

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As for the p-step forecast, with $p \in \mathbb{N}$, we have

$$\mathbb{E}_t\left[x_{t+p}\right] = \sum_{h=0}^{+\infty} \alpha_{h+p}\varepsilon_{t-h}$$

and the variable $\mathbb{E}_t\left[x_{t+p}\right]$ has Extended Wold Decomposition

$$\mathbb{E}_t\left[x_{t+p}\right] = \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty} \beta_{k,p}^{(j)}\varepsilon_{t-k2^j}^{(j)},$$

where, for any $j \in \mathbb{N}$ and any $k \in \mathbb{N}_0$, the coefficients $\beta_{k,p}^{(j)}$ are

$$\beta_{k,p}^{(j)} = \frac{1}{\sqrt{2^j}}\left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i+p} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i+p}\right).$$

Accordingly, we define $\beta_{k,0}^{(j)} = \beta_k^{(j)}$. We see that, once the Extended Wold Decomposition of x_t is known, forecasts do not require a large additional effort. In fact, they are driven by the detail processes $\boldsymbol{\varepsilon^{(j)}}$ too and the computation of coefficients $\beta_{k,p}^{(j)}$ is easily achieved.

Now, let us focus on the conditional variance. By exploiting the properties of the Classical Wold Decomposition of x_{t+1}, it may be shown that

$$\mathbb{E}_t\left[x_{t+1}^2\right] = \mathbb{E}_t\left[\left(\sum_{h=0}^{+\infty}\alpha_h\varepsilon_{t+1-h}\right)^2\right] = \alpha_0^2 + \left(\sum_{h=0}^{+\infty}\alpha_{h+1}\varepsilon_{t-h}\right)^2 = \alpha_0^2 + \left(\mathbb{E}_t\left[x_{t+1}\right]\right)^2$$

so that $\mathrm{Var}_t\left(x_{t+1}\right) = \alpha_0^2$. Thanks to Theorem 1.6, the classical Wold coefficient α_0 may be retrieved from the extended Wold coefficients $\beta_k^{(j)}$, therefore

$$\mathrm{Var}_t\left(x_{t+1}\right) = \alpha_0^2 = \left(\sum_{j=1}^{+\infty}\frac{\beta_0^{(j)}}{\sqrt{2^j}}\right)^2.$$

In general, for any $p \in \mathbb{N}$,

$$\mathbb{E}_t\left[x_{t+p}^2\right] = \alpha_0^2 + \cdots + \alpha_{p-1}^2 + \left(\mathbb{E}_t\left[x_{t+p}\right]\right)^2$$

and, similarly, the p-step conditional variance is

$$\mathrm{Var}_t\left(x_{t+p}\right) = \alpha_0^2 + \cdots + \alpha_{p-1}^2,$$

where coefficients α_h can be derived from coefficients $\beta_k^{(j)}$ by Theorem 1.6.

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1.D Simulations: $AR(1)$ and $ARMA(1,1)$

In this section we analyse the nature of coefficients $\gamma_k^{(j)}$ and $\beta_k^{(j)}$ of a weakly stationary purely non-deterministic $ARMA(1,1)$ process $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$, namely

$$x_t = \rho x_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1},$$

where $|\rho| < 1$, $\rho + \theta \neq 0$ and $\boldsymbol{\varepsilon} = \{\varepsilon\}_{t\in\mathbb{Z}}$ is a unit variance white noise. Later, by setting $\theta = 0$ we will retrieve an $AR(1)$ process as a special case.

By using the lag operator $\mathbf{L}$ and the identity map I, we can rewrite the previous equation as

$$(I - \rho\mathbf{L})x_t = (I + \theta\mathbf{L})\varepsilon_t.$$

The process $\mathbf{x}$ admits a moving average representation that we can find by exploiting the invertibility of the operator $I - \rho\mathbf{L}$. In fact,

$$x_t = (I - \rho\mathbf{L})^{-1}(I + \theta\mathbf{L})\varepsilon_t = \varepsilon_t + (\rho + \theta)\sum_{l=1}^{+\infty}\rho^{l-1}\varepsilon_{t-l} = \sum_{h=0}^{+\infty}\alpha_h\varepsilon_{t-h},$$

where we define

$$\alpha_h = \begin{cases} 1 & \text{if } h = 0, \\ (\rho+\theta)\rho^{h-1} & \text{if } h \geqslant 1. \end{cases}$$

By Proposition 1.2, we easily find the Extended Wold Decomposition of x_t by using the coefficients α_h. Let us fix a scale $j \in \mathbb{N}$. The coefficient $\gamma_0^{(j)}$ turns out to be

$$\gamma_0^{(j)} = \frac{1}{\sqrt{2^j}}\left\{1 + (\rho+\theta)\frac{1-\rho^{2^j-1}}{1-\rho}\right\},$$

while, for any $k \in \mathbb{N}$,

$$\gamma_k^{(j)} = (\rho+\theta)\frac{1-\rho^{2^j}}{\sqrt{2^j}(1-\rho)}\rho^{k2^j-1}.$$

Afterwards, $\beta_0^{(j)}$ is given by

$$\beta_0^{(j)} = \frac{1}{\sqrt{2^j}}\left\{1 + (\rho+\theta)\frac{1-2\rho^{2^{j-1}-1}+\rho^{2^j-1}}{1-\rho}\right\}$$

and, for each $k \in \mathbb{N}$,

$$\beta_k^{(j)} = (\rho+\theta)\frac{\left(1-\rho^{2^{j-1}}\right)^2}{\sqrt{2^j}(1-\rho)}\rho^{k2^j-1}.$$

Observe, inter alia, that the multiscale impulse responses $\beta_k^{(j)}$ are decreasing in k and that $\lim_{j\to\infty}\beta_k^{(j)} = 0$ for all $k \in \mathbb{N}_0$.

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By setting $\theta = 0$ we find the expressions of coefficients $\gamma_k^{(j)}$, $\beta_k^{(j)}$ in case the process $\mathbf{x}$ is $AR(1)$. In particular, for any $k \in \mathbb{N}_0$, the coefficients $\beta_k^{(j)}$ turn out to be

$$\beta_k^{(j)} = \frac{\rho^{k2^j}\left(1-\rho^{2^{j-1}}\right)^2}{\sqrt{2^j}(1-\rho)}.$$

In the first panel of Figure 1.D.1 we plot the autocorrelation function (ACF) of an $AR(1)$ process with $\rho = 0.99$, while in the second panel we see the multiscale impulse responses of the same process at increasing scales.

In Figure 1.D.2 we plot the ACF and the coefficients $\beta_k^{(j)}$ of an $ARMA(1,1)$ process with $\rho = 0.965$ and $\theta = -0.85$ at increasing scales. The multiscale impulse response functions are zoomed in to have a better understanding of their behaviour. By choosing the parameter ρ close to $-\theta$, we study the case in which the time series $\mathbf{x}$ is approximately white noise. Indeed, for such a choice of parameters, in the representation $(I - \rho\mathbf{L})x_t = (I + \theta\mathbf{L})\varepsilon_t$ we have the so-called *root cancellation*. As we see in the top panel, the autocorrelation does not cross the value $1/2$ and so the concerned process is usually classified as not persistent. As for the multiscale impulse responses, at scales $j = 4, 5$ we can spot a relatively large positive coefficient $\beta_1^{(j)}$, that we were not able to identify at finer scales.

1.E Further illustrative examples for Theorems 1.5 and 1.6

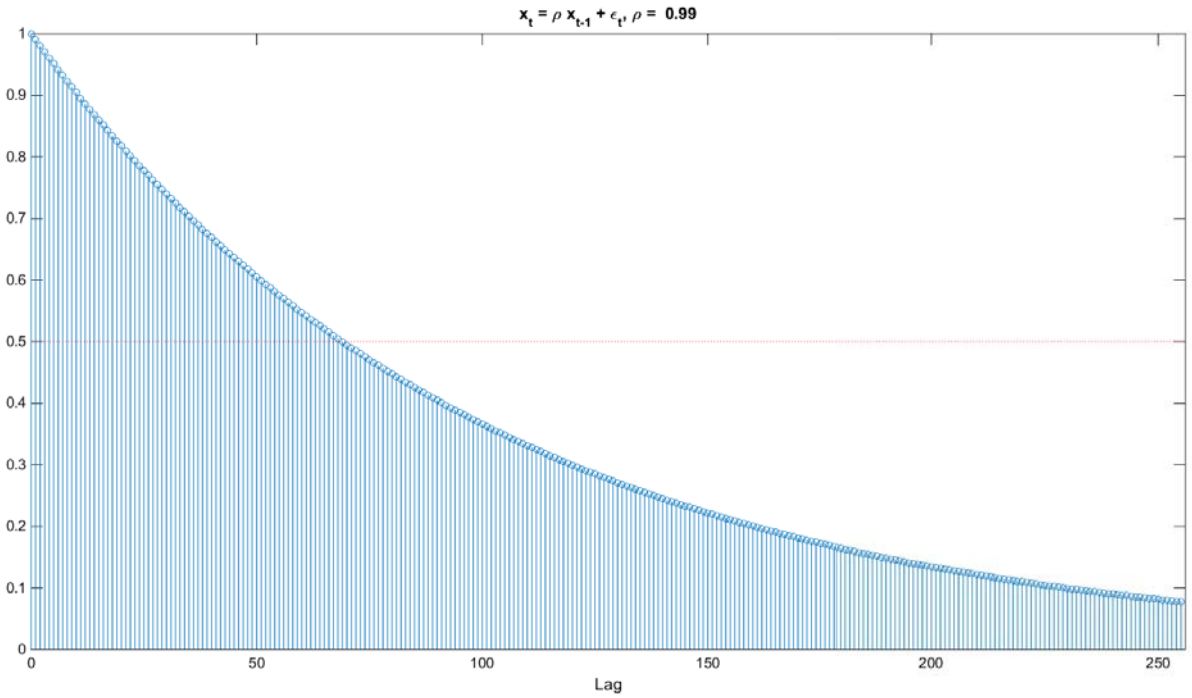
We provide some complementary simulations related to the examples discussed in Subsection 1.3.2, where $\mathbf{x}$ is a standard $AR(1)$ process and $\mathbf{y}$ is an $AR(1)$ with horizon 2^J.

First, we focus on the choice of the scale J. Indeed, as J increases, the process $\mathbf{y}$ concentrates on more and more persistent time scales. In Figure 1.E.1 we plot the ACF of $\mathbf{z}$ in case $J = 3, 4, 5$ by setting the parameters $\rho_x = 0.7$ and $\rho_y = 0.9$. We see that the tail of the ACF increases as J goes up, revealing more and more persistence.

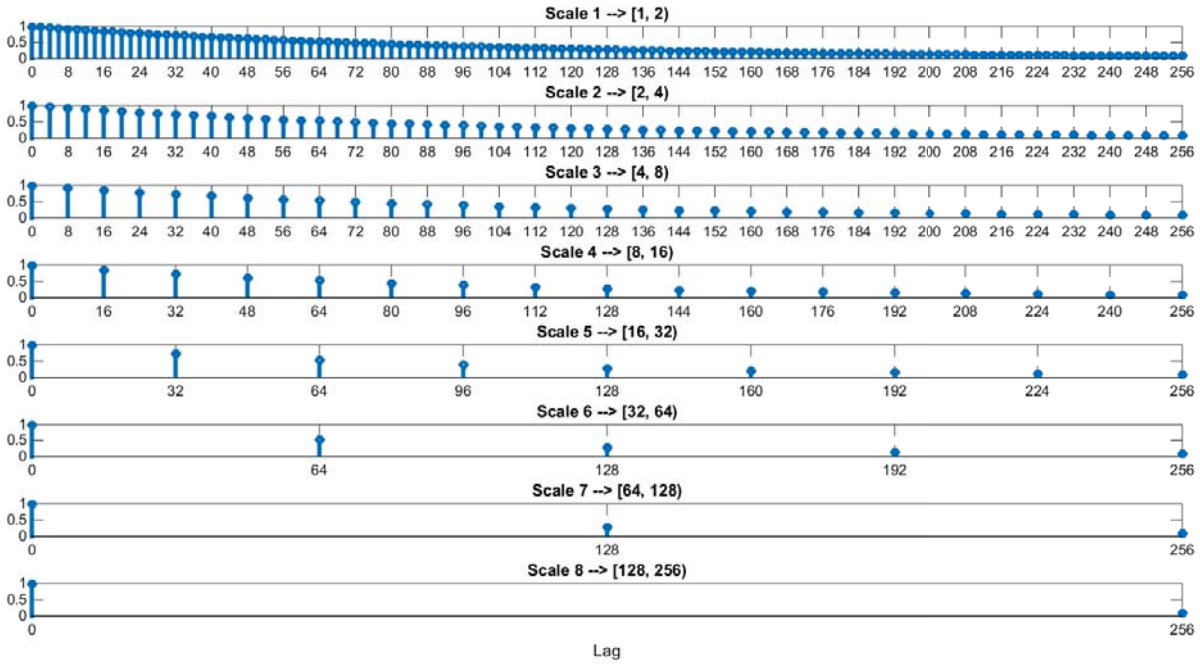
These examples illustrate that the persistence of a standard $AR(1)$ process may increase when we sum an $AR(1)$ with horizon 2^J. However, other choices of ρ_y allow to modify the short lags of the ACF of $\mathbf{x}$, while keeping the long lags unchanged. Indeed, by setting $\rho_y = \rho_x^{2^J}$ we have, for any $j \geqslant J+1$ and $k \in \mathbb{N}_0$,

$$\beta_{y,k}^{(j)} = \frac{\rho_x^{k2^j}}{\sqrt{2^j}} \frac{\left(1-\rho_x^{2^{j-1}}\right)^2}{1-\rho_x^{2^J}}.$$

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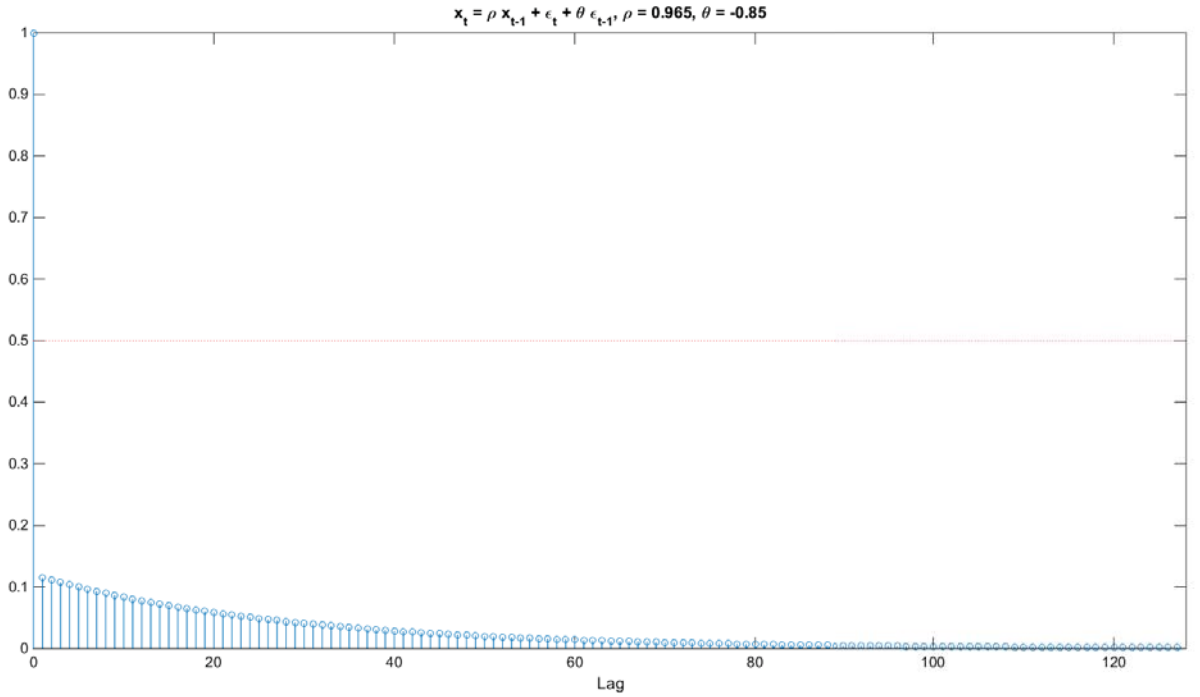
(a) ACF



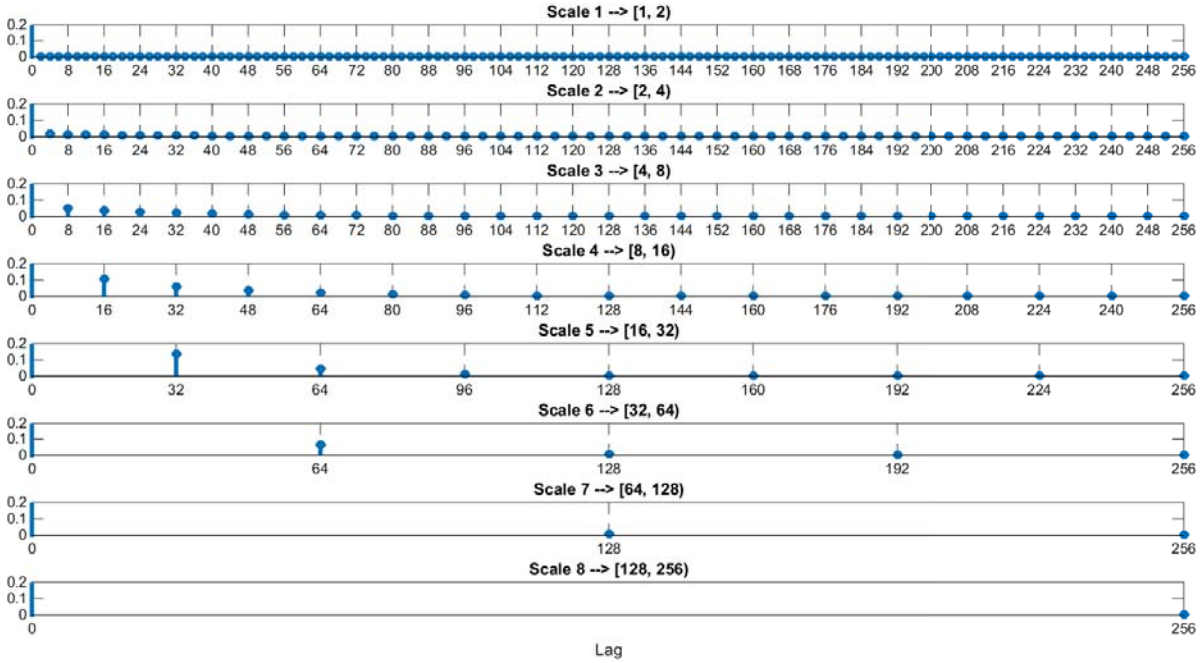
(b) Multiscale impulse responses

Figure 1.D.1: ACF and multiscale impulse responses of an $AR(1)$ process with $\rho = 0.99$.

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(a) ACF



(b) Multiscale impulse responses

Figure 1.D.2: ACF and multiscale impulse responses of an $ARMA(1,1)$ process with parameters $\rho = 0.965$ and $\theta = -0.85$.

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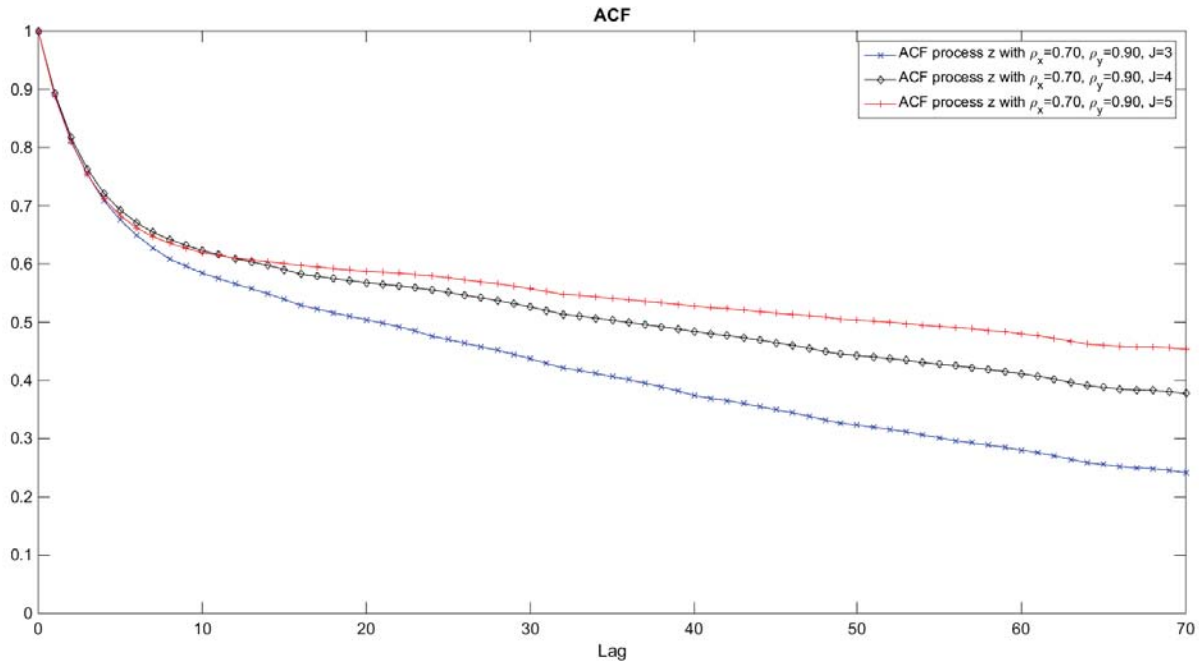


Figure 1.E.1: ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = 0.9$ and $J = 3, 4, 5$.

Then, for all $j \geqslant J+1$ and $k \in \mathbb{N}_0$, the multiscale impulse responses of $\mathbf{z}$ are

$$\beta_{z,k}^{(j)} = \frac{\rho_x^{k2^j}}{\sqrt{2^j}} \frac{\left(1-\rho_x^{2^{j-1}}\right)^2}{1-\rho_x} \left\{1 + \frac{\sqrt{2^J}\,(1-\rho_x)}{1-\rho_x^{2^J}}\right\}.$$

As $1 + \frac{\sqrt{2^J}(1-\rho_x)}{1-\rho_x^{2^J}}$ is not dependent on j or k, at scales $j \geqslant J+1$ we see that $\beta_{z,k}^{(j)}$ are proportional to the extended Wold coefficients of an $AR(1)$ with parameter ρ_x. Differently stated, at scales $j \geqslant J+1$, the coefficients $\beta_{z,k}^{(j)}$ coincide with the multiscale impulse responses of an $AR(1)$ with parameter ρ_x, whose error variance is

$$1 + \frac{\sqrt{2^J}\,(1-\rho_x)}{1-\rho_x^{2^J}}.$$

As the ACF of an $AR(1)$ does not depend on the variance of the innovations and the scales $j \geqslant J+1$ characterize the long-run behaviour of $\mathbf{z}$, we find that the long lags of the ACF of $\mathbf{z}$ coincide with the ones of the process $\mathbf{x}$. Nevertheless, the short lags of the ACF of $\mathbf{z}$ are modified, as we see in Figure 1.E.2.

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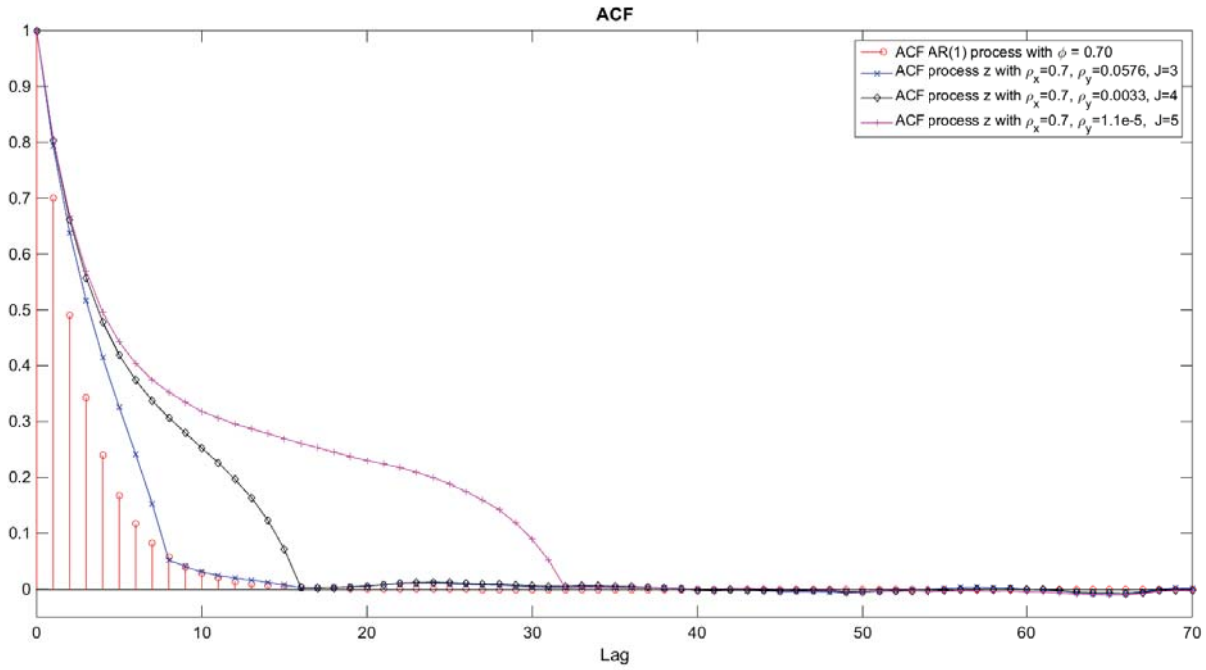


Figure 1.E.2: ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = \rho_x^{2^J}$ and $J = 3, 4, 5$.

1.F Supplement to Realized Volatility decomposition

1.F.1 Additional figures about Realized Volatility

Figure 1.F.1 depicts the sample of daily Realized Volatility of USD/UCH exchange rate from December 1989 to December 2003, while Figure 1.F.2 contains the related sample ACF. Finally, Figure 1.F.3 collects the plots of first ten estimated persistent components of daily Realized Volatility.

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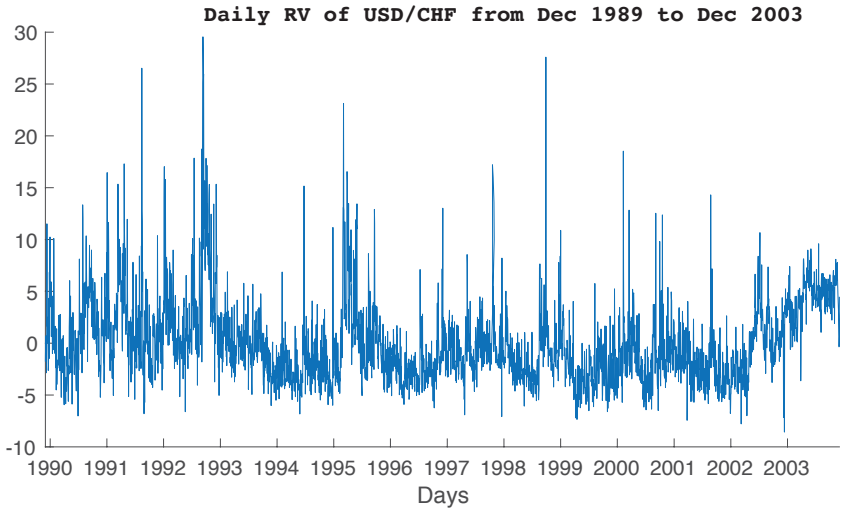


Figure 1.F.1: Realizations of daily Realized Volatility of USD/UCH exchange rate from December 1989 to December 2003.

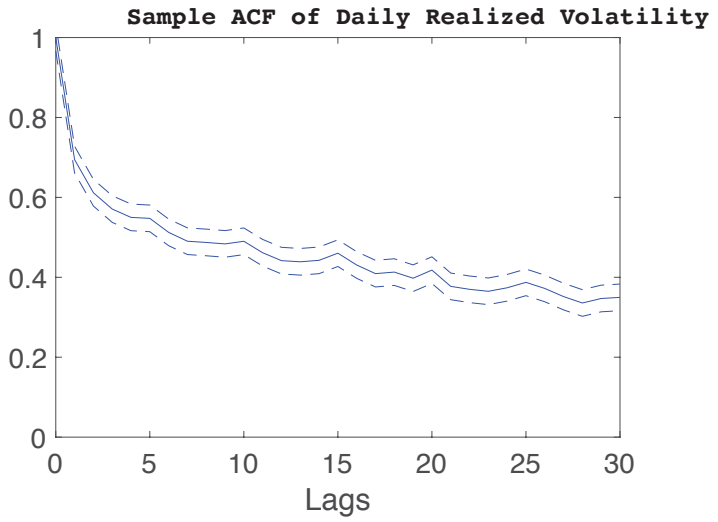


Figure 1.F.2: Sample ACF of daily Realized Volatility of USD/UCH exchange rate.

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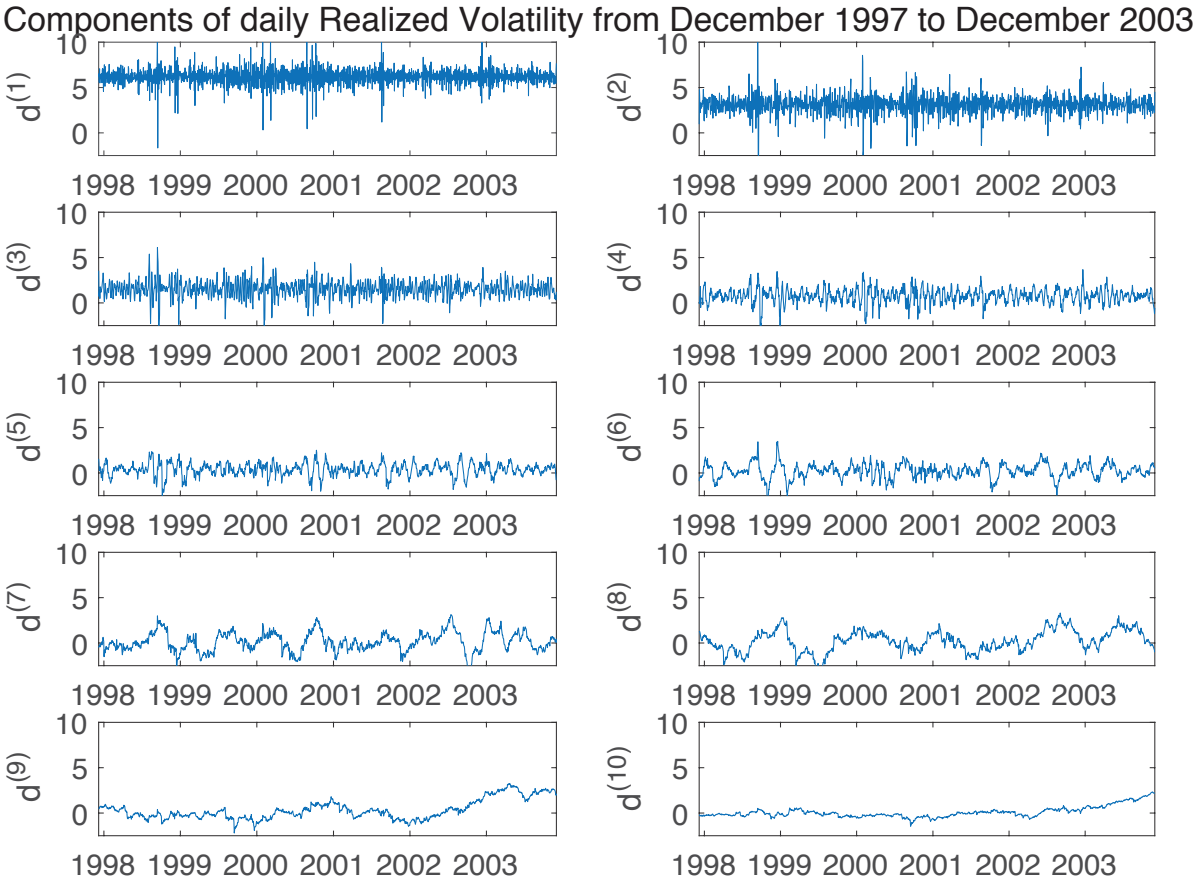


Figure 1.F.3: Estimated persistent components of daily Realized Volatility from scale 1 to 10.

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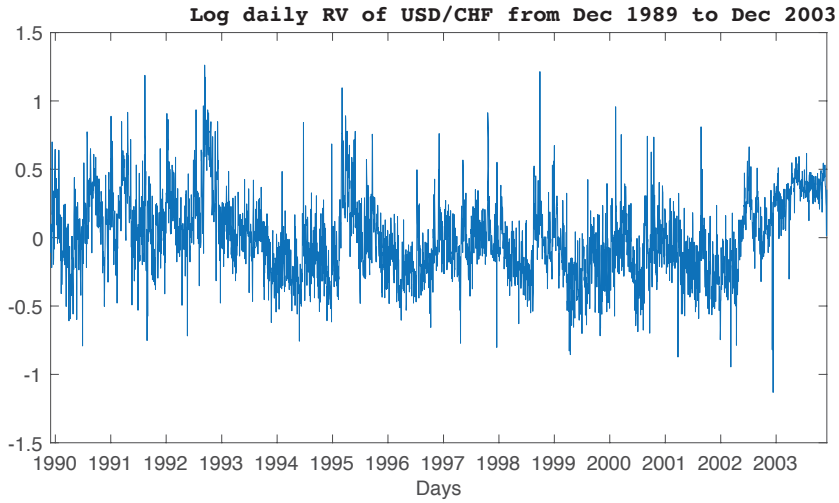


Figure 1.F.4: Realizations of log daily Realized Volatility of USD/UCH exchange rate from December 1989 to December 2003.

1.F.2 Analysis of log Realized Volatility

In this section we repeat the main steps of Section 1.5 for the time series of logarithmic daily Realized Volatility. All results are consistent with findings of Section 1.5. We start with some figures about the sample, the ACF and persistent components up to scale 10, which explain around 94% of total sample variance.

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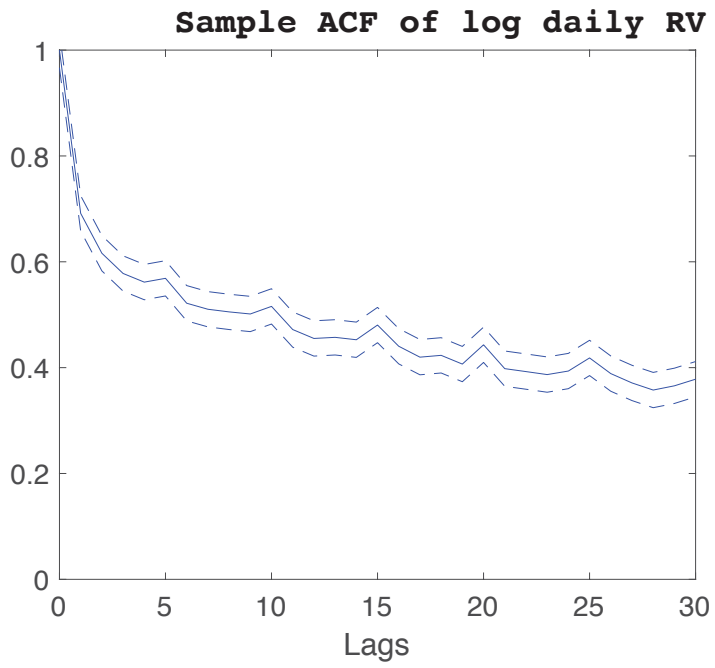


Figure 1.F.5: Sample ACF of log daily Realized Volatility of USD/UCH exchange rate.

We compute weekly and monthly log volatility by averaging log Realized Volatility. In Figure 1.F.7 we plot the variance decomposition (across scales) of daily, weekly and monthly log Realized Volatility.

We find again some evidence of the Heterogeneous Market Hypothesis, as well as long memory of daily Realized Volatility in logarithmic terms. Scales 7, 8 and 9 still explain the most variance. High scales are even more important for weekly and monthly log Realized Volatility, in agreement with the previous results of Figure 1.5. The comparison of autocorrelation functions in Figure 1.F.8 replicates, in fact, the analysis of Figure 1.6 of Subsection 1.5.4.

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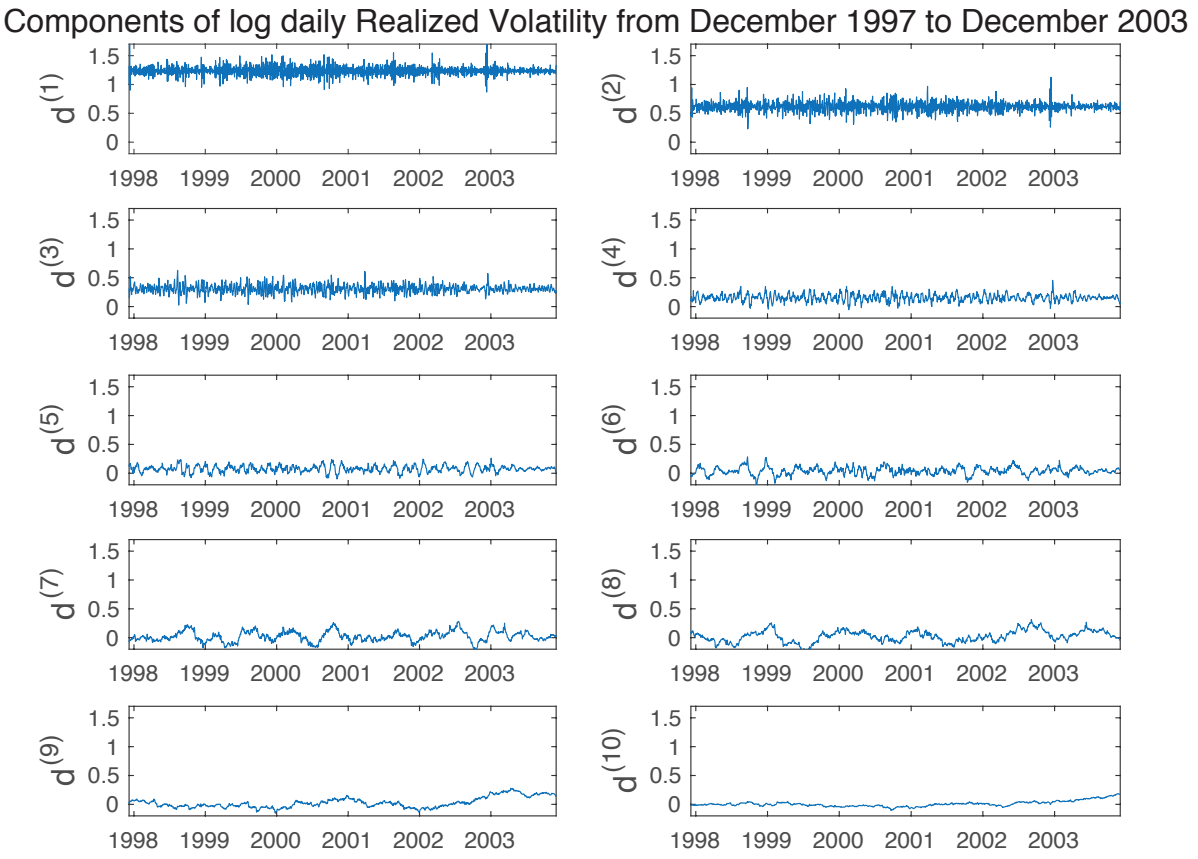


Figure 1.F.6: Estimated persistent components of log daily Realized Volatility from scale 1 to 10.

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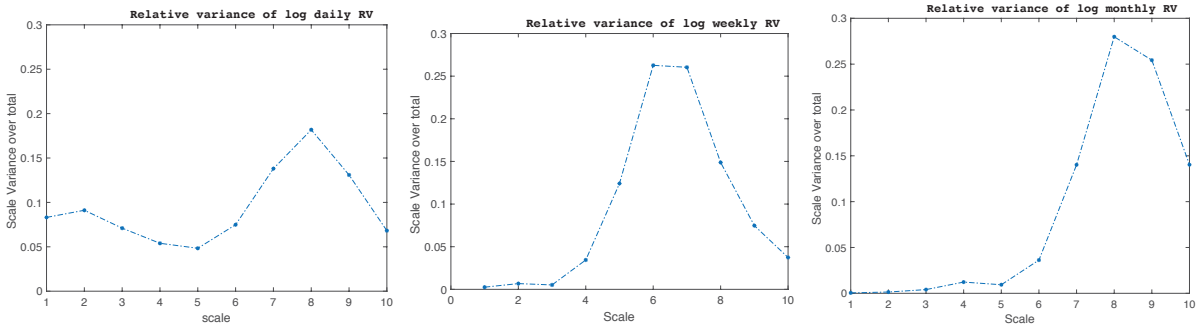


Figure 1.F.7: Variance ratio explained by each scale for daily, weekly and monthly log Realized Volatility.

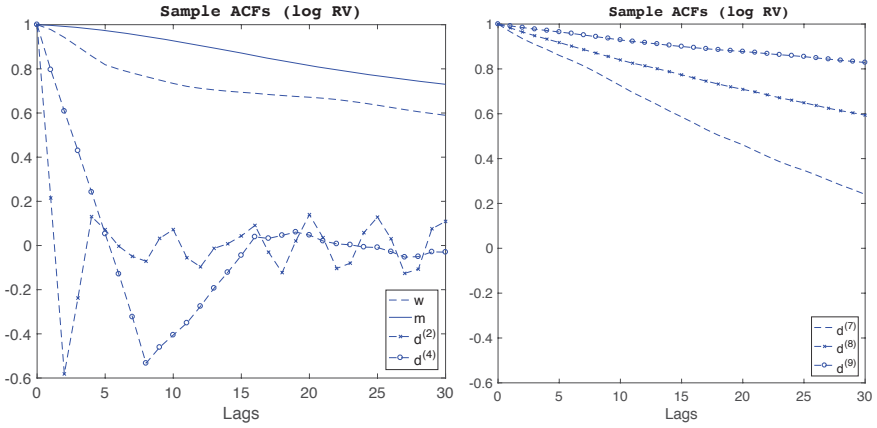


Figure 1.F.8: Left panel: comparison between sample ACFs of weekly and monthly log Realized Volatility (dashed and solid line respectively) with ACFs of persistent components of log daily Realized Volatility at scales 2 and 4 (lines marked with x and circles respectively). Right panel: sample ACFs of persistent components of log daily Realized Volatility at scales 7 (dashed line), 8 (line marked with x) and 9 (line marked with circles).

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We now implement the forecasting method based on the components $d_t^{(7)}$, $d_t^{(8)}$ and $d_t^{(9)}$ of log daily Realized Volatility, following Eq. (1.6). From Figure 1.F.9 it is apparent that the use of three components as regressors provides a good trade-off between performance and parsimony. As shown in Table 1.F.1, the persistence-based forecasting method is competitive with the HAR model.

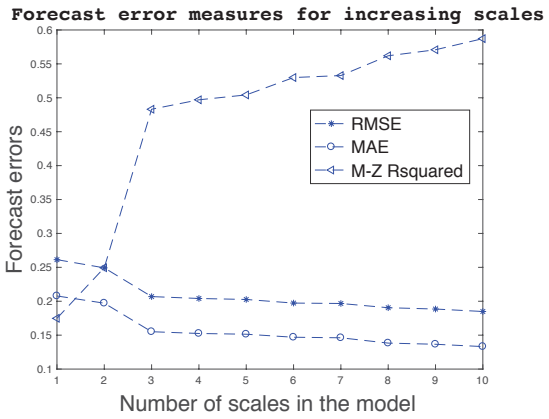


Figure 1.F.9: Forecasting performance of persistence-based models for log Realized Volatility with increasing number of components as regressors. Persistent components are added according to their explained variance. RMSE, MAE and R^2 denote Root Mean Square Error, Mean Absolute Error and R^2 of Mincer-Zarnowitz regression.

Table 1.F.1: Forecasting performance of HAR model and persistence-based models for log Realized Volatility with 3 and 10 component. RMSE, MAE and R^2 denote Root Mean Square Error, Mean Absolute Error and R^2 of Mincer-Zarnowitz regression.

	RMSE	MAE	R^2
HAR	0.192	0.141	0.557
Extended Wold (3)	0.207	0.155	0.483
Extended Wold (10)	0.185	0.133	0.587

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1.G Extended Wold Decomposition in base N

The derivation of the Extended Wold Decomposition exploits 2 as reference number. The construction of Section 1.2 is, in fact, reminiscent of the dyadic procedure of the Discrete Haar Transform. However, it is possible to deduce alternative orthogonal decompositions by using any *base* $N \in \mathbb{N}$. The given weakly stationary time series $\mathbf{x}$ is decomposed into the sum of uncorrelated components whose durations are now classified according to the time grid N^j at any scale j.

Fixed $N \in \mathbb{N}$, consider the operator $\mathbf{R} : \mathcal{H}_t(\boldsymbol{\varepsilon}) \longrightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ defined by

$$\mathbf{R}: \quad \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \quad \longmapsto \quad \sum_{k=0}^{+\infty} \frac{a_k}{\sqrt{N}} \sum_{i=0}^{N-1} \varepsilon_{t-Nk-i} = \sum_{k=0}^{+\infty} \frac{a_{\lfloor \frac{k}{N} \rfloor}}{\sqrt{N}} \varepsilon_{t-k}.$$

$\mathbf{R}$ is isometric on $\mathcal{H}_t(\boldsymbol{\varepsilon})$ and its adjoint is the operator $\mathbf{R}^* : \mathcal{H}_t(\boldsymbol{\varepsilon}) \longrightarrow \mathcal{H}_t(\boldsymbol{\varepsilon})$ such that

$$\mathbf{R}^*: \quad \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \quad \longmapsto \quad \sum_{k=0}^{+\infty} \frac{1}{\sqrt{N}} \left(\sum_{i=0}^{N-1} a_{Nk+i} \right) \varepsilon_{t-k}.$$

The wandering subspace is

$$\mathcal{L}_t^{\mathbf{R}} = \ker(\mathbf{R}^*) = \left\{ \sum_{k=0}^{+\infty} a_k \varepsilon_{t-k} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \quad a_{Nk+N-1} = -\sum_{i=0}^{N-2} a_{Nk+i} \quad \forall k \in \mathbb{N}_0 \right\}.$$

In addition, by defining the innovations

$$\hat{\varepsilon}_t^{(j)} = \frac{1}{\sqrt{N^{j-1}}} \sum_{k=0}^{N^{j-1}-1} \varepsilon_{t-k}$$

for all $j \in \mathbb{N}$ and $t \in \mathbb{N}_0$, we have

$$\mathbf{R}^{j-1}\mathcal{L}_t^{\mathbf{R}} = \left\{ \sum_{k=0}^{+\infty} \left\{ \sum_{i=0}^{N-2} a_{Nk+i} \hat{\varepsilon}_{t-N^j k - N^{j-1} i} - \left(\sum_{i=0}^{N-2} a_{Nk+i} \right) \hat{\varepsilon}_{t-N^j k - N^{j-1}(N-1)} \right\} \in \mathcal{H}_t(\boldsymbol{\varepsilon}) : \; a_h \in \mathbb{R} \right\}.$$

Following the same steps of Proposition 1.2, we find the decompositions of x_t

$$x_t = \pi_t^{(J)} + \sum_{j=1}^{J} g_t^{(j)} \qquad \text{or} \qquad x_t = \sum_{j=1}^{+\infty} g_t^{(j)},$$

where persistent components $g_t^{(j)}$ are now defined by

$$g_t^{(j)} = \sum_{k=0}^{+\infty} \sum_{q=0}^{N-1} \beta_{k,q}^{(j)} \hat{\varepsilon}_{t-kN^j - qN^{j-1}}$$

with

$$\beta_{k,q}^{(j)} = \frac{1}{\sqrt{N^{j-1}}} \left(\sum_{i=0}^{N^{j-1}-1} \alpha_{N^j k + N^{j-1} q + i} - \frac{1}{N} \sum_{i=0}^{N^j - 1} \alpha_{N^j k + i} \right).$$

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Moreover, the residual component at scale j is

$$\pi_t^{(j)} = \sum_{k=0}^{+\infty} \gamma_k^{(j)} \left(\frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \hat{\varepsilon}^{(j)}_{t-N^j k - N^{j-1} q} \right)$$

with

$$\gamma_k^{(j)} = \frac{1}{\sqrt{N^j}} \sum_{i=0}^{N^j-1} \alpha_{N^j k + i}.$$

The Extended Wold Decomposition in base 2 arises as a special case of this decomposition.

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2

Weak time-derivatives and no arbitrage pricing

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2.1 Introduction

The[1] no arbitrage pricing formula for the valuation of random payoffs is a milestone of asset pricing theory. For instance, it states that the *proper* price of a European option at time t is given by the conditional expectation of the discounted future payoff under a risk-neutral probability Q. If the derivative has payoff h_T at maturity T, a risk-neutral price $\pi(t)$ is

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-r(T-t)} h_T \right],$$

where r is a constant interest rate. From this formulation it is clear that the instantaneous rate r plays a fundamental role in the derivative pricing. Indeed, price dynamics are determined by the same interest rate process, whatever the terminal payoff of the option. This intuition goes back to Cox and Ross (1976), who derived the Black and Scholes (1973) result by exploiting the accounting relations between bonds, stocks and options. This line of reasoning actually stems from the original approach of Modigliani and Miller (1958).

In this paper we formalize the intuition that risk-neutral valuation is driven by the process of interest rates by generalizing the ordinary differential equation satisfied by the (deterministic) risk-free bond price $B(t) = e^{\int_0^t r(s)ds}$, namely

$$\frac{dB}{dt} = rB.$$

When the price process under consideration is stochastic, the issue of properly differentiating a random process arises. We address this problem by introducing the notion of *weak time-derivative* for adapted processes - denoted by $\mathcal{D}$ - and showing the basic rules of its calculus. By using this differential instrument, we prove that the no arbitrage pricing formula is the unique solution of the equation

$$\mathcal{D}\pi = r\pi \tag{2.1}$$

with the terminal condition $\pi(T) = h_T$. We prove this result in Theorems 2.15 and 2.17, where we suppose interest rates to be stochastic. The relation formalized by (2.1) stems from the foundations of asset pricing theory since it rephrases the equality between the instantaneous return of a risky asset - namely $\mathcal{D}\pi/\pi$ - and the instantaneous interest rate r, that needs to be valid in arbitrage-free markets. Indeed, by employing weak time-derivatives we are able to characterize no arbitrage - in particular, equivalent martingale measures - for a wide class of semimartingale price processes (see Proposition 2.14).

The definition of weak time-derivative requires a suitable set of test functions (see Definitions 2.1 and 2.2) and involves the conditional expectation of an adapted process. This instrument

[1] This is a joint work with Massimo Marinacci.

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provides a handy characterization of martingales with respect to a given probability measure. Indeed, as claimed in Proposition 2.3, the weak time-derivative of a stochastic process is null if and only if the process is a martingale. Hence, the weak time-derivative allows us to qualify martingales as the stochastic counterpart of constant sequences in deterministic settings.

The parallel between calculus of weak time-derivatives - that we develop in Subsections 2.2.3 and 2.2.4 - and deterministic differential calculus is even deeper. For instance, submartingality and supermartingality are monotonicity properties related to positive or negative signs of weak time-derivatives. In terms of interpretation, the weak time-derivative provides an indication of the upward or downward growth rate of the conditional expectation of a random process.

More generally, by Theorem 2.6 the set of weakly time-differentiable processes coincides with a large class of special semimartingales. Specifically, weak time-derivatives capture the drift of processes under consideration, a useful property when canonical decompositions are unknown. Furthermore, the identification of the drift and of the martingale component reminds of the logic behind Girsanov Theorem and, in fact, is crucial for the analysis of eq. (2.1).

A nice feature of the weak time-derivative is that it applies to any adapted processes, without requiring the Feller property (as the *infinitesimal generator*) or Markovianity (as the *extended infinitesimal generator*). In addition, under suitable assumptions, the weak time-derivative specializes to both these notions, as we illustrate in Corollaries 2.10 and 2.11. In fact, the weak time-derivative allows us to deal with generalized formulations of problems that are usually formalized by these instruments.

As anticipated, the main results of the paper are summarized by Theorems 2.15 and 2.17, which show existence and uniqueness of the solution of eq. (2.1) under both deterministic and stochastic interest rates. In addition, Proposition 2.16 provides an interesting generalization of eq. (2.1) to cashflows.

When the risk-free rate is constant, by rewriting eq. (2.1) in operator form, we obtain a reformulation of the eigenvalue-eigenvector problem, analyzed by Hansen and Scheinkman (2009), which employs the weak time-derivative in place of the extended infinitesimal generator. Following Hansen and Scheinkman, we obtain in our setting a decomposition of the stochastic discount factor into a martingale and a transient component.

Our paper combines different areas of mathematical analysis and stochastic calculus. The overall approach comes from variational calculus, it exploits the theory of Sobolev spaces and weak formulations of differential equations. See, for example, Brezis (2010), Adams and Fournier (2003) and Lions (1971) for a comprehensive introduction to variational calculus, and Revuz and Yor (1999) for stochastic calculus.

From a financial point of view, our work builds directly on the foundations of no arbitrage

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asset pricing theory illustrated, for instance, in Björk (2004), Hansen and Richard (1987), Föllmer and Schied (2011) and Delbaen and Schachermayer (1998). In addition, our eigenvalue formulation refers to the long-term risk literature, in particular to Hansen and Scheinkman (2009) and related works, like Alvarez and Jermann (2005).

The paper is organized as follows. Section 2.2 develops the mathematical formalism of the weak time-derivative, proves its main properties (Subsections 2.2.3 and 2.2.4) and relates it to infinitesimal generators (Subsections 2.2.5 and 2.2.6). In Sections 2.3 and 2.5 we solve the no arbitrage pricing equation with deterministic and stochastic interest rates, respectively. A brief discussion of the special case of Black-Scholes model is presented in Subsection 2.3.1, while Subsection 2.3.2 discusses the risk-neutral pricing of cashflows. Section 2.4 deals with the eigenvalue-eigenvector problem and the decomposition of the stochastic discount factor. In particular, Subsection 2.4.1 compares the roles of weak time-derivative and infinitesimal generator in the eigenvalue-eigenvector formulation. Finally, all proofs are included in Appendix.

2.2 The weak time-derivative

After describing the technical framework, we define weak time-derivatives and illustrate their peculiar features. Then, in the last subsections we contrast weak time-derivatives with infinitesimal generators of Feller processes and with extended infinitesimal generators of Markov processes.

2.2.1 Filtration, measurability and identification

Given a probability space $(\Omega, \mathcal{F}, P)$ with strictly positive P, we fix $T > 0$, $N \in \mathbb{N}$ and consider a vectorial process $X = \{X_t\}_{t\in[0,T]}$ such that $X : [0,T] \times \Omega \longrightarrow \mathbb{R}^N$. In particular $X_t = \left[X_t^{(1)}, \ldots, X_t^{(N)}\right]'$ for all $t \in [0,T]$. For example, X_t may be a bunch of primary asset prices at time t. By convenience we will equivalently use both the notations X_t and $X(t)$. Moreover, $\mathbb{R}^N$ is endowed with the Borel σ-algebra and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ denotes the filtration generated by X.

We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfies the usual conditions, namely $\mathbb{F}$ is complete and right-continuous. Accordingly, we write $\mathcal{F}_t = \mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$. In addition, we assume left-continuity in T, i.e. $\mathcal{F}_T = \mathcal{F}_{T^-}$.

Throughout the paper, random variables are identified almost surely. Also inequalities between random variables are meant almost surely. Moreover, we identify stochastic processes up to modifications.

On the given filtered probability space, we consider processes $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ that are

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progressively measurable (progressive, for brevity), namely, for all $t \in [0,T]$, the function $[0,t] \times \Omega \longrightarrow \mathbb{R}$ that maps (τ,ω) to the valuation $u_\omega(\tau)$ is Borel$([0,t]) \otimes \mathcal{F}_t$-measurable (see Revuz and Yor (1999)). In particular, such processes are adapted. Since any measurable and adapted u admits a progressive modification - see Karatzas and Shreve (2012) - and we identify stochastic processes up to modifications, we will simply require adaptability.

Seen as function of time, our processes take values in $L^1(\mathcal{F}_T)$ and, with a small abuse of notation, we simply write $u : [0,T] \longrightarrow L^1(\mathcal{F}_T)$. We will also often require that u is L^1-continuous, meaning that, for every $t \in [0,T]$, $\mathbb{E}\left[\left|u(\tau) - u(t)\right|\right]$ tends to zero whenever τ goes to t.

Moreover, we denote as $C([0,T],\mathbb{R})$ the space of continuous functions $f : [0,T] \longrightarrow \mathbb{R}$ while $C_c^n([0,T],\mathbb{R})$ contains n-times continuously differentiable functions from $[0,T]$ to $\mathbb{R}$ with compact support.

We define the space $\mathcal{V}$ as

$$\mathcal{V} = \Bigg\{ \text{adapted } u : [0,T] \longrightarrow L^1(\mathcal{F}_T), \quad L^1\text{-right-continuous in } [0,T), \\ L^1\text{-left-continuous in } T, \quad \int_0^T \mathbb{E}\left[\left|u(\tau)\right|\right] d\tau < +\infty \Bigg\}. \tag{2.2}$$

As a result, any $u \in \mathcal{V}$ is Bochner integrable and the Bochner integral of u is the element of $L^1(\mathcal{F}_T)$ denoted by $\int_0^T u(\tau)\, d\tau$. Indeed, condition (2.2) is necessary and sufficient for Bochner integrability of u. In addition, progressive measurability ensures the joint measurability of u on $[0,T] \times \Omega$. This property guarantees that the Bochner integral of u coincides almost surely with the pathwise Lebesgue integral, that we will employ at some point. See Diestel and Uhl (1977) and Aliprantis and Border (2006) as general references.

Observe that the space $\mathcal{V}$ contains all martingales defined on $(\Omega,\mathcal{F},\mathbb{F},P)$. The one requirement to check is L^1-continuity. In fact, for any $t \in [0,T)$, if $\tau \to t^+$,

$$\mathbb{E}\left[\left|u(\tau) - u(t)\right|\right] = \mathbb{E}\left[\left|\mathbb{E}_\tau\left[u(T)\right] - \mathbb{E}_t\left[u(T)\right]\right|\right] = \mathbb{E}\left[\left|\mathbb{E}_\tau\left[u(T)\right] - \mathbb{E}_{t^+}\left[u(T)\right]\right|\right] \longrightarrow 0$$

by Lévy's Downward Theorem. Moreover, if $\tau \to T^-$,

$$\mathbb{E}\left[\left|u(\tau) - u(T)\right|\right] = \mathbb{E}\left[\left|\mathbb{E}_\tau\left[u(T)\right] - \mathbb{E}_{T^-}\left[u(T)\right]\right|\right] \longrightarrow 0$$

by Lévy's Upward Theorem. See, for instance, Williams (1991).

2.2.2 Weak time-differentiability

We have now all the instruments to introduce the concept of *weak time-differentiability* for processes in $\mathcal{V}$.

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Definition 2.1 Given $u \in \mathcal{V}$, we say that u is *weakly time-differentiable* when there exists $w \in \mathcal{V}$ such that for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = -\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T],\mathbb{R}\right).$$

In this case we call w a *weak time-derivative* of u.

Similarly, $u \in \mathcal{V}$ is said to be *twice weakly time-differentiable* when there exists $z \in \mathcal{V}$ such that for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[z\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = \int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi''(\tau)d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^2\left([t,T],\mathbb{R}\right).$$

Observe that, if $u \in \mathcal{V}$, the integrals

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \quad \text{and} \quad \int_t^T \mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau$$

are finite for any choice of $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$. As a result, Definition 2.1 is well-posed.

The definition of weak time-derivative combines a variational approach, employed for weak solutions of (deterministic) partial differential equations, with the information structure due to the filtered probability space. Hence, the weak time-derivative builds a bridge between the *information-free* setting of calculus of variations and the adaptability concerns of stochastic process.

The presence of indicator functions reveals that weak time-differentiability actually involves the conditional expectation of random processes. The definition is stated in terms of the physical measure P. However, weak time-differentiability may be established also with respect to other probability measures. This is the case of asset pricing applications in which risk-neutral measures equivalent to P are employed for no arbitrage pricing, as illustrated in Section 2.3. Moreover, measure changes associated with the choice of a convenient numéraire are common practices in option pricing and interest rate theory. See, e.g., Margrabe (1978) and Geman, El Karoui, and Rochet (1995), among the others. Therefore, the weak time-derivative may be a fruitful instrument for asset pricing applications also in these settings.

The weak time-derivative is unique, up to modifications.

Proposition 2.1 *Let $u \in \mathcal{V}$ be weakly time-differentiable. Then, the weak time-derivative of u is unique.*

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We denote by $\mathcal{D}u$ the weak time-derivative of u. Moreover, we introduce the space

$$\mathcal{W} = \{\text{weakly time-differentiable } u \in \mathcal{V}\}$$

with norm

$$\|u\|_{\mathcal{W}} = \int_0^T \mathbb{E}\left[\left|u(\tau)\right|\right] d\tau + \int_0^T \mathbb{E}\left[\left|\mathcal{D}u(\tau)\right|\right] d\tau.$$

When we consider deterministic processes, the weak time-derivative coincides with the classical derivative of calculus.

Proposition 2.2 *Let $u \in \mathcal{V}$, $g \in C\left([0,T],\mathbb{R}\right)$ and, for every $t \in [0,T]$*

$$u(t) = \int_0^t g(s)ds.$$

Then, $\mathcal{D}u = g$.

2.2.3 Calculus of the weak time-derivative

We start by establishing the equivalence between processes with null weak time-derivative and martingales.

Proposition 2.3 *A process u belongs to $\mathcal{W}$ and it has $\mathcal{D}u = 0$ if and only if it is a martingale.*

A simple corollary of Proposition 2.3 shows that, given a weak time-derivative w, all processes $u \in \mathcal{W}$ such that $\mathcal{D}u = w$ differ by a martingale.

Corollary 2.4 *Let $w \in \mathcal{V}$ be the weak time-derivative of $u_1 \in \mathcal{W}$. w is also the weak time-derivative of $u_2 \in \mathcal{W}$ if and only if*

$$u_2 = u_1 + m,$$

where m is a martingale

Moreover, the following result holds.

Proposition 2.5 *Let u be defined, for all $t \in [0,T]$, by*

$$u(t) = \int_0^t g(s)\, ds + m(t),$$

where $g \in \mathcal{V}$ and m is a martingale. Then, u belongs to $\mathcal{W}$ and $\mathcal{D}u = g$.

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Example 2.1 Assume that for every $t \in [0, T]$

$$u(t) = \alpha t + m(t) \tag{2.3}$$

with $\alpha \in \mathbb{R}$ and m a martingale. Then $\mathcal{D}u = \alpha$. In addition, by Corollary 2.4 all processes $u \in \mathcal{W}$ such that $\mathcal{D}u = \alpha$ may be written as in eq. (2.3). In other words, a process in $\mathcal{W}$ is the sum of a deterministic trend and a martingale if and only if it has constant weak time-derivative. In this case, the value of $\mathcal{D}u$ identifies the linear drift.

If $N = 1$ this feature may be retrieved, for instance, in the Black-Scholes model, where the stock price satisfies

$$X_t = X_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

with $r \in \mathbb{R}, \sigma > 0$ and W denoting a Wiener process under the risk-neutral measure Q. In fact, log prices are the sum of a deterministic drift and a martingale process, namely

$$log(X_t) = \log(X_0) + \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Here, the weak time-derivative of log prices properly captures the drift coefficient $r - \sigma^2/2$ under Q.

Example 2.2 When $N = 1$ another illustration of Proposition 2.5 comes from continuous Itô semimartingales (or generalized diffusions) as, for instance, the process X in $\mathcal{V}$ such that

$$X_t = X_0 + \int_0^t g(s)\, ds + \int_0^t h(s) dW_s,$$

where $g \in \mathcal{V}$, h is adapted and $\int_0^T \mathbb{E}\left[h^2(s)\right] ds$ finite. The stochastic differential of the process is usually written as

$$dX_t = g(t)dt + h(t)dW_t.$$

The Itô integral of h with respect to the Wiener process W is a martingale and it belongs to $\mathcal{W}$. As a result, the weak time-derivative of X is the drift g:

$$\mathcal{D}X = g.$$

Moreover, any process defined by $v(t) = f(X_t)$ for all $t \in [0, T]$ with f twice continuously differentiable is a continuous Itô semimartingale, too. The stochastic differential of v can, then, be derived by Itô's formula and the weak time-derivative of v is identified as the drift in the stochastic differential of v.

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Inspired by Example 2.2, we provide a characterization of weakly time-differentiable processes in terms of special semimartingales (see Protter (2004)). Although by Definition 2.1 weak time-differentiability is established through an integration by parts relation, this notion identifies a class of stochastic processes widely employed in dynamic asset pricing modelling.

Theorem 2.6 *$\mathcal{W}$ is the space of special semimartingales u such that*

$$u = a + m,$$

with $a(t) = \int_0^t g(s)ds$, $g \in \mathcal{V}$ and m a martingale.

It is now apparent that weak time-differentiability captures a class of special semimartingales that feature a (unique) absolutely continuous finite variation term and a (unique) local martingale term which is actually a martingale. The innovation of our approach relies on the fact that this class of processes is characterized via a synthetic differentiability condition that does not require the knowledge, ex-ante, of the canonical decomposition of the semimartingale into consideration. In other words, the weak time-derivative allows the identification of the drift term of semimartingales in $\mathcal{W}$ even when the canonical decomposition is not available.

In the next example we dig into the martingale term of Theorem 2.6. Indeed, in general contexts m combines a continuous martingale term with a pure-jump martingale.

Example 2.3 Set $N = 1$ and consider in the time interval $[0, T]$ the process X driven by the dynamics

$$\frac{dX_t}{X_{t^-}} = \mu dt + \sigma dW_t + dH_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, W is a Wiener process and H is a compound Poisson process. In particular, $H_t = \sum_{k=1}^{N_t} z_k$, i.e. $dH_t = z_{N_{t^-}+1} dN_t$, where N is a Poisson process with intensity λ and z_k are i.i.d. random variables independent of W and N such that $\mathbb{E}[z_k] = z$ and $z_k \geqslant -1$. This jump-diffusion process has been, firstly, employed by Merton (1976) for option pricing. A taxonomy of similar SDEs applied in asset valuation can be retrieved, for instance, in Platen and Bruti-Liberati (2010).

Although H is not a martingale in general, the *compensated* Poisson process $\hat{H}$ defined by $\hat{H}_t = Y_t - \lambda z t$ for all $t \in [0, T]$ is a martingale. Therefore, we can rewrite the dynamics of X as

$$\frac{dX_t}{X_{t^-}} = (\mu + \lambda z)dt + \sigma dW_t + d\hat{H}_t.$$

In the last formulation, the finite variation term - eventually captured by the weak time-derivative - is driven by the deterministic drift coefficient $\mu + \lambda z$. However, the martingale term is the sum of two process: a continuous martingale given by the Wiener process and a pure-jump martingale individuated by the compensated Poisson process.

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As Example 2.3 suggests, the decomposition of Theorem 2.6 holds for a wide class of Lévy processes. Beyond Merton's model, such processes are extensively employed for modelling asset price dynamics. Relevant examples are provided by Kou (2002), Barndorff-Nielsen (1997) and Carr, Geman, Madan, and Yor (2002), among the others.

2.2.4 Monotonicity and convexity

We now discuss the connection between monotonicity and sign of the weak time-derivative. In conditional terms, we can associate the positivity of $\mathcal{D}u$ with the increasing monotonicity of u. Similarly, a negative weak time-derivative reveals the decreasing monotonicity of u. Interestingly, these features can be related to *submartingality* and *supermartingality* respectively.

Proposition 2.7 *Let $u \in \mathcal{W}$. We have $\mathcal{D}u \geqslant 0$ if and only if, for every $t \in [0,T]$ and $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 \geqslant \tau_1$,*

$$\mathbb{E}_t\left[u\left(\tau_2\right)\right] \geqslant \mathbb{E}_t\left[u\left(\tau_1\right)\right].$$

In this case, u is a submartingale.

An analogous result holds with $\leqslant$ instead of $\geqslant$.

Next we focus on the increments of weak time-derivatives, i.e. we deal with convexity (or concavity). The following result shows that a process $u \in \mathcal{W}$ satisfies a convexity property when $\mathcal{D}u$ is increasing, after taking the conditional expectation.

Proposition 2.8 *Let $u \in \mathcal{W}$. For every $t \in [0,T]$ and $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$*

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right] \leqslant \mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]$$

if and only if, for every $t \in [0,T]$ and $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right] \leqslant \frac{\mathbb{E}_t\left[u\left(\tau_2\right)\right] - \mathbb{E}_t\left[u\left(\tau_1\right)\right]}{\tau_2 - \tau_1} \leqslant \mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right].$$

An analogous result holds with $\geqslant$ instead of $\leqslant$.

Summing up, the calculus of weak time-derivatives generalizes in a natural way some key results of standard differential calculus.

2.2.5 Comparison with the infinitesimal generator

We relate the notion of weak time-derivative with the one of *infinitesimal generator*, widely employed in stochastic calculus. A further comparison of the applications of both instruments in option pricing is discussed in Subsection 2.4.1.

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We begin by considering the infinitesimal incremental ratios of conditional expectations. These quantities converge to the weak time-derivative.

Proposition 2.9 *Let $u \in \mathcal{W}$ and $t \in [0,T]$. If for any $\tau \in [t,T]$*

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}$$

is convergent in L^1 when $h \longrightarrow 0^+$, then

$$\frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h} \xrightarrow{L^1} \mathcal{D}u\left(t\right) \qquad h \longrightarrow 0^+.$$

Fixing $t \in [0,T]$, the outcome of Proposition 2.9 can be restated as

$$\mathbb{E}_t\left[u\left(\tau\right)\right]-u\left(t\right)-\mathcal{D}u\left(t\right)\left(\tau-t\right) \xrightarrow{L^1} 0 \qquad \tau \longrightarrow t^+,$$

which is a first-order expansion of $\mathbb{E}_t\left[u\left(\tau\right)\right]$ in a right neighbourhood of t, with the limit taken in L^1.

As described in Revuz and Yor (1999), the infinitesimal generator of a Feller process X is the operator $\mathcal{A}$ that maps any continuous bounded function f belonging to the domain of $\mathcal{A}$ into the function $\mathcal{A}f$ such that, for any $t \in [0,T]$,

$$\mathcal{A}f\left(X_t\right) = \lim_{h \to 0^+} \frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h}.$$

The limit, here, is in the uniform topology over all states $\omega \in \Omega$ and $\mathcal{A}f$ is continuous and bounded.

Proposition 2.9 shows that the weak time-derivative in $\mathcal{W}$ generally works as the infinitesimal generator with the limit $h \longrightarrow 0^+$ taken in the L^1-norm without requiring the Feller property of the underlying process. We now show that the weak time-derivative and the infinitesimal generator coincide when they are both well-defined.

Corollary 2.10 *Let X be a Feller process. Let $u \in \mathcal{W}$ be such that, for every $t \in [0,T]$, $u\left(t\right) = f\left(X_t\right)$ with f continuous and bounded in the domain of $\mathcal{A}$. Then, for every $t \in [0,T]$,*

$$\mathcal{D}u\left(t\right) = \mathcal{A}f\left(X_t\right).$$

As we will see in Subsection 2.4.1, weak time-derivatives provide more general formulations of operator equations that are usually expressed through infinitesimal generators, such as the eigenvalue-eigenvector problem $\mathcal{A}f = rf$. This kind of generalization is made possible by the fact that both instruments provide a similar characterization of martingales. Indeed, the process $\left\{f\left(X_t\right)\right\}_{t\in[0,T]}$ is a martingale when the infinitesimal generator of f is null, as ensured by Proposition 1.6 in Chapter VII of Revuz and Yor (1999). This result actually shares the same insight of Proposition 2.3 that relates the martingale property to the nullity of weak time-derivatives.

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2.2.6 Comparison with the extended infinitesimal generator

To illustrate the relation between the weak time-derivative and the extended infinitesimal generator, we begin with a corollary of Proposition 2.5.

Corollary 2.11 *Let $u \in \mathcal{W}$. Then, the process v defined, for all $t \in [0,T]$, by*

$$v(t) = u(t) - u(0) - \int_0^t \mathcal{D}u(\tau)d\tau$$

is a martingale.

This result rephrases Dynkin's formula for Markov processes. See, e.g., Protter (2004). In particular, it ensures that, for all $t \in [0,T]$ and $\tau \in [t,T]$,

$$\mathbb{E}_t\left[u(\tau)\right] = u(t) + \mathbb{E}_t\left[\int_t^\tau \mathcal{D}u(s)ds\right].$$

In case X is a Markov process and $u(t) = f(X_t)$ for all $t \in [0,T]$, the martingality of v implied by Corollary 2.11 guarantees that the weak time-derivative of u coincides with the *extended infinitesimal generator* of f. In fact, the extended infinitesimal generator of a measurable function f of X_t is a measurable function g such that $g(X_t)$ is integrable over time and the process v defined, for all $t \in [0,T]$, by

$$v(t) = f(X_t) - f(X_0) - \int_0^t g(X_\tau)\, d\tau$$

is a martingale (see Revuz and Yor (1999)). We summarize this observation in a proper statement.

Corollary 2.12 *Let X_t be a Markov process. Let $u \in \mathcal{W}$ be such that, for every $t \in [0,T]$, $u(t) = f(X_t)$. Then, $\mathcal{D}u$ is the extended infinitesimal generator of f.*

Up to minor adjustments, the extended infinitesimal generator has been used by Hansen and Scheinkman (2009) for the analysis of pricing operators. In fact, the extended infinitesimal generator keeps the most relevant features of the infinitesimal generator (in particular the nullity for martingales) but it does not require the Feller property of the underlying process.

To recap, the weak time-derivative is a general instrument that allows the differentiation of adapted processes and it nicely compares with both the infinitesimal generator and the extended infinitesimal generator. First, the infinitesimal generator applies to Feller processes and its extended version involves Markov processes, while the weak time-derivative is defined for a wide class of special semimartingales. Note, moreover, that the weak time-derivative mostly relies on measure theoretical assumptions, while the infinitesimal generator requires more restrictive topological assumptions.

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2.3 No arbitrage pricing

We consider a continuous-time market over the time window $[0,T]$ with N risky assets, whose prices are collected in the vectorial process X, where $X_t = \left[X_t^{(1)}, \dots, X_t^{(N)}\right]'$. A risk-free security with price B such that $B_t = e^{rt}$ for all $t \in [0,T]$ is also traded. Following for instance Björk (2004), for any $t \in [0,T]$ we define the vector of relative asset prices $Z_t = X_t/B_t$. We then consider as environment the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by Z and P is the physical probability.

We assume that our price system satisfies the no free lunch with vanishing risk (NFLVR) condition and that relative asset prices Z are semimartingales. The dynamics of these processes are, indeed, compatible with NFLVR restrictions, as explained by Delbaen and Schachermayer (1994).

A portfolio strategy is given by any adapted $(N+1)$-dimensional process ϑ and its discounted value process is V^ϑ defined by

$$V_t^\vartheta = \vartheta_t^{(0)} + \sum_{i=1}^N \vartheta_t^{(i)} Z_t^{(i)}, \qquad t \in [0,T].$$

All portfolios under consideration are supposed to be *admissible*, namely there is $w \geqslant 0$ such that

$$\int_0^t \sum_{i=1}^N \vartheta_t^{(i)} dZ_t^{(i)} \geqslant -w$$

for all $t \in [0,T]$, and *self-financing*, i.e.

$$dV_t^\vartheta = \sum_{i=1}^N \vartheta_t^{(i)} dZ_t^{(i)}, \qquad t \in [0,T].$$

According to the First Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer (1998), NFLVR ensures that there exists a probability measure Q equivalent to P such that Z is a *sigma-martingale*. In other words, Z is the martingale transform of some martingale via an integrable predictable process (see Émery (1980)). Moreover, the measure Q does not need to be unique.

Remark 2.13 *We assume that at least one of the measures Q inferred by Delbaen and Schachermayer (1998) is an* equivalent martingale measure*, i.e. it makes Z a martingale process.*

From now on we consider one of these risk-neutral measures Q for the valuation of securities in the market. In particular, we move to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Integrability conditions, expectations and convergences are computed with respect to Q.

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Since relative prices are martingales under Q, each $Z^{(i)}$ belongs to $\mathcal{W}$. In the following, we look for discounted price processes of attainable payoffs in this space of special semimartingales. Moreover, any process u belongs to $\mathcal{W}$ if and only if u/B belongs to $\mathcal{W}$, a fact that will be apparent from the proofs of Proposition 2.14 and Theorem 2.15. Therefore, we concentrate directly on (non-discounted) payoff prices in $\mathcal{W}$. In general, special semimartingale prices are compatible with no free lunch conditions, as in Föllmer and Schweizer (1991) and Ansel and Stricker (1992), and they are included in the generic semimartingales required by Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998).

We consider a payoff h_T in $L^1(\mathcal{F}_T)$ associated, for instance, with a traded European derivative. We want to determine a price process π for h_T in $\mathcal{W}$, which is consistent with our arbitrage-free market, namely π/B is a martingale under Q.

Proposition 2.14 *Let Q be an equivalent martingale measure for Z and let $\pi \in \mathcal{W}$. The following are equivalent:*

(i) π is a no arbitrage price process;

(ii) $\mathcal{D}(\pi/B) = 0$;

(iii) $\mathcal{D}\pi = r\pi$.

In fact, Proposition 2.14 may be re-read as a characterization of equivalent martingale measures. Specifically, Q is an equivalent martingale measure if and only if it is equivalent to P and, employed in the weak time-derivative, it ensures that $\mathcal{D}\pi = r\pi$ for all asset prices π in the market. Intuitively, this property is reminiscent of the outcome of Girsanov Theorem in Black-Scholes model (illustrated in Subsection 2.3.1), which makes the drift of the stock price proportional to r under Q.

When the price process is deterministic, point (iii) constitutes the usual differential equation solved by the price of a riskless bond. In particular, the bond price satisfies the boundary problem

$$\begin{cases} \frac{dB}{dt}(t) = rB(t) & t \in [0,T) \\ B(T) = e^{rT} \end{cases}$$

where the classical time-derivative is employed. In words, with continuous compounding, the rate of change of $B(t)$ is proportional to $B(t)$ and the coefficient of proportionality coincides with r.

Proposition 2.14, in fact, establishes a differential relation which is satisfied by the no arbitrage pricing function π of any risky payoff. For instance, in the case of a European derivative, we

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formulate the boundary problem

$$\begin{cases} \mathcal{D}\pi(t) = r\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases} \tag{2.4}$$

where $h_T \in L^1(\mathcal{F}_T)$.

The financial interpretation of the problem is straightforward once we recall that $\mathcal{D}\pi$ is an adapted process and that the filtration $\mathbb{F}$ is right-continuous. In fact, for any $t \in [0,T]$, the infinitesimal variation $\mathcal{D}\pi(t)$ is known at time t and so the no arbitrage condition imposes that $\mathcal{D}\pi$ instantaneously behaves as the deterministic bond. The riskiness of π locally becomes immaterial. In other words, the rate of change of $\pi(t)$ must be proportional to $\pi(t)$ as it is for the riskless asset price. Equivalently, instantaneous returns of h_T, i.e. $\mathcal{D}\pi/\pi$, coincide with the risk-free rate r when arbitrages are forbidden.

We now show that there exists a unique solution of Problem (2.4) that coincides with the risk-neutral valuation formula of a payoff at time T under the measure Q.

Theorem 2.15 *There exists a unique solution π of Problem (2.4) in $\mathcal{W}$ and, for every t in $[0,T]$*

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_t^Q[h_T]. \tag{2.5}$$

To be coherent with the no arbitrage setting, Q must be an equivalent martingale measure for the extended market made by the securities with discounted prices Z and π/B. Therefore, the only possible no arbitrage price process is given by eq. (2.5) because it satisfies

$$\frac{\pi(t)}{B_t} = \mathbb{E}_t^Q\left[\frac{h_T}{B_T}\right] \tag{2.6}$$

for all $t \in [0,T]$. We refer to $\pi(t)$ as the *no arbitrage pricing function* (or *risk-neutral pricing function*) of any payoff h_T under Q.

The additional value of Theorem 2.15 with respect to the existing theory consists in the fact that π belongs to the space $\mathcal{W}$ and is characterized by its dynamics $\mathcal{D}\pi = r\pi$.

Suppose now that the interest rate is deterministic but time-dependent. Moreover, $r(t)$ is assumed to be Lebesgue measurable (over time) and bounded. Then, under additional technical assumptions, the no arbitrage pricing function

$$\pi(t) = e^{-\int_t^T r(s)ds}\mathbb{E}_t^Q[h_T]$$

is the unique solution of problem

$$\begin{cases} \mathcal{D}\pi(t) = r(t)\pi(t) & t \in [0,T) \\ \pi(T) = h_T. \end{cases}$$

We provide a detailed solution of the problem in Section 2.5, in the more general case of stochastic interest rates.

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2.3.1 Example: Black-Scholes model

Black and Scholes (1973) model involves a continuous-time financial market with a riskless bond with price B and a risky asset with price X. In the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the filtration $\mathbb{F}$ is generated by a P-Wiener process $\mathbf{W}$. The bond and stock prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t d\overline{W}_t,$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and $r \in \mathbb{R}$ is the risk-free rate. Girsanov Theorem ensures that there exists a probability measure Q equivalent to P under which the discounted stock price process is a martingale. According to the First Fundamental Theorem of Asset Pricing the market is, then, arbitrage-free. In particular, the dynamics of the stock price under Q are

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

where W is a Q-Wiener process. Hence, in this geometric Brownian motion setting, the risky security and the bond must share the same drift coefficient given by the interest rate r in order to exclude any arbitrage possibility (see, e.g., Björk (2004)). Specifically, the stock price is

$$X_t = X_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

and it is included in $\mathcal{V}$. Its integral representation is

$$X_t = X_0 + \int_0^t rX_s ds + \int_0^t \sigma X_s dW_s,$$

which individuates a continuous Itô semimartingale. Then, as in Example 2.2, the weak time-derivative captures the drift, that is

$$\mathcal{D}X_t = rX_t.$$

This relation turns out to be a restatement of the no arbitrage pricing equation of Problem (2.4) for the stock price under Q.

As described in the initial part of the section, the no arbitrage price process of a European derivative with $\mathcal{F}_T$-measurable payoff h_T is π given by eq. (2.5). In addition, in Black-Scholes model $\pi(t)$ is a deterministic function of t and X_t. Since the discounted price process $\left\{e^{-rt}\pi(t)\right\}_{t\in[0,T]}$ is also a Q-martingale, the drift of π is equal to r, too. This is the crucial outcome of no arbitrage, which is captured, more in general, by Problem (2.4), where no specific price dynamics are assumed. This is also the intuition that drives Cox and Ross (1976) derivation of Black-Scholes equation, which is based on a hedging strategy that exploits a locally riskless portfolio.

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2.3.2 Valuation of cashflows

The payoff of a European derivative with maturity T can be seen as a special cashflow in which there is a unique random payment at time T. Indeed, the no arbitrage theory described so far generalizes to the pricing of payoff streams.

In particular, we consider an adapted cashflow h such that $h : [0,T] \longrightarrow L^1(\mathcal{F}_T)$. We assume that h is Bochner integrable with respect to a finite measure μ on $[0,T]$ that weighs cashflows over time.

Given an equivalent martingale measure Q, the no arbitrage price process π of h is the expected discounted value of future cashflows under Q, i.e.

$$\pi(t) = \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m \mu(dm) \right] \tag{2.7}$$

for all $t \in [0,T]$. For example, if μ is a counting measure, the previous formula evaluates a finite number or a sequence of future payments. In case μ is absolutely continuous, we are pricing instead a continuous stream of payoffs. In the next statement we assume that μ is absolutely continuous with respect to Lebesgue measure on $[0,T]$ and we write $\mu(dm) = p_m dm$, denoting the Radon-Nikodym derivative by p_m.

We also assume that the process $h \cdot p$ belongs to $\mathcal{V}$. Hence, we are able to show that the risk-neutral pricing formula for cashflows satisfies the differential equation

$$\mathcal{D}\pi(t) = r\pi(t) - h_t p_t \qquad t \in [0,T],$$

where $\mathcal{D}\pi$ is the weak time-derivative of π under Q.

Proposition 2.16 *Let $h \cdot p \in \mathcal{V}$ and $\pi(t) = \mathbb{E}_t^Q \left[\int_t^T e^{-r(m-t)} h_m p_m dm \right]$. Then, π belongs to $\mathcal{W}$ and it solves the equation*

$$\mathcal{D}\pi(t) = r\pi(t) - h_t p_t \qquad t \in [0,T].$$

Observe that a term analogous to $-h_t p_t$ is added in Feynman-Kac equation when a stream of dividends is present (see, e.g., Duffie (2010)).

Intuitively, if h_t is null except for the time T and μ has mass concentrated in T, we retrieve as special case the differential equation of Problem (2.4) about individual payoffs T. However, a formal claim about this case requires the theory of distributions and it is beyond the scope of Proposition 2.16.

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2.4 An operator approach

In this section we define the spaces and operators that allow us to formalize Problem (2.4) as an eigenvalue-eigenvector problem.

We first introduce some notation, still in the framework of Section 2.3. We denote by L_T the Radon-Nikodym derivative of the risk-neutral measure Q with respect to the physical measure P. Setting $L_t = \mathbb{E}_t [L_T]$ for all $t \in [0, T]$, we rewrite the no arbitrage price at any time t as

$$\pi(t) = e^{-r(T-t)} \mathbb{E}_t \left[\frac{L_T}{L_t} h_T \right].$$

We can also restate the martingale property of discounted prices under Q by saying that the process $\left\{ e^{-rt} L_t \pi(t) \right\}_{t \in [0,T]}$ is a martingale under the physical measure. In addition, the measure Q induces a stochastic discount factor process S that, at any $t \in [0, T]$, takes the form $S_t = e^{-rt} L_t$. See, for instance, Hansen and Richard (1987) and Björk (2004) as general references on risk-neutral pricing.

The starting point of our derivation is the observation that the no arbitrage pricing function π is weakly time-differentiable infinitely many times. Indeed, $\mathcal{D}\pi$ belongs to $\mathcal{V}$ and it equals the original π except for the multiplicative constant r. Hence, $\mathcal{D}\pi$ is weakly time-differentiable, too. By defining the subspace of $\mathcal{W}$

$$\mathcal{Z} = \{\text{infinitely weakly time-differentiable } u \in \mathcal{V}\},$$

we have that $\pi \in \mathcal{Z}$. Moreover, the weak time-derivative defines a linear operator $\mathcal{D} : \mathcal{Z} \longrightarrow \mathcal{Z}$ that maps any $u \in \mathcal{Z}$ to $\mathcal{D}u$. Therefore, the differential equation of Problem (2.4) delivers the eigenvalue-eigenvector problem

$$\mathcal{D}\pi = r\pi, \qquad \pi \in \mathcal{Z}, \tag{2.8}$$

which rephrases the one faced by Hansen and Scheinkman (2009) where, instead of $\mathcal{D}$, the *extended generator* of the underlying Markov process is involved. In our setting Markovianity is not required and the no arbitrage pricing function π is an eigenfunction of the operator $\mathcal{D}$, defined through weak time-derivatives. Moreover, the process $\left\{ e^{-rt} L_t \pi(t) \right\}_t$ is a martingale under P.

Following Hansen and Scheinkman (2009), we choose a positive payoff h_T. The positivity of h_T is related to the requirement of π to be an eigenfunction related to the *principal eigenvalue* in Hansen and Scheinkman (2009). Indeed, Hansen and Scheinkman generalize the Perron-Frobenius theory (see Meyer (2000)) from the finite-state Markov chain setting to more abstract frameworks.

Then, we define

$$\hat{L}_t = e^{-rt} L_t \frac{\pi(t)}{\pi(0)},$$

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which still satisfies the martingale property. In addition, the stochastic discount factor S_t decomposes as

$$S_t = \hat{L}_t \frac{\pi(0)}{\pi(t)} = e^{-rt} \hat{L}_t \frac{\tilde{\pi}(0)}{\tilde{\pi}(t)},$$

where we define $\tilde{\pi}(t) = \mathbb{E}_t\left[L_{t,T} h_T\right]$. In the last decomposition $-r$ is referred to as the *growth rate* of S_t, $\hat{L}_t$ is the *martingale component* and $\tilde{\pi}(0)/\tilde{\pi}(t)$ is the *transient component*. However, the decomposition is not unique.

This kind of results has proved to be fruitful in the macro-financial literature. For instance, Alvarez and Jermann (2005) employ the last decomposition to quantify the dynamics of stochastic discount factors. Moreover, an application to the study of long-term risk-return trade-off for the valuation of cash flows is described in Hansen, Heaton, and Li (2008).

The use of weak time-derivatives allows to extend the applicability of Hansen-Scheinkman decomposition to a wide class of special semimartingales. The crucial point of the construction is the identification of martingales through null weak time-derivatives.

2.4.1 Comparison with the infinitesimal generator

As we saw in Proposition 2.9, the weak time-derivative provides a way to differentiate random processes which generalizes the infinitesimal generator for Feller processes X and the extended infinitesimal generator for Markov processes. By focusing on the first one, if the infinitesimal generator of f is null, then the process $\{f(X_t)\}_{t\in[0,T]}$ is a martingale, a fact that parallels Proposition 2.3. In particular, simple computations show that the no arbitrage pricing function of eq. (2.5) satisfies the eigenvalue-eigenvector problem $\mathcal{A}\pi = r\pi$. Hence, we can refer to $\mathcal{A}\pi = r\pi$ as a *strong form* eigenvalue-eigenvector problem, while Problem (2.4), rewritten as (2.8), defines a *generalized form*.

In addition, it holds that $\mathcal{A}\left(e^{-rt}\pi(t)\right) = 0$, hence the discounted price process $\left\{e^{-rt}\pi(t)\right\}_{t\in[0,T]}$ is a martingale under Q. By exploiting the terminal condition $\pi(T) = h_T$, this fact ensures that π is the unique solution of the problem in strong form.

In fact, we followed a parallel path of reasoning with weak time-derivatives, but with relevant differences: the class of processes involved and the continuity required (L^1 instead of uniform topology). An analogous remark is valid for the extended infinitesimal generator.

2.5 No arbitrage pricing with stochastic interest rates

We provide a refinement of our theory in order to solve the no arbitrage pricing differential equation when interest rates are stochastic.

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In this case we have two sources of randomness described by processes X and r defined on the probability space $(\Omega, \mathcal{F}, P)$ over the time interval $[0, T]$. As in Section 2.3, X is associated with the N-dimensional process of underlying stock prices. In addition, r is one-dimensional and represents stochastic instantaneous rates. Here, $B_t = e^{\int_0^t r(s)ds}$ for all $t \in [0, T]$ and normalized prices are defined by $Z_t = e^{-\int_0^t r(s)ds} X_t$. Hence, we consider the filtration generated by the pair (Z, r), which we denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$.

In this section we assign a stronger meaning to weak time-differentiability. To distinguish the new definition from that of Section 2.2 we add r-, which stays for *robust*. Indeed, *r-weak time-differentiability* involves a larger set of test functions than weak time-differentiability of Definition 2.1. Specifically, we employ as test functions all adapted processes in $C_c^1\left([t, T], L^\infty(\mathcal{F}_T)\right)$. This space is composed of functions $\varphi : [0, T] \longrightarrow L^\infty(\mathcal{F}_T)$ with compact support that are continuously differentiable in the following sense: for any $t \in [0, T]$, $\varphi(t) = \int_0^t \psi(\tau)d\tau$ with $\psi : [0, T] \longrightarrow L^\infty(\mathcal{F}_T)$ adapted, continuous and with compact support. Hence, r-weak time-differentiability is a stronger requirement than weak time-differentiability.

Definition 2.2 Given $u \in \mathcal{V}$, we say that u is *r-weakly time-differentiable* when there exists $w \in \mathcal{V}$ such that for every $t \in [0, T]$

$$\int_t^T \mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau = -\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau$$

$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t, T], L^\infty(\mathcal{F}_T)\right) \quad \textit{adapted.}$$

In this case we call w a *r-weak time-derivative* of u.

Definition 2.2 is well-posed because the integrals

$$\int_t^T \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau \quad \text{and} \quad \int_t^T \mathbb{E}\left[w(\tau)\,\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau$$

are finite for any choice of A_t and φ, as required. Indeed, φ and φ' are continuous functions that take values in $L^\infty(\mathcal{F}_T)$, so their image is bounded.

We finally define the space

$$\mathcal{W}_r = \{\text{r-weakly time-differentiable } u \in \mathcal{V}\}.$$

If u is r-weakly time-differentiable, it is also weakly time-differentiable because the test functions φ, that are random processes, may specialize to deterministic functions. This simple observation allows us to inherit some of the results of Section 2.2. For instance, the r-weak time-derivative is still unique. Moreover, if a process $u \in \mathcal{V}$ is r-weakly time-differentiable with $\mathcal{D}u = 0$, then it is a martingale.

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In our financial application, we assume that instantaneous rates define an adapted process $r : [0,T] \longrightarrow L^{\infty}(\mathcal{F}_T)$. In addition, we impose that interest rates are uniformly bounded over time, i.e. there is a positive $\tilde{R}$ such that

$$|r(t)| \leqslant \tilde{R} \qquad \forall t \in [0,T]$$

and that they are L^2-right-continuous in any $t \in [0,T)$ and L^2-left-continuous in T. Progressive measurability (which holds up to modifications) and boundedness ensure the Bochner integrability of r. As a result, the Bochner integral $\int_0^T r(\tau)d\tau$ is a well-defined object in $L^{\infty}(\mathcal{F}_T)$.

By following the same line of reasoning of Section 2.3, from NFLVR we infer the existence of a sigma-martingale measure Q, which we assume to be an equivalent martingale measure. Thus, we move to the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Since now interest rates are stochastic, under the measure Q the no arbitrage pricing differential equation is

$$\begin{cases} \mathcal{D}\pi(t) = r(t)\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases} \tag{2.9}$$

where we assume $h_T \in L^2(\mathcal{F}_T)$. Differently from Problem (2.4), now each $r(t) \in L^{\infty}(\mathcal{F}_t)$ and $\mathcal{D}$ represents the r-weak time-derivative. We now show the unique solution of this problem in $\mathcal{W}_r$.

Theorem 2.17 *Under the previous assumptions on r, there exists a unique solution π of Problem (2.9) in $\mathcal{W}_r$ and, for every t in $[0,T]$*

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} h_T \right]. \tag{2.10}$$

2.6 Conclusions

We introduced the weak time-derivative, a novel mathematical tool that allows us to differentiate stochastic processes in a more general way than infinitesimal generators. It provides easy characterizations of martingales and permits to formulate differential equations for random processes in weak form. Therefore, we expect this instrument to be suitable for different kinds of differential problems, beyond the ones discussed in this work.

As we described in the paper, a fruitful application of the weak time-derivative involves the solution of the no arbitrage pricing equation for random payoffs. In particular, the generalized form that we solve clarifies the central role of interest rates in driving the asset prices, with both deterministic and stochastic short-term rates. In addition, constant interest rates deliver an eigenvalue-eigenvector formulation of the risk-neutral pricing equation in full agreement with

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the long-term risk literature. Nevertheless, how to set up the analogous eigenvalue-eigenvector problem when interest rates are time-varying or stochastic still remains an open problem. Indeed, the candidate eigenvalue would be a function or a random process. Moreover, such a formulation should be able to generate a term structure of interest rates. We leave this question for future research.

Another promising direction of research comes from the analysis of price dynamics through the lenses of different risk-neutral measures, associated with specific numéraire changes (see Geman, El Karoui, and Rochet (1995)). Indeed, the risk-free rate r is an eigenvalue when the measure Q is employed. Under a different equivalent martingale measure (as, for instance, the forward measure in the context of stochastic rates), the first question to answer is whether a differential equation with the structure of eq. (2.4) is still valid. A plausible possibility is that the same dynamics are present, but the driving parameter is not r. Hence, the second question is the identification of the proper eigenvalue according to the employed measure. This approach, made possible by the flexibility of weak time-derivatives, opens a promising perspective on payoff valuation in various contexts, while remaining within the boundaries of no arbitrage theory. We will deal with these aspects in future works.

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Appendix

This Appendix includes all the proofs.

2.A Proofs

Lemma 2.18 *Let $f : [t, T] \longrightarrow \mathbb{R}$.*

(i) If f is bounded, nonnegative, with compact support and $\int_t^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c([t, T], \mathbb{R})$, then $f = 0$ a.e.

(ii) If f is measurable and $\int_t^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c([t, T], \mathbb{R})$, then $f = 0$ a.e.

Proof (i) If f is strictly positive on a set A with positive measure, consider the indicator function $\mathbf{1}_A$ and a sequence $\{U_n\}_n$ of continuous positive approximations of $\mathbf{1}_A$, obtained by convolution with a smooth positive kernel. As U_n converges to $\mathbf{1}_A$ in L^2,

$$0 \leqslant \int_t^T f(\tau)\mathbf{1}_A(\tau)d\tau = \lim_n \int_t^T f(\tau)U_n(\tau)d\tau = 0.$$

In consequence, f is null a.e.

(ii) Suppose that f is positive with compact support. For any $N > 0$ consider $f_N(s) = \min\{f(\tau), N\}$. Then

$$0 \leqslant \int_t^T f_N(\tau)g(\tau)d\tau \leqslant \int_t^T f(\tau)g(\tau)d\tau = 0.$$

Therefore, each f_N is null a.e. by (i) and so f is.

■

Proof of Proposition 2.1

Proof Let w and $\hat{w}$ be two weak time-derivatives of u. Then, for every $t \in [0, T]$

$$\int_t^T \mathbb{E}\left[\left\{w(\tau) - \hat{w}(\tau)\right\}\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = 0$$

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$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T],\mathbb{R}\right).$$

By Lemma 2.18 in Appendix, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}\left[\left\{w\left(\tau\right)-\hat{w}\left(\tau\right)\right\}\mathbf{1}_{A_t}\right]=\mathbb{E}\left[0\mathbf{1}_{A_t}\right]=0 \qquad \forall A_t \in \mathcal{F}_t.$$

Hence, the null process fits the definition of conditional expectation of $w\left(\tau\right)-\hat{w}\left(\tau\right)$ with respect to $\mathcal{F}_t$. Therefore, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}_t\left[w\left(\tau\right)\right]=\mathbb{E}_t\left[\hat{w}\left(\tau\right)\right].$$

Now, consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,\tau]$ such that $\tau_i \longrightarrow t^+$ and $\mathbb{E}_t\left[w\left(\tau_i\right)\right]=\mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ for all i. Since w is L^1-continuous from the right, $\mathbb{E}_t\left[w\left(\tau_i\right)\right]$ converges in L^1 to $w(t)$ as τ_i approaches t because

$$\mathbb{E}\left[\left|\mathbb{E}_t\left[w\left(\tau_i\right)\right]-w(t)\right|\right]=\mathbb{E}\left[\left|\mathbb{E}_t\left[w\left(\tau_i\right)-w(t)\right]\right|\right]\leqslant\mathbb{E}\left[\left|w\left(\tau_i\right)-w(t)\right|\right]\longrightarrow 0.$$

Simultaneously, $\mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ converges in L^1 to $\hat{w}(t)$. However, $\mathbb{E}_t\left[w\left(\tau_i\right)\right]$ and $\mathbb{E}_t\left[\hat{w}\left(\tau_i\right)\right]$ coincide a.s. Therefore, by uniqueness of the L^1-limit, $w\left(t\right)=\hat{w}\left(t\right)$. ■

Proof of Proposition 2.2

Proof As g is deterministic and continuous, $g \in \mathcal{V}$. Taken any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$, for every $t \in [0,T]$ we have

$$\begin{aligned}
\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \mathbb{E}\left[\int_0^\tau g(s)ds\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= \mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T\left(\int_0^\tau g(s)ds\right)\varphi'(\tau)d\tau \\
&= -\mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T g(\tau)\varphi(\tau)d\tau \\
&= -\int_t^T \mathbb{E}\left[g(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau
\end{aligned}$$

because φ has compact support. Hence, g is the weak time-derivative of u. ■

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Proof of Proposition 2.3

Proof Assume that u is a martingale. As observed in Subsection 2.2.1, u belongs to $\mathcal{V}$. Fixed $t \in [0,T]$, for all $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$,

$$\begin{aligned}\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \mathbb{E}\left[\mathbb{E}_t\left[u\left(\tau\right)\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \int_t^T \mathbb{E}\left[u(t)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \mathbb{E}\left[u(t)\mathbf{1}_{A_t}\right]\int_t^T \varphi'(\tau)d\tau \\ &= 0\end{aligned}$$

because φ is a function in $C_c^1\left([t,T],\mathbb{R}\right)$. As a result, $w(t)=0$ for all $t\in[0,T]$ satisfies the definition of weak time-derivative of u, i.e. $\mathcal{D}u = 0$.

Conversely, suppose that $u \in \mathcal{W}$ has $\mathcal{D}u = 0$. We first show that, given $t \in [0,T]$, $\mathbb{E}_t\left[u\left(\tau\right)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take a continuous function $\eta : [t,T] \longrightarrow \mathbb{R}$ with compact support such that $\int_t^T \eta(\tau)d\tau = 1$. Given a continuous function $\xi : [t,T] \longrightarrow \mathbb{R}$ with compact support, we define the function $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(s) = \xi(s) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(s).$$

k_ξ is continuous with compact support and $\int_t^T k_\xi(\tau)d\tau = 0$. Hence, k_ξ has a primitive K_ξ which is continuous with compact support. As $K_\xi \in C_c^1\left([t,T],\mathbb{R}\right)$, we employ it as a test function in the definition of weak time-derivative of u. Since $\mathcal{D}u = 0$, for all $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}0 &= \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\left\{\xi(s) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(s)\right\}ds \\ &= \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\xi(s)ds - \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\left(\int_t^T \xi(\tau)d\tau\right)\eta(s)ds \\ &= \int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\xi(\tau)d\tau - \int_t^T \left\{\int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds\right\}\xi(\tau)d\tau \\ &= \int_t^T \left\{\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds\right\}\xi(\tau)d\tau.\end{aligned}$$

By the density of continuous functions ξ, Lemma 2.18 in Appendix ensures that for a.e. $\tau \in [t,T]$

$$\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\eta(s)ds.$$

Since $\int_t^T \eta(s)ds = 1$, we can rewrite the left-hand side as $\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\eta(s)ds$ so that

$$\int_t^T \left\{\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] - \mathbb{E}\left[u\left(s\right)\mathbf{1}_{A_t}\right]\right\}\eta(s)ds = 0.$$

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As the last equality holds for any continuous function η with compact support in $[t,T]$, we find that, for a.e. $s \in [t,T]$,

$$\mathbb{E}\left[u(s)\,\mathbf{1}_{A_t}\right] = \mathbb{E}\left[u(\tau)\,\mathbf{1}_{A_t}\right]$$

and so $\mathbb{E}_t\left[u(s)\right] = \mathbb{E}_t\left[u(\tau)\right]$. As a result, $\mathbb{E}_t\left[u(\tau)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and we can state that $\mathbb{E}_t\left[u(\tau)\right] = f_t$ for some $f_t \in L^1(\mathcal{F}_t)$.

Since u is L^1-right-continuous,

$$\mathbb{E}_t\left[u(\tau)\right] \xrightarrow{L^1} u(t) \qquad \tau \longrightarrow t^+.$$

Since for a.e. $\tau \in [t,T]$, $\mathbb{E}_t\left[u(\tau)\right]$ coincides a.s. with f_t, which is not dependent on τ, the uniqueness of the L^1-limit implies that $f_t = u(t)$. Therefore, for any $t \in [0,T]$, for a.e. $\tau \in [t,T]$

$$\mathbb{E}_t\left[u(\tau)\right] = u(t).$$

This property is actually satisfied by any $\tau \in [t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,T]$ such that $\tau_i \longrightarrow \tau^+$ and $\mathbb{E}_t\left[u(\tau_i)\right] = u(t)$. Since u is L^1-right-continuous, the L^1-limit of $\mathbb{E}_t\left[u(\tau_i)\right]$ is $\mathbb{E}_t\left[u(\tau)\right]$. Nevertheless, $\mathbb{E}_t\left[u(\tau_i)\right] = u(t)$ for all i and so, by uniqueness of the L^1-limit, $\mathbb{E}_t\left[u(\tau)\right] = u(t)$. ■

Proof of Corollary 2.4

Proof Assume that w is also the weak time-derivative of $u_2 \in \mathcal{W}$ and consider the process $m = u_2 - u_1 \in \mathcal{W}$. The weak time-derivative of m is null and so, by Proposition 2.3, m is a martingale.

The converse implication follows from the fact that the weak time-derivative of any martingale is null. ■

Proof of Proposition 2.5

Proof As $g \in \mathcal{V}$, g is Bochner integrable. Thus, for all $t \in [0,T]$, the process $G(t) = \int_0^t g(s)ds$ is well-defined and adapted.

Firstly, observe that

$$\begin{aligned}\int_0^T \mathbb{E}\left[\left|G(\tau)\right|\right] d\tau &\leqslant \int_0^T \mathbb{E}\left[\int_0^\tau \left|g(s)\right| ds\right] d\tau \leqslant \int_0^T \mathbb{E}\left[\int_0^T \left|g(s)\right| ds\right] d\tau \\ &= T\mathbb{E}\left[\int_0^T \left|g(s)\right| ds\right] = T\int_0^T \mathbb{E}\left[\left|g(s)\right|\right] ds,\end{aligned}$$

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which is finite because $g \in \mathcal{V}$. Here, the exchange between the order of expectation (which is a bounded operator) and Bochner integral is made possible by Lemma 11.45 in Aliprantis and Border (2006).

Secondly, for any $t, \tau \in [0,T]$,

$$\mathbb{E}\left[\left|G(\tau) - G(t)\right|\right] \leqslant \mathbb{E}\left[\int_{\min\{\tau,t\}}^{\max\{\tau,t\}} \left|g(s)\right| ds\right] = \int_{\min\{\tau,t\}}^{\max\{\tau,t\}} \mathbb{E}\left[\left|g(s)\right|\right] ds$$

and the last quantity tends to zero as τ approaches t, ensuring L^1-continuity. As a result, G belongs to $\mathcal{V}$.

We now show that $\mathcal{D}G = g$. Given $t \in [0,T]$, consider any $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$. Then

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau = \int_t^T \mathbb{E}\left[\int_0^\tau g\left(s\right)ds\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau.$$

We exchange the order of expectation and Bochner integral and, later, we apply Fubini's Theorem and we exploit the compact support of φ:

$$\begin{aligned}
\int_t^T \mathbb{E}\left[G\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \left(\int_0^\tau \mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\varphi'(\tau)\right]ds\right)d\tau \\
&= \int_0^t \left(\int_t^T \mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\right)ds \\
&\quad + \int_t^T \left(\int_s^T \mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\right)ds \\
&= \int_0^t \left(\mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\right]\int_t^T \varphi'(\tau)d\tau\right)ds \\
&\quad + \int_t^T \left(\mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\right]\int_s^T \varphi'(\tau)d\tau\right)ds \\
&= -\int_t^T \mathbb{E}\left[g\left(s\right)\mathbf{1}_{A_t}\right]\varphi(s)ds.
\end{aligned}$$

In consequence, g is the weak time-derivative of G.

As for m, this process belongs to $\mathcal{W}$ and $\mathcal{D}m = 0$ (see Proposition 2.3). Therefore, by additivity, $u \in \mathcal{W}$ and $\mathcal{D}u = g$. ■

Proof of Theorem 2.6

Proof Take $u \in \mathcal{W}$ and let $\mathcal{D}u$ denote its weak time-derivative. For any $t \in [0,T]$ define

$$a(t) = \int_0^t \mathcal{D}u(s)ds.$$

$a \in \mathcal{W}$ and it has the same weak time-derivative of u. Therefore, by Corollary 2.4, $m = u - a$ is a martingale. As a result, u decomposes into the sum $u = a + m$. Here, a has finite variation

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(because $\mathcal{D}u$ is integrable), it is càdlàg (because its paths are continuous) and adapted. Moreover, m is a local martingale (because it is a martingale). These features make u a semimartingale. In addition, a is also predictable (because it is left-continuous and adapted). As a result, u is a special semimartingale.

Conversely, let u be a special semimartingale such that, for all $t \in [0,T]$

$$u(t) = \int_0^t g(s)ds + m(t),$$

with $g \in \mathcal{V}$ and m a martingale. By Proposition 2.5, u belongs to $\mathcal{W}$ and $\mathcal{D}u = g$. ■

Proof of Proposition 2.7

Proof As in the proof of Proposition 2.3, we consider the continuous with compact support functions $\eta, \xi : [t,T] \longrightarrow \mathbb{R}$, with $\int_t^T \eta(s)ds = 1$ and we define $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(\tau) = \xi(\tau) - \left(\int_t^T \xi(s)ds\right)\eta(\tau).$$

The primitive

$$K_\xi(\tau) = \int_t^\tau \xi(s)ds - \left(\int_t^T \xi(s)d\tau\right)\int_t^\tau \eta(s)ds$$

belongs to $C_c^1\left([t,T],\mathbb{R}\right)$ and we employ it as a test function in the definition of weak time-derivative of u. In addition, we require that $\int_t^T \xi(s)ds = 1$, so that we consider

$$K_\xi(\tau) = \int_t^\tau \left(\xi(s) - \eta(s)\right)ds, \qquad\qquad k_\xi(\tau) = \xi(\tau) - \eta(\tau).$$

As $u \in \mathcal{W}$, for all $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$ and $A_t \in \mathcal{F}_t$, we have

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\left(\xi(\tau) - \eta(\tau)\right)d\tau$$
$$= -\int_t^T \mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]\left(\int_t^\tau \left(\xi(s) - \eta(s)\right)ds\right)d\tau.$$

By Fubini's Theorem,

$$\int_t^T \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\left(\xi(\tau) - \eta(\tau)\right)d\tau$$
$$= -\int_t^T \left(\int_s^T \mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]d\tau\right)\left(\xi(s) - \eta(s)\right)ds$$

so that

$$\int_t^T \left(\mathbb{E}\Big[u\left(\tau\right)\mathbf{1}_{A_t}\Big] + \int_\tau^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds\right)\xi(\tau)d\tau$$
$$= \int_t^T \left(\mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{A_t}\right] + \int_\tau^T \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right]ds\right)\eta(\tau)d\tau.$$

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By the density of continuous functions ξ, η, Lemma 2.18 in Appendix implies that for a.e. $\tau_1, \tau_2 \in [t,T]$

$$\mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{A_t}\right] + \int_{\tau_1}^{T} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds = \mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{A_t}\right] + \int_{\tau_2}^{T} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds,$$

namely

$$\mathbb{E}\left[\left(u\left(\tau_2\right) - u\left(\tau_1\right)\right)\mathbf{1}_{A_t}\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds. \tag{2.11}$$

If $\mathcal{D}u \geqslant 0$, then, for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 \geqslant \tau_1$,

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds \geqslant 0$$

for any $A_t \in \mathcal{F}_t$ and so $\mathbb{E}\left[\left(u\left(\tau_2\right) - u\left(\tau_1\right)\right)\mathbf{1}_{A_t}\right] \geqslant 0$. Since this holds for any $\mathcal{F}_t$-measurable set A_t, we infer that

$$\mathbb{E}_t\left[u\left(\tau_2\right) - u\left(\tau_1\right)\right] \geqslant 0$$

for a.e. $\tau_1, \tau_2 \in [t,T]$ with $\tau_2 \geqslant \tau_1$.

We are left to show that the last inequality is satisfied by all $\tau_1, \tau_2 \in [t,T]$ with $\tau_2 \geqslant \tau_1$. Indeed, suppose by contradiction that there exists a pair $\tau_1, \tau_2 \in [t,T]$ such that $\tau_2 > \tau_1$ and

$$\mathbb{E}_t\left[u\left(\tau_2\right)\right]\mathbf{1}_{E_t} < \mathbb{E}_t\left[u\left(\tau_1\right)\right]\mathbf{1}_{E_t},$$

where E_t is a $\mathcal{F}_t$-measurable set with positive measure. Then,

$$\mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{E_t}\right] < \mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{E_t}\right].$$

Now take into consideration a sequence $\{t_i\}_{i\in\mathbb{N}} \subset [\tau_1, \tau_2]$ such that $t_i \longrightarrow \tau_1^+$ and

$$\mathbb{E}_t\left[u\left(\tau_2\right)\right] \geqslant \mathbb{E}_t\left[u\left(t_i\right)\right].$$

This ensures that

$$\mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{E_t}\right] \geqslant \mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{E_t}\right].$$

Since u is L^1-right-continuous,

$$\mathbb{E}_t\left[u\left(t_i\right)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u\left(\tau_1\right)\right] \qquad t_i \longrightarrow \tau_1^+$$

and so

$$\mathbb{E}_t\left[u\left(t_i\right)\right]\mathbf{1}_{E_t} \xrightarrow{L^1} \mathbb{E}_t\left[u\left(\tau_1\right)\right]\mathbf{1}_{E_t} \qquad t_i \longrightarrow \tau_1^+.$$

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As a result,

$$\mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{E_t}\right] \longrightarrow \mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{E_t}\right] \qquad t_i \longrightarrow \tau_1^+,$$

where we deal with a sequence of real numbers. Since $\mathbb{E}\left[u\left(\tau_1\right)\mathbf{1}_{E_t}\right] > \mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{E_t}\right]$, by permanence of sings we can find an index $\hat{i}$ such that

$$\mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{E_t}\right] > \mathbb{E}\left[u\left(\tau_2\right)\mathbf{1}_{E_t}\right]$$

for all $i > \hat{i}$. Therefore, we obtain a contradiction.

Conversely, if $\mathbb{E}_t\left[u\left(\tau_2\right) - u\left(\tau_1\right)\right] \geqslant 0$ for every $\tau_1, \tau_2 \in [t, T]$ such that $\tau_2 \geqslant \tau_1$, then

$$\int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds \geqslant 0.$$

Since this holds for any $\tau_1, \tau_2 \in [t, T]$, it follows that, for a.e. $\tau \in [t, T]$

$$\mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right] \geqslant 0.$$

As A_t is any $\mathcal{F}_t$-measurable set, we have

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \geqslant 0.$$

By L^1-right-continuity of du/dt, as $\tau \longrightarrow t^+$,

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \xrightarrow{L^1} \mathcal{D}u\left(t\right).$$

As a result, $\mathcal{D}u \geqslant 0$ for all $t \in [0, T]$.

We are left to show that, in this case, u is a submartingale: for every $t \in [0, T]$ and $\tau \in [t, T]$

$$\mathbb{E}_t\left[u(\tau)\right] \geqslant u(t).$$

By contradiction, suppose that there exists $\tau \in (t, T]$ and a set $F_t \in \mathcal{F}_t$ with $P\left(F_t\right) > 0$ such that

$$\mathbb{E}_t\left[u(\tau)\right]\mathbf{1}_{F_t} < u(t)\mathbf{1}_{F_t}$$

This implies that

$$\mathbb{E}\left[u(\tau)\mathbf{1}_{F_t}\right] < \mathbb{E}\left[u(t)\mathbf{1}_{F_t}\right].$$

Now, consider a sequence $\{t_i\}_{i\in\mathbb{N}} \subset [t, \tau]$ such that $t_i \longrightarrow t^+$. For all i we have

$$\mathbb{E}_t\left[u(\tau)\right] \geqslant \mathbb{E}_t\left[u\left(t_i\right)\right].$$

It follows that

$$\mathbb{E}\left[u(\tau)\mathbf{1}_{F_t}\right] \geqslant \mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{F_t}\right].$$

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Since u is L^1-continuous from the right,

$$\mathbb{E}_t\left[u\left(t_i\right)\right] \xrightarrow{L^1} u\left(t\right) \qquad t_i \longrightarrow t^+$$

and so

$$\mathbb{E}_t\left[u\left(t_i\right)\right]\mathbf{1}_{F_t} \xrightarrow{L^1} u\left(t\right)\mathbf{1}_{F_t} \qquad t_i \longrightarrow t^+.$$

In consequence,

$$\mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{F_t}\right] \longrightarrow \mathbb{E}\left[u\left(t\right)\mathbf{1}_{F_t}\right] \qquad t_i \longrightarrow t^+,$$

which is the limit of a sequence of real numbers. As observed at the beginning,

$$\mathbb{E}\left[u\left(t\right)\mathbf{1}_{F_t}\right] > \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{F_t}\right]$$

and so, by permanence of signs, it is possible to find an index $\hat{i}$ such that, for all $i > \hat{i}$,

$$\mathbb{E}\left[u\left(t_i\right)\mathbf{1}_{F_t}\right] > \mathbb{E}\left[u\left(\tau\right)\mathbf{1}_{F_t}\right].$$

Hence, we meet a contradiction. ■

Proof of Proposition 2.8

Proof Following the proof of Proposition 2.7, in particular eq. (2.11), given $t \in [0,T]$, for a.e. $\tau_1, \tau_2 \in [t,T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[u\left(\tau_2\right) - u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(s\right)\right]\mathbf{1}_{A_t}\right] ds.$$

Let $\tau_1 \leqslant \tau_2$. If

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right] \leqslant \mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right],$$

we have

$$\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] \leqslant \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right]$$

and this monotonicity ensures that

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] &\leqslant \frac{\int_{\tau_1}^{\tau_2}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(s\right)\right]\mathbf{1}_{A_t}\right]ds}{\tau_2 - \tau_1} \\ &\leqslant \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

By the initial equality,

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right] &\leqslant \frac{\mathbb{E}\left[\mathbb{E}_t\left[u\left(\tau_2\right) - u\left(\tau_1\right)\right]\mathbf{1}_{A_t}\right]}{\tau_2 - \tau_1} \\ &\leqslant \mathbb{E}\left[\mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right]\mathbf{1}_{A_t}\right].\end{aligned}$$

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As this holds for any $A_t \in \mathcal{F}_t$, we deduce that

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right] \leqslant \frac{\mathbb{E}_t\left[u\left(\tau_2\right)\right] - \mathbb{E}_t\left[u\left(\tau_1\right)\right]}{\tau_2 - \tau_1} \leqslant \mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right].$$

Conversely, if the last inequality holds, it is clear that

$$\mathbb{E}_t\left[\mathcal{D}u\left(\tau_1\right)\right] \leqslant \mathbb{E}_t\left[\mathcal{D}u\left(\tau_2\right)\right].$$

Although the results written so far hold for a.e. $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$, by exploiting L^1-continuity as in the proof of Proposition 2.7, we obtain that all conclusions are satisfied for every $\tau_1, \tau_2 \in [t,T]$ such that $\tau_1 \leqslant \tau_2$. ■

Proof of Proposition 2.9

Proof We first show that, for a.e. $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right) - u\left(\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \qquad h \longrightarrow 0^+. \tag{2.12}$$

By following the same steps of the proof of Proposition 2.7 we find that, for a.e. $\tau, \hat{\tau} \in [t,T]$, for every $\mathcal{F}_t$-measurable set A_t

$$\mathbb{E}\left[\mathbb{E}_t\left[u\left(\hat{\tau}\right) - u\left(\tau\right)\right]\mathbf{1}_{A_t}\right] = \int_\tau^{\hat{\tau}} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds.$$

By setting $\hat{\tau} = \tau + h$ for some $h > 0$, we have

$$\mathbb{E}\left[\frac{\mathbb{E}_t\left[u\left(\tau+h\right) - u\left(\tau\right)\right]}{h}\mathbf{1}_{A_t}\right] = \frac{1}{h}\int_\tau^{\tau+h} \mathbb{E}\left[\mathcal{D}u\left(s\right)\mathbf{1}_{A_t}\right] ds.$$

Now we take the limit as $h \longrightarrow 0^+$. By Lebesgue Differentiation Theorem, the right-hand side converges to $\mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]$. Moreover, if $w\left(\tau\right)$ denotes the $\mathcal{F}_t$-measurable L^1-limit of $\frac{\mathbb{E}_t\left[u(\tau+h)-u(\tau)\right]}{h}$, the left-hand side converges to $\mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right]$. Consequently,

$$\mathbb{E}\left[w\left(\tau\right)\mathbf{1}_{A_t}\right] = \mathbb{E}\left[\mathcal{D}u\left(\tau\right)\mathbf{1}_{A_t}\right]$$

for every $\mathcal{F}_t$-measurable set A_t. Hence, by definition of conditional expectation,

$$w\left(\tau\right) = \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right].$$

As a result, the convergence in eq. (2.12) is proved.

Now recall that, since u and $\mathcal{D}u$ are L^1-right-continuous, as $\tau \longrightarrow t^+$,

$$\mathbb{E}_t\left[u\left(\tau\right)\right] \xrightarrow{L^1} u\left(t\right), \qquad \mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right] \xrightarrow{L^1} \mathcal{D}u\left(t\right).$$

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Also, the fact that, for any $\tau \in [t,T]$,

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h} \xrightarrow{L^1} w\left(\tau\right) \qquad h \longrightarrow 0^+$$

ensures that, for any $\tau \in [t,T]$,

$$\mathbb{E}_t\left[u\left(\tau+h\right)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u\left(\tau\right)\right] \qquad h \longrightarrow 0^+.$$

Indeed, for every $h>0$,

$$\begin{aligned}\mathbb{E}\Big[\Big|\mathbb{E}_t[u\left(\tau+h\right)-u\left(\tau\right)]\Big|\Big] &= h\mathbb{E}\left[\frac{\left|\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]\right|}{h}\right] \\ &\leqslant h\left\{\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}-w\left(\tau\right)\right|\right]+\mathbb{E}\left[\left|w\left(\tau\right)\right|\right]\right\}\end{aligned}$$

and this quantity converges to 0 as $h \longrightarrow 0^+$ since $w(\tau)$ is in L^1. In particular, for any fixed $h>0$, we have

$$\mathbb{E}_t\left[u\left(\tau+h\right)\right] \xrightarrow{L^1} \mathbb{E}_t\left[u\left(t+h\right)\right] \qquad \tau \longrightarrow t^+$$

and so

$$\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h} \xrightarrow{L^1} \frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h} \qquad \tau \longrightarrow t^+.$$

Putting things together, for any $\tau \in [t,T], h>0$ we have

$$\begin{aligned}&\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h}-\mathcal{D}u\left(t\right)\right|\right] \\ &\quad\leqslant \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h}-\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}\right|\right] \\ &\qquad+\mathbb{E}\left[\left|-\mathcal{D}u\left(t\right)+\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right]\right|\right] \\ &\qquad+\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u\left(\tau+h\right)-u\left(\tau\right)\right]}{h}-\mathbb{E}_t\left[\mathcal{D}u\left(\tau\right)\right]\right|\right].\end{aligned}$$

The previous convergences allow us to choose $\tau \in [t,T]$ in a way that the first two terms in the right-hand side are arbitrarily small and the convergence in eq. (2.12) allows us to choose h so that the last term is arbitrarily little. Hence, when $h \longrightarrow 0^+$,

$$\frac{\mathbb{E}_t\left[u\left(t+h\right)\right]-u\left(t\right)}{h} \xrightarrow{L^1} \mathcal{D}u\left(t\right).$$

■

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Proof of Corollary 2.10

Proof The function f is continuous and bounded, $\mathcal{A}f$ is continuous and bounded and $\mathcal{A}f(X_t)$ belongs to L^1. Since, for every $\tau \in [t,T]$,

$$\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}$$

converges to $\mathcal{A}f(X_\tau)$ as $h \longrightarrow 0^+$ in the uniform topology,

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right] \qquad h \longrightarrow 0^+.$$

Indeed, since f is in the domain of the infinitesimal generator $\mathcal{A}$, we can find an arbitrary small $\varepsilon > 0$ such that

$$\begin{aligned}\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right| &\leqslant \left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}-\mathcal{A}f\left(X_\tau\right)\right|+\left|\mathcal{A}f\left(X_\tau\right)\right| \\ &\leqslant \sup_{\omega\in\Omega}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}-\mathcal{A}f\left(X_\tau\right)\right|+\left|\mathcal{A}f\left(X_\tau\right)\right| \\ &\leqslant \varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|.\end{aligned}$$

By the Conditional Dominated Convergence Theorem, when $h \longrightarrow 0^+$

$$\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right] \xrightarrow{a.s.} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right],$$

that is

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{a.s.} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right].$$

Moreover,

$$\begin{aligned}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\right| &= \left|\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right]\right| \\ &\leqslant \mathbb{E}_t\left[\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right|\right] \\ &\leqslant \mathbb{E}_t\left[\varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|\right] \\ &= \varepsilon+\mathbb{E}_t\left[\left|\mathcal{A}f\left(X_\tau\right)\right|\right].\end{aligned}$$

Therefore, by the Dominated Convergence Theorem, for every $t \in [0,T]$, for every $\tau \in [t,T]$

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right] \qquad h \longrightarrow 0^+.$$

In particular,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \mathcal{A}f\left(X_t\right) \qquad h \longrightarrow 0^+.$$

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Since $\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f(X_\tau)}{h}$ is convergent in L^1 as $h \longrightarrow 0^+$ for every $t \in [0,T]$ and every $\tau \in [t,T]$, Proposition 2.9 applies. In consequence,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h} \xrightarrow{L^1} \mathcal{D}u\left(t\right) \qquad h \longrightarrow 0^+.$$

By uniqueness of the L^1-limit, we infer that

$$\mathcal{D}u\left(t\right) = \mathcal{A}f\left(X_t\right).$$

■

Proof of Corollary 2.11

Proof v belongs to $\mathcal{V}$ as discussed in the proof of Proposition 2.5. Still by Proposition 2.5, the weak time-derivative of the process $U(t) = \int_0^t \mathcal{D}u(s)ds$ is $\mathcal{D}u$. Since the weak time-derivative of $u(0)$ is null, by additivity we conclude that

$$\mathcal{D}v = \mathcal{D}u - 0 - \mathcal{D}u = 0.$$

In consequence, by Proposition 2.3, v is a martingale. ■

Proof of Corollary 2.12

Proof By Corollary 2.11, the process v defined by

$$v(t) = u(t) - u(0) - \int_0^t \mathcal{D}u(\tau)d\tau$$

is a martingale. Therefore, $\mathcal{D}u$ satisfies the definition of extended infinitesimal generator of f.

■

Proof of Proposition 2.14

Proof We first prove $(i) \Leftrightarrow (ii)$ and then $(ii) \Leftrightarrow (iii)$.

$(i) \Leftrightarrow (ii)$ π is a no arbitrage price process if and only if π/B is a martingale under Q. By Proposition 2.3, this is equivalent to claim that the weak time-derivative of π/B is null.

$(ii) \Leftrightarrow (iii)$ For any $t \in [0,T]$, $\pi(t)/B_t = e^{-rt}\pi(t)$. For any $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1\left([t,T],\mathbb{R}\right)$,

$$\begin{aligned}\int_t^T \mathbb{E}^Q\left[\mathcal{D}\pi\left(\tau\right)\mathbf{1}_{A_t}\right]e^{-r\tau}\varphi(\tau)d\tau &= -\int_t^T \mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\left(e^{-rt}\varphi(t)\right)'(\tau)d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\right]e^{-r\tau}\varphi'(\tau)d\tau \\ &\quad + \int_t^T \mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\right]e^{-r\tau}r\varphi(\tau)d\tau,\end{aligned}$$

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namely

$$\int_t^T \mathbb{E}^Q\left[e^{-r\tau}\left(\mathcal{D}\pi\left(\tau\right)-r\pi\left(\tau\right)\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\ = -\int_t^T \mathbb{E}^Q\left[e^{-r\tau}\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau.$$

As a result, the weak time-derivative of $e^{-rt}\pi\left(t\right)$ is such that

$$\mathcal{D}\left(e^{-rt}\pi\left(t\right)\right)=e^{-rt}\left(\mathcal{D}\pi\left(t\right)-r\pi\left(t\right)\right).$$

Therefore, the weak time-derivative of π/B is null if and only if $\mathcal{D}\pi=r\pi$.

■

Proof of Theorem 2.15

Proof • EXISTENCE

In order to show that $\pi\in\mathcal{W}$, we prove that π belongs to $\mathcal{V}$ and that it is weakly time-differentiable.

First, for all $\tau\in[0,T]$, $\pi(\tau)\in L^1\left(\mathcal{F}_\tau\right)$ because

$$\mathbb{E}^Q\left[\left|\pi\left(\tau\right)\right|\right]=e^{-r(T-\tau)}\mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[h_T\right]\right|\right]\leqslant e^{-r(T-\tau)}\mathbb{E}^Q\left[\left|h_T\right|\right],$$

which is finite since $h_T\in L^1\left(\mathcal{F}_T\right)$. In addition,

$$\int_0^T \mathbb{E}^Q\left[\left|\pi\left(\tau\right)\right|\right]d\tau\leqslant\int_0^T e^{-r(T-\tau)}\mathbb{E}^Q\left[\left|h_T\right|\right]d\tau \\ =\left(\int_0^T e^{-r(T-\tau)}d\tau\right)\mathbb{E}^Q\left[\left|h_T\right|\right],$$

which is finite, too.

As for the L^1-continuity, we check that, for any $t\in[0,T)$, $\mathbb{E}^Q\left[\left|\pi(\tau)-\pi(t)\right|\right]$ tends to zero as

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$\tau \longrightarrow t^+$. We have

$$\begin{aligned}
\mathbb{E}^Q\left[\left|\pi(\tau)-\pi(t)\right|\right] &= \mathbb{E}^Q\left[\left|e^{-r(T-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-e^{-r(T-t)}\mathbb{E}^Q_t\left[h_T\right]\right|\right] \\
&= e^{-r(T-t)}\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right] \\
&\leqslant e^{-r(T-t)}\Bigg\{\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_\tau\left[h_T\right]\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right]\Bigg\} \\
&\leqslant e^{-r(T-t)}\Bigg\{\left|e^{-r(t-\tau)}-1\right|\mathbb{E}^Q\left[\left|h_T\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-\mathbb{E}^Q_t\left[h_T\right]\right|\right]\Bigg\}.
\end{aligned}$$

In the last expression, both addends go to zero as τ approaches t from the right. In particular, the convergence of the second one is ensured by Lévy's Downward Theorem, which guarantees that

$$\mathbb{E}^Q_\tau\left[h_T\right] \xrightarrow{L^1} \mathbb{E}^Q_{t^+}\left[h_T\right] = \mathbb{E}^Q_t\left[h_T\right] \qquad \tau \longrightarrow t^+.$$

Similarly, when $\tau \longrightarrow T^-$,

$$\mathbb{E}^Q\left[\left|\pi(\tau)-h_T\right|\right] \leqslant \left\{\left|e^{-r(T-\tau)}-1\right|\mathbb{E}^Q\left[\left|h_T\right|\right] + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[h_T\right]-h_T\right|\right]\right\}.$$

Here, the convergence of the second term is due to Lévy's Upward Theorem:

$$\mathbb{E}^Q_\tau\left[h_T\right] \xrightarrow{L^1} \mathbb{E}^Q_{T^-}\left[h_T\right] = \mathbb{E}^Q_T\left[h_T\right] = h_T \qquad \tau \longrightarrow T^-.$$

Therefore π belongs to $\mathcal{V}$.

Now we look for the weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and $\varphi \in C^1_c\left([t,T],\mathbb{R}\right)$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$, we have

$$\begin{aligned}
-\int_t^T \mathbb{E}^Q\left[\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= -\int_t^T \mathbb{E}^Q\left[e^{-r(T-\tau)}\mathbb{E}^Q_\tau\left[h_T\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= -\int_t^T e^{-r(T-\tau)}\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\int_t^T e^{-r(T-\tau)}\varphi'(\tau)d\tau \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\right]\int_t^T re^{-r(T-\tau)}\varphi(\tau)d\tau \\
&= \int_t^T r\mathbb{E}^Q\left[e^{-r(T-\tau)}h_T\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\
&= \int_t^T \mathbb{E}^Q\left[r\pi\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}$$

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Therefore, the candidate weak time-derivative of π is $r\pi$. Since $r\pi$ belongs to $\mathcal{V}$, $r\pi$ is effectively the weak time-derivative of π:

$$\mathcal{D}\pi = r\pi.$$

Of course, $\pi(T) = h_T$ and so $\pi \in \mathcal{W}$ solves Problem (2.4).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}$ be two solutions of Problem (2.4), that is

$$\mathcal{D}\pi_i = r\pi_i \qquad i = 1, 2,$$

with the boundary condition $\pi_i(T) = h_T$. By defining $z = \pi_1 - \pi_2 \in \mathcal{W}$, we have that

$$\mathcal{D}z = rz$$

and $z(T) = 0$. As in the proof of Proposition 2.14, the weak time-derivative of $e^{-rt}z(t)$ is

$$e^{-rt}\left(\mathcal{D}z(t) - rz(t)\right).$$

However this process is null. Therefore, $e^{-rt}z(t)$ has null weak time-derivative. Consequently, by Proposition 2.3, $e^{-rt}z(t)$ defines a martingale and so, for any $t \in [0, T]$ and $\tau \in [t, T]$

$$\mathbb{E}_t^Q\left[z(\tau)\right] = e^{r(\tau - t)}z(t).$$

Letting τ go to T^-, we have that

$$\mathbb{E}_t^Q\left[z(\tau)\right] \longrightarrow e^{r(T-t)}z(t) \qquad \textit{pointwise.}$$

In addition, $z(\tau)$ converges to $z(T) = 0$ in $L^1(Q)$ as τ approaches T^- and so $\mathbb{E}_t^Q\left[z(\tau)\right]$ tends to zero in $L^1(Q)$. By uniqueness of the L^1-limit, we infer that $z(t) = 0$ for all $t \in [0, T]$. This proves uniqueness of the solution of Problem (2.4). ■

Proof of Proposition 2.16

Proof We denote $\mu(dm) = p_m dm$. We show that $\pi(t) = \mathbb{E}_t^Q\left[\int_t^T e^{-r(m-t)}h_m\mu(dm)\right]$ belongs to $\mathcal{V}$ and it is weakly time-differentiable.

First, for all $\tau \in [0, T]$, $\pi(\tau) \in L^1(\mathcal{F}_\tau)$:

$$\begin{aligned}
\mathbb{E}\left[\left|\pi(\tau)\right|\right] &= \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)}h_m\mu(dm)\right]\right|\right] \\
&\leqslant \mathbb{E}^Q\left[\int_\tau^T e^{-r(m-\tau)}\left|h_m\right|\mu(dm)\right] \\
&\leqslant K\mathbb{E}^Q\left[\int_\tau^T \left|h_m\right|\mu(dm)\right] < +\infty
\end{aligned}$$

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with $K > 0$. The last quantity is finite because h is Bochner integrable with respect to μ. Similarly,

$$\int_0^T \mathbb{E}^Q\left[\left|\pi\left(\tau\right)\right|\right] d\tau \leqslant TK\mathbb{E}^Q\left[\int_0^T \left|h_m\right| \mu(dm)\right] < +\infty.$$

To establish L^1-continuity, for any $t \in [0,T)$, consider $\tau \longrightarrow t^+$. We have

$$\begin{aligned}
&\mathbb{E}^Q\left[\left|\pi(\tau) - \pi(t)\right|\right] \\
&\quad = \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm)\right] - \mathbb{E}_t^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\quad \leqslant \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)} h_m \mu(dm)\right] - \mathbb{E}_\tau^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\qquad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_t^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\quad = \mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_\tau^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\qquad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_t^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right]
\end{aligned}$$

In the last sum, the second term converges to zero as τ approaches t by Lévy's Downward Theorem. Next, we show that also the first term vanishes:

$$\begin{aligned}
&\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-t)} h_m \mu(dm)\right] - \mathbb{E}_\tau^Q\left[\int_t^T e^{-r(m-t)} h_m \mu(dm)\right]\right|\right] \\
&\quad \leqslant \left|e^{-r(t-\tau)} - 1\right| \mathbb{E}^Q\left[\int_\tau^T e^{-r(m-t)} \left|h_m\right| \mu(dm)\right] \\
&\qquad + \mathbb{E}^Q\left[\int_t^\tau e^{-r(m-t)} \left|h_m\right| \mu(dm)\right] \\
&\quad \leqslant \left|e^{-r(t-\tau)} - 1\right| \mathbb{E}^Q\left[\int_t^T e^{-r(m-t)} \left|h_m\right| \mu(dm)\right] \\
&\qquad + \mathbb{E}^Q\left[\int_t^\tau e^{-r(m-t)} \left|h_m\right| \mu(dm)\right].
\end{aligned}$$

Bochner integrability of h ensures that the last quantities are well-defined and convergent to zero as $\tau \longrightarrow t^+$.

As for L^1-convergence from the left in T, we consider $\tau \longrightarrow T^-$. Since $\pi(T) = 0$, we get

$$\begin{aligned}
\mathbb{E}^Q\left[\left|\pi(\tau) - \pi(T)\right|\right] &= \mathbb{E}^Q\left[\left|e^{-r(T-\tau)}\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-T)} h_m \mu(dm)\right]\right|\right] \\
&\leqslant e^{-r(T-\tau)}\mathbb{E}^Q\left[\int_\tau^T e^{-r(m-T)} \left|h_m\right| \mu(dm)\right],
\end{aligned}$$

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where the last quantity goes to zero as τ tends to T from the left because of Bochner integrability of h.

In consequence, π belongs to $\mathcal{V}$.

Now we compute the weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1([t,T],\mathbb{R})$. The indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$ and so

$$\begin{aligned}
&-\int_t^T \mathbb{E}^Q\left[\pi(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&\qquad = -\int_t^T \mathbb{E}^Q\left[\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)}h_m\mu(dm)\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&\qquad = -\int_t^T \mathbb{E}^Q\left[\int_\tau^T e^{-r(m-\tau)}h_m\mathbf{1}_{A_t}p_m dm\right]\varphi'(\tau)d\tau.
\end{aligned}$$

because $\mu(dm) = p_m dm$. Since the expectation is a bounded operator, by Lemma 11.45 in Aliprantis and Border (2006) we can exchange it with the integral. Later we apply integration by parts:

$$\begin{aligned}
&-\int_t^T \mathbb{E}^Q[\pi(\tau)\mathbf{1}_{A_t}]\varphi'(\tau)d\tau \\
&= -\int_t^T\left(\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi'(\tau)d\tau \\
&= 0 + \int_t^T \frac{d}{d\tau}\left(\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi(\tau)d\tau \\
&= \int_t^T\left(0 - e^{-r(\tau-\tau)}\mathbb{E}^Q\left[h_\tau p_\tau \mathbf{1}_{A_t}\right] + r\int_\tau^T e^{-r(m-\tau)}\mathbb{E}^Q\left[h_m p_m \mathbf{1}_{A_t}\right]dm\right)\varphi(\tau)d\tau \\
&= \int_t^T\left(\mathbb{E}^Q\left[(-h_\tau p_\tau)\mathbf{1}_{A_t}\right] + \mathbb{E}^Q\left[r\int_\tau^T e^{-r(m-\tau)}h_m p_m \mathbf{1}_{A_t}dm\right]\right)\varphi(\tau)d\tau \\
&= \int_t^T \mathbb{E}^Q\left[(-h_\tau p_\tau)\mathbf{1}_{A_t} + r\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(m-\tau)}h_m\mu(dm)\right]\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau \\
&= \int_t^T \mathbb{E}^Q\left[\left(r\pi(\tau) - h_\tau p_\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}$$

Since both π and $h \cdot p$ belong to $\mathcal{V}$, it follows that $\pi - h \cdot p$ is included in $\mathcal{V}$. Therefore, the latter is the weak time-derivative of π:

$$\mathcal{D}\pi(t) = r\pi(t) - h_t p_t, \qquad t \in [0,T].$$

■

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Proof of Theorem 2.17

Proof • EXISTENCE

In order to show that $\pi \in \mathcal{W}_r$, we prove that π belongs to $\mathcal{V}$ and that it is r-weakly time-differentiable.

First, for all $\tau \in [0,T]$, $\pi(\tau) \in L^1(\mathcal{F}_\tau)$ because r is uniformly bounded. Hence, for some $K > 0$,

$$\mathbb{E}^Q\left[\left|\pi(\tau)\right|\right] \leqslant \mathbb{E}^Q\left[e^{-\int_\tau^T r(s)ds}\left|h_T\right|\right] \leqslant K\mathbb{E}^Q\left[\left|h_T\right|\right]$$

and the last quantity is finite because $h_T \in L^1(\mathcal{F}_T)$. In addition,

$$\int_0^T \mathbb{E}^Q\left[\left|\pi(\tau)\right|\right] d\tau \leqslant KT\mathbb{E}^Q\left[\left|h_T\right|\right],$$

which is finite.

As for L^1-continuity, for all $t \in [0,T)$, consider $\tau \longrightarrow t^+$. Then,

$$\begin{aligned}
\mathbb{E}^Q\left[\left|\pi(\tau)-\pi(t)\right|\right] &= \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_\tau^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\leqslant \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_\tau^T r(s)ds}h_T\right] - \mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&= \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\left(e^{-\int_\tau^t r(s)ds}-1\right)\right]\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\leqslant \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds}\left|h_T\right|\left|e^{-\int_\tau^t r(s)ds}-1\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|e^{-\int_\tau^t r(s)ds}-1\right|\right] \\
&\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]
\end{aligned}$$

because r is uniformly bounded by $\tilde{R}$. Moreover, we can apply Lagrange's Theorem to the continuously differentiable function $\sigma \longmapsto e^{-\int_\sigma^t r(s)ds}$ for all $\sigma \in [t,\tau]$. Hence, we find $\hat{t} \in (t,\tau)$ such

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that

$$e^{-\int_\tau^t r(s)ds} - 1 = r\left(\hat{t}\right) e^{-\int_{\hat{t}}^t r(s)ds}(\tau - t), \tag{2.13}$$

so that

$$\left|e^{-\int_\tau^t r(s)ds} - 1\right| = \left|r\left(\hat{t}\right)\right| e^{-\int_{\hat{t}}^t r(s)ds}(\tau - t).$$

As a result,

$$\begin{aligned} e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|e^{-\int_\tau^t r(s)ds} - 1\right|\right] &= e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r\left(\hat{t}\right)\right| e^{-\int_{\hat{t}}^t r(s)ds}\right](\tau - t) \\ &\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\right]\tilde{R}e^{(t-\hat{t})\tilde{R}}(\tau - t). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}^Q\left[\left|\pi(\tau) - \pi(t)\right|\right] &\leqslant e^{(T-\hat{t})\tilde{R}}\mathbb{E}^Q\left[|h_T|\right](\tau - t)\tilde{R} \\ &\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right]. \end{aligned}$$

In the last expression, both terms tend to zero as τ reaches t^+. Specifically, the convergence of the second one is guaranteed by Lévy's Downward Theorem.

Now we concentrate on L^1-convergence in T from the left and we take $\tau \longrightarrow T^-$. Similarly to before, we get

$$\begin{aligned} \mathbb{E}^Q\left[\left|\pi(\tau) - \pi(T)\right|\right] &\leqslant e^{(T-\hat{t})\tilde{R}}\mathbb{E}^Q\left[|h_T|\right](T - \tau)\tilde{R} \\ &\quad + \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[h_T\right] - h_T\right|\right] \end{aligned}$$

and the convergence of the last term is due to Lévy's Upward Theorem.

Therefore π belongs to $\mathcal{V}$.

Now we look for the r-weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and any adapted $\varphi \in C_c^1\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right)$. Recall that indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since φ' is adapted too, we deduce that

$$\begin{aligned} -\int_t^T \mathbb{E}^Q\Big[\pi(\tau)\mathbf{1}_{A_t}\varphi'(\tau)\Big]d\tau & \\ &= -\int_t^T \mathbb{E}^Q\left[\mathbb{E}_\tau^Q\left[e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[e^{-\int_\tau^T r(s)ds}h_T\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau. \end{aligned}$$

$e^{-\int_\tau^T r(s)ds}\varphi'(\tau)$ is a continuous function of $\tau \in [t,T]$, hence it is Bochner integrable. The expectation is a bounded operator, so Lemma 11.45 in Aliprantis and Border (2006) allows us to exchange

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expectation and integral. Therefore,

$$\begin{aligned}
-\int_t^T \mathbb{E}^Q\left[\pi(\tau)\,\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau &= -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T e^{-\int_\tau^T r(s)ds}\varphi'(\tau)d\tau\right] \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r(s)ds}\right)\varphi'(\tau)d\tau\right] \\
&\quad -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\varphi'(\tau)d\tau\right] \\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r(s)ds}\right)\varphi'(\tau)d\tau\right]
\end{aligned}$$

because φ has compact support. Now consider the function $u \longmapsto r(u)e^{-\int_u^T r(s)ds}$. This function is Bochner integrable (because r is uniformly bounded) and its Bochner integral coincides almost surely with the pathwise Lebesgue integral. For any state $\omega \in \Omega$ the restriction r_ω of r satisfies:

$$\int_\tau^T r_\omega(u)e^{-\int_u^T r_\omega(s)ds}du = \left[e^{-\int_u^T r_\omega(s)ds}\right]_\tau^T = 1-e^{-\int_\tau^T r_\omega(s)ds}.$$

In consequence, the Bochner integral is

$$\int_\tau^T r(u)e^{-\int_u^T r(s)ds}du = 1-e^{-\int_\tau^T r(s)ds}.$$

By exploiting integration by parts (see Craven (1970)), we obtain

$$\begin{aligned}
&-\int_t^T \mathbb{E}^Q\left[\pi(\tau)\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau \\
&\qquad = \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(\int_\tau^T r(u)e^{-\int_u^T r(s)ds}du\right)\varphi'(\tau)d\tau\right] \\
&\qquad = \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)d\tau\right] \\
&\qquad = \int_t^T \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}r(\tau)e^{-\int_\tau^T r(s)ds}\varphi(\tau)\right]d\tau \\
&\qquad = \int_t^T \mathbb{E}^Q\left[r(\tau)\mathbb{E}_\tau^Q\left[e^{-\int_\tau^T r(s)ds}h_T\right]\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\
&\qquad = \int_t^T \mathbb{E}^Q\left[r(\tau)\pi(\tau)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau.
\end{aligned}$$

Therefore, the candidate r-weak time-derivative of π is $r\pi$ and $r(t)\pi(t)$ belongs to $L^1(\mathcal{F}_t)$ for all

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t because r is bounded. As for L^1-continuity, let τ go to t^+ for any $t \in [0,T)$. Then,

$$\begin{aligned}
&\mathbb{E}^Q\left[\left|r(\tau)\pi(\tau) - r(t)\pi(t)\right|\right] \\
&\qquad = \mathbb{E}^Q\left[\left|r(\tau)\mathbb{E}^Q_\tau\left[e^{-\int_\tau^T r(s)ds}h_T\right] - r(t)\mathbb{E}^Q_t\left[e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad \leqslant \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(\tau)e^{-\int_\tau^T r(s)ds}h_T\right] - \mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}^Q_t\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right] \\
&\qquad \leqslant \mathbb{E}^Q\left[\left|r(\tau)e^{-\int_\tau^T r(s)ds}h_T - r(t)e^{-\int_t^T r(s)ds}h_T\right|\right] \\
&\qquad\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}^Q_t\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right].
\end{aligned}$$

By exploiting the uniform boundedness of r and eq. (2.13), we find that the first addend in the last expression satisfies

$$\begin{aligned}
&\mathbb{E}^Q\left[\left|r(\tau)e^{-\int_\tau^T r(s)ds}h_T - r(t)e^{-\int_t^T r(s)ds}h_T\right|\right] \\
&\qquad = \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds}\left|h_T\right|\left|r(\tau)e^{-\int_\tau^t r(s)ds} - r(t)\right|\right] \\
&\qquad \leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau)e^{-\int_\tau^t r(s)ds} - r(t)\right|\right] \\
&\qquad = e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau) + r(\tau)r\left(\hat{t}\right)e^{-\int_{\hat{t}}^t r(s)ds}(\tau - t) - r(t)\right|\right] \\
&\qquad \leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau) - r(t)\right|\right] \\
&\qquad\quad + e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[\left|h_T\right|\left|r(\tau)r\left(\hat{t}\right)\right|e^{-\int_{\hat{t}}^t r(s)ds}\right]|\tau - t|.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\mathbb{E}^Q\left[\left|r(\tau)\pi(\tau) - r(t)\pi(t)\right|\right] \\
&\qquad \leqslant e^{(T-t)\tilde{R}}\left(\mathbb{E}^Q\left[h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^Q\left[\left|r(\tau) - r(t)\right|^2\right]\right)^{\frac{1}{2}} \\
&\qquad\quad + e^{(T-\hat{t})\tilde{R}}\tilde{R}^2\mathbb{E}^Q\left[\left|h_T\right|\right]|\tau - t| \\
&\qquad\quad + \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r(t)e^{-\int_t^T r(s)ds}h_T\right] - \mathbb{E}^Q_t\left[r(t)e^{-\int_t^T r(s)ds}h_T\right]\right|\right].
\end{aligned}$$

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As τ approaches t from the right, the first term goes to zero because $h_T \in L^2(\mathcal{F}_T)$ and r is L^2-right-continuous; the second term tends to zero because r is uniformly bounded; the last term is convergent to zero by Lévy's Downward Theorem. Therefore, the L^1-right-continuity is proved.

In order to establish L^1-left-continuity in T, we observe that, by analogous steps, we obtain

$$\begin{aligned}\mathbb{E}^Q\left[\left|r(\tau)\pi(\tau) - r(T)\pi(T)\right|\right] \\ \leqslant \left(\mathbb{E}^Q\left[h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^Q\left[\left|r(\tau)-r(T)\right|^2\right]\right)^{\frac{1}{2}} \\ + \tilde{R}^2\mathbb{E}^Q\left[|h_T|\right]|\tau - T|,\end{aligned}$$

which goes to zero by L^2-left-continuity of r in T.

Hence, $r\pi$ belongs to $\mathcal{V}$ and it is the r-weak time-derivative of π:

$$\mathcal{D}\pi(t) = r(t)\pi(t).$$

Summing up, we showed that $\pi \in \mathcal{W}_r$ and it solves Problem (2.9).

• UNIQUENESS

Let $\pi_1, \pi_2 \in \mathcal{W}_r$ be two solutions of Problem (2.9), that is for every $t \in [0,T]$

$$\mathcal{D}\pi_i(t) = r(t)\pi_i(t) \qquad i = 1,2,$$

with $\pi_i(T) = h_T$. By defining $z = \pi_1 - \pi_2 \in \mathcal{W}_r$, we have that, for every $t \in [0,T]$,

$$\mathcal{D}z(t) = r(t)z(t)$$

and $z(T) = 0$.

The process r is Bochner integrable over time. Reasoning state by state, we have

$$\int_t^T r_\omega(s)ds = R_\omega(T) - R_\omega(t),$$

where R_ω is a primitive of r_ω. By denoting with R the random variable that collects all R_ω, it follows that the Bochner integral of r is

$$\int_t^T r(s)ds = R(T) - R(t).$$

Now we show that the r-weak time-derivative of the process $e^{-R(t)}z(t)$ is

$$e^{-R(t)}\left(\mathcal{D}z(t) - r(t)z(t)\right).$$

For any adapted $\varphi \in C_c^1\left([t,T], L^\infty(\mathcal{F}_T)\right)$, consider the function

$$u \longmapsto e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\varphi'(u).$$

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Since r is bounded, this function is Bochner integrable. By reasoning pathwise, it follows that

$$\int_\tau^T \left(e^{-R(u)}r(u)\varphi(u) - e^{-R(u)}\varphi'(u)\right) du = e^{-R(\tau)}\varphi(\tau).$$

Hence, $e^{-R}\varphi$ is adapted, it belongs to $C_c^1\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right)$ and so we can use it as test function in the definition of r-weak time-derivative of z:

$$\begin{aligned}\int_t^T \mathbb{E}^Q\left[\mathcal{D}z(\tau)\mathbf{1}_{A_t}e^{-R(\tau)}\varphi(\tau)\right]d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[z\left(\tau\right)\mathbf{1}_{A_t}\left(e^{-R(\tau)}\varphi'(\tau) - e^{-R(\tau)}r(\tau)\varphi(\tau)\right)\right]d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[z\left(\tau\right)\mathbf{1}_{A_t}e^{-R(\tau)}\varphi'(\tau)\right]d\tau \\ &\quad + \int_t^T \mathbb{E}^Q\left[z\left(\tau\right)\mathbf{1}_{A_t}e^{-R(\tau)}r(\tau)\varphi(\tau)\right]d\tau\end{aligned}$$

that is

$$\begin{aligned}\int_t^T \mathbb{E}^Q\left[e^{-R(\tau)}\left(\mathcal{D}z\left(\tau\right) - r(\tau)z\left(\tau\right)\right)\mathbf{1}_{A_t}\varphi(\tau)\right]d\tau \\ &= -\int_t^T \mathbb{E}^Q\left[e^{-R(\tau)}z\left(\tau\right)\mathbf{1}_{A_t}\varphi'(\tau)\right]d\tau.\end{aligned}$$

This means that the r-weak time-derivative of $e^{-R(t)}z\left(t\right)$ is

$$e^{-R(t)}\left(\mathcal{D}z\left(t\right) - r(t)z\left(t\right)\right).$$

However this process is null. Therefore, $e^{-R(t)}z\left(t\right)$ has null r-weak time-derivative. Consequently, by following the proof of Proposition 2.3 for test functions in $C_c^1\left([t,T], L^\infty\left(\mathcal{F}_T\right)\right)$, $e^{-R(t)}z(t)$ constitutes a martingale: for every $t \in [0,T]$ and $\tau \in [t,T]$

$$\mathbb{E}_t^Q\left[e^{-R(\tau)}z\left(\tau\right)\right] = e^{-R(t)}z(t).$$

As τ approaches T from the left, $\mathbb{E}_t^Q\left[e^{-R(\tau)}z\left(\tau\right)\right]$ goes to zero in $L^1(Q)$. Indeed, since $e^{-R(\tau)}$ is bounded,

$$\begin{aligned}\mathbb{E}^Q\left[\left|\mathbb{E}_t^Q\left[e^{-R(\tau)}z(\tau)\right] - 0\right|\right] &\leqslant \mathbb{E}^Q\left[\mathbb{E}_t^Q\left[\left|z(\tau)\right|e^{-R(\tau)}\right]\right] \\ &\leqslant C\mathbb{E}^Q\left[\left|z(\tau)\right|\right]\end{aligned}$$

for some $C > 0$. However, the last term converges to zero because $z(\tau)$ tends to $z(T) = 0$ in $L^1(Q)$ as τ approaches T^-.

By uniqueness of the L^1-limit, we infer that $e^{-R(t)}z(t) = 0$. As a result, $z(t) = 0$ for all $t \in [0,T]$ and uniqueness of the solution of Problem (2.9) is established. ■

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3

Long-term risk with stochastic interest rates

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3.1 Motivations and main results

A deep analysis of the term structure of interest rates is essential for dynamic asset pricing theory. Indeed, once fixed-income securities are issued by an institution, they are likely to display time-varying yields to maturity in the secondary market. In fact, different yields embody the changeable expectations of market participants about future events that could impinge on the credit market at different horizons. Monetary policy interventions may directly affect short-term rates while maturities far away are sensitive, for instance, to firms investment decisions, households attitudes and output growth. Moreover, the relation between short-term and long-term bond yields is an indicator of the overall performance of the economy. Usually distant maturities require more risk compensation since investors need to hedge from numerous potentially negative events and their wealth is bound for a long period. On the other hand, inverted yield curves may constitute a predictor for recessions. See, for instance, Ang, Piazzesi, and Wei (2006) among the others.

Understanding the economic mechanisms behind the term structure of rates has been a challenge since the last century. The coherent compounding of credit risks associated with heterogeneous maturities pertains to the essentials of fixed-income modelling. A substantial stream of literature involves the expectation theory, introduced by Fisher (1930) and developed by Keynes (1930). Inspired by intertemporal substitution principles, the expectation hypothesis is based on the equivalence between forward rates and expected spot rates in fixed-income security pricing. Some subsequent elaborations of the theory are due to Lutz (1940) and Hicks (1946), who highlights the role of risk aversion on the term structure. Alternative explanations to the expectation theory are the market segmentation of Culbertson (1957) and the preferred habitat of Modigliani and Sutch (1966). Purposeful surveys on this topic are provided by Cox, Ingersoll, and Ross (1985) and by Russell (1992).

Similarly, in dynamic asset pricing theory, investors with different trading horizons require compensation for the exposures to sources of randomness with heterogeneous duration. As a consequence, long-term premia are the outcome of the intertemporal aggregation of local risk exposures. Importantly, the components of economic variables that survive in the long run are free from inaccuracies due to temporary risk adjustments and are more likely to reflect economic fundamentals. A seminal paper in this branch of literature is Bansal and Yaron (2004) who explain how a small (but persistent) perturbation on consumption and dividend growth may affect long-term returns and propose a potential explanation to the equity premium puzzle of Mehra and Prescott (1985). Relevant empirical assessment of this theory are documented by Bansal, Kiku, and Yaron (2009) and Beeler and Campbell (2009), while Hansen, Heaton, and Li (2008) provide a theoretical analysis of risk-return trade-offs across increasing maturities.

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Hansen and Scheinkman (2009) develop a general operator framework in order to connect short-run and long-run risk compensations. The valuation of cashflows exploits the evolution of both stochastic growth and stochastic discount functionals. In particular, in arbitrage-free markets every stochastic discount factor features a multiplicative decomposition in a deterministic growth term, a permanent (or martingale) component and a transient factor. In particular, the martingale component induces a probability measure change in the long run and it is responsible for a large variance explanation. In equilibrium this decomposition has impact on the asymptotic dynamics of aggregate consumption and wealth through investors' marginal utility, as assessed by Alvarez and Jermann (2005) and Hansen (2012).

Stochastic discount factors reflect the risk-neutral valuation of risky payoffs in viable markets. In particular, the deterministic growth rate elicited by Hansen-Scheinkman decomposition provides a proper discounting which is adjusted for those risks that are relevant in the long run. If interest rates in the market are constant over time, the pricing kernel growth rate coincides with the same instantaneous rate of interest at any maturity under scrutiny. Instead, when rates are stochastic, their variability is captured by the interplay of stochastic discounting and stochastic growth, as discussed in Borovička, Hansen, Hendricks, and Scheinkman (2011). In this case, a synthetic characterization of pricing kernel growth rates (in terms of real financial variables) is missing.

In this paper we provide a bridge between the fixed-income and the long-term risk literature in showing the extent to which the intertemporal aggregation of instantaneous random rates contributes to pricing kernel growth rates at finite (but large) horizons. Moreover, by shifting maturity farther and farther over time, we show how to retrieve the asymptotic deterministic growth rate inferred by Qin and Linetsky (2017). Notably, to derive our results we do not need to assume any specific dynamics for rates of interest.

3.1.1 Main results

We consider a continuous-time arbitrage-free market with horizon T and stochastic instantaneous rates Y_t. We concentrate on risk-neutral price dynamics and the evolution of stochastic discount factors. We present all the results in a conditional setting, where we take into consideration two sources of randomness: prices of primary assets and instantaneous rates.

As for price dynamics, we consider the no arbitrage price $\pi(t)$ at time $0 \leqslant t \leqslant T$ of an attainable payoff with maturity T. When rates of interest are constant over time, the no arbitrage condition delivers the equality

$$\text{instantaneous asset return} = \text{instantaneous risk-free rate}.$$

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In the language of Hansen and Scheinkman (2009) and Hansen (2012), this relation is formalized by the eigenvalue-eigenvector problem

$$\mathcal{A}\pi(t) = r\pi(t), \qquad t \in [0, T] \tag{3.1}$$

where $\mathcal{A}$ is the extended infinitesimal generator of the Markov price process. Here, $\mathcal{A}$ works as a differential operator and generalizes the ordinary differential equation satisfied by risk-free bonds with interest rate r. The dynamics described by eq. (3.1) are supposed to hold under a risk-neutral probability Q. Moreover, the eigenvalue r has a prominent role in the dynamics of the stochastic discount factor because it captures the deterministic growth rate. Indeed, the pricing kernel in any time interval $[s, t]$ is $M_{s,t} = e^{-r(t-s)}L_{s,t}$, where $L_{s,t}$ is the ratio of Radon-Nikodym densities at time t and s of the measure Q with respect to the physical probability.

Hence, when rates are constant over time, the instantaneous rate simultaneously determines risk-neutral price dynamics and the pricing kernel growth rate. However, in a stochastic rate setting, the action of instantaneous rates is more convoluted. Indeed, eq. (3.1) may be rephrased at any instant t by employing the random rate Y_t instead of r - as described by Karatzas and Shreve (1998) - but the sole Y_t is unable to capture the stochastic discount factor growth rate. In fact, the pricing kernel in $[s, t]$ takes the form $M_{s,t} = e^{-\int_s^t Y_\tau d\tau}L_{s,t}$.

A comparison of the expressions of $M_{s,t}$ in the two contexts gives an immediate idea of the heavy measurability requirement of the growth term $e^{-\int_s^t Y_\tau d\tau}$ with respect to the deterministic $e^{-r(t-s)}$. The definition of a growth rate for $M_{s,t}$ turns out to be a challenge when rates of interest are floating over time. Qin and Linetsky (2017) individuate a *long-term yield* as deterministic asymptotic growth rate for stochastic discount factors. Nevertheless, this quantity is not suitable for characterizing instantaneous returns in the sense of eq. (3.1). In a nutshell, the conceptual reason beyond these difficulties rests upon the dual nature of stochastic rates. On the one hand, they represent a proxy for (absent) risk-free rates. On the other, they constitute a source of randomness per se.

The main contribution of this paper is to provide a generalization of eq. (3.1) for stochastic rates settings where the employed eigenvalue is, at the same time, the determinant of the pricing kernel growth rate. Our construction delivers the correct relations when rates are constant over time. In addition, it is consistent with the asymptotic results of the long-term risk literature when the horizon T goes farther and farther.

Our generalization of eq. (3.1) under stochastic rates accounts for conditioning information at some intermediate time s between 0 and T. To do so, we introduce a mathematical instrument that allows us to differentiate stochastic processes on the time window $[s, T]$ by disentangling the role of known information at instant s: the weak time-derivative in $[s, T]$, denoted by $\mathcal{D}$. This tool

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is a generalization of the weak time-derivative of Marinacci and Severino (2017), which applies to a wide class of semimartingale processes, and it is more suitable for conditional settings. Similarly to the original instrument, the weak time-derivative in $[s, T]$ provides a handy characterization of conditional martingales with respect to some fixed probability measure and so it is essential for the identification of no arbitrage discounted prices, forward and future prices. Indeed, we prove in Proposition 3.3 that the weak time-derivative is null when the process under investigation is a conditional martingale.

In Theorem 3.5 we show that, when short-term rates are stochastic, the dynamics described by eq. (3.1) can be retrieved in the differential relation

$$\mathcal{D}\pi_s(t) = r_{s,T}\pi_s(t), \qquad t \in [s, T], \tag{3.2}$$

where the weak time-derivative in $[s, T]$ is involved and several departures from the deterministic case are present.

First, the constant rate is replaced by $r_{s,T}$, the *yield to maturity* in the time interval $[s, T]$ of a pure discount bond that delivers one unit of account at maturity. When the horizon T becomes arbitrarily large, $r_{s,T}$ converges in probability to the long-term yield.

Second, the dynamics are defined under the forward measure instead of the risk-neutral one. This probability measure, which is risk-adjusted, is due to a convenient change of numéraire that is widely employed in security pricing with floating rates. In fact, the price of a zero-coupon bond is employed to discount payoffs, instead of the money market account: see, for instance, Geman, El Karoui, and Rochet (1995). Due to their certain payoff, pure discount bonds play, in fact, the role of risk-free assets in a context in which the money market account is unforeseeable.

Third, the solution π_s is closely related to the no arbitrage price π via

$$\pi_s(t) = e^{r_{s,T}(t-s)}\frac{p(s, T)}{p(t, T)}\pi(t), \qquad t \in [s, T],$$

where $p(\tau, T)$ denotes the no arbitrage price of the discount bond with maturity T at time τ. In case rates of interest are constant, π_s coincides with π. In general, the two variables are equal at the initial instant s and at the final one. π_s can be interpreted as a *rate-adjusted price* or *indifference price*, that provides hedging from the variability of rates. In fact, π_s is the no arbitrage price of the payoff under consideration in a parallel market with constant interest rates equal to the bond yield $r_{s,T}$. When the expiration date goes to infinity, the ratio between $\pi_s(t)$ and $\pi(t)$ stabilizes both in probability and in expectation.

Fourth, the dynamics of eq. (3.2) are conditional on the information structure available at date s. We are actually considering an eigenvalue-eigenvector problem in which the eigenvalue is

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a random variable, known at the beginning of the trading interval. Hence, measurable scenarios at date s affect the dynamics under scrutiny at any finite future horizon.

In Section 3.6 we show that yields to maturity evaluated at time s provide a random growth rate for the pricing kernel in the time window $[s,t]$ included in $[0,T]$. Indeed, we argue that, when rates are stochastic, the growth term of $M_{s,t}$ is the quantity

$$\frac{e^{-r_{s,T}(T-2s)}}{e^{-r_{s,T-t}(T-t-s)}},$$

which is measurable at time s. This expression reduces to $e^{-r(t-s)}$ when rates are constant and, in general, its logarithm is convergent to the long-term yield when date T goes to infinity. As a result, pure discount bond yields play both the role of eigenvalues (for prices) and that of growth rates (for pricing kernels), generalizing the features of the constant rate case.

Our theory pertains to linear asset pricing. Indeed, the weak time-derivative allows us to characterize the space $H_{s,T}$ of forward prices, which are conditional martingales with respect to the forward measure. Moreover, $H_{s,T}$ is endowed with an inner product that exploits the conditional expectation at time s. This inner product is inspired by Hansen and Richard (1987) and the overall structure is termed Hilbert module by Cerreia-Vioglio, Maccheroni, and Marinacci (2017). In particular, the no arbitrage pricing rule induces a linear functional on $H_{s,T}$ which has an inner product representation, in agreement with the asset pricing literature since Harrison and Kreps (1979).

The paper is organized as follows. Section 3.2 introduces the no arbitrage pricing framework and describes all the numéraire changes and the related risk-adjusted measures that are employed in the body of the article. In Section 3.3 we define the weak time-derivative in $[s,T]$ and we prove its main properties, in particular the homogeneity with respect to information measurable at time s and the characterization of conditional martingales. Section 3.4 describes the Hilbert module structure of $H_{s,T}$ and the representation of linear pricing functionals. In Section 3.5 we solve the differential equation (3.2) and we discuss the role of π_s with respect to no arbitrage prices. In Section 3.6 we present a decomposition of the pricing kernel in $[s,t]$ which highlights the random growth term, while Section 3.7 is devoted to an illustration of our results in affine models. Section 3.8 concludes. The Appendix contains complementary results about measure changes, further simulations and all the proofs.

3.2 Asset pricing framework

We begin with introducing a filtered probability space that satisfies usual technical assumptions and we build a continuous-time arbitrage-free market upon it. After defining risk-neutral measures

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and stochastic discount factors, we describe several changes of numéraire that will be employed throughout the paper.

3.2.1 Arbitrage-free market

Let $(\Omega, \mathcal{F}, P)$ be a probability space with positive P. We fix $T > 0$ and we consider stochastic processes on the time interval $[0, T]$. We will endow the probability space with a filtration, denoted $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, that is supposed to satisfy the usual conditions, namely $\mathbb{F}$ is complete and right-continuous: $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \in [0, T)$. We additionally assume left-continuity in T, that is $\mathcal{F}_T = \mathcal{F}_{T^-}$.

Throughout the article, we identify random variables that coincide almost surely and we identify stochastic processes up to modifications. Moreover, on the given filtered probability space, we will consider processes $u : [0, T] \times \Omega \longrightarrow \mathbb{R}$ that are progressively measurable (progressive, for brevity). This requirement is equivalent to adaptability, up to modifications - see, for instance, Karatzas and Shreve (2012). Finally, for typographical convenience we will interchangeably use the notations u_t and $u(t)$ to denote the realizations of u at time t.

We build a financial market on $(\Omega, \mathcal{F}, P)$ by considering two random processes $X = \{X_t\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$. The process X is N-dimensional and consists of prices of N primary risky assets that generate the market, i.e. $X_t = \left[X_t^{(1)}, \ldots, X_t^{(N)}\right]'$ with $t \in [0, T]$. Instead, the process Y is one-dimensional and represents the stochastic instantaneous interest rate. By computing the integral $\int_0^T Y_\tau d\tau$ pathwise, we are allowed to define the *money market account* price process $\left\{e^{\int_0^t Y_\tau d\tau}\right\}_{t\in[0,T]}$. In our market, pure discount bonds with any possible maturity are traded, too. For computational convenience, we assume that the face value of these instruments is one unit.

At any instant $t \in [0, T]$ we define the vector of relative prices $Z_t = e^{-\int_0^t Y_\tau d\tau} X_t$, namely asset prices discounted by the value of the money market account. We, then, consider as environment the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by the pair (Z, Y).

We assume that our price system satisfies the no free lunch with vanishing risk (NFLVR) condition and that relative asset prices Z are semimartingales. See Delbaen and Schachermayer (1994) as a reference. We represent portfolio strategies by adapted $(N + 1)$-dimensional processes ϑ. The portfolio discounted value is given by the process V^ϑ, where

$$V_t^\vartheta = \vartheta_t^{(0)} + \sum_{i=1}^{N} \vartheta_t^{(i)} Z_t^{(i)}$$

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for all $t \in [0,T]$. All portfolios that we consider are supposed to be *admissible*, i.e.

$$\int_0^t \sum_{i=1}^N \vartheta_t^{(i)} dZ_t^{(i)} \geqslant -w$$

for all $t \in [0,T]$ and some $w \geqslant 0$, and *self-financing*, that is

$$dV_t^{\vartheta} = \sum_{i=1}^N \vartheta_t^{(i)} dZ_t^{(i)}, \qquad t \in [0,T].$$

The NFLVR assumption ensures that there exists a probability measure Q equivalent to P such that Z is a *sigma-martingale*: see the First Fundamental Theorem of Asset Pricing in Delbaen and Schachermayer (1998). As explained by Émery (1980), Z turns out to be the martingale transform of some martingale, via an integrable predictable process. Notably, the measure Q does not need to be unique.

Remark 3.1 *We assume that at least one of the existing sigma-martingale measures is an* equivalent martingale measure*, i.e. it makes Z a martingale process.*

From now on we consider one of these equivalent martingale measures Q for the valuation of securities in the market. We denote by L_T the Radon-Nikodym derivative of Q with respect to P and $L_t = \mathbb{E}_t[L_T]$ for all $t \in [0,T]$. We also define $L_{t,T} = L_T/L_t$. Hence, the conditional expectation of any measurable random variable V at time t under Q is $\mathbb{E}_t^Q[V] = \mathbb{E}_t[L_{t,T}V]$. Coherently with Harrison and Kreps (1979), we assume that, for any $t \in [0,T]$, $e^{-\int_0^t Y_\tau d\tau} L_t$ belongs to $L^2(\mathcal{F}_t)$.

We denote by $S = \{S_t\}_{t\in[0,T]}$ the strictly positive *stochastic discount factor* process associated with the measure Q, which has $S_0 = 1$. In addition, we define the related pricing kernel in the time interval $[t,T]$ by $M_{t,T} = S_T/S_t$, so that a no arbitrage price at time t of an attainable $\mathcal{F}_T$-measurable payment h_T is $\pi(t) = \mathbb{E}_t[M_{t,T}h_T]$. Since interest rates are stochastic, with continuous compounding the stochastic discount factor and the pricing kernel in $[t,T]$ satisfy

$$S_t = e^{-\int_0^t Y_\tau d\tau} L_t, \qquad M_{t,T} = e^{-\int_t^T Y_\tau d\tau} L_{t,T}.$$

We now consider a bond with maturity T and we denote its no arbitrage price (induced by Q) at time t by $p(t,T)$. A relevant adapted process for our construction is $\{r_{t,T}\}_{t\in[0,T]}$ defined by

$$r_{t,T} = -\frac{\log p(t,T)}{T-t},$$

which is the *yield* of the pure discount bond maturing at T computed at time t. In any time interval $[t,T]$, the yield to maturity satisfies the relation

$$e^{-r_{t,T}(T-t)} = p(t,T) = \mathbb{E}_t^Q\left[e^{-\int_t^T Y_\tau d\tau}\right]$$

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and this quantity coincides with $\mathbb{E}_t\left[M_{t,T}\right]$. We also define $r_{T,T}$ as the a.s. limit of $r_{t,T}$ when t approaches T. $r_{t,T}$ may be considered as an $\mathcal{F}_t$-measurable rate of interest equivalent to the compounding of forthcoming instantaneous rates Y_τ when τ varies from t to T. It is also referred to as *internal rate of return*.

If interest rates are constant over time (and deterministic), the yield coincides with the instantaneous rate. Hence, the different role of yields and interest rates can be appreciated just when a term structure of floating rates is considered.

3.2.2 Forward measure

From the seminal work of Margrabe (1978), the use of different numéraires has become quite a common practice in asset pricing, especially in option valuation. Such changes of unit of measurements have proved to be convenient especially when dealing with stochastic interest rates, as Geman (1989) and Jamshidian (1989) discuss. A comprehensive treatment is given by Geman, El Karoui, and Rochet (1995) who show, in an arbitrage-free market, how to build a probability measure that makes the prices of all traded securities, discounted by a given numéraire, martingale processes. Among the others, Girotto and Ortu (1996) provide a characterization of numéraire portfolios. The no arbitrage discussion in Delbaen and Schachermayer (1995) is also remarkable.

In our framework, by exploiting as numéraire the no arbitrage price of a zero-coupon bond with maturity T, we construct the forward measure with horizon T or, simply, *T-forward measure* and we denote it by F. This probability measure is equivalent to the risk-neutral measure Q and we denote its Radon-Nikodym derivative with respect to P by G_T. Importantly, G_T belongs to $L^2\left(\mathcal{F}_T\right)$ because $e^{-\int_0^T Y_\tau d\tau} L_T$ is included in $L^2\left(\mathcal{F}_T\right)$. Moreover, we set $G_t = \mathbb{E}_t\left[G_T\right]$ for any $t \in [0,T]$ and we define $G_{t,T} = G_T/G_t$. See Appendix 3.A for a detailed derivation.

Using this measure, the stochastic discount factor S_t and the pricing kernel in $[t,T]$ associated with Q may be expressed as

$$S_t = e^{r_{t,T}(T-t)-r_{0,T}T} G_t, \qquad M_{t,T} = e^{-r_{t,T}(T-t)} G_{t,T}.$$

In the time interval $[s,t]$ with $s \leqslant t \leqslant T$, the pricing kernel becomes

$$M_{s,t} = S_t/S_s = e^{r_{t,T}(T-t)-r_{s,T}(T-s)} G_{s,t}.$$

Although $M_{s,t}$ refers to the time window $[s,t]$, its expression depends on the ultimate horizon T under consideration when the density of the forward measure is employed.

In the special case in which rates of interest are constant over time, the forward probability F coincides with the measure Q.

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We focus on the forward measure because it provides a handy representation of risk-neutral asset prices. Consider, for instance, an attainable $\mathcal{F}_T$-measurable payoff h_T. Its no arbitrage price according to Q at any intermediate time t is

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T Y_\tau d\tau} h_T \right].$$

However, we can express $\pi(t)$ by employing the change of numéraire associated with the forward measure, so that

$$\pi(t) = e^{-r_{t,T}(T-t)} \mathbb{E}_t^F \left[h_T \right]. \tag{3.3}$$

This expression will prove to be crucial in the formulation of the no arbitrage pricing equation (3.7) described in Subsection 3.5.1 because it makes a fruitful bridge between constant-rate and stochastic-rate asset valuation.

Finally, we define the yield of bonds with maturity $t < T$ in the same way as T-bond yield is constructed. This family of bond yields satisfies the following compounding rule: for any $s \leqslant t \leqslant T$,

$$e^{r_{s,T}(T-s)} = e^{r_{s,t}(t-s)} \mathbb{E}_s^F \left[e^{r_{t,T}(T-t)} \right],$$

which is proved in Lemma 3.9 in Appendix 3.A. This simple result additionally shows that $\mathbb{E}_s \left[M_{s,t} \right] = e^{-r_{s,t}(t-s)}$.

We finally introduce another numéraire change that will be useful for our construction. We take as numéraire the no arbitrage price process of a (virtual) security generating the payoff G_T and we denote by $\overline{Q}$ the related risk-adjusted measure. Its Radon-Nikodym derivative with respect to P equals

$$I_T = \frac{G_T^2}{\mathbb{E}\left[G_T^2\right]}.$$

For all $\tau \in [0, T]$ we define $I_\tau = \mathbb{E}_\tau \left[I_T \right]$ and $I_{\tau,T} = I_T / I_\tau$. As a result, for any random variable V,

$$\mathbb{E}_\tau^{\overline{Q}} \left[V \right] = \mathbb{E}_\tau \left[I_{\tau,T} V \right] = \mathbb{E}_\tau \left[\frac{G_T^2}{\mathbb{E}_\tau \left[G_T^2 \right]} V \right].$$

3.3 Weak time-derivative in $[s, T]$

In this section we describe the weak time-derivative in $[s, T]$, a differential tool that enables the formulation of the no arbitrage pricing equation with stochastic rates. The theory is presented under the physical measure. However, all definitions and results can be restated under any other probability measure. In fact, in our financial application, we will employ the forward measure.

We fix an instant $s \in [0, T]$ and we develop the necessary tools to deal with conditioning information in $\mathcal{F}_s$. We start by defining the conditional L^1-space

$$L_s^1 \left(\mathcal{F}_T \right) = \left\{ f \in L^0 \left(\mathcal{F}_T \right) : \quad \mathbb{E}_s \left[|f| \right] \in L^0 \left(\mathcal{F}_s \right) \right\}.$$

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Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth (2016) show that $L^1_s(\mathcal{F}_T)$ is an L^0-module with the multiplicative decomposition $L^1_s(\mathcal{F}_T) = L^0(\mathcal{F}_s)\,L^1(\mathcal{F}_T)$. Clearly, $L^1_s(\mathcal{F}_T)$ contains all functions f in $L^1(\mathcal{F}_T)$: in this case $\mathbb{E}_s[|f|] \in L^1(\mathcal{F}_s)$. In general, however, the conditional expectation is defined for random variables that are merely in $L^0(\mathcal{F}_T)$. See, for instance, Shreve (2004).

In $L^1_s(\mathcal{F}_T)$ we use the L^0-valued metric $d(f,g) = \mathbb{E}_s[|f-g|]$. Accordingly, we say that a stochastic process $u : [0,T] \longrightarrow L^1_s(\mathcal{F}_T)$ is L^1_s-continuous if and only if, for all $t \in [0,T]$, $\mathbb{E}_s\left[\left|u(\tau)-u(t)\right|\right] \longrightarrow 0$ a.s. when $\tau \longrightarrow t$. This property is weaker than standard L^1-continuity and it is implied by that.

Then, inspired by Marinacci and Severino (2017), we define the L^0-module

$$\mathcal{V}_{s,T} = \Bigg\{ \text{adapted } u : [s,T] \longrightarrow L^1_s(\mathcal{F}_T), \\ L^1_s - \text{right} - \text{continuous in } [s,T), \\ L^1_s - \text{left} - \text{continuous in } T \Bigg\}.$$

As a consequence of Tonelli's Theorem, all processes in $\mathcal{V}_{s,T}$ satisfy

$$\int_s^T \mathbb{E}_s\left[\left|u(\tau)\right|\right] d\tau \in L^0(\mathcal{F}_s),$$

where the integral is computed trajectory by trajectory.

Similarly to Marinacci and Severino (2017), $\mathcal{V}_{s,T}$ contains all martingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ defined on the time window $[s,T]$. However, more is actually true. Indeed, $\mathcal{V}_{s,T}$ includes all *conditional* (or *generalized*) *martingales*. By this terminology we mean processes u defined in the time interval $[s,T]$ with all the properties of martingales except for integrability, which is replaced by the weaker condition $\mathbb{E}_s[|u(t)|] \in L^0(\mathcal{F}_s)$ for all $t \in [s,T]$. See, for instance, Shiryaev (1996) or Kabanov and Safarian (2009).

We, then, define the weak time-derivative for processes in $\mathcal{V}_{s,T}$. The definition employs the space $C^1_c\left([0,T], L^0(\mathcal{F}_s)\right)$ of functions $\varphi : [0,T] \longrightarrow L^0(\mathcal{F}_s)$ that have compact support in $[0,T]$ and are continuously differentiable over time in the following sense: there exists a continuous function $\psi : [0,T] \longrightarrow L^0(\mathcal{F}_s)$ with compact support such that the pathwise integral $\int_0^t \psi(\tau)d\tau$ equals $\varphi(t)$ for all $t \in [0,T]$. In this case, we simply write $\varphi' = \psi$.

Definition 3.1 Given $u \in \mathcal{V}_{s,T}$, we say that u is *weakly time-differentiable in* $[s,T]$ when there exists $w \in \mathcal{V}_{s,T}$ such that for every $t \in [s,T]$

$$\int_t^T \mathbb{E}_s\left[w(\tau)\,\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = -\int_t^T \mathbb{E}_s\left[u(\tau)\,\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau$$

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$$\forall A_t \in \mathcal{F}_t, \qquad \forall \varphi \in C_c^1\left([t,T], L^0\left(\mathcal{F}_s\right)\right).$$

In this case we call w a *weak time-derivative* of u in $[s,T]$.

The integrals in Definition 3.1 are pathwise integrals of processes in $L^0\left(\mathcal{F}_s\right)$. In fact, Definition 3.1 generalizes the notion of weak time-derivative introduced by Marinacci and Severino (2017), where weakly time-differentiable processes constitute a class of special semimartingales. The case of Marinacci and Severino (2017) arises when $s = 0$ and the whole time frame $[0,T]$ is taken into account. Accordingly, test functions employed there are deterministic, namely $\varphi \in C_c^1\left([0,T],\mathbb{R}\right)$. Therefore, in the applications, the construction in Marinacci and Severino (2017) is suitable for unconditional dynamic frameworks, while Definition 3.1 provides a suitable instrument for conditioning upon incoming information at time s: see the following eq. (3.4).

We first prove that the weak time-derivative in $[s,T]$ is unique, up to modifications.

Proposition 3.2 *Let $u \in \mathcal{V}_{s,T}$ be weakly time-differentiable. Then, the weak time-derivative of u is unique.*

Hence, we denote $\mathcal{D}u$ one of the versions of the weak time-derivative of u in $[s,T]$. Then, we introduce the L^0-submodules of $\mathcal{V}_{s,T}$

$$\mathcal{W}_{s,T} = \left\{\text{weakly time} - \text{differentiable } u \in \mathcal{V}_{s,T}\right\}$$

and

$$\mathcal{Z}_{s,T} = \left\{\text{infinitely weakly time} - \text{differentiable } u \in \mathcal{W}_{s,T}\right\}.$$

$\mathcal{F}_s$-measurable functions play the role of multiplicative constants for the weak time-derivative in $[s,T]$. Indeed, taken any process $u \in \mathcal{W}_{s,T}$ and any $\lambda_s \in L^0\left(\mathcal{F}_s\right)$, the process v defined by $v(t) = \lambda_s u(t)$ for all $t \in [s,T]$ belongs to $\mathcal{W}_{s,T}$, too, and

$$\mathcal{D}\left(\lambda_s u\right) = \lambda_s \mathcal{D}u. \tag{3.4}$$

Indeed, thanks to the take-out-what-is-known property, it is immediate to verify that v belongs to $\mathcal{V}_{s,T}$ and that it is weakly time-differentiable with $\mathcal{D}v = \lambda_s \mathcal{D}u$. Hence, the weak time-derivative is homogeneous when the multiplicative coefficient taken in $L^0\left(\mathcal{F}_s\right)$.

We now prove that a weakly time-differentiable function u in $[s,T]$ has null weak time-derivative if and only if it is a conditional martingale.

Proposition 3.3 *A process u belongs to $\mathcal{W}_{s,T}$ and it has $\mathcal{D}u = 0$ if and only if it is a conditional martingale.*

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When the risk-neutral probability Q is employed, a simple example of conditional martingale is provided by *future prices* of a claim with expiry T at previous instants in the time window $[s,T]$. Instead, under the forward measure, *forward prices* in $[s,T]$ for the settlement date T are conditional martingales. See, for instance, Musiela and Rutkowski (2005). Both these processes exhibit null weak time-derivatives with respect to different probability measures.

In general, the weak time-derivative in $[s,T]$ turns out to be a useful tool for the analysis of no arbitrage prices because, after a proper discounting, they constitute martingale processes under some risk-neutral measure. Moreover, the weak time-derivative captures the drift of semimartingale processes and so it applies to a wide range of continuous-time asset pricing models. See Marinacci and Severino (2017) and the example in Subsection 3.5.1 about Black-Scholes market.

3.4 The module $H_{s,T}$ and linear pricing functionals

In this section we use as reference measure the forward measure and we move to the filtered probability space $(\Omega,\mathcal{F},\mathbb{F},F)$. The weak time-derivative allows us to define an L^0-module of forward price processes. The properties of this structure ensure an inner product representation of linear pricing functionals, presented in Subsection 3.4.2.

3.4.1 The Hilbert module $H_{s,T}$

Considering $(\Omega,\mathcal{F},\mathbb{F},F)$ as environment, we define the vector space $H_{s,T}$

$$H_{s,T}=\left\{u\in\mathcal{W}_{s,T}:\quad \mathcal{D}u=0,\qquad \mathbb{E}_s^{\overline{Q}}\left[u_T^2\right]\in L^0\left(\mathcal{F}_s\right)\right\}.$$

By Proposition 3.3, $H_{s,T}$ is made up of conditional F-martingales u (or conditional martingales Gu under the physical measure) with conditionally square integrable terminal value according to $\overline{Q}$. As described in Subsection 3.2.2, the probability measure $\overline{Q}$ is equivalent to P and it has Radon-Nikodym derivative proportional to G_T^2.

From the financial perspective, $H_{s,T}$ is a set of *forward prices* for traded assets with settlement date T. More precisely, $H_{s,T}$ is composed of forward prices $u\in\mathcal{W}_{s,T}$ such that $\mathbb{E}_s^{\overline{Q}}\left[u_T^2\right]\in L^0\left(\mathcal{F}_s\right)$. Indeed, forward prices are (conditional) martingales under the forward measure F and so they belong to $H_{s,T}$ once few technical conditions are met.

$H_{s,T}$ is a pre-Hilbert module on the algebra $L^0\left(\mathcal{F}_s\right)$ when we define the outer product $\cdot: L^0\left(\mathcal{F}_s\right)\times H_{s,T}\longrightarrow H_{s,T}$ by

$$a_s\cdot u=a_s u$$

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and the L^0-valued inner product $\langle , \rangle_{s,T} : H_{s,T} \times H_{s,T} \longrightarrow L^0(\mathcal{F}_s)$ by

$$\langle u, v \rangle_{s,T} = \mathbb{E}_s^{\overline{Q}} [u_T v_T] .$$

Moreover, $H_{s,T}$ is selfdual, a property that enables an inner product representation of all L^0-linear bounded functionals. See, for instance, Cerreia-Vioglio, Maccheroni, and Marinacci (2017) for a recent treatment of Hilbert module theory.

Proposition 3.4 *$H_{s,T}$ is a selfdual pre-Hilbert module on $L^0(\mathcal{F}_s)$.*

Since $H_{s,T}$ is a selfdual pre-Hilbert module, to simplify the terminology we call it *Hilbert module*. The inner product that we introduced above mimics the conditional Hilbert structure of Hansen and Richard (1987), who build up an effective conditional asset pricing framework under the physical measure. Nevertheless, here we use a different probability measure and we work directly on terminal values of martingale processes.

Moreover, the homogeneity of the inner product with respect to $\mathcal{F}_s$-measurable variables, i.e.

$$\langle a_s \cdot u, v \rangle_{s,T} = a_s \langle u, v \rangle_{s,T}$$

for any u, v in $H_{s,T}$ and a_s in $L^0(\mathcal{F}_s)$, is consistent with the homogeneity of the weak time-derivative in $[s, T]$ formalized in eq. (3.4).

Finally, the inner product structure delivers a natural notion of orthogonality. In fact, two processes u, v in $H_{s,T}$ are orthogonal when

$$\langle u, v \rangle_{s,T} = \mathbb{E}_s^{\overline{Q}} [u_T v_T] = 0.$$

This notion may be useful to discuss mean-variance optimality of traded asset returns.

3.4.2 Linear pricing functionals in $H_{s,T}$

Consider an $\mathcal{F}_T$-measurable payoff h_T at maturity such that $\mathbb{E}_s^{\overline{Q}} \left[h_T^2 \right]$ belongs to $L^0(\mathcal{F}_s)$. As anticipated by eq. (3.3) in Subsection 3.2.2, the no arbitrage price of h_T at time s can be expressed by exploiting both the risk-neutral and the forward measure. It is interesting to notice that the payoff under consideration may be seen as the terminal value of the forward price process $h = \{h_t\}_{t \in [s,T]}$ defined by $h_t = \mathbb{E}_t^F [h_T]$ at any instant t. As a result, h belongs to $H_{s,T}$. This simple observation suggests that the no arbitrage pricing formula actually exploits an L^0-valued functional on the Hilbert module $H_{s,T}$. We denote this functional by Π_s:

$$\Pi_s : \quad H_{s,T} \longrightarrow L^0(\mathcal{F}_s)$$

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$$\Pi_s : \quad u \longmapsto e^{-r_{s,T}(T-s)}\mathbb{E}_s^F\left[u_T\right].$$

In fact, $\Pi_s(h) = e^{-r_{s,T}(T-s)}\mathbb{E}_s^F\left[h_T\right] = \pi(s)$.

Note that Π_s is L^0-linear because, for all $a_s, b_s \in L^0\left(\mathcal{F}_s\right)$ and $u, v \in H_{s,T}$,

$$\Pi_s\left(a_s \cdot u + b_s \cdot v\right) = a_s\Pi_s\left(u\right) + b_s\Pi_s\left(v\right)$$

and Π_s is also positive. Importantly, Π_s is a bounded functional, i.e. there exists a positive $c_s \in L^0\left(\mathcal{F}_s\right)$ such that

$$\Pi_s(u)^2 \leqslant c_s\left\langle u, u\right\rangle_{s,T} \qquad \forall u \in H_{s,T}.$$

See the definitions in Cerreia-Vioglio, Maccheroni, and Marinacci (2017). Indeed, by conditional Jensen's inequality,

$$\Pi_s(u)^2 = \frac{e^{-2r_{s,T}(T-s)}}{G_s^2}\left(\mathbb{E}_s\left[G_T u_T\right]\right)^2 \leqslant \frac{e^{-2r_{s,T}(T-s)}\mathbb{E}_s\left[G_T^2\right]}{G_s^2}\left\langle u, u\right\rangle_{s,T}.$$

For selfdual pre-Hilbert L^0-modules a generalization of Riesz' Theorem holds, as shown in Cerreia-Vioglio, Maccheroni, and Marinacci (2017). Therefore, any L^0-linear bounded functional of $H_{s,T}$ has a inner product representation. This is the case of Π_s, which is represented by the L^0-valued inner product:

$$\Pi_s(u) = \left\langle z, u\right\rangle_{s,T}, \qquad z_t = \frac{e^{-r_{s,T}(T-s)}\mathbb{E}_s\left[G_T^2\right]}{G_s}\frac{1}{G_t},$$

for any $u \in H_{s,T}$. Indeed,

$$\left\langle z, u\right\rangle_{s,T} = \frac{e^{-r_{s,T}(T-s)}}{G_s}\mathbb{E}_s\left[G_T^2\frac{u_T}{G_T}\right] = e^{-r_{s,T}(T-s)}\mathbb{E}_s\left[G_{s,T}u_T\right].$$

Since, for all $t \in [s,T]$, G_t/G_t equals 1, which is a martingale, it is apparent that the process $1/G$ belongs to $H_{s,T}$. Hence, the L^0-valued inner product representation is well-defined. This fact generalizes the standard representation of linear pricing functionals in Hilbert spaces and it is consistent with the asset pricing literature.

In fact, the use of linear prices is widespread in financial economics since the introduction of state-price securities by Arrow (1964). Applications to option pricing may be retrieved in later works by Ross (1976) and Rubinstein (1976). More specifically, the explicit representation of linear pricing functionals for the valuation of contingent claims is due to Harrison and Kreps (1979), while Ross (1978) employs a similar approach for risky return streams. Moreover, in a rational expectation setting, Hansen (1994) exploits the inner product structure of pricing functionals to compute equilibrium prices in a permanent-income model. Finally, as for conditional asset pricing, an exhausting treatment is provided by Hansen and Richard (1987), who first introduced conditional inner products. As a result, our linear pricing functional on $H_{s,T}$ does not overstep the scope of this literature.

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3.5 No arbitrage pricing equation

By employing weak time-derivatives, we formulate and solve an analogous of the no arbitrage pricing equation for the valuation of random payments in a context of stochastic interest rates. We, then, compare the solution of the equation with the usual risk-neutral pricing formula, allowing also for infinite horizons.

3.5.1 No arbitrage pricing

In the time frame $[s, T]$ we face the problem of evaluating an $\mathcal{F}_T$-measurable payoff h_T associated, for instance, with a European derivative with exercise date T. In line with the framework of Subsection 3.2.1, we postulate the existence of an equivalent martingale measure Q. Following Subsection 3.2.2, we can express the time t no arbitrage price of h_T associated with Q by using both the measure Q and the forward measure F, namely

$$\pi(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T Y_\tau d\tau} h_T \right] = e^{-r_{t,T}(T-t)} \mathbb{E}_t^F \left[h_T \right].$$

This writing conveys the intuition that security pricing with stochastic interest rates is not so dissimilar from the case of deterministic rates. Up to replacing the risk-neutral measure with the forward one and employing bond yields instead of instantaneous rates, the pricing formula is formally the same.

As discussed by Hansen and Scheinkman (2009), in an arbitrage-free market with constant rate of interest r, asset prices satisfy the relation

$$\mathcal{A}\pi = r\pi,$$

where $\mathcal{A}$ is the extended generator of an underlying Markov process and the risk-neutral measure Q is employed. This eigenvalue-eigenvector problem has its roots in the Perron-Frobenius theory and it captures the essence of no arbitrage. Indeed, it relates infinitesimal price increments of a possibly risky asset with the interest rate deriving from a locally riskless investment. This approach is known in the financial literature since Cox and Ross (1976) derivation of Black-Scholes formula. Moreover, the rate $-r$ defines the growth rate of the stochastic discount factor in Hansen-Scheinkman decomposition.

Marinacci and Severino (2017) formalize the eigenvalue-eigenvector problem mentioned above in a semimartingale framework by exploiting weak time-derivatives on the time interval $[0, T]$ with the measure Q. In particular, they show that the risk-neutral price process π is the unique solution

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of the no arbitrage pricing equation

$$\begin{cases} \mathcal{D}\pi(t) = r\pi(t) & t \in [0,T) \\ \pi(T) = h_T \end{cases} \tag{3.5}$$

where $h_T \in L^1(\mathcal{F}_T, Q)$. In this problem the interest rate r is supposed to be constant.

Example 3.1 As an illustrative example we consider a Black-Scholes market with a riskless bond and a risky security whose prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t dW_t^P$$

under the physical measure. Here, $r, \mu \in \mathbb{R}$, $\sigma > 0$ and W_t^P is a standard Wiener process. It is apparent that the bond price satisfies Problem (3.5) while the risky asset price does not. In fact, W_t^P exhibits null weak time-derivative with respect to P because of martingality, but the drift of X_t differs from rX_t. However, moving to the risk-neutral measure Q, the dynamics of X_t becomes

$$dX_t = rX_t dt + \sigma X_t dW_t^Q, \tag{3.6}$$

where W_t^Q is a Q-Wiener process, and so $\mathcal{D}X_t = rX_t$. Hence, the differential equation in Problem (3.5) is satisfied under Q. In this arbitrage-free market both the risk-free and the risky security share the same drift, determined by the instantaneous rate.

The dynamics presented so far are generalized to stochastic rates settings by replacing r with the instantaneous rate Y_t. For example, Karatzas and Shreve (1998) show how to obtain similar dynamics to eq. (3.6) in a semimartingale pricing framework with random rates. A parallel generalization is made by Marinacci and Severino (2017) in terms of weak time-derivatives. However, they assume uniformly bounded instantaneous rates, in contrast with common fixed-income modelling.

Nevertheless, there are two major flaws in these generalizations. First, the drift coefficient Y_t is unable to capture the stochastic discount factor growth rate over any time period, differently from the constant rate case. Second, Y_t is not known ex ante (since it is floating over time) and so it forbids an eigenvalue-eigenvector formulation of the problem in the spirit of Hansen and Scheinkman (2009).

Our generalization of Problem (3.5), instead, solves these issues for any given dynamics of rates of interest. In particular, we employ the forward measure instead of Q and T-bond yields in place of short-term rates in order to be able to formulate a proper eigenvalue-eigenvector problem.

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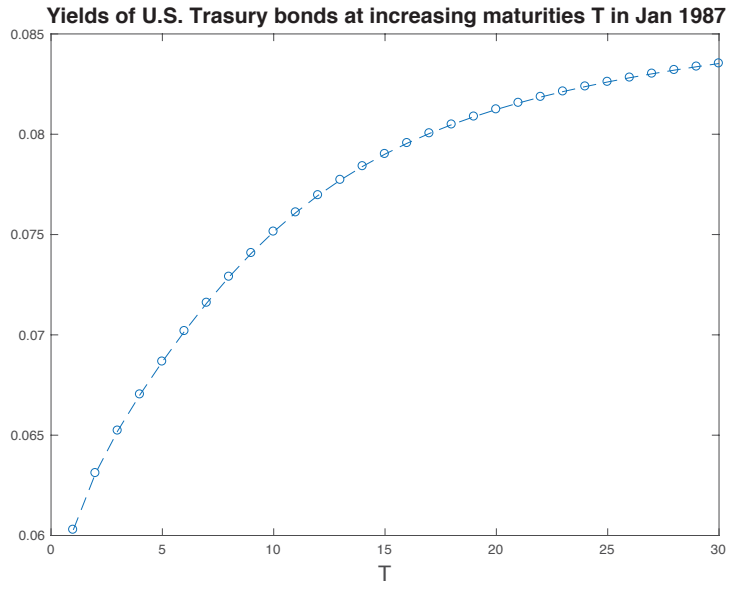


Figure 3.1: Term structure of U.S. Treasury bonds in January 1987. Annualized yields to maturity $r_{s,T}$ where $s = 0$ is January 1987 and T varies yearly from 1988 to 2017.

Moreover, we provide a conditional version of the differential problem for pricing functionals defined on any time window $[s, T]$ by employing the weak time-derivative in $[s, T]$ of Definition 3.1. In our case, the arising eigenvalue is an $\mathcal{F}_s$-measurable random variable (known at the beginning of the trading interval) and, coherently, the growth rate of the stochastic discount factor inherits this randomness. When $s = 0$ the eigenvalue is actually deterministic. Figure 3.1 depicts the term structure of U.S. Treasury bonds yields $r_{s,T}$ by fixing $s = 0$ at January 1987 and letting T vary annually from 1988 to 2017. Data are provided by the Federal Reserve Board at daily frequency: see Gürkaynak, Sack, and Wright (2007).

We now enter the details of our construction. We place ourselves in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, F)$, where we employ the forward measure. We want to solve in $\mathcal{W}_{s,T}$ the following pricing differential equation with random coefficient given by the yield to maturity $r_{s,T} \in L^0(\mathcal{F}_s)$, i.e.

$$\begin{cases} \mathcal{D}\pi_s(t) = r_{s,T}\pi_s(t) & t \in [s, T) \\ \pi_s(T) = h_T \end{cases} \tag{3.7}$$

with $h_T \in L^1_s(\mathcal{F}_T, F)$. The unique solution is

$$\pi_s(t) = e^{-r_{s,T}(T-t)}\mathbb{E}^F_t[h_T].$$

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Theorem 3.5 *There exists a unique solution π_s of Problem (3.7) in $\mathcal{W}_{s,T}$ and, for every t in $[0,T]$*

$$\pi_s(t) = e^{-r_{s,T}(T-t)}\mathbb{E}_t^F[h_T]. \tag{3.8}$$

Theorem 3.5 is substantially an assessment of the conditional martingality of the process $\left\{e^{r_{s,T}(T-t)}\pi_s(t)\right\}_{t\in[s,T]}$ under F. This process is, in fact, the collection of forward prices for h_T in the period $[s,T]$.

For any $t \in [s,T]$, $\pi_s(t)$ relies fundamentally on the information available at the initial date s because $r_{s,T}$ is the only rate taken into consideration. Indeed, $\pi_s(t)$ aims at pricing at time t the lump sum h_T according to the $\mathcal{F}_s$-measurable yield $r_{s,T}$. The one variable in which the valuation instant is synchronous with the information set is actually $\pi_s(s)$. Specifically, $\pi_s(s)$ coincides with the no arbitrage valuation formula $\pi(s)$. The two coincide also at the terminal date:

$$\pi_s(s) = \pi(s), \qquad \pi_s(T) = \pi(T) = h_T.$$

In general, when $s < t < T$ the function $\pi_s(t)$ is different from the no arbitrage price $\pi(t)$. In particular, fixed a date t, a bunch of valuations $\pi_s(t)$ are available, obtained by solving several problems as (3.7) defined on different time intervals $[s,T]$. However, $\pi_s(t)$ with different s are somewhat coherent within them. Indeed, the martingality of forward prices under F ensures that, for any $s_1 \leqslant s_2 \leqslant t$,

$$\mathbb{E}_{s_1}^F\left[e^{r_{s_2,T}(T-t)}\pi_{s_2}(t)\right] = \mathbb{E}_{s_1}^F\left[e^{r_{s_1,T}(T-t)}\pi_{s_1}(t)\right].$$

Up to a proper adjustment, although $\pi_{s_2}(t)$ is based on a richer information set, its conditional expectation at time s_1 equals the one of $\pi_{s_1}(t)$, which exploits less information.

The proper link between π_s and π is given by

$$\pi_s(t) = e^{-\left(r_{s,T}-r_{t,T}\right)(T-t)}\pi(t) = e^{r_{s,T}(t-s)}\frac{p(s,T)}{p(t,T)}\pi(t).$$

This equality can also be read as a parity relation between π_s and π:

$$e^{r_{s,T}(T-t)}\pi_s(t) = e^{r_{t,T}(T-t)}\pi(t),$$

where both sides deliver the price at time t of a forward contract on h_T for date T.

We observe that the discrepancy between π_s and the no arbitrage price is genuinely due to the term structure of interest rates in the market. In case rates are constant, it is apparent that the distortion between π_s and π disappears. As a result, Problem (3.7) generalizes the dynamics of no arbitrage prices from the well-known case of constant rates of interest. Indeed, π_s solves in $[s,T]$ the analogous differential equation of π, where interest rates are replaced by $\mathcal{F}_s$-measurable bond

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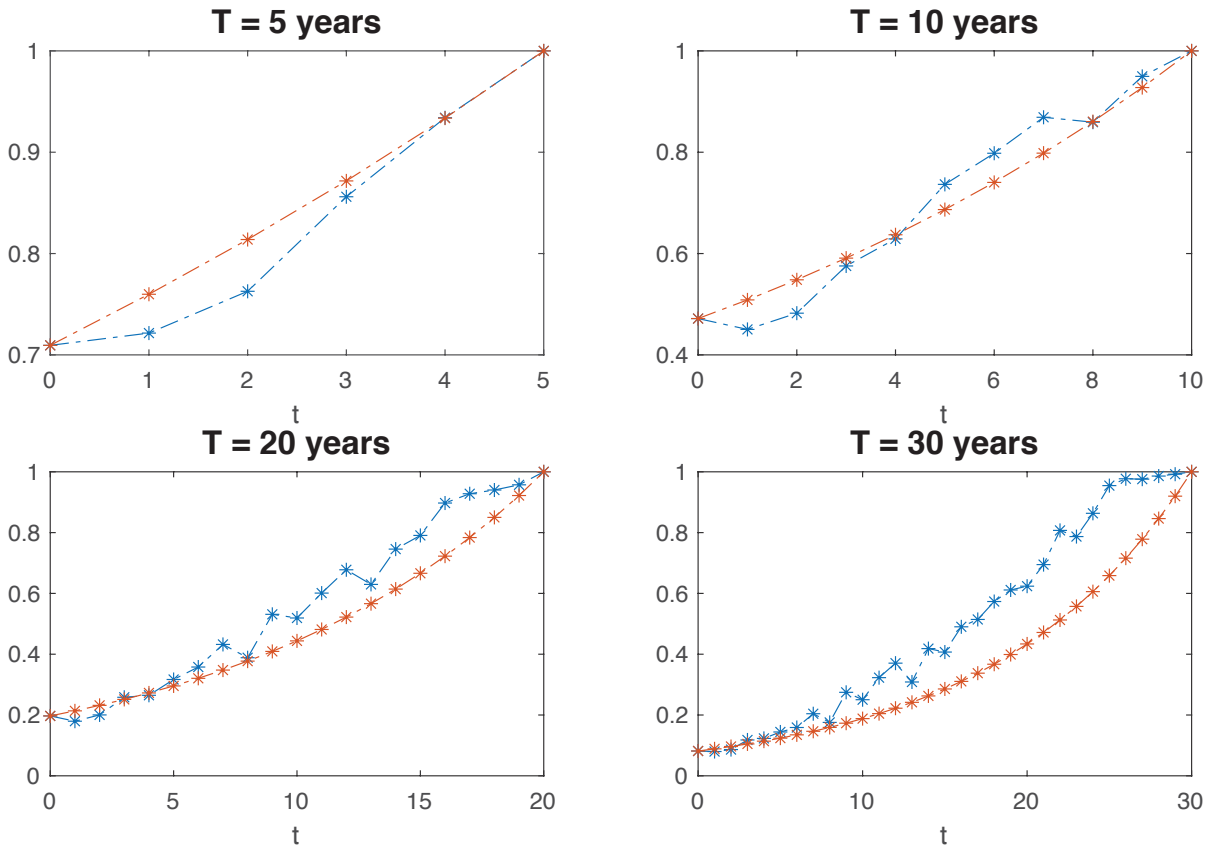


Figure 3.2: We fix $s = 0$ at January 1987 and we consider increasing annual maturities T of U.S. Treasury bonds. Red lines depict the annual values of $\pi_0(t)$ for these assets in $[0, T]$. Accordingly, blue lines represent the (ex-post) realizations of no arbitrage prices $p(t, T)$ at any year t until expiration.

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yields and the forward measure is employed instead of the risk-neutral one. In this perspective π_s may be seen as the analogous of the no arbitrage price in a floating rate setting.

Figure 3.2 considers U.S. Treasury bonds in January 1987 with increasing expiry T of $5, 10, 20$ and 30 years. We plot the values $\pi_0(t)$ of these securities by fixing $s = 0$ at January 1987 and considering annual t up to maturity. In the same graphs we depict the ex-post realizations of no arbitrage prices $p(t, T)$, observable in later years. $\pi_0(t)$ and $p(t, T)$ are indistinguishable both in the short-term and near maturity. Different values of $r_{0,T}$ may, however, induce an overestimation or underestimation of bond prices, that can be evaluated in future periods according to the actual realizations of bond prices in the market.

Additional comparisons between π_s and the no arbitrage price π are illustrated in Section 3.7, where payoffs of European call options are also considered.

3.5.2 π_s as risk-adjusted price

π_s is the no arbitrage price of the payoff h_T in a parallel market in which interest rates are fixed and equal to the bond yield $r_{s,T}$ over $[s, T]$. In Problem (3.7), indeed, we retrieve the dynamics of the constant-rate case up to considering the *average rate* $r_{s,T}$.

In fact, π_s can be interpreted as a *rate-adjusted price* or *indifference price*, that hedges from interest rates variability. We bolster this intuition by describing a self-financing strategy in distinct markets that delivers the same cashflows.

We consider an investor who incurs an expenditure of $p(s, T)$ at time s in order to buy a self-financing portfolio that delivers some units of h_T at maturity. However, the agent has access to two different arbitrage-free markets. In the first one, rates are stochastic and h_T is priced by π. In the second one, instead, instantaneous rates are constantly equal to $r_{s,T}$ and the price of h_T is provided by π_s.

In the first market, at date s the investor buys a zero-coupon bond with expiration T. At a later time t she rebalances her allocation, worth $p(t, T)$, by buying the amount $p(t, T)/\pi(t)$ of derivative at the no arbitrage price $\pi(t)$. Then, she holds this portfolio until maturity.

In the second market, the agent still buys a zero-coupon T-bond at date s. However, at time t her position is worth $e^{-r_{s,T}(T-t)}$. She liquidates this amount to buy $e^{-r_{s,T}(T-t)}/\pi_s(t)$ units of derivative.

In both markets the investor ends up with the same quantity of derivatives at maturity: terminal portfolio values coincide. Hence, this description substantiates the hedging nature of π_s. Indeed, although in the first market the agent benefits from the variability of interest rates, in the second one she enjoys a flat term structure. Given an identical initial expenditure, the same

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outcome obtains in both contexts.

3.5.3 Role of $r_{s,T}$ in the differential problem

Problem (3.7) is not the sole way to generalize the differential relation $\mathcal{D}\pi = r\pi$ of Problem (3.5). Indeed, under the forward measure, any process $\zeta \in \mathcal{W}_{s,T}$ defined by $\zeta(t) = e^{-\rho_s(T-t)}\mathbb{E}_t^F[h_T]$ with $\rho_s \in L^0(\mathcal{F}_s)$ solves

$$\begin{cases} \mathcal{D}\zeta(t) = \rho_s \zeta(t) & t \in [s,T) \\ \zeta(T) = h_T \end{cases} \tag{3.9}$$

where $h_T \in L_s^1(\mathcal{F}_T, F)$.

For instance, assume that the pure discount bond price $p(s,T)$ is differentiable with respect to maturity T. We can set ρ_s as the instantaneous forward rate with settlement date T contracted at time s, i.e.

$$\rho_s = -\frac{\partial \log p(s,T)}{\partial T} = r_{s,T} + \frac{\partial r_{s,T}}{\partial T}(T-s).$$

The associated process ζ satisfies the parity relation with the no arbitrage price π

$$e^{\rho_s(T-t)}\zeta(t) = e^{r_{t,T}(T-t)}\pi(t)$$

and both terms of the equality match the forward price of h_T at t. In addition, if case interest rates are constant, ρ_s collapses to the instantaneous rate and ζ specializes to the no arbitrage price. However, although conceptually attracting, the instantaneous forward rate has an important drawback. Indeed, at the initial date s, $\zeta(s)$ is different from the no arbitrage price $\pi(s)$ when rates are time-varying.

Another interesting parametrization is provided by of $\rho_s = \mathbb{E}_s[Y_T]$. This quantity actually coincides with the instantaneous forward rate if the expectation hypothesis of Fisher (1930) holds. The solution ζ of Problem (3.9) can be easily determined. As in the previous example, ρ_s reduces to the instantaneous rate in case rates are constant and, accordingly, ζ becomes indistinguishable from π. Nevertheless, when interest rates are floating over time, there is an initial mispricing because $\zeta(s)$ is different from $\pi(s)$.

In fact, the initial $\zeta(s)$ equals the no arbitrage price $\pi(s)$ if and only if $\rho_s = r_{s,T}$. This fact motivates the choice of the yield to maturity $r_{s,T}$ in the formalization of Problem (3.7) instead of alternative generalizations of the instantaneous rate. Indeed, the arising solution π_s exactly provides the no arbitrage price at instant s. Furthermore, π_s is the unique process - among all possible ζ that solve problems as (3.9) - that features this correctness property.

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3.5.4 Long-term relations between π_s and π

Our semimartingale framework is compatible with the setting of Qin and Linetsky (2017), which formalize the convergence of pure discount bond yields, forward measures and stochastic discount factors when the horizon T becomes larger and larger. Under their regularity assumptions, for any positive s, the yield to maturity $r_{s,T}$ converges in probability to a *long-term yield* $\lambda > 0$. This result is coherent with the persistence of the yield curve for large maturities widely documented empirically and discussed, for instance, in Diebold and Li (2006). Notably, λ is not dependent on s, consistently with the impossibility of falling long-term rates illustrated by Dybvig, Ingersoll Jr, and Ross (1996) and Hubalek, Klein, and Teichmayn (2002). Moreover, the convergences in probability holds with respect to any measure equivalent to the physical one (we will employ P for simplicity).

Qin and Linetsky (2017) assume that:

- the stochastic discount factor S_t is a strictly positive semimartingale such that $\mathbb{E}\left[S_T/S_t\right]$ is finite for all $0 \leqslant t < T$,
- for all $t > 0$, when T goes to infinity, $\mathbb{E}_t\left[S_T\right]/\mathbb{E}\left[S_T\right]$ converges in L^1 to a positive $\mathcal{F}_t$-measurable random variable M_t^∞.

The long-term yield λ is associated with a long bond, obtained by a roll-over portfolio over zero-coupon bonds across increasing maturities. The time t value of this portfolio is denoted by B_t^∞, while π_t^∞ is the related long-term discounted value, namely

$$\pi_t^\infty = e^{-\lambda t} B_t^\infty.$$

Our setting contributes to the interpretation of λ as eigenvalue for the long run. Indeed, λ is the limit in probability of eigenvalues $r_{s,T}$ in our eigenvalue-eigenvector problems

$$\mathcal{D}\pi_s = r_{s,T}\pi_s$$

as T goes to infinity. The long-term yield is, then, characterized as a limit of eigenvalues. However, the eigenvector associated with each $r_{s,T}$ is π_s and generally differs from the no arbitrage price π. Then, it is important to investigate the behaviour of the discrepancy between $\pi_s(t)$ and $\pi(t)$ when maturity T increases, for any given $s > 0$ and $t > s$.

The relation

$$\pi_s(t) = e^{-\left(r_{s,T} - r_{t,T}\right)(T-t)}\pi(t)$$

may convey the intuition that $\pi_s(t)$ and $\pi(t)$ approach each other when the horizon under scrutiny is far enough. Indeed, $r_{s,T} - r_{t,T}$ converges in probability to zero since $r_{s,T} \xrightarrow{P} \lambda$ irrespectively of s. Nevertheless, the asymptotic relations between $\pi_s(t)$ and $\pi(t)$ are more delicate. The next proposition provides two results both in levels and in logarithmic terms.

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Proposition 3.6 *Under the assumptions of Qin and Linetsky (2017), for all $s > 0$ and $t > s$*

$$\frac{\pi_s(t) - \pi(t)}{\mathbb{E}_t^F [h_T]} \xrightarrow{P} 0, \qquad T \longrightarrow +\infty$$

and

$$\frac{\log \pi_s(t) - \log \pi(t)}{T - t} \xrightarrow{P} 0, \qquad T \longrightarrow +\infty.$$

The asymptotic behaviours of Proposition 3.6 are valid for any choice of s and $t > s$, which are fixed before taking the limit across increasing horizons. However, convergences are taken just after a proper rescaling of $\pi_s(t)$ and the no arbitrage price $\pi(t)$. The normalization of the first asymptotic result is actually irrelevant when pure discount bonds with farther and farther expiries are under scrutiny. The second result, instead, employs the tenor to rescale the difference between $\pi_s(t)$ and $\pi(t)$ in logs.

Note that the speed of convergence of $r_{s,T} - r_{t,T}$ may improve the above asymptotic trends. For instance, if $\left|r_{s,T} - r_{t,T}\right| \leqslant (T-t)^\theta$ definitely with $\theta < -1$, then

$$\log \pi_s(t) - \log \pi(t) \xrightarrow{P} 0, \qquad T \longrightarrow +\infty,$$

namely

$$\frac{\pi_s(t)}{\pi(t)} \xrightarrow{P} 1, \qquad T \longrightarrow +\infty.$$

In fact, the behaviour of the product $\left(r_{s,T} - r_{t,T}\right)(T-t)$ actually determines the convergence.

We concentrate on the long-run behaviour of the ratio π_s/π in the following proposition.

Proposition 3.7 *Under the assumptions of Qin and Linetsky (2017), for all $s > 0$ and $t > s$*

$$\mathbb{E}_s^F \left[\frac{\pi_s(t)}{\pi(t)}\right] \xrightarrow{P} e^{\left(\lambda - r_{s,t}\right)(t-s)}, \qquad T \longrightarrow +\infty.$$

Moreover,

$$\frac{\pi_s(t)}{\pi(t)} \xrightarrow{P} \frac{\pi_s^\infty}{\pi_t^\infty}, \qquad T \longrightarrow +\infty.$$

The difference between the long-term yield and the bond yield $r_{s,t}$ determines the limit for the expected misalignment between $\pi_s(t)$ and the no arbitrage price at t, under the forward measure. In addition, the ratio π_s/π turns out to have a well-defined long-term limit in probability. As a result, any $\pi_s(t)$ differs from $\pi(t)$ by a multiplicative $\mathcal{F}_t$-measurable factor associated with the discounted long bond.

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3.6 Pricing kernel growth

From the definition of Problem (3.7) it is clear that $\mathcal{D}\pi_s$ is weakly time-differentiable, too. The same also holds for weak time-derivatives of higher orders. As a result, we can claim that π_s is infinitely weakly time-differentiable and so $\pi_s \in \mathcal{Z}_{s,T}$.

Now fix some instant t in the time window $[s,T]$. Following eq. (3.4), the weak time-derivative defines an L^0-linear operator $\mathcal{D} : \mathcal{Z}_{s,T} \longrightarrow \mathcal{Z}_{s,T}$ and π_s satisfies the eigenvalue-eigenvector problem

$$\mathcal{D}\pi_s = r_{s,T}\pi_s,$$

where the eigenvalue $r_{s,T}$ belongs to $L^0(\mathcal{F}_s)$.

As in Hansen and Scheinkman (2009) we suppose that h_T is positive so that $\pi_s(t)$ is positive. Hence, $\pi_s(t)$ is a *principal eigenvector* related to $r_{s,T}$, a fact in line with the Perron-Frobenius theory, usually employed in Markovian environments and successfully applied by Ross (2015). We now show that the eigenvalue $r_{s,T}$ plays a fundamental role in individuating a growth rate for the pricing kernel in any time interval $[s,t]$, based on the information available at instant s.

As introduced in Subsection 3.2.2, the pricing kernel $M_{s,t}$ satisfies

$$M_{s,t} = \frac{e^{-r_{s,T}(T-s)}}{e^{-r_{t,T}(T-t)}} G_{s,t} = \frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} \, \frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)} \, G_{s,t}$$

for all $T > t+s$. In the next proposition we assess the asymptotic behaviours of the last three factors. Note, in particular, that the ratio $p(s,T)/p(s,T-t)$ is the return on a forward rate agreement between $T-t$ and T contracted at date s.

Proposition 3.8 *Under the assumptions of Qin and Linetsky (2017), for all $s > 0$ and $t > s$*

$$\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} \xrightarrow{P} e^{-\lambda(t-s)}, \qquad T \longrightarrow +\infty$$

$$\frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)} \xrightarrow{P} \frac{\pi_s^\infty}{\pi_t^\infty}, \qquad T \longrightarrow +\infty$$

$$G_{s,t} \xrightarrow{P} \frac{M_t^\infty}{M_s^\infty}, \qquad T \longrightarrow +\infty$$

where M_t^∞ is a P-martingale.

In a Markovian environment, M_t^∞/M_s^∞ defines the *permanent* (or *martingale*) *component* of Hansen-Scheinkman decomposition, while $\pi_s^\infty/\pi_t^\infty$ constitutes the *transient component*. For the empirical estimation of these terms see the recent Christensen (2017).

In addition to being a limit of bond yields, the long-term yield λ is also a limit of continuously compounded forward rates contracted at s, as the convergence of $p(s,T)/p(s,T-t)$ suggests.

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This approach is reminiscent of the construction of Backus, Gregory, and Zin (1989) and of the methodology of Alvarez and Jermann (2005), who highlight the informativeness of prices of bonds with distant maturities on pricing kernel persistence. The term $e^{-\lambda(t-s)}$ determines, in fact, the growth of the pricing kernel in the time interval $[s,t]$ embedded in an arbitrarily large time horizon.

Coherently with the long-run convergence, when the horizon is finite we call *growth term* of $M_{s,t}$ the first ratio of Proposition 3.8, i.e.

$$\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} = \frac{e^{-r_{s,T}(T-2s)}}{e^{-r_{s,T-t}(T-t-s)}}. \tag{3.10}$$

In case interest rates are constant, this quantity actually reduces to $e^{-r(t-s)}$, which characterizes the growth of the pricing kernel $e^{-r(t-s)}L_{s,t}$.

We can, then, claim that the pricing kernel $M_{s,t}$ features a random growth term determined by bond yields computed at time s. Moreover, $r_{s,T}$ is an $\mathcal{F}_s$-measurable eigenvalue for the operator $\mathcal{D}$ with eigenvector π_s, which generally differs from the no arbitrage price π. When interest rates are constant, the eigenvalue r coincides with the growth rate of $M_{s,t}$ and π_s equals π, while the situation is more involved when rates are time-varying. However, a connection between the eigenvalue of $\mathcal{D}$ and the growth rate of the pricing kernel is still present since they are both driven by yields to maturity, known ex ante. This relation, which holds for finite horizons, is also robust for increasing maturities in the long run. Hence, $r_{s,T}$ arises as a risk-neutral price determinant which is compatible with both short and long-term risk exposures.

3.7 Affine interest rate models

We provide an illustration of the relations described so far in an arbitrage-free market with diffusive short-term rates. After specializing to affine term structure models, we illustrate the differences between π_s and the no arbitrage price by considering, in particular, the payoffs of pure discount bonds and bond European call options. We also devote some attention to the Feynman-Kaç partial differential equations satisfied by the two prices. We finally highlight the expression of the pricing kernel growth term in this class of models.

3.7.1 Short-term rate models under the forward measure

We assume that instantaneous rates follow the diffusion process

$$dY_t = \mu\left(t, Y_t\right) dt + \sigma\left(t, Y_t\right) dW_t^P$$

in the time window $[0,T]$. Here, μ and σ are measurable functions of t and Y_t and W_t^P denotes a Wiener process under the physical measure. We assume that the pure discount T-bond price at

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time t is function of t and Y_t. Therefore, Itô's formula allows us to determine drift and diffusion coefficient of $p(t,T)$, so that

$$\frac{dp(t,T)}{p(t,T)} = \tilde{\mu}\left(t, Y_t\right) dt + \tilde{\sigma}\left(t, Y_t\right) dW_t^P.$$

We denote by ν the market price of risk process

$$\nu_t = \frac{\tilde{\mu}\left(t, Y_t\right) - Y_t}{\tilde{\sigma}\left(t, Y_t\right)}.$$

By Girsanov Theorem, we are able to build a Wiener process under the risk-neutral measure Q through the stochastic differential

$$dW_t^Q = dW_t^P + \nu_t dt.$$

The equivalent martingale measure Q triggers a drift change in bond price dynamics:

$$\frac{dp(t,T)}{p(t,T)} = Y_t dt + \tilde{\sigma}\left(t, Y_t\right) dW_t^Q.$$

Hence, the drift of no arbitrage bond prices coincides with the instantaneous rate at any instant. In case $p(t,T)$ is weakly time-differentiable, its weak time-derivative under Q coincides with $Y_t p(t,T)$ at any instant t.

When changing the numéraire from the money market account to the pure discount T-bond, the dynamics of $p(t,T)$ under the forward measure become

$$\frac{dp(t,T)}{p(t,T)} = \left[Y_t + \tilde{\sigma}^2\left(t, Y_t\right)\right] dt + \tilde{\sigma}\left(t, Y_t\right) dW_t^F, \tag{3.11}$$

where W_t^F denotes the Wiener process, under the forward measure, satisfying

$$dW_t^F = dW_t^Q - \tilde{\sigma}\left(t, Y_t\right) dt.$$

A clear derivation can be found in Brigo and Mercurio (2007). In eq. (3.11) the drift change associated with the forward measure is apparent. Differently from the Q-dynamics, the drift under F depends on the diffusion coefficient of the bond price itself. This feature contrasts with the drift of the price $\pi_s(t)$ of the same pure discount T-bond, whose differential satisfies

$$\frac{d\pi_s(t)}{\pi_s(t)} = r_{s,T} dt.$$

In terms of weak time-differentiability, under the forward measure the weak time-derivative of $p(t,T)$ turns out to be $\left[Y_t + \tilde{\sigma}^2\left(t, Y_t\right)\right] p(t,T)$. Hence, the coefficient of $p(t,T)$ in the drift is floating over time, even in the simple example of security with unitary payoff. For this reason, it is unlikely to find an eigenvalue-eigenvector formulation by explicitly employing no arbitrage prices when rates of interest are stochastic.

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3.7.2 Affine models

In one-factor affine term structure models - as Vasicek (1977), Cox, Ingersoll, and Ross (1985), Ho and Lee (1986) or Hull and White (1990) - the price of a zero-coupon bond with expiration T depends on the instantaneous rate Y_t through the exponential relation

$$p(t,T) = e^{A(t,T)-B(t,T)Y_t},$$

where A, B are deterministic functions. Accordingly, the yield to maturity is

$$r_{t,T} = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t}Y_t.$$

By Itô's differential rule, the diffusion coefficient of dp/p is

$$\tilde{\sigma}\left(t, Y_t\right) = -B(t,T)\sigma\left(t, Y_t\right)$$

and so, under the forward measure, the drift of dp/p depends on the volatility of the underlying instantaneous rate:

$$\frac{dp(t,T)}{p(t,T)} = \left[Y_t + B^2(t,T)\sigma^2\left(t, Y_t\right)\right]dt - B(t,T)\sigma\left(t, Y_t\right)dW_t^F.$$

However, the dynamic of the related π_s is

$$\frac{d\pi_s(t)}{\pi_s(t)} = \left(-\frac{A(s,T)}{T-s} + \frac{B(s,T)}{T-s}Y_s\right)dt.$$

As an example, we consider the Vasicek (1977) model, in which the coefficients of the instantaneous rate are

$$\mu\left(t, Y_t\right) = k\theta - (k - \sigma\xi)Y_t, \qquad \sigma\left(t, Y_t\right) = \sigma$$

where $k, \theta, \sigma, \xi > 0$ and the market price of risk is ξY_t. Therefore, the dynamics under Q are

$$dY_t = k\left[\theta - Y_t\right]dt + \sigma dW_t^Q. \tag{3.12}$$

The short-term rate is mean-reverting towards the value θ in the long term at a speed dictated by k. In addition, volatility is constant over time. Under this specification,

$$A(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4k}B^2(t,T)$$

and

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k}.$$

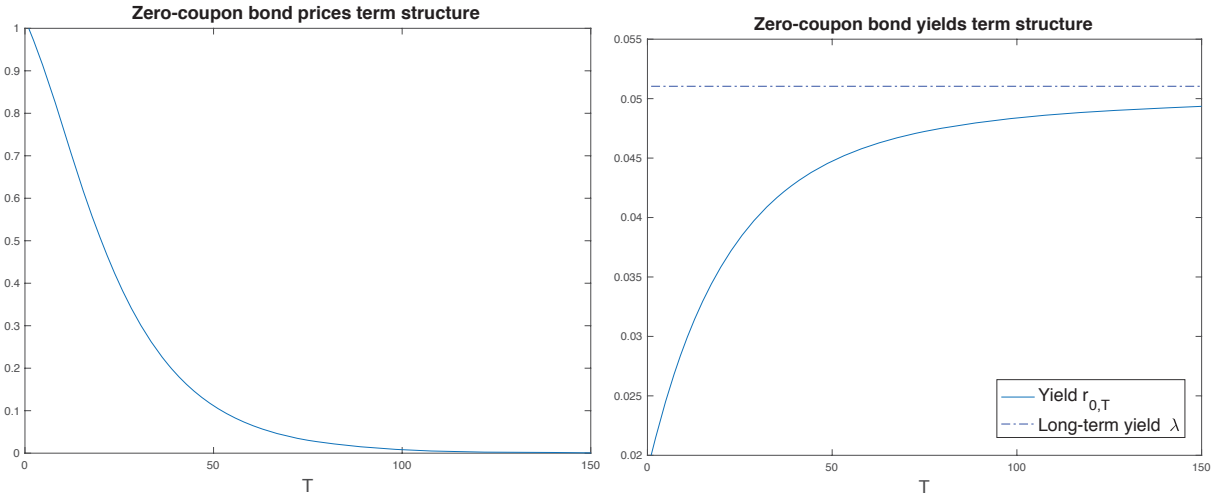
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Therefore, the drift coefficient of the risk-neutral bond price under the forward measure is

$$Y_t + \frac{\sigma^2}{k^2}\left(1 - e^{-k(T-t)}\right)^2.$$

When the horizon T becomes infinitely large, this term converges a.s. to $Y_t + \sigma^2/k^2$. On the other hand, the drift parameter $r_{0,T}$ of the price $\pi_0(t,T)$ converges a.s. to the long-term yield

$$\lambda = \theta - \frac{\sigma^2}{2k^2}. \tag{3.13}$$



(a) Zero-coupon bond prices term structure

(b) Yield-to-maturity term structure

Figure 3.1: The left and right panel depict no arbitrage pure discount bond prices (at time zero) and yields to maturity with respect to increasing horizons T under Vasicek specifications. The dashed horizontal line represents the long-term yield λ of eq. (3.13).

Figure 3.1 displays the term structure of no arbitrage prices and yields of a pure discount bond in a simulated Vasicek model, computed at time zero. For the numerical exercise we fix the parameters $k = 0.0311$, $\theta = 0.0967$, $\sigma = 0.01$ as estimated by Nowman (1997) in Table III with monthly data and we set $\xi = 0.2$ and $Y_0 = 0.02$. When horizon T gets larger and larger we visualize the convergence of yields to maturity to the long-term yield of eq. (3.13). In Figure 3.2, instead, the horizon is fixed and prices $p(t,T)$ and $\pi_0(t)$ of pure discount T-bonds are plotted. We observe that $\pi_0(t)$ overlaps the no arbitrage price both in the short-term and in the proximity of maturity, coherently with the discussion in Subsection 3.5.3. A departure from $p(t,T)$ is sizeable in the middle of the trading window. Moreover, due to its simpler expression, $\pi_0(t)$ turns out to be generally smoother than $p(t,T)$.

In Appendix 3.C we repeat the exercise by simulating a Cox, Ingersoll, and Ross (1985) model for short-term rates.

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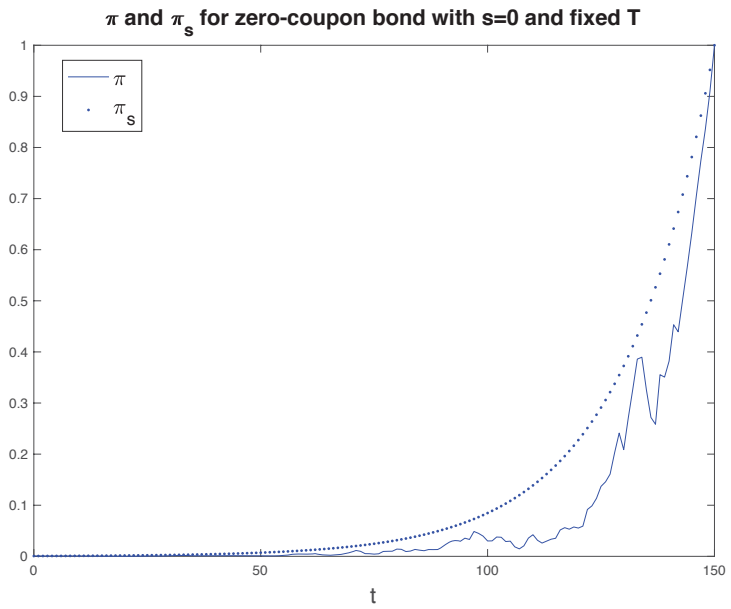


Figure 3.2: Values of $\pi_0(t)$ and of a realization of the no arbitrage price $p(t,T)$ of a pure discount bond with fixed maturity T for any t in $[0,T]$ in Vasicek. The two prices are represented by a dotted and a solid line, respectively.

3.7.3 Call options in Vasicek

So far we discussed no arbitrage prices of zero-coupon bonds under Vasicek assumptions. However, also derivative pricing has been extensively studied in this setting by employing forward measures. A germane example of payoff valuation under these measures is given by Jamshidian (1989), which extends Black and Scholes (1973) to options on pure discount bonds.

Thus, consider a European call option with expiration τ and strike price c over a zero coupon bond with maturity $T > \tau$. When short-term rates follow Vasicek dynamics, the no arbitrage price of the option at any time t between 0 and τ is

$$\pi(t) = p(t,T)\mathcal{N}(q) - cp(t,\tau)\mathcal{N}(q-\hat{\sigma}),$$

where $\mathcal{N}$ denotes the cumulative distribution function of a standard Gaussian,

$$q = \frac{\log\left(\frac{p(t,T)}{cp(t,\tau)}\right) + \frac{\hat{\sigma}^2}{2}}{\hat{\sigma}}$$

and

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2(k-\sigma\xi)(\tau-t)}}{2(k-\sigma\xi)} \frac{\left(1 - e^{-(k-\sigma\xi)(T-\tau)}\right)^2}{(k-\sigma\xi)^2}.$$

See the derivation by Jamshidian (1989).

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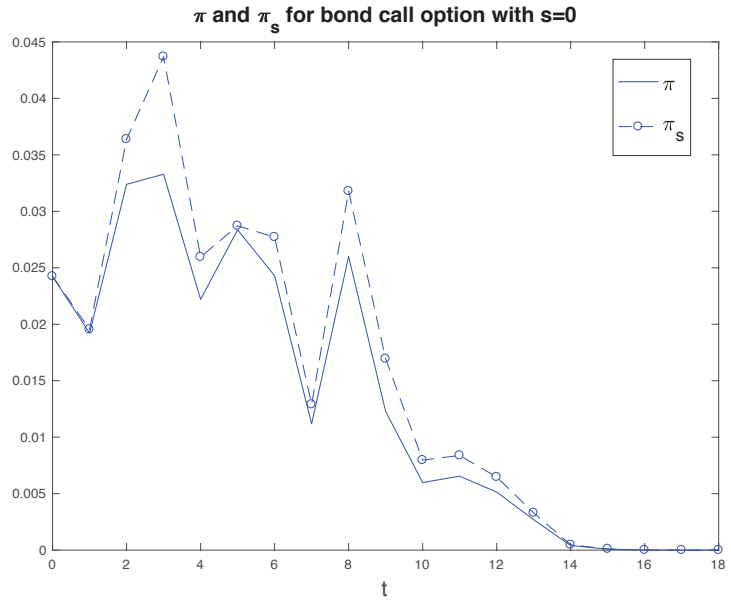


Figure 3.3: Values of $\pi_0(t)$ and of a realization of the no arbitrage price $\pi(t)$ of a European call option on a pure discount bond as described in the text. The two prices are represented by a dotted and a solid line, respectively.

As illustrated in Subsection 3.5.1, the price π_0 of the call option can be easily obtained from the no arbitrage price through the relation

$$\pi_0(t) = e^{-\left(r_{0,\tau} - r_{t,\tau}\right)(\tau - t)} \pi(t).$$

We plot both price processes in Figure 3.3, where we set $T = 36$ months, $\tau = 18$ months and $c = 0.5$. The two quantities almost coincide in the first months and near the option expiration τ. The difference between the two may be appreciated just during intermediate months and only in case the overall trading period is sufficiently long.

3.7.4 Feynman-Kaç PDE in affine models

The situation is even more involved when the market is generated by an additional set of primary risky assets with prices $X_t^{(1)}, \dots X_t^{(N)}$. Many sources of randomness are present, beyond short-term rates. Although all the drift coefficients of no arbitrage prices of such securities coincide with Y_t under the measure Q, price dynamics become awkward when the forward measure is employed. In particular, the drifts of these prices under the measure F are additionally affected by the correlation between the instantaneous rate and the idiosyncratic random component of the asset under consideration. Rabinovitch (1989) - that incorporates the previous Merton (1973)

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- discusses this point at length. On the other hand, the drift of the corresponding prices π_s is always equal to the yield to maturity of zero-coupon bonds, for any traded security. To further elucidate the issue, we borrow the dynamics of rates and stock prices from the Appendix of Brigo and Mercurio (2007). In the end we determine and contrast the Feynman-Kaç partial differential equations satisfied by π and π_s.

We assume that short-term rates move as in Vasicek model, with dynamics described by eq. (3.12) in the time interval $[s, T]$. Then, we consider a stock price X_t that follows a geometric Brownian motion with volatility $\eta > 0$, correlated with interest rates shocks. The instantaneous correlation between the two underlying Wiener processes is ρ. We can orthogonalize the two sources of randomness and consider, without loss of generality, the dynamics

$$\begin{cases} dX_t = X_t Y_t dt + \eta X_t \left[\rho dW_t^Q + \sqrt{1-\rho^2} dZ_t^Q\right] \\ dY_t = k\left(\theta - Y_t\right) dt + \sigma dW_t^Q, \end{cases}$$

where W_t^Q and Z_t^Q are independent Wiener processes. Under the forward measure, we get

$$\begin{cases} dX_t = X_t \left[Y_t - \frac{\rho\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right] dt + \eta X_t \left[\rho dW_t^F + \sqrt{1-\rho^2} dZ_t^F\right] \\ dY_t = \left[k\left(\theta - Y_t\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right] dt + \sigma dW_t^F. \end{cases}$$

From the differential of X_t it is apparent that the correlation parameter ρ deeply impacts on the drift of X_t under the measure F.

Consider now an attainable contingent claim h_T whose no arbitrage price $\pi(t)$ is a regular function of t, X_t and Y_t, so that Itô's Lemma applies. By omitting negligible higher order terms in the differential of the product, we get

$$d\left(e^{-\int_s^t Y_\tau d\tau}\pi(t)\right) = e^{-\int_s^t Y_\tau d\tau}\left(d\pi(t) - Y_t\pi(t)dt\right).$$

Moreover,

$$\begin{aligned} d\pi(t) = {} & \frac{\partial\pi}{\partial t}dt + \frac{\partial\pi}{\partial x}X_t\left[Y_t dt + \eta\rho dW_t^Q + \eta\sqrt{1-\rho^2}dZ_t^Q\right] \\ & + \frac{\partial\pi}{\partial y}\left[k\left(\theta - Y_t\right)dt + \sigma dW_t^Q\right] \\ & + \frac{\partial^2\pi}{\partial x^2}\frac{\eta^2 X_t^2}{2}dt + \frac{\partial^2\pi}{\partial y^2}\frac{\sigma^2}{2}dt + \frac{\partial^2\pi}{\partial x\partial y}\rho\sigma\eta X_t dt. \end{aligned}$$

Since $e^{-\int_s^t Y_\tau d\tau}\pi(t)$ defines a Q-martingale when t varies in $[s, T]$, the drift of its differential is null. As a result, by setting $X_t = x$ and $Y_t = y$, we obtain the Feynman-Kaç partial differential equation for the no arbitrage price π:

$$\frac{\partial\pi}{\partial t} + xy\frac{\partial\pi}{\partial x} + k\left(\theta - y\right)\frac{\partial\pi}{\partial y} + \frac{\eta^2 x^2}{2}\frac{\partial^2\pi}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2\pi}{\partial y^2} + \rho\sigma\eta x\frac{\partial^2\pi}{\partial x\partial y} = y\pi \tag{3.14}$$

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with terminal condition $\pi(T) = h_T$. Note the role of instantaneous correlation ρ in weighing the cross derivative of π. In case $\rho = 0$ and interest rates are constant, the usual Black-Scholes partial differential equation appears.

We now develop the parallel analysis for the price $\pi_s(t)$ of the payoff h_T. We have

$$d\left(e^{r_{s,T}(T-t)}\pi_s(t)\right) = e^{r_{s,T}(T-t)}\left(d\pi_s(t) - r_{s,T}\pi_s(t)dt\right)$$

and

$$\begin{aligned} d\pi_s(t) = {} & \frac{\partial \pi_s}{\partial t}dt + \frac{\partial \pi_s}{\partial x}X_t\left[\left(Y_t - \frac{\rho\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right)dt + \eta\rho dW_t^F + \eta\sqrt{1-\rho^2}dZ_t^F\right] \\ & + \frac{\partial \pi_s}{\partial y}\left[\left(k\left(\theta - Y_t\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right)dt + \sigma dW_t^F\right] \\ & + \frac{\partial^2 \pi_s}{\partial x^2}\frac{\eta^2 X_t^2}{2}dt + \frac{\partial^2 \pi_s}{\partial y^2}\frac{\sigma^2}{2}dt + \frac{\partial^2 \pi_s}{\partial x \partial y}\rho\sigma\eta X_t dt. \end{aligned}$$

By equating to zero the drift of the differential of the forward price process $e^{r_{s,T}(T-t)}\pi_s(t)$, which is an F-martingale, we obtain the Feynman-Kaç partial differential equation for π_s:

$$\begin{aligned} & \frac{\partial \pi_s}{\partial t} + x\left(y - \frac{\rho\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \pi_s}{\partial x} + \left(k\left(\theta - y\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \pi_s}{\partial y} \\ & \quad + \frac{\eta^2 x^2}{2}\frac{\partial^2 \pi_s}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \pi_s}{\partial y^2} + \rho\sigma\eta x\frac{\partial^2 \pi_s}{\partial x \partial y} = r_{s,T}\pi_s \end{aligned} \tag{3.15}$$

with $\pi_s(T) = h_T$.

Observe the different right-hand side in eq. (3.14) and (3.15): the instantaneous rate y for π and the yield to maturity $r_{s,T}$ for π_s. The coefficients of *spatial* first-order derivatives are also different but the dissimilarity reduces when the convergence speed k increases. In addition, the coefficients of $\partial\pi/\partial x$ and $\partial\pi_s/\partial x$ coincide when correlation ρ is absent.

In general, we can claim that the disparity between π and π_s can also be seized through the solution of different partial differential equations that, however, share the same parabolic structure.

3.7.5 Long-term relations in affine models

Let us denote, as usual, by h_T a generic attainable payoff in a market with exponential affine interest rates. The no arbitrage price of h_T at time t is

$$\pi(t) = e^{A(t,T) - B(t,T)Y_t}\mathbb{E}_t^F\left[h_T\right],$$

while Problem (3.7) features the solution

$$\pi_s(t) = e^{\left\{A(s,T) - B(s,T)Y_s\right\}\frac{T-t}{T-s}}\mathbb{E}_t^F\left[h_T\right].$$

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As a result, the ratio $\pi_s(t)/\pi(t)$ depends only on the instantaneous rate at instants s and t:

$$\frac{\pi_s(t)}{\pi(t)} = e^{\left\{A(s,T)-B(s,T)Y_s\right\}\frac{T-t}{T-s}-A(t,T)+B(t,T)Y_t}.$$

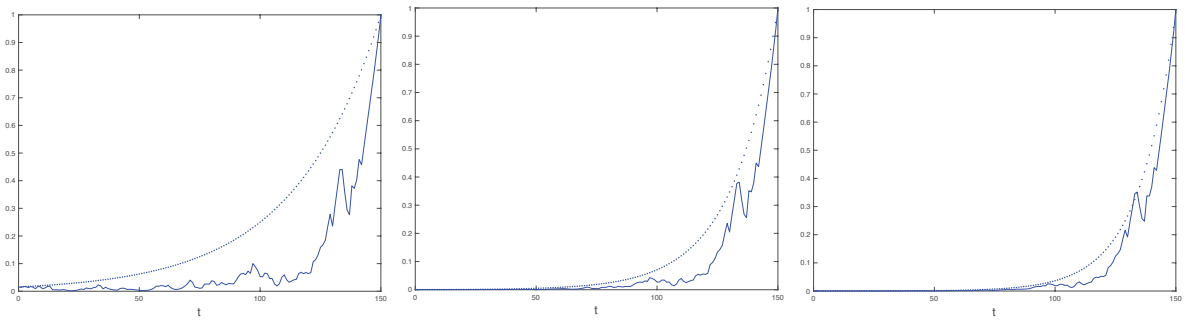
In addition, the long-term relations between π_s and π can be determined explicitly under Vasicek assumptions. Indeed, when T goes to infinity,

$$\mathbb{E}_s^F\left[\frac{\pi_s(t)}{\pi(t)}\right] \xrightarrow{a.s.} e^{\left(\lambda - r_{s,t}\right)(t-s)}$$

and

$$\frac{\pi_s(t)}{\pi(t)} \xrightarrow{a.s.} e^{-\frac{1}{k}(Y_s - Y_t)},$$

where λ is provided by eq. (3.13). The limit of the ratio $\pi_s(t)/\pi(t)$ is determined by the speed parameter k. If k is high, the two prices coincide when maturities are large.



(a) $k = 0.025$ (b) $k = 0.035$ (c) $k = 0.045$

Figure 3.4: Values of $\pi_0(t)$ and of a realization of the no arbitrage price $p(t,T)$ of a zero-coupon bond with fixed maturity T for any t in $[0,T]$. In all panels, the two prices are represented by a dotted and a solid line, respectively. The employed Vasicek parameters are the ones described in the text except for k, which takes increasing values.

More is actually true. In fact, for any (finite) horizon T, $\pi_s(t)/\pi(t)$ converges to 1 a.s. when k grows to infinity. Indeed, fast mean reversion and reduced long-run variance make Vasicek instantaneous rates behave as if they were constant, letting $\pi_s(t)$ collapse to $\pi(t)$. To illustrate this point, in Figure 3.4 we plot the two prices for increasing values of k, while keeping $\theta = 0.0967$, $\sigma = 0.01$, $\xi = 0.2$ and $Y_0 = 0.02$ fixed.

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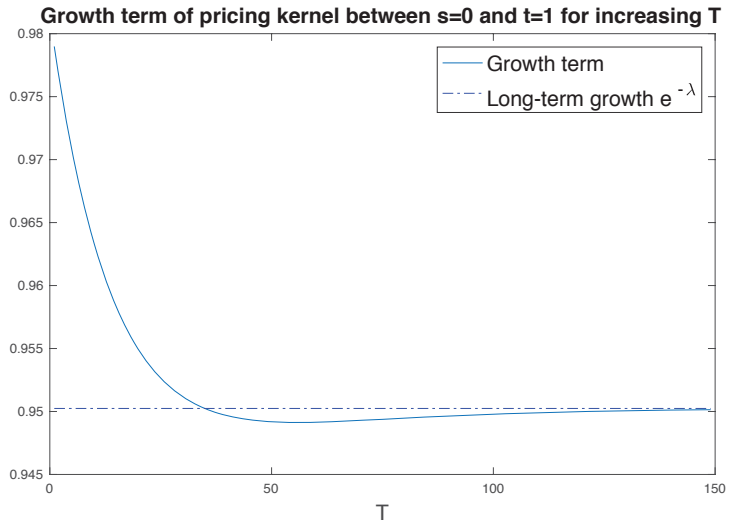


Figure 3.5: Term structure of growth terms of the pricing kernel $M_{0,1}$ in Vasicek model. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed horizontal line is the long-term growth $e^{-\lambda}$ with λ given by eq. (3.13).

3.7.6 Pricing kernel growth in affine models

We finally focus on the pricing kernel between s and t in case yields to maturity are affine. Then, the growth term of the pricing kernel is an exponential function of Y_s, namely

$$\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} = e^{A(s,T)\left(\frac{T-2s}{T-s}\right)-A(s,T-t)-\left\{B(s,T)\left(\frac{T-2s}{T-s}\right)-B(s,T-t)\right\}Y_s}.$$

When the terminal date T goes to infinity, this term converges a.s. to $e^{-\lambda(t-s)}$, as expected. In Figure 3.5 we display this convergence under our Vasicek specifications by setting $s = 0$ and $t = 1$. The pricing kernel growth term converges to the value $e^{-\lambda}$, where the long-term yield is given by eq. (3.13). In addition, Figure 3.C.3 in Appendix 3.C depicts the analogous convergence when short-term rates follow the dynamics of Cox, Ingersoll, and Ross (1985).

3.8 Conclusions

Several known facts of asset pricing theory assume a more general form when interest rates are supposed to be stochastic instead of deterministic. Although the generalizations are not immediate at a first glance, the overall picture clarifies after making the proper substitutions. Indeed, no arbitrage pricing is better understood when the forward measure is employed instead of the risk-neutral one and zero-coupon bond yields replace instantaneous rates. As expected, the discrepancy disappears when instantaneous rates are constant.

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These elements allow us to isolate a random (but ex ante known) growth term from any pricing kernel, driven by yields to maturity, which are also present in the no arbitrage pricing differential equation. The dual role of bond yields reconciles short-term and long-term properties of risk-neutral pricing and sheds light on the aggregation of increasing risk-exposures over time.

Introducing further specific dynamics of interest rates may constitute an interesting avenue for future research, with the purposes of quantifying numerically the difference between π_s and no arbitrage prices, as well as the pricing kernel growth term. In a broader context, it could also be desirable to characterize short rates dynamics through exogenous - or, even, endogenous - factors that enrich the information structure.

From a theoretical perspective, instead, it would be challenging to better understand the implications of random eigenvalues in the Perron-Frobenius theory, which underlies the stochastic discount factor decomposition. Indeed, random dominating eigenvalues may be an indicator of non-deterministic steady states for the dynamics of diverse economic variables.

Last but not least, the notion of orthogonality introduced in Subsection 3.4.1 may be exploited to discuss mean-variance optimality of asset returns. We expect a feasible derivation of the mean-variance frontier in the spirit of Hansen and Richard (1987), with a relevant role played by the measure change.

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Appendix

This Appendix is composed of three main sections. The first one contains a brief summary about forward measures, Section 3.B lists all the proofs and the last section provides some simulations with Cox, Ingersoll, and Ross (1985) model of interest rates.

3.A Forward measure

The *T-forward measure* F is constructed by employing as numéraire the no arbitrage price of a zero-coupon bond with maturity T. F is equivalent to the risk-neutral measure Q and its Radon-Nikodym derivative with respect to Q is

$$F_T = \frac{e^{-\int_0^T Y_\tau d\tau}}{\mathbb{E}\left[L_T e^{-\int_0^T Y_\tau d\tau}\right]} = e^{r_{0,T}T - \int_0^T Y_\tau d\tau}.$$

Moreover,

$$F_t = \mathbb{E}_t\left[L_{t,T}F_T\right] = e^{r_{0,T}T - r_{t,T}(T-t) - \int_0^t Y_\tau d\tau}$$

and we set $F_{t,T} = F_T/F_t$. The Radon-Nikodym derivative of F with respect to P is $G_T = F_T L_T$, which belongs to $L^2(\mathcal{F}_T)$. From $F_t = \mathbb{E}_t\left[L_{t,T}F_T\right]$, we have

$$G_t = \mathbb{E}_t\left[G_T\right] = \mathbb{E}_t\left[L_T F_T\right] = L_t F_t$$

and we define $G_{t,T} = G_T/G_t$.

If rates of interest are constant over time, $F_t = 1$ for any t so that $G_{t,T} = L_{t,T}$.

As for t-bond yields, their relation with T-bond yields is expressed in the following lemma.

Lemma 3.9 *For any $s \leqslant t \leqslant T$,*

$$e^{r_{s,T}(T-s)} = e^{r_{s,t}(t-s)}\mathbb{E}_s^F\left[e^{r_{t,T}(T-t)}\right].$$

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Proof For any $s \leqslant T$, we have $G_{s,T}e^{-r_{s,T}(T-s)} = L_{s,T}F_{s,T} = L_{s,T}e^{-\int_s^T Y_\tau d\tau}$, so that

$$\begin{aligned} e^{-r_{s,t}(t-s)} &= \mathbb{E}_s\left[L_{s,t}e^{-\int_s^t Y_\tau d\tau}\right] = \mathbb{E}_s\left[\frac{L_{s,T}e^{-\int_s^T Y_\tau d\tau}}{L_{t,T}e^{-\int_t^T Y_\tau d\tau}}\right] \\ &= \mathbb{E}_s\left[\frac{G_{s,T}e^{-r_{s,T}(T-s)}}{G_{t,T}e^{-r_{t,T}(T-t)}}\right] = e^{-r_{s,T}(T-s)}\mathbb{E}_s\left[G_{s,t}e^{r_{t,T}(T-t)}\right] \\ &= e^{-r_{s,T}(T-s)}\mathbb{E}_s^F\left[e^{r_{t,T}(T-t)}\right]. \end{aligned}$$

■

We now introduce a numéraire change. $\overline{Q}$ is the risk-adjusted measure associated with the no arbitrage price process of a security generating the payoff G_T, employed as numéraire. Its Radon-Nikodym derivative with respect to F equals $G_T/\mathbb{E}\left[G_T^2\right]$ and, at previous times τ, it becomes

$$\frac{\mathbb{E}_\tau\left[G_T^2\right]}{G_\tau\mathbb{E}\left[G_T^2\right]}.$$

We denote by I_T the Radon-Nikodym density of $\overline{Q}$ with respect to P and by I_τ its conditional expectation at a previous time τ. They are obtained via multiplication by G_T and G_τ respectively:

$$I_T = \frac{G_T^2}{\mathbb{E}\left[G_T^2\right]}, \qquad I_\tau = \frac{\mathbb{E}_\tau\left[G_T^2\right]}{\mathbb{E}\left[G_T^2\right]}.$$

3.B Proofs

Lemma 3.10 *Let $f : [s,T] \longrightarrow \mathbb{R}$.*

i) If f is bounded, nonnegative, with compact support and $\int_s^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c\left([s,T],\mathbb{R}\right)$, then $f = 0$ a.e.

ii) If f is measurable and $\int_s^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c\left([s,T],\mathbb{R}\right)$, then $f = 0$ a.e.

Proof See the Appendix of Marinacci and Severino (2017). ■

Proof of Proposition 3.2

Proof Follow the proof of Proposition 2.1 in Marinacci and Severino (2017) by replacing the unconditional expectation with the conditional expectation with respect to $\mathcal{F}_s$, and the convergence in L^1 with that in L_s^1. ■

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Proof of Proposition 3.3

Proof Assume that u is a conditional martingale. As observed before, u belongs to $\mathcal{V}_{s,T}$. Fixed $t \in [s,T]$, for all $\varphi \in C_c^1\left([t,T], L^0(\mathcal{F}_s)\right)$ and $A_t \in \mathcal{F}_t$,

$$\begin{aligned}\int_t^T \mathbb{E}_s\left[u(\tau)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \int_t^T \mathbb{E}_s\left[\mathbb{E}_t\left[u(\tau)\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \int_t^T \mathbb{E}_s\left[u(t)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\ &= \mathbb{E}_s\left[u(t)\mathbf{1}_{A_t}\right]\int_t^T \varphi'(\tau)d\tau \\ &= 0\end{aligned}$$

because φ is a function in $C_c^1\left([t,T], L^0(\mathcal{F}_s)\right)$. As a result, $w(t) = 0$ for all $t \in [s,T]$ satisfies the definition of weak time-derivative of u, i.e. $\mathcal{D}u = 0$.

Conversely, suppose that $u \in \mathcal{W}_{s,T}$ has $\mathcal{D}u = 0$. First, u is adapted and any $u(t) \in L_s^1(\mathcal{F}_s)$. Therefore, $\mathbb{E}_s\left[|u(t)|\right] \in L^0(\mathcal{F}_s)$ for all $t \in [s,T]$. We need to show that u satisfies the martingale property. We begin with proving that, given $t \in [s,T]$, $\mathbb{E}_t\left[u(\tau)\right]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take a continuous function $\eta : [t,T] \longrightarrow \mathbb{R}$ with compact support such that $\int_t^T \eta(\tau)d\tau = 1$. Given a continuous function $\xi : [t,T] \longrightarrow \mathbb{R}$ with compact support, we define the function $k_\xi : [t,T] \longrightarrow \mathbb{R}$ by

$$k_\xi(\sigma) = \xi(\sigma) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma).$$

k_ξ is continuous with compact support and $\int_t^T k_\xi(\tau)d\tau = 0$. Hence, k_ξ has a primitive K_ξ which is continuous with compact support. As $K_\xi \in C_c^1([t,T],\mathbb{R})$, it is included in $C_c^1\left([t,T], L^0(\mathcal{F}_s)\right)$ and so we employ it as a test function in the definition of weak time-derivative of u. Since $\mathcal{D}u = 0$, for all $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}0 &= \int_t^T \mathbb{E}_s\left[u(\sigma)\mathbf{1}_{A_t}\right]\left\{\xi(\sigma) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)\right\}d\sigma \\ &= \int_t^T \mathbb{E}_s\left[u(\sigma)\mathbf{1}_{A_t}\right]\xi(\sigma)d\sigma - \int_t^T \mathbb{E}_s\left[u(\sigma)\mathbf{1}_{A_t}\right]\left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)d\sigma \\ &= \int_t^T \mathbb{E}_s\left[u(\tau)\mathbf{1}_{A_t}\right]\xi(\tau)d\tau - \int_t^T \left\{\int_t^T \mathbb{E}_s\left[u(\sigma)\mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right\}\xi(\tau)d\tau \\ &= \int_t^T \left\{\mathbb{E}_s\left[u(\tau)\mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}_s\left[u(\sigma)\mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right\}\xi(\tau)d\tau.\end{aligned}$$

By the density of continuous functions ξ, Lemma 3.10 in Appendix ensures that for a.e. $\tau \in$

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$[t,T]$

$$\mathbb{E}_s\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]=\int_t^T\mathbb{E}_s\left[u\left(\sigma\right)\mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma.$$

Since $\int_t^T\eta(\sigma)d\sigma=1$, we can rewrite the left-hand side as $\int_t^T\mathbb{E}_s\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma$ so that

$$\int_t^T\left\{\mathbb{E}_s\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]-\mathbb{E}_s\left[u\left(\sigma\right)\mathbf{1}_{A_t}\right]\right\}\eta(\sigma)d\sigma=0.$$

As the last equality holds for any continuous function η with compact support in $[t,T]$, we find that, for a.e. $\sigma\in[t,T]$,

$$\mathbb{E}_s\left[u\left(\sigma\right)\mathbf{1}_{A_t}\right]=\mathbb{E}_s\left[u\left(\tau\right)\mathbf{1}_{A_t}\right]$$

and so $\mathbb{E}_t\left[u\left(\sigma\right)\right]=\mathbb{E}_t\left[u\left(\tau\right)\right]$. As a result, $\mathbb{E}_t\left[u\left(\tau\right)\right]$ is not dependent on τ for a.e. $\tau\in[t,T]$ and we can state that $\mathbb{E}_t\left[u\left(\tau\right)\right]=f_t$ for some $f_t\in L^1\left(\mathcal{F}_t\right)$.

Since u is L^1_s-right-continuous,

$$\mathbb{E}_t\left[u\left(\tau\right)\right]\xrightarrow{L^1_s}u\left(t\right)\qquad\qquad\tau\longrightarrow t^+.$$

Since for a.e. $\tau\in[t,T]$, $\mathbb{E}_t\left[u\left(\tau\right)\right]$ coincides a.s. with f_t, which is not dependent on τ, the uniqueness of the L^1_s-limit implies that $f_t=u\left(t\right)$. Therefore, for any $t\in[0,T]$, for a.e. $\tau\in[t,T]$

$$\mathbb{E}_t\left[u\left(\tau\right)\right]=u\left(t\right).$$

This property is actually satisfied by any $\tau\in[t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}}\subset[t,T]$ such that $\tau_i\longrightarrow\tau^+$ and $\mathbb{E}_t\left[u\left(\tau_i\right)\right]=u\left(t\right)$. Since u is L^1_s-right-continuous, the L^1_s-limit of $\mathbb{E}_t\left[u\left(\tau_i\right)\right]$ is $\mathbb{E}_t\left[u\left(\tau\right)\right]$. Nevertheless, $\mathbb{E}_t\left[u\left(\tau_i\right)\right]=u\left(t\right)$ for all i and so, by uniqueness of the L^1_s-limit, $\mathbb{E}_t\left[u\left(\tau\right)\right]=u\left(t\right)$. ■

Proof of Proposition 3.4

Proof The algebra $L^0\left(\mathcal{F}_s\right)$ is endowed with the pointwise sum and product between random variables. The outer product $\cdot:L^0\left(\mathcal{F}_s\right)\times H_{s,T}\longrightarrow H_{s,T}$ is well-defined because, for any $a_s\in L^0\left(\mathcal{F}_s\right)$ and $u\in H_{s,T}$, a_su belongs to $\mathcal{V}_{s,T}$ too and $\mathcal{D}\left(a_su\right)=a_s\mathcal{D}u=0$ by eq. (3.4). The conditional square integrability requirement of $H_{s,T}$ is still fulfilled and so a_su belongs to $H_{s,T}$.

Moreover, for each $a_s,b_s\in L^0\left(\mathcal{F}_s\right)$ and $u,v\in H_{s,T}$ the following properties hold.

(1) $a_s\cdot(u+v)=a_s\cdot u+a_s\cdot v$.

(2) $(a_s+b_s)\cdot u=a_s\cdot u+b_s\cdot u$.

(3) $a_s\cdot(b_s\cdot u)=(a_sb_s)\cdot u$.

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(4) If e_s denotes the $\mathcal{F}_s$-measurable random variable equal to one, $e_s \cdot u = u$.

These features make $H_{s,T}$ a module over $L^0(\mathcal{F}_s)$.

Now consider the inner product $\langle , \rangle_{s,T} : H_{s,T} \times H_{s,T} \longrightarrow L^0(\mathcal{F}_s)$. For all $u \in H_{s,T}$, $\mathbb{E}_s^{\overline{Q}}\left[u_T^2\right] \in L_s^0(\mathcal{F}_s)$. Therefore, by Footnote 3 in Hansen and Richard (1987), $\langle u, v\rangle_{s,T} = \mathbb{E}_s^{\overline{Q}}[u_T v_T]$ belongs to $L^0(\mathcal{F}_s)$.

In addition, for each $a_s \in L^0(\mathcal{F}_s)$ and $u, v, z \in H_{s,T}$ the following properties are satisfied.

(5) $\langle u, u\rangle_{s,T} = \mathbb{E}_s^{\overline{Q}}\left[u_T^2\right] = \mathbb{E}_s\left[G_T^2 u_T^2\right] / \mathbb{E}_s\left[G_T^2\right] \geqslant 0$ with equality if and only if $G_T u_T = 0$. This implies that, for any $\sigma \in [s,T]$, $\mathbb{E}_\sigma[G_T u_T] = G_\sigma u_\sigma = 0$. As a result, $u = 0$.

(6) $\langle u, v\rangle_{s,T} = \langle v, u\rangle_{s,T}$.

(7) $\langle u + v, z\rangle_{s,T} = \langle u, z\rangle_{s,T} + \langle v, z\rangle_{s,T}$.

(8) $\langle a_s \cdot u, v\rangle_{s,T} = a_s \mathbb{E}_s^{\overline{Q}}[u_T v_T] = a_s \langle u, v\rangle_{s,T}$.

As a result, $H_{s,T}$ is a pre-Hilbert module.

We now prove that $H_{s,T}$ is selfdual. First, note that $L^0(\mathcal{F}_s)$ is endowed with the Lévy metric $d(f,g) = \mathbb{E}^F[\min\{|f-g|, 1\}]$ for all $f, g \in L^0(\mathcal{F}_s)$. As described in Cerreia-Vioglio, Maccheroni, and Marinacci (2017), in a pre-Hilbert L^0-module a metric, denoted by d_H, is given by the composition of d with the L^0-valued norm induced by the L^0-valued inner product. Hence, the d_H distance between two processes u, v in $H_{s,T}$ is

$$d_H(u,v) = d\left(\sqrt{\langle u-v, u-v\rangle_{s,T}}, 0\right) = \mathbb{E}^F\left[\min\left\{\sqrt{\mathbb{E}_s^{\overline{Q}}\left[(u_T - v_T)^2\right]}, 1\right\}\right].$$

Cerreia-Vioglio, Maccheroni, and Marinacci (2017) show that selfduality of a pre-Hilbert L^0-module is equivalent to d_H-completeness of the module. Therefore, we verify this property in $H_{s,T}$.

In addition, we observe that the metric d_H actually involves just terminal values u_T and v_T and so $d_H(u,v)$ actually coincides with the distance between random variables u_T, v_T belonging to the L^0-module

$$L_s^2\left(\mathcal{F}_T, \overline{Q}\right) = \left\{f \in L^0(\mathcal{F}_T) : \quad \mathbb{E}_s^{\overline{Q}}\left[f^2\right] \in L^0(\mathcal{F}_s)\right\},$$

which is complete: see Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth (2016). This fact makes d_H-completeness of $H_{s,T}$ straightforward.

Therefore, consider a Cauchy sequence $\left\{u^{(n)}\right\}_{n\in\mathbb{N}} \subset H_{s,T}$: for all $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that, for all $n, m > N_\varepsilon$,

$$d_H\left(u^{(n)}, u^{(m)}\right) = \mathbb{E}^F\left[\min\left\{\sqrt{\mathbb{E}_s^{\overline{Q}}\left[\left(u_T^{(n)} - u_T^{(m)}\right)^2\right]}, 1\right\}\right] < \varepsilon.$$

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Thus, we obtain a Cauchy sequence $\left\{u_T^{(n)}\right\}_{n\in\mathbb{N}} \subset L_s^2\left(\mathcal{F}_T, \overline{Q}\right)$, which is complete. As a result, this sequence has limit $u_T \in L_s^2\left(\mathcal{F}_T, \overline{Q}\right)$. From u_T we define the process $u = \{u_t\}_{t\in[s,T]}$ by setting $u_t = \mathbb{E}^F[u_T]$. This process is a conditional F-martingale and so it belongs to $H_{s,T}$. To establish this fact, we check that $\mathbb{E}_s^F[|u_t|] \in L^0(\mathcal{F}_s)$ for all $t \in [s,T]$.

Since any $|u(t)|$ is nonnegative, its conditional expectation is always defined as an extended real random variable. Moreover, by the conditional Cauchy-Schwartz' inequality,

$$\begin{aligned}\mathbb{E}_s^F[|u_t|] &\leqslant \mathbb{E}_s^F\left[\left|\mathbb{E}_s^F[u_T]\right|\right] \leqslant \mathbb{E}_s^F[|u_T|] \\ &= \frac{\mathbb{E}_s[G_T|u_T|]}{G_s} \leqslant \frac{\sqrt{\mathbb{E}_s\left[G_T^2 u_T^2\right]}}{G_s} \\ &= \frac{\sqrt{\mathbb{E}_s^{\overline{Q}}[u_T^2]\,\mathbb{E}_s[G_T^2]}}{G_s},\end{aligned}$$

where the last quantity belongs to $L^0(\mathcal{F}_s)$. Consequently, $\mathbb{E}_s^F[|u_t|] \in L^0(\mathcal{F}_s)$ for all $t \in [s,T]$.

We, then, determined a process $u \in H_{s,T}$ such that

$$d_H\left(u^{(n)}, u\right) = \mathbb{E}^F\left[\min\left\{\sqrt{\mathbb{E}_s^{\overline{Q}}\left[\left(u_T^{(n)} - u_T\right)^2\right]}, 1\right\}\right]$$

is arbitrarily small. Since $u^{(n)}$ goes to u in d_H, $H_{s,T}$ is d_H-complete and so selfdual. ■

Proof of Theorem 3.5

Proof • EXISTENCE

In order to show that $\pi_s \in \mathcal{W}_{s,T}$, we prove that $e^{r_{s,T}T}\pi_s$ belongs to $\mathcal{V}_{s,T}$ and that it is weakly time-differentiable.

First, for all $\tau \in [s,T]$, $e^{r_{s,T}T}\pi(\tau) \in L_s^1(\mathcal{F}_\tau)$. Indeed, $\left|e^{r_{s,T}T}\pi(\tau)\right|$ is nonnegative and so its conditional expectation at time s is an extended real random variable. However,

$$\mathbb{E}_s^F\left[\left|e^{r_{s,T}T}\pi(\tau)\right|\right] = e^{r_{s,T}\tau}\mathbb{E}_s^F\left[\left|\mathbb{E}_\tau^F[h_T]\right|\right] \leqslant e^{r_{s,T}\tau}\mathbb{E}_s^F[|h_T|],$$

which is in $L^0(\mathcal{F}_s)$. Therefore, $\mathbb{E}_s^F\left[\left|e^{r_{s,T}T}\pi(\tau)\right|\right]$ belongs to $L^0(\mathcal{F}_s)$.

As for the L_s^1-continuity, we check that, for any $t \in [s,T)$, $\mathbb{E}_s^F\left[\left|e^{r_{s,T}T}\pi_s(\tau) - e^{r_{s,T}T}\pi_s(t)\right|\right]$

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tends to zero when $\tau \longrightarrow t^+$. We have

$$\begin{aligned}
\mathbb{E}_s^F\left[\left|e^{r_{s,T}T}\pi_s(\tau) - e^{r_{s,T}T}\pi_s(t)\right|\right] &= \mathbb{E}_s^F\left[\left|e^{r_{s,T}\tau}\mathbb{E}_\tau^F\left[h_T\right] - e^{r_{s,T}t}\mathbb{E}_t^F\left[h_T\right]\right|\right] \\
&= e^{r_{s,T}t}\mathbb{E}_s^F\left[\left|e^{r_{s,T}(\tau-t)}\mathbb{E}_\tau^F\left[h_T\right] - \mathbb{E}_t^F\left[h_T\right]\right|\right] \\
&\leqslant e^{r_{s,T}t}\Bigg\{\mathbb{E}_s^F\left[\left|e^{r_{s,T}(\tau-t)}\mathbb{E}_\tau^F\left[h_T\right] - \mathbb{E}_\tau^F\left[h_T\right]\right|\right] \\
&\quad + \mathbb{E}_s^F\left[\left|\mathbb{E}_\tau^F\left[h_T\right] - \mathbb{E}_t^F\left[h_T\right]\right|\right]\Bigg\} \\
&\leqslant e^{r_{s,T}t}\Bigg\{\left|e^{r_{s,T}(\tau-t)} - 1\right|\mathbb{E}_s^F\left[\left|h_T\right|\right] \\
&\quad + \mathbb{E}_s^F\left[\left|\mathbb{E}_\tau^F\left[h_T\right] - \mathbb{E}_t^F\left[h_T\right]\right|\right]\Bigg\}.
\end{aligned}$$

In the last expression, both addends go to zero a.s. when τ approaches t from the right. In particular, the convergence of the first term is due to the fact that almost every realization of $r_{s,T}$ is a real number (fixed for the convergence). As for the second term, its convergence is ensured by Lévy's Downward Theorem (employed with $\mathcal{F}_s$ instead of the trivial sigma-algebra), which guarantees that

$$\mathbb{E}_\tau^F\left[h_T\right] \xrightarrow{L_s^1} \mathbb{E}_{t^+}^F\left[h_T\right] = \mathbb{E}_t^F\left[h_T\right] \qquad \tau \longrightarrow t^+.$$

Similarly, when $\tau \longrightarrow T^-$,

$$\mathbb{E}^F\left[\left|\pi_s(\tau) - h_T\right|\right] \leqslant e^{r_{s,T}T}\Bigg\{\left|e^{r_{s,T}(\tau-T)} - 1\right|\mathbb{E}^F\left[\left|h_T\right|\right] + \mathbb{E}^F\left[\left|\mathbb{E}_\tau^F\left[h_T\right] - h_T\right|\right]\Bigg\}.$$

Here, the convergence of the second term is due to Lévy's Upward Theorem:

$$\mathbb{E}_\tau^F\left[h_T\right] \xrightarrow{L_s^1} \mathbb{E}_{T^-}^F\left[h_T\right] = \mathbb{E}_T^F\left[h_T\right] = h_T \qquad \tau \longrightarrow T^-.$$

Therefore π_s belongs to $\mathcal{V}_{s,T}$.

Now we look for the weak time-derivative of $e^{r_{s,T}T}\pi_s$. We consider any $A_t \in \mathcal{F}_t$ and $\varphi \in$

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$C_c^1\left([t,T],L^0\left(\mathcal{F}_s\right)\right)$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau\in[t,T]$, we have

$$\begin{aligned}
-\int_t^T \mathbb{E}_s^F\left[e^{r_{s,T}T}\pi_s\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= -\int_t^T \mathbb{E}_s^F\left[e^{r_{s,T}\tau}\mathbb{E}_\tau^F\left[h_T\right]\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\\
&= -\int_t^T e^{r_{s,T}\tau}\mathbb{E}_s^F\left[h_T\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau\\
&= -\mathbb{E}_s^F\left[h_T\mathbf{1}_{A_t}\right]\int_t^T e^{r_{s,T}\tau}\varphi'(\tau)d\tau\\
&= \mathbb{E}_s^F\left[h_T\mathbf{1}_{A_t}\right]\int_t^T r_{s,T}e^{r_{s,T}\tau}\varphi(\tau)d\tau\\
&= \int_t^T r_{s,T}\mathbb{E}_s^F\left[e^{r_{s,T}\tau}h_T\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau\\
&= \int_t^T \mathbb{E}_s^F\left[r_{s,T}e^{r_{s,T}T}\pi_s\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}$$

The integral of the function $\sigma\longmapsto e^{r_{s,T}\sigma}\varphi'(\sigma)$ is computed pathwise in $L^0\left(\mathcal{F}_s\right)$, exploiting the compact support of φ.

Therefore, the candidate weak time-derivative of $e^{r_{s,T}T}\pi_s$ is $r_{s,T}e^{r_{s,T}T}\pi_s$. Since $r_{s,T}e^{r_{s,T}T}\pi_s$ belongs to $\mathcal{V}_{s,T}$, $r_{s,T}e^{r_{s,T}T}\pi_s$ is effectively the weak time-derivative of $e^{r_{s,T}T}\pi_s$ and so

$$\mathcal{D}\pi_s = r_{s,T}\pi_s.$$

Of course, $\pi(T)=h_T$ since almost all realizations of $r_{T,T}$ are real numbers and so $\pi\in\mathcal{W}_{s,T}$ solves Problem (3.7).

• UNIQUENESS

Let $\pi_s^{(1)},\pi_s^{(2)}\in\mathcal{W}_{s,T}$ be two solutions of Problem (3.7), that is

$$\mathcal{D}\pi_s^{(i)} = r_{s,T}\pi_s^{(i)}\qquad i=1,2,$$

with the boundary condition $\pi_s^{(i)}\left(T\right)=h_T$. By defining $z=\pi_s^{(1)}-\pi_s^{(2)}\in\mathcal{W}_{s,T}$, we have that

$$\mathcal{D}z = r_{s,T}z$$

and $z\left(T\right)=0$. The weak time-derivative of $e^{-rt}z\left(t\right)$ is

$$e^{-rt}\left(\mathcal{D}z\left(t\right)-rz\left(t\right)\right).$$

Indeed, fix $t\in[s,T]$. For any $\varphi\in C_c^1\left([t,T],L^0\left(\mathcal{F}_s\right)\right)$, consider the function

$$\theta\longmapsto e^{-r_{s,T}\theta}r_{s,T}\varphi(\theta)-e^{-r_{s,T}\theta}\varphi'(\theta)$$

that takes values in $L^0\left(\mathcal{F}_s\right)$. By integrating pathwise, it follows that

$$\int_\tau^T\left(e^{-r_{s,T}\theta}r_{s,T}\varphi(\theta)-e^{-r_{s,T}\theta}\varphi'(\theta)\right)d\theta = e^{-r_{s,T}\tau}\varphi(\tau)\qquad a.s.$$

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Hence, $e^{-r_{s,T}\tau}\varphi(\tau)$ belongs to $C_c^1\left([t,T],L^0\left(\mathcal{F}_s\right)\right)$ and so we can use it as test function in the definition of weak time-derivative of z:

$$\begin{aligned}\int_t^T \mathbb{E}_s^F\left[\mathcal{D}z\left(\tau\right)\mathbf{1}_{A_t}e^{-r_{s,T}\tau}\right]\varphi(\tau)d\tau &= -\int_t^T \mathbb{E}_s^F\left[z\left(\tau\right)\mathbf{1}_{A_t}\right]\left(e^{-r_{s,T}\tau}\varphi'(\tau)-e^{-r_{s,T}\tau}r_{s,T}\varphi(\tau)\right)d\tau\\ &= -\int_t^T \mathbb{E}_s^F\left[z\left(\tau\right)\mathbf{1}_{A_t}e^{-r_{s,T}\tau}\right]\varphi'(\tau)d\tau\\ &\quad +\int_t^T \mathbb{E}_s^F\left[z\left(\tau\right)\mathbf{1}_{A_t}e^{-r_{s,T}\tau}r_{s,T}\right]\varphi(\tau)d\tau\end{aligned}$$

that is

$$\int_t^T \mathbb{E}_s^F\left[e^{-r_{s,T}\tau}\left(\mathcal{D}z\left(\tau\right)-r_{s,T}z\left(\tau\right)\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = -\int_t^T \mathbb{E}_s\left[e^{-r_{s,T}\tau}z\left(\tau\right)\mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau.$$

This means that the weak time-derivative of $e^{-r_{s,T}t}z\left(t\right)$ is

$$e^{-r_{s,T}t}\left(\mathcal{D}z\left(t\right)-r_{s,T}z\left(t\right)\right).$$

However this process is null. Therefore, $e^{-r_{s,T}t}z\left(t\right)$ has null weak time-derivative. Consequently, by Proposition 3.3, $e^{-r_{s,T}t}z\left(t\right)$ defines a conditional F-martingale and so, for any $t\in[s,T]$ and $\tau\in[t,T]$

$$\mathbb{E}_t^F\left[z\left(\tau\right)\right] = e^{r_{s,T}(\tau-t)}z\left(t\right).$$

Letting τ go to T^-, we have that

$$\mathbb{E}_t^F\left[z\left(\tau\right)\right]\longrightarrow e^{r_{s,T}(T-t)}z(t)\qquad \textit{pointwise.}$$

In addition, $z\left(\tau\right)$ converges to $z(T)=0$ in $L_s^1(F)$ as τ approaches T^- and so $\mathbb{E}_t^F\left[z\left(\tau\right)\right]$ tends to zero in $L_s^1(F)$. By uniqueness of the L_s^1-limit, we infer that $z(t)=0$ for all $t\in[s,T]$. This proves uniqueness of the solution of Problem (3.7). ■

Proof of Proposition 3.6

Proof Fix any positive s and consider the limit in probability when T goes to infinity. The fact that $r_{s,T}\xrightarrow{P}\lambda$ ensures that $e^{-r_{s,T}(T-t)}\xrightarrow{P}0$ for all $s>0$ and $t>s$. Therefore, $e^{-r_{s,T}(T-t)}-e^{-r_{t,T}(T-t)}\xrightarrow{P}0$, so that

$$\frac{\pi_s(t)-\pi(t)}{\mathbb{E}_t^F\left[h_T\right]}\xrightarrow{P}0,\qquad T\longrightarrow+\infty.$$

As for the second convergence, since $r_{s,T}\xrightarrow{P}\lambda$ for all $s>0$ as T goes to infinity, the difference $r_{s,T}-r_{t,T}$ converges in probability to zero for all $s>0$ and $t>s$. In addition, for any positive ε,

$$P\left(\frac{\left|\log\pi_s(t)-\log\pi(t)\right|}{T-t}>\varepsilon\right)=P\left(\left|r_{s,T}-r_{t,T}\right|>\varepsilon\right)$$

and this quantity goes to zero as T increases because $r_{s,T}-r_{t,T}\xrightarrow{P}0$. ■

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Proof of Proposition 3.7

Proof By Lemma 3.9, for any $t > s$

$$\mathbb{E}_s^F\left[\frac{\pi_s(t)}{\pi(t)}\right] = e^{r_{s,T}(t-s)}\mathbb{E}_s^F\left[\frac{p(s,T)}{p(t,T)}\right] = e^{r_{s,T}(t-s)}p(s,t) \xrightarrow{P} e^{\left(\lambda - r_{s,t}\right)(t-s)}$$

as T goes to infinity.

As for the second convergence, consider the expression

$$\frac{\pi_s(t)}{\pi(t)} = e^{r_{s,T}(t-s)}\frac{p(s,T)}{p(t,T)} = e^{r_{s,T}(t-s)}\frac{p(0,T-s)}{p(0,T-t)}\frac{p(s,T)}{p(0,T-s)}\frac{p(0,T-t)}{p(t,T)}.$$

Qin and Linetsky (2017) ensure that, at T goes to infinity,

- $e^{r_{s,T}(t-s)}$ converges to $e^{\lambda(t-s)}$ in probability;
- $p(0,T-s)/p(0,T-s-(t-s))$ converges to $e^{-\lambda(t-s)}$ in probability;
- $p(s,T)/p(0,T-s)$ converges to π_s^∞ in the semimartingale topology of Émery (1979);
- $p(0,T-s)/p(t,T)$ converges to $1/\pi_t^\infty$ in the semimartingale topology.

In the semimartingale topology the product of convergent processes converges to the product of the respective limit processes. Moreover, the convergence in the semimartingale topology necessarily entails the convergence in probability for any fixed t. Therefore, by Slutsky Theorem we can conclude that

$$\frac{\pi_s(t)}{\pi(t)} \xrightarrow{P} \frac{e^{\lambda(t-s)}}{e^{\lambda(t-s)}}\frac{\pi_s^\infty}{\pi_t^\infty} = \frac{\pi_s^\infty}{\pi_t^\infty}$$

as T goes to infinity. ■

Proof of Proposition 3.8

Proof We exploit the asymptotic results of Qin and Linetsky (2017) as in the proof of Proposition 3.7. Since T goes to infinity, we assume that $T > t + s$ without loss of generality.

As for the first convergence, consider

$$\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} = e^{r_{s,T}s}\frac{p(0,T-s)}{p(0,T-t-s)}\frac{p(s,T)}{p(0,T-s)}\frac{p(0,T-t-s)}{p(s,T-t)}.$$

We have

- $e^{r_{s,T}s}$ converges to $e^{\lambda s}$ in probability;
- $p(0,T-s)/p(0,T-t-s)$ converges to $e^{-\lambda t}$ in probability;

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- $p(s,T)/p(0,T-s)$ converges to π_s^∞ in the semimartingale topology;
- $p(0,T-t-s)/p(s,T-t)$ converges to $1/\pi_s^\infty$ in the semimartingale topology.

As a result, when T goes to infinity,

$$\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)} \xrightarrow{P} e^{-\lambda(t-s)}.$$

Similarly,

$$\frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)} = e^{-r_{s,T}s}\frac{p(0,T-t-s)}{p(0,T-t)}\frac{p(s,T-t)}{p(0,T-t-s)}\frac{p(0,T-t)}{p(t,T)},$$

where

- $e^{-r_{s,T}s}$ converges to $e^{-\lambda s}$ in probability;
- $p(0,T-t-s)/p(0,T-t)$ converges to $e^{\lambda s}$ in probability;
- $p(s,T-t)/p(0,T-t-s)$ converges to π_s^∞ in the semimartingale topology;
- $p(0,T-t)/p(t,T)$ converges to $1/\pi_t^\infty$ in the semimartingale topology.

In consequence, as T goes to infinity,

$$\frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)} \xrightarrow{P} \frac{\pi_s^\infty}{\pi_t^\infty}.$$

In addition, Qin and Linetsky (2017) prove that the stochastic discount factor S_t satisfies $S_t = M_t^\infty/B_t^\infty$ for all $t>0$, where M_t^∞ is a P-martingale. Therefore,

$$M_{s,t} = \frac{S_t}{S_s} = \frac{M_t^\infty}{M_s^\infty}\frac{B_s^\infty}{B_t^\infty} = e^{-\lambda(t-s)}\frac{\pi_s^\infty}{\pi_t^\infty}\frac{M_t^\infty}{M_s^\infty}.$$

Hence,

$$\begin{aligned} G_{s,t} &= \left(\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)}\right)^{-1}\left(\frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)}\right)^{-1} M_{s,t} \\ &= \left(\frac{e^{r_{s,T}s}p(s,T)}{p(s,T-t)}\right)^{-1}\left(\frac{e^{-r_{s,T}s}p(s,T-t)}{p(t,T)}\right)^{-1} e^{-\lambda(t-s)}\frac{\pi_s^\infty}{\pi_t^\infty}\frac{M_t^\infty}{M_s^\infty}. \end{aligned}$$

Since the first factor on the right-hand side converges in probability to $e^{\lambda(t-s)}$ and the second one converges to $\pi_t^\infty/\pi_s^\infty$, we deduce that

$$G_{s,t} \xrightarrow{P} \frac{M_t^\infty}{M_s^\infty}, \qquad T \longrightarrow +\infty.$$

■

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3.C Cox, Ingersoll, Ross (1985) dynamics

We enhance the discussion of Subsection 3.7.2 about affine interest rate models by illustrating our theory when short-term rates follow Cox, Ingersoll, and Ross (1985) dynamics.

Specifically, the coefficients of the instantaneous rate are

$$\mu\left(t, Y_t\right) = k\theta - (k - \sigma\xi)Y_t, \qquad \sigma\left(t, Y_t\right) = \sigma\sqrt{Y_t}$$

with $k, \theta, \sigma, \xi > 0$ and $\sigma^2 < 2k\theta$. Here, the market price of risk is supposed to be $\xi\sqrt{Y_t}$. Under the risk-neutral measure Q, we get the dynamics

$$dY_t = k\left[\theta - Y_t\right]dt + \sigma\sqrt{Y_t}dW_t^Q. \tag{3.16}$$

This specification ensures that interest rates are positive and their volatility raises when they increase. Moreover,

$$A(t,T) = \left[\frac{2\sqrt{k^2+2\sigma^2}\exp\left\{\left(k+\sqrt{k^2+2\sigma^2}\right)(T-t)/2\right\}}{2\sqrt{k^2+2\sigma^2}+\left(k+\sqrt{k^2+2\sigma^2}\right)\left(\exp\left\{(T-t)\sqrt{k^2+2\sigma^2}\right\}-1\right)}\right]^{2k\theta/\sigma^2}$$

and

$$B(t,T) = \frac{2\left(\exp\left\{(T-t)\sqrt{k^2+2\sigma^2}\right\}-1\right)}{2h+\left(k+\sqrt{k^2+2\sigma^2}\right)\left(\exp\left\{(T-t)\sqrt{k^2+2\sigma^2}\right\}-1\right)}$$

as mentioned in Brigo and Mercurio (2007).

Under the forward measure the drift coefficient of the risk-neutral bond price turns out to be

$$\left(1+\sigma^2 B^2(t,T)\right)Y_t$$

and it converges a.s. to

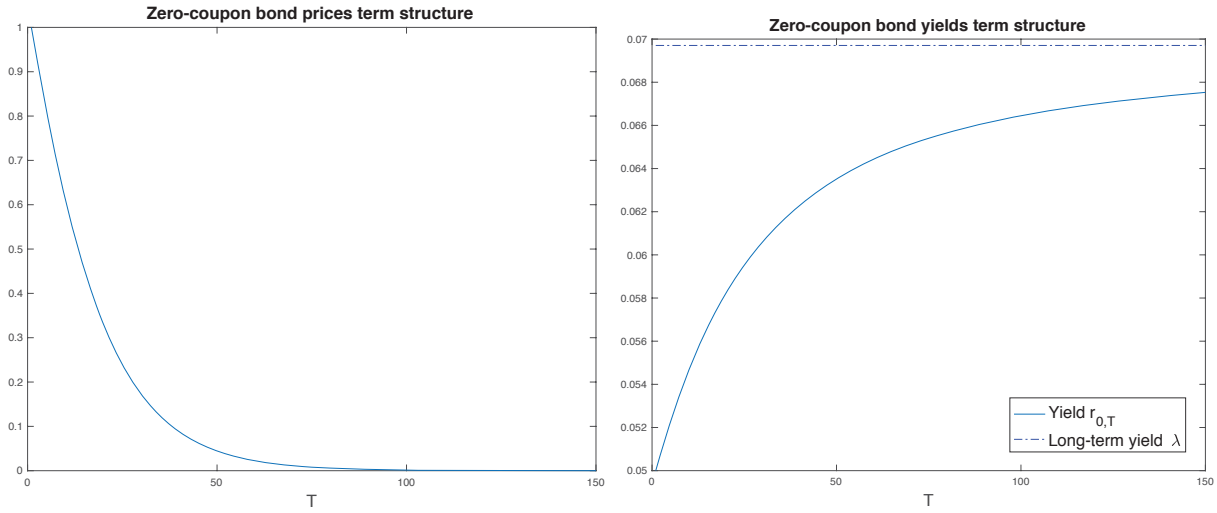
$$\left(1+\sigma^2\frac{4}{\left(k+\sqrt{k^2+2\sigma^2}\right)^2}\right)Y_t$$

when the trading horizon T becomes infinitely large. Differently, the drift parameter $r_{0,T}$ of the price $\pi_0(t,T)$ converges a.s. to the long-term yield

$$\lambda = \frac{k\theta}{\sigma^2}\left(\sqrt{k^2+2\sigma^2}-k\right). \tag{3.17}$$

We plot in Figure 3.C.1 the term structures of no arbitrage prices and yields of a zero-coupon bond in a simulated Cox, Ingersoll and Ross model, computed at time zero. We set the parameters

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(a) Zero-coupon bond prices term structure

(b) Yield-to-maturity term structure

Figure 3.C.1: The left and right panel depict no arbitrage pure discount bond prices (at time zero) and yields to maturity with respect to increasing horizons T under Cox, Ingersoll and Ross specifications. The dashed horizontal line represents the long-term yield λ of eq. (3.17).

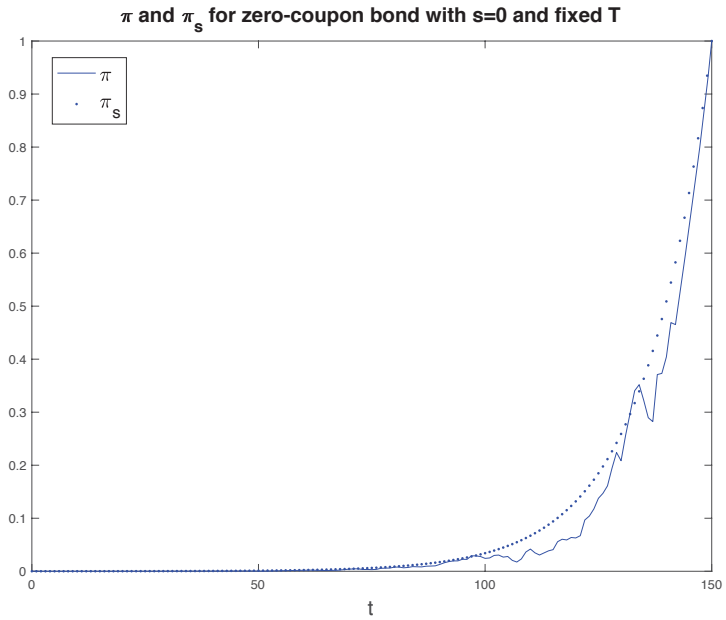


Figure 3.C.2: Values of $\pi_0(t)$ and of a realization of the no arbitrage price $p(t,T)$ of a pure discount bond with fixed maturity T for any t in $[0,T]$ in Cox, Ingersoll and Ross. The two prices are represented by a dotted and a solid line, respectively.

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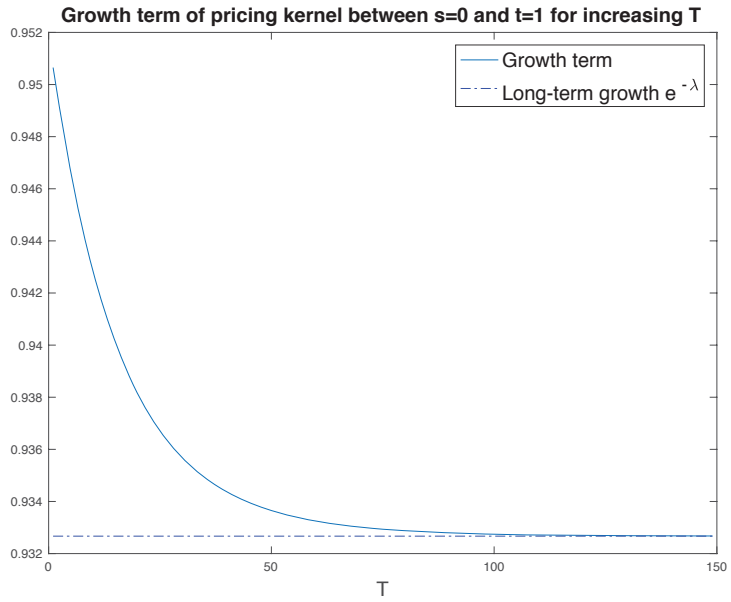


Figure 3.C.3: Term structure of growth terms of the pricing kernel $M_{0,1}$ in Cox, Ingersoll and Ross model. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed horizontal line is the long-term growth $e^{-\lambda}$ with λ given by eq. (3.17).

$k = 0.0332$, $\theta = 0.0874$, $\sigma = 0.0265$ following Table III in Nowman (1997). We also fix $\xi = 0.2$ and $Y_0 = 0.05$. When maturity T becomes large, we notice that yields to maturity converge to the long-term yield provided by eq. (3.17), as expected. In Figure 3.C.2, on the contrary, we fix T and we consider prices $p(t,T)$ and $\pi_0(t)$ of pure discount T-bonds. As discussed in the body of the paper, $p(t,T)$ and $\pi_0(t)$ are almost indistinguishable in the short-term as well as near maturity.

Finally, we illustrate the growth term of the pricing kernel between time $s = 0$ and time $t = 1$, following the description in Subsection 3.7.6. Figure 3.C.3 depicts the convergence of the pricing kernel growth term to $e^{-\lambda(t-s)}$ when T goes to infinity under Cox, Ingersoll and Ross dynamics. In this case, λ is the long-term yield provided by eq. (3.17).

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