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AN HENSTOCK-KURZWEIL TYPE INTEGRAL  
ON A MEASURE METRIC SPACE

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## Abstract

We consider an Henstock-Kurzweil type integral defined on a complete measure metric space  $X = (X, d)$  endowed with a Radon measure  $\mu$  and with a family  $\mathcal{F}$  of “intervals” that satisfies, besides usual conditions, the Vitali covering theorem.

In particular, for such integral, we obtain extensions of the descriptive characterization of the classical Henstock-Kurzweil integral on the real line, in terms of  $ACG_*$  functions and in terms of variational measures.

Moreover we show that, besides the usual Henstock-Kurzweil integral on the real line, such integral includes also the dyadic Henstock-Kurzweil integral [37], the  $GP$ -integral [30] and the  $s$ -HK integral ([3] and [4]).

For this last integral we prove a better version of the Fundamental Theorem of Calculus since the classical one is not true in this setting .

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# Introduction

The theory of measure and integration developed in 1902 by H. Lebesgue is a key tool in several branches of Mathematical Analysis. It is well-known that the Lebesgue integral removes the defects of the Riemann integral, but it does not have a very intuitive approach and his techniques require a considerable foundation in measure theory. Moreover, as Lebesgue himself remarked, his integral does not integrate all unbounded derivatives [22]. In fact, it is trivial to observe that the following function

$$F(x) = \begin{cases} x^2 \sin(1/x^2), & x \in (0, 1] \\ 0, & x = 0 \end{cases} \quad (1)$$

is differentiable everywhere on  $[0, 1]$ , but its derivative, denoted by  $F'(x)$ , is not Lebesgue integrable on  $[0, 1]$  since  $F(x)$  is not absolutely continuous on  $[0, 1]$ . This means that the Lebesgue integral does not solve the problem of the primitives, that is the problem of recovering a function from its derivative. Therefore, it was natural to find an integration process for which the following theorem holds:

**The Fundamental Theorem of Calculus** *If  $F : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$ , then the function  $F'(x)$  is integrable on  $[a, b]$  and  $\int_a^x F'(t)dt = F(x) - F(a)$  for all  $x \in [a, b]$ .*

A first solution to this problem was given in 1912 by A. Denjoy [7], shortly followed by N. Luzin [28] and in 1914 by O. Perron [32]. Denjoy developed a new method of integration, called totalization, that obtained the primitives of a function through a transfinite process of Lebesgue integrations and limit operations. The Denjoy integral is technical and quite complicated and it includes the Lebesgue integral. A few months later, Luzin connected the Denjoy integral with the notion of generalized absolute continuity in the restrictive sense (briefly  $ACG_*$ ) as follows:

**Theorem** *A function  $f$  is Denjoy integrable on a closed interval  $[a, b]$  of the real line if and only if there exists a function  $F$  which is  $ACG_*$  on  $[a, b]$  such that  $F'(x) = f(x)$  almost everywhere [36, Chap. VIII, §1].*

On the other hand, Perron developed another approach which it is proved

to be equivalent to the Denjoy integral [36, Theorems 3.9 and 3.11]. He uses families of major and minor functions instead of single primitives. In literature, the previous theorem is known also as the descriptive definition of the Denjoy-Perron integral.

A further approach to the problem of the primitives was introduced in 1957 by J. Kurzweil [21] and in 1963 by R. Henstock [12], independently. They defined a generalized version of the Riemann integral that is known as the Henstock-Kurzweil integral, also abbreviated as the HK-integral<sup>1</sup>. The advantage of the HK-integral is that it is very similar in construction and in simplicity to the Riemann integral and it has the power of the Lebesgue integral. Moreover, in the real line, the HK-integral solves the problem of the primitives. The definition of the HK-integral is constructive, as in the Riemann integral, and the value of the HK-integral is defined as the limit of Riemann sums over suitable partitions of the domain of integration. The main difference between the two definitions is that, in the HK-integral, a positive function, called gauge, is used, instead of the constant utilized in the Riemann integral to measure the fineness of a partition. This gives a better approximation of the integral near singular points of the function.

Subsequently, Henstock [12] and other mathematicians ([20] and [24]) showed that the Denjoy-Perron integral is equivalent to the HK-integral, by proving the following descriptive characterizations of the HK-integral:

**Theorem A** *A function  $f : [a, b] \rightarrow \mathbb{R}$  is Henstock-Kurzweil integrable on  $[a, b]$  if and only if there exists a function  $F : [a, b] \rightarrow \mathbb{R}$  such that  $F$  is  $ACG_*$  and  $F'(x) = f(x)$  almost everywhere in  $[a, b]$ .*

The main tool of the proof of this characterization relies heavily on the validity of the Vitali covering theorem, so the extension of the above theorem to the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , requires the regularity of the intervals used in the definition.

It is well-known that the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , satisfies almost all the properties of the one dimensional HK-integral ([12] and [13]). However, in contrast with the one-dimensional case, this integral does not integrate all derivatives. Indeed, there are differentiable vector fields with a non-integrable divergence [33, Example 5.7]. Therefore the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , does not satisfy the Divergence theorem, that, as it is well-known, it is the extension of the Fundamental Theorem of Calculus to higher-dimensions. This deficiency was removed, firstly, in 1981 by J. Mawhin [30] who defined a multidimensional Henstock-Kurzweil type integral, called the  $GP$ -integral, where a condition

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<sup>1</sup>In some references such integral is called the Henstock integral, the Kurzweil integral, the generalized Riemann integral or the Riemann complete integral.



of regularity of the partition is used. Nevertheless, this integral failed the property to be additive. Subsequently, in 1983, J. Jarník, J. Kurzweil and Š Schwabik [18] introduced the  $M_1$ -integral that not only preserves the good properties of the Mawhin's integral but also satisfies the additivity property. Over the years other highly technical modifications of the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , were provided, among others, by Jarník and Kurzweil ([16] and [17]) who introduced the  $PU$ -integral which is based on partitions of unity and by W. F. Pfeffer [34] who defined new integrals as the *gage*-integral, the  $\mathcal{F}$ -integral and the  $\mathcal{BV}$ - integral.

In 1995, B. Bongiorno, L. Di Piazza and V. A. Skvortsov [1, Theorem 3] proved a real-line independent descriptive characterization of the HK-integral, by using a variational measure associated with the integrable function, introduced by B. S. Thomson [38], as follows:

**Theorem B** *A function  $f : [a, b] \rightarrow \mathbb{R}$  is Henstock-Kurzweil integrable on  $[a, b]$  if and only if there exists a function  $F : [a, b] \rightarrow \mathbb{R}$  such that its variational measure is absolutely continuous with respect to the Lebesgue measure and  $F'(x) = f(x)$  almost everywhere in  $[a, b]$ .*

Moreover, this result was extended to various multidimensional Henstock-Kurzweil type integrals. Precisely, in 1996, Bongiorno, Pfeffer and Thomson [2] characterized the indefinite *gage*-integral by using suitable modification of the variational measure. One year later, Z. Buczolic and W. F. Pfeffer [5] obtained a generalization of the previous result to the indefinite  $\mathcal{F}$ -integral and to the indefinite  $\mathcal{BV}$ -integral. In 2001, Di Piazza [8] characterized in terms of variational measure the primitives of the  $GP$ -integral. In 2003, L. Tuo-Yeong [39] extended Theorem B to the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ . His proof is very deep and technical since in the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , the regularity of the intervals is not required.

Moreover theorems of type A were given by P. Y. Lee and N. W. Leng [23] and by J. Lu and P. Y. Lee [27] for the  $n$ -dimensional Henstock-Kurzweil integral, for  $n > 1$ , They used different generalizations of the notion of  $ACG_*$  function and made some additional conditions on the primitive function  $F$ .

In the more general setting of a generic metric measure space, it is well known that the biggest difficulty in the definition of an Henstock-Kurzweil type integral is the definition of a suitable family of measurable sets which plays the role of "intervals". N. W. Leng and L. P. Yee in [26], by a modification of the notion of a division space introduced by Henstock in [14], defined an Henstock-Kurzweil type integral on a complete metric measure space, called the  $H$ -integral. In [26] the family of "*intervals*" is defined as the class of all finite intersection of sets each of which is the difference of two closed balls. The  $H$ -integral includes the HK-integral on the real line. Later one,

Leng [25] proved a theorem of type A for the  $H$ -integral by improving the results showed in [23]. Unfortunately, such characterization required, besides an  $ACG_*$  type notion, some strong additional conditions on the primitive function  $F$  ([25, Theorem 19]).

In this thesis, we introduce an Henstock-Kurzweil type integral, called the  $\mu$ -HK integral, defined on a complete measure metric space  $(X, d)$  endowed with a Radon measure  $\mu$  and with a family  $\mathcal{F}$  of “intervals” that satisfies, besides usual conditions, the Vitali covering theorem. The  $\mu$ -HK integral includes the usual HK-integral on the real line, the dyadic HK-integral [37], the  $GP$ -integral [30] and the  $s$ -HK integral ([3] and [4]).

This last integral was defined to overcome the problem of the inapplicability of the Fundamental Theorem of Calculus for functions defined on a closed fractal subsets of the real line.

It is well-known that the standard methods of ordinary calculus are usually inapplicable to fractal sets. In 1998, H. Jiang and W. Su [19] and more recently A. Parvate and A. D. Gangal [31] introduced, independently, a Riemann type integration process for functions defined on a closed fractal subset of the real line of positive  $s$ -Hausdorff measure, with  $0 < s < 1$ . Here, we denote by  $s$ -R integral such integral. Both authors proved that the usual elementary properties of the classical Riemann integral are still valid for the  $s$ -R integral and they provided conditions for the validity of the reformulation of the Fundamental Theorem of Calculus, in which the notion of  $s$ -derivative ([6] and [19]) is used. The necessity of additional conditions is due to the existence of non constant singular functions on the fractal sets.

For example, on the ternary Cantor set  $C \subset [0, 1]$  the following function

$$F_C(x) = \begin{cases} 0, & x \leq \frac{1}{3} \\ 3x - 1, & \frac{1}{3} < x < \frac{2}{3} \\ 1, & x \geq \frac{2}{3} \end{cases}$$

has the  $s$ -derivative  $(F_C)'_s$  null on  $C$ . Then  $(F_C)'_s$  is  $s$ -R integrable but

$$(R) \int_C (F_C)'_s(t) d\mathcal{H}^s(t) = 0 \neq F_C(1) - F_C(0) = 1.$$

In the sequel, we provide the general outline of this thesis and the main results.

In Chapter 1, we recall the terminology, notations and some basic results that will be used throughout the thesis.

In Chapter 2, we define the  $\mu$ -HK integral, we describe some basic properties of such integral and we study its relation with the Lebesgue integral.

Moreover, we pay our attention to the descriptive characterization of the primitives of a  $\mu$ -HK integrable function, by proving the natural extensions of Theorems A and B, called the Main Theorem 1 and the Main Theorem 2, respectively. This is the principal result of this thesis. We remark that in the formulation of Main theorem 1 we don't need any additional hypothesis on the primitive function  $F$ , as it happens in Lee and Leng [23, Theorem 11], in Lu and Lee [27, Theorem 4.5 and Theorem 5.2] and in Leng [25, Theorem 19].

In Chapter 3, we study some basic properties of the  $s$ -HK integral and we formulate the better version of the Fundamental Theorem of Calculus, improving the results of Jiang and Su [19, Theorem 2.3] and that of Parvate and Gangal [31, Theorem 57].

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# Chapter 1

## Preliminaries

In this chapter, we provide some general measure-theoretic concepts, notations and some basic results that will be used throughout the thesis. The most part of our notations and terminology is standard and all the theorems presented here are without proofs.

### 1.1 Some basic notations

We denote by  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{R}$  the set of all natural, integer and real numbers, respectively. The  $n$ -dimensional Euclidean space is denoted by  $\mathbb{R}^n$ .

Let  $(X, d)$  be a metric space. The diameter of a non-empty subset  $A$  of  $X$  is defined as

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

For each  $x \in X$  and for each  $A$  and  $B$  non-empty subsets of  $X$ , the distance from  $x$  to  $A$  and the distance between  $A$  and  $B$  are defined as

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

respectively.

The closure, the interior, the boundary and the characteristic function of a non-empty subsets of  $A$  of  $X$  are denoted by  $\overline{A}$ ,  $A^\circ$ ,  $\partial A$  and  $\chi_A$ , respectively.

### 1.2 Measures

Let  $X$  be any set. A collection  $\mathcal{M}$  of subsets of  $X$  is called a  $\sigma$ -algebra if:

1.  $X \in \mathcal{M}$ ;

2.  $A \in \mathcal{M}$ , implies  $X \setminus A \in \mathcal{M}$ ;
3.  $A_n \in \mathcal{M}$ , for  $n = 1, 2, \dots$ , implies  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ .

Let  $\mathcal{C}$  be an arbitrary collection of subsets of  $X$ . The smallest  $\sigma$ -algebra  $\sigma(\mathcal{C})$  containing  $\mathcal{C}$  is called the  $\sigma$ -algebra generated by  $\mathcal{C}$ . Such a  $\sigma$ -algebra is the intersection of all  $\sigma$ -algebras in  $X$  which contain  $\mathcal{C}$ .

Let  $\mathcal{M}$  be a  $\sigma$ -algebra of subsets of a set  $X$ . A non-negative function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a *measure* if

1.  $\mu(\emptyset) = 0$ ;
2.  $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$  for each sequences  $\{A_j\}_j$  of pairwise disjoint sets from in  $\mathcal{M}$ .

It is easy to see that each measure is monotone, i.e. if  $A, B \in \mathcal{M}$  and  $A \subset B$ , then  $\mu(A) \leq \mu(B)$ . The triplet  $(X, \mathcal{M}, \mu)$  is called a *measure space*.

An *outer measure*  $\mu$  on  $X$  is a function defined for each subset of  $X$  taking value in  $[0, \infty]$  such that

1.  $\mu(\emptyset) = 0$ ;
2.  $\mu(A) \leq \mu(B)$  if  $A \subset B$ ;
3.  $\mu(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$  for each sequences  $\{A_j\}_j$  of subsets of  $X$ .

Condition 3 is called the  $\sigma$ -subadditivity of the outer measure  $\mu$ .

Given an outer measure  $\mu$  on  $X$  a subset  $E$  of  $X$  is called  $\mu$ -measurable or measurable with respect to  $\mu$  (in the sense of Carathéodory) if

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E), \quad \text{for each test set } A \subset X.$$

Now, let  $(X, d)$  be a metric space.

The sets belonging to the  $\sigma$ -algebra generated by the open subsets of  $X$  are called the *Borel sets* of the space. The Borel sets include the closed sets, the countable intersections of open sets ( $G_\delta$ -sets), the countable unions of closed sets ( $F_\sigma$ -sets), etc.

Let  $\mu$  be an outer measure on  $X$ .

- $\mu$  is *metric* if  $\mu(A \cup B) = \mu(A) + \mu(B)$  for each pair  $A, B \subset X$  such that  $d(A, B) > 0$ .

- $\mu$  is a *locally finite measure* if for every  $x \in X$  there is  $r > 0$  such that  $\mu(B(x, r)) < \infty$ .
- $\mu$  is a *Borel measure* if each Borel subset of  $X$  is  $\mu$ -measurable.
- $\mu$  is a *Borel regular measure* if it is a Borel measure and if for each  $E \subset X$  there exists a Borel subset  $B$  of  $X$  such that  $E \subset B$  and  $\mu(B) = \mu(E)$ .
- $\mu$  is a *Radon measure* if  $\mu$  is a Borel measure and if
  - (i)  $\mu(K) < \infty$  for each compact set  $K \subset X$ ;
  - (ii)  $\mu(V) = \sup\{\mu(K) : K \subset V, K \text{ is compact}\}$  for open sets  $V \subset X$ ;
  - (iii)  $\mu(A) = \inf\{\mu(V) : A \subset V, V \text{ is open}\}$  for  $A \subset X$ .

From now on, we denote by  $\mathcal{L}^n$  the  $n$ -dimensional Lebesgue measure.  $\mathcal{L}^n$  is a Radon measure.

Radon measures are always Borel regular by definition, but in general the converse is not true. Indeed, for example, the counting measure  $n$  on  $X$ , defined by letting  $n(A)$  be the number of elements in  $A$ , is Borel regular on any metric space  $X$  but it is a Radon measure only if every compact subset of  $X$  is finite, that is,  $X$  is discrete.

The relevant of the metric outer measures is due to the following Carathéodory criterion ([10], Theorem 1.5):

*If  $\mu$  is a metric outer measure on a metric space  $X$ , then  $\mu$  is Borel measure.*

Let  $\mu$  be an outer measure. A set  $E \subset X$  is called  $\mu$ -negligible or  $\mu$ -null if  $\mu(E) = 0$ . We say that a certain property  $P$  is satisfied  $\mu$ -almost everywhere in a set  $E \subset X$  if  $P$  is true for all points of  $E$  except at most the points of a  $\mu$ -negligible subset  $N$  of  $E$ . Moreover, we say that  $\mu$  is *non-atomic* if each singleton  $\{x\}$  in  $X$  is  $\mu$ -measurable and  $\mu(\{x\}) = 0$ .

### 1.3 Vitali covering theorem

The Vitali covering theorem is one of the most useful tools of measure theory. Given a collection of sets that cover some set  $E$ , the Vitali theorem selects a disjoint subcollection that covers almost all of  $E$ . Here, we recall the Vitali's covering theorem for Radon measures ([29], pag 34).

**Theorem 1.3.1.** *Let  $\mu$  be a Radon measure on  $\mathbb{R}^n$ ,  $A \subset \mathbb{R}^n$  and  $\mathcal{B}$  a family of closed balls such that each point of  $A$  is the centre of arbitrarily small balls of  $\mathcal{B}$ , that is,*

$$\inf\{r : B(x, r) \in \mathcal{B}\} = 0, \quad \text{for each } x \in A.$$

*Then there are disjoint balls  $B_i \in \mathcal{B}$ ,  $i = 1, 2, \dots$  such that*

$$\mu\left(A \setminus \bigcup_i B_i\right) = 0.$$

## 1.4 Hausdorff measures

Let  $X$  be a separable metric space and let  $0 \leq s < \infty$ . Given a subset  $A$  of  $X$ , we define the  $s$ -dimensional Hausdorff measure of  $A$  as follows:

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A),$$

where

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(A_i))^s : A \subset \bigcup_{i=1}^{\infty} A_i, \text{diam}(A_i) \leq \delta \right\},$$

and  $\{A_i\}_i$  is a countable family of subset of  $X$ .

If  $s = 0$ , then  $\mathcal{H}^0$  is the counting measure, i.e.  $\mathcal{H}^0(A)$  is the number of points in  $A$ . If  $s = n$ , then

$$\mathcal{H}^n(A) = C_n \mathcal{L}^n(A),$$

where  $C_n$  is a constant depending only on  $n$ .

Moreover, it can be proved that the  $s$ -dimensional Hausdorff measure,  $\mathcal{H}^s$ , is Borel regular. Usually  $\mathcal{H}^s$  is not a Radon measure, since it need not be locally finite. But taking any  $\mathcal{H}^s$  measurable set  $A$  in  $\mathbb{R}^n$  with  $\mathcal{H}^s(A) < \infty$ , the restriction of  $\mathcal{H}^s$  to  $A$  is a Radon measure ([29, Theorem 1.9(2) and Corollary 1.11]).

The *Hausdorff dimension* of  $A$  is defined as the unique number  $s$  for which

$$\begin{aligned} t < s &\text{ implies } \mathcal{H}^t(A) = \infty, \\ t > s &\text{ implies } \mathcal{H}^t(A) = 0. \end{aligned}$$

A set  $A \subset \mathbb{R}^n$  is called  $s$ -set ( $0 < s \leq n$ ) if it is measurable with respect to the  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s$  (briefly  $\mathcal{H}^s$ -measurable) and  $0 < \mathcal{H}^s(A) < \infty$ .



## 1.5 Self similar sets and fractals

An important class of  $s$ -set is formed by the self-similar sets. A subset of  $\mathbb{R}^n$  is said to self-similar if it can be split into parts which are geometrically similar to the whole set. Many of the classical fractal sets are “self-similar” [9].

### 1.5.1 The ternary Cantor set

The ternary Cantor set is one of the best known and most easily constructed fractals. It is constructed from a unit interval by a sequence of deletion operations.

Let  $E_0$  be the interval  $[0, 1]$  in  $\mathbb{R}$ . Let  $E_1$  be the set obtained by removing the open middle third of  $E_0$ , so that  $E_1$  consists of the two intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$ . Deleting the middle thirds of these intervals gives  $E_2$ . By this way,  $E_k$  is obtained by removing the (open) middle third of each interval in  $E_{k-1}$ . Thus  $E_k$  is the union of  $2^k$  intervals of length  $3^{-k}$ .

This procedure yields a sequence of compact sets  $E_k$  with  $k = 0, 1, \dots$  such that  $E_0 \supset E_1 \supset \dots \supset E_j \supset E_{j+1} \supset \dots$ .

The *ternary Cantor set*  $C$  is defined as follows

$$C = \bigcap_{k=0}^{\infty} E_k. \quad (1.1)$$

The ternary Cantor set  $C$  is self-similar. Indeed, it contains copies of itself at many different scales. Moreover,  $C$  is an uncountable compact set with zero Lebesgue measure. The Hausdorff dimension of the Cantor set  $C$  is  $s = \log_3 2$  and  $\mathcal{H}^s(C) = 1$  ([10, Theorem 1.14]).



Figure 1.1: Construction of the ternary Cantor set

### 1.5.2 The Sierpinski triangle

The Sierpinski triangle (or Sierpinski gasket) is another example of fractal set constructed using recursive procedures.

Let  $A_0$  be an equilateral triangle with side length 1. Divide  $A_0$  into four equilateral triangles obtained by joining the middle points of sides of  $A_0$ .

Let  $A_1$  be the set obtained by removing the interior of the middle triangle of  $A_0$ , so that  $A_1$  consists of the three equilateral triangles of side length  $\frac{1}{2}$  each. Continuing in this way with every one of the three equilateral triangles constituting  $A_1$ , at the  $k$ -th step we arrive at a compact set  $A_k$  consisting of  $3^k$  equilateral triangles of side length  $2^{-k}$ .

The Sierpinski triangle  $T$  is defined as follows

$$T = \bigcap_{j=0}^{\infty} A_j.$$

The Sierpinski triangle  $T$  is self-similar and it is a compact set of two-dimensional Lebesgue measure zero. Moreover, the Hausdorff dimension of the Sierpinski triangle  $T$  is  $s = \log_2 3$ .

The construction used above may be applied to any initial triangle in place of  $A_0$ .

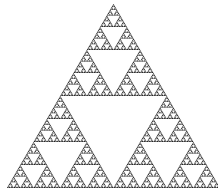


Figure 1.2: Sierpinski gasket

### 1.5.3 Iterated function systems

The iterated function systems which are an efficient way to define self-similar sets.

Let  $D$  be a closed subset of  $\mathbb{R}^n$ . A mapping  $\phi : D \rightarrow D$  is called a *contraction* on  $D$  if there exists  $c \in \mathbb{R}$ , with  $0 < c < 1$ , such that  $|\phi(x) - \phi(y)| \leq c|x - y|$  for all  $x, y \in D$ .

If  $|\phi(x) - \phi(y)| = c|x - y|$ , then  $\phi$  transforms sets into geometrically similar sets and we call  $\phi$  a *contracting similarity*. A finite family of contractions  $\{\phi_1, \dots, \phi_m\}$  with  $m \geq 2$  is called an *iterated function system* or IFS. A non-empty compact subset  $K$  of  $D$  is an *attractor* for the IFS if

$$K = \bigcup_{i=1}^m \phi_i(K).$$

The fundamental property of an iterated function system is that it determines a unique attractor, which is usually a fractal. More precisely, we have the

following result ([10, Theorem 9.1]):

Let  $\{\phi_1, \dots, \phi_m\}$  be a IFS on  $D \subset \mathbb{R}^n$  and  $\mathcal{K}$  the family of non-empty compact sets of  $D$ . Then there exists a unique attractor  $K$  such that  $K = \bigcup_{i=1}^m \phi_i(K)$ . Moreover, if we define a transformation  $\Phi : \mathcal{K} \rightarrow \mathcal{K}$  by  $\Phi(E) = \bigcup_{i=1}^m \phi_i(E)$ , for each  $E \in \mathcal{K}$  and we denote by  $\Phi^k$  the  $k$ -th iterate of  $\Phi$  such that  $\Phi_0(E) = E$  and  $\Phi^k(E) = \Phi(\Phi^{k-1}(E))$  for  $k \geq 1$ , then

$$K = \bigcap_{i=0}^{\infty} \Phi^i(E), \quad (1.2)$$

for each set  $E \in \mathcal{K}$  such that  $\phi_i(E) \subset E$  for each  $i = 1, 2, \dots, m$ .

The previous transformation  $\Phi$  is the key to computing the attractor of an IFS; indeed (1.2) already provides a method for doing so. In fact, the sequence of iterates  $\Phi^k(E)$  converges to the attractor  $K$  for any initial set  $E$  in  $\mathcal{K}$ . Thus the  $\Phi^k(E)$  provide increasingly good approximations to  $K$ . If  $K$  is a fractal, these approximations are sometimes called pre-fractals for  $K$ . Moreover, for each  $k$

$$\Phi^k(E) = \bigcup_{I_k} \phi_{i_1} \circ \dots \circ \phi_{i_k}(E),$$

where the union is over the set  $I_k$  of all  $k$ -term sequences  $(i_1, \dots, i_k)$  with  $1 \leq i_j \leq m$ . The pre-fractals  $\Phi^k(E)$  provide the usual construction of many fractals for a suitably chosen initial set  $E$ ; the  $\phi_{i_1} \circ \dots \circ \phi_{i_k}(E)$  are called the *level- $k$  sets* of the construction.

### Examples of self-similar sets

**Example 1.5.1.** Let  $C$  be the ternary Cantor set and let  $\phi_1, \phi_2 : [0, 1] \rightarrow [0, 1]$  be given by

$$\phi_1(x) = \frac{1}{3}x, \quad \phi_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then  $\phi_1(C) = [0, \frac{1}{3}] \cap C$  and  $\phi_2(C) = [\frac{2}{3}, 1] \cap C$ , so that  $C = \phi_1(C) \cup \phi_2(C)$ . Thus the ternary Cantor set  $C$  is an attractor of the IFS consisting of the contractions  $\{\phi_1, \phi_2\}$ . Moreover, if  $E = [0, 1]$  then  $\Phi^k(E) = E_k$ , i.e. the union of  $2^k$  intervals of length  $3^{-k}$  obtained at the  $k$ -th stage of the usual Cantor set construction (see 1.1).

**Example 1.5.2.** Self-similarity properties are not only properties of the fractal sets, but also of other sets. For example, the dyadic intervals are self-similar sets. A dyadic interval  $I_{k,m} \subset \mathbb{R}$  is a bounded interval of the form

$$I_{k,m} = \left[ \frac{m}{2^k}, \frac{m+1}{2^k} \right],$$

where  $m$  and  $k$  are integers.

Let  $I = [0, 1]$  be a dyadic interval of  $\mathbb{R}$  and let  $\phi_1, \phi_2 : [0, 1] \rightarrow [0, 1]$  be given by

$$\phi_1(x) = \frac{1}{2}x, \quad \phi_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Thus the dyadic interval  $I$  is an attractor of the IFS consisting of the contractions  $\{\phi_1, \phi_2\}$ .

**Example 1.5.3.** Let  $T$  be the Sierpinski triangle, let  $D \subset \mathbb{R}^2$  be the starting equilateral triangle of vertices  $\vec{a} = (0, 0)$ ,  $\vec{b} = (1, 0)$ ,  $\vec{c} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$  and let  $\phi_1, \phi_2, \phi_3 : D \rightarrow D$  be given by

$$\begin{aligned} \phi_1(\vec{x}) &= \frac{1}{2}\vec{x}, \\ \phi_2(\vec{x}) &= \frac{1}{2}\vec{x} + (\frac{1}{2}, 0), \\ \phi_3(\vec{x}) &= \frac{1}{2}\vec{x} + (\frac{1}{4}, \frac{\sqrt{3}}{4}), \end{aligned}$$

with  $\vec{x} \in D$ .

Thus the Sierpinski triangle  $T$  is an attractor of the IFS consisting of the contractions  $\{\phi_1, \phi_2, \phi_3\}$ .

## 1.6 The $s$ -R integral

It is well-known that the ordinary integral of functions with fractal support  $F \subset \mathbb{R}$  is zero or undefined depending on the definition of integral (Lebesgue or Riemann) and the nature of the support. The  $s$ -R integral introduced by Jiang and Su [19] and by Parvate and Gangal [31] suits the needs of integration of such functions.

In this section, we denote by  $E$  a closed  $s$ -set of  $\mathbb{R}$ , with  $0 < s < 1$ , and by  $a = \min E$  and  $b = \max E$ .

**Definition 1.6.1.** We say that a subset  $\tilde{A}$  of  $E$  is an  $E$ -interval whenever there exists an interval  $A \subset [a, b]$  such that  $\tilde{A} = A \cap E$ .

**Definition 1.6.2.** Given a finite collection  $P = \{(\tilde{A}_i, x_i)\}_{i=1}^p$  of pairwise disjoint  $E$ -intervals  $\tilde{A}_i$  and points  $x_i \in \tilde{A}_i$ , we say that  $P$  is a *partition of  $E$*  if  $E = \bigcup_{i=1}^p \tilde{A}_i$ .

Let  $P = \{(\tilde{A}_i, x_i)\}_{i=1}^p$  be a partition of  $E$ , let  $f : E \rightarrow \mathbb{R}$  be a function and let us consider the following Riemann-type sum

$$S(f, P) = \sum_{i=1}^p f(x_i) \mathcal{H}^s(\tilde{A}_i).$$

As mentioned above, Jiang and Su in [19] and Parvate and Gangal in [31], introduced for functions defined on a closed  $s$ -set of the real line, an extension of the Riemann integral. The following definition is due to Jiang and Su.

**Definition 1.6.3.** Let  $f : E \rightarrow \mathbb{R}$ . We say that  $f$  is  $s$ -R integrable on  $E$ , if there exists a real number  $I$  such that for each  $\varepsilon > 0$  there is a  $\delta > 0$  with

$$|S(f, P) - I| < \varepsilon,$$

for each partition  $P = \{(\tilde{A}_i, x_i)\}_{i=1}^p$  of  $E$  with  $\mathcal{H}^s(\tilde{A}_i) < \delta$ , for  $i = 1, 2, \dots, p$ .

The number  $I$  is called the  $s$ -R integral of  $f$  on  $E$  and we write

$$I = (R) \int_E f(t) d\mathcal{H}^s(t).$$

The collection of all  $s$ -R integrable functions on  $E$  will be denoted by  $s$ -R( $E$ ).

In [19] and [31] the authors proved that the usual elementary properties of the classical Riemann integral are still valid for the  $s$ -R integral and they provided conditions for the validity of the natural reformulation of the Fundamental Theorem of Calculus, in which the notion of  $s$ -derivative ([6] and [19]) is used. Below, we recall this definition.

**Definition 1.6.4.** Let  $F : E \rightarrow \mathbb{R}$  and let  $x_0 \in E$ . The  $s$ -derivatives of  $F$  at the point  $x_0$ , on the right and on the left, are defined, respectively, as follows:

$$F_s^+(x_0) = \lim_{\substack{x \rightarrow x_0^+ \\ x \in E}} \frac{F(x) - F(x_0)}{\mathcal{H}^s([x_0, x] \cap E)}, \text{ if } \mathcal{H}^s([x_0, x] \cap E) > 0 \text{ for all } x > x_0,$$

$$F_s^-(x_0) = \lim_{\substack{x \rightarrow x_0^- \\ x \in E}} \frac{F(x_0) - F(x)}{\mathcal{H}^s([x, x_0] \cap E)}, \text{ if } \mathcal{H}^s([x, x_0] \cap E) > 0 \text{ for all } x < x_0,$$

when these limits exist.

The  $s$ -derivative of  $F$  at  $x_0$  exists if  $F_s^+(x_0) = F_s^-(x_0)$  or if the  $s$ -derivative of  $F$  on the right (resp. on the left) at  $x_0$  exists and for some  $\varepsilon > 0$  we have  $\mathcal{H}^s([x_0, x_0 + \varepsilon] \cap E) = 0$  (resp.  $\mathcal{H}^s([x_0 - \varepsilon, x_0] \cap E) = 0$ ). The  $s$ -derivative of  $F$  at  $x_0$ , when it exists, will be denoted by  $F_s'(x_0)$ .

**Remark 1.** If  $F$  is  $s$ -derivable at  $x_0$ , then  $F$  is continuous at  $x_0$  according to the topology induced on  $E$  by the usual topology of  $\mathbb{R}$ .

By standard techniques, in [31, Theorem 39] the authors show that if  $F$  is continuous on  $E$  with respect to the induced topology, then  $F \in s\text{-}R(E)$ .

However, the Fundamental Theorem of Calculus on an  $s$ -set does not hold. The following example illustrates this statement.

**Example 1.6.1.** Let  $C \subset [0, 1]$  be the ternary Cantor set. We know that  $C$  is an  $s$ -set for  $s = \log_3 2$ . Let  $F : [0, 1] \rightarrow \mathbb{R}$  be the following function

$$F(x) = \begin{cases} 0, & x \leq \frac{1}{3} \\ 3x - 1, & \frac{1}{3} < x < \frac{2}{3} \\ 1, & x \geq \frac{2}{3}. \end{cases}$$

Let us denote by  $F_C$  the restricted function of  $F$  to  $C$ . It easily follows that  $(F_C)'_s$  is null on  $C$ . Then

$$(R) \int_C (F_C)'_s(t) d\mathcal{H}^s(t) = 0 \neq F_C(1) - F_C(0) = 1.$$

Jiang and Su in [19] announced, without proof, a modified version of the Fundamental Theorem of Calculus on an  $s$ -set, by using the following extension of the ordinary definition of absolutely continuous function.

**Definition 1.6.5.** (see [19, Theorem 2.3]) A function  $F : E \rightarrow \mathbb{R}$  is said to be  $\mathcal{H}^s$ -absolutely continuous if for each  $\varepsilon > 0$  there exists a constant  $\eta > 0$  such that

$$\sum_{k=1}^n |F(b_k) - F(a_k)| < \varepsilon,$$

whenever  $\sum_{k=1}^n \mathcal{H}^s(\widetilde{[a_k, b_k]}) < \eta$  with  $a_k, b_k \in E$  for  $k = 1, 2, \dots, n$  and  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$ .

**Theorem 1.6.1.** (see [19]) Let  $f : E \rightarrow \mathbb{R}$  be continuous on  $E$  with respect to the topology induced by the usual topology of  $\mathbb{R}$ . If  $F : E \rightarrow \mathbb{R}$  is  $\mathcal{H}^s$ -absolutely continuous on  $E$  and  $F'_s(x) = f(x)$  at  $\mathcal{H}^s$ -almost each point  $x \in E$ , then

$$(R) \int_E f(t) d\mathcal{H}^s(t) = F(b) - F(a).$$

Parvate and Gangal in [31], on the other hand, introduced a second condition as follows.

**Definition 1.6.6.** (see [31]) Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . A point  $x \in \mathbb{R}$  is said to be a *point of change of  $F$* , if the function  $F$  is not constant over any open interval  $(c, d)$  containing  $x$ . The set of all points of change of  $F$  is denoted by  $\text{Sch}(F)$ .

**Theorem 1.6.2.** (see [31, Theorem 57]) *If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $s$ -differentiable at each point  $x \in E$  with  $F'_s$  continuous and if  $\text{Sch}(F) \subseteq E$ , then*

$$(R) \int_E F'_s(t) d\mathcal{H}^s(t) = F(b) - F(a).$$

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# Chapter 2

## The $\mu$ -HK integral

Let  $X = (X, d)$  be a complete metric space endowed with a non atomic Radon measure  $\mu$ .

### 2.1 The family $\mathcal{F}$ of $\mu$ -cells

In this section, we define a suitable family of measurable sets which play the role of the “intervals”. To this end, we introduce the following definitions.

Let  $\mathcal{F}$  be a family of non-empty closed subsets of  $X$ .

**Definition 2.1.1.** Let  $P, Q \in \mathcal{F}$ . We say that  $P$  and  $Q$  are *non-overlapping* if the interiors of  $P$  and  $Q$  are disjoint.

**Definition 2.1.2.** Let  $Q \in \mathcal{F}$ . A finite collection  $\{Q_1, \dots, Q_m\}$  of pairwise non-overlapping elements of  $\mathcal{F}$  is called a *division* of  $Q$  if  $\bigcup_{i=1}^m Q_i = Q$ .

**Definition 2.1.3.** Let  $E \subset X$  and let  $\mathcal{G}$  be a subfamily of  $\mathcal{F}$ . We say that  $\mathcal{G}$  is a *fine cover* of  $E$  if

$$\inf\{\text{diam } Q : Q \in \mathcal{G}, Q \ni x\} = 0,$$

for each  $x \in E$ .

**Definition 2.1.4.** We say that  $\mathcal{F}$  is a  $\mu$ -*Vitali* family if for each subset  $E$  of  $X$  and for each subfamily  $\mathcal{G}$  of  $\mathcal{F}$  that is a fine cover of  $E$ , there exists a countable system  $\{Q_1, Q_2, \dots, Q_j, \dots\}$  of pairwise non-overlapping elements of  $\mathcal{G}$  such that

$$\mu(E \setminus \bigcup Q_j) = 0.$$

**Definition 2.1.5.** Let  $\mathcal{F}$  be a  $\mu$ -Vitali family. We say that  $\mathcal{F}$  is a family of  $\mu$ -*cells* if it satisfies the following conditions:

- (a) Given  $Q \in \mathcal{F}$  and a constant  $\delta > 0$ , there exist a division  $\{Q_1, \dots, Q_m\}$  of  $Q$  such that  $\text{diam}(Q_i) < \delta$  for  $i = 1, 2, \dots, m$ ;
- (b) Given  $A, Q \in \mathcal{F}$  and  $A \subset Q$ , there exists a division  $\{Q_1, \dots, Q_m\}$  of  $Q$  such that  $A = Q_1$ .
- (c)  $\mu(\partial Q) = 0$  for each  $Q \in \mathcal{F}$ .

**Example 2.1.1.** Let  $X$  be the interval  $[0, 1]$  of the real line endowed with the Euclidean distance in  $\mathbb{R}$  and with the one-dimensional Lebesgue measure  $\mathcal{L}$ . The system  $\mathcal{F}$  of all non-empty closed subintervals of  $X$  is the simplest example of  $\mathcal{L}$ -cells in  $[0, 1]$ .

In fact,  $\mathcal{F}$  is a  $\mathcal{L}$ -Vitali family by the well known Vitali covering theorem on the real line [36, Chapter IV, § 3] and Conditions (a), (b) and (c) are trivially satisfied.

**Example 2.1.2.** Let  $X$  be the interval  $[0, 1]$  of the real line endowed with the Euclidean distance in  $\mathbb{R}$  and with the one-dimensional Lebesgue measure  $\mathcal{L}$ . It is easy to see that the system  $\mathcal{F}_d$  of all non-empty closed dyadic subintervals of  $[0, 1]$  is also a family of  $\mathcal{L}$ -cells in  $[0, 1]$ .

**Example 2.1.3.** Let  $n > 1$  and let  $X$  be the unit cube  $[0, 1]^n$  of  $\mathbb{R}^n$  endowed with the Euclidean distance in  $\mathbb{R}^n$  and with the  $n$ -dimensional Lebesgue measure  $\mathcal{L}^n$ . For a fixed  $\alpha \in (0, 1]$ , the system  $\mathcal{F}_\alpha$  of all non-empty closed subintervals  $Q$  of  $[0, 1]^n$  such that  $\mathcal{L}^n(Q) \geq \alpha \mathcal{L}^n(B)$ , for some ball  $B$  containing  $Q$ , is a family of  $\mathcal{L}^n$ -cells.

In fact,  $\mathcal{F}_\alpha$  is a  $\mathcal{L}^n$ -Vitali family by [36, Chapter IV, § 3] and Conditions (a), (b) and (c) are trivially satisfied.

**Example 2.1.4.** Let  $X$  be the interval  $[0, 1]$  of the real line endowed with the Euclidean distance in  $\mathbb{R}$  and let  $E \subset [0, 1]$  be an  $s$ -set; i.e. a closed fractal subset of  $[0, 1]$  of positive  $s$ -Hausdorff measure  $\mathcal{H}^s$ , with  $0 < s < 1$ . The system  $\mathcal{F}_E$  of all non-empty closed subintervals of  $[0, 1]$  is a family of cells with respect to the measure  $\mu_E(\cdot) = \mathcal{H}^s(\cdot \cap E)$ .

In fact,  $\mu_E$  is a Radon measure by [29, Theorem 1.9 (2) and Corollary 1.11],  $\mathcal{F}_E$  is a  $\mu_E$ -Vitali family by [29, Theorem 2.8] and Conditions (a), (b) and (c) are trivially satisfied.

## 2.2 $\mu$ -HK integral

Throughout this thesis, we assume that  $X = (X, d)$  is a fixed complete metric space endowed with a non atomic Radon measure  $\mu$  and with a family  $\mathcal{F}$  of

$\mu$ -cells. To simplify the notation, from now on we use the name cell instead of the name  $\mu$ -cell each time there is no ambiguity.

**Definition 2.2.1.** A *gauge* on a cell  $Q$  is any positive real function  $\delta$  defined on  $Q$ .

**Definition 2.2.2.** Let  $Q \in \mathcal{F}$ , let  $E \subset Q$  and let  $\delta$  be a gauge on  $Q$ . A collection  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of finite ordered pairs of points and cells is said to be

- a *partition* of  $Q$  if  $\{Q_1, \dots, Q_m\}$  is a division of  $Q$  and  $x_i \in Q_i$  for  $i = 1, 2, \dots, m$ ;
- a *partial partition* of  $Q$  if  $\{Q_1, \dots, Q_m\}$  is a subsystem of a division of  $Q$  and  $x_i \in Q_i$  for  $i = 1, 2, \dots, m$ ;
- $\delta$ -*fine* if  $\text{diam}(Q_i) < \delta(x_i)$  for  $i = 1, 2, \dots, m$ ;
- $E$ -*anchored* if the points  $x_1, \dots, x_m$  belong to  $E$ .

The following Cousin's type lemma addresses the existence of  $\delta$ -fine partitions of a given cell  $Q$ .

**Lemma 2.2.1.** *If  $\delta$  is a gauge on  $Q$ , then there exists a  $\delta$ -fine partition of  $Q$ .*

*Proof.* Let us observe that if  $\{Q_1, \dots, Q_m\}$  is a division of  $Q$  and if  $\mathcal{P}_1, \dots, \mathcal{P}_m$  are  $\delta$ -fine partitions of cells  $Q_1, \dots, Q_m$  respectively, then  $\bigcup_{i=1}^m \mathcal{P}_i$  is a  $\delta$ -fine partition of  $Q$ . Using this observation, we proceed by contradiction.

By condition (a) of Definition 2.1.5 there exists a division  $\{Q_1, \dots, Q_m\}$  of  $Q$  such that  $\text{diam}(Q_i) < \text{diam}(Q)/2$  for  $i = 1, 2, \dots, m$ . Let us suppose that  $Q$  does not have a  $\delta$ -fine partition, then there exists an index  $i \in \{1, 2, \dots, m\}$  such that  $Q_i$  does not have a  $\delta$ -fine partition. Let us say  $i = 1$ . By repeating indefinitely this argument we obtain a sequence of nested cells:

$$Q \supset Q_1 \supset \dots \supset Q_k \supset \dots$$

such that  $\text{diam}(Q_k) \leq \text{diam}(Q)/2^k$  and  $Q_k$  does not have a  $\delta$ -fine partition. Since  $\text{diam}(Q_k) \rightarrow 0$  and the cells are closed sets, then there exists a point  $\xi \in Q$  such that

$$\bigcap_{k=1}^{\infty} Q_k = \{\xi\}.$$

Then, by  $\delta(\xi) > 0$ , we can find a natural  $k$  such that  $\text{diam}(Q_k) < \delta(\xi)$ . Thus  $\{(\xi, Q_k)\}$  is a  $\delta$ -fine partition of  $Q_k$ , contrarily to our assumption.  $\square$

Given a partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q \in \mathcal{F}$  and a function  $f : Q \rightarrow \mathbb{R}$  we set

$$S(f, \mathcal{P}) = \sum_{i=1}^m f(x_i) \mu(Q_i).$$

**Definition 2.2.3.** We say that a function  $f : Q \rightarrow \mathbb{R}$  is *HK-integrable* on  $Q$  with respect to  $\mu$  if there exists a real number  $I$  such that for each  $\varepsilon > 0$  there is a gauge  $\delta$  on  $Q$  with

$$|S(f, \mathcal{P}) - I| < \varepsilon,$$

for each  $\delta$ -fine partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

The number  $I$  is called the *HK-integral* of  $f$  on  $Q$  with respect to  $\mu$  (or  $\mu$ -HK integral) and we write

$$I = \int_Q f d\mu.$$

The collection of all  $\mu$ -HK integrable functions on  $Q$  (with respect to  $\mu$ ) will be denoted by  $\mu$ -HK( $Q$ ).

**Observation 1.** The number  $I$  from Definition 2.2.3 is unique. Suppose that  $I$  and  $J$  satisfies Definition 2.2.3 and assume that  $J \neq I$ . Let  $\varepsilon = |I - J|$ . Then there exist two gauges  $\delta_1$  and  $\delta_2$  on  $Q$  such that

$$|S(f, \mathcal{P}_1) - I| < \frac{\varepsilon}{2},$$

for each  $\delta_1$ -fine partition  $\mathcal{P}_1$  of  $Q$  and

$$|S(f, \mathcal{P}_2) - J| < \frac{\varepsilon}{2},$$

for each  $\delta_2$ -fine partition  $\mathcal{P}_2$  of  $Q$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ . By Lemma 2.2.1, there exists a  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ . Since  $\mathcal{P}$  is both  $\delta_1$ -fine and  $\delta_2$ -fine it follows that

$$|I - J| \leq |I - S(f, \mathcal{P})| + |S(f, \mathcal{P}) - J| < \varepsilon = |I - J|,$$

which is a contradiction. Therefore  $I = J$ .

**Observation 2.** The  $\mu$ -HK integral includes the classical HK-integral on the real line and other Henstock-Kurzweil type integrals. Indeed, if  $X$ ,  $\mu$  and  $\mathcal{F}$  are defined as in

- Example 2.1.1, then the  $\mu$ -HK integral is the usual Henstock-Kurzweil integral on  $[0, 1]$ ;
- Example 2.1.2, then the  $\mu$ -HK integral is the dyadic Henstock-Kurzweil integral on  $[0, 1]$ ;
- Example 2.1.3, then the  $\mu$ -HK integral is the Mawhin integral on  $[0, 1]^n$ ;
- Example 2.1.4, then the  $\mu$ -HK integral is the  $s$ -HK integral on a  $s$ -set studied in [3] and [4].

## 2.3 Simple properties

In this section we prove some basic properties of the  $\mu$ -HK integral.

**Theorem 2.3.1.** *If  $f, g \in \mu\text{-HK}(Q)$ , then  $f + g \in \mu\text{-HK}(Q)$  and*

$$\int_Q (f + g) d\mu = \int_Q f d\mu + \int_Q g d\mu.$$

*Proof.* Let  $\varepsilon > 0$  be given and let  $I$  and  $J$  be the  $\mu$ -HK integrals of  $f$  and  $g$  on  $Q$ , respectively. Since  $f \in \mu\text{-HK}(Q)$ , there exists a gauge  $\delta_1$  on  $Q$  such that

$$|S(f, \mathcal{P}_1) - I| < \frac{\varepsilon}{2},$$

for each  $\delta_1$ -fine partition  $\mathcal{P}_1$  of  $Q$ .

In a similar way, there exists a gauge  $\delta_2$  on  $Q$  such that

$$|S(g, \mathcal{P}_2) - J| < \frac{\varepsilon}{2},$$

for each  $\delta_2$ -fine partition  $\mathcal{P}_2$  of  $Q$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$  be a gauge on  $Q$ . By Lemma 2.2.1, there exists a  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ . Since  $\mathcal{P}$  is both  $\delta_1$ -fine and  $\delta_2$ -fine it follows that

$$|S(f + g, \mathcal{P}) - (I + J)| \leq |S(f, \mathcal{P}) - I| + |S(g, \mathcal{P}) - J| < \varepsilon.$$

By arbitrariness of  $\varepsilon$ , we obtain that  $f + g \in \mu\text{-HK}(Q)$ . □

**Theorem 2.3.2.** *If  $f \in \mu\text{-HK}(Q)$  and  $k \in \mathbb{R}$ , then  $kf \in \mu\text{-HK}(Q)$  and*

$$\int_Q kf d\mu = k \int_Q f d\mu.$$

*Proof.* Let  $\varepsilon > 0$  be given and let  $I$  be the  $\mu$ -HK integral of  $f$  on  $Q$ . Since  $f \in \mu\text{-HK}(Q)$ , there exists a gauge  $\delta$  on  $Q$  such that

$$|S(f, \mathcal{P}) - I| < \varepsilon,$$

for each  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ . Therefore, for each  $k \in \mathbb{R}$ , we have

$$|S(kf, \mathcal{P}) - kI| = |k| |S(f, \mathcal{P}) - I| < |k| \varepsilon.$$

By arbitrariness of  $\varepsilon$ , we obtain that  $kf \in \mu\text{-HK}(Q)$ . □

**Theorem 2.3.3.** *If  $f \in \mu\text{-HK}(Q)$  and  $f(x) \geq 0$  for each  $x \in Q$ , then*

$$\int_Q f \, d\mu \geq 0.$$

*Proof.* Let  $\varepsilon > 0$  be given. Since  $f \in \mu\text{-HK}(Q)$ , there exists a gauge  $\delta$  on  $Q$  such that

$$\left| S(f, \mathcal{P}) - \int_Q f \, d\mu \right| < \varepsilon,$$

for each  $\delta$ -fine partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

Since  $f(x) \geq 0$  for each  $x \in Q$ , we have

$$S(f, \mathcal{P}) = \sum_{i=1}^m f(x_i) \mu(Q_i) \geq 0.$$

Therefore

$$-\varepsilon \leq S(f, \mathcal{P}) - \varepsilon < \int_Q f \, d\mu < S(f, \mathcal{P}) + \varepsilon.$$

By the arbitrariness of  $\varepsilon$ , we obtain that  $\int_Q f \, d\mu \geq 0$ . □

**Corollary 2.3.4.** *Let  $f, g \in \mu\text{-HK}(Q)$ . If  $f \leq g$  for each  $x \in Q$ , then*

$$\int_Q f \, d\mu \leq \int_Q g \, d\mu.$$

*Proof.* Let  $h := g - f$ . By Theorem 2.3.1, we have  $h \in \mu\text{-HK}(Q)$  and

$$\int_Q h \, d\mu = \int_Q g \, d\mu - \int_Q f \, d\mu.$$

Since  $f \leq g$ , then  $h(x) \geq 0$  for each  $x \in Q$  and, by Theorem 2.3.3, we obtain that  $\int_Q h \, d\mu \geq 0$ . Therefore  $\int_Q f \, d\mu \leq \int_Q g \, d\mu$ . □

**Theorem 2.3.5** (The Cauchy Criterion). *A function  $f : Q \rightarrow \mathbb{R}$  is  $\mu$ -HK integrable on  $Q$  if and only if for each  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $Q$  such that*

$$|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| < \varepsilon,$$

for each pair  $\delta$ -fine partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $Q$ .

*Proof.* Assume first that  $f : Q \rightarrow \mathbb{R}$  is  $\mu$ -HK integrable on  $Q$ . Given  $\varepsilon > 0$ , there exists a gauge  $\delta$  on  $Q$  such that

$$\left| S(f, \mathcal{P}) - \int_Q f \, d\mu \right| < \frac{\varepsilon}{2},$$

for each  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ . If  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are two  $\delta$ -fine partitions of  $Q$ , we have

$$|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| \leq \left| S(f, \mathcal{P}_1) - \int_Q f \, d\mu \right| + \left| S(f, \mathcal{P}_2) - \int_Q f \, d\mu \right| < \varepsilon.$$

Viceversa, for each  $n \in \mathbb{N}$  let  $\delta_n$  be a gauge on  $Q$  such that

$$|S(f, \mathcal{A}_n) - S(f, \mathcal{B}_n)| < \frac{1}{n}$$

for each pair  $\delta_n$ -fine partitions  $\mathcal{A}_n$  and  $\mathcal{B}_n$  of  $Q$ .

Let  $\Delta_n(x) = \min\{\delta_1(x), \dots, \delta_n(x)\}$  be a gauge on  $Q$ . By Lemma 2.2.1, there exists a  $\Delta_n$ -fine partition  $\mathcal{P}_n$  of  $Q$ , for each  $n \in \mathbb{N}$ .

Let  $\varepsilon > 0$  be given and choose a positive natural  $N$  such that  $\frac{1}{N} < \frac{\varepsilon}{2}$ . If  $m$  and  $n$  are positive natural ( $n < m$ ) such that  $n \geq N$ , then  $\mathcal{P}_n$  and  $\mathcal{P}_m$  are  $\Delta_n$ -fine partitions on  $Q$ ; hence

$$|S(f, \mathcal{P}_n) - S(f, \mathcal{P}_m)| < \frac{1}{n} < \frac{\varepsilon}{2}.$$

Consequently,  $\{S(f, \mathcal{P}_n)\}_{n=1}^{\infty}$  is a Cauchy sequence of real numbers and hence converges. If  $A = \lim_{n \rightarrow \infty} S(f, \mathcal{P}_n)$ , then

$$|S(f, \mathcal{P}_n) - A| < \frac{\varepsilon}{2},$$

for each  $n \geq N$ . Let  $\mathcal{P}$  be a  $\Delta_N$ -fine partitions on  $Q$ , then

$$|S(f, \mathcal{P}) - A| \leq |S(f, \mathcal{P}) - S(f, \mathcal{P}_N)| + |S(f, \mathcal{P}_N) - A| < \varepsilon.$$

Thus  $f \in HK(Q)$  and  $A = \int_Q f \, d\mu$ . □

In the following theorem, we prove that the  $\mu$ -HK integrability of  $f$  on a cell  $Q$  implies its  $\mu$ -HK integrability on each subcell of  $Q$ .

**Theorem 2.3.6.** *Let  $Q \in \mathcal{F}$ . If  $f \in \mu\text{-HK}(Q)$ ,  $\mathcal{D}$  is a division of  $Q$  and  $A \in \mathcal{D}$ , then  $f \in \mu\text{-HK}(A)$  and  $\int_A f d\mu = \int_Q f \chi_A d\mu$ .*

*Proof.* Given  $\varepsilon > 0$ , by Theorem 2.3.5, there exists a gauge  $\delta$  on  $Q$  such that

$$|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)| < \varepsilon,$$

for each pair  $\delta$ -fine partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $Q$ . Given that  $A \subset Q$ , by Condition (b) of Definition 2.1.5, there exists a division  $\mathcal{D} = \{Q_1, \dots, Q_m\}$  of  $Q$  such that  $A = Q_1$ .

For each  $k \in \{2, \dots, m\}$  we fix a  $\delta$ -fine partition  $\mathcal{P}_k$  of  $Q_k$ . If  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are  $\delta$ -fine partitions of  $A$ , then  $\mathcal{R}_1 \cup \bigcup_{k=2}^m \mathcal{P}_k$  and  $\mathcal{R}_2 \cup \bigcup_{k=2}^m \mathcal{P}_k$  are  $\delta$ -fine partitions of  $Q$ . Thus

$$\begin{aligned} & |S(f, \mathcal{R}_1) - S(f, \mathcal{R}_2)| \\ &= \left| S(f, \mathcal{R}_1) + \sum_{k=2}^m S(f, \mathcal{P}_k) - S(f, \mathcal{R}_2) - \sum_{k=2}^m S(f, \mathcal{P}_k) \right| \\ &= \left| S\left(f, \mathcal{R}_1 \cup \bigcup_{k=2}^m \mathcal{P}_k\right) - S\left(f, \mathcal{R}_2 \cup \bigcup_{k=2}^m \mathcal{P}_k\right) \right| < \varepsilon. \end{aligned}$$

Therefore, by Theorem 2.3.5, it follows that  $f \in \mu\text{-HK}(A)$ .  $\square$

**Definition 2.3.1.** Let  $\pi : \mathcal{F} \rightarrow \mathbb{R}$  be a function. We say that  $\pi$  is an *additive function of cell*, if for each  $Q \in \mathcal{F}$  and for each division  $\{Q_1, \dots, Q_m\}$  of  $Q$  we have

$$\pi(Q) = \sum_{i=1}^m \pi(Q_i).$$

**Observation 3.** Let  $f : Q \rightarrow \mathbb{R}$  be a  $\mu$ -HK integrable function on  $Q$ . If  $\{Q_1, \dots, Q_m\}$  is a division of  $Q$ , then  $f \in \mu\text{-HK}(Q_1) \cap \dots \cap \mu\text{-HK}(Q_m)$  and

$$\int_Q f d\mu = \sum_{i=1}^m \int_{Q_i} f d\mu.$$

*Proof.* Given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $Q$  such that

$$\left| S(f, \mathcal{P}) - \int_Q f d\mu \right| < \varepsilon,$$



for each  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ .

By theorem 2.3.6 it follows that  $f \in \mu\text{-HK}(Q_i)$  for  $i = 1, \dots, m$ . Then there exists a gauge  $\delta_i$  on  $Q_i$  for  $i = 1, 2, \dots, m$  such that  $\delta_i(x) < \delta(x)$  for each  $x \in Q_i$  and such that  $\left| S(f, \mathcal{P}_i) - \int_{Q_i} f d\mu \right| < \frac{\varepsilon}{m}$ , for each  $\delta_i$ -fine partition  $\mathcal{P}_i$  of  $Q_i$ . Therefore  $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_m$  is a  $\delta$ -fine partition of  $Q$ . Consequently

$$\begin{aligned} & \left| S(f, \mathcal{P}) - \sum_{i=1}^m \int_{Q_i} f d\mu \right| \\ & \leq \left| S(f, \mathcal{P}_1) - \int_{Q_1} f d\mu \right| + \dots + \left| S(f, \mathcal{P}_m) - \int_{Q_m} f d\mu \right| < \varepsilon. \end{aligned}$$

Therefore  $\int_Q f d\mu = \sum_{i=1}^m \int_{Q_i} f d\mu$ . □

**Definition 2.3.2.** Let  $Q \in \mathcal{F}$  and let  $f : Q \rightarrow \mathbb{R}$  be a  $\mu$ -HK integrable function on  $Q$ . We say that the map

$$F \rightarrow A \rightsquigarrow \int_A f d\mu,$$

defined on each subcell of  $Q$  is the *indefinite  $\mu$ -HK integral* of  $f$ .

By Observation 3, it follows that  $F$  is an additive function of cells.

## 2.4 The Saks-Henstock Lemma

Now, we prove the following Saks-Henstock type Lemma.

**Lemma 2.4.1.** *A function  $f : Q \rightarrow \mathbb{R}$  is  $\mu$ -HK integrable on  $Q$  if and only if there exists an additive cell function  $\pi$  defined on the family of all subcells of  $Q$  such that, for each  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $Q$  with*

$$\sum_{(x_i, Q_i) \in \mathcal{P}} \left| \pi(Q_i) - f(x_i)\mu(Q_i) \right| < \varepsilon, \quad (2.1)$$

for each  $\delta$ -fine partial partition  $\mathcal{P}$  of  $Q$ .

In this situation,  $\pi$  is the *indefinite  $\mu$ -HK integral* of  $f$  on  $Q$ .

*Proof.* Assume first that  $f \in \mu\text{-HK}(Q)$ , then for each  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $Q$  such that

$$\left| \int_Q f d\mu - S(f, \mathcal{P}) \right| < \frac{\varepsilon}{3},$$

for each  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ .

Fix a partition  $\mathcal{P}_0$  of  $Q$  and let  $\mathcal{P} \subset \mathcal{P}_0$  be a  $\delta$ -fine partial partition of  $Q$ . Then  $\mathcal{P}_0 \setminus \mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$ .

Moreover, by Theorem 2.3.6, it follows that  $f \in \mu\text{-HK}(Q_j)$  for  $j = 1, 2, \dots, m$ . Therefore, given  $\eta > 0$  and for  $j \in \{1, \dots, m\}$  there exists a gauge  $\delta_j$  on  $Q_j$  such that  $\delta_j(x) < \delta(x)$  for each  $x \in Q_j$  and such that

$$\left| \int_{Q_j} f \, d\mu - S(f, \mathcal{P}_j) \right| < \frac{\eta}{m},$$

for each  $\delta_j$ -fine partition  $\mathcal{P}_j$  of  $Q_j$ .

Therefore  $\mathcal{P}_0 = \mathcal{P} \cup \bigcup_{j=1}^m \mathcal{P}_j$  is a  $\delta$ -fine partition of  $Q$  and

$$\sum_{(x_i, Q_i) \in \mathcal{P}_0} f(x_i) \mu(Q_i) = \sum_{(x_i, Q_i) \in \mathcal{P}} f(x_i) \mu(Q_i) + \sum_{j=1}^m \sum_{(x_i, Q_i) \in \mathcal{P}_j} f(x_i) \mu(Q_i).$$

Let us denote by  $\pi$  the indefinite  $\mu$ -HK integral of  $f$  on  $Q$ . Then, by Observation 3, we have

$$\pi(Q) = \sum_{(x_i, Q_i) \in \mathcal{P}} \pi(Q_i) + \sum_{j=1}^m \pi(Q_j).$$

Consequently

$$\begin{aligned} & \left| \sum_{(x_i, Q_i) \in \mathcal{P}} \left( \pi(Q_i) - f(x_i) \mu(Q_i) \right) \right| \leq \left| \pi(Q) - \sum_{(x_i, Q_i) \in \mathcal{P}_0} f(x_i) \mu(Q_i) \right| + \\ & + \sum_{j=1}^m \left| \pi(Q_j) - \sum_{(x_i, Q_i) \in \mathcal{P}_j} f(x_i) \mu(Q_i) \right| < \frac{\varepsilon}{3} + m \frac{\eta}{m} = \frac{\varepsilon}{3} + \eta. \end{aligned}$$

So, by the arbitrariness of  $\eta$ , we have

$$\left| \sum_{(x_i, Q_i) \in \mathcal{P}} \left( \pi(Q_i) - f(x_i) \mu(Q_i) \right) \right| < \frac{\varepsilon}{2}, \quad (2.2)$$

for each  $\delta$ -fine partial partition  $\mathcal{P}$  of  $Q$ .

Let

$$\mathcal{P}^+ = \{(x_i, Q_i) \in \mathcal{P} : \pi(Q_i) - f(x_i) \mu(Q_i) \geq 0\},$$

and

$$\mathcal{P}^- = \{(x_i, Q_i) \in \mathcal{P} : \pi(Q_i) - f(x_i) \mu(Q_i) < 0\}.$$

Note that both  $\mathcal{P}^+$  and  $\mathcal{P}^-$  are  $\delta$ -fine partial partition of  $Q$ , so they satisfy (2.2). Thus

$$\begin{aligned} & \sum_{(x_i, Q_i) \in \mathcal{P}} \left| \pi(Q_i) - f(x_i)\mu(Q_i) \right| \\ &= \sum_{(x_i, Q_i) \in \mathcal{P}^+} \left( \pi(Q_i) - f(x_i)\mu(Q_i) \right) - \sum_{(x_i, Q_i) \in \mathcal{P}^-} \left( \pi(Q_i) - f(x_i)\mu(Q_i) \right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Now assume that there exists an additive cell function  $\pi$ , defined on the family of all subcells of  $Q$ , such that for each  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $Q$  with

$$\sum_{(x_i, Q_i) \in \mathcal{P}} \left| \pi(Q_i) - f(x_i)\mu(Q_i) \right| < \varepsilon,$$

for each  $\delta$ -fine partial partition  $\mathcal{P}$  of  $Q$ . In particular, this inequality holds for a  $\delta$ -fine partition  $\mathcal{P}_0 = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  of  $Q$ .

Since  $\pi$  is an additive cell function, we have

$$\begin{aligned} & \left| \pi(Q) - \sum_{i=1}^m f(x_i)\mu(Q_i) \right| = \left| \sum_{i=1}^m \pi(Q_i) - \sum_{i=1}^m f(x_i)\mu(Q_i) \right| \\ & \leq \sum_{i=1}^m \left| \pi(Q_i) - f(x_i)\mu(Q_i) \right| < \varepsilon. \end{aligned}$$

Therefore  $f \in \mu\text{-HK}(Q)$  and  $\int_Q f d\mu = \pi(Q)$ .  $\square$

## 2.5 Relation with the Lebesgue integral

It is useful to remark that each Lebesgue integrable function on a cell  $Q$  is  $\mu$ -HK integrable and the two integrals coincide. In order to do this we recall the following theorem ([35, Theorem 2.25]), where we denote by  $(L)\int_Q f d\mu$  the Lebesgue integral of  $f$  on  $Q$  with respect to  $\mu$ .

**Theorem 2.5.1** (The Vitali-Carathéodory Theorem). *Let  $f$  be a real function defined on a cell  $Q$ . If  $f$  is Lebesgue integrable on  $Q$  with respect to  $\mu$  and  $\varepsilon > 0$ , then there exists functions  $u$  and  $v$  on  $Q$  such that  $u \leq f \leq v$ ,  $u$*

is upper semicontinuous and bounded above,  $v$  is lower semicontinuous and bounded below, and

$$(L)\int_Q (v - u) d\mu < \varepsilon.$$

**Theorem 2.5.2.** *Let  $f : Q \rightarrow \mathbb{R}$  be a function. If  $f$  is Lebesgue integrable on  $Q$ , with respect to  $\mu$ , then  $f$  is  $\mu$ -HK integrable on  $Q$  and*

$$(L)\int_Q f d\mu = \int_Q f d\mu.$$

*Proof.* By Vitali-Carathéodory Theorem, given  $\varepsilon > 0$  there exist functions  $u$  and  $v$  on  $Q$  that are upper and lower semicontinuous respectively such that  $-\infty \leq u \leq f \leq v \leq +\infty$  and  $(L)\int_Q (v - u) d\mu < \varepsilon$ . Define on  $Q$  a gauge  $\delta$  so that

$$u(t) \leq f(x) + \varepsilon \quad \text{and} \quad v(t) \geq f(x) - \varepsilon,$$

for each  $t \in Q$  with  $d(x, t) < \delta(x)$ .

Let  $\mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  be a  $\delta$ -fine partition of  $Q$ . Then, for each  $i \in \{1, 2, \dots, p\}$ , we have

$$(L)\int_{Q_i} u d\mu \leq (L)\int_{Q_i} f d\mu \leq (L)\int_{Q_i} v d\mu. \quad (2.3)$$

Moreover, by  $u(t) \leq f(x_i) + \varepsilon$  for each  $t \in Q_i$ , it follows

$$(L)\int_{Q_i} (u - \varepsilon) d\mu \leq (L)\int_{Q_i} f(x_i) d\mu$$

and therefore

$$(L)\int_{Q_i} u d\mu - \varepsilon \mu(Q_i) \leq f(x_i) \mu(Q_i).$$

Similarly, by  $v(t) \geq f(x_i) - \varepsilon$  for each  $t \in Q_i$ , it follows

$$f(x_i) \mu(Q_i) \leq (L)\int_{Q_i} v d\mu + \varepsilon \mu(Q_i).$$

So, for  $i = 1, 2, \dots, p$ , we have

$$(L)\int_{Q_i} u d\mu - \varepsilon \mu(Q_i) \leq f(x_i) \mu(Q_i) \leq (L)\int_{Q_i} v d\mu + \varepsilon \mu(Q_i).$$

Hence,

$$(L)\int_Q u d\mu - \varepsilon \leq S(f, \mathcal{P}) \leq (L)\int_Q v d\mu + \varepsilon,$$

and, by (2.3),

$$(L)\int_Q u \, d\mu \leq (L)\int_Q f \, d\mu \leq (L)\int_Q v \, d\mu.$$

Thus

$$\left| S(f, \mathcal{P}) - (L)\int_Q f \, d\mu \right| \leq (L)\int_E (v - u) \, d\mu + 2\varepsilon < 3\varepsilon,$$

and the theorem is proved.  $\square$

**Remark 2.** It is well known that on  $\mathbb{R}^n$  there exist functions that are  $\mu$ -HK integrable with respect to the Lebesgue measure, but that are not Lebesgue integrable. The following example shows that the same holds on the ternary Cantor set with respect to the  $s$ -dimensional Hausdorff measure, with  $s = \log_3 2$ .

**Example 2.5.1.** Let  $E$  be the ternary Cantor set, let  $\mu$  be the  $\log_3 2$ -dimensional Hausdorff measure and let  $f : E \rightarrow \mathbb{R}$  be the function defined as follow

$$f(x) = \begin{cases} \frac{(-1)^{n+1}3^n}{n}, & \text{for } x \in \left[ \frac{2}{3^n}, \frac{1}{3^{n-1}} \right] \cap E, \quad n = 1, 2, 3, \dots \\ 0, & \text{for } x = 0. \end{cases}$$

We will prove that  $f \in \mu\text{-HK}(E)$ , but that  $f$  is not Lebesgue integrable on  $E$  with respect to  $\mu$ .

*Proof.* Fixed  $\varepsilon > 0$ , let  $k \in \mathbb{N}$  such that  $\varepsilon k \geq 2$  and

$$\left| \sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}}{i} \right| < \frac{\varepsilon}{2}, \quad \text{for each } n \geq k. \quad (2.4)$$

Define a gauge  $\delta$  on  $E$  such that

- if  $x \in E$  and  $x \neq 0$ ,  $f(x)$  is constant on  $(x - \delta(x), x + \delta(x)) \cap E$ ;
- $\delta(0) = \frac{1}{3^{k-1}}$ ,

and let us consider  $\mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  a  $\delta$ -fine partition of  $E$  such that  $Q_1 = [0, c] \cap E$ . Our choice of  $\delta$  implies that  $x_1 = 0$  and  $c < \frac{1}{3^{k-1}}$ . Let  $n \in \mathbb{N}$  such that  $\frac{1}{3^n} < c < \frac{1}{3^{n-1}}$ , then  $n \geq k$ . Moreover, by the definition of  $\delta$  it follows that:

$$\bigcup_{i=2}^m Q_i = \begin{cases} ([c, \frac{1}{3^{n-1}}] \cup [\frac{2}{3^{n-1}}, 1]) \cap E, & \text{if } c \geq \frac{2}{3^n} \\ [\frac{2}{3^n}, 1] \cap E, & \text{if } c < \frac{2}{3^n}. \end{cases}$$

So

$$\sum_{i=1}^m f(x_i) \mu(Q_i) = \begin{cases} \sum_{i=1}^n \frac{(-1)^{i+1} 3^i}{i} \mu \left( \left[ \frac{2}{3^i}, \frac{1}{3^{i-1}} \right] \cap E \right) + \\ \quad + \frac{(-1)^{n+1} 3^n}{n} \mu \left( \left[ c, \frac{1}{3^{n-1}} \right] \cap E \right), & \text{if } c \geq \frac{2}{3^n} \\ \sum_{i=1}^n \frac{(-1)^{i+1} 3^i}{i} \mu \left( \left[ \frac{2}{3^i}, \frac{1}{3^{i-1}} \right] \cap E \right), & \text{if } c < \frac{2}{3^n}. \end{cases}$$

Hence

$$|S(f, \mathcal{P}) - \log 2| = \begin{cases} \left| \sum_{i=1}^n \frac{(-1)^{i+1}}{i} - \log 2 \right| + \left| \frac{1}{n} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, & \text{if } c \geq \frac{2}{3^n} \\ \left| \sum_{i=1}^n \frac{(-1)^{i+1}}{i} - \log 2 \right| < \frac{\varepsilon}{2}, & \text{if } c < \frac{2}{3^n}. \end{cases}$$

Consequently  $f \in \mu\text{-HK}(E)$ .

Now, if  $f$  were Lebesgue integrable on  $E$  with respect to  $\mu$ ,  $|f|$  would be Lebesgue integrable on  $E$  with respect to  $\mu$ . But

$$(L) \int_E |f| d\mu = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

hence  $f$  is not Lebesgue integrable on  $E$  with respect to  $\mu$ .  $\square$

## 2.6 Absolutely $\mu$ -HK integrable functions

**Definition 2.6.1.** Let  $Q$  be a cell. We say that a function  $f : Q \rightarrow \mathbb{R}$  is *absolutely  $\mu$ -HK integrable* on  $Q$  if  $|f|$  is  $\mu$ -HK integrable on  $Q$ .

In this section we study the absolutely  $\mu$ -HK integrable functions, in particular we prove that these functions are Lebesgue integrable and their primitives are differentiable  $\mu$ -almost everywhere.

**Definition 2.6.2.** Let  $F$  be a cell function defined on  $\mathcal{F}$  and let  $x \in X$ . We say that the *upper derivative* of  $F$  at  $x$ , with respect to  $\mu$ , is defined as

$$\overline{D}F(x) = \limsup_{\mathcal{F} \ni B \rightarrow x} \frac{F(B)}{\mu(B)},$$

where the limit superior is taken over all sequences of cells  $B$  such that  $x \in B$  and  $\text{diam}(B) \rightarrow 0$ . Similarly, the *lower derivative*  $\underline{D}F(x)$  is the lower limit of that same ratio  $\frac{F(B)}{\mu(B)}$ . Whenever  $\underline{D}F(x) = \overline{D}F(x) \neq \infty$ , then  $F$  is said to be *differentiable* at  $x$  and their common value is called the *derivative* of  $F$  at  $x$  and it is denoted by  $F'(x)$ .

**Remark 3.** By previous definition, it is easy to see that  $\underline{D}F(x) \leq \overline{D}F(x)$ .

**Theorem 2.6.1.** *If  $f$  is a non-negative  $\mu$ -HK integrable function on a cell  $Q$  and  $F$  is its indefinite  $\mu$ -HK integral, then  $F$  is differentiable  $\mu$ -almost everywhere in  $Q$  and  $F' = f$   $\mu$ -almost everywhere in  $Q$ .*

*Proof.* To prove that  $F' = f$   $\mu$ -almost everywhere in  $Q$ , it is enough to show that  $\overline{D}F \leq f \leq \underline{D}F$   $\mu$ -almost everywhere in  $Q$ , since  $\underline{D}F \leq \overline{D}F$  everywhere. To this end, we consider rational numbers  $p, q$  such that  $q > p$  and we set

$$A_{p,q} = \{x \in Q : \overline{D}F(x) > q > p > f(x)\}.$$

If we prove that  $\mu(A_{p,q}) = 0$  for each  $p$  and  $q$ , then  $\overline{D}F(x) \leq f(x)$   $\mu$ -almost everywhere in  $Q$ . Similarly, we can prove that  $\underline{D}F(x) \geq f(x)$   $\mu$ -almost everywhere in  $Q$ .

Given  $\varepsilon > 0$ , by Lemma 2.4.1 there exists a gauge  $\delta$  on  $Q$  such that

$$\sum_{j=1}^m |F(Q_j) - f(x_j)\mu(Q_j)| < \varepsilon,$$

for each  $\delta$ -fine partial partition  $\{(x_j, Q_j)\}_{j=1}^m$  of  $Q$ .

Let  $\mathcal{V}$  be the system of all cells  $B \subset Q$  such that  $F(B) > q\mu(B)$  and there exists  $x \in B \cap A_{p,q}$  with  $\text{diam}(B) < \delta(x)$ . It is easy to see that this system  $\mathcal{V}$  is a fine cover of  $A_{p,q}$ , therefore ( $\mathcal{F}$  being a  $\mu$ -Vitali family) there exists a system of pairwise non-overlapping cells  $\{B_j\}_{j=1}^m \subset \mathcal{V}$  such that

$$\mu(A_{p,q}) \leq \sum_{j=1}^m \mu(B_j) + \varepsilon. \quad (2.5)$$

For  $j = 1, 2, \dots, m$ , let  $x_j \in B_j \cap A_{p,q}$  such that  $\text{diam}(B_j) < \delta(x_j)$ . Since  $\{(x_j, B_j)\}_{j=1}^m$  is a  $\delta$ -fine partial partition of  $Q$ , we get

$$\begin{aligned} q \sum_{j=1}^m \mu(B_j) &< \sum_{j=1}^m F(B_j) \leq \sum_{j=1}^m |F(B_j) - f(x_j)\mu(B_j)| + \sum_{j=1}^m f(x_j)\mu(B_j) \\ &< \varepsilon + p \sum_{j=1}^m \mu(B_j). \end{aligned}$$

Therefore  $(q - p) \sum_{j=1}^m \mu(B_j) < \varepsilon$ .

By the arbitrariness of  $\varepsilon$  we have that  $\sum_{j=1}^m \mu(B_j) = 0$  and by (2.5) we obtain  $\mu(A_{p,q}) = 0$ .  $\square$

Now, we prove that each absolutely  $\mu$ -HK integrable function is Lebesgue integrable. To this end we need the following Monotone Convergence type Theorem.

**Theorem 2.6.2.** *Let  $\{f_k\}_k$  be an increasing sequence of functions that are  $\mu$ -HK integrable on a cell  $Q$  and let  $f = \lim_k f_k$ . If*

$$\lim_{k \rightarrow \infty} \int_Q f_k d\mu < \infty,$$

*then  $f$  is  $\mu$ -HK integrable on  $Q$  and*

$$\int_Q f d\mu = \lim_{k \rightarrow \infty} \int_Q f_k d\mu.$$

*Proof.* We observe that, since  $\{f_k\}_k$  is an increasing sequence of functions and since  $\{\int_Q f_k d\mu\}_k$  is bounded on  $Q$ , therefore  $\{\int_Q f_k d\mu\}_k$  is an increasing sequence and it converges to a number  $A \in \mathbb{R}$ . Then, given  $\varepsilon > 0$  there exists  $K \in \mathbb{N}$  such that for  $k \geq K$  we have

$$0 \leq A - \int_Q f_k d\mu \leq \varepsilon. \quad (2.6)$$

Since  $f_k$  is  $\mu$ -HK integrable on  $Q$  for each  $k$ , therefore by Lemma 2.4.1 there exists an additive function  $\pi$  on the subcells of  $Q$  such that, for the previous  $\varepsilon$ , there exists a gauge  $\delta_k$  on  $Q$  with

$$\sum_{(x_i, Q_i) \in \mathcal{P}} \left| \pi(Q_i) - f_k(x_i) \mu(Q_i) \right| < \frac{\varepsilon}{2^k}, \quad (2.7)$$

for each  $\delta_k$ -fine partial partition  $\mathcal{P}$  of  $Q$  and  $\pi(Q_i) = \int_{Q_i} f_k d\mu$ .

By the pointwise convergence of  $\{f_k\}_k$  to  $f$ , it follows that for each  $x \in Q$  we can choose a natural number  $n(x) \geq K$  so that

$$|f(x) - f_k(x)| < \varepsilon, \quad (2.8)$$

whenever  $k \geq n(x) \geq K$ .

We define  $\delta(x) = \delta_{n(x)}(x)$  for  $x \in Q$ , so that  $\delta$  is a gauge on  $Q$ . We use this gauge to show that  $f$  is  $\mu$ -HK integrable on  $Q$  with integral  $A$ . Let  $\mathcal{P} = \{(Q_1, x_1), \dots, (Q_p, x_p)\}$  be a  $\delta$ -fine partition of  $Q$  and we consider the difference  $|S(f, \mathcal{P}) - A|$ . Adding and subtracting  $\sum_{i=1}^p f_{n(x_i)}(x_i) \mu(Q_i)$  and



$\sum_{i=1}^p \int_{Q_i} f_{n(x_i)} d\mu$  we have that

$$\begin{aligned}
|S(f, \mathcal{P}) - A| &\leq \left| \sum_{i=1}^p f(x_i) \mu(Q_i) - \sum_{i=1}^p f_{n(x_i)}(x_i) \mu(Q_i) \right| \\
&\quad + \left| \sum_{i=1}^p f_{n(x_i)}(x_i) \mu(Q_i) - \sum_{i=1}^p \int_{Q_i} f_{n(x_i)} d\mu \right| \\
&\quad + \left| \sum_{i=1}^p \int_{Q_i} f_{n(x_i)} d\mu - A \right|.
\end{aligned} \tag{2.9}$$

By (2.8) the first term on the right of (2.9) is dominated by

$$\sum_{i=1}^p |f(x_i) - f_{n(x_i)}(x_i)| \mu(Q_i) < \varepsilon \mu(Q).$$

The second term on the right of (2.9) is dominated by

$$\sum_{i=1}^p \left| f_{n(x_i)}(x_i) \mu(Q_i) - \int_{Q_i} f_{n(x_i)} d\mu \right|.$$

Let  $S = \max\{n(x_1), n(x_2), \dots, n(x_m)\} \geq K$ . Then, the previous sum can be written as

$$\sum_{k=K}^S \sum_{n(x_i)=k} \left| f_{n(x_i)}(x_i) \mu(Q_i) - \int_{Q_i} f_{n(x_i)} d\mu \right|,$$

in which we have grouped together all terms corresponding to  $f_k$  for a fixed  $k$ .

We have that  $\{(Q_i, x_i) : n(x_i) = k\}$  is a  $\delta_k$ -fine partition of  $Q$ . Therefore, by (2.7), we have

$$\sum_{n(x_i)=k} \left| f_{n(x_i)}(x_i) \mu(Q_i) - \int_{Q_i} f_{n(x_i)} d\mu \right| \leq \frac{\varepsilon}{2^k}.$$

Summing over  $k$ , we find that second term of (2.9) is dominated by

$$\sum_{k=K}^S \frac{\varepsilon}{2^k} < \varepsilon.$$

Finally, we consider the third term on the right of (2.9). Since the sequence  $\{f_k\}_k$  is increasing and  $n(x_i) \geq K$  for  $i = 1, 2, \dots, p$ , then  $f_{n(x_i)} \geq f_K$  implies

$$\int_Q f_K d\mu = \sum_{i=1}^p \int_{Q_i} f_K d\mu \leq \sum_{i=1}^p \int_{Q_i} f_{n(x_i)} d\mu$$

and therefore, by (2.6), we have

$$0 \leq A - \sum_{i=1}^p \int_{Q_i} f_{n(x_i)} d\mu \leq A - \int_Q f_K d\mu < \varepsilon.$$

Combining these three estimates we obtain that

$$|S(f, \mathcal{P}) - A| \leq \varepsilon \mu(Q) + \varepsilon + \varepsilon,$$

for each  $\delta$ -fine partition  $\mathcal{P}$  of  $Q$ .

By the arbitrariness of  $\varepsilon$ , then  $f$  is  $\mu$ -HK integrable on  $Q$  with integral  $A$ .  $\square$

**Theorem 2.6.3.** *If  $f$  is a non-negative  $\mu$ -HK integrable function on a cell  $Q$  and  $F$  is its indefinite  $\mu$ -HK integral, then  $f$  is  $\mu$ -measurable.*

*Proof.* For  $k \in \mathbb{N}$ , let  $\mathcal{P}_k$  be a  $1/k$ -fine partial partition of  $Q$  and let  $f_k$  be the  $\mu$ -simple function defined as follows

$$f_k(x) = \sum_{(x,B) \in \mathcal{P}_k} \frac{F(B)}{\mu(B)}.$$

We set  $C = \bigcup_{k=1}^{\infty} \bigcup_{B \in \mathcal{P}_k} \partial B$  and we set

$$D = \{x \in Q : F'(x) \text{ does not exist, or } F'(x) \text{ exists and } F'(x) \neq f(x)\}.$$

By Condition (c) of Definition 2.1.5 and by Theorem 2.6.1 the set  $E = C \cup D$  is  $\mu$ -null. Now let  $x \in Q \setminus E$ . For each  $k \in \mathbb{N}$  there exists  $Q_{k,x} \in \mathcal{F}$  such that  $(x, Q_{k,x}) \in \mathcal{P}_k$ ,  $\text{diam}(Q_{k,x}) < 1/k$  and  $f_k(x) = F(Q_{k,x})/\mu(Q_{k,x})$ . Then, by  $F'(x) = f(x)$ , we obtain  $f_k(x) \rightarrow f(x)$ .

Thus the claim follows by the  $\mu$ -measurability of  $f_k$  for each  $k \in \mathbb{N}$ .  $\square$

**Theorem 2.6.4.** *If  $f$  is absolutely  $\mu$ -HK integrable on a cell  $Q$ , then  $f$  is Lebesgue integrable on  $Q$ .*

*Proof.* For  $k \in \mathbb{N}$ , let  $f_k(x) = \min\{|f(x)|, k\}$ , for each  $x \in Q$ . By Theorem 2.6.3,  $|f|$  is Lebesgue measurable, therefore  $f_k$  is Lebesgue measurable and bounded, then it is Lebesgue integrable. Thus, by Theorem 2.5.2,  $f_k$  is  $\mu$ -HK integrable on  $Q$ .

Hence, since  $\{f_k\}_k$  is an increasing sequence of non-negative functions convergent to  $|f|$ , by Theorem 2.6.2, we have

$$(L) \int_Q |f| d\mu = (L) \lim_{k \rightarrow \infty} \int_Q f_k d\mu = \lim_{k \rightarrow \infty} \int_Q f_k d\mu = \int_Q |f| d\mu < \infty,$$

and the proof is complete.  $\square$

## 2.7 Characterization of $\mathcal{F}$ -additive functions

Hereafter, we denote by  $\pi$  a fixed additive function defined on the family of all subcells of  $Q$ .

**Definition 2.7.1.** Given  $E$  an arbitrary subset of  $Q$ , we set

$$V^\delta \pi(E) = \sup \left\{ \sum_{i=1}^m |\pi(Q_i)| \right\},$$

where the supremum is taken over all the  $\delta$ -fine  $E$ -anchored partial partition  $\mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  of  $Q$ .

The *critical variation* of  $\pi$  on  $E$  is given by

$$V\pi(E) = \inf V^\delta \pi(E),$$

where the infimum is taken over all gauges  $\delta$  on  $E$ .

Now, we prove that the extended real-valued function  $V\pi : E \rightsquigarrow V\pi(E)$  is a metric outer measures on  $Q$ . Therefore, by Carathéodory criterion ([10], Theorem 1.5),  $V\pi$  is a Borel measure.

**Theorem 2.7.1.** *A critical variation  $V\pi$  is a metric outer measures on a cell  $Q$ .*

*Proof.* To verify that  $V\pi$  is an outer measures on  $Q$ , we only need to prove that  $V\pi$  is  $\sigma$ -subadditive. Indeed, it is easy to prove that  $V\pi(\emptyset) = 0$  and that  $V\pi(A) \leq V\pi(B)$  if  $A \subset B$ .

Let  $\{A_j\}_j$  be a family of pairwise disjoint subsets of  $Q$ . For each  $j \in \mathbb{N}$ , given  $\varepsilon > 0$ , there exists a gauge  $\delta_j$  on  $A_j$  such that

$$V^{\delta_j} \pi(A_j) \leq V\pi(A_j) + \frac{\varepsilon}{2^j}, \quad (2.10)$$

for each  $\delta_j$ -fine  $A_j$ -anchored partial partition of  $Q$ .

Therefore, the function  $\delta$  defined by setting  $\delta(x) = \delta_j(x)$  for  $x \in A_j$  and  $j \in \mathbb{N}$  is a gauge on  $A = \bigcup_{j=1}^{\infty} A_j$ .

Let  $\mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  be a  $\delta$ -fine  $A$ -anchored partial partition of  $Q$ . Then  $\mathcal{P}_j = \{(x_k, Q_k) : x_k \in A_j\}$  for each  $j \in \mathbb{N}$  is  $\delta_j$ -fine  $A_j$ -anchored partial partition of  $Q$ .

By decomposition of every  $\delta$ -fine  $A$ -anchored partial partition in the union of  $\delta_j$ -fine  $A_j$ -anchored partial partition we have

$$\sum_{i=1}^m |\pi(Q_i)| \leq \sum_{j=1}^{\infty} \sum_{x_k \in A_j} |\pi(Q_k)| \leq \sum_{j=1}^{\infty} V^{\delta_j} \pi(A_j).$$

By arbitrariness of  $\mathcal{P}$  and by (2.10) it follows that

$$V^\delta \pi(A) \leq \sum_{j=1}^{\infty} V^{\delta_j} \pi(A_j) \leq \sum_{j=1}^{\infty} V \pi(A_j) + \varepsilon.$$

Moreover  $V \pi(A) \leq V^\delta \pi(A)$ . By arbitrariness of  $\varepsilon$ , it follows that  $V \pi(A) \leq \sum_{j=1}^{\infty} V \pi(A_j)$ .

Now, let  $A_1, A_2 \in \mathcal{Q}$  such that  $d(A_1, A_2) > 0$ . To prove that  $V \pi$  is a metric outer measures it is enough to show that  $V \pi(A_1) + V \pi(A_2) \leq V \pi(A)$  where  $A = A_1 \cup A_2$ .

If  $d(A_1, A_2) > 0$ , then there exist open sets  $G_1$  and  $G_2$  such that  $A_1 \subset G_1$ ,  $A_2 \subset G_2$  and  $G_1 \cap G_2 = \emptyset$ .

Given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $A$  such that

$$V^\delta \pi(A) \leq V \pi(A) + \varepsilon,$$

for each  $\delta$ -fine  $A$ -anchored partial partition  $\mathcal{P} = \{(x_1, Q_1), \dots, (x_m, Q_m)\}$  of  $Q$ . We define  $\delta_j(x) = \inf\{\delta(x), d(x, \partial G_j)\}$  for  $x \in A_j$ . Let  $\mathcal{P}_j$  be a  $\delta_j$ -fine  $A_j$ -anchored partial partition of  $Q$ . Moreover, by Definition 2.7.1, for  $j \in \{1, 2\}$  we have

$$V^{\delta_j} \pi(A_j) \leq \sum_{x_k \in A_j} |\pi(Q_k)| + \varepsilon.$$

Therefore  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$  and the cells  $Q_1, \dots, Q_m$  are contained in  $G_1$  or in  $G_2$ . Hence

$$\begin{aligned} V \pi(A_1) + V \pi(A_2) &\leq V^{\delta_1} \pi(A_1) + V^{\delta_2} \pi(A_2) \\ &\leq \sum_{x_k \in A_1} |\pi(Q_k)| + \sum_{x_k \in A_2} |\pi(Q_k)| + 2\varepsilon = \sum_{i=1}^m |\pi(Q_i)| + 2\varepsilon \\ &\leq V^\delta \pi(A) + 2\varepsilon \leq V \pi(A) + 3\varepsilon. \end{aligned}$$

Therefore the proof is complete. □

**Definition 2.7.2.** We say that the measure  $V \pi$  is *absolutely continuous* with respect to  $\mu$  (or  $\mu$ -AC) if for each  $E \subset Q$  with  $\mu(E) = 0$ , then  $V \pi(E) = 0$ .

**Theorem 2.7.2.** *If  $f$  is  $\mu$ -HK integrable on a cell  $Q$  and  $F$  is its indefinite  $\mu$ -HK integral, then the critical variation  $V F$  is  $\mu$ -AC on  $Q$ .*

*Proof.* Let  $E \subset Q$  be  $\mu$ -null. We set

$$h(x) = \begin{cases} f(x), & \text{for } x \in Q \setminus E \\ 0, & \text{for } x \in E. \end{cases}$$

It is clear that  $F$  is also the indefinite  $\mu$ -HK integral of  $h$ . Then, by Lemma 2.4.1, given  $\varepsilon > 0$  we can find a gauge  $\delta$  on  $Q$  such that

$$\sum_{i=1}^m |F(Q_i) - h(x_i)\mu(Q_i)| < \varepsilon,$$

for each  $\delta$ -fine partial partition  $\mathcal{P} = \{(x_1, Q_1), (x_2, Q_2), \dots, (x_m, Q_m)\}$  of  $Q$ . In particular, if  $\mathcal{P}$  is anchored in  $E$ , then we have

$$\sum_{i=1}^m |F(Q_i)| < \varepsilon.$$

Hence, by the arbitrariness of  $\varepsilon$ , it follows  $VF(E) = 0$ . Thus  $VF$  is  $\mu$ -AC on  $Q$ .  $\square$

**Theorem 2.7.3.** *If  $\pi$  is differentiable  $\mu$ -almost everywhere on a cell  $Q$  and  $V\pi$  is  $\mu$ -AC, then  $\pi'$  is  $\mu$ -HK integrable on  $Q$  and  $\pi$  is the indefinite  $\mu$ -HK integral of  $\pi'$ .*

*Proof.* We denote by  $E$  the  $\mu$ -negligible set of all  $x \in Q$  for which  $\pi$  is not differentiable at  $x$  and we define

$$f(x) = \begin{cases} \pi'(x), & \text{for } x \in Q \setminus E \\ 0, & \text{for } x \in E. \end{cases}$$

It suffices to show that  $f$  is  $\mu$ -HK integrable on  $Q$  and that  $\pi$  is the indefinite  $\mu$ -HK integral of  $f$ . Since  $V\pi$  is  $\mu$ -AC, given  $\varepsilon > 0$  there exists a gauge  $\delta_1$  on  $E$  such that  $\sum_{i=1}^p |\pi(A_i)| < \varepsilon/2$  for each  $\delta_1$ -fine  $E$ -anchored partial partition  $\{(y_1, A_1), \dots, (y_p, A_p)\}$  of  $Q$ .

Moreover, given  $x \in Q \setminus E$ , by the differentiability of  $\pi$ , there exists  $\delta_2(x) > 0$  such that

$$|\pi(B) - f(x)\mu(B)| < \frac{\varepsilon}{2\mu(Q)}\mu(B),$$

for each subset  $B$  of  $Q$  such that  $B \in \mathcal{F}$ ,  $x \in B$  and  $\text{diam}(B) < \delta_2(x)$ . Now, we define a gauge  $\delta$  on  $Q$  by setting

$$\delta(x) = \begin{cases} \delta_1(x), & \text{for } x \in E \\ \delta_2(x), & \text{for } x \in Q \setminus E \end{cases}$$

and we choose a  $\delta$ -fine  $E$ -anchored partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ . Then

$$\begin{aligned} \sum_{i=1}^m |\pi(Q_i) - f(x_i)\mu(Q_i)| &\leq \sum_{x_i \in E} |\pi(Q_i)| + \sum_{x_i \notin E} |\pi(Q_i) - f(x_i)\mu(Q_i)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\mu(Q)} \sum_{x_i \notin E} \mu(Q_i) = \varepsilon, \end{aligned}$$

since  $f(x_i) = 0$  for  $x_i \in E$  and  $\sum_{x_i \notin E} \mu(Q_i) = \mu(Q \setminus E) = \mu(Q)$ . Therefore  $f$  is  $\mu$ -HK integrable on  $Q$  and  $\pi$  is the indefinite  $\mu$ -HK integral of  $f$ .  $\square$

**Definition 2.7.3.** Let  $Q$  be a cell and let  $E$  be an arbitrary subset of  $Q$ . We say that  $\pi$  is  $BV^\Delta$  on  $E$  if there exists a gauge  $\delta$  on  $E$  such that  $V^\delta \pi(E) < \infty$ .

We say that  $\pi$  is  $BVG^\Delta$  on  $Q$  if there exists a countable sequence of closed sets  $\{E_k\}_k$  such that  $\bigcup_k E_k = Q$  and  $\pi$  is  $BV^\Delta$  on  $E_k$ , for each  $k \in \mathbb{N}$ .

**Definition 2.7.4.** Let  $Q$  be a cell and let  $E$  be an arbitrary subset of  $Q$ . We say that  $\pi$  is  $AC^\Delta$  on  $E$  if for  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  and a positive constant  $\eta$  such that the condition  $\sum_{i=1}^m \mu(Q_i) < \eta$ , implies  $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$ , for each  $\delta$ -fine  $E$ -anchored partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

We say that  $\pi$  is  $ACG^\Delta$  on  $Q$  if there exists a countable sequence of closed sets  $\{E_k\}_k$  such that  $\bigcup_k E_k = Q$  and  $\pi$  is  $AC^\Delta$  on  $E_k$ , for each  $k \in \mathbb{N}$ .

**Theorem 2.7.4.** *Let  $E$  be a compact subset of  $Q$ . If  $\pi$  is  $AC^\Delta$  on  $E$ , then  $\pi$  is  $BV^\Delta$  on  $E$ .*

*Proof.* Since  $\pi$  is  $AC^\Delta$  on  $E$ , given  $\varepsilon = 1$ , then there exists a gauge  $\delta$  on  $Q$  and a positive constant  $\eta$  such that  $\sum_{i=1}^m |\pi(Q_i)| < 1$  whenever  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  is a  $\delta$ -fine  $E$ -anchored partial partition of  $Q$  such that  $\sum_{i=1}^m \mu(Q_i) < \eta$ .

Moreover, since  $\mu$  is non-atomic, for each  $x \in Q$  there exists an open neighborhood  $G$  of  $x$  such that  $\mu(G) < \eta$ . Then, by compactness of  $E$ , there exist open sets  $G_1, G_2, \dots, G_p$  such that  $\mu(G_j) < \eta$ , for  $j = 1, 2, \dots, p$  and  $E \subset \bigcup_{j=1}^p G_j$ . Given  $x \in E$ , there exists  $j \in \{1, \dots, p\}$  such that  $x \in G_j$  and define  $\delta_1(x) = \min\{\delta(x), d(x, \partial G_j)\}$ .

Let  $\{(x_i, Q_i)\}_{i=1}^m$  be an arbitrary  $\delta_1$ -fine  $E$ -anchored partial partition and let  $I_j = \{i : Q_i \subset G_j\}$ . Therefore we have

$$\sum_{i=1}^m |\pi(Q_i)| \leq \sum_{j=1}^p \sum_{i \in I_j} |\pi(Q_i)| \leq p < \infty,$$

since  $\mu\left(\bigcup_{i \in I_j} Q_i\right) \leq \mu(G_j) < \eta$ . Hence  $V^{\delta_1} \pi(E) < \infty$  and the proof is complete.  $\square$

**Theorem 2.7.5.** *If  $f$  is  $\mu$ -HK integrable on a cell  $Q$  and  $F$  is its indefinite  $\mu$ -HK integral, then there exists a sequence  $\{E_k\}$  of closed sets such that  $Q = \bigcup_{k=1}^\infty E_k$  and such that  $f$  is Lebesgue integrable on  $E_k$  for each  $k \in \mathbb{N}$ .*

*Proof.* By Theorem 2.6.3,  $|f|$  is  $\mu$ -measurable. For each positive natural  $m$ , let

$$A_m = \{x \in Q : |f(x)| \leq m\}.$$

Since  $\mu$  is a Radon measure, we have  $A_m = N_m \cup \bigcup_{i=1}^{\infty} A_{m,i}$  where  $N_m$  is  $\mu$ -null and the  $A_{m,i}$ , for  $i = 1, 2, \dots$  are closed sets.

Now let  $N = \bigcup_{m=1}^{\infty} N_m$  and let  $\{C_k\}_k$  be a rearrangement of  $\{A_{m,i}\}_i$ . Moreover, let

$$Q = N \cup \bigcup_{k=1}^{\infty} C_k$$

and let

$$h(x) = \begin{cases} f(x), & \text{for } x \in \bigcup_{k=1}^{\infty} C_k \\ 0, & \text{for } x \in N. \end{cases}$$

Remark that  $h$  is still  $\mu$ -HK integrable on  $Q$  and that  $F$  is its indefinite  $\mu$ -HK integral. Therefore, by Lemma 2.4.1, given  $\varepsilon = 1$ , there exists a gauge  $\delta$  on  $Q$  such that

$$\sum_{i=1}^m |F(Q_i) - h(x_i)\mu(Q_i)| < 1, \quad (2.11)$$

for each  $\delta$ -fine partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ . Then, in particular,

$$\sum_{i=1}^m |F(Q_i)| < 1, \quad (2.12)$$

for each  $\delta$ -fine  $N$ -anchored partial partition  $\mathcal{P} = \{(\xi_i, Q_i)\}_{i=1}^m$  of  $Q$ .

For each positive natural  $k$ , let

$$W_k = \left\{ x \in N : \delta(x) \geq \frac{1}{k} \right\}.$$

It is clear that  $N = \bigcup_{i=1}^{\infty} W_k$ , hence  $N \subset \bigcup_k \overline{W}_k$ . So  $Q = \bigcup_k \overline{W}_k \cup \bigcup_k C_k = \bigcup_{k=1}^{\infty} E_k$ , where  $\{E_k\}_k$  are closed sets obtained through a rearrangement of  $\{\overline{W}_k\}_k$  and  $\{C_k\}_k$ .

The function  $h$  is Lebesgue integrable on  $C_k$ , for  $k = 1, 2, \dots$ , since it is measurable and bounded, then to complete the proof it is enough to show that  $h$  is Lebesgue integrable on  $\overline{W}_k$ , for  $k = 1, 2, \dots$ .

To this aim, for each  $q \in \mathbb{N}$  we remark that the function  $h_q(x) = \min\{|h(x)|, q\}$  is measurable and bounded, therefore  $h_{q,k} := h_q \chi_{\overline{W}_k}$  is Lebesgue integrable on  $Q$ . Hence, by Theorem 2.5.2, it is  $\mu$ -HK integrable on  $Q$ .

Let  $F_{q,k}$  be the indefinite  $\mu$ -HK integral of  $h_{q,k}$  with respect to  $\mu$  (or the indefinite  $\mu$ -HK integral of  $h_q$  with respect to  $\mu_k$ , with  $\mu_k(E) = \mu(E \cap \overline{W}_k)$ ); then, by Lemma 2.4.1, there exists a gauge  $\delta_1$  on  $Q$  such that  $\delta_1(x) < \inf\{\delta(x), 1/k\}$ , for each  $x \in Q$  and

$$\sum_i |F_{q,k}(Q_i) - h_q(x_i)\mu_k(Q_i)| < 1,$$

for each  $\delta_1$ -fine partial partition  $\{(x_i, Q_i)\}_i$ .

Let  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  be a fixed  $\delta_1$ -fine partition of  $Q$  and let  $I = \{i : W_k \cap Q_i^\circ \neq \emptyset\}$ . Then

- If  $i \notin I$ , we have  $(Q_i \cap \overline{W}_k) \subseteq \partial Q_i$ ; so, by Condition (c),

$$0 \leq \sum_{i \notin I} F_{q,k}(Q_i) = \sum_{i \notin I} \int_{Q_i \cap \overline{W}_k} h_q d\mu \leq \sum_{i \notin I} \int_{\partial Q_i} h_q d\mu = 0.$$

- If  $i \in I$ , there exists  $\xi \in Q_i \cap W_k$ ; so  $\{(\xi_i, Q_i)\}_i$  is a  $\delta_1$ -fine  $W_k$ -anchored partial partition.

Thus by (2.11) and (2.12) we have

$$\begin{aligned} \sum_{i \in I} |h_q(\xi_i) \mu_k(Q_i)| &\leq \sum_{i \in I} |h(\xi_i) \mu(Q_i)| \\ &\leq \sum_{i \in I} |h(\xi_i) \mu(Q_i) - F(Q_i)| + \sum_{i \in I} |F(Q_i)| \leq 1 + 1 = 2. \end{aligned}$$

Hence

$$\begin{aligned} F_{q,k}(Q) &= \sum_{i=1}^m |F_{q,k}(Q_i)| = \sum_{i \in I} |F_{q,k}(Q_i)| \\ &\leq \sum_{i \in I} |F_{q,k}(Q_i) - h_q(\xi_i) \mu_k(Q_i)| + \sum_{i \in I} |h_q(\xi_i) \mu_k(Q_i)| \leq 1 + 2 = 3. \end{aligned}$$

Thus  $0 \leq \int_Q h_q d\mu_k = F_{q,k}(Q) \leq 3$ ; i.e.  $h_q$  is Lebesgue integrable on  $Q$ .

In conclusion, since  $h_q \rightarrow |h|$ , by the Monotone Convergence Theorem, we have

$$(L) \int_Q |h| d\mu_k = \lim_{k \rightarrow \infty} (L) \int_Q h_q d\mu_k \leq 3,$$

i.e.  $h$  is Lebesgue integrable on  $\overline{W}_k$ . □

**Theorem 2.7.6.** *Let  $f$  be  $\mu$ -HK integrable on a cell  $Q$  and let  $F$  be its indefinite  $\mu$ -HK integral. If  $f$  is Lebesgue integrable on a closed subset  $A$  of  $Q$ , then  $F$  is  $AC^\Delta$  on  $A$ .*

*Proof.* By Lemma 2.4.1, for each  $\varepsilon > 0$  there exists a gauge  $\delta_1$  on  $Q$  such that

$$\sum_{i=1}^m |F(Q_i) - f(x_i) \mu(Q_i)| < \frac{\varepsilon}{3}, \quad (2.13)$$



for each  $\delta_1$ -fine partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

Moreover, since  $f$  is Lebesgue integrable on  $A$  then  $f\chi_A$  is  $\mu$ -HK integrable on  $Q$ . We set  $f_A := f\chi_A$  and we denote by  $F_A(Q)$  the indefinite  $\mu$ -HK integral of  $f_A$  on  $Q$ . Therefore, by Lemma 2.4.1, there exists a gauge  $\delta_2$  on  $Q$  such that

$$\sum_{i=1}^m |F_A(Q_i) - f_A(\xi_i)\mu(Q_i)| = \sum_{i=1}^m |F_A(Q_i) - f(\xi_i)\mu(Q_i)| < \frac{\varepsilon}{3}, \quad (2.14)$$

for each  $\delta_2$ -fine  $A$ -anchored partial partition  $\{(\xi_i, Q_i)\}_{i=1}^m$  of  $Q$ .

Now, since  $f$  is Lebesgue integrable on  $A$ , the function  $F_A$  is  $\mu$ -AC on  $A$ .

Consequently, we can find a positive constant  $\eta$  such that the condition  $\mu(\bigcup_{i=1}^m Q_i) = \sum_{i=1}^m \mu(Q_i) < \eta$  implies

$$\sum_{i=1}^m |F_A(Q_i)| \leq \sum_{i=1}^m \int_{Q_i \cap A} |f| d\mu \leq \int_{\bigcup_{i=1}^m Q_i \cap A} |f| d\mu < \frac{\varepsilon}{3}. \quad (2.15)$$

Let  $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$  be a gauge on  $Q$ . Therefore, by (2.13), (2.14) and (2.15), we infer

$$\begin{aligned} \sum_{i=1}^m |F(Q_i)| &\leq \sum_{i=1}^m |F(Q_i) - f(\xi_i)\mu(Q_i)| \\ &\quad + \sum_{i=1}^m |f(\xi_i)\mu(Q_i) - F_A(Q_i)| + \sum_{i=1}^m |F_A(Q_i)| < \varepsilon, \end{aligned}$$

for each  $\delta$ -fine  $A$ -anchored partial partition  $\{(\xi_i, Q_i)\}_{i=1}^m$ .

Hence  $F$  is  $AC^\Delta$  on  $A$ . □

**Theorem 2.7.7.** *If  $f$  is  $\mu$ -HK integrable on a cell  $Q$  and  $F$  is its indefinite  $\mu$ -HK integral, then  $F$  is  $ACG^\Delta$  on  $Q$ .*

*Proof.* By Theorem 2.7.5, there exists a sequence  $\{E_k\}_k$  of closed sets such that  $Q = \bigcup_{k=1}^\infty E_k$  and  $f$  is Lebesgue integrable on  $E_k$ , for each  $k \in \mathbb{N}$ . Moreover, by Theorem 2.7.6,  $F$  is  $AC^\Delta$  on  $E_k$ , for each  $k \in \mathbb{N}$ . Therefore  $F$  is  $ACG^\Delta$  on  $Q$ . □

**Theorem 2.7.8.** *If  $\pi$  is  $ACG^\Delta$  on  $Q$ , then  $V\pi$  is  $\mu$ -AC.*

*Proof.* By hypothesis, there exists a sequence of closed sets  $\{E_k\}_k$  such that  $\bigcup_k E_k = Q$  and  $\pi$  is  $AC^\Delta$  on  $E_k$  for each  $k \in \mathbb{N}$ . Therefore, for  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E_k$  and a positive constant  $\eta$  such that the condition

$\sum_{i=1}^m \mu(Q_i) < \eta$  implies  $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$ , for each  $\delta$ -fine  $E_k$ -anchored partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

Let  $E \subset Q$  be  $\mu$ -null. Since  $E \cap E_k$  is  $\mu$ -null, for each  $k \in \mathbb{N}$ , there exists an open set  $G_k$  such that  $E \cap E_k \subset G_k$  and  $\mu(G_k) < \eta$ .

For each  $x \in E \cap E_k$ , we define  $\delta_1(x) = \min\{\delta(x), d(x, \partial G_k)\}$ . So, if  $\{(x_i, Q_i)\}_{i=1}^m$  is a  $\delta_1$ -fine  $E \cap E_k$ -anchored partial partition of  $Q$ , we have  $Q_i \subset G_k$ , for each  $i$ . Then  $\sum_{i=1}^m \mu(Q_i) \leq \mu(G_k) < \eta$ , that implies  $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$ . Therefore  $V^{\delta_1} \pi(E \cap E_k) \leq \varepsilon$  and  $V\pi(E \cap E_k) \leq \varepsilon$ . By the arbitrariness of  $\varepsilon$ , it follows that  $V\pi(E \cap E_k) = 0$ . Hence, since  $V\pi$  is an outer measure and  $E = \bigcup_{k=1}^{\infty} (E \cap E_k)$ , we have

$$V\pi(E) \leq \sum_{k=1}^{\infty} V\pi(E \cap E_k) = 0.$$

Thus  $V\pi$  is  $\mu$ -AC. □

**Definition 2.7.5.** Let  $\lambda$  a signed measure defined on the  $\sigma$ -algebra of all  $\mu$ -measurable sets of  $Q$ . We recall that  $\lambda$  is *absolutely continuous* with respect to  $\mu$  on  $A$  and we write  $\lambda \ll \mu$  if the condition  $\mu(E) = 0$ , implies  $|\lambda|(E) = 0$ , for each  $\mu$ -measurable subset  $E$  of  $A$ . Here  $|\lambda|(E)$  denotes the variation of  $\lambda$  on  $E$ .

**Lemma 2.7.9.** *Let  $A$  be a closed subset of a cell  $Q$  and let  $\lambda$  be a signed measure on  $Q$  such that  $\lambda \ll \mu$ . Then  $\lambda$  is  $AC^\Delta$  on  $A$ .*

The proof follows easily by [35, Theorem 6.11].

**Lemma 2.7.10.** *If  $\pi$  is  $AC^\Delta$  on a closed subset  $A$  of a cell  $Q$ , then*

$$E = \left\{ x \in A : \lim_{\mathcal{F} \ni B \rightarrow x} \frac{|\pi(B)|}{\mu(B)} \neq 0 \right\} \text{ is } \mu\text{-null.}$$

*Proof.* Let

$$E_n = \left\{ x \in E : \text{there exists } \{B_k^x\} \rightarrow x, \text{ with } \frac{|\pi(B_k^x)|}{\mu(B_k^x)} > \frac{1}{n} \text{ for each } k \in \mathbb{N} \right\}.$$

It is trivial to remark that  $E = \bigcup_n E_n$ , therefore to end the proof it is enough to show that  $\mu(E_n) = 0$ , for each  $n \in \mathbb{N}$ . Proceeding towards a contradiction, we can suppose that there exists a natural  $\bar{n} \in \mathbb{N}$  such that  $\mu(E_{\bar{n}}) > 0$ . Thus there exists a compact set  $K \subset E_{\bar{n}}$  for which  $\mu(K) > 0$ . Less than subtracting from  $K$  a  $\mu$ -null relatively open subset, we can assume that  $\mu(K \cap U) > 0$ , for each open set  $U \subset X$  with  $K \cap U \neq \emptyset$ .

Since  $K$  is compact there exists a countable dense subset  $C$  of  $K$ . Let  $H \supset C$  be a  $\mu$ -null  $G_\delta$  set, therefore  $K \cap H$  is a  $\mu$ -null  $G_\delta$  subset of  $K$  that is dense on  $K$ . We show that  $V_\pi(K \cap H) > 0$ , contradicting Theorem 2.7.8.

Set  $D = K \cap H$ , and let  $\delta$  be a gauge on  $D$ . We define  $D_m = \{x \in D : \delta(x) > 1/m\}$ , for  $m \in \mathbb{N}$ . Then, by  $D = \bigcup_m D_m$ , and by the Baire Category theorem, there exists an open set  $U$  such that  $D \cap U \neq \emptyset$  and there exists a natural  $\bar{m}$  such that  $D_{\bar{m}}$  is dense on  $D \cap U$ , hence on  $K \cap U$ .

Let  $\mathcal{B}$  be the system of all cells  $B$  such that  $|\pi(B)| > \mu(B)/\bar{m}$  and  $\text{diam}(B) < 1/\bar{m}$ . Therefore  $\mathcal{B}$  is a fine cover of  $K \cap U$ . Moreover, since  $\mu(K \cap U) > 0$  and since  $\mathcal{F}$  is a  $\mu$ -Vitali family, by the previous remark on the choice of  $K$ , there exists a nonoverlapping system of cells  $\{B_i \in \mathcal{B}\}_i$  that covers  $K \cap U$  up to a  $\mu$ -null set. Then

$$\sum_{i=1}^{\infty} |\pi(B_i)| > \frac{1}{\bar{m}} \sum_{i=1}^{\infty} \mu(B_i) > \frac{1}{\bar{m}} \mu(K \cap U) = M.$$

So, there exists an integer  $p \geq 1$  such that  $\sum_{i=1}^p |\pi(B_i)| > M$  and since  $\mu$  does not charge boundaries of the cells (Condition (c)), the interior of each  $B_i$  meets  $K \cap U$ . Thus, by the density of  $D_{\bar{m}}$  on  $K \cap U$  we have  $D_{\bar{m}} \cap B_i \neq \emptyset$ , therefore we can select  $x_i \in D_{\bar{m}} \cap B_i$ , for each  $i \in \mathbb{N}$ . So  $\{(x_1, B_1), (x_2, B_2), \dots, (x_p, B_p)\}$  is a  $\delta$ -fine  $D_{\bar{m}}$ -anchored partial partitions of  $K \cap U$ , consequently  $V_\pi^\delta(D_{\bar{m}}) \geq M$ . Then, by the arbitrariness of  $\delta$ , we have  $V_\pi(D_{\bar{m}}) \geq M$ , the required contradiction.  $\square$

**Theorem 2.7.11.** *If  $\pi$  is  $AC^\Delta$  on a closed subset  $A$  of a cell  $Q$ , then  $\pi$  is differentiable  $\mu$ -almost everywhere in  $A$ .*

*Proof.* Given an arbitrary subset  $Y$  of  $Q$ , we define the functions

$$V_+^\delta \pi(Y) = \sup \left\{ \sum_{i=1}^m (\pi(Q_i))^+ \right\} \quad \text{and} \quad V_-^\delta \pi(Y) = \sup \left\{ \sum_{i=1}^m (\pi(Q_i))^- \right\},$$

where  $(\pi(Q_i))^+ = \max\{\pi(Q_i), 0\}$  and  $(\pi(Q_i))^- = \max\{-\pi(Q_i), 0\}$  are the positive and the negative parts of  $\pi$ , respectively, and the supremum is taken over all  $\delta$ -fine  $Y$ -anchored partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

As for the definition of  $V\pi$ , we can define  $V_+\pi$  and  $V_-\pi$  by

$$V_+\pi(Y) = \inf V_+^\delta \pi(Y) \quad \text{and} \quad V_-\pi(Y) = \inf V_-^\delta \pi(Y),$$

where the infimum are taken over all gauges  $\delta$  on  $E$ . It is easy to prove that  $V_+\pi$  and  $V_-\pi$  are finite measures.

For each measurable set  $E$  of  $Q$ , we define  $\nu^+(E) = V_+\pi(E \cap A)$  and  $\nu^-(E) =$

$V_- \pi(E \cap A)$ .

Since  $\pi$  is  $AC^\Delta$  on  $A$ , given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $A$  and  $\eta > 0$  such that condition  $\sum_{i=1}^m \mu(Q_i) < \eta$ , implies  $\sum_{i=1}^m |\pi(Q_i)| < \varepsilon$ , for each  $\delta$ -fine  $A$ -anchored partial partition  $\mathcal{P} = \{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ .

Let  $E \subset Q$  be  $\mu$ -null. Therefore  $E \cap A$  is  $\mu$ -null, thus there exists an open set  $G$  such that  $E \cap A \subset G$  and  $\mu(G) < \eta$ . By the argument used in the proof of Theorem 2.7.8 we have that  $\sum_{i=1}^m \mu(Q_i) \leq \mu(G) < \eta$ , that implies  $\sum_{i=1}^m (\pi(Q_i))^+ \leq \sum_{i=1}^m |\pi(Q_i)| < \varepsilon$  for each  $\delta_1$ -fine  $(E \cap A)$ -anchored partial partition  $\{(x_i, Q_i)\}_{i=1}^m$  of  $Q$ . Therefore  $V_+^{\delta_1} \pi(E \cap A) \leq \varepsilon$  and  $\nu^+(E) = V_+ \pi(E \cap A) \leq \varepsilon$ . Thus  $\nu^+ \ll \mu$ .

Similarly, we can prove that  $\nu^- \ll \mu$ .

So, by the Radon-Nikodym Theorem ([15, Theorem 19.23]), there exist non-negative Lebesgue integrable functions  $f^+$  and  $f^-$  on  $Q$  such that

$$\nu^+(E) = (L) \int_E f^+ d\mu \quad \text{and} \quad \nu^-(E) = (L) \int_E f^- d\mu,$$

for every  $\mu$ -measurable subset  $E$  of  $Q$ .

We set  $f = f^+ - f^-$  and we remark that  $f$  is Lebesgue integrable on  $Q$ . Therefore, by Theorem 2.5.2,  $f$  is  $\mu$ -HK integrable on  $Q$  and  $\nu = \nu_+ - \nu_-$  is the indefinite  $\mu$ -HK integral of  $f$ . Since  $f$  is the Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$ , we have

$$\lim_{\mathcal{F} \ni B \rightarrow x} \frac{\nu(B)}{\mu(B)} = f(x), \quad (2.16)$$

$\mu$ -almost everywhere on  $A$ .

Now by Lemma 2.7.9 the signed measure  $\nu$  is  $AC^\Delta$  on  $A$ , then also  $\pi - \nu$  is  $AC^\Delta$  on  $A$ . Hence, by Lemma 2.7.10 we have  $\lim_{B \rightarrow x} (\pi(B) - \nu(B)) / \mu(B) = 0$   $\mu$ -almost everywhere in  $A$  and, by (2.16), we have  $\lim_{B \rightarrow x} \pi(B) / \mu(B) = f(x)$   $\mu$ -almost everywhere in  $A$ ; i.e.  $\pi'(x) = f(x)$   $\mu$ -almost everywhere in  $A$ .  $\square$

**Theorem 2.7.12.** *If  $\pi$  is  $ACG^\Delta$  on a cell  $Q$ , then  $\pi$  is differentiable  $\mu$ -almost everywhere in  $Q$ .*

*Proof.* Since  $\pi$  is  $ACG^\Delta$  on  $Q$ , then there exists a countable sequence of closed sets  $\{E_k\}_k$  such that  $\bigcup_k E_k = Q$  and  $\pi$  is  $AC^\Delta$  on  $E_k$ , for each  $k \in \mathbb{N}$ . So, by Theorem 2.7.11,  $\pi$  is differentiable  $\mu$ -almost everywhere in  $E_k$  for each  $k \in \mathbb{N}$ . Thus it is differentiable  $\mu$ -almost everywhere on  $Q$ .  $\square$

Now, it is possible to obtain the natural extensions of Theorems A and B, called the Main Theorem 1 and the Main Theorem 2, respectively.

**Main Theorem 1** (of Type A). *Let  $Q$  be a cell. A function  $f : Q \rightarrow \mathbb{R}$  is  $\mu$ -HK integrable on  $Q$  if and only if there exists an additive cell function  $F$  that is  $ACG^\Delta$  on  $Q$  and such that  $F'(x) = f(x)$   $\mu$ -almost everywhere in  $Q$ .*

*Proof.* Let  $f : Q \rightarrow \mathbb{R}$  be  $\mu$ -HK integrable on  $Q$  and let  $F$  be its  $\mu$ -HK primitive. By Theorem 2.7.7,  $F$  is  $ACG^\Delta$  on  $Q$ . So, by Theorem 2.7.12,  $F$  is differentiable  $\mu$ -almost everywhere on  $Q$ . Moreover, by Theorem 2.7.8,  $VF$  is  $\mu$ -AC. Then, by Theorem 2.7.3,  $F'(x) = f(x)$   $\mu$ -almost everywhere in  $Q$ . Viceversa, let  $F$  be an additive cell function that is  $ACG^\Delta$  on  $Q$  and such that  $F'(x) = f(x)$   $\mu$ -almost everywhere in  $Q$ . By Theorem 2.7.8,  $VF$  is  $\mu$ -AC on  $Q$ , then, by Theorem 2.7.3,  $F$  is the  $\mu$ -HK primitive of  $F'$ . Thus, the condition  $f(x) = F'(x)$   $\mu$ -almost everywhere on  $Q$ , implies the  $\mu$ -HK integrability of  $f$  on  $Q$ . □

**Main Theorem 2** (of Type B). *Let  $Q$  be a cell. A function  $f : Q \rightarrow \mathbb{R}$  is  $\mu$ -HK integrable on  $Q$  if and only if there exists an additive cell function  $F$  such that  $VF$  is  $\mu$ -AC and  $F'(x) = f(x)$   $\mu$ -almost everywhere in  $Q$ .*

*Proof.* Let  $f : Q \rightarrow \mathbb{R}$  be  $\mu$ -HK integrable on  $Q$  and let  $F$  be its  $\mu$ -HK primitive. By Theorems 2.7.7 and 2.7.8,  $VF$  is  $\mu$ -AC. Moreover by Theorems 2.7.7 and 2.7.12,  $F$  is differentiable  $\mu$ -almost everywhere on  $Q$  and by Theorem 2.7.3,  $F'(x) = f(x)$   $\mu$ -almost everywhere on  $Q$ . Viceversa, let  $F$  be an additive cell function such that  $VF$  is  $\mu$ -AC and  $F'(x) = f(x)$   $\mu$ -almost everywhere on  $Q$ . Then, by Theorem 2.7.3,  $F$  is the  $\mu$ -HK primitive of  $F'$  on  $Q$ . Thus, the condition  $f(x) = F'(x)$   $\mu$ -almost everywhere on  $Q$ , implies the  $\mu$ -HK integrability of  $f$ . □

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# Chapter 3

## The $s$ -HK integral

As mentioned in Observation 2, Example 2.1.4, the  $\mu$ -HK integral coincide with the  $s$ -HK integral when the domain of integration is a  $s$ -set on the real line ([3] and [4]). The  $s$ -HK integral, generalizes the  $s$ -R-integral defined by Jiang and Su [19] and Parvate and Gangal [31].

In this chapter, we formulate a better version of the Fundamental Theorem of Calculus which is false in our context as mentioned in Example 1.6.1.

We denote by  $a = \min E$  and  $b = \max E$ , by  $\int_E f d\mathcal{H}^s$  the  $s$ -HK integral of  $f$  on  $E$  and finally by  $s\text{-HK}(E)$  the collection of all HK-integrable functions on  $E$  with respect to  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s$ .

All the properties shown in the previous chapter, of course, are valid for the  $s$ -HK integral. Here, we recall some properties that we use in this chapter.

(I) If  $f \in s\text{-R}(E)$ , then  $f \in s\text{-HK}(E)$  and

$$\int_E f d\mathcal{H}^s = (R) \int_E f d\mathcal{H}^s.$$

(II) If  $f \in L^1_{\mathcal{H}^s}(E)$ , then  $f \in s\text{-HK}(E)$  and

$$\int_E f d\mathcal{H}^s = (L) \int_E f d\mathcal{H}^s;$$

(III) If  $f \in s\text{-HK}(E)$  and  $a = \min E < x < b = \max E$ , then the function

$$F(x) = \int_{E \cap [a,x]} f d\mathcal{H}^s$$

is continuous and

$$\int_E f d\mathcal{H}^s = \int_{E \cap [a,x]} f d\mathcal{H}^s + \int_{E \cap [x,b]} f d\mathcal{H}^s.$$

### 3.1 The Fundamental Theorem of Calculus

It is well known that if  $F : [a, b] \rightarrow \mathbb{R}$  is a differentiable function on  $[a, b]$ , then its derivative  $F'$  is Henstock-Kurzweil integrable on  $[a, b]$  with  $\int_a^b F' dx = F(b) - F(a)$ . This theorem is false in our context (see Example 1.6.1). The following theorem gives the best positive answer to this problem.

**Theorem 3.1.1.** *Let  $\{(a_j, b_j)\}_{j \in \mathbb{N}}$  be the contiguous intervals of  $E$ . If  $F : E \rightarrow \mathbb{R}$  is  $s$ -differentiable at each point  $x \in E$  and if*

$$\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| < +\infty, \quad (3.1)$$

then  $F'_s \in s\text{-HK}(E)$  and

$$\int_E F'_s(t) d\mathcal{H}^s(t) = F(b) - F(a) - \sum_{j=1}^{\infty} (F(b_j) - F(a_j)). \quad (3.2)$$

*Proof.* Given  $\varepsilon > 0$  and choose  $N \in \mathbb{N}$  such that

$$\sum_{j=N}^{\infty} |F(b_j) - F(a_j)| < \frac{\varepsilon}{2}.$$

We set  $m = \inf\{|b_j - a_j| : j = 1, \dots, N-1\}$ .

Given  $x \in E$ , by the  $s$ -differentiability of  $F$  at the point  $x$ , there exists  $0 < \delta(x) < m$  such that

$$\left| F(u) - F(x) - F'_s(x) \mathcal{H}^s(\widetilde{[u, x]}) \right| \leq \varepsilon \frac{\mathcal{H}^s(\widetilde{[u, x]})}{2 \mathcal{H}^s(E)}, \quad (3.3)$$

for each  $u \in E \cap (x - \delta(x), x + \delta(x))$ .

Now let  $\{\widetilde{[u_i, v_i]}, x_i\}_{i=1}^n$  be a  $\delta$ -fine partition of  $E$ . By (3.3) we have

$$\begin{aligned} & \left| F(v_i) - F(u_i) - F'_s(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) \right| \leq \\ & \leq \left| F(v_i) - F(x_i) - F'_s(x_i) \mathcal{H}^s(\widetilde{[x_i, v_i]}) \right| + \\ & + \left| F(x_i) - F(u_i) - F'_s(x_i) \mathcal{H}^s(\widetilde{[u_i, x_i]}) \right| \leq \\ & \leq \varepsilon \left( \frac{\mathcal{H}^s(\widetilde{[u_i, x_i]})}{2 \mathcal{H}^s(E)} + \frac{\mathcal{H}^s(\widetilde{[x_i, v_i]})}{2 \mathcal{H}^s(E)} \right) = \varepsilon \frac{\mathcal{H}^s(\widetilde{[u_i, v_i]})}{2 \mathcal{H}^s(E)}. \end{aligned} \quad (3.4)$$



Therefore

$$\begin{aligned} \left| \sum_{i=1}^n F'_s(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \sum_{i=1}^n (F(v_i) - F(u_i)) \right| &\leq \\ &\leq \frac{\varepsilon}{2 \mathcal{H}^s(E)} \sum_{i=1}^n \mathcal{H}^s(\widetilde{[u_i, v_i]}) = \frac{\varepsilon}{2}. \end{aligned} \quad (3.5)$$

Since  $u_i$  and  $v_i$  belong to  $E$  and  $\{(a_j, b_j)\}_{j \in \mathbb{N}}$  is the sequence of all contiguous intervals of  $E$  then

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^n (F(v_i) - F(u_i)) \\ &+ \sum_{(a_j, b_j) \notin \bigcup_{i=1}^n [u_i, v_i]} (F(b_j) - F(a_j)) \end{aligned} \quad (3.6)$$

Moreover, by the definition of  $\delta$ , the condition  $[a_j, b_j] \subset [u_i, v_i]$  implies

$$|b_j - a_j| \leq |v_i - u_i| < 2\delta(x_i) < m.$$

Consequently it is  $j \geq N$ , hence

$$\sum_{[a_j, b_j] \subset \bigcup_{i=1}^n [u_i, v_i]} |F(b_j) - F(a_j)| \leq \sum_{i=N}^{\infty} |F(b_j) - F(a_j)| \leq \frac{\varepsilon}{2}. \quad (3.7)$$

Finally by (3.5), (3.6) and (3.7) we have:

$$\begin{aligned} &\left| \sum_{i=1}^n F'_s(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \left( F(b) - F(a) - \sum_{j=1}^{\infty} (F(b_j) - F(a_j)) \right) \right| \\ &\leq \left| \sum_{i=1}^n F'_s(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \sum_{i=1}^n (F(v_i) - F(u_i)) \right| \\ &+ \left| \sum_{i=1}^n (F(v_i) - F(u_i)) - \left( F(b) - F(a) - \sum_{j=1}^{\infty} (F(b_j) - F(a_j)) \right) \right| \\ &\leq \frac{\varepsilon}{2} + \left| \sum_{[a_j, b_j] \subset \bigcup_{i=1}^n [u_i, v_i]} (F(b_j) - F(a_j)) \right| \\ &\leq \frac{\varepsilon}{2} + \sum_{[a_j, b_j] \subset \bigcup_{i=1}^n [u_i, v_i]} |F(b_j) - F(a_j)| \leq \varepsilon. \end{aligned}$$

□

The following corollary is an extension of Theorem 1.6.2.

**Corollary 3.1.2.** *If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $s$ -differentiable on  $E$  and if  $\text{Sch}(F) \subseteq E$ , then*

$$\int_E F'_s(t) d\mathcal{H}^s(t) = F(b) - F(a).$$

*Proof.* Condition  $\text{Sch}(F) \subseteq E$  implies that  $F$  is constant on each contiguous interval  $(a_k, b_k)$  of  $E$ . Then  $F(a_k) = F(b_k)$  for  $k \in \mathbb{N}$ , since  $F$  is continuous. Thus condition (3.1) is satisfied and Theorem 3.1.1 can be applied.  $\square$

**Remark 4.** If we assume, like in Theorem 1.6.2, that  $F'_s$  is continuous, then it is  $s$ -R integrable ([31, Theorem 39]) and by Property (I) we have

$$(R) \int_E F'_s(t) d\mathcal{H}^s(t) = F(b) - F(a).$$

**Remark 5.** *Condition (3.1) is necessary for the  $s$ -HK integrability of  $F'_s$ .*

In fact on the ternary Cantor set  $C$  we can define a function  $F$  such that

- (a)  $\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| = +\infty$ , where  $\{(a_j, b_j)\}_{j=1}^{\infty}$  is the sequence of all contiguous intervals of  $C$ ;
- (b)  $F'_s$  exists everywhere on  $C$  for  $s = \log_3 2$ ;
- (c)  $F'_s \notin s\text{-HK}(C)$ .

Let  $\varphi : C \rightarrow [0, 1]$  be the Cantor function, defined by  $\varphi(x) = \mathcal{H}^s(\widetilde{[0, x]})$  for each  $x \in C$ . Fix a decreasing sequence  $\{\gamma_k\}_{k=1}^{\infty}$  of dyadic-rational numbers of  $[0, 1]$  such that  $\gamma_k \rightarrow 0$  and

$$\frac{1}{k} < \gamma_{2^k}^2, \quad k = 2, 3, \dots \quad (3.8)$$

For a fixed  $k \in \mathbb{N}$ , there exists a contiguous interval of  $C$ , say  $(\alpha_k, \beta_k)$ , such that  $\gamma_k \in (\alpha_k, \beta_k)$  and

$$\varphi(\alpha_k) = \varphi(\beta_k) = \gamma_k. \quad (3.9)$$

Let  $s = \log_3 2$  and let

$$g_k(x) = \frac{\mathcal{H}^s(\widetilde{[\beta_{2^k}, x]})}{k \mathcal{H}^s(\widetilde{[\beta_{2^k}, \alpha_{2^{k-1}}]})}. \quad (3.10)$$

We define

$$F(x) = \begin{cases} g_k(x), & \text{if } x \in [\beta_{2k}, \alpha_{2k-1}], \quad k = 1, 2, \dots \\ 0, & \text{elsewhere.} \end{cases}$$

It is trivial to observe that

$$\sum_{k=1}^{\infty} |F(\beta_k) - F(\alpha_k)| = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty,$$

then

$$\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| = +\infty.$$

Now let  $x \in C \setminus \{0\}$ . Therefore  $x \in [\beta_1, 1]$  or  $x \in [\beta_{k+1}, \alpha_k]$  for a unique  $k \in \mathbb{N}$ , since the intervals  $(\alpha_k, \beta_k)$  are contiguous to  $C$ .

So,

- if  $x \in [\beta_{2k+1}, \alpha_{2k}] \cup [\beta_1, 1]$ , then  $F'_s(x) = 0$ .
- if  $x \in [\beta_{2k}, \alpha_{2k-1}]$ , then

$$F'_s(x) = \frac{1}{k \mathcal{H}^s([\beta_{2k}, \alpha_{2k-1}])}.$$

Moreover

$$F'_s(0) = F_s^+(0) = \lim_{\substack{x \rightarrow 0^+ \\ x \in C}} \frac{F(x) - F(0)}{\mathcal{H}^s([0, x])} = \lim_{\substack{k \rightarrow +\infty \\ x \in [\beta_{2k}, \alpha_{2k-1}]}} \frac{g_k(x)}{\varphi(x)}.$$

Consequently  $F'_s(0) = 0$ , since by (3.8), (3.9) and (3.10) we obtain

$$\frac{g_k(x)}{\varphi(x)} \leq \frac{1}{k \gamma_{2k}} < \frac{\gamma_{2k}^2}{\gamma_{2k}} = \gamma_{2k}, \quad \text{for each } x \in [\beta_{2k}, \alpha_{2k-1}].$$

Now, if we assume that  $f \in s\text{-HK}(C)$ , by Property (III), we get the following contradiction

$$\begin{aligned} \int_C F'_s(t) d\mathcal{H}^s(t) &= \sum_{k=1}^{\infty} \int_{[\beta_{2k}, \alpha_{2k-1}]} \frac{1}{k \mathcal{H}^s([\beta_{2k}, \alpha_{2k-1}])} d\mathcal{H}^s \\ &= \sum_{k=1}^{\infty} \frac{1}{k \mathcal{H}^s([\beta_{2k}, \alpha_{2k-1}])} \mathcal{H}^s([\beta_{2k}, \alpha_{2k-1}]) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} = +\infty. \end{aligned}$$

**Remark 6.** Condition (3.1) is necessary for the validity of some formulation of the Fundamental Theorem of Calculus.

In fact, for each  $u, v \in \mathbb{R}$ , we can define on the ternary Cantor set  $C$  a function  $F$  such that

(a)  $\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| = +\infty$ , where  $\{(a_j, b_j)\}_{j=1}^{\infty}$  is the sequence of all contiguous intervals of  $C$ ;

(b) for  $s = \log_3 2$  and  $x \in C$  it is  $F'_s(x) = 0$ , consequently

$$(R) \int_C F'_s(t) d\mathcal{H}^s(t) = 0;$$

(c)  $F(1) - F(0) = u$ ;

(d)  $F(1) - F(0) - \sum_{j=1}^{\infty} (F(b_j) - F(a_j)) = v$ .

Let  $\varphi$ ,  $\{\gamma_k\}_{k=1}^{\infty}$  and  $\{(\alpha_k, \beta_k)\}_{k=1}^{\infty}$  be defined as in the proof of the previous Remark and let  $k_0 \in \mathbb{N}$  be such that  $(a_{k_0}, b_{k_0}) \subset (\beta_1, 1)$ . We define

$$F(x) = \begin{cases} 1/k, & \text{if } x \in [\beta_{2k}, \alpha_{2k-1}], \quad k = 1, 2, \dots \\ v, & \text{if } x \in (\beta_1, a_{k_0}] \\ u, & \text{if } x \in [b_{k_0}, 1] \\ 0, & \text{elsewhere.} \end{cases}$$

Then we have

$$\sum_{k=1}^{\infty} |F(\beta_k) - F(\alpha_k)| = 2 \sum_{k=1}^{\infty} \frac{1}{k} = +\infty,$$

therefore

$$\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| = +\infty.$$

Moreover, by the definition of  $F$ , we have

$$\sum_{j=1}^{\infty} (F(b_j) - F(a_j)) = F(b_{k_0}) - F(a_{k_0}) = u - v.$$

It is trivial to observe that, for  $s = \log_3 2$ , we have  $F'_s(x) = 0$  at each  $x \in C \setminus \{0\}$ . Now, for  $x \in C \setminus \{0\}$  it is

$$\frac{F(x) - F(0)}{\mathcal{H}^s(\widetilde{[0, x]})} = \begin{cases} \frac{1}{k\varphi(x)}, & \text{if } \beta_{2k} \leq x \leq \alpha_{2k-1}, \quad k = 1, 2, \dots \\ 0, & \text{if } \beta_{2k+1} \leq x \leq \alpha_{2k}, \quad k = 1, 2, \dots \end{cases}$$

Moreover, for  $\beta_{2k} \leq x \leq \alpha_{2k-1}$ , by (3.8) and (3.9) we infer

$$\frac{1}{k \varphi(x)} \leq \frac{1}{k \gamma_{2k}} < \frac{\gamma_{2k}^2}{\gamma_{2k}} = \gamma_{2k},$$

therefore we have  $F_s^+(0) = 0$ .

So, the function  $F$  satisfies the required conditions.

## 3.2 Extension to $\mathcal{H}^s$ -almost $s$ -derivatives

The function  $F$  defined in (1) of the introduction is differentiable everywhere in  $[0, 1]$ , and  $F'$  is HK-integrable but not Lebesgue integrable. Therefore the positive part of  $F'$ , say  $(F')^+$ , is not HK-integrable. Now, for  $x \neq \sqrt{2/(4k \pm 1)\pi}$ ,  $k \in \mathbb{Z}$ , the function  $(F')^+$  is the derivative of

$$G(x) = \begin{cases} x^2 \sin(1/x^2), & \text{if } (2k - 1/2)\pi \leq 1/x^2 \leq (2k + 1/2)\pi \\ 0, & \text{elsewhere.} \end{cases}$$

By [36, Chapter VII, Theorem 2.3 (Lusin's Theorem)],  $(F')^+$  is also the derivative of a continuous function almost everywhere on  $[0, 1]$ .

This implies that the Fundamental theorem of Calculus usually fails for functions that are not differentiable at each point. However it works for functions that are absolutely continuous or  $ACG_\delta$  [11].

In this section we prove an extension of Theorem 1.6.1 that gives an answer to this problem.

**Theorem 3.2.1.** *Let  $F : E \rightarrow \mathbb{R}$  be a function  $\mathcal{H}^s$ -absolutely continuous on  $E$  such that  $F_s'$  exists  $\mathcal{H}^s$ -almost everywhere in  $E$ . Then*

$$\int_E f(t) d\mathcal{H}^s(t) = F(b) - F(a),$$

where  $f : E \rightarrow \mathbb{R}$  is such that  $f(x) = F_s'(x)$  if  $F_s'(x)$  exists.

*Proof.* Let  $\{(a_j, b_j)\}_{j=1}^\infty$  be the sequence of all contiguous intervals of  $E$ . Since  $F$  is  $\mathcal{H}^s$ -absolutely continuous on  $E$ , for each  $\varepsilon > 0$  there exists a constant  $\eta > 0$  such that  $\mathcal{H}^s(\widetilde{[x, y]}) < \eta$  for each  $x, y \in E$ , implies  $|F(x) - F(y)| < \varepsilon$ . Then, by the arbitrariness of  $\varepsilon$  and by  $\mathcal{H}^s(\widetilde{[a_j, b_j]}) = 0$ , we have

$$F(a_j) = F(b_j), \quad \text{for } j = 1, 2, \dots \quad (3.11)$$

Let  $T = \{x \in E : F_s'(x) \text{ does not exist}\}$ , therefore  $\mathcal{H}^s(T) = 0$ .

Now define

$$f(x) = \begin{cases} F_s'(x), & \text{if } x \in E \setminus T \\ 0, & \text{if } x \in T. \end{cases}$$

The condition  $\mathcal{H}^s(T) = 0$  implies that the characteristic function  $\chi_T$  is  $\mathcal{H}^s$ -Lebesgue integrable on  $E$  with  $(L)\int_E \chi_T(t) d\mathcal{H}^s(t) = \mathcal{H}^s(T) = 0$ . Then by Property (II) we have  $\int_E \chi_T(t) d\mathcal{H}^s(t) = 0$ . This implies that, given  $\eta > 0$ , we can find a gauge  $\delta_1(x)$  such that

$$\sum_{\substack{j=1 \\ y_j \in T}}^p \mathcal{H}^s(\widetilde{[a_j, b_j]}) < \eta,$$

for each  $\delta_1$ -fine partition  $\{\widetilde{[a_j, b_j]}, y_j\}_{j=1}^p$  of  $E$ .

Now, given  $\varepsilon > 0$ , take  $\eta$  from the  $\mathcal{H}^s$ -absolute continuity of  $F$  on  $E$  and define  $\delta(x)$ , for  $x \in E \setminus T$ , like in the proof of Theorem 3.1.1. Moreover, for  $x \in T$  define  $\delta(x) = \delta_1(x)$ .

Let  $\{\widetilde{[u_i, v_i]}, x_i\}_{i=1}^n$  be a  $\delta$ -fine partition of  $E$ . Therefore, by (3.6) and (3.11), we have

$$F(b) - F(a) = \sum_{i=1}^n (F(v_i) - F(u_i)),$$

and, by (3.5), we obtain

$$\begin{aligned} & \left| \sum_{x_i \in E \setminus T} f(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \sum_{x_i \in E \setminus T} (F(v_i) - F(u_i)) \right| \\ & \leq \frac{\varepsilon}{2\mathcal{H}^s(E)} \sum_{x_i \in E \setminus T} \mathcal{H}^s(\widetilde{[u_i, v_i]}) = \frac{\varepsilon}{2}. \end{aligned}$$

Moreover, by the definition of  $\delta$  on  $T$  and by the  $\mathcal{H}^s$ -absolute continuity of  $F$  on  $E$  it follows

$$\sum_{x_i \in T} |F(u_i) - F(v_i)| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} & \left| \sum_{i=1}^n f(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - (F(b) - F(a)) \right| \\ & \leq \left| \sum_{x_i \in E \setminus T} f(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \sum_{x_i \in E \setminus T} (F(v_i) - F(u_i)) \right| \\ & \quad + \sum_{x_i \in T} |F(u_i) - F(v_i)| < \varepsilon. \end{aligned}$$

This implies  $f \in s\text{-HK}(E)$  and  $\int_E f(t) d\mathcal{H}^s(t) = F(b) - F(a)$ . □

**Remark 7.** If we assume, like in Theorem 1.6.1, that  $f$  is continuous, then it is  $s$ -R integrable ([19, Theorem 2.2]) and by Property (I) we have

$$(R) \int_E f(t) d\mathcal{H}^s(t) = F(b) - F(a).$$

A small modification in the proof of the last theorem gives a further extension of Theorem 1.6.1.

Now, we provide the following definition which is a special case of Definition 2.2.2 of the previous chapter.

**Definition 3.2.1.** Let  $D$  be an arbitrary subset of  $E$  and let  $\delta$  be a gauge on  $E$ . A collection  $P = \{(\tilde{A}_i, x_i)\}_{i=1}^p$  of finite ordered pairs of points and cells is said to be

- a *partition* of  $E$  if  $\{\tilde{A}_1, \dots, \tilde{A}_p\}$  is a collection of pairwise non-overlapping  $E$ -intervals such that  $\bigcup_{i=1}^p \tilde{A}_i = E$  and  $x_i \in \tilde{A}_i$ , for  $i = 1, \dots, p$ ;
- a *partial partition* of  $E$  if  $\{\tilde{A}_1, \dots, \tilde{A}_p\}$  is a collection of pairwise non-overlapping  $E$ -intervals such that  $\bigcup_{i=1}^p \tilde{A}_i \subset E$  and  $x_i \in \tilde{A}_i$ , for  $i = 1, \dots, p$ ;
- $\delta$ -*fine* if  $\tilde{A}_i \subseteq ]x_i - \delta(x_i), x_i + \delta(x_i)[$ , for  $i = 1, \dots, p$ ;
- $D$ -*anchored* if the points  $x_1, \dots, x_m$  belong to  $D$ .

**Definition 3.2.2.** Let  $D \subset E$  and let  $F : E \rightarrow \mathbb{R}$ . We say that  $F$  is  $\mathcal{H}^s$ - $AC_\delta$  on  $D$  if for each  $\varepsilon > 0$  there exist a gauge  $\delta$  on  $D$  and a constant  $\eta > 0$  such that if  $\sum_{i=1}^n \mathcal{H}^s(\widetilde{[u_i, v_i]}) < \eta$  then

$$\sum_{i=1}^n |F(u_i) - F(v_i)| < \varepsilon,$$

for each  $\delta$ -fine  $D$ -anchored partial partition  $\{\widetilde{[u_i, v_i]}, x_i\}_{i=1}^n$  of  $E$ .

We say that  $F$  is  $\mathcal{H}^s$ - $ACG_\delta$  on  $E$  if there exists a countable sequence of sets  $\{E_k\}$  such that  $\bigcup_k E_k = E$  and  $F$  is  $\mathcal{H}^s$ - $AC_\delta$  on  $E_k$ , for each  $k \in \mathbb{N}$ .

**Theorem 3.2.2.** Let  $\{(a_j, b_j)\}_{j \in \mathbb{N}}$  be the contiguous intervals of  $E$ . If  $F : E \rightarrow \mathbb{R}$  is a function  $\mathcal{H}^s$ - $ACG_\delta$  on  $E$  such that  $F'_s$  exists  $\mathcal{H}^s$ -almost everywhere in  $E$  and if

$$\sum_{j=1}^{\infty} |F(b_j) - F(a_j)| < +\infty, \quad (3.12)$$

then the function  $f : E \rightarrow \mathbb{R}$ , defined as  $f(x) = F'_s(x)$  at all points  $x \in E$  where  $F'_s(x)$  exists and  $f(x) = 0$  elsewhere, is HK-integrable on  $E$  and

$$\int_E f(t) d\mathcal{H}^s(t) = F(b) - F(a) - \sum_{j=1}^{\infty} (F(b_j) - F(a_j)). \quad (3.13)$$

*Proof.* Let  $T \subset E$  such that  $\mathcal{H}^s(T) = 0$  and  $F'_s(x)$  exists for all  $x \in E \setminus T$ . Then

$$f(x) = \begin{cases} F'_s(x), & \text{if } x \in E \setminus T \\ 0, & \text{if } x \in T. \end{cases}$$

Moreover let  $E = \bigcup_k E_k$  be a decomposition of  $E$  such that  $F$  is  $\mathcal{H}^s$ - $AC_\delta$  on  $E_k$ , for  $k = 1, 2, \dots$ .

Now, let  $\varepsilon > 0$  and  $k \in \mathbb{N}$ . By the  $\mathcal{H}^s$ - $AC_\delta$  absolute continuity of  $F$  on  $T \cap E_k$ , there is a constant  $\eta_k > 0$  and a gauge  $\delta_k$  on  $T \cap E_k$  such that

$$\sum_{i=1}^n |F(u_i) - F(v_i)| < \frac{\varepsilon}{2^{k+1}}, \quad (3.14)$$

for each  $\delta_k$ -fine partial partition  $\{\widetilde{[a_j, b_j]}, y_j\}_{j=1}^p$  anchored on  $T \cap E_k$  with

$$\sum_{j=1}^p \mathcal{H}^s(\widetilde{[a_j, b_j]}) < \eta_k. \quad (3.15)$$

Since  $\mathcal{H}^s(T \cap E_k) = 0$  we can repeat the argument used in the proof of Theorem 3.2.1 and assume that for each  $\delta_k$ -fine partial partition  $\{\widetilde{[a_j, b_j]}, y_j\}_{j=1}^p$  anchored on  $T \cap E_k$  condition (3.15) is satisfied. Then also (3.14) is satisfied. Define  $N$  and  $\delta(x)$  for  $x \in E \setminus T$ , like in the proof of Theorem 3.1.1. Moreover, for  $x \in T \cap E_k$  define  $\delta(x) = \delta_k(x)$ , for  $k = 1, 2, \dots$ .

So, if  $\{\widetilde{[u_i, v_i]}, x_i\}_{i=1}^n$  is a  $\delta$ -fine partition of  $E$ , by (3.4) and (3.14), we have

$$\begin{aligned} & \left| \sum_{i=1}^n f(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - \sum_{i=1}^n (F(v_i) - F(u_i)) \right| \\ & \leq \sum_{x_i \in E \setminus T} \left| f(x_i) \mathcal{H}^s(\widetilde{[u_i, v_i]}) - (F(v_i) - F(u_i)) \right| \\ & \quad + \sum_{k=1}^{\infty} \sum_{x_i \in T \cap E_k} |F(u_i) - F(v_i)| \\ & < \frac{\varepsilon}{2 \mathcal{H}^s(E)} \sum_{x_i \in E \setminus T} \mathcal{H}^s(\widetilde{[u_i, v_i]}) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$



Therefore we can conclude the proof repeating the argument used in the proof of Theorem 3.1.1.  $\square$

**Remark 8.** *Even in Theorem 3.2.2, the absolute convergence of the series  $\sum_{j=1}^{\infty} (F(b_j) - F(a_j))$  is necessary both for the  $s$ -HK integrability of  $f$  and for the validity of some formulation of the Fundamental Theorem of Calculus.*

In fact, it is easy to check that the functions defined in the Remark 5 and in the Remark 6 are  $\mathcal{H}^s$ -ACG $_{\delta}$  on the ternary Cantor set.

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