UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Dipartimento di Matematica F. Enriques

Dottorato di Ricerca in Scienze Matematiche

XXXVI ciclo

Tesi di Dottorato

Symplectic action of groups of order four on K3 surfaces and $K3^{[2]}$-type manifolds

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A.A. 2023–2024

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Introduction

An irreducible holomorphic symplectic (IHS) manifold is a compact, simply connected complex manifold X whose space of holomorphic 2-forms has complex dimension one, being generated by a symplectic (nowhere degenerate) form ω_X; IHS manifolds have been introduced by Beauville [6] as one of the blocks appearing in the decomposition theorem of Kähler manifolds with trivial canonical bundle. The smallest IHS manifolds are K3 surfaces, whose rich geometry has been studied for more than a century. On the second integral cohomology of K3 surfaces there is an interplay of two structures: the cup product, which is bilinear and nondegenerate, gives a lattice structure $H^2(X, \mathbb{Z}) \simeq \Lambda$, the same for any K3 surface up to isometries; on the other hand, the Hodge decomposition on $H^2(X, \mathbb{C})$ gives $H^2(X, \mathbb{Z})$ a pure weight 2 Hodge structure. The geometry of X is encoded in the relation between these two structures: once we fix a marking $\varphi : H^2(X, \mathbb{Z}) \simeq \Lambda$ we can parametrize K3 surfaces through the period map which, roughly speaking, maps each X to the image of its symplectic form ω_X via the marking in the lattice Λ; the Torelli theorem for K3 surfaces states that X and Y are isomorphic if and only if $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are Hodge isometric, i.e. they are isomorphic both as lattices and as Hodge structures; moreover, it allows us to tell if an isometry of the lattice Λ comes from an automorphism on X just by looking at its action on the Hodge structure.

All of this can be generalized to higher dimensions: if X is an IHS manifold of dimension $2n > 2$, the role of the cup product is taken by the Beauville-Bogomolov-Fujiki form, which gives again to $H^2(X, \mathbb{Z})$ a lattice structure invariant by deformation [6], [25]; there is a period map, and a Torelli theorem holds [93], [42], [47]. Moreover, in recent years it has been found that these results can be generalized if we allow some mild singularities [50] [4]; many definitions of singular analogues to IHS manifolds have been proposed [79], and the theory of symplectic varieties has become its own research subject.
Indeed, one particular issue with IHS manifolds is that there are very few examples: in every dimension, one has two deformation classes, $\mathrm{K3}^{[n]}$ and Km_n, already described in the same paper by Beauville in which the concept of IHS manifold was introduced; other than these, there are only two known examples, of dimension 6 and 10, discovered by O'Grady [76], [75]. It is unknown if this list is complete, but no other smooth example has been found yet, despite the numerous attempts: to extend the theory of IHS manifolds to singular examples seems therefore a logical step forward.

One way to obtain new symplectic varieties is to take quotients by symplectic automorphisms: an automorphism α of X is symplectic if $\alpha^*\omega_X = \omega_X$. On K3 surfaces, symplectic automorphisms fix only points: therefore, if a finite group G acts symplectically on X, the quotient X/G is a singular surface that always admits a K3 surface as resolution of the singularities; the moduli spaces of K3 surfaces with a given symplectic action of G, and K3 surfaces which arise as a quotient by such action, are in bijection [71]. If X is an IHS manifold of higher dimension, the codimension of the fixed locus of a symplectic action can vary, and the quotient X/G does not always admit a symplectic resolution of the singularities: the terminalization $\tilde{Y} \to X/G$ produces, in general, a symplectic variety [24].
Using quotients, one can also define symplectic automorphisms on some symplectic varieties: indeed, if $H \subset G$ is a normal subgroup, the terminalization of X/H admits a symplectic action of G/H. However, this obviously requires G be non simple, and in the literature we find the description of the induced action of G on the cohomology of an IHS manifold only if G has prime order (see for instance [32], [26] for K3 surfaces; [56], [49], [43] for higher dimensions).

In this thesis, we study the first cases of non simple symplectic actions on IHS manifolds: we describe the action of a group G of order 4, both $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^2$, on $K3^{[2]}$-type manifolds. This action is always standard, i.e. it behaves in cohomology as the induced action of G on $S^{[2]}$, the Hilbert scheme of two points on a K3 surface S that also admits a symplectic action of G [39]. Therefore we begin by providing, in the first chapters, a complete account of the symplectic action of G on a K3 surface S: in particular, we give a lattice-theoretic characterization of the intermediate quotient surface $\tilde{Z}$, i.e. the resolution of the singularities of S/i, with $i \in G$ an element of order 2, and compare its moduli space to that of S and the resolution of the singularities of S/G. In the projective case, these moduli spaces are not irreducible: the many different irreducible components of the moduli spaces of S and the resolution of the singularities of S/G are still in bijection, but this does not extend to $\tilde{Z}$.
We then apply our knowledge of the action of G on K3 surfaces to $K3^{[2]}$-type manifolds. If X is a $K3^{[2]}$-type manifold with a symplectic action of G the role of the intermediate quotient surface is taken by the Nikulin orbifold Y, the symplectic variety that arises as partial resolution of the quotient X/i, whose BBF-form was first described in [48]: we establish the correspondence between the moduli spaces of X with a symplectic action of G and Y in the projective case, and we describe the two induced involutions on Nikulin orbifolds. Moreover, we give lattice theoretic conditions under which each of these involutions persist under deformations $\tilde{Y}$ of Y: unlike what happens with IHS manifolds, the co-invariant lattice Ω admits more than one primitive embedding in $H^2(\tilde{Y}, \mathbb{Z})$ with the same orthogonal complement, and the involution exists on $\tilde{Y}$ only for a specific one of them.

What has been described above is the core material of the thesis. Moreover, the reader will find two chapters in which we explore the relation between manifolds with the same transcendental sublattice $T(X) \subset H^2(X, \mathbb{Z})$. The interest in this topic stems from

the fact that, by Lefschetz's theorem, algebraic classes are identified with $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$: therefore, the Hodge structure on $H^2(X, \mathbb{Z})$ which, as we already alluded to, is what encodes most of the geometric information, is always trivial on the algebraic sublattice, but not on the transcendental $T(X) := (H^{1,1}(X) \cap H^2(X, \mathbb{Z}))^\perp$.
In chapter four we study how to define an analogue to Shioda-Inose structures for symplectic automorphisms of order 4: given any abelian surface A, if X is a K3 surface such that $T(X)$ and $T(A)$ are Hodge isometric then X has a symplectic involution ι such that the resolution of the singularities of X/ι is isomorphic to $Kum(A)$ (and the converse also holds) [63]. We prove that, if A has a symplectic automorphism of order 4, the same condition on transcendental lattices is not enough to get a symplectic action of a group of order 4 on a K3 surface X: finding that it is not possible to fully extend the classical construction, we propose some partial generalizations. The last chapter presents a joint work with Ángel David Ríos Ortiz, about IHS manifolds whose transcendental lattice is Hodge isometric to that of a K3 or an abelian surface, and whether they are birationally equivalent to moduli spaces over said surfaces or not. We find that this holds if X is an IHS manifold of $\mathrm{K3}^{[n]}$-type or Km_n-type, but not always if X is of OG6-type or OG10-type: in these cases, we give some partial results using lattice theory.

* * *

Chapter 1 is devoted to preliminaries: we introduce irreducible holomorphic symplectic manifolds and recall the results on lattice theory, moduli spaces and symplectic automorphisms that are needed in the rest of the thesis.

In **Chapters 2 and 3** we study the symplectic action of a group of order four G on a K3 surface X, $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^2$ respectively, following the same steps employed for symplectic actions of order 2 and 3 in [32], [26]. The main results of these two chapters are the lattice-theoretic characterization of $\tilde{Z}$, the resolution of the singularities of the intermediate quotient surface X/i, and the comparison between its moduli space and those of X and $\tilde{Y}$, the minimal resolution of X/τ. In particular, we know from [71] that $\tilde{Z}$, admitting a symplectic involution and being itself the resolution of a quotient by a symplectic involution, has to admit a primitive embedding in its Néron-Severi lattice of both the lattices $\Omega_{\mathbb{Z}/2\mathbb{Z}}$ and $M_{\mathbb{Z}/2\mathbb{Z}}$ (the first characterizes K3 surfaces with a symplectic involution, the second those that are the quotient of one); however, they cannot be in direct sum in $NS(\tilde{Z})$ because of their rank: we find the lattices that characterize $\tilde{Z}$, which are negative definite of rank 14 and 12, the same as the lattices that characterize X with a symplectic action of $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^2$ respectively.
To describe the action of G on the second integral cohomology of a K3 surface, we use a K3 surface X with high Picard rank and a Jacobian fibration $\pi : X \to \mathbb{P}^1$ such that $MW(\pi) \simeq G$: the resulting description on $H^2(X, \mathbb{Z})$ holds actually true for any K3 surface thanks to [71, Thm. 4.7]. We then study the maps induced in cohomology by the rational quotient maps $X \dashrightarrow \tilde{Z}$, $X \dashrightarrow \tilde{Y}$, $\tilde{Z} \dashrightarrow \tilde{Y}$, and give our lattice-theoretic characterization of $\tilde{Z}$.
The moduli space of projective K3 surfaces with a symplectic action of G splits in irreducible components (here called *projective families*), and there is a bijection between

projective families of X and $\tilde{Y}$. However, this bijection does not extend to projective families of $\tilde{Z}$, and we find different phenomena: either two families of X and $\tilde{Y}$ collide on the same family of $\tilde{Z}$, or (only if G is not cyclic) there can be up to three different families of $\tilde{Z}$ associated to the same family of X and $\tilde{Y}$.
The correspondence between projective families and the knowledge of the maps induced by the rational quotients also allows us to find explicit examples of X and its quotients in projective spaces of small dimension.

Chapter 4 represents the first detour from the main stream of the thesis.
Shioda-Inose structures relate Abelian and K3 surfaces admitting a symplectic involution: by [63, Thm. 6.3], for every abelian surface A, if X is a K3 surface such that there is a Hodge isometry $T(A) \simeq T(X)$, then X is a (rational) double cover of $Kum(A)$. In this chapter, we study the action of a symplectic automorphism of order 4 α on $H^2(A, \mathbb{Z})$ and explore possible generalizations of Shioda-Inose structures.
We find however that if A has a symplectic automorphism of order four and $T(A) \simeq T(X)$, X does not in general admit a symplectic action of a group of order four G, such that X/G is birational to $Kum_4(A)$: we therefore propose two different generalizations of Shioda-Inose structures. Strong structures are triples (X, A, τ), with A as above and τ a symplectic automorphism of order four on X that acts as a cycle on four copies of the lattice D_4 in $NS(X)$, such that the resolution of the singularities of $X/\tau^2, X/\tau$ are $Kum(A), Kum_4(A)$ respectively: they are a true generalization of Shioda-Inose structure, because $T(A) \simeq T(X)$, but they do not exist for any A (it depends on the lattice structure of $T(A)$). Weak structures are triples (X, A, τ), with τ a symplectic automorphism of order four on X that acts as a cycle on four copies of the lattice D_4 in $NS(X)$, such that the resolution of the singularities of X/τ is $Kum_4(A)$: these exist for any choice of A with a symplectic automorphism of order 4, but $T(A)$ does not uniquely determine $T(X)$. However, we're still able to give a correspondence between the transcendental lattices.

In **Chapter 5** we move from K3 surfaces to $K3^{[2]}$-type manifolds. Thanks to the classification of finite symplectic actions of [39], we know that a symplectic action of a group of order 4 G on a $K3^{[2]}$-type manifold X is always standard, meaning that a pair (X, G) can be deformed to a natural pair $(S^{[2]}, G)$, where the action of G is induced from that on S. This means not only that the action on $H^2(X, \mathbb{Z})$ depends only on G, but also that the locus of points of X with non-trivial stabilizer is homeomorphic to that of the natural pair.
We start with the classification of the projective families: thanks to the standardness of the action, for every $K3^{[2]}$-type manifold X with a symplectic action of G there exists a K3 surface S with the same action such that $NS(X) \simeq NS(S)$; however, unlike what happens for S, $T(X)$ is not (in general) uniquely determined by $NS(X)$. For some of the families we found we are also able to describe the general member (that the corresponds to a general point in the moduli space): either as Fano manifold over a cubic fourfold, or as Hilbert scheme of two points of a quartic surface with a mixed (partially non-symplectic) action of G, or as double cover of a cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

We then turn our attention to Nikulin orbifolds and their deformation class: if X admits a symplectic action of G, and $i \in G$ is an element of order 2, then the Nikulin orbifold Y obtained as terminalization of X/i admits a symplectic involution induced by the quotient G/i. The two groups of order four induce two very different involutions on Y: indeed, we can see from the action on X that the one induced by $\mathbb{Z}/4\mathbb{Z}$ fixes only points on Y, while the locus of the one induced by $(\mathbb{Z}/2\mathbb{Z})^2$ has codimension 2. We describe the action of these involutions on $H^2(Y,\mathbb{Z})$, using the quotient maps introduced in the second and third chapters. We then prove that our involutions extend to any Nikulin-type orbifold that satisfy a given lattice-theoretic condition: differently than what happens on the known irreducible holomorphic symplectic manifolds, it is not enough that a Nikulin-type orbifold Y be polarized with the correct anti-invariant lattice Ω_ι for it to admit a standard involution ι, but there is also a gluing datum between invariant and co-invariant lattices that has to be respected, i.e. a condition on the embedding $\Omega_\iota \hookrightarrow H^2(Y,\mathbb{Z})$.
We conclude with the lattice-theoretic classification of projective Nikulin orbifolds Y that are terminalization of X/i, where X is a $\mathrm{K3}^{[2]}$-type manifold with a symplectic action of a group of order 4 G, and $i \in G$ is an element of order 2, therefore completely describing the correspondence between the moduli spaces of X and Y. After noticing that standard involutions on Nikulin-type orbifolds commute with the non-standard involution described in [52], we classify also projective Nikulin-type orbifolds that admit a mixed action of $(\mathbb{Z}/2\mathbb{Z})^2$, where one of the generators is standard, and the other is not.

Chapter 6 presents a joint work with Ángel David Ríos Ortiz: in Chapter 4 we discussed K3 surfaces whose transcendental lattice is isometric to that of an abelian surface. In this chapter, we discuss irreducible holomorphic symplectic manifolds that share the transcendental lattice of an abelian or K3 surface: in particular, we're interested to find whether or not such a manifold X is birationally equivalent to a moduli space over the corresponding surface.
We prove that if X is a manifold of Km_n-type, $T(X) \simeq T(A)$ if and only if X is birationaly equivalent to a moduli space on an abelian surface A, and an analogous result holds with X is a $\mathrm{K3}^{[n]}$-type manifold and S a K3 surface such that $T(X) \simeq T(S)$. If X is of OG6-type or OG10-type however, the condition on the transcendental lattices is not enough, and lattice theory is needed. We give lattice-theoretic criteria and construct many examples of Hodge structures of manifolds in O'Grady's families which have the same transcendental lattice of symplectic surfaces but do not arise as resolution of the singularities of a moduli space. We highlight the very different behavior of those that which are or aren't moduli spaces, and we point out connections with the non-modular construction of O'Grady's 10 dimensional example due to Laza-Saccà-Voisin [44].

Ringraziamenti

Grazie Alice, per aver accettato di prendermi come tesista prima ancora di conoscermi, e per avermi accompagnato durante tutto il percorso dalla tesi di magistrale a questa tesi di dottorato.
Grazie Bert, per avermi incoraggiato a fare ricerca quando io non ero sicura che fosse la strada giusta per me.
Grazie Stevell, per avermi aiutato più volte a fare i conti con i reticoli. Grazie Angel, per aver accettato di farmi includere in questa tesi il lavoro che abbiamo scritto insieme.
Grazie a Annalisa, Valeria, Grégoire per le interessanti conversazioni.
Grazie ai miei colleghi di dottorato: ai miei fratellini Martina e Lucas, con cui ho potuto condividere la frustrazione di quando la matematica non collaborava, e i momenti felici quando invece un nuovo pezzetto andava al suo posto; a Ji, Yanyan, Filipa, e poi Davide, Giobo, Shula, Cami, Giulia e tutti gli altri con cui ho condiviso pranzi, cene, aperitivi, serate giochi, gite e chiacchierate.
In ultimo, grazie a Stefano e alla mia famiglia, che mi sono sempre stati vicini.

A warm thank you goes also to the referees of this thesis, for the time and care they spent reading it and their nice comments about my work.

To future readers, if any, I hope my work proves useful to your research. I'll be always available for comments and questions, just send me an email anytime.

Chapter 1

Preliminaries

1.1 An overview on IHS manifolds

Irreducible holomorphic symplectic (IHS) manifolds appear as building blocks of compact complex Kähler manifolds with trivial canonical bundle, which by Yau's solution to Calabi conjecture are exactly those that admit a Kähler metric with trivial Ricci curvature: the scope of the interest in this class of manifolds therefore extends not only to algebraic geometry, but also to differential geometry and theoretical physics.

Theorem 1.1.0.1 ([6, §5])**.** *Let X be a compact Kähler manifold with vanishing first Chern class: then there exists a finite étale covering of X isomorphic to the product $T \times \prod V_i \times \prod X_j$, where T is a complex torus, V_i are irreducible Calabi-Yau manifolds, and X_j are IHS manifolds.*

Definition 1.1.0.2. A Calabi-Yau manifold is a smooth Kähler manifold Y that is compact and simply-connected and whose canonical bundle is trivial, such that $h^{i,0}(Y) = 1$ only for $i \in \{0, dim_{\mathbb{C}}(Y)\}$, and 0 otherwise.

Definition 1.1.0.3 ([34, Def. 21.1]). An IHS manifold is a Kähler manifold X that is compact and simply connected, such that $H^{2,0}(X)$ is generated by a symplectic form, i.e. a nowhere degenerate holomorphic 2-form ω_X.

Definition 1.1.0.4. A *deformation* of X is a smooth proper morphism $\mathcal{X} \to B$ where $\mathcal{X}$ is smooth and B is connected, and a distinguished point $0 \in B$ such that $\mathcal{X}_0 \simeq X$.
A deformation $\mathcal{X} \to B$ of X is called *universal* if any other deformation $\mathcal{X}' \to B'$ is isomorphic to the pull-back under a uniquely determined morphism $\beta : B' \to B$ with $\beta(0) = 0$. The universal deformation family is unique up to isomorphisms, if it exists, and it will be denoted by $\mathcal{X} \to \mathrm{Def}(X)$.

Theorem 1.1.0.5 ([34, Prop. 23.14])**.** *Let X be an IHS manifold of dimension $2n$. There exists a quadratic form q_X on $H^2(X, \mathbb{C})$ and a unique constant $c_X \in \mathbb{Q}_{>0}$ such that*

1. *for all* $\alpha \in H^2(X,\mathbb{C})$*, it holds* $\int_X \alpha^{2n} = c_X q_X(\alpha)^n$*;*
2. *the form* q_X *is a primitive integral quadratic form on* $H^2(X,\mathbb{Z})$*.*

Proposition 1.1.0.6 ([34, Lemma 22.9, Cor. 23.11])**.** *Some properties of* q_X*:*

1. *it is invariant under deformations;*
2. *it holds* $q_X(\omega_X) = 0,\ q_X(\omega_X + \overline{\omega}_X) > 0$*;*
3. *it has signature* $(3, b_2(X) - 3)$ *on* $H^2(X,\mathbb{R})$*;*
4. *it is positive definite on Kähler classes and on* $(H^{2,0}(X) \oplus H^{0,2}(X)) \otimes \mathbb{R}$*.*

Few examples of IHS manifolds are known. There are two classes for each dimension $2n$, $n \in \mathbb{N}$, already described by Beauville [6, §6-7]: $\mathrm{K3}^{[n]}$-type manifolds, deformation equivalent to the Hilbert scheme of n points on a K3 surface, and Km_n-type manifolds, deformations of the n-th generalized Kummer variety of an abelian surface. Every attempt to find new examples of IHS manifolds has produced elements in these two classes, with the exception of two examples, of dimension 6 and 10 respectively, constructed by O'Grady as a symplectic resolution of some singular moduli spaces of semistable sheaves on an abelian surface or on a projective K3 surface ([76], [75] respectively): their deformations are called OG6-type and OG10-type manifolds.

Example 1.1.0.7. Let S be a K3 surface. The *Hilbert scheme of n points* $S^{[n]}$ is the moduli space of subschemes of S of dimension 0 and length n. We can construct it as resolution of the singularities of the symmetric product $Sym^n(S)$ via the Hilbert-Chow morphism [22, Cor. 2.6]. Indeed the number of times a point $p \in S$ appears in a given $Z \in S^{[n]}$ is determined by $dim_{\mathbb{C}}\mathcal{O}_{Z,p}$, so we can define the cycle

$$|Z| \coloneqq \sum_{p \in S} dim_{\mathbb{C}}\mathcal{O}_{Z,p} \cdot p \ \in \ Sym^n(S) \coloneqq \frac{\overbrace{S \times S \times \cdots \times S}^{n \text{ times}}}{Sym(n)}$$

and the Hilbert-Chow morphism $S^{[n]} \to Sym^n(S),\ Z \mapsto |Z|$. If X is an IHS manifolds, and there exists a K3 surface S such that X is deformation equivalent to $S^{[n]}$, we call X a $\mathrm{K3}^{[n]}$-type manifold.

Example 1.1.0.8. Let A be an abelian surface: if we compose the Hilbert-Chow morphism with the sum operation s in A, we get the map

$$\begin{array}{cccccc} \phi: & A^{[n+1]} & \longrightarrow & Sym^{n+1}(A) & \longrightarrow & A \\ & Z & \longmapsto & |Z| & \longmapsto & \sum_{p \in A} dim_{\mathbb{C}}\mathcal{O}_{Z,p}\ . \end{array}$$

The fiber over 0 of the map ϕ defined above is the n-th generalized Kummer variety $\mathrm{Km}_n(A)$. If X is an IHS manifolds, and there exists an abelian surface A such that X is deformation equivalent to $\mathrm{Km}_n(A)$, we call X a Km_n-type manifold.
The manifold $\mathrm{Km}_n(A)$ is birational to the quotient variety $A^n/Sym(n+1)$: indeed,

points $(a_1, \ldots, a_{n+1}) \in s^{-1}(0)$ satisfy the relation $a_{n+1} = -\sum_{i=1}^{n} a_i$, so $s^{-1}(0) \simeq A^n$; the action of $Sym(n+1)$ is naturally induced by the one on A^{n+1}. If $n = 2$, the quotient $A^2/Sym(3)$ is singular along a surface Δ isomorphic to A, which contains 81 "more singular" points, image in the quotient variety of the set $\Delta_3 = \{(a, a) \in A \times A \mid 3a = 0\}$; the singularity of $A^2/Sym(3)$ are resolved by a $\mathbb{P}^1$-bundle over $\Delta \setminus \Delta_3$, and over each point of Δ_3 a surface isomorphic to the cone in $\mathbb{P}^4$ described by the equations $\{w_0 w_2 = w_1^2,\ w_0 w_3 = w_1 w_2,\ w_1 w_3 = w_2^2\}$.

1.2 Lattice theory

In this section, we're going to recall some fundamental results on lattices and discriminant forms; most of them are due to Nikulin, and are exposed in [72, §1].

Definition 1.2.0.1. An *even lattice* is a free $\mathbb{Z}$-module S of finite rank, equipped with a nondegenerate quadratic form $q : S \to 2\mathbb{Z}$. Working in characteristic different than two, this is equivalent to an integral nondegenerate bilinear symmetric even form $b : S \times S \to \mathbb{Z}$; we will refer to b as *intersection form* of S. The *discriminant* $d(S)$ is defined as $|\det(B)|$, where B is any matrix that represents b.
An *isomorphism* between lattices (or *isometry*) is an isomorphism of $\mathbb{Z}$-modules that preserves the intersection form. Denote $O(S)$ the group of isometries of S into itself.

Example 1.2.0.2. The lattice U is the unique even unimodular lattice of rank 2. The ADE lattices are the negative definite even lattices whose Gram matrix is the Cartan matrix of the Dynkin ADE diagrams: each vertex has self-intersection -2, and two verteces have intersection 1 if they are adjacent, 0 otherwise.

$$A_n : \quad a_1 - a_2 - \ldots - a_n \qquad\qquad D_n : \quad \begin{array}{c} d_2 - d_3 - \ldots - d_n \\ \quad\ | \qquad\qquad\quad \\ \quad d_1 \qquad\qquad\quad \end{array}$$

$$E_n : \quad \begin{array}{c} e_2 - e_3 - e_4 - e_5 - e_6 - \ldots - e_n, \\ | \qquad\qquad\qquad\qquad \\ e_1 \qquad\qquad\qquad\qquad \end{array}$$

Example 1.2.0.3. The quadratic form q_X in Theorem 1.1.0.5 is known as the *Beauville-Bogomolov-Fujiki form* on X. The integrality of q_X gives to $H^2(X, \mathbb{Z})$ a lattice structure that depends only on the deformation type, and that has been determined in [6], [84], [85]: the results are summarized in the following table.

X	$\dim(X)$	$b_2(X)$	$(H^2(X,\mathbb{Z}),\, q_X)$
$\mathrm{K3}^{[n]}$	$2n$	23	$\Lambda_{\mathrm{K3}^{[n]}} := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$
Km_n	$2n$	7	$\Lambda_{\mathrm{Km}_n} := U^{\oplus 3} \oplus \langle -2(n+1) \rangle$
OG6	6	8	$\Lambda_{\mathrm{OG6}} := U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$
OG10	10	24	$\Lambda_{\mathrm{OG10}} := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$

Definition 1.2.0.4. Define in particular the *K3 lattice* $\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8^{\oplus 2}$. For any K3 surface X, the second integral cohomology group $H^2(X,\mathbb{Z})$ equipped with the cup product $H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$ is isometric to the K3 lattice.

Definition 1.2.0.5. Let X be an IHS manifold. Define the *Néron-Severi lattice* $NS(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$. Define the *transcendental lattice* $T(X)$ its orthogonal complement in $H^2(X,\mathbb{Z})$.

Remark 1.2.0.6.

1. The Néron-Severi lattice is the image of the first Chern class $c_1 : Pic(X) \to H^2(X,\mathbb{Z})$: therefore it holds $rk(NS(X)) = \rho$ the Picard rank of X.
2. Since $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$ with respect to the cup product of $H^2(X,\mathbb{C})$, $T(X)$ is the smallest sublattice of $H^2(X,\mathbb{Z})$ containing ω_X.

Remark 1.2.0.7 ([71, §3.2]). Let X be an IHS manifold, $\Lambda \simeq H^2(X,\mathbb{Z})$. The following cases are possible:

1. $NS(X)$ is nondegenerate of signature $(1, \rho - 1)$ and $T(X)$ is nondegenerate of signature $(2, rk(\Lambda) - 2 - \rho)$: in this case, X is projective.
2. $NS(X)$ has a one-dimensional kernel K_1, and $NS(X)/K_1$ is negative definite of rank $\rho - 1$; then, also $T(X)$ has a one-dimensional kernel K_2, and $T(X)/K_2$ has signature $(2, rk(\Lambda) - 3 - \rho)$.
3. $NS(X)$ is nondegenerate and negative definite and $T(X)$ is nondegenerate of signature $(3, rk(\Lambda) - 3 - \rho)$.

Definition 1.2.0.8. Let S be an even lattice: define the *dual lattice* $S^* = \{\ x \in S \otimes \mathbb{Q} \mid \forall s \in S,\ b_{\mathbb{Q}}(x,s) \in \mathbb{Z}\ \}$ where $b_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear extension of b.
Denote *discriminant group* of S $A_S := S^*/S$, where $S \hookrightarrow S^*$ via $s \mapsto b(s,-)$: it is a finite group of cardinality $d(S)$. An invariant of the discriminant group is its *length* ℓ, that is defined as the minimum number of generators of A_S; we are going to write

$$\lambda(S) = \lambda(A_S) = (n_1, n_2, \ldots, n_\ell)$$

if the first of the generators in a set that satisfies the minimum has order n_1, the second n_2 and so on, with $n_1 \leq n_2 \leq, \ldots, \leq n_\ell$.
Define the *discriminant (quadratic) form* $q_S : A_S \to \mathbb{Q}/2\mathbb{Z}$, induced on A_S by the

quadratic form q of S. A subgroup $H \subset A_S$ is said to be *isotropic* if it is annihilated by the discriminant form q_S.

Definition 1.2.0.9. Two torsion quadratic forms $q, \tilde{q}$ defined on a finite abelian group G are *isomorphic* (or *equivalent*) if there is an automorphism γ of G such that $\tilde{q}(x) = q(\gamma(x))$ for every $x \in G$.

Proposition 1.2.0.10. *Two torsion quadratic forms defined on a finite abelian group G are isomorphic if and only if for every prime p they are p-equivalent, i.e. equivalent when restricted to the maximal p group A_p contained in G.*

The criteria for p-equivalence of torsion quadratic forms are given in [72, §1.7-9] and in [54, §IV]. In particular, we refer the reader to Lemma 1.4, Cor. 2.5, Prop. 3.2 of the latter. Moreover, one can compare two torsion quadratic form by their *normal form decomposition*.

Proposition 1.2.0.11 ([54, Prop. IV.2.4, Prop. IV.4.8])**.** *If p is odd, any nondegenerate torsion quadratic form on a finite abelian p-group has a unique normal form decomposition. On a finite abelian 2-group, every torsion quadratic form has a unique normal form decomposition. Two torsion quadratic forms are equivalent if and only if they share the same normal form.*

Definition 1.2.0.12. The *genus* of a lattice S is the set of all lattices with the same signature of S and discriminant form equivalent to q_S.

Remark 1.2.0.13. Lattices in the same genus may not be isomorphic: for instance, this applies to the (negative definite) lattices $N \oplus E_8$ and K_2 (see Proposition 4.2.1.2).

1.2.1 Overlattices and primitive embeddings

Definition 1.2.1.1. An *embedding* of (even) lattices $(S, q) \overset{\iota}{\hookrightarrow} (M, \tilde{q})$ is an injective homomorphism of $\mathbb{Z}$-modules such that $\tilde{q}(\iota(s)) = q(s)$ for all $s \in S$. In this case, we say that M is an *overlattice* of S. An embedding is *primitive* if $M/\iota(S)$ is free; an overlattice is *of index n* if $M/\iota(S)$ is a (abelian) group of order n, and it is a *cyclic overlattice* if $M/\iota(S)$ is cyclic. Given an embedding $(S, q) \overset{\iota}{\hookrightarrow} (M, \tilde{q})$, the *primitive saturation* of S is the smallest overlattice of $\iota(S)$ that is primitive in M.

Remark 1.2.1.2. Let M be an overlattice of index n of S. The discriminant of M is related to that of S by

$$d(S)/d(M) = n^2.$$

Theorem 1.2.1.3 ([72, Prop. 1.4.1.a)])**.** *Let S be an even lattice, let M be an overlattice of finite index of S, let $H_M = M/S$. The correspondence $M \to H_M$ determines a bijection between overlattices of finite index of S and isotropic subgroups of A_S.*

Definition 1.2.1.4. Two embeddings $S \hookrightarrow M$, $S \hookrightarrow M'$ are *isomorphic* if there is an isometry between M and M' that restricted to S is the identity.

Two overlattices of finite index of S, Q and Q' are *isomorphic* if there is an isometry $\alpha \in O(S)$ that extends $\mathbb{Q}$-linearly to an isometry between Q and Q'.

Theorem 1.2.1.5 ([72, Prop. 1.6.1]). *A primitive embedding of an even lattice S into an even unimodular lattice L, in which the orthogonal complement of S is isomorphic to K, is determined by an isomorphism $\gamma : A_S \xrightarrow{\sim} A_K$ for which the discriminant forms satisfy $q_K \circ \gamma = -q_S$. Two such isomorphisms γ and γ' determine isomorphic primitive embeddings if and only if they are conjugate via an automorphism of K.*

Corollary 1.2.1.6. *Let L, S and K be as in Theorem 1.2.1.5. The isomorphism classes of overlattices Q of $S \oplus K$ in L, such that $Q/(S \oplus K)$ is cyclic, are in bijection with the isometry classes for the action of $O(S)$ induced on A_S (equivalently on A_K via γ).*

Proof. Using the notation of the previous theorem, fix the isomorphism $\gamma : A_S \simeq A_K$; let $s \in A_S$ such that $q_S(s) = d \in \mathbb{Q}/2\mathbb{Z}$, let $k = \gamma(s)$: then $q_K(k) = -d$, and the cyclic subgroup generated by $s + k$ is isotropic in $A_S \oplus A_K$, so it determines an overlattice of finite index Q of $S \oplus K$ that is by construction a sublattice of L. Consider an isometry $\alpha \in O(S)$, denote $\overline{\alpha}$ the induced isometry on A_S, and call $s' = \overline{\alpha}(s)$: then $q_S(s') = d$, and $\beta := \gamma \circ \overline{\alpha} \circ \gamma^{-1} \in O(A_K)$, hence the subgroup generated by $s' + \beta(k)$ determines an overlattice of $S \oplus K$ isomorphic to Q thanks to the previous theorem.
On the other hand, consider Q such that $S \oplus K \hookrightarrow Q \hookrightarrow L$ and $Q/(S \oplus K)$ is cyclic: then $Q/(S \oplus K)$ is generated by an isotropic element in $A_{S \oplus K} = A_S \oplus A_K$, that is by construction of the form $s + k$, with $q_S(s) = d = -q_K(k)$. □

Remark 1.2.1.7. With S, K, L as above, consider a primitive sublattice $H \subset K$, with $H = \langle h \rangle$, and suppose that $h/n \in K^*$ for some integer n: then, L contains a cyclic overlattice Q of $S \oplus K$, corresponding to the isotropic subgroup $\langle \gamma^{-1}(h/n) + h/n \rangle \subset A_{S \oplus K}$, as in Corollary 1.2.1.6. It contains also an overlattice R of $S \oplus H$, $R \subseteq Q$, generated over $\mathbb{Z}$ by a $\mathbb{Z}$-basis of S and the element $(s + h)/n$, where s/n is a representative in S^* of $\gamma^{-1}(h/n) \in A_S$. Consider now an isometry $\psi \in O(K)$, and let $\tilde{H} = \psi(H)$. Let $\tilde{H} = \langle \tilde{h} \rangle$: then $\tilde{h}/n = \psi(h)/n$ belongs to the same isometry class of h/n in A_K, and $\langle \gamma^{-1}(\tilde{h}/n) + \tilde{h}/n \rangle$ is isotropic in A_K, so it defines an overlattice $\tilde{R}$ of $S \oplus \tilde{H}$ as above: the relation between H and $\tilde{R}$ consists in the fact that $\tilde{R}$ is isomorphic to R as a sublattice of L.

Theorem 1.2.1.8 ([72, Prop. 1.14.1]). *For an even lattice S of signature (s_+, s_-) and discriminant form q_S, and an even unimodular indefinite lattice L of signature (l_+, l_-), all primitive embeddings of S into L are isomorphic if and only if the lattice T with signature $(l_+ - s_+, l_- - s_-)$ and discriminant form $q_T = -q_S$ is unique in its genus and the homomorphism $O(T) \to O(q_T)$ is surjective.*

Remark 1.2.1.9. Using the notation of the theorem, if $\ell(S) > rk(L) - rk(S)$, no primitive embedding of S in L exists: indeed, if it existed, then T would satisfy $rk(T) < \ell(T)$, which is impossible.

We are going to give some conditions on the invariants of S, L and T under which it is possible to apply Theorem 1.2.1.8.

Theorem 1.2.1.10 ([72, Thm. 1.14.2]). *Let T be an even, indefinite lattice; for any prime number p, define ℓ_p the minimum number of generators of $(A_T)_p$. Suppose the following conditions hold:*

1. *for all primes $p \neq 2$, $rk(T) \geq \ell_p + 2$;*
2. *if $rk(T) = \ell_2$, then $(A_T)_2$ has discriminant form isomorphic to one of the following, for some torsion quadratic form q':*

$$\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \oplus q', \text{ or } \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \oplus q'.$$

Then T is unique in its genus, and the map $O(T) \to O(q_T)$ is surjective.

Proposition 1.2.1.11 ([54, Cor. VIII.7.8]). *Let T be an indefinite lattice such that $rk(T) \geq 3$. Write $A_T = \mathbb{Z}/d_1\mathbb{Z} \oplus, \ldots, \oplus \mathbb{Z}/d_r\mathbb{Z}$ with $d_i \geq 1$ and $d_i \mid d_{i+1}$. Suppose that one of the following holds:*

1. *$d_1 = d_2 = 2$;*
2. *$d_1 = 2, d_2 = 4$ and $d_3 =_8 4$;*
3. *$d_1 = d_2 = 4$.*

Then T is unique in its genus, and the map $O(T) \to O(q_T)$ is surjective.

Theorem 1.2.1.12 ([63, Thm. 2.8], after [72, Thm. 1.14.4]). *Let S be an even lattice of signature (s_+, s_-) and discriminant form q_S, and L an even unimodular lattice of signature (l_+, l_-). Suppose that $s_+ < l_+$, $s_- < l_-$, $\ell(S) \leq rk(L) - rk(S) - 2$. Then there exists a unique primitive embedding of S into L.*

Corollary 1.2.1.13 ([72, Rem. 1.14.5]). *If $A_S \simeq (\mathbb{Z}/2\mathbb{Z})^3 \oplus A'$, the conditions of the previous theorem are satisfied.*

The following theorem describes primitive embeddings when the ambient lattice is not unimodular.

Theorem 1.2.1.14 ([72, Prop. 1.15.1]). *The primitive embeddings of a lattice S with signature (s_+, s_-) and discriminant form q_S into an even lattice M with signature (m_+, m_-) and discriminant form q_M are determined by the sets $(H_S, H_M, \gamma, T, \gamma_T)$, where:*

1. *$H_S \subset A_S$ and $H_M \subset A_M$ are subgroups and $\gamma : q_S|_{H_S} \to q_M|_{H_M}$ is an isomorphism of the subgroups, preserving the restrictions of the forms;*
2. *T, which will be the orthogonal complement to S in M, is an even lattice with signature $(m_+ - s_+, m_- - s_-)$ and discriminant form q_T;*

3. $\gamma_T : q_T \to -\delta$ *is an isomorphism of discriminant forms, where* $\delta \simeq ((q_S \oplus -q_M)|_{\Gamma^\perp})/\Gamma$*,* $\Gamma = H_S \oplus H_M$ *(notice that* $(q_S \oplus -q_M)|_\Gamma = 0$ *by choice of* γ*).*

Two such sets, $(H_S, H_M, \gamma, T, \gamma_T)$ *and* $(H'_S, H'_M, \gamma', T', \gamma_{T'})$ *determine isomorphic primitive embeddings if and only if* $H_S \simeq H'_S$ *and there exist* $\xi \in O(A_M)$ *and* $\psi : T \to T'$ *isometries for which* $\gamma' = \xi \circ \gamma$ *and* $\bar{\xi} \circ \gamma_T = \gamma_{T'} \circ \bar{\psi}$*, where* $\bar{\xi}$ *is the isomorphism of discriminant forms* δ *and* δ' *induced by* $id \oplus \xi$*, and* $\bar{\psi}$ *is the isomorphism of discriminant forms* q_T *and* $q_{T'}$ *induced by* ψ*.*

1.3 Moduli spaces and Torelli theorems

As the lattice structure on the second cohomology group of an IHS manifold is invariant under deformation, to distinguish between non-isomorphic manifolds of the same deformation type we have to compare their Hodge structure: to parametrize IHS manifolds deformation equivalent to X we use their *period*, which is ultimately determined by ω_X. Indeed, recall from Remark 1.2.0.6 that the Hodge structure on $NS(X)$ is trivial, while that on $T(X)$ is determined by ω_X, the generator of $H^{2,0}(X)$.

Definition 1.3.0.1 (see [41, Def. 3.2.3], [18, Def. 4.1]). A pure integral Hodge structure V is said to be *of* K3*-type* if V has weight two and

$$\dim_{\mathbb{C}}(V^{2,0}) = 1 \qquad \text{and} \qquad V^{p,q} = 0 \quad \text{for} \quad |p-q| > 2.$$

A *Beauville-Bogomolov form* on V is a non-degenerate quadratic form q on V that induces a morphism of integral Hodge structures $q : V \otimes V \to \mathbb{Z}$ and is positive definite on the real part of $V^{2,0} \oplus V^{0,2}$.

A *Hodge isometry* is an isomorphism of Hodge structures of K3-type which is an isometry with respect to their Beauville-Bogomolov forms.

Example 1.3.0.2. - If X is a IHS manifold, then $H^2(X, \mathbb{Z})$ endowed with its Beauville-Bogomolov form is a Hodge structure of K3-type. The transcendental lattice $T(X)$ with the restriction of the Beauville-Bogomolov form is also a Hodge structure of K3-type.

- Let A be an abelian surface, then $H^2(A, \mathbb{Z})$ endowed with the cup product is a Hodge structure of K3-type whose lattice structure is $H^2(A, \mathbb{Z}) = U^{\oplus 3}$. The transcendental lattice $T(A) = NS(A)^\perp$ with the restriction of the cup product is also a Hodge structure of K3-type.

- Let $Y \subseteq \mathbb{P}^5$ be a cubic fourfold, $h \in H^2(Y, \mathbb{Z})$ the hyperplane class. Then $H^4_{prim}(Y, \mathbb{Z}) = (h^2)^{\perp_{H^4(Y,\mathbb{Z})}}$ endowed with the cup product is a Hodge structure of K3-type: its lattice structure is $H^4_{prim}(Y, \mathbb{Z}) = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$. The transcendental lattice $H^4_{tr}(Y, \mathbb{Z}) := (H^{2,2}(Y) \cap H^4_{prim}(Y, \mathbb{Z}))^\perp$ with the restriction of the cup product is also a Hodge structure of K3-type.

Definition 1.3.0.3 ([34, Def. 25.4]). A *marked* IHS manifold is a pair (X, φ) where φ is an isometry between $H^2(X, \mathbb{Z})$ and a given lattice Λ. Define $\mathcal{M}_\Lambda$ the set $\{(X, \varphi)\}/\sim$ of marked IHS manifolds, where $(X, \varphi) \sim (X', \varphi')$ if and only if there exists an isomorphism $f : X \to X'$ such that $f^* = \varphi^{-1} \circ \varphi'$.

Remark 1.3.0.4. Given $\mathcal{X} \to B$ a flat deformation of the marked IHS (X, φ), there exists a neighborhood of 0 and a family of markings $\Phi_t : \mathcal{X}_t \to \Lambda$ such that $\varphi = \Phi_0$. In particular, this holds for the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ (see Definition 1.1.0.4).

Definition 1.3.0.5 ([34, Def. 25.3]). Let Λ be a given lattice of signature $(3, rk\Lambda - 3)$. Define the *period domain*

$$\Omega_\Lambda = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle x, x \rangle = 0, \langle x, \overline{x} \rangle > 0\},$$

so it is an open subset of a smooth quadric hypersurface in $\mathbb{P}(\Lambda \otimes \mathbb{C})$. By Theorem 1.1.0.5, given X and IHS manifold and $\varphi : H^2(X, \mathbb{Z}) \to \Lambda$ a marking, we can define the *period map*

$$\mathcal{P} : (X, \varphi) \mapsto \varphi(H^{2,0}(X)) = \varphi(\mathbb{C}\omega_X).$$

Theorem 1.3.0.6 ([6, Thm. 5])**.** *Let (X, φ) be a marked IHS manifold. The period map $\mathcal{P} :$ $\mathrm{Def}(X) \to \Omega_\Lambda$ is a local isomorphism.*

Remark 1.3.0.7. Notice that Ω_Λ has complex dimension $rk(\Lambda) - 2 = h^{1,1}(X)$: by the Theorem above, $\mathcal{M}_\Lambda$ has the same dimension.

Theorem 1.3.0.6 is known as the *local Torelli theorem* for IHS manifolds. A global Torelli theorem holds for K3 surfaces, and a weaker version also holds for IHS manifolds; moreover, similar statements hold for complex tori of dimension 2 (abelian surfaces if projective) and cubic hypersurfaces in $\mathbb{P}^5$. We collect here the Torelli theorems for smooth manifolds we're going to refer to in this work.

Theorem 1.3.0.8 (Torelli theorem for K3 surfaces, [41, Thm. 5.3])**.** *Two complex K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. Moreover, for any Hodge isometry $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ that maps some Kähler class on X to a Kähler class on Y, there exists a (unique) isomorphism $\tilde{f} : Y \to X$, such that $f = \tilde{f}^*$.*

Lemma 1.3.0.9 ([74, Prop. I.6.2])**.** *If two IHS manifolds X and Y are bimeromorphic, then there exists a Hodge isometry $H^2(X, \mathbb{Z}) \cong_{\mathrm{Hdg}} H^2(Y, \mathbb{Z})$. In particular their transcendental lattices are Hodge isometric.*

Definition 1.3.0.10 ([47, Def. 1.1]). Let X and Y be IHS manifolds, which are deformation equivalent. A lattice isometry $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is a *parallel transport operator* if there exists a smooth and proper family $\pi : \mathcal{X} \to B$ and a continuous path $\gamma : [0, 1] \to B$ such that $X \simeq \mathcal{X}_{\gamma(0)}$ and $Y \simeq \mathcal{X}_{\gamma(1)}$ and f is induced by parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along γ. A parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is called a *monodromy operator* of X.

Theorem 1.3.0.11 (Torelli theorem for IHS manifolds, [47, Thm. 1.3])**.** *Let X and Y be IHS manifolds which are deformation equivalent.*

1. *X and Y are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$, which is an isomorphism of integral Hodge structures.*

2. *Let $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f} : Y \to X$, such that $f = \tilde{f}^*$, if and only if f maps some Kähler class on X to a Kähler class on Y.*

Remark 1.3.0.12. Monodromy operators have been characterized for all known deformation types of IHS manifolds: see [47] for $K3^{[n]}$-type and [59] for Km_n-type manifolds, [60] for OG6-type and [77] for OG10-type manifolds.

Remark 1.3.0.13. Consider the map $\psi : Aut(X) \to O(H^2(X, \mathbb{Z}))$, $f \mapsto f^*$. Thanks to [38, Thm. 2.1], the kernel of this map is invariant by deformation, so the following results extend to the respective deformation type.

- if $X = S^{[n]}$ with S a K3 surface, then then $ker(\psi) = id_X$ [7, Prop. 10];
- if $X = \mathrm{Km}_n(A)$, $ker(\psi) = A[n] \rtimes \langle -id \rangle$, where $A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^4$ is the group of n-torsion points of A [14, Cor. 5];
- if $X = \widetilde{K_{2v}}(A, H)$ (OG6-type), $ker(\psi) = \langle A[2], A^\vee[2] \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^8$, the group generated by two-torsion points in $A \times A^\vee$ [62, Thm. 5.2];
- if $X = \widetilde{M_{2v}}(S, H)$ (OG10-type), $ker(\psi) = id_X$ [62, Thm. 3.1].

Theorem 1.3.0.14 (Torelli theorem for abelian surfaces, [91, Thm. 1])**.** *Let X, Y be complex tori of dimension 2 such that there exists an isometry $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ of determinant 1, such that $\omega_X \circ \varphi = c\omega_X$ (i.e. φ preserves the periods) for some constant c. Then φ is induced by an isomorphism $f : Y \to X$. If $\det(\varphi) = -1$ instead, then $f : Y \to X^\vee$.*

Theorem 1.3.0.15 (Torelli theorem for cubic fourfolds, [95])**.** *Let $X, Y \subset \mathbb{P}^5$ be smooth cubic fourfolds such that there exists a Hodge isometry $\varphi : H^4(X, \mathbb{Z}) \to H^4(Y, \mathbb{Z})$ preserving h^2, where $h \in H^2(Y, \mathbb{Z})$ is the hyperplane class: then φ is induced by an isomorphism $f : Y \to X$.*

1.4 Symplectic automorphisms

Definition 1.4.0.1. An automorphism α of an IHS manifold X is symplectic if it preserves its symplectic form, i.e. if $\alpha^*\omega_X = \omega_X$. Define $Aut^s(X) \subset Aut(X)$ the group of symplectic automorphisms of X.

Consider $G \subset O(H^2(X, \mathbb{Z}))$ be the group of isometries induced by a finite group $G \subset Aut^s(X)$ via $g \mapsto g^*$. Fix a marking $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$: then we can define the *invariant* lattice $\Lambda^G \subseteq \Lambda$ and the *co-invariant* lattice $\Omega_G = (\Lambda^G)^{\perp_\Lambda}$. Notice that G does not, in general, act uniquely in cohomology: to the same G may correspond more than one pair (Λ^G, Ω_G).

Proposition 1.4.0.2 (see [71, Thm. 3.1.b)]). *A finite group G acts symplectically on an IHS manifold X if and only if $\Omega_G \subseteq \varphi(NS(X))$ or, equivalently, $\varphi(T(X)) \subseteq \Lambda^G$.*

There are constraints on the invariant and co-invariant lattices for the symplectic action of G on IHS manifolds: for $K3^{[n]}$-type manifolds these are contained in [58]. All symplectic actions on $K3^{[2]}$-type manifolds have been subsequently classified in [39]. Symplectic automorphisms of Km_n-type manifolds and their invariant and co-invariant lattices are explored in [61]. On OG6-type and OG10-type manifolds all symplectic automorphisms act trivially in cohomology [36], [33].

We can describe the moduli space of marked IHS manifolds of a given deformation type with a symplectic action of G via the notion of *lattice polarized* manifolds.

Definition 1.4.0.3 ([20, §1]). Let M be an even lattice. An IHS manifold X is *M-polarized* if there is a primitive embedding $M \hookrightarrow NS(X)$.

Remark 1.4.0.4. Let X be an IHS manifold and fix the marking $\Lambda \simeq (H^2(X,\mathbb{Z}), q_X)$. Let M be an even lattice: the moduli space of M-polarized IHS manifolds deformation equivalent to X has dimension $rk(\Lambda) - 2 - rk(M)$.

Remark 1.4.0.5. The moduli space of marked IHS manifolds of a given deformation type admitting a symplectic action of a finite abelian group G and a polarization of degree $2d$ can split in irreducible components, that we'll usually refer to as *projective families with a symplectic action of* G. Each of them is determined by the Néron-Severi lattice of its general member X, and the isomorphism class of the primitive embedding of $NS(X)$ in Λ (see Theorem 1.2.1.14). Since $NS(X)$ is an overlattice of finite index of $\Omega_G \oplus \langle 2d \rangle$, for each d one always finds a finite number of projective families.

1.4.1 On K3 surfaces

Let X be a K3 surface, and $G \subseteq Aut^s(X)$ a finite group. Since symplectic automorphisms of K3 surfaces only fix a finite number of points [71, §5], the surface $Y = X/G$ has only ADE singularities, and its resolution $\tilde{Y}$ is again a K3 surface.

The main result of Nikulin's paper [71] is that there are 14 finite abelian groups which can act symplectically on K3 surfaces, and the action of each one of these groups on Λ_{K3} is unique up to isometries [71, Thm. 4.7]. The lattices Ω_G are known for all groups G acting symplectically on a K3 surface: see [63] and [32] for $G = \mathbb{Z}/2\mathbb{Z}$, [28] for $G = \mathbb{Z}/p\mathbb{Z}$, $p = 3, 5, 7$, [30] for the remaining abelian cases.
The classification of finite group actions on K3 surfaces has been completed in works of Mukai and Xiao [66], [98]. A list of all the invariant and co-invariant lattices for G finite is provided in [37]: there are exactly five (non abelian) groups which can act in two different ways, for all the other groups the uniqueness result still holds.

Nikulin also characterized K3 surfaces $\tilde{Y}$ which are resolution of X/G, G finite abelian, by the existence of a primitive embedding of another lattice M_G in $NS(\tilde{Y})$ [71]. The first explicit description of the map that relates the lattices $H^2(X,\mathbb{Z})$ and $H^2(\tilde{Y},\mathbb{Z})$ was given by Morrison [63] for a symplectic involution, and subsequent works by Garbagnati, van Geemen and Sarti [32], [29] produced a complete description of the correspondence between families of projective K3 surfaces that admit a symplectic involution, and those that arise as resolution of the singularities of their quotient. Garbagnati and Prieto in [26] obtained a similar result for symplectic automorphisms of order 3.

1.4.2 On $K3^{[n]}$-type manifolds

Let S be a K3 surface with a symplectic action of a finite group G, let $X = S^{[n]}$ the Hilbert scheme of n points on S. The action of G lifts to X via the Hilbert-Chow morphism, so we can give the following definition.

Definition 1.4.2.1. Let S be a K3 surface and $X = S^{[n]}$, let G act symplectically on X. If this action is induced by an action of G on S we call it *natural*. A similar definition applies when $X = \mathrm{Km}_n(A)$.

Definition 1.4.2.2 ([57, Def. 2.1]). Let X be a manifold and let $G \subseteq Aut(X)$ a finite group. A deformation of the pair (X, G) consists of the following data:

- A flat family $\mathcal{X} \to B$, B connected and $\mathcal{X}$ smooth, and a distinguished point $0 \in B$ such that $\mathcal{X}_0 \simeq X$.
- A faithful action of the group G on $\mathcal{X}$ inducing fiberwise faithful actions, restricting to the given action of G on $\mathcal{X}_0$.

Definition 1.4.2.3. Let X be a $K3^{[n]}$-type (or Km_n-type) manifold with a symplectic action of a finite group G. We call (X, G) a *standard pair* when (X, G) can be deformed to a pair $(S^{[n]}, H)$ (or $(\mathrm{Km}_n(A), H)$) where $H \simeq G$ and the action of H is natural.

Remark 1.4.2.4. If (X, G) is a standard pair its fixed locus (and, more generally, the locus of points with non-trivial stabilizer) will be diffeomorphic to that of the natural action of H on $S^{[n]}$ (or $\mathrm{Km}_n(A)$). Indeed for any symplectic action of a group G, the fixed locus for any element of G is smooth and consists of points, abelian and K3 surfaces [24, Prop. 2.6], but smooth deformations of a K3 (resp. abelian) surface are still K3 (resp. abelian) surfaces, and smooth deformations of points are points. The number of points, K3 and abelian surfaces in the locus with non-trivial stabilizer under the action of H, and their intersections, are therefore preserved under deformations.

When X is a manifold of $K3^{[n]}$-type and G acts symplectically, sometimes (depending on n) we can say whether the action of G is standard or not just by looking at the cohomological action.

Definition 1.4.2.5 ([57, Def. 2.4]). Let X be a manifold of $K3^{[n]}$-type and let $G \subseteq Aut^s(X)$. The group G is *numerically standard* if there exists a K3 surface S and some $H \subseteq Aut^s(S)$ such that $H \simeq G$ such that the following conditions hold:

- the co-invariant lattices for the actions of G and H are isometric;

- the invariant lattices for the two actions satisfy $\Lambda^G_{\mathrm{K3}^{[n]}} \simeq \Lambda^H_{\mathrm{K3}} \oplus \langle\delta\rangle$, with $\delta^2 = -2(n-1)$ and $(\delta, H^2(X,\mathbb{Z})) = 2(n-1)\mathbb{Z}$.

Remark 1.4.2.6. Notice that for any standard pair (X, G), G is numerically standard: indeed, (X, G) can be deformed to a natural pair $(S^{[n]}, H)$, and by construction the natural action of H on $S^{[n]}$ fixes the class δ of the exceptional divisor of the Hilbert-Chow morphism, and acts on $\langle\delta\rangle^\perp \simeq H^2(S,\mathbb{Z})$ as expected.

Theorem 1.4.2.7 ([57, Thm. 2.5])**.** *Let X be a manifold of K3$^{[n]}$-type and let $n-1$ be a prime power. Let $G \subseteq Aut^s(X)$ be a finite group of numerically standard automorphisms. Then (X, G) is a standard pair.*

Remark 1.4.2.8. Symplectic automorphisms of K3$^{[2]}$-type manifolds have been classified in [39]. In chapter 5 of this thesis, we're going to study symplectic actions of groups G of order four: from the classification we know that in this case the action of G is always numerically standard, and therefore standard.

1.5 Symplectic varieties and quotients of IHS manifolds

Unlike the case of K3 surfaces, if G acts symplectically on an IHS manifold X the quotient X/G may not admit a symplectic resolution of the singularities. However, there is not just one way to define singular varieties that generalize the concept of IHS manifolds (see for instance Perego's account [79]).

The first definition we are going to give applies to singular varieties which arise in a broader context, and cannot always be obtained from quotients of IHS manifolds: the theory of singularities and teminalizations is indeed used more in general for the purposes of the minimal model program.

Definition 1.5.0.1 ([3, Def. 3.1]). A *symplectic variety* is a pair (X, σ) consisting of a normal variety X and a closed holomorphic symplectic form $\sigma \in H^{2,0}(X_{reg})$ such that there is a resolution of the singularities $\pi : Y \to X$ for which $\pi^*\sigma$ extends to a holomorphic form on Y. A *primitive symplectic variety* is a normal compact Kähler variety X such that $H^1(X, \mathcal{O}_X) = 0$ and $H^{2,0}(X_{reg}) = \mathbb{C}\sigma$, such that (X, σ) is a symplectic variety.

Remark 1.5.0.2. If X is an IHS manifold and G is a finite group acting symplectically on X, then the quotient X/G is a primitive symplectic variety.

Definition 1.5.0.3 ([68, Cor. 1]). Let X be a symplectic variety, $\Sigma := X\setminus X_{reg}$ its singular locus. Then X has *terminal singularities* if and only if $codim(\Sigma) \geq 4$.

Definition 1.5.0.4. Let X be an IHS manifold, G a finite group acting symplectically on X. We call $\phi : Y \to X/G$ a *terminalization* if Y is a $\mathbb{Q}$-factorial symplectic variety with terminal singularities and ϕ is a crepant proper birational morphism. Recall that $\mathbb{Q}$-*factorial* means that for every Weil divisor D on Y there is an $m \in \mathbb{N}$ such that mD is Cartier, and *crepant* means that $\mathcal{K}_Y = \phi^*\mathcal{K}_X$.

Theorem 1.5.0.5 ([3, Thm. 9.1]). *Let Z be a primitive symplectic variety with second Betti number at least 5. Then there exist a primitive symplectic variety Z' which is inseparable from Z in (locally trivial) moduli and a terminalization $Y \to Z'$.*

We will now introduce another definition, more appropriate in the context of quotients of IHS manifolds (see Section 1.5.1).

Definition 1.5.0.6 ([49], after [24]). An *orbifold* is a compact analytic complex space with at worst finite quotient singularities. A compact Kähler orbifold is called *symplectic* if its singularities are in codimension 4 and its smooth locus is endowed with an everywhere non-degenerate holomorphic 2-form. In addition, a symplectic orbifold is said *primitively symplectic* if the holomorphic 2-form is unique up to scaling, and *irreducible holomorphic symplectic* (IHSO) if moreover its smooth locus is simply connected.

Remark 1.5.0.7. Irreducible symplectic orbifolds appear as building blocks of compact Kähler orbifolds with trivial canonical class [17] and a Torelli theorem holds for them [50], [51].

To deal with O'Grady's examples in Section 6.4, we need yet another definition: indeed, they are obtained as resolution of the singularities of some moduli spaces of sheaves on an abelian or K3 surface (see Theorem 6.4.1.1).

Definition 1.5.0.8 ([79, Def. 2.16]). A *resolvable symplectic variety* is a normal, compact Kähler space X whose smooth locus has a holomorphic symplectic form σ, and which has a symplectic resolution of the singularities $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is a IHS manifold.

Remark 1.5.0.9. The singularities of resolvable symplectic varieties are, by definition, not terminal: terminalizations of resolvable symplectic varietes are smooth. If X is an IHS manifold with a symplectic action of a group G, the quotient X/G is a resolvable symplectic variety if and only if it admits a smooth terminalization: the known cases of this particular phenomenon happening are collected in Example 1.5.1.5. We remark that new cases can only be obtained by considering non-natural actions, as natural actions are classified in [10].

Lemma 1.5.0.10 ([4, Lemma 3.5]). *Let $\pi : \widetilde{X} \to X$ be a symplectic resolution of a resolvable symplectic variety: then the pull-back map $\pi^* : H^2(X, \mathbb{Z}) \to H^2(\widetilde{X}, \mathbb{Z})$ is injective. The restriction of the BBF form $q_{\widetilde{X}}$ to $\pi^*(H^2(X, \mathbb{Z}))$ is non-degenerate, and it induces an equality of the transcendental lattices (i.e. the exceptional classes introduced with the resolution are algebraic).*

A Torelli theorem also holds for resolvable symplectic varieties [4, Thm. 1.3].

1.5.1 Nikulin orbifolds and other quotients of IHS manifolds

If X is an IHS manifold and G is a finite group acting symplectically on X, then the terminalization Y of the quotient X/G is a primitively symplectic orbifold (see Definition 1.5.0.6). In [10] a classification of orbifolds obtained this way is provided, under the assumption that (X, G) be a natural pair: for each Y the authors give the second Betti number and the group $\pi_1(Y_{reg})$. Moreover, they prove the following result.

Proposition 1.5.1.1 ([10, Prop. 7.1])**.** *Let X be an IHS manifold, let G be a finite group acting symplectically on X, let $Y \to X/G$ be a terminalization of X/G. Then $\pi_1(Y_{reg}) = G/N$, where N is the normal subgroup of G generated by elements with fixed locus of codimension 2.*

Example 1.5.1.2. **Nikulin orbifolds:** Let X be a $K3^{[2]}$-type manifold with a symplectic involution ι: the pair (X, ι) is always standard [56], so by Remark 1.4.2.4 its fixed locus is diffeomorphic to that of the natural involution, that is, a K3 surface Σ and 28 isolated points. Let $\pi : X \to X/\iota$ be the quotient map: X/ι is singular in codimension 2. Blowing up $\pi(\Sigma)$ we obtain an IHSO Y with 28 isolated singularities, which in [16] is called a Nikulin orbifold in analogy with Nikulin surfaces.

Nikulin-type orbifolds (which are IHSOs deformation equivalent to Nikulin orbifolds) are one of the most studied examples of quotients of IHS manifold: their second integral cohomology group (i.e. that of their smooth locus) is endowed with a symplectic form, which gives it a lattice structure [48]

$$H^2(Y, \mathbb{Z}) \simeq \Lambda_{\mathrm{N}} := E_8 \oplus U(2)^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}.$$

Moreover, there are results about their Kähler cone in [52], and projective families of Nikulin orbifolds have been classified in [16].

Remark 1.5.1.3 ([48]). Let Y be a Nikulin orbifold: then its Néron-Severi lattice will always contain the class of the divisor $\tilde{\Sigma}$ obtained by blowing up $\pi(\Sigma)$; it holds $\tilde{\Sigma}^2 = -4$, so $NS(Y)$ is polarized with the lattice $\langle -4 \rangle$.
Moreover, suppose that Y is obtained from a natural pair $(S^{[2]}, \iota)$: then, $NS(S^{[2]})$ contains half the class of the exceptional divisor (the blow-up of the diagonal of $Sym^2(S)$) $\delta = \Delta/2$, that satisfies $\delta^2 = -2$, $\delta H^2(S^{[2]}) = 2\mathbb{Z}$. Let now $\tilde{\delta} = \pi_* \delta$: then $\tilde{\delta}^2 = -4$, and $(\tilde{\delta} + \tilde{\Sigma})/2 \in NS(Y)$ because Y admits a 2:1 cover branched over $\tilde{\delta} \cup \tilde{\Sigma}$, given by $(S \times S)/\langle \sigma \circ (\iota \times \iota) \rangle$, where σ switches the two copies of S. In this case, we see that the orthogonal component $\langle -2 \rangle^{\oplus 2}$ of $H^2(Y, \mathbb{Z})$ is generated by $(\tilde{\delta} \pm \tilde{\Sigma})/2$.
Since the second integral cohomology is invariant by deformation, and symplectic involutions on $K3^{[2]}$-type manifolds are standard, we conclude that even for a general $K3^{[2]}$-type manifold X admitting a symplectic involution, where the class $\mu \in \Lambda_{K3^{[2]}}$ of square -2 such that $\mu^\perp \simeq \Lambda_{K3}$ does not have a geometric meaning, still $(\pi_* \mu \pm \tilde{\Sigma})/2$ is integral in $H^2(Y, \mathbb{Z})$.

Let X be an IHS manifold, G a finite group acting symplectically on X and Y be the terminalization of X/G. Assume that Y has terminal singularities: we know the lattice structure of $H^2(Y, \mathbb{Z})$ for a very short list of cases, all of them obtained terminalizing quotients of IHS fourfolds.

Example 1.5.1.4.

1. if X is a $K3^{[2]}$-type manifold and $G = \mathbb{Z}/2\mathbb{Z}$ then Y is a Nikulin orbifold, and $H^2(Y, \mathbb{Z}) \simeq E_8 \oplus U(2)^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$;

2. if X is a K3$^{[2]}$-type manifold, $G = \mathbb{Z}/3\mathbb{Z}$ and (X, G) is standard, then Y has 27 isolated singularities, and $H^2(Y, \mathbb{Z}) \simeq U(3) \oplus U^2 \oplus A_2^2 \oplus \langle -6 \rangle$ [49];
3. if X is a Km$_2$-type manifold and $G = \mathbb{Z}/2\mathbb{Z}$, then Y is an IHSO with 36 isolated singularities and $H^2(Y, \mathbb{Z}) \simeq U(3)^3 \oplus \begin{bmatrix} -5 & -4 \\ -4 & -5 \end{bmatrix}$ [43];
4. if X is a K3$^{[2]}$-type manifold and $G = \mathbb{Z}/11\mathbb{Z}$ there are two different actions of G on $H^2(X, \mathbb{Z})$ that share the same co-invariant lattice Ω_G, but with different invariant lattices [55, §7.4.4]. The orbifolds Y_1 and Y_2 have 5 isolated singularities, and $H^2(Y_1, \mathbb{Z}) \simeq \begin{bmatrix} 2 & -1 & 3 \\ -1 & 8 & -1 \\ 3 & -1 & 6 \end{bmatrix}$ while $H^2(Y_2, \mathbb{Z}) \simeq \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ [49].

We can add to this list that of the known cases in which Y is smooth:

Example 1.5.1.5.

1. if X is a K3$^{[2]}$-type manifold, $G = \mathbb{Z}/3\mathbb{Z}$ and (X, G) is not standard, then Y is a Km$_2$-type manifold [67, Ex. 1.7.iv];
2. if X is a K3$^{[2]}$-type manifold, $G = (\mathbb{Z}/2\mathbb{Z})^4$ and (X, G) is standard, then Y is a K3$^{[2]}$-type manifold [10, Prop. 8.1];
3. if X is a Km$_2$-type manifold and $G = (\mathbb{Z}/3\mathbb{Z})^3$ is such that each nontrivial element fixes a surface [10, Lemma 4.6], then Y is a K3$^{[2]}$-type manifold [10, Rem. 9.2];
4. if X is a Km$_3$-type manifold, the action of $G = (\mathbb{Z}/2\mathbb{Z})^5$ is such that each non trivial element fixes a surface [10, Lemma 4.6], then Y is a K3$^{[3]}$-type manifold [10, Rem. 9.2].

In this thesis, we study the action of groups of order 4 on K3$^{[2]}$-type manifolds: this allows us to describe in Chapter 5 the two different involutions induced on Nikulin orbifolds. Moreover, we can state the following results about their quotients.

Proposition 1.5.1.6.

1. *Let $G = \mathbb{Z}/4\mathbb{Z}$ act symplectically on a K3$^{[2]}$-type manifold X. The terminalization Y_4 of the quotient X/G is a primitively symplectic orbifold, $\pi_1(Y_4)_{reg} = \mathbb{Z}/2\mathbb{Z}$ and $H^2(Y_4, \mathbb{Z})$ is an overlattice of finite index of the lattice $U(4) \oplus U^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$.*
2. *Let $G = (\mathbb{Z}/2\mathbb{Z})^2$ act symplectically on a K3$^{[2]}$-type manifold X. The terminalization $Y_{2,2}$ of the quotient X/G is an IHSO, and $H^2(Y_{2,2}, \mathbb{Z})$ has rank 14.*

Chapter 2

Action of $\mathbb{Z}/4\mathbb{Z}$ on a K3 surface

2.1 Introduction

In this chapter we study K3 surfaces X with a symplectic automorphism of order four τ; most of the material here presented can be also found in [82].
The order of τ being non-prime allows us to analyze a new type of surfaces: those that are minimal resolution of an intermediate quotient, as is in our case the K3 surface $\tilde{Z}$, the minimal resolution of X/τ^2. The main results of this chapter are the lattice-theoretic characterization of $\tilde{Z}$, and the comparison between its moduli space and those of X and $\tilde{Y}$, the minimal resolution of X/τ.
From Section 1.4.1 we expect that $\tilde{Z}$, admitting a symplectic involution and being itself the resolution of a quotient by a symplectic involution, be polarized with both lattices $\Omega_2 := \Omega_{\mathbb{Z}/2\mathbb{Z}}$, and $N := M_{\mathbb{Z}/2\mathbb{Z}}$. However, these lattices are not in direct sum in $NS(\tilde{Z})$: the negative definite, rank 14 lattice Γ_4 that characterizes $\tilde{Z}$ is introduced in Definition 2.4.2.3, where the primitive embedding $N \hookrightarrow \Gamma_4$ is also described; the primitive embedding $\Omega_2 \hookrightarrow \Gamma_4$ can be found in Remark 2.4.3.1.

Theorem 2.1.0.1 (see Thm. 2.4.5.1)**.** *A K3 surface $\tilde{Z}$ is the minimal resolution of X/τ^2, for some K3 surface X with a symplectic automorphism of order 4 τ, if and only if $\tilde{Z}$ is Γ_4-polarized.*

Corollary 2.1.0.2. *Let X be a general K3 surface with a symplectic automorphism of order four τ, let $\tilde{Z}$ and $\tilde{Y}$ be respectively the minimal resolution of X/τ^2 and X/τ. The moduli spaces of $X, \tilde{Z}$ and $\tilde{Y}$ are all irreducible of dimension 6: more precisely, it holds $NS(X) = \Omega_4, NS(\tilde{Z}) = \Gamma_4, NS(\tilde{Y}) = M_4$.*

Theorem 2.1.0.3 (see Thm. 2.5.2.3, 2.5.3.3)**.** *Let X be a general projective K3 surface with a symplectic automorphism of order four τ; let $\tilde{Z}$ and $\tilde{Y}$ be respectively the minimal resolution of X/τ^2 and X/τ. Then, using the notation introduced for overlattices in Remark 2.5.0.5, we have the following correspondence between $NS(X)$, $NS(\tilde{Z})$ and $NS(\tilde{Y})$ depending on the value of d modulo 4. For $d \equiv 2 \pmod 4$ the two possible*

$NS(X)$ are not isomorphic, and the same holds for $NS(\tilde{Y})$.

	$NS(X)$	$NS(\tilde{Z})$	$NS(\tilde{Y})$
$\forall d$	$\Omega_4 \oplus \langle 2d \rangle$	$(\Gamma_4 \oplus \langle 4d \rangle)'$	$(M_4 \oplus \langle 8d \rangle)^\star$
$d =_4 2$	$(\Omega_4 \oplus \langle 2d \rangle)'^{(1)}$ $(\Omega_4 \oplus \langle 2d \rangle)'^{(2)}$	$(\Gamma_4 \oplus \langle 4d \rangle)^\star$	$(M_4 \oplus \langle 2d \rangle)'^{(1)}$ $(M_4 \oplus \langle 2d \rangle)'^{(2)}$
$d =_4 3$	$(\Omega_4 \oplus \langle 2d \rangle)'$	$(\Gamma_4 \oplus \langle 4d \rangle)^\star$	$(M_4 \oplus \langle 2d \rangle)'$
$d =_4 0$	$(\Omega_4 \oplus \langle 2d \rangle)'$ $(\Omega_4 \oplus \langle 2d \rangle)^\star$	$(\Gamma_4 \oplus \langle d \rangle)'$	$(M_4 \oplus \langle 2d \rangle)'$ $M_4 \oplus \langle d/2 \rangle$

Notice that there is a 1:1 correspondence between families of projective surfaces X and $\tilde{Y}$, as it happens for automorphisms of order 2 and 3; however, when d is even two different families of X (or $\tilde{Y}$) can correspond to the same family of $\tilde{Z}$.

The correspondence above is given by the maps induced in cohomology by the quotient maps from X to $X/\tau^2, X/\tau$: to define these maps one needs firstly to understand the isometry τ^* induced by τ on $H^2(X, \mathbb{Z})$. By [71, Thm. 4.7] this depends neither on X, nor on τ, but only on its order: therefore, we can use as a starting point a projective K3 surface on which τ^* is easy to describe.

2.2 A symplectic automorphism τ of order 4 on the surface X_4

In this section, we introduce the surface X_4 (see [94]), which has Picard rank 20, and admits a Jacobian fibration that provides us with a presentation of τ^* as permutation action on a certain sublattice $W \subset H^2(X, \mathbb{Z})$ of finite index: this action can be then extended uniquely to the whole lattice $H^2(X, \mathbb{Z})$.

2.2.1 Jacobian fibrations

Definition 2.2.1.1. Define *Jacobian fibration* a fibration $p : X \to \mathbb{P}^1$ whose generic fiber is a genus 1 curve, and that admits a global section $s : \mathbb{P}^1 \to X$ (it holds $p \circ s = id_X$), denoted *zero section*: the fiber over a generic point $F = p^{-1}(x)$ is an elliptic curve with the zero for the group law defined as $s(x)$.
The *Mordell-Weil group* $MW(p)$ of a Jacobian fibration is the group generated by all the sections, with the group law induced by that of the generic fiber.

Remark 2.2.1.2. The group $MW(p)$ acts on X by translation on each fiber, therefore it acts as the identity on the symplectic form ω_X.

Given a Jacobian fibration, the Mordell-Weil group is linked to the Néron-Severi group of the surface by the following isomorphism [89, Thm. 6.3]:

$$MW(p) \simeq NS(X)/\mathcal{T}(p) \tag{2.2.1.1}$$

where the *trivial lattice* $\mathcal{T}(p)$ is the sublattice of $NS(X)$ generated by the generic fiber, the image of the zero section $s = s(\mathbb{P}^1)$ and the irreducible components of the reducible fibers which do not intersect the curve s.

The Mordell-Weil group is endowed with the *height pairing* [89, §11.6 et seq.], a symmetric $\mathbb{Q}$-valued bilinear form induced by the projection of the intersection form of $NS(X)$ onto $NS(X)/\mathcal{T}(p)$. In particular, for any $t \in MW(p)$ one gets $h(t) = 2\chi(X) - 2ts - K$, where K depends on the intersection of t whith the reducible fibers of p according to [89, §11.8, Table 4], and ts is the usual intersection product of t and s in $NS(X)$. The height of t is 0 if and only if t is a torsion section.

2.2.2 The surface X_4

The surface X_4 is the unique K3 surface with transcendental lattice $T(X_4) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$: it arises as resolution of the singularities of the quotient surface $A/\langle\sigma\rangle$, where A is the abelian surface $E_i \times E_i$, E_i is the elliptic curve of lattice $\langle 1, i\rangle$, and σ is the automorphism of A defined by $\sigma(e_1, e_2) = (ie_1, -ie_2)$; this surface is well known, see for instance [92], [94].

A description of all the possible Jacobian fibrations on X_4 is given by Nishiyama [73, Table 1.2]: in particular there exists a fibration

$$\pi : X_4 \to \mathbb{P}^1 \quad \text{s.t.} \quad MW(\pi) \simeq \mathbb{Z}/4\mathbb{Z}$$

which provides a symplectic automorphism τ of order four on X_4 by means of a section t_1 that generates $MW(\pi)$. Moreover, by (2.2.1.1) the curves of the trivial lattice form a $\mathbb{Q}$-basis for $NS(X_4)$.

The reducible fibers of π are one of type I_4 and one of type I_{16} [53, Table IV.3.1]. Call B_0 (respectively C_0) the component of I_4 (resp. I_{16}) intersected by s, and number the other components so that $B_{[i]_4}$ intersects only $B_{[i+1]_4}$ and $B_{[i-1]_4}$, and $C_{[j]_{16}}$ intersects only $C_{[j+1]_{16}}$ and $C_{[j-1]_{16}}$ where $[a]_n$ is the class of a modulo n. Thus $\mathcal{T}(\pi)$ is generated by the class F of the generic fiber of π, the curve s and the components $B_i, C_j,\ i = 1, 2, 3,\ j = 1, \ldots, 15$ of the reducible fibers: except for F, these curves are rational, so they have self-intersection -2; F satisfies $F^2 = 0$. The curves B_1, B_2, B_3 span the lattice A_3 [89, §6.5], and $C_1, \ldots, C_{15}$ span the lattice A_{15}.

Using the height pairing we can determine the components B_i, C_j of the reducible fibers that have non-trivial intersection with a non-zero section $t \in MW(\pi)$. It holds $\chi(X_4) = 2$ because X_4 is a K3 surface, and since t is a torsion section it holds $ts = 0$: therefore i, j satisfy the equation

$$0 = h(t) = 4 - \left(\frac{i(4-i)}{4} + \frac{j(16-j)}{16}\right) \quad (\textit{height formula}).$$

We will choose the following notation for the elements of $MW(\pi)$: the zero section s intersects the components B_0 and C_0; the section t_1 intersects the components B_2 and C_4; the section t_2 intersects the components B_0 and C_8; the section t_3 intersects the components B_2 and C_{12}. Notice that each of t_1 and t_3 generates $MW(\pi)$, whereas t_2 has order 2.
We can write t_1, t_2, t_3 in function of the basis of the trivial lattice $\mathcal{T}(\pi)$ using the information about their intersections:

$$\begin{aligned} t_1 &= 2F + s - \frac{B_1 + 2B_2 + B_3}{2} - \frac{3C_1 + 6C_2 + 9C_3 + \sum_{j=1}^{12} jC_{16-j}}{4} \\ t_2 &= 2F + s - \frac{\sum_{j=1}^{7} j(C_j + C_{16-j}) + 8C_8}{2} \\ t_3 &= 2F + s - \frac{B_1 + 2B_2 + B_3}{2} - \frac{\sum_{j=1}^{12} jC_j + 9C_{13} + 6C_{14} + 3C_{15}}{4}. \end{aligned}$$

Since the discriminant group of $NS(X_4)$ is $(\mathbb{Z}/2\mathbb{Z})^2$ (it is indeed the orthogonal complement to $T(X_4)$ in Λ_{K3}), from (2.2.1.1) and the equations above it can be readily seen that $NS(X_4)$ admits as a $\mathbb{Z}$-basis $\mathcal{B} = \{F, s, t_1, B_1, B_2, B_3, C_1, \ldots, C_{14}\}$.

2.2.3 The action of τ^* on the second cohomology of X_4

The symplectic automorphism τ induces an isometry τ^* on $NS(X_4)$ such that

$$\begin{aligned} &F \overset{\tau^*}{\longmapsto} F \qquad s \overset{\tau^*}{\longmapsto} t_1 \overset{\tau^*}{\longmapsto} t_2 \overset{\tau^*}{\longmapsto} t_3 \overset{\tau^*}{\longmapsto} s \\ &\tau^*(C_j) = C_{[j+4]_{16}} \qquad \tau^*(B_i) = B_{[i+2]_4}. \end{aligned}$$

where $[a]_n$ is the class of a modulo n. Therefore, we can easily identify two copies of A_1, $\langle B_1 \rangle$ and $\langle B_3 \rangle$, exchanged by the action of τ^*, and a set of four copies of D_4 on which τ^* acts as a cycle of order 4: $\{\langle s, C_{15}, C_0, C_1 \rangle, \langle t_1, C_3, C_4, C_5 \rangle, \langle t_2, C_7, C_8, C_9 \rangle, \langle t_3, C_{11}, C_{12}, C_{13} \rangle\}$. All these lattices are pairwise orthogonal, and the orthogonal complement in $NS(X)$ of the direct sum $D_4^{\oplus 4} \oplus A_1^{\oplus 2}$ is generated over $\mathbb{Q}$ by the vectors

$$\begin{aligned} R_1 = &-8F - 4s + 8t_1 + 4C_1 + 8C_2 + 13C_3 + 18C_4 + 15C_5 + 12C_6 + 10C_7 + 8C_8 + \\ &+ 6C_9 + 4C_{10} + 3C_{11} + 2C_{12} + C_{13} + 3B_1 + 6B_2 + 3B_3, \\ R_2 = &-4F - 2s + 2t_1 + 2C_1 + 4C_2 + 5C_3 + 6C_4 + 5C_5 + 4C_6 + 4C_7 + 4C_8 + \\ &+ 4C_9 + 4C_{10} + 3C_{11} + 2C_{12} + C_{13} + B_1 + 2B_2 + B_3, \end{aligned}$$

whose intersection form satisfies $R_1^2 = 4,\ R_2^2 = -4,\ R_1R_2 = 0$. It can be also verified that $\tau^* R_1 = R_1$, while $\tau^* R_2 = -R_2$. Therefore, we have the following description:

Proposition 2.2.3.1. *Consider the sublattice $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$ of $NS(X_4)$ generated as above. The isometry τ^* acts on this sublattice as the cyclic permutation of order 4 on $D_4 \oplus D_4 \oplus D_4 \oplus D_4$, as the cyclic permutation of order 2 on $A_1 \oplus A_1$, as id (the identity) on $\langle 4 \rangle$ and as $-id$ on $\langle -4 \rangle$.*

The lattice $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$ has discriminant group

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

so it has index 2^6 in $NS(X_4)$; the latter can be obtained by adding the following generators to the generators of $D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle 4 \rangle \oplus \langle -4 \rangle$:

$$\begin{aligned}
R &:= (R_1 + R_2)/2;\\
a &:= R_1/4 + R_2/4 - (s + C_{15})/2 - (t_1 + C_5)/2 = C_0 + C_1 + C_2 + C_3 + C_4;\\
b &:= R_1/4 - R_2/4 - (t_1 + C_3)/2 - (t_2 + C_9)/2 = C_4 + C_5 + C_6 + C_7 + C_8;\\
c &:= R_1/4 + R_2/4 - (t_2 + C_7)/2 - (t_3 + C_{13})/2 = C_8 + C_9 + C_{10} + C_{11} + C_{12};\\
d &:= R_1/4 - R_2/4 - (t_3 + C_{11})/2 - (s + C_1)/2 = C_{12} + C_{13} + C_{14} + C_{15} + C_0;\\
e &:= R_1/2 - (C_3 + C_5)/2 - (C_{11} + C_{13})/2 - B_1/2 - B_3/2 = t_1 + t_3 + C_4 + C_{12} + B_2.
\end{aligned} \tag{2.2.3.1}$$

Now, $H^2(X_4, \mathbb{Z})$ is an overlattice of index 2^2 of the lattice $NS(X_4) \oplus T(X_4)$. Since $H^2(X_4, \mathbb{Z})$ is unimodular, following Theorem 1.2.1.3 we have to find an isotropic subgroup of $A_{NS(X_4) \oplus T(X_4)}$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Denoting $\{\omega_1, \omega_2\}$ the $\mathbb{Z}$-basis of $T(X_4)$ for which the intersection matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, the elements we have to add are:

$$\begin{aligned}
f &= (s + t_1 + C_1 + C_3 + B_1 + \omega_1)/2,\\
g &= (s + t_1 + C_1 + C_3 + B_3 + \omega_2)/2.
\end{aligned} \tag{2.2.3.2}$$

2.3 The action of τ^* and $(\tau^2)^*$ on the K3 lattice

Nikulin's uniqueness result [71, Thm. 4.7] enables us to deduce the action of any symplectic automorphism of order 4 (and of its square) on the second cohomology group of any K3 surface X by looking at the surface X_4 with the action of the automorphism τ introduced in the previous section.

2.3.1 A convenient description of the K3 lattice

The isometry τ^* induced on Λ_{K3} by an automorphism τ of order 4 acts on the sublattice of finite index of Λ_{K3} $W := D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \langle -4 \rangle \oplus \langle 4 \rangle \oplus \langle 2 \rangle \oplus \langle 2 \rangle$ as the cycle $(1, 2, 3, 4)$ on the four copies of D_4, as $(1, 2)$ on the two copies of A_1, as $-id$ on $\langle -4 \rangle$ and as id on the remaining orthogonal components. The following diagram describes the situation:

$$W := \quad D_4 \oplus D_4 \oplus D_4 \oplus D_4 \oplus A_1 \oplus A_1 \oplus \overset{-id}{\langle -4 \rangle} \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \circlearrowleft id \tag{2.3.1.1}$$

Denote $e_1, \dots, e_4,\ f_1, \dots, f_4,\ g_1, \dots, g_4,\ h_1, \dots, h_4$ the generators of the four copies of D_4, such that $e_3e_1 = e_3e_2 = e_3e_4 = 1$, and $\tau^* : e_i \mapsto f_i \mapsto g_i \mapsto h_i \mapsto e_i$ for $i = 1, \dots, 4$; a_1 and a_2 the generators of the two copies of A_1; σ the generator of $\langle -4 \rangle$, ρ the generator of $\langle 4 \rangle$, ω_1 and ω_2 the generators of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then, Λ_{K3} is obtained by adding to W the elements (cf. (3.2.2.1) and (3.2.2.2))

$$\begin{aligned}
\chi &= (\rho + \sigma)/2; \\
\alpha &= (\rho + \sigma)/4 + (e_1 + e_2 + f_1 + f_4)/2; \\
\beta &= (\rho - \sigma)/4 + (f_1 + f_2 + g_1 + g_4)/2; \\
\gamma &= (\rho + \sigma)/4 + (g_1 + g_2 + h_1 + h_4)/2; \\
\delta &= (\rho - \sigma)/4 + (h_1 + h_2 + e_1 + e_4)/2; \\
\varepsilon &= (\rho + f_2 + f_4 + h_2 + h_4 + a_1 + a_2)/2; \\
\zeta &= (e_1 + f_1 + e_4 + f_2 + a_1 + \omega_1)/2; \\
\eta &= (e_1 + f_1 + e_4 + f_2 + a_2 + \omega_2)/2;
\end{aligned} \tag{2.3.1.2}$$

the action of τ^* and $(\tau^2)^*$ on these elements is deduced by the one on the sublattice W described above by $\mathbb{Q}$-linear extension: notice that $\tau^* : \alpha \mapsto \beta \mapsto \gamma \mapsto \delta \mapsto \alpha$, and that $(\tau^2)^*$ fixes ε and χ.

A note on the action of τ and τ^2 on the K3 lattice

Any symplectic involution on a K3 surface acts on Λ_{K3} exchanging two copies of E_8, as was first observed by Morrison [63, proof of Thm. 5.7]; on the other hand, the lattice E_8 can be abstractly described as an overlattice of finite index of $D_4 \oplus D_4$ (Nishiyama, see [73, Lemma 4.3] and the discussion in §5.4): given the $\mathbb{Z}$-bases of the lattices involved (with the elements numbered as in Example 1.2.0.2) $\{d_1^{(i)}, \dots, d_4^{(i)}\}$ $\{e_1, \dots, e_8\}$, we can define

$$\begin{aligned}
&e_1 = d_1^{(1)}, \quad && e_2 = (d_1^{(2)} - d_2^{(2)} - d_1^{(1)} - d_4^{(1)})/2 - d_2^{(1)} - d_3^{(1)}, \quad && e_3 = d_2^{(1)}, \quad && e_4 = d_3^{(1)}, \\
&e_5 = d_4^{(1)}, \quad && e_6 = (d_2^{(2)} - d_4^{(2)} - d_1^{(1)} - d_2^{(1)})/2 - d_3^{(1)} - d_4^{(1)}, \quad && e_7 = d_4^{(2)}, \quad && e_8 = d_3^{(2)}.
\end{aligned}$$

Moreover, we could construct the lattice Λ_{K3} as overlattice of W by gluing pairwise the four copies of D_4 as above, and then adding as generators also the elements $(a_1 + \omega_1)/2, (a_2 + \omega_2)/2, (\sigma + \rho)/4$. However, doing so the action of τ^{2*} on W described in (2.3.1.1) does not extend to an action on the lattice Λ_{K3} that exchanges the two copies of E_8 we built.

Proposition 2.3.1.1. *The gluing of the four copies of D_4 in W to two copies of E_8 does not extend the action of τ^* on W to an action on Λ_{K3} such that $(\tau^2)^*$ exchanges the two copies of E_8.*

Proof. We follow Nishiyama's construction above (using the same notation) to glue pairwise four copies of D_4 to two copies of E_8: denote $D_4^{(k)}$ the k-th copy of D_4, and similarly $E_8^{(j)}$. The isometry τ^* acts on the four copies of D_4 as the permutation $(1,2,3,4)$ of the apices, and its action on the elements can be written as $d_i^{(1)} \mapsto d_i^{(2)} \mapsto d_i^{(3)} \mapsto d_i^{(4)}$ without loss of generality. To have $(\tau^2)^*$ exchange the two copies of E_8 we construct, we have to glue either $D_4^{(1)}$ to $D_4^{(2)}$ and $D_4^{(3)}$ to $D_4^{(4)}$, or $D_4^{(1)}$ to $D_4^{(3)}$ and $D_4^{(2)}$ to $D_4^{(4)}$: suppose the gluing be

$$\begin{aligned} D_4^{(1)} \oplus D_4^{(2)} &\rightsquigarrow E_8^{(1)}, \\ D_4^{(3)} \oplus D_4^{(4)} &\rightsquigarrow E_8^{(2)}. \end{aligned}$$

Consider the element $e_2^{(1)}$: if τ^* were an isometry of Λ_{K3}, then also $\tau^* e_2^{(1)}$ would belong to Λ_{K3}, and their intersection would be integer: however, we get

$$\begin{aligned} e_2^{(1)} &= (d_1^{(2)} - d_2^{(2)} - d_1^{(1)} - d_4^{(1)})/2 - d_2^{(1)} - d_3^{(1)}, \\ \tau^* e_2^{(1)} &= (d_1^{(3)} - d_2^{(3)} - d_1^{(2)} - d_4^{(2)})/2 - d_2^{(2)} - d_3^{(2)}; \end{aligned}$$

therefore $e_2^{(1)} \cdot \tau^* e_2^{(1)} = -1/2$. The other choice of gluing gives a similar result. □

Remark 2.3.1.2. The isometry $(\tau^2)^*$ does indeed exchange two copies of E_8 in Λ_{K3}, it being a symplectic involution: we can easily see this in $NS(X_4)$ (see Section 2.2.2), for instance using the two copies of E_8 $\langle s, C_{14}, C_{15}, C_0, C_1, C_2, C_3, C_4\rangle$ and $\langle t_2, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}\rangle$. However, the elements a, b, c, d that actually glue the four copies of D_4 in $NS(X_4)$ all require a contribution of R_1 and R_2, which is lost in the abstract construction.

2.3.2 Invariant and co-invariant lattices for the action of τ and τ^2

From now on, denote $\Lambda_{\mathrm{K3}}^{\langle\tau\rangle}$ the invariant lattice, and Ω_4 the co-invariant lattice for the action on Λ_{K3} induced by the automorphism of order four τ. The lattice Ω_4 is a negative definite lattice of rank 14 and discriminant group $(\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ [71, §10].

The following elements are invariant for the action of τ^* on W (see (2.3.1.1)): for $i = 1, \ldots, 4$, $\kappa_i = e_i + f_i + g_i + h_i$; $\kappa_5 = a_1 + a_2$, $\kappa_6 = \rho$, $\kappa_7 = \omega_1$, $\kappa_8 = \omega_2$. These elements span a sublattice of $\Lambda_{\mathrm{K3}}^{\langle\tau\rangle}$ of index 2^3: to obtain $\Lambda_{\mathrm{K3}}^{\langle\tau\rangle}$, add the generators $(\kappa_2+\kappa_4)/2$ (that is, $\alpha+\beta+\gamma+\delta-\kappa_1-\kappa_6$), $(\kappa_5+\kappa_7+\kappa_8)/2$ and $(\kappa_1+\kappa_4+\kappa_6+\kappa_7+\kappa_8)/2$ (these are easily verified to be, in fact, integral elements in Λ_{K3}).

Its orthogonal complement Ω_4 is an overlattice of index 2^4 of the lattice

$$\Delta = \left[\begin{array}{c|c|c|c} D_4(2) & D_4 & D_4 & 0 \\ \hline D_4 & D_4(2) & D_4 & 0 \\ \hline D_4 & D_4 & D_4(2) & 0 \\ \hline 0 & 0 & 0 & \begin{array}{cc} -4 & 0 \\ 0 & -4 \end{array} \end{array}\right]$$

spanned by the elements $e_i - f_i,\ e_i - g_i,\ e_i - h_i$ for $i = 1, \ldots, 4$, $a_1 - a_2$ and σ: to obtain Ω_4, we shall add to this lattice $\alpha - \beta,\ \alpha - \gamma,\ \alpha - \delta$ and one more class in $\frac{1}{2}\Delta$ integral in Λ_{K3}, that we can choose to be $(e_2 - g_2 + e_4 - g_4 + a_1 - a_2 + \sigma)/2$.
The discriminant groups of $\Lambda_{K3}^{\langle\tau\rangle}$ and Ω_4 satisfy $A_{\Lambda_{K3}^{\langle\tau\rangle}} \simeq (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2 \simeq A_{\Omega_4}$.

The co-invariant lattice for a symplectic involution $\Omega_2 := \Omega_{\mathbb{Z}/2\mathbb{Z}}$ is isometric to the lattice $E_8(2)$ (see [32, §1.3] and the proof of [63, Thm. 5.7]). Considering the involution τ^2, Ω_2 is obviously contained in the co-invariant lattice for τ, Ω_4: in the basis of Ω_4 described above, Ω_2 is generated by the elements $\alpha - \gamma,\ \beta - \delta,\ e_1 - g_1,\ e_3 - g_3,\ f_1 - h_1,\ f_2 - h_2,\ f_3 - h_3,\ f_4 - h_4$.
Denote R the orthogonal complement to Ω_2 in Ω_4: then Ω_4 is an overlattice of $\Omega_2 \oplus R$ such that $\Omega_4/(\Omega_2 \oplus R) = (\mathbb{Z}/2\mathbb{Z})^4$.

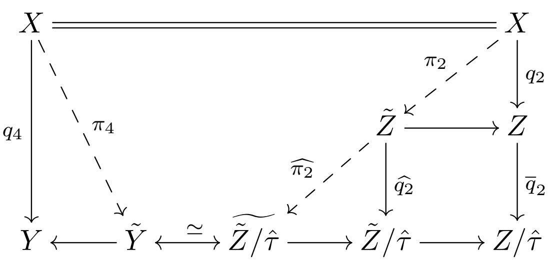
2.4 Quotients

For each of the abelian groups G that act symplectically on a K3 surface X, Nikulin provides in [71, §5-7] a description of the singular locus of the quotient surface X/G, and of the *exceptional lattice* M_G: this is the minimal primitive sublattice of Λ_{K3} containing all the exceptional curves of the minimal resolution $\widetilde{X/G}$ of X/G. Denoting $q : X \to X/G$ the quotient map, $H^2(\widetilde{X/G}, \mathbb{Z})$ is an overlattice of finite index of $q_* H^2(X, \mathbb{Z}) \oplus M_G$.

Remark 2.4.0.1. The lattices Ω_G and M_G are closely related via the quotient map: this fact allowed Nikulin to compute the rank and the discriminant group of Ω_G starting from the (simpler) exceptional lattice [71, Lemma 10.2]. When G is non-cyclic, Nikulin's results have been corrected by Garbagnati and Sarti in [30]; for a complete account of the relation between Ω_G and M_G, see Whitcher's paper [96].

Consider a K3 surface X that admits a symplectic automorphism τ of order 4, and the (singular) quotient surfaces $Y = X/\tau,\ Z = X/\tau^2$; resolve the singularities of Y and Z to obtain the K3 surfaces $\tilde{Y}$, $\tilde{Z}$: then τ induces an involution $\hat{\tau}$ on Z such that $Z/\hat{\tau} \simeq Y$, and this involution can be extended to $\tilde{Z}$, as we're going to show in the following sections.

Denote the maps between these surfaces as in the following diagram:



(2.4.0.1)

Remark 2.4.0.2. The surfaces $\tilde{Y}$ and $\widetilde{\tilde{Z}/\hat{\tau}}$ are isomorphic, because they are birationally equivalent K3 surfaces.

We can describe the maps

$$\pi_{4*} : \Lambda_{\mathrm{K3}} \simeq H^2(X,\mathbb{Z}) \xrightarrow{q_{4*}} q_{4*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Y},\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$$
$$\pi_{2*} : \Lambda_{\mathrm{K3}} \simeq H^2(X,\mathbb{Z}) \xrightarrow{q_{2*}} q_{2*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Z},\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$$

by defining them on the sublattice W (see (2.3.1.1)) in the first place, and subsequently on all of Λ_{K3} by $\mathbb{Q}$-linear extension to the elements presented in (2.3.1.2).

The description of

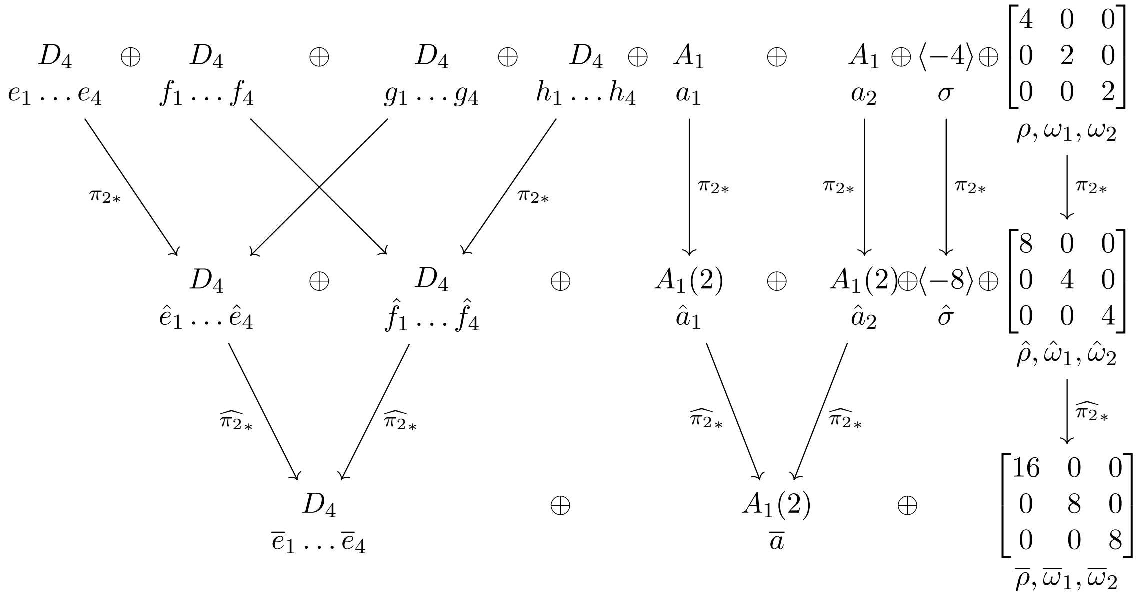
$$\widehat{\pi_2}_* : H^2(\tilde{Z},\mathbb{Z}) \to H^2(\tilde{Y},\mathbb{Z})$$

will require some more effort: in fact, $H^2(\tilde{Z},\mathbb{Z})$ is an overlattice of finite index of $q_{2*}H^2(X,\mathbb{Z}) \oplus M_{\mathbb{Z}/2\mathbb{Z}}$, while $H^2(\tilde{Y},\mathbb{Z})$ is an overlattice of finite index of $q_{4*}H^2(X,\mathbb{Z}) \oplus M_{\mathbb{Z}/4\mathbb{Z}} = (\overline{q}_2 \circ q_2)_* H^2(X,\mathbb{Z}) \oplus M_{\mathbb{Z}/4\mathbb{Z}}$; for now, notice that

$$\widehat{\pi_2}_*|_{M_{\mathbb{Z}/2\mathbb{Z}}} : M_{\mathbb{Z}/2\mathbb{Z}} \to M_{\mathbb{Z}/4\mathbb{Z}},$$
$$\widehat{\pi_2}_*|_{q_{2*}H^2(X,\mathbb{Z})} : q_{2*}H^2(X,\mathbb{Z}) \to q_{4*}H^2(X,\mathbb{Z}). \qquad (2.4.0.2)$$

2.4.1 The image of $H^2(X,\mathbb{Z})$ via the maps π_{4*} and π_{2*}

Proposition 2.4.1.1. *The maps $\pi_{2*}, \widehat{\pi_2}_*$ and $\pi_{4*} = \widehat{\pi_2}_* \circ \pi_{2*}$ act in the following way on W and its image in $\pi_{2*}H^2(X,\mathbb{Z})$:*



Proof. The action of τ^* on W is given in (2.3.1.1): we can use it to compute the intersection form of $\pi_{4*}W$ via the push-pull formula. Since π_4 is a finite morphism of degree 4, we get

$$\pi_{4*}x \cdot \pi_{4*}y = \frac{1}{4}(\pi_4^*\pi_{4*}x \cdot \pi_4^*\pi_{4*}y)$$

where $\pi_4^*\pi_{4*}x = \sum_{k=0}^{3}(\tau^k)^*(x)$. The intersection form of $\pi_{2*}W$ can be similarly determined. □

Corollary 2.4.1.2. *The embedding $q_{4*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Y},\mathbb{Z})$ is unique up to isometries of the latter, and the same holds for $q_{2*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Z},\mathbb{Z})$.*

Proof. Computing $q_{4*}H^2(X,\mathbb{Z})$ and $q_{2*}H^2(X,\mathbb{Z})$ by $\mathbb{Q}$-linear extension of the maps in the proposition above to the elements in (2.3.1.2), it can be seen that the lattice $q_{4*}H^2(X,\mathbb{Z})$ is even, indefinite, and it has $rk = 8$ and $\ell = 4$, so it satisfies the conditions of Theorem 1.2.1.10; similarly, $q_{2*}H^2(X,\mathbb{Z})$ is even, indefinite, it has $rk = 14$ and $\ell = 6$, and the same result can be applied. □

2.4.2 The resolution of $Z = X/\tau^2$, and the lattice Γ_4

For any symplectic involution ι on a K3 surface X, the quotient surface X/ι has 8 isolated singularities, that are ordinary double points [71, §5]: to resolve it, it is sufficient to blow up these points once.

Definition 2.4.2.1 ([71, Def. 6.2, case 1a]). Denote *Nikulin lattice* the lattice $N \coloneqq M_{\mathbb{Z}/2\mathbb{Z}}$: if $\{n_1, \dots, n_8\}$ is a $\mathbb{Z}$-basis of $A_1^{\oplus 8}$, then a set of generators over $\mathbb{Z}$ for N can be obtained by adding to this list the element $\nu = (n_1 + \dots + n_8)/2$.

The second integral cohomology of the K3 surface $\tilde{Z}$, the minimal resolution of the quotient $Z = X/\tau^2$, can be described as an overlattice of index 2^6 of $\pi_{2*}H^2(X,\mathbb{Z}) \oplus N_Z$, where N_Z is a copy of the Nikulin lattice: this is done via an isomorphism of the discriminant groups of $\pi_{2*}H^2(X,\mathbb{Z})$ and N_Z, as described in Theorem 1.2.1.5. The generators that we need to add are the following:

$$\begin{aligned}
z_1 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{a}_2 + \hat{\eta})/2 + (n_2 + n_8)/2 \\
z_2 &= \hat{\varepsilon}/2 + (n_3 + n_8)/2 \\
z_3 &= \hat{a}_2/2 + (n_4 + n_8)/2 \\
z_4 &= (\hat{\varepsilon} + \hat{a}_2 + \hat{\rho})/2 + (n_5 + n_8)/2 \\
z_5 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{\rho} + \hat{\chi} + \hat{\eta})/2 + (n_6 + n_8)/2 \\
z_6 &= (\hat{\beta} + \hat{f}_1 + \hat{f}_2 + \hat{\varepsilon} + \hat{a}_2 + \hat{\rho} + \hat{\zeta})/2 + (n_7 + n_8)/2,
\end{aligned} \tag{2.4.2.1}$$

where $\hat{\beta} = \pi_{2*}\beta$ $(= \pi_{2*}\delta)$, and similarly the cap over the other elements of Λ_{K3} denotes their image via π_{2*}. These generators were already known in the general case of a symplectic involution [32, Lemma 1.10].

Remark 2.4.2.2. The lattice $\pi_{2*}\Omega_4$ is isomorphic to $D_6(2)$ with the following generators: $d_1 = -(\hat{a}_1 - \hat{a}_2 + \hat{\sigma})/2$, $d_2 = (\hat{a}_1 - \hat{a}_2 - \hat{\sigma})/2$, $d_3 = \hat{\alpha} - \hat{\beta}$, $d_4 = \hat{e}_4 - \hat{f}_4$, $d_5 = \hat{e}_3 - \hat{f}_3$, $d_6 = \hat{e}_1 - \hat{f}_1$.

Definition 2.4.2.3. Define Γ_4 the lattice, of rank 14 and discriminant group $(\mathbb{Z}/2\mathbb{Z})^6 \times (\mathbb{Z}/4\mathbb{Z})^2$, obtained as an overlattice of $\pi_{2*}\Omega_4 \oplus N_Z$ by adding to the list of generators the elements

$$x_1 = \frac{n_3 + n_4 + n_5 + n_8 + d_1 + d_2}{2}, \qquad x_2 = \frac{n_3 + n_4 + d_1 + d_4 + d_6}{2}. \tag{2.4.2.2}$$

The lattice Γ_4 can be primitively embedded in $H^2(\tilde{Z},\mathbb{Z})$ with the lattice $S = \langle \hat{e}_1 + \hat{f}_1,\ \hat{\alpha} + \hat{\beta},\ \hat{e}_3 + \hat{f}_3,\ \hat{e}_2 + \hat{f}_2,\ \hat{\rho},\ (\hat{a}_1 + \hat{a}_2 + \hat{\rho})/2, (\hat{\rho} + \hat{\omega}_1 + \hat{\omega}_2)/2, \hat{\omega}_2 \rangle$ as orthogonal complement; indeed, we can obtain $H^2(\tilde{Z},\mathbb{Z})$ as overlattice of finite index of $S \oplus \Gamma_4$ by adding the generators:

$$\begin{aligned}
z_1' &= (\hat{\alpha} + \hat{\beta})/2 + (\hat{\alpha} - \hat{\beta})/2, \\
z_2' &= (\hat{e}_3 + \hat{f}_3)/2 + (\hat{e}_3 - \hat{f}_3)/2, \\
z_3' &= (\hat{e}_2 + \hat{f}_2)/2 + (\hat{e}_4 - \hat{f}_4)/2 + (n_3 + n_4 + n_5 + n_8)/2, \\
z_4' &= (\hat{a}_1 + \hat{a}_2 + \hat{\rho})/4 + (n_2 + n_3 + n_4 + n_6)/2 + (\hat{e}_1 - \hat{f}_1 + \hat{e}_4 - \hat{f}_4)/2, \\
z_5' &= (\hat{\rho} + \hat{\omega}_1 + \hat{\omega}_2)/4 + (n_2 + n_7)/2 + (\hat{e}_1 - \hat{f}_1 + \hat{e}_4 - \hat{f}_4)/2, \\
z_6' &= \hat{\omega}_2/2 + (n_2 + n_6)/2, \\
z_7' &= (\hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + (\hat{a}_1 + \hat{a}_2 + \hat{\rho})/2 + \hat{\omega}_2)/4+ \\
&\quad + (\hat{\alpha} - \hat{\beta})/2 + (3n_5 + 2n_6 + n_8)/4 + (x_1 + x_2)/2, \\
z_8' &= \hat{\rho}/4 + (2n_2 + n_3 + 3n_4 + n_5 + 2n_6 + 3n_8)/4 + x_1/2.
\end{aligned} \tag{2.4.2.3}$$

Remark 2.4.2.4. The primitive embedding $\Gamma_4 \hookrightarrow \Lambda_{K3}$ is unique up to isometries of Λ_{K3}, because Theorem 1.2.1.8 holds: in fact the orthogonal complement S of Γ_4 satisfies the first condition of Proposition 1.2.1.11.

2.4.3 The map $\widehat{\pi_2}_*$ and the resolution of $Y = X/\tau$

The action of a symplectic automorphism of order 4 τ on a K3 surface X has always exactly eight isolated points on X with non trivial stabilizer: four of them are fixed by τ, and four more exchanged by τ (so they are fixed by τ^2) [71, §5, case 2]; therefore the singular locus of the quotient X/τ consists of six isolated points, two of which are resolved by blowing up once (thus introducing two rational curves in the quotient surface), and each of the other four introducing three curves in A_3 configuration.
The exceptional lattice $M_4 := M_{\mathbb{Z}/4\mathbb{Z}}$ therefore satisfies $M_4 \otimes \mathbb{Q} = (A_3^{\oplus 4} \oplus A_1^{\oplus 2}) \otimes \mathbb{Q}$; calling $\tilde{m}^1$, $\tilde{m}^2$ the generators of the two copies of A_1, and m_1^i, m_2^i, m_3^i the generators of the i-th copy of A_3 numbered as in Example 1.2.0.2, then a set of $\mathbb{Z}$-generators for M_4 consists of all these elements, and (see [71, §6, Def. 6.2, case 1b]) the class

$$\mu = \frac{\sum_{i=1}^4 (m_1^i + 2m_2^i + 3m_3^i)}{4} + \frac{\tilde{m}^1 + \tilde{m}^2}{2}. \qquad (2.4.3.1)$$

We can now describe the second integral cohomology of the K3 surface $\tilde{Y}$, the minimal resolution of the quotient $Y = X/\tau$: the discriminant group of each of the orthogonal summands M_4 and $\pi_{4*}H^2(X, \mathbb{Z})$ is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$, and the elements that define Λ_{K3} as an overlattice of $M_4 \oplus \pi_{4*}H^2(X, \mathbb{Z})$ are:

$$\begin{aligned}
y_1 &= \left(m_1^2 + 2m_2^2 + 3m_3^2 + m_1^3 + 2m_2^3 + 3m_3^3\right)/4 + \tilde{m}^2/2 + (\overline{e}_1 + \overline{e}_4 + \overline{\zeta} + \overline{\eta})/2 + (\overline{a} + \overline{\chi})/4 \\
y_2 &= (m_1^3 + 2m_2^3 + 3m_3^3 + m_1^4 + 2m_2^4 + 3m_3^4)/4 + \tilde{m}^2/2 + (\overline{a} + \overline{\zeta} + 3\overline{\eta})/4 \\
y_3 &= (m_1^4 + m_3^4 + \tilde{m}^2)/2 + (\overline{e}_1 + \overline{e}_4 + \overline{\alpha} + \overline{\chi} + \overline{\zeta})/2 \qquad (2.4.3.2) \\
y_4 &= (\tilde{m}^1 + \tilde{m}^2)/2 + \overline{a}/2,
\end{aligned}$$

where $\overline{e}_i = \pi_{4*}e_i = \pi_{4*}f_i = \pi_{4*}g_i = \pi_{4*}h_i$, $\overline{a} = \pi_{4*}a_1 = \pi_{4*}a_2$, and similarly $\overline{\star} = \pi_{4*}(\star)$.

The exceptional lattice M_4 can be also computed from the image of N_Z via $\widehat{\pi_2}_*$ (see Rmk. 2.4.0.2) with the resolution of the singularities that arise from the quotient: in fact, the involution $\hat{\tau}$ (that is induced on $\tilde{Z}$ by the action of τ on X) acts by fixing two points on each of the the four exceptional curves of $\tilde{Z}$ corresponding to the four points of X fixed by τ, and by exchanging pairwise the remaining four exceptional curves (these correspond to the four points fixed only by τ^2). Therefore, the invariant lattice for the action of $\hat{\tau}^*$ on N_Z is the sublattice spanned by the four invariant curves, and the sum of the pairs of exchanged curves.

Remark 2.4.3.1. In the lattice Γ_4 (see Def. 2.4.2.3) the orthogonal complement of the invariant lattice for the action of $\hat{\tau}^*$ is a copy of Ω_2: this is indicative of the fact that the surface $\tilde{Z}$ admits a symplectic involution, which is indeed $\hat{\tau}$. The curves of N_Z

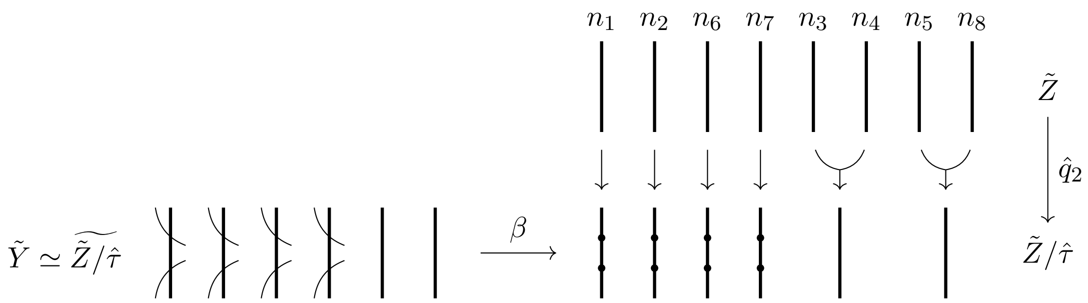
were numbered such that the gluing between N_Z and $\pi_{2*}H^2(X,\mathbb{Z})$ be described by the elements in (2.4.2.1): since the action of $\hat{\tau}^*$ on $\pi_{2*}H^2(X,\mathbb{Z})$ determines the action of $\hat{\tau}^*$ on N_Z via this gluing, we find accordingly that $\hat{\tau}^*$ fixes n_1, n_2, n_6, n_7 and exchanges n_3 with n_4, n_5 with n_8.

Proposition 2.4.3.2. *It holds $\widehat{\pi_2}_*(\Gamma_4) \subset M_4$: more precisely, $\widehat{\pi_2}_*$ annihilates $\hat{R}$, because $\widehat{\pi_2}_*\hat{R} = \widehat{\pi_2}_*\pi_{2*}\Omega_4 = \pi_{4*}\Omega_4 = 0$, and is defined on the $\mathbb{Q}$-generators of N_Z as follows:*

$$\begin{aligned}
&\overline{n}_1 := \widehat{\pi_2}_* n_1 = m_1^1 + 2m_2^1 + m_3^1, \quad &&\overline{n}_2 := \widehat{\pi_2}_* n_2 = m_1^3 + 2m_2^3 + m_3^3,\\
&\overline{n}_6 := \widehat{\pi_2}_* n_6 = m_1^2 + 2m_2^2 + m_3^2, \quad &&\overline{n}_7 := \widehat{\pi_2}_* n_7 = m_1^4 + 2m_2^4 + m_3^4,\\
&\overline{n}_3 := \widehat{\pi_2}_* n_3 = \widehat{\pi_2}_* n_4 = \tilde{m}^1, \quad &&\overline{n}_5 := \widehat{\pi_2}_* n_5 = \widehat{\pi_2}_* n_8 = \tilde{m}^2,
\end{aligned}$$

so that $\widehat{\pi_2}_* A_1^{\oplus 8} = A_1(2)^{\oplus 4} \oplus A_1^{\oplus 2}$.

Proof. Let $k = 1, 2, 6, 7$, $j = 1, \ldots, 8$. The surface $\tilde{Z}/\hat{\tau}$ is singular in eight points, two on each of the curves $\hat{q}_{2*}n_k$; consider the blow-up $\beta : \tilde{Y} \to \tilde{Z}/\hat{\tau}$ of the singular points: then, the curve $\overline{n}_j := \widehat{\pi_2}_* n_j$ is the pullback of $\hat{q}_{2*}n_j$. By push-pull we have $\overline{n}_k^2 = -4$, $\overline{n}_3^2 = -2 = \overline{n}_5^2$. Consider the following diagram:



Firstly, notice that either $(\overline{n}_3, \overline{n}_5) = (\tilde{m}^1, \tilde{m}^2)$ or $(\overline{n}_3, \overline{n}_5) = (\tilde{m}^2, \tilde{m}^1)$.
The eight exceptional curves $\{m_1^i,\ m_3^i\}_{i=1}^4$ introduced with the blow-up β, together with the element

$$\nu_Y = \frac{\sum_{i=1}^4 (m_1^i + m_3^i)}{2},$$

span a new copy of the Nikulin lattice $N_Y \subset M_4$; the class $\overline{n}_k$ is by definition orthogonal to the exceptional curves: thus we find that $\overline{n}_k = m_1^i + 2m_2^i + m_3^i$ for some i. To determine which copy of A_3, A_1 in M_4 each $\overline{n}_j$ corresponds to, we still have to require that the image of the elements z_i defined in (2.4.2.1) be integral in $H^2(\tilde{Y},\mathbb{Z})$; this forces the definition of $\widehat{\pi_2}_*$ as stated. □

Corollary 2.4.3.3. *The lattice M_4 is an overlattice of index 2^5 of the lattice $\widehat{\pi_2}_* N_Z \oplus N_Y$, obtained by adding as generators the elements $m_2^1 = (\overline{n}_1 - m_1^1 - m_3^1)/2$, $m_2^2 = (\overline{n}_6 - m_1^2 - m_3^2)/2$, $m_2^3 = (\overline{n}_2 - m_1^3 - m_3^3)/2$, $m_2^4 = (\overline{n}_7 - m_1^4 - m_3^4)/2$ and μ (see (4.1.2.1)).*

Proof. The element μ defined in (4.1.2.1) is

$$\mu = \frac{\nu_Y + m_2^1 + m_2^2 + m_2^3 + m_2^4 + m_3^1 + m_3^2 + m_3^3 + m_3^4 + \tilde{m}^1 + \tilde{m}^2}{2},$$

so it gives an overlattice (of index 2) of the overlattice (of index 2^4) of $\widehat{\pi_2}_* N_Z \oplus N_Y$ generated by the m_2^i, $i = 1, \dots, 4$. □

Notice that we have the following equalities:

$$\begin{aligned} H^2(\tilde{Y}, \mathbb{Q}) &= (\pi_{4*} H^2(X, \mathbb{Z}) \oplus M_4) \otimes \mathbb{Q} = (\pi_{4*} H^2(X, \mathbb{Z}) \oplus \widehat{\pi_2}_* N_Z \oplus N_Y) \otimes \mathbb{Q} = \\ &= (\widehat{\pi_2}_*(\pi_{2*} H^2(X, \mathbb{Z}) \oplus N_Z) \oplus N_Y) \otimes \mathbb{Q} = (\widehat{\pi_2}_* H^2(\tilde{Z}, \mathbb{Z}) \oplus N_Y) \otimes \mathbb{Q}. \end{aligned}$$

Working on $\mathbb{Z}$, we recover the first equality using the y_i's in (2.4.3.2), and the second one as in Proposition 2.4.3.3; the next one is trivial, and for the last one we use the z_i's in (2.4.2.1). Thus, the lattice $H^2(\tilde{Y}, \mathbb{Z})$ can be also described directly as an overlattice of finite index of $\widehat{\pi_2}_* H^2(\tilde{Z}, \mathbb{Z}) \oplus N_Y$, allowing for an easier computation of the map $\widehat{\pi_2}^*$ in Section 2.4.4: to do this, use for $\widehat{\pi_2}_* H^2(\tilde{Z}, \mathbb{Z})$ the $\mathbb{Z}$-basis $\{\overline{e}_1, \overline{\alpha}, \overline{e}_3, \overline{e}_4, \overline{a}, \overline{\chi}, \overline{\zeta}, \overline{\eta}, \widehat{\pi_2}_* n_6, \widehat{\pi_2}_* \nu, \widehat{\pi_2}_* z_1, \widehat{\pi_2}_* z_2, \widehat{\pi_2}_* z_5, \widehat{\pi_2}_* z_6\}$, with the z_i's defined in (2.4.2.1), and for N_Y the $\mathbb{Z}$-basis $\{\nu_Y, m_3^1, m_1^2, m_3^2, m_1^3, m_3^3, m_1^4, m_3^4\}$; then, the gluing elements are

$$\begin{aligned} y'_1 &= (\overline{a} + \overline{\chi} + \widehat{\pi_2}_* \nu + m_3^1 + m_1^2 + m_1^3 + m_3^4)/2 \\ y'_2 &= (\overline{\alpha} + \overline{\zeta} + \widehat{\pi_2}_* z_1 + \widehat{\pi_2}_* z_5 + m_3^2 + m_3^3)/2 \\ y'_3 &= (\overline{\chi} + \overline{\zeta} + \overline{\eta} + \widehat{\pi_2}_* n_6 + m_1^4 + m_3^4)/2 \\ y'_4 &= (\overline{e}_4 + \overline{a} + \overline{\eta} + \widehat{\pi_2}_* z_1 + \widehat{\pi_2}_* z_6 + m_3^3 + m_3^4)/2 \\ y'_5 &= (\overline{\alpha} + \overline{a} + \overline{\eta} + \widehat{\pi_2}_* z_1 + \widehat{\pi_2}_* z_5 + m_3^2 + m_1^3 + m_1^4 + m_3^4)/2 \\ y'_6 &= (\widehat{\pi_2}_* n_6 + m_1^2 + m_3^2)/2. \end{aligned} \tag{2.4.3.3}$$

2.4.4 The dual maps

We're now going to define the dual maps

$$\begin{aligned} \pi_4^* &: H^2(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \\ \widehat{\pi_2}^* &: H^2(\tilde{Y}, \mathbb{Z}) \to H^2(\tilde{Z}, \mathbb{Z}) \\ \pi_2^* &: H^2(\tilde{Z}, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \end{aligned}$$

using the descriptions of $H^2(\tilde{Y}, \mathbb{Z})$ as an overlattice respectively of $\pi_{4*}\Lambda_{\mathrm{K3}} \oplus M_4$ and $\widehat{\pi_2}_*\Lambda_{\mathrm{K3}} \oplus N_Y$, and of $H^2(\tilde{Z}, \mathbb{Z})$ as an overlattice of $\pi_{2*}\Lambda_{\mathrm{K3}} \oplus N_Z$. These maps are used in Section 2.6 to find the dimension of the eigenspaces for the action induced by the automorphism τ on $H^0(X, L)$ for any possible choice of polarization L on X, and they are given here explicitly for completeness of our exposition.

Proposition 2.4.4.1. *1. The map π_4^* annihilates M_4, and acts on $\pi_{4*}W \subset \pi_{4*}\Lambda_{\mathrm{K3}}$ as*

$$\begin{array}{ccccccc} \pi_4{}^*: & D_4 \oplus A_1(2) \oplus \begin{bmatrix} 16 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} & \longrightarrow & D_4^{\oplus 4} & \oplus & A_1^{\oplus 2} \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} \begin{matrix} \overline{e}_1 \\ \overline{e}_2 \\ \overline{e}_3 \\ \overline{e}_4 \end{matrix}, & \overline{a}, & \overline{\rho}, \overline{\omega}_1, \overline{\omega}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \begin{matrix} e_1+f_1+g_1+h_1 \\ e_2+f_2+g_2+h_2 \\ e_3+f_3+g_3+h_3 \\ e_4+f_4+g_4+h_4 \end{matrix}, & 2a_1+2a_2, & 4\rho, 4\omega_1, 4\omega_2 \end{pmatrix}$$

Its action can be extended to $\pi_{4*}\Lambda_{\mathrm{K3}}$ adding these elements (and their respective images to the image lattice): $\overline{\alpha} = \overline{\rho}/4 + \overline{e}_1 + (\overline{e}_2 + \overline{e}_4)/2$, $\overline{\chi} = \overline{\rho}/2$, $\overline{\zeta} = \overline{e}_1 + (\overline{e}_2 + \overline{e}_4 + \overline{a} + \overline{\omega}_1)/2$, $\overline{\eta} = \overline{e}_1 + (\overline{e}_2 + \overline{e}_4 + \overline{a} + \overline{\omega}_2)/2$; to extend the action to $H^2(\tilde{Y}, \mathbb{Z})$, add also $y_1, \ldots, y_4$ (see (2.4.3.2)).

2. The map π_2^* annihilates N, and acts on $\pi_{2*}W \subset \pi_{2*}\Lambda_{\mathrm{K3}}$ as

$$\begin{array}{ccccccc} \pi_2{}^*: & D_4^{\oplus 2} \oplus A_1(2)^{\oplus 2} \oplus \langle -8 \rangle \oplus \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} & \longrightarrow & D_4^{\oplus 4} & \oplus & A_1^{\oplus 2} \oplus \langle -4 \rangle \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} \begin{matrix} \hat{e}_1, \hat{f}_1 \\ \hat{e}_2, \hat{f}_2 \\ \hat{e}_3, \hat{f}_3 \\ \hat{e}_4, \hat{f}_4 \end{matrix}, & \hat{a}_1, \hat{a}_2, & \hat{\sigma}, & \hat{\rho}, \hat{\omega}_1, \hat{\omega}_2 \end{pmatrix} \mapsto \begin{pmatrix} \begin{matrix} e_1+g_1, f_1+h_1 \\ e_2+g_2, f_2+h_2 \\ e_3+g_3, f_3+h_3 \\ e_4+g_4, f_4+h_4 \end{matrix}, & 2a_1, 2a_2, & 2\sigma, & 2\rho, 2\omega_1, 2\omega_2 \end{pmatrix}$$

Its action can be extended to $\pi_{2*}\Lambda_{\mathrm{K3}}$ adding the following elements (and their respective image to the image lattice): $\hat{\alpha} = (\hat{\rho} + \hat{\sigma})/4 + (\hat{e}_1 + \hat{e}_2 + \hat{f}_1 + \hat{f}_4)/2$, $\hat{\beta} = (\hat{\rho} - \hat{\sigma})/4 + (\hat{e}_1 + \hat{e}_4 + \hat{f}_1 + \hat{f}_2)/2$, $\hat{\varepsilon} = \hat{\rho}/2 + \hat{f}_2 + \hat{f}_4 + (\hat{a}_1 + \hat{a}_2)/2$, $\hat{\chi} = (\hat{\rho} + \hat{\sigma})/2$, $\hat{\zeta} = (\hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_2 + \hat{a}_1 + \hat{\omega}_1)/2$, $\hat{\eta} = (\hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_2 + \hat{a}_2 + \hat{\omega}_2)/2$; to extend the action to $H^2(\tilde{Z}, \mathbb{Z})$, add also $z_1, \ldots, z_6$ (see (2.4.2.1)).

3. Recall from Corollary 2.4.3.3 that $H^2(\tilde{Y}, \mathbb{Z})$ is an overlattice of finite index of $\widehat{\pi_2}_* H^2(\tilde{Z}, \mathbb{Z}) \oplus N_Y$. The lattice $N_Y \subseteq H^2(\tilde{Y}, \mathbb{Z})$ is annihilated by $\widehat{\pi_2}^*$, and for the generators of $\widehat{\pi_2}_* H^2(\tilde{Z}, \mathbb{Z})$ the map $\widehat{\pi_2}^*$ is defined as follows:

$$\begin{aligned}
&\widehat{\pi_2}^* \overline{e}_i = \hat{e}_i + \hat{f}_i \text{ (for } i = 1, \ldots, 4, \text{ but } \overline{e}_2 \text{ is not needed as generator)}, \\
&\widehat{\pi_2}^* \overline{\zeta} = \hat{e}_1 + \hat{f}_1 + (\hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + 2\hat{\omega}_1)/2, \\
&\widehat{\pi_2}^* \overline{\eta} = \hat{e}_1 + \hat{f}_1 + (\hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + 2\hat{\omega}_2)/2, \\
&\widehat{\pi_2}^* \overline{\alpha} = \hat{\alpha} + \hat{\beta}, \quad \widehat{\pi_2}^* \overline{a} = \hat{a}_1 + \hat{a}_2, \quad \widehat{\pi_2}^* \overline{\chi} = \widehat{\pi_2}^* \overline{\rho}/2 = \hat{\rho}, \\
&\widehat{\pi_2}^* \widehat{\pi_2}_* n_6 = 2n_6, \quad \widehat{\pi_2}^* \widehat{\pi_2}_* \nu = 2\nu, \\
&\widehat{\pi_2}^* \widehat{\pi_2}_* z_1 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + \hat{a}_1 + \hat{a}_2 + \widehat{\pi_2}^* \overline{\eta} + 2n_2 + n_5 + n_8)/2, \\
&\widehat{\pi_2}^* \widehat{\pi_2}_* z_2 = (\hat{\rho} + \hat{e}_2 + \hat{f}_2 + \hat{e}_4 + \hat{f}_4 + \hat{a}_1 + \hat{a}_2 + n_3 + n_4 + n_5 + n_8)/2, \\
&\widehat{\pi_2}^* \widehat{\pi_2}_* z_5 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_2 + \hat{f}_2 + \hat{\rho} + \widehat{\pi_2}^* \overline{\eta} + 2n_6 + n_5 + n_8)/2 + \hat{\rho},
\end{aligned}$$

$$\widehat{\pi_2}^*\widehat{\pi_2}_* z_6 = (\hat{\alpha} + \hat{\beta} + \hat{e}_1 + \hat{f}_1 + \hat{e}_4 + \hat{f}_4 + \hat{\rho} + \widehat{\pi_2}^*\overline{\zeta} + 2n_7 + n_5 + n_8)/2+ \\ + \hat{a}_1 + \hat{a}_2 + \hat{e}_2 + \hat{f}_2 + \hat{\rho}.$$

Notice that $\widehat{\pi_2}^*\widehat{\pi_2}_* z_i = z_i + \hat{\tau}^* z_i$, *and that to obtain the whole image of* $\widehat{\pi_2}^*$ *the images of the elements* y_i' *of* (2.4.3.3) *are also to be considered.*

Proof. We are going to prove only that the map π_4^* acts on $\pi_{4*}W$ as stated above; the other cases are similar.
Since π_4^* and π_{4*} are dual maps, $\pi_4^* a = b$ if and only if $(b \cdot c)_X = (a \cdot \pi_{4*}c)_Y$ for every $a \in \pi_{4*}W$, $c \in W$: hence $\pi_4^* a \cdot c = a \cdot \pi_{4*}c$. Take $e_j = (e_j, 0, 0, 0) \in D_4^{\oplus 4}$: then $\pi_4^*\overline{e}_i \cdot e_j = \overline{e}_i \cdot \pi_{4*}e_j = \overline{e}_i \cdot \overline{e}_j$, but it holds also $\overline{e}_i \cdot \overline{e}_j = \overline{e}_i \cdot \pi_{4*}f_j = \overline{e}_i \cdot \pi_{4*}g_j = \overline{e}_i \cdot \pi_{4*}h_j$; therefore $\pi_4^*\overline{e}_i = e_i + f_i + g_i + h_i$. Take $a_1 = (a_1, 0) \in A_1^{\oplus 2}$: then $\pi_4^*\overline{a} \cdot a_1 = \overline{a} \cdot \pi_{4*}a_1 = 2(\overline{a} \cdot \overline{a})$ because π_{4*} doubles the intersection form on $A_1 \oplus A_1$; moreover, $\pi_{4*}a_1 = \overline{a} = \pi_{4*}a_2$, therefore we get $\pi_4^*\overline{a} = 2a_1 + 2a_2$. Similarly, since π_{4*} multiplies by 4 the intersection form of the sublattice of W invariant for the action of τ, we can conclude that $\pi_4^*\overline{\rho} = 4\rho_1$, $\pi_4^*\overline{\omega}_1 = 4\omega_1, \pi_4^*\overline{\omega}_2 = 4\omega_2$. □

Corollary 2.4.4.2. *The image of* $H^2(\tilde{Y}, \mathbb{Z})$ *via the map* π_4^* *coincides with the invariant lattice* $\Lambda_{\mathrm{K3}}^{\langle\tau\rangle}$ *described in Section 2.3.2. In other words, it holds*

$$\pi_4^* H^2(\tilde{Y}, \mathbb{Z}) = \Omega_4^{\perp_{\Lambda_{\mathrm{K3}}}}.$$

Similarly, we obtain:

$$\pi_2^* H^2(\tilde{Z}, \mathbb{Z}) = \Omega_2^{\perp_{H^2(X,\mathbb{Z})}} = \Omega_2^{\perp_{\Lambda_{\mathrm{K3}}}},$$
$$\widehat{\pi_2}^* H^2(\tilde{Y}, \mathbb{Z}) = \Omega_2^{\perp_{H^2(\tilde{Z},\mathbb{Z})}} = \Omega_2^{\perp_{\Lambda_{\mathrm{K3}}}}.$$

Proof. It holds $\pi_4^*\overline{e}_i = e_i + f_i + g_i + h_i$ for $i = 1, \ldots, 4$, $\pi_4^*\overline{a} = 2(a_1 + a_2)$, $\pi_4^*\overline{\rho} = 4\rho$, $\pi_4^*\overline{\omega}_1 = 4\omega_1$, $\pi_4^*\overline{\omega}_2 = 4\omega_2$; however, these elements generate only a sublattice of finite index of $\pi_4^* H^2(\tilde{Y}, \mathbb{Z})$: adding as generators the images via π_4^* of the elements $\overline{\alpha}, \overline{\chi}, \overline{\zeta}, \overline{\eta}$ and $y_1, \ldots, y_4$, we obtain the whole invariant lattice for the action of τ on Λ_{K3}. □

2.4.5 Characterization of the surface $\tilde{Z}$: the non-projective case

Nikulin's seminal work [71] provides a lattice theoretic characterization of K3 surfaces X that admit a symplectic action of a cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ (for $n = 2, \ldots, 8$), and of surfaces $\tilde{Y}$ that are the resolution of the quotient X/G, by providing a relation between the lattices Ω_G and M_G that have to be primitively embedded in their respective Néron-Severi lattices; we want to show that similarly, in the case $G = \mathbb{Z}/4\mathbb{Z}$ (generated by an automorphism τ of X), the lattice Γ_4 characterizes the surface $\tilde{Z}$ that is the resolution of $Z \coloneqq X/\tau^2$. For simplicity, we are going to state our result for the most general K3 surface X: in this case, both $NS(X) = \Omega_4$ and $NS(\tilde{Y}) = M_4$ have rank 14.

Theorem 2.4.5.1. *Let $\tilde{Z}$ be a K3 surface such that $rk(NS(\tilde{Z})) = 14$. There exists a pair (X, τ) where X is a K3 surface and τ is a symplectic automorphism of order 4 such that $\tilde{Z}$ is birationally equivalent to the quotient X/τ^2 if and only if $NS(\tilde{Z}) = \Gamma_4$ (see Def. 2.4.2.3).*

Proof. The "only if" is true by construction (see Section 2.4.2). Conversely, suppose $NS(\tilde{Z}) = \Gamma_4$. The embedding $\Omega_2 \subset \Gamma_4$ described in Remark 2.4.3.1 defines a symplectic involution $\hat{\tau}$ on $\tilde{Z}$, and the Néron-Severi lattice of the resolution $\widetilde{\tilde{Z}/\hat{\tau}}$ is naturally a copy of M_4, as proved in Corollary 2.4.3.3; therefore, by the results of Nikulin the surface $\widetilde{\tilde{Z}/\hat{\tau}}$ is the resolution of the quotient of a K3 surface X for a symplectic automorphism τ of order 4, and it holds $NS(X) = \Omega_4$. The action of τ on Ω_4 naturally defines a copy of $\Omega_2 \subset \Omega_4$ by $\Omega_2 = (\Omega_4^{\tau^2})^{\perp_{\Omega_4}}$, as described in Section 2.3.2; taking the quotient map $\pi_2 : X \to X/\tau^2$ and the resolution $\widetilde{X/\tau^2}$, it holds $NS(\widetilde{X/\tau^2}) \simeq NS(\tilde{Z})$. □

2.5 Projective families of K3 surfaces with a symplectic automorphism of order 4 and their quotients

It was already known by Nikulin that the correspondence between surfaces X that admit a symplectic action of an abelian group G, and surfaces $\tilde{Y}$ that are the resolution of X/G, is actually a moduli spaces correspondence [71, Prop. 2.9]; the same idea was later generalized to the non-abelian case by Whitcher [96, §3].
We can therefore refine the characterization of X, $\tilde{Z}$ and $\tilde{Y}$ by their Néron-Severi to the projective case. The approach we follow mimics the one used in [32],[29] for symplectic involutions, and in [26] for symplectic automorphisms of order 3.

Proposition 2.5.0.1 (see [71, Prop. 2.9]; also [32, Prop. 2.2])**.** *Projective K3 surfaces X that admit a symplectic action of an abelian group G are polarized with a lattice of rank $1 + rk(\Omega_G)$ that contains primitively both the lattice Ω_G and a class L of square $2d$; projective K3 surfaces that are the resolution of X/G are polarized with a lattice of rank $1 + rk(M_G)$ that contains primitively both the lattice M_G and a class H of square $2e$.*
Moreover, from Theorem 2.4.5.1 we deduce that in the projective case $\tilde{Z}$ is polarized with a lattice of rank 15 that contains primitively both the lattice Γ_4 and a class K of square $2f$.

Lemma 2.5.0.2. *Let X be a general projective K3 surface admitting a symplectic action of G, so $NS(X)$ has signature $(1, rk(\Omega_G))$. Then we may assume that $L = \Omega_G^{\perp_{NS(X)}}$ is ample.*

Proof. We may assume that L is effective up to a sign change, because $L^2 > 0$. Then, since there are no (-2)-classes in $L^\perp = \Omega_G$ [71, Thm. 4.3], any (-2)-curve has class of the form $nL + w$ with $n \in \mathbb{N}$ and $w \in \Omega_G$: classes of this form have positive intersection with L, so L is ample by the Nakai-Moishezon criterion. □

Remark 2.5.0.3. Let S be either Ω_G or M_G: the only lattices that satisfy the proposition above are $S \oplus \langle 2d \rangle$ and its cyclic overlattices of finite index [30, Prop. 6.1]. Each non-isomorphic primitive embedding of any of these lattices in Λ_{K3} gives a different irreducible component of a moduli space of projective K3 surfaces: either of surfaces X that admit a symplectic action of G (if $S = \Omega_G$), or of surfaces $\tilde{Y}$ that are the minimal resolution of X/G (if $S = M_G$).
For $G = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ it is already known that there exists a bijection between the irreducible components of the moduli spaces of X and $\tilde{Y}$ ([32], [26]). We are going to show that this also holds for $\mathbb{Z}/4\mathbb{Z}$, but not when considering irreducible components of the moduli spaces of X (or $\tilde{Y}$) and of intermediate quotient surfaces $\tilde{Z}$.

Remark 2.5.0.4. If $G = \mathbb{Z}/4\mathbb{Z}$, we have to study $S = \Omega_4,\ M_4,\ \Gamma_4$ (see Sections 2.3.2, 2.4.3, 2.4.2). The moduli spaces of projective K3 surfaces X that admit a symplectic automorphism τ of order 4, and projective K3 surfaces are the resolution of X/τ^2 or X/τ all have dimension 5.

Remark 2.5.0.5. **Notation.** Consider the lattice $S \oplus \langle k \rangle$, where S is a negative definite even lattice and $\langle k \rangle$ is an even positive definite lattice with intersection matrix $[k]$.
Denote $(S \oplus \langle k \rangle)'$ and $(S \oplus \langle k \rangle)^\star$ any cyclic overlattices of $S \oplus \langle k \rangle$ obtained by adding to the list of generators a class of the form $(s+\kappa)/2,\ (s+\kappa)/4$ respectively, with $s \in S$ and κ the generator of $\langle k \rangle$. When two overlattices of index 2 of $S \oplus \langle k \rangle$ as above are not isomorphic as abstract lattices, they will be denoted as $(S \oplus \langle k \rangle)'^{(i)},\ i = 1, 2$.

2.5.1 Projective families of K3 surfaces with a symplectic automorphism τ of order 4

We are going to find all the non isomorphic overlattices of finite index of $\Omega_4 \oplus \langle 2d \rangle$ as follows.
In Proposition 2.5.1.2 we look at the orbits for the induced action of $O(\Omega_4)$ on A_{Ω_4}, and we fix a representative $s \in A_{\Omega_4}$ for each of them in Corollary 2.5.1.3, using the embedding of Ω_4 in Λ_{K3} defined in Section 2.3.2. In Theorems 2.5.1.4 and 2.5.1.5 we find all the overlattices of $\Omega_4 \oplus \langle 2d \rangle$, and prove that each one admits a unique primitive embedding in Λ_{K3}. In Example 2.5.1.6 we then give for each s defined in Corollary 2.5.1.3 a primitive classe $L \in \Lambda_{\mathrm{K3}}^\tau$ of square $2d, d \in \mathbb{Z}_{>0}$ such that $L/m + s$ is an integral class in Λ_{K3}: the maximum m for which this happens is the index of the overlattice of $\Omega_4 \oplus \langle 2d \rangle$ that we obtain choosing L as generator of $\langle 2d \rangle$. According to Corollary 1.2.1.6 and Remark 1.2.1.7, elements $\tilde{s}$ in the same orbit of s for the action of $O(S)$, and the corresponding $\tilde{L}$, will give the same irreducible component of the moduli space.

Definition 2.5.1.1. Consider an even lattice S, its group of isometries $O(S)$ and its discriminant group A_S with discriminant form q_S. We define two equivalence relations on A_S:

- *by order and square*: two elements $r, s \in A_S$ are in relation ($r \sim_S s$) if they have the same order and square, i.e. $\langle r \rangle = \langle s \rangle = \mathbb{Z}/k\mathbb{Z} \subset A_S$ and $q_S(r) = q_S(s) = g \in \mathbb{Q}/2\mathbb{Z}$; we will denote the equivalence classes for this relation with the pair (k, g);

- *by (induced) isometry*: two elements $r, s \in A_S$ are in relation ($r \approx_S s$) if there exists an isometry $\overline{\varphi} \in O(A_S)$ induced by an isometry $\varphi \in O(S)$ such that $\overline{\varphi}(r) = s$; we will denote the equivalence classes for this relation with the triple (k, g, n), where k, g are as above, and n is the cardinality of the class (in our case, this is sufficient to uniquely identify each of them).

Proposition 2.5.1.2. *The relation $\sim_{\Omega_4}$ divides A_{Ω_4} in 7 non-trivial equivalence classes (plus the trivial one $\{0\}$), whose cardinality is displayed below. Each of them corresponds to an equivalence class for $\approx_{\Omega_4}$, except for $(2,1)$ which is the union of two classes: $(2,1,6)$ e $(2,1,10)$.*

k \ g	0	1/2	1	3/2
2	15	32	16	0
4	240	240	240	240

Proof. The equivalence classes for $\sim_{\Omega_4}$ can be computed from a basis of A_{Ω_4} and its discriminant form. The generators of $O(\Omega_4)$ can be computed using the Integral Lattices package in SAGE [87]: then, choosing for each of the classes (k, g) of $\sim_{\Omega_4}$ a representative element $x_{(k,g)}$, their orbit for the induced action of $O(\Omega_4)$ on A_{Ω_4} is computed recursively [40, Algorithm I.4]. □

Corollary 2.5.1.3. *We give a representative element $x_{(k,g,n)}$ for each non-trivial equivalence class (k, g, n) for the relation $\approx_{\Omega_4}$, in terms of the generators of Ω_4 introduced in Section 2.3.2.*

class (k, g, n)	*representative* $x_{(k,g,n)}$
$(2, 0, 15)$	$\frac{3e_3-f_3-g_3-h_3}{2}$
$(2, 1/2, 32)$	$\frac{2e_1-f_1-g_1+e_4-f_4+\alpha-\gamma+\sigma+(e_2-g_2+e_4-g_4+a_1-a_2+\sigma)/2}{2}$
$(2, 1, 6)$	$\frac{\sigma}{2}$
$(2, 1, 10)$	$\frac{\sigma+3e_3-f_3-g_3-h_3}{2}$
$(4, 0, 240)$	$\frac{3(3e_3-f_3-g_3-h_3)+2(2e_1-f_1-g_1+e_4-f_4+\alpha-\gamma)+e_2-g_2+e_4-g_4+a_1-a_2+\sigma}{4}$
$(4, 1/2, 240)$	$\frac{2(e_1-g_1+e_2-g_2)+a_1-a_2+\sigma}{4}$
$(4, 1, 240)$	$\frac{3(3e_1-f_1-g_1-h_1+3\alpha-\beta-\gamma-\delta)+2(e_2-g_2+e_4-g_4+\sigma)}{4}$
$(4, 3/2, 240)$	$\frac{3(3e_1-f_1-g_1-h_1)}{4}$

Theorem 2.5.1.4. *Let X be a projective K3 surface that admits a symplectic automorphism of order 4, such that $rk(NS(X)) = 15$. Then, using the notation introduced in*

Remark 2.5.0.5, $NS(X)$ is one of the following lattices:

1. *for every $d \in \mathbb{N}$, $NS(X) = \Omega_4 \oplus \langle 2d \rangle$;*
2. *for $d \neq_4 1$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'$; this is uniquely determined by d and the index of the overlattice for $d \neq_4 2$, while for $d =_4 2$ there are two non isometric possibilities.*
3. *For $d =_4 0$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^\star$, uniquely determined by d and the index of the overlattice.*

Proof. By Corollary 1.2.1.6 overlattices of index 2 of $\Omega_4 \oplus \langle 2d \rangle$ correspond to isotropic elements in $A_{\Omega_4 \oplus \langle 2d \rangle}$ of the form $(L+v)/2$, where L generates $\langle 2d \rangle$ and $v \in \Omega_4$ is chosen up to the action of $O(\Omega_4)$ on A_{Ω_4}. Requiring

$$\left(\frac{L+v}{2}\right)^2 = \frac{d}{2} + \left(\frac{v}{2}\right)^2 = 0 \quad \text{in} \quad \frac{\mathbb{Q}}{2\mathbb{Z}}$$

we see that for each value of d modulo 4, $v/2$ belongs to one of the classes of $\approx_{\Omega_4}$ described in Proposition 2.5.1.2 containing elements of order 2. Therefore, for $d = 4h+1$ no overlattice of index 2 of $\Omega_4 \oplus \langle 2d \rangle$ exists; for $d = 4h$ or $d = 4h+3$ there exists one overlattice of index 2 of $\Omega_4 \oplus \langle 2d \rangle$; for $d = 4h+2$ there are two equivalence classes for $\approx_{\Omega_4}$, and the corresponding overlattices of $\Omega_4 \oplus \langle 2d \rangle$ are not in the same genus. This can be proved using [72, Prop. 1.15.1]: the overlattices corresponding to $(2,1,10)$ and $(2,1,6)$ have discriminant group $(\mathbb{Z}/4\mathbb{Z})^4 \times \mathbb{Z}/2d\mathbb{Z}$ and discriminant form respectively

$$\begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \oplus \begin{bmatrix} \frac{d+1}{2d} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix} \oplus \begin{bmatrix} \frac{d+1}{2d} \end{bmatrix}.$$

By a similar argument, overlattices of type $(\Omega_4 \oplus \langle 2d \rangle)^\star$ exist only for $d \equiv 0 \pmod 4$; to each value of d modulo 16 corresponds one equivalence class of $\approx_{\Omega_4}$ of those containing elements of order 4. □

Theorem 2.5.1.5. *Each of the lattices presented in Theorem 2.5.1.4 admits a unique primitive embedding in Λ_{K3} up to isometries of Λ_{K3}.*

Proof. Let $NS(X) = \Omega_4 \oplus \langle 2d \rangle$, let $T(X) = NS(X)^{\perp_{\Lambda_{\mathrm{K3}}}}$; it holds $\lambda(A_{T(X)}) = \lambda(A_{NS(X)}) = (2,2,4,4,4,4,2d)$ ($(2,2,2,4,4,4,4)$ for $d=1$); if d is odd, then $\mathbb{Z}/2d\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, so we have $A_{NS(X)} = (\mathbb{Z}/2\mathbb{Z})^3 \times A'$ and $NS(X)$ satisfies Corollary 1.2.1.13; if $d = 2d'$, then $\lambda(A_{T(X)}) = (2,2,4,4,4,4,4d')$, so $T(X)$ satisfies Proposition 1.2.1.11, and therefore $NS(X)$ satisfies Theorem 1.2.1.8.
Let $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'$, let $T(X) = NS(X)^{\perp_{\Lambda_{\mathrm{K3}}}}$: for $d =_4 2,3$ $A_{NS(X)}$ has length 5, so there exists a unique primitive embedding of $NS(X)$ in Λ_{K3} thanks to Theorem 1.2.1.12; for $d = 4d'$, $T(X)$ satisfies Proposition 1.2.1.11, for $\lambda(A_{T(X)}) = (2,2,2,2,4,4,8d')$; therefore $NS(X)$ satisfies Theorem 1.2.1.8.
Let $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^\star$: then $A_{NS(X)}$ has length 5, so it satisfies Theorem 1.2.1.12. □

Example 2.5.1.6. We now use Remark 1.2.1.7 (and the notation of Section 3.3.1) to provide for each $x_{(k,g,n)}$ in Corollary 2.5.1.3 primitive classes $L \in \Omega_4^{\perp_{H^2(X,\mathbb{Z})}}$ such that $L^2 = 2d$ and $L/k + x_{(k,g,n)}$ is an integral class in $H^2(X,\mathbb{Z})$. By Theorems 2.5.1.4 and 2.5.1.5 the general member of each irreducible compoment of the moduli space of projective K3 surfaces with a symplectic automorphism of order four is obtained as one of these examples.

1. For every $d \in \mathbb{N} \setminus \{0\}$, the class

$$L_0 = L_0(d) = \frac{a_1 + a_2 + \omega_1 + \omega_2}{2} + d\left(\frac{-a_1 - a_2 + \omega_1 - \omega_2}{2}\right)$$

generates the lattice $\langle 2d \rangle$ such that $\Omega_4 \oplus \langle 2d \rangle$ is primitively embedded in $H^2(X,\mathbb{Z})$.

2. For $d = 4(h-1),\ h \in \mathbb{N} \setminus \{0,1\}$ the class

$$L_{2,0}(h) = 2L_0(h) + e_3 + f_3 + g_3 + h_3$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X,\mathbb{Z})$; $L_{2,0}/2 + x_{(2,0,15)}$ is in fact an integral class in $H^2(X,\mathbb{Z})$.

3. For $d = 4h+2,\ h \in \mathbb{N}$, the class

$$L_{2,2}^{(1)}(h) = 2L_0(h) + \rho$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X,\mathbb{Z})$; $L_{2,2}^{(1)}/2 + x_{(2,1,6)}$ is in fact an integral class in $H^2(X,\mathbb{Z})$.

4. For $d = 4(h-1)+2,\ h \in \mathbb{N} \setminus \{0\}$, the class

$$L_{2,2}^{(2)}(h) = 2L_0(h) + \rho + e_3 + f_3 + g_3 + h_3$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X,\mathbb{Z})$; $L_{2,2}^{(2)}/2 + x_{(2,1,10)}$ is in fact an integral class in $H^2(X,\mathbb{Z})$.

5. For $d = 4h+3,\ h \in \mathbb{N}$, the class

$$L_{2,3}(h) = 2L_0(h) + \omega_2 + \frac{e_1 + f_1 + g_1 + h_1 + e_4 + f_4 + g_4 + h_4 + a_1 + a_2 + 3\rho}{2}$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X,\mathbb{Z})$; $L_{2,3}/2 + x_{(2,1/2,32)}$ is in fact an integral class in $H^2(X,\mathbb{Z})$.

6. For $d = 16(h-1),\ h \in \mathbb{N} \setminus \{0,1\}$, the class

$$L_{4,0}(h) = 4L_0(h)+e_1+f_1+g_1+h_1+3(e_3+f_3+g_3+h_3)+e_4+f_4+g_4+h_4+a_1+a_2+\rho+2\omega_2$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X,\mathbb{Z})$; $L_{4,0}/4 + x_{(4,0,240)}$ is in fact an integral class in $H^2(X,\mathbb{Z})$.

7. For $d = 16h + 4,\ h \in \mathbb{N}$, the class

$$L_{4,4}(h) = 4L_0(h) + a_1 + a_2 + \rho + 2\omega_2$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{4,4}/4 + x_{(4,3/2,240)}$ is in fact an integral class in $H^2(X, \mathbb{Z})$.

8. For $d = 16(h-4) + 8,\ h \in \mathbb{N} \setminus \{0, 1, 2, 3\}$, the class

$$L_{4,8}(h) = 4L_0(h) + \rho + \frac{3(e_2 + f_2 + g_2 + h_2) + 7(e_4 + f_4 + g_4 + h_4)}{2}$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{4,8}/4 + x_{(4,1,240)}$ is in fact an integral class in $H^2(X, \mathbb{Z})$.

9. For $d = 16h + 12,\ h \in \mathbb{N}$, the class

$$L_{4,12}(h) = 4L_0(h) + e_1 + f_1 + g_1 + h_1 + 2(e_4 + f_4 + g_4 + h_4 + \rho + \omega_1 + \omega_2)$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_4 \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{4,12}/4 + x_{(4,1/2,240)}$ is in fact an integral class in $H^2(X, \mathbb{Z})$.

Projective families with the action of τ^2 compared to projective families with a general symplectic involution

Let X be a general projective K3 surface which admits a symplectic automorphism of order 4 τ: then X, having Picard number 15, is a special member of one of the families of projective K3 surfaces that admit a symplectic involution, whose general element has Picard number 9. Indeed, each of the $NS(X)$ presented in Theorem 2.5.1.4 admits a primitive embedding of $\Omega_2 \oplus \langle 2d \rangle$ or some overlattice of it: to compare the families, we need to see how each of the ample classes L of Example 2.5.1.6 glues to the lattice $\Omega_2 \subset \Omega_4$ associated to the action of τ^2 on X (see Section 2.3.2).

Theorem 2.5.1.7 (see [32, Prop. 2.2])**.** *Let X be a projective K3 surface with a symplectic involution ι, such that $rk(NS(X)) = 9$. Then we have the following possible cases for $NS(X)$:*

(a) for every d, $NS(X) = \Omega_2 \oplus \langle 2d \rangle$;

(b) for d even, $NS(X) = (\Omega_2 \oplus \langle 2d \rangle)'$.

Remark 2.5.1.8. The action of $O(\Omega_2)$ on A_{Ω_2} has two non-trivial orbits, that consist of elements of order 2 and square respectively 0 or 1. Depending on the value of d (mod 4), the overlattice $(\Omega_2 \oplus \langle 2d \rangle)'$ is obtained using one or the other (compare to the proof of Theorem 2.5.1.4).

To determine which of these families our Néron-Severi groups belong to, we fix the embedding of Ω_2 in Ω_4 as in Section 2.3.2, so that the symplectic involution we're considering is indeed τ^2; as class L of square $2d$, we take again those defined in Example 2.5.1.6. We recall that, denoting R the orthogonal complement of Ω_2 in Ω_4, the latter is an overlattice of index 2^4 of $\Omega_2 \oplus R$.

Theorem 2.5.1.9. *1. For every $d \in \mathbb{N}$, $NS(X) = \Omega_4 \oplus \langle 2d \rangle$ corresponds to case (a) of Thm 2.5.1.7.*

2. For $d =_4 2, 3$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'$ corresponds to case (a) of Thm 2.5.1.7.

3. For $d =_4 0$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)'$ corresponds to case (b) of Thm 2.5.1.7.

4. For $d =_4 0$, $NS(X) = (\Omega_4 \oplus \langle 2d \rangle)^\star$ corresponds to case (b) of Thm 2.5.1.7

The following table describes the situation: saying that L glues to Ω_2 (and similarly for R) we mean there exists an element in $NS(X)$ of the form $(L + v)/2, v \in \Omega_2$.

$NS(X)$		L *glues to* Ω_2	L *glues to* R	$NS(X)/(\Omega_2 \oplus R \oplus \langle 2d \rangle)$
$\forall d$	$\Omega_4 \oplus \langle 2d \rangle$	*No*	*No*	$(\mathbb{Z}/2\mathbb{Z})^4$
$d =_4 2, 3$	$(\Omega_4 \oplus \langle 2d \rangle)'$	*No*	*Yes*	$(\mathbb{Z}/2\mathbb{Z})^5$
$d =_4 0$	$(\Omega_4 \oplus \langle 2d \rangle)'$	*Yes*	*Yes*	$(\mathbb{Z}/2\mathbb{Z})^5$
	$(\Omega_4 \oplus \langle 2d \rangle)^\star$	*Yes*	*Yes*	$(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$

Proof. The class L_0 doesn't glue to Ω_4, so it cannot glue neither to Ω_2 nor to R. Since $L_{2,3}^2 = 2d$ with $d =_4 3$, this case corresponds necessarily to case (a) of Thm 2.5.1.7; $\Omega_2 \oplus R \oplus \langle 2d \rangle$ has index 2^5 in $NS(X)$, because there exists $r \in R$ such that $(L_{2,3} + r)/2 \in NS(X)$: $r = \sigma + g_1 - f_1 + e_4 - f_4 + \alpha - \gamma - (e_2 - g_2 + e_4 - g_4 + a_1 - a_2 + \sigma)/2$. There are no elements in Ω_2 that glue to $L_{2,2}^{(1)}$ or $L_{2,2}^{(2)}$, so we are again in case (a): this means that a symplectic involution that satisfies case (b) of Theorem 2.5.1.7, with $L^2 =_8 4$, cannot be the square of a symplectic automorphism of order 4.
The gluings for the cases in which $d = 4k$ are described as follows: $(L_{2,0} + \tilde{v})/2$, $(L_{2,0} + r)/2 \in NS(X)$ for $\tilde{v} = e_3 - g_3 + f_3 - h_3 \in \Omega_2$, $r = e_3 - f_3 + g_3 - h_3 \in R$; since $(\tilde{v} + r)/2$ is one of the elements that glue Ω_2 to R, the index of $\Omega_2 \oplus R \oplus \langle 2d \rangle$ in $NS(X)$ is still 2^5; $(L_{4,0} + \tilde{v})/2$, $(L_{4,0} + r)/2 \in NS(X)$ for the same $\tilde{v} \in \Omega_2$, $r \in R$ as $L_{2,0}$; $(L_{4,4} + \tilde{v})/2$, $(L_{4,4} + r)/2 \in NS(X)$ for $\tilde{v} = f_2 - h_2 + f_4 - h_4 \in \Omega_2$, $r = a_1 - a_2 + \sigma \in R$; $(L_{4,8} + \tilde{v})/2$, $(L_{4,8} + r)/2 \in NS(X)$ for $\tilde{v} = \alpha - \gamma + \beta - \delta + e_1 - g_1 + f_1 - h_1 \in \Omega_2$, $r = e_1 - f_1 + g_1 - h_1 + \alpha - \beta + \gamma - \delta \in R$; $(L_{4,12} + \tilde{v})/2, (L_{4,12} + r)/2 \in NS(X)$ for $\tilde{v} = e_1 - g_1 + f_1 - h_1 \in \Omega_2$, $r = e_1 - f_1 + g_1 - h_1 \in R$. Thus, these cases correspond to case (b) of Theorem 2.5.1.7; $NS(X)$ is obtained as overlattice of $\Omega_2 \oplus R \oplus \langle 2d \rangle$ by gluing firstly Ω_2 with R to get Ω_4, and then $L_{4,k}$ with Ω_4 as in Example 2.5.1.6. □

2.5.2 Projective families of K3 surfaces which arise as resolution of the singularities of X/τ

Projective K3 surfaces $\tilde{Y}$ that are the resolution of X/τ have to primitively contain in their Néron-Severi both the exceptional lattice M_4 described in Section 2.4.3 [71, §5-7], and a positive class H of square $2e$ that generates $M_4^{\perp_{NS(\tilde{Y})}}$: therefore, $\tilde{Y}$ is polarized with the lattice $M_4 \oplus \langle 2e \rangle$ or one of its cyclic overlattices.

Theorem 2.5.2.1. *The relation $\sim_{M_4}$ (see Def. 2.5.1.1) divides A_{M_4} in 7 non-trivial equivalence classes (plus the trivial one $\{0\}$):*

$k \backslash g$	0	1/2	1	3/2
2	3	8	4	0
4	12	12	12	12

Each of them corresponds to an equivalence class for $\approx_{M_4}$, except for $(2,1)$ which is the union of two classes: $(2,1,1)$ and $(2,1,3)$. We give a representative element $x_{(k,g,n)}$ for each non-trivial equivalence class (k,g,n) in terms of the generators of M_4 introduced in Section 2.4.3.

class (k,g,n)	representative $x_{(k,g,n)}$
$(2,0,3)$	$\frac{m_1^2+2m_2^2+3m_3^2+m_1^3+2m_2^3+3m_3^3}{2}$
$(2,1/2,8)$	$\frac{m_1^4+m_3^4+\tilde{m}^2}{2}$
$(2,1,1)$	$\frac{\tilde{m}^1+\tilde{m}^2}{2}$
$(2,1,3)$	$\frac{m_1^2+2m_2^2+3m_3^2+m_1^3+2m_2^3+3m_3^3+\tilde{m}^1+\tilde{m}^2}{2}$
$(4,0,12)$	$\frac{m_1^2+2m_2^2+3m_3^2+m_1^3+2m_2^3+3m_3^3}{4}+\frac{\tilde{m}^2}{2}$
$(4,1/2,12)$	$\frac{m_1^3+2m_2^3+3m_3^3+m_1^4+2m_2^4+3m_3^4}{4}+\frac{m_1^4+m_3^4}{2}$
$(4,1,12)$	$\frac{m_1^2+2m_2^2+3m_3^2+m_1^4+2m_2^4+3m_3^4}{4}+\frac{m_1^3+m_3^3+m_1^4+m_3^4+\tilde{m}^1}{2}$
$(4,3/2,12)$	$\frac{m_1^2+2m_2^2+3m_3^2+m_1^4+2m_2^4+3m_3^4}{4}+\frac{m_1^3+m_3^3}{2}$

Theorem 2.5.2.2. *Let $\tilde{Y}$ be a projective K3 surface such that $rk(NS(\tilde{Y})) = 15$ and $NS(\tilde{Y})$ contains primitively M_4 and $\langle 2e \rangle$, $e \in \mathbb{N} \setminus \{0\}$. Then, using the notation introduced in Remark 2.5.0.5, $NS(\tilde{Y})$ is one of the following:*

1. *for every e, $NS(\tilde{Y}) = M_4 \oplus \langle 2e \rangle$;*
2. *for e $\not\equiv_4 1$, $NS(\tilde{Y}) = (M_4 \oplus \langle 2e \rangle)'$; this is uniquely determined by e and the index of the overlattice for e $\not\equiv_4 2$, while for e $\equiv_4 2$ there are two non isometric possibilities.*
3. *For e $\equiv_4 0$, $NS(\tilde{Y}) = (M_4 \oplus \langle 2e \rangle)^\star$, uniquely determined by e and the index of the overlattice.*

Each of these lattices admits a unique primitive embedding in Λ_{K3}.

Proof. The overlattices of $M_4 \oplus \langle 2e\rangle$ are in bijection with the equivalence classes for $\approx_{M_4}$. Fix the primitive embedding $M_4 \hookrightarrow \Lambda_{\mathrm{K3}}$ as in Section 2.4.3: since the orthogonal complement of M_4 is the lattice $\pi_{4*}H^2(X,\mathbb{Z})$, we can use as generators of the lattice $\langle 2e\rangle$ the primitive classes $\overline{L}$ in $H^2(\tilde{Y},\mathbb{Z})$ obtained from $\pi_{4*}L$ (with L as in Example 2.5.1.6) as follows. Refer to Section 2.4.1 for the computation of $\pi_{4*}L$, and see also the corresponding classes D_1 in Section 2.6.1 for the explicit gluing element $(\overline{L}+m)/k,\ m \in A_{M_4},\ k = 2,4$:

$NS(\tilde{Y})$	e	$\overline{L}$
$M_4 \oplus \langle 2e\rangle$	$4(h-1)$	$\pi_{4*}L_{4,0}(h)/4$
	$4h+1$	$\pi_{4*}L_{4,4}(h)/4$
	$4(h-4)+2$	$\pi_{4*}L_{4,8}(h)/4$
	$4h+3$	$\pi_{4*}L_{4,0}(h)/4$
$(M_4 \oplus \langle 2e\rangle)'$	$4(h-1)$	$\pi_{4*}L_{2,0}(h)/2$
	$4h+2$	$\pi_{4*}L^{(1)}_{2,2}(h)/2$
	$4(h-1)+2$	$\pi_{4*}L^{(2)}_{2,2}(h)/2$
	$4h+3$	$\pi_{4*}L_{2,3}(h)/2$
$(M_4 \oplus \langle 2e\rangle)^\star$	$4h$	$\pi_{4*}L_0(h)$

Notice that for every choice of $e =_4 2$ there are two non isomorphic realizations of $(M_4 \oplus \langle 2e\rangle)'$, using alternatively $\pi_{4*}L^{(i)}_{2,2}(h)/2,\ i = 1,2$: indeed, for h odd both of them glue to the class $(2,1,1)$, while for h even they both glue to $(2,1,3)$. The resulting lattices belong to different genera.
Each of the possible lattices $NS(\tilde{Y})$ admits a unique primitive embedding in $H^2(X,\mathbb{Z})$, because $\ell(A_{NS(\tilde{Y})}) \leq 5$, so Theorem 1.2.1.12 holds. □

Theorem 2.5.2.3. *There is a 1-1 correspondence between families of K3 surfaces X with $NS(X)$ as in 2.5.1.4, and families of K3 surfaces $\tilde{Y}$ with $NS(\tilde{Y})$ as in Theorem 3.5.2.2. The primitive classes $\overline{L} \in NS(\tilde{Y})$ that generate the sublattices $\langle nd\rangle$ as stated are indicated in curly brackets. For $d =_4 2$ the lattices $S^{(1)}, S^{(2)}$ ar not isometric.*

	$NS(X)$	$NS(\tilde{Y})$	
$\forall d$	$\Omega_4 \oplus \langle 2d \rangle$	$(M_4 \oplus \langle 8d \rangle)^{\star}$	$\{\overline{L} = \pi_{4*}L\}$
$d =_4 2$	$(\Omega_4 \oplus \langle 2d \rangle)'^{(1)}$	$(M_4 \oplus \langle 2d \rangle)'^{(1)}$	$\{\overline{L} = \frac{\pi_{4*}L}{2}\}$
	$(\Omega_4 \oplus \langle 2d \rangle)'^{(2)}$	$(M_4 \oplus \langle 2d \rangle)'^{(2)}$	$\{\overline{L} = \frac{\pi_{4*}L}{2}\}$
$d =_4 3$	$(\Omega_4 \oplus \langle 2d \rangle)'$	$(M_4 \oplus \langle 2d \rangle)'$	$\{\overline{L} = \frac{\pi_{4*}L}{2}\}$
$d =_4 0$	$(\Omega_4 \oplus \langle 2d \rangle)'$	$(M_4 \oplus \langle 2d \rangle)'$	$\{\overline{L} = \frac{\pi_{4*}L}{2}\}$
	$(\Omega_4 \oplus \langle 2d \rangle)^{\star}$	$M_4 \oplus \langle d/2 \rangle$	$\{\overline{L} = \frac{\pi_{4*}L}{4}\}$

Proof. The map π_{4*} kills Ω_4, so the possible Néron-Severi groups for the general smooth quotient surface $\tilde{Y} = \widetilde{X/\tau}$ are determined by how the image of the ample class L glues to the exceptional lattice M_4. For each of the L in Example 2.5.1.6 we compute $\pi_{4*}L$ using the description of the image lattice $\pi_{4*}H^2(X,\mathbb{Z})$ given in Section 2.4.1; we find the unique integral and primitive $\overline{L} = \pi_{4*}L/k$, where k can be $1, 2$ or 4 depending on the case, and we then compare $\overline{L}^2$ to L^2. □

2.5.3 Projective families of K3 surfaces which arise as resolution of the singularities of X/τ^2

The process used in the previous section can be also applied to describe the K3 surfaces $\tilde{Z}$ that are resolution of X/τ^2, and the relations between $NS(X)$ and $NS(\tilde{Z})$; for the general symplectic involution, this was already done by Garbagnati and Sarti:

Theorem 2.5.3.1 (see [29, Cor. 2.2])**.** *Let X be an algebraic K3 surface with $rk(NS(X)) = 9$ admitting a Nikulin involution ι, and let $\tilde{Z}$ be the resolution of the singularities of the quotient X/ι. Then:*

(a) $NS(X) = \Omega_2 \oplus \langle 2d \rangle$ if and only if $NS(\tilde{Z}) = (N \oplus \langle 4d \rangle)'$;

(b) $NS(X) = (\Omega_2 \oplus \langle 2d \rangle)'$ if and only if $NS(\tilde{Z}) = N \oplus \langle d \rangle$.

When $\iota = \tau^2$, we have the following theorem.

Theorem 2.5.3.2. *Let $\tilde{Z}$ be a K3 surface such that $rk(NS(\tilde{Z})) = 15$; suppose $NS(\tilde{Z})$ admits a primitive embedding of Γ_4 (see Def. 2.4.2.3), and contains a class of positive square $2d$ that generates $\Gamma_4^{\perp_{NS(\tilde{Z})}}$. Then $d = 2x$, and $NS(\tilde{Z})$ is one of the following:*

1. *for every x, $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)'$, uniquely determined by x and the index of the overlattice.*

2. *for* $x =_4 2, 3$, $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)^\star$, *uniquely determined by* x *and the index of the overlattice.*

Moreover, there exists a unique primitive embedding of these lattices in $H^2(\tilde{Z}, \mathbb{Z})$ *up to isometries of the latter.*

Proof. The lattice $\Gamma_4 \oplus \langle 2d \rangle$ cannot be the Néron-Severi of a K3 surface, since $\lambda(A_{\Gamma_4 \oplus \langle 2d \rangle}) = (2, 2, 2, 2, 2, 2, 4, 4, 2d))$, so its length is $9 > 22 - rk(\Gamma_4 \oplus \langle 2d \rangle)$ (see Rem. 1.2.1.9). Consider the table of non-trivial equivalence classes for $\sim_{\Gamma_4}$:

k \ g	0	1/4	1/2	3/4	1	7/4	3/2	9/4
2	127	0	0	0	128	0	0	0
4	0	256	144	0	0	256	112	0

an element of the form $(E + \gamma)/2$, with $E^2 = 2d$ and $\gamma \in \Gamma_4$, has integer, even self-intersection only if d is even, and an element of the form $(E + \gamma)/4$ only if $d = 2x$ with $x =_4 2, 3$. The non-trivial equivalence classes for $\approx_{\Gamma_4}$ are presented in the following table; the corresponding overlattice of $\Gamma_4 \oplus \langle 4x \rangle$ can be realized having fixed the embedding $\Gamma_4 \in H^2(\tilde{Z}, \mathbb{Z})$ as in Def. 2.4.2.3, using as positive class $\hat{L} = \pi_{2*}L$ for $L \in \{L_0, L^{(1)}_{2,2}, L^{(2)}_{2,2}, L_{2,3}\}$, and $\hat{L} = \pi_{2*}L/2$ for $L \in \{L_{2,0}, L_{4,0}, L_{4,4}, L_{4,8}, L_{4,12}\}$.

class (k, g, n)	representative $x_{(k,g,n)}$	glues to:
$(2, 0, 1)$	$(n_3 + n_4 + n_5 + n_8)/2$	$\hat{L}^{(i)}_{2,2}/2$ for $i = 1, 2$
$(2, 0, 6)$	$(n_5 + n_6 + n_7 + n_8)/2$	$\hat{L}_0(h)/2$, $h =_2 0$
$(2, 0, 30)$	$\frac{(\hat{e}_3 - \hat{f}_3) + n_3 + n_4}{2}$ $\frac{(\hat{\alpha} - \hat{\beta}) + (\hat{e}_1 - \hat{f}_1) + n_3 + n_4 + n_5 + n_8}{2}$	$\hat{L}_{4,0}/2$ $\hat{L}_{4,8}/2$
$(2, 0, 90)$	$\frac{(\hat{e}_3 - \hat{f}_3) + n_2 + n_3 + n_4 + n_5 + n_6 + n_8}{2}$	$\hat{L}_{2,0}(h)/2$, $h =_2 1$
$(2, 1, 2)$	$(n_5 + n_8)/2$	$\hat{L}_{2,3}/2$
$(2, 1, 6)$	$(n_2 + n_3 + n_4 + n_5 + n_6 + n_8)/2$	$\hat{L}_0(h)/2$, $h =_2 1$
$(2, 1, 30)$	$\frac{(\hat{e}_1 - \hat{f}_1) + (\hat{e}_4 - \hat{f}_4) + n_3 + n_4}{2}$ $(\hat{e}_1 - \hat{f}_1)/2$	$\hat{L}_{4,4}$ $\hat{L}_{4,12}$
$(2, 1, 90)$	$\frac{(\hat{e}_3 - \hat{f}_3) + n_5 + n_6 + n_7 + n_8}{2}$	$\hat{L}_{2,0}(h)/2$, $h =_2 0$
$(4, 1/4, 256)$	$\frac{(\hat{e}_1 - \hat{f}_1) + (\hat{e}_4 - \hat{f}_4) + x_1 + x_2 + n_2}{2} + \frac{3n_5 + n_8}{4}$	$\hat{L}_{2,3}(h)/4$, $h =_2 1$
$(4, 1/2, 24)$	$\frac{x_1}{2} + \frac{3n_3 + n_4 + 3n_5 + n_8}{4}$	$\hat{L}^{(1)}_{2,2}(h)/4$, $h =_2 1$

$(4,1/2,120)$	$\frac{(\hat{e}_3-\hat{f}_3)+x_1+n_2+n_7}{2}+\frac{n_3+3n_4+3n_5+n_8}{4}$	$\hat{L}^{(2)}_{2,2}(h)/4,\ h=_2 0$
$(4,5/4,256)$	$\frac{x_1+x_2+n_3+n_4+n_7+(\hat{e}_1-\hat{f}_1)+(\hat{e}_4-\hat{f}_4)}{2}+\frac{3n_5+n_8}{4}$	$\hat{L}_{2,3}(h)/4,\ h=_2 0$
$(4,3/2,40)$	$\frac{(\hat{e}_3-\hat{f}_3)+x_1}{2}+\frac{3n_3+n_4+3n_5+n_8}{4}$	$\hat{L}^{(2)}_{2,2}(h)/4,\ h=_2 1$
$(4,3/2,72)$	$\frac{x_1+n_2+n_7}{2}+\frac{n_3+3n_4+3n_5+n_8}{4}$	$\hat{L}^{(1)}_{2,2}(h)/4,\ h=_2 0$

Now, the classes $(2,0,1)$ and $(2,1,2)$ produce overlattices of $\Gamma_4\oplus\langle 4x\rangle$ that are not admissible as Néron-Severi of a K3 surfaces, because they have $\ell=9$ (see Rem. 1.2.1.9). For the remaining classes (k,g,n), those contained in the same equivalence class (k,g) for the relation $\sim_{\Gamma_4}$ give rise to isomorphic lattices: indeed, having fixed x, it can be proved that all the lattices of the form $(\Gamma_4\oplus\langle 4x\rangle)'$ are in the same genus, and the same holds for all the lattices of the form $(\Gamma_4\oplus\langle 4x\rangle)^\star$; however, since $\lambda(A_{(\Gamma_4\oplus\langle 4x\rangle)'})=(2,2,2,2,4,4,4x)$, and $\lambda(A_{(\Gamma_4\oplus\langle 4x\rangle)^\star})=(2,2,2,2,2,2,4x)$, actually $(\Gamma_4\oplus\langle 4x\rangle)'$ and $(\Gamma_4\oplus\langle 4x\rangle)^\star$ are unique in their genus by Proposition 1.2.1.11. Furthermore, they admit a unique primitive embedding in Λ_{K3}, as we can apply Proposition 1.2.1.11 to the corresponding trascendental lattices, both of signature $(2,5)$ and length 7, $T'=((\Gamma_4\oplus\langle 4x\rangle)')^\perp$, $T^\star=((\Gamma_4\oplus\langle 4x\rangle)^\star)^\perp$: indeed $\lambda(A_{T'})=\lambda(A_{(\Gamma_4\oplus\langle 4x\rangle)'})$, and $\lambda(A_{T^\star})=\lambda(A_{(\Gamma_4\oplus\langle 4x\rangle)^\star})$. □

We are now ready to describe the correspondence between irreducible components of the moduli space of X and of $\tilde{Z}$ in the projective case. Moreover, similarly to Theorem 2.5.1.9, we check whether the positive class orthogonal to Γ_4 in $NS(\tilde{Z})$ glues to the lattices N_Z,Ω_2 embedded in Γ_4 as described in Definition 2.4.2.3 and Remark 2.4.3.1 respectively. This will be used in Corollary 2.5.3.4 to find the irreducible component of the moduli space of (respectively) projective K3 surfaces S with a symplectic involution ι, and projective K3 surfaces that are resolution of a quotient S/ι, that $\tilde{Z}$ belongs to.

Theorem 2.5.3.3. *Let τ be a symplectic automorphism of order 4 on a projective K3 surface X such that $rk(NS(X))=15$, and consider $\tilde{Z}$ that is the resolution of the singularities of the quotient X/τ^2: the following table describes the correspondence between $NS(X)$ and $NS(\tilde{Z})$. The primitive classes $\hat{L}$ in $NS(\tilde{Z})$ that generate the sublattices $\langle nd\rangle$ as stated are indicated in curly brackets.*

$NS(X)$		$NS(\tilde{Z})$	$\hat{L}$ *glues to* N_Z	$\hat{L}$ *glues to* Ω_2
$\forall d$	$\Omega_4\oplus\langle 2d\rangle$	$(\langle 4d\rangle\oplus\Gamma_4)'$ $\{\hat{L}=\pi_{2*}L\}$	*Yes*	*No*
$d=_4 2$	$(\Omega_4\oplus\langle 2d\rangle)'^{(1)}$ $(\Omega_4\oplus\langle 2d\rangle)'^{(2)}$	$(\langle 4d\rangle\oplus\Gamma_4)^\star$ $\{\hat{L}=\pi_{2*}L\}$	*Yes*	*Yes*

$d =_4 3$	$(\Omega_4 \oplus \langle 2d \rangle)'$	$(\langle 4d \rangle \oplus \Gamma_4)^\star$ $\{\hat{L} = \pi_{2*}L\}$	*Yes*	*Yes*
$d =_4 0$	$(\Omega_4 \oplus \langle 2d \rangle)'$ $(\Omega_4 \oplus \langle 2d \rangle)^\star$	$(\langle d \rangle \oplus \Gamma_4)'$ $\{\hat{L} = \frac{\pi_{2*}L}{2}\}$	*No*	*No* *Yes*

Proof. Recall from Theorem 2.5.1.9 the possible Néron-Severi groups of X. We use the map π_{2*} (see Section 2.4.1) to compute $\hat{L}$ for each of the L in Example 2.5.1.6, and we check their eventual gluing to N following Section 2.4.2.
Fix $d = 4h + 2 = 4(k-1) + 2$, and consider the ample classes $L^{(1)}_{2,2}(h)$, $L^{(2)}_{2,2}(k)$ of X that generate the two non isomorphic overlattices of index 2 of $\Omega_4 \oplus \langle 2d \rangle$: denote these lattices $NS(X)^{(1)}$ and $NS(X)^{(2)}$. Now take $\tilde{Z}$ the resolution of X/τ^2: from the previous theorem, we have $NS(\tilde{Z})^{(1)} = \langle \Gamma_4, \hat{L}^{(1)}_{2,2}(h) \rangle \simeq NS(\tilde{Z})^{(2)} = \langle \Gamma_4, \hat{L}^{(2)}_{2,2}(k) \rangle$. Therefore, for $d =_4 2$ there is a 2-1 correspondence between $(\Omega_4 \oplus \langle 2d \rangle)'$-polarized families of X and $(\Gamma_4 \oplus \langle 4d \rangle)^\star$-polarized families of $\tilde{Z}$. Similar considerations apply to $d =_4 0$. □

Projective families of $\tilde{Z}$ and $\tilde{Y}$ compared to projective families of K3 surfaces with a symplectic involution, or that are a quotient of one

From Theorem 2.5.3.3 we can see whether $\hat{L}$ glues to Ω_2 and N, or not: this determines the general member of the projective family $\tilde{Z}$ belongs to respectively as a surface S admitting a symplectic involution ι, and as resolution of a quotient S/ι, according to Theorems 2.5.1.7, 2.5.3.1. However, if we remove the information about X and we only consider $NS(\tilde{Z})$ abstractly, we see that $\tilde{Z}$ can belong to the intersection of two families of S (or two families of quotients).

Corollary 2.5.3.4. *Consider $\tilde{Z}$ as a projective K3 surface with a symplectic involution:*

1. *if $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)^*$, then $\tilde{Z}$ belongs to the deformation family whose general member has Néron-Severi $(\Omega_2 \oplus \langle 2d \rangle)'$;*
2. *if $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)'$ then $\tilde{Z}$ belongs to the intersection between the two families of Theorem 2.5.1.7.*

Consider $\tilde{Z}$ as the resolution of a quotient of a K3 surface by a symplectic involution:

1. *if $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)^*$, then $\tilde{Z}$ belongs to the deformation family whose general member has Néron-Severi $(N \oplus \langle 4d \rangle)'$.*
2. *if $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)'$ then $\tilde{Z}$ belongs to the intersection between the two quotient families of Theorem 2.5.3.1.*

Proof. If $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)^*$, from the previous theorem we see that $\tilde{Z}$ belongs to the intersection of the deformation families whose general members have Néron-Severi $(\Omega_2 \oplus \langle 2d \rangle)'$ and $(N \oplus \langle 4d \rangle)'$.

If $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4x \rangle)'$, for the same x we can choose the embedding of Γ_4 in $NS(\tilde{Z})$ that has orthogonal complement generated respectively by $\pi_{2*}L_0$ or $\{\pi_{2*}L_{2,0}/2$, that don't glue to Ω_2, or by one among $\{\pi_{2*}L_{4,i}/2\}_{i=0,4,8,12}$, that do. Similarly, considering $\tilde{Z}$ as a quotient surface we can choose the embedding of Γ_4 in $NS(\tilde{Z})$ that has orthogonal complement generated respectively by $\pi_{2*}L_0$, that glues to N, or by one among $\{\pi_{2*}L_{2,0}/2, \pi_{2*}L_{4,i}/2\}$, $i = 0, 4, 8, 12$, that don't. □

The surface $\tilde{Y}$ is also a special member of one of the projective families of K3 surfaces that arise as resolution of the singularities of a quotient S/ι, and we have the following result.

Corollary 2.5.3.5. *Let $\tilde{Y}$ be a general projective K3 surface polarized with $M_4 \oplus \langle 2e \rangle$ or one of its overlattice, according to Theorem 3.5.2.2, and define $H = M_4^{\perp}$. Consider the primitive embedding $N_Y \subset M_4$ described in the proof of Proposition 2.4.3.2, and the double cover $\tilde{Z} \dashrightarrow \tilde{Y}$ with ramification divisor ν_Y. Then:*

1. *If H glues to N_Y, then $e =_4 0$ and $NS(\tilde{Y})$ is an overlattice of index 2 or 4 of $M_4 \oplus \langle 2e \rangle$; moreover, $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle e \rangle)'$.*
2. *If H does not glue to N_Y, then we distinguish two cases: either $NS(\tilde{Y}) = M_4 \oplus \langle 2e \rangle$, or, only if $e =_4 2, 3$, $NS(\tilde{Y}) = (M_4 \oplus \langle 2e \rangle)'$; in the former case, $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4e \rangle)'$; in the latter, $NS(\tilde{Z}) = (\Gamma_4 \oplus \langle 4e \rangle)^*$.*

2.6 Projective models

Given a nef and big divisor L on X, there is a natural map $\phi_{|L|} : X \to \mathbb{P}(H^0(X, L)^*) \simeq \mathbb{P}^n$, with $n = L^2/2+1$. Any automorphism σ of X that preserves L induces an action on $H^0(X, L)$: in particular, if σ is finite of order m, we can split $H^0(X, L)$ in eigenspaces corresponding to the m-roots of unity.

Remark 2.6.0.1. Notice that the action of σ on $H^0(X, L)$ could have order km for some integer $k > 1$: indeed, if

$$\sigma^m : (x_0, \ldots, x_n) \mapsto \xi_k(x_0, \ldots, x_n)$$

for ξ_k a root of unity, then on $\mathbb{P}(H^0(X, L)^*)$ it will hold $\sigma^m = id$; but this is irrelevant when we study the action of the cyclic group $\langle \sigma \rangle$ on the projective surface X, as we can just take the action of σ^k on $H^0(X, L)$ without loss of generality.

Thus, considering a symplectic automorphism τ of order 4 on a K3 surface X, we have

$$H^0(X, L) = V_1 \oplus V_i \oplus V_{-1} \oplus V_{-i} = W_+ \oplus W_-$$

where $V_\bullet$ are the eigenspaces relative to the action of τ^*, and $W_\bullet$ are relative to $(\tau^2)^*$, so that $W_+ = V_1 \oplus V_{-1}$, and $W_- = V_i \oplus V_{-i}$.

2.6.1 Eigenspaces of τ^* and classes in $NS(\tilde{Y})$

The purpose of this section is to prove the following proposition:

Proposition 2.6.1.1 (see [32, Prop. 2.7] and [26, Thm. 5.6])**.** *There exist divisors $D_1, \dots, D_4 \in NS(\tilde{Y})$ such that*

$$H^0(X, L) = \pi_4^* H^0(\tilde{Y}, D_1) \oplus \pi_4^* H^0(\tilde{Y}, D_2) \oplus \pi_4^* H^0(\tilde{Y}, D_3) \oplus \pi_4^* H^0(\tilde{Y}, D_4)$$

and every $\pi_4^* H^0(\tilde{Y}, D_i)$ corresponds to one of the eigenspaces for the action of τ^* on $H^0(X, L)$.

We start by defining some divisors $D_1, \dots, D_4$ associated to each L of Example 2.5.1.6. The proof of the proposition can be found below, and amounts to show that these divisors are indeed the ones in the statement.

Consider the following elements in M_4^*, for $i = 1, \dots, 4$, $j = 1, 2$ (see also Section 2.4.3):

$$\alpha^i = \frac{3m_1^i + 2m_2^i + m_3^i}{4}, \quad \beta^i = \frac{m_1^i + 2m_2^i + m_3^i}{2}, \quad \gamma^i = \frac{m_1^i + 2m_2^i + 3m_3^i}{4}, \quad \delta^j = \frac{\tilde{m}^j}{2};$$

notice that $(\alpha^i)^2 = (\gamma^i)^2 = -3/4$, $(\beta^i)^2 = -1$, $(\delta^j)^2 = -1/2$ with respect to the intersection form of M_4 extended $\mathbb{Q}$-linearly to M_4^*.

– Consider $L_0(d)$; depending on the value of d mod 4, we define $D_1, \dots, D_4$ as follows:

$L_0(d)$	$d =_4 0$	$d =_4 1$
D_1	$\pi_{4*}L_0/4 - \gamma^2 - \gamma^4 - \delta^2$	$\pi_{4*}L_0/4 - \gamma^2 - \alpha^3 - \delta^1 - \delta^2$
D_2	$\pi_{4*}L_0/4 - \alpha^1 - \alpha^3 - \delta^1$	$\pi_{4*}L_0/4 - \alpha^1 - \beta^3 - \alpha^4$
D_3	$\pi_{4*}L_0/4 - \beta^1 - \alpha^2 - \beta^3 - \alpha^4 - \delta^2$	$\pi_{4*}L_0/4 - \beta^1 - \alpha^2 - \gamma^3 - \beta^4 - \delta^1 - \delta^2$
D_4	$\pi_{4*}L_0/4 - \gamma^1 - \beta^2 - \gamma^3 - \beta^4 - \delta^1$	$\pi_{4*}L_0/4 - \gamma^1 - \beta^2 - \gamma^4$

$L_0(d)$	$d =_4 2$	$d =_4 3$
D_1	$\pi_{4*}L_0/4 - \gamma^2 - \beta^3 - \alpha^4 - \delta^2$	$\pi_{4*}L_0/4 - \gamma^2 - \gamma^3 - \beta^4 - \delta^1 - \delta^2$
D_2	$\pi_{4*}L_0/4 - \alpha^1 - \gamma^3 - \beta^4 - \delta^1$	$\pi_{4*}L_0/4 - \alpha^1 - \gamma^4$
D_3	$\pi_{4*}L_0/4 - \beta^1 - \alpha^2 - \gamma^4 - \delta^2$	$\pi_{4*}L_0/4 - \beta^1 - \alpha^2 - \alpha^3 - \delta^1 - \delta^2$
D_4	$\pi_{4*}L_0/4 - \gamma^1 - \beta^2 - \alpha^3 - \delta^1$	$\pi_{4*}L_0/4 - \gamma^1 - \beta^2 - \beta^3 - \alpha^4$

– Consider $L_{2,0}(h)$, whose square is $2d = 8(h-1)$; depending on the value of h mod 2, we define $D_1, \dots, D_4$ as follows.

$L_{2,0}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\pi_{4*}L_{2,0}/4 - \beta^2 - \beta^4$	$\pi_{4*}L_{2,0}/4 - \beta^2 - \beta^3$
D_2	$\pi_{4*}L_{2,0}/4 - \alpha^1 - \gamma^2 - \alpha^3 - \gamma^4 - \delta^1 - \delta^2$	$\pi_{4*}L_{2,0}/4 - \alpha^1 - \gamma^2 - \gamma^3 - \alpha^4 - \delta^1 - \delta^2$
D_3	$\pi_{4*}L_{2,0}/4 - \beta^1 - \beta^3$	$\pi_{4*}L_{2,0}/4 - \beta^1 - \beta^4$
D_4	$\pi_{4*}L_{2,0}/4 - \gamma^1 - \alpha^2 - \gamma^3 - \alpha^4 - \delta^1 - \delta^2$	$\pi_{4*}L_{2,0}/4 - \gamma^1 - \alpha^2 - \alpha^3 - \gamma^4 - \delta^1 - \delta^2$

– Consider $L_{2,2}^{(j)}(h)$, $j = 1, 2$; recall that $L_{2,2}^{(1)}(h)^2 = 8h + 4$, while $L_{2,2}^{(2)}(h)^2 = 8h - 4$: thus, any value of $d =_4 2$ can be realized *both* with h even *and* h odd, using one between $L_{2,2}^{(1)}, L_{2,2}^{(2)}$ alternatively, giving two non-isomorphic cases.

$L_{2,2}^{(j)}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\pi_{4*}L_{2,2}^{(j)}/4 - \beta^3 - \beta^4 - \delta^1 - \delta^2$	$\pi_{4*}L_{2,2}^{(j)}/4 - \delta^1 - \delta^2$
D_2	$\pi_{4*}L_{2,2}^{(j)}/4 - \alpha^1 - \alpha^2 - \gamma^3 - \gamma^4$	$\pi_{4*}L_{2,2}^{(j)}/4 - \alpha^1 - \alpha^2 - \alpha^3 - \alpha^4$
D_3	$\pi_{4*}L_{2,2}^{(j)}/4 - \beta^1 - \beta^2 - \delta^1 - \delta^2$	$\pi_{4*}L_{2,2}^{(j)}/4 - \beta^1 - \beta^2 - \beta^3 - \beta^4 - \delta^1 - \delta^2$
D_4	$\pi_{4*}L_{2,2}^{(j)}/4 - \gamma^1 - \gamma^2 - \alpha^3 - \alpha^4$	$\pi_{4*}L_{2,2}^{(j)}/4 - \gamma^1 - \gamma^2 - \gamma^3 - \gamma^4$

– Consider $L_{2,3}(h)$, whose square is $2d = 2(4h + 3)$; depending on the value of h mod 2, we define $D_1, \ldots, D_4$ as follows.

$L_{2,3}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\pi_{4*}L_{2,3}/4 - \beta^4 - \delta^2$	$\pi_{4*}L_{2,3}/4 - \beta^3 - \delta^2$
D_2	$\pi_{4*}L_{2,3}/4 - \alpha^1 - \alpha^2 - \alpha^3 - \gamma^4 - \delta^1$	$\pi_{4*}L_{2,3}/4 - \alpha^1 - \alpha^2 - \gamma^3 - \alpha^4 - \delta^1$
D_3	$\pi_{4*}L_{2,3}/4 - \beta^1 - \beta^2 - \beta^3 - \delta^2$	$\pi_{4*}L_{2,3}/4 - \beta^1 - \beta^2 - \beta^4 - \delta^2$
D_4	$\pi_{4*}L_{2,3}/4 - \gamma^1 - \gamma^2 - \gamma^3 - \alpha^4 - \delta^1$	$\pi_{4*}L_{2,3}/4 - \gamma^1 - \gamma^2 - \alpha^3 - \gamma^4 - \delta^1$

– Consider $L_{4,j}$ for $j = 0, 4, 8, 12$; in this case $\pi_{4*}L_{4,j}/4$ is primitive in $NS(\tilde{Y})$, and we can define $D_1, \ldots, D_4$ simultaneously for any j and any value of h, as follows:

$L_{4,j}(h)$	any h
D_1	$\pi_{4*}L_{4,j}/4$
D_2	$\pi_{4*}L_{4,j}/4 - \alpha^1 - \alpha^2 - \alpha^3 - \alpha^4 - \delta^1 - \delta^2$
D_3	$\pi_{4*}L_{4,j}/4 - \beta^1 - \beta^2 - \beta^3 - \beta^4$
D_4	$\pi_{4*}L_{4,j}/4 - \gamma^1 - \gamma^2 - \gamma^3 - \gamma^4 - \delta^1 - \delta^2$

Proof of Proposition 2.6.1.1. Consider L any of the ample divisors of X presented in Example 2.5.1.6, and the corresponding $D_1, \ldots, D_4$ as in the tables above. Notice that for every i the relation $\pi_4^*(D_i) = L$ is satisfied (since $\pi_4^* M = 0$): therefore we always have $\pi_4^* H^0(\tilde{Y}, D_i) \subset H^0(X, L)$. Moreover, $\pi_4^* H^0(\tilde{Y}, D_i)$ is all contained in one of the eigenspaces $V_\bullet(L)$ (the sections of D_i are in fact well defined on the quotient surface $\tilde{Y}$) and, for $i \neq j$, $\pi_4^* H^0(\tilde{Y}, D_i)$ and $\pi_4^* H^0(\tilde{Y}, D_j)$ are in different eigenspaces, since D_i and D_j intersect differently the exceptional lattice for $i \neq j$: this proves that $H^0(X, L) \supseteq \bigoplus_{i=1}^4 \pi_4^* H^0(\tilde{Y}, D_i)$.
To show the other inclusion, it is enough to chech the dimensions, and we can actually work with the Euler characteristics: $h^0(L) = \chi(L)$ because L is ample, and on the other hand we get that $h^0(D_i) \geq \chi(D_i)$. Indeed, since $D_i^2 \geq -2$, either $H^0(\tilde{Y}, D_i)$ or $H^0(\tilde{Y}, D_i^*)$ is trivial [41, §1.4], and the fact that D_i intersects positively $\pi_{4*} L$ excludes the former case; moreover, by Serre's duality we get $H^2(\tilde{Y}, D_i) \simeq H^0(\tilde{Y}, D_i^*) = 0$.
So we only need to check that

$$\chi(L) = \sum_i \chi(D_i) :$$

by Riemann-Roch, for any divisor C on a K3 surface we have $\chi(C) = C^2/2 + 2$, so in particular for any L of square $2d$, $\chi(L) = d + 2$. The values for each $\chi(D_i)$ are displayed in the following table, thus proving the statement.

Table 2.7: Euler characteristics

	no.	L	$\chi(D_1)$	$\chi(D_2)$	$\chi(D_3)$	$\chi(D_4)$
$d =_4 1$	1	L_0	$(d+3)/4$	$(d+3)/4$	$(d-1)/4$	$(d+3)/4$
$d =_4 2$	2	L_0	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$
	3	$L_{2,2}^{(i)}$	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$
	4	$L_{2,2}^{(j)}$	$(d+6)/4$	$(d+2)/4$	$(d-2)/4$	$(d+2)/4$
$d =_4 3$	5	L_0	$(d+1)/4$	$(d+5)/4$	$(d+1)/4$	$(d+1)/4$
	6	$L_{2,3}$	$(d+5)/4$	$(d+1)/4$	$(d+1)/4$	$(d+1)/4$
$d =_4 0$	7	L_0	$d/4+1$	$d/4+1$	$d/4$	$d/4$
	8	$L_{2,0}$	$d/4+1$	$d/4$	$d/4+1$	$d/4$
	9	$L_{4,j}$	$d/4+2$	$d/4$	$d/4$	$d/4$

□

2.6.2 Eigenspaces of τ^{2*}

The automorphism τ^2 is a symplectic involution on X: the following proposition describes the eigenspaces of a general symplectic involution.

Proposition 2.6.2.1 ([32, Prop. 2.7]). *Let ι be a symplectic involution on a K3 surface X such that $rk(NS(X)) = 9$, let $Z = \widetilde{X/\iota}$ the resolution of the quotient surface, and let $\pi : X \dashrightarrow Z$ the induced rational map. Let L be an ample divisor on X, such that $L^2 = 2d$, that generates $\Omega_2^{\perp_{NS(X)}}$. Then $H^0(X, L) \simeq \pi^* H^0(Z, E_1) \oplus \pi^* H^0(Z, E_2)$, with E_1, E_2 described as follows, for suitable numbering $n_1, \ldots, n_8$ of the exceptional curves of Z:*

1. *if $NS(X) = \Omega_2 \oplus \langle L \rangle$, and $d =_2 0$, then $E_1 = \pi_* L/2 - (n_1 + n_2 + n_3 + n_4)/2$, $E_2 = \pi_* L/2 - (n_5 + n_6 + n_7 + n_8)/2$;*
2. *if $NS(X) = \Omega_2 \oplus \langle L \rangle$, and $d =_2 1$, then $E_1 = \pi_* L/2 - (n_1 + n_2)/2$, $E_2 = \pi_* L/2 - (n_3 + n_4 + n_5 + n_6 + n_7 + n_8)/2$;*
3. *if $NS(X) = (\Omega_2 \oplus \langle L \rangle)'$ (this case occurs only if $d =_2 0$), then $E_1 = \pi_* L/2$, $E_2 = \pi_* L/2 - \sum_{i=1}^8 n_i/2$.*

If X admits an automorphism τ of order 4 and $\tilde{Z}$ is the minimal resolution of X/τ^2, taking L ample that generates $\Omega_4^{\perp_{NS(X)}}$ we have the following relation between the eigenspaces of τ^* and τ^{2*}:

$$H^0(X, L) = \bigoplus_{i=1}^{4} \pi_4^* H^0(\tilde{Y}, D_i) = \pi_2^* H^0(\tilde{Z}, E_1) \oplus \pi_2^* H^0(\tilde{Z}, E_2);$$

the Nef divisors $E_1, E_2 \in NS(\tilde{Z})$ that satisfy this equality for the examples of ample classes introduced in Example 2.5.1.6 are defined in the following tables, with the exceptional curves numbered as in Sections 2.4.2, 2.4.3:

$L_0(d)$	$d =_2 0$	$d =_2 1$
E_1	$\pi_{2*} L_0/2 - (n_5 + n_6 + n_7 + n_8)/2$	$\pi_{2*} L_0/2 - (n_2 + n_3 + n_4 + n_5 + n_6 + n_8)/2$
E_2	$\pi_{2*} L_0/2 - (n_1 + n_2 + n_3 + n_4)/2$	$\pi_{2*} L_0/2 - (n_1 + n_7)/2$

$L_{2,0}(h)$	any h
E_1	$\pi_{2*} L_{2,0}/2$
E_2	$\pi_{2*} L_{2,0}/2 - \sum_{i=1}^8 n_i/2$

$L_{2,2}^{(j)}(h)$	any h, $j = 1, 2$
E_1	$\pi_{2*} L_{2,2}^{(j)}/2 - (n_3 + n_4 + n_5 + n_8)/2$
E_2	$\pi_{2*} L_{2,2}^{(j)}/2 - (n_1 + n_2 + n_6 + n_7)/2$

$L_{2,3}(h)$	any h
E_1	$\pi_{2*}L_{2,3}/2 - (n_5 + n_8)/2$
E_2	$\pi_{2*}L_{2,3}/2 - (n_1+n_2+n_3+n_4+n_6+n_7)/2$

$L_{4,j}(h)$	any h, $j = 0, 4, 8, 12$
E_1	$\pi_{2*}L_{4,j}/2$
E_2	$\pi_{2*}L_{4,j}/2 - \sum_{i=1}^{8} n_i/2$

Proposition 2.6.2.2. *It holds* $\pi_2^*H^0(\tilde{Z}, E_1) = \pi_4^*H^0(\tilde{Y}, D_1) \oplus \pi_4^*H^0(\tilde{Y}, D_3)$*, while* $\pi_2^*H^0(\tilde{Z}, E_2) = \pi_4^*H^0(\tilde{Y}, D_2) \oplus \pi_4^*H^0(\tilde{Y}, D_4)$*.*

Proof. It's easy to see that the dimensions agree using Euler's characteristic (compare to the proof of Proposition 2.6.1.1). Moreover, notice that if E_1 intersects positively $n_i \in \{n_1, n_2, n_6, n_7\}$ (classes fixed by $\hat{\tau}^*$), then $\phi_{|E_1|}(n_i)$ is a curve $\mathcal{C}$ in $\phi_{|E_1|}(\tilde{Z})$; consider now the induced automorphism $\hat{\tau}$ on $\phi_{|E_1|}(\tilde{Z})$: it fixes two points p_1, p_2 on $\mathcal{C}$, each belonging to one (or the other) of the eigenspaces for the action of $\hat{\tau}^*$ on $H^0(\tilde{Z}, E_1)$, that are

$$H^0(\tilde{Z}, E_1) = \widehat{\pi_2}^*H^0(\widetilde{\tilde{Z}/\hat{\tau}}, F_1) \oplus \widehat{\pi_2}^*H^0(\widetilde{\tilde{Z}/\hat{\tau}}, F_2)$$

for some divisors F_1, F_2 of $\widetilde{\tilde{Z}/\hat{\tau}}$. Therefore, $F_1 + F_2$ intersects positively the two curves $\mathcal{C}_1, \mathcal{C}_2$, that resolve the singular points image of p_1, p_2 in $\tilde{Z}/\hat{\tau}$. If E_1 intersects trivially n_i, then $\phi_{|E_1|}(n_i)$ is a point p in $\phi_{|E_1|}\tilde{Z}$, which is fixed by $\hat{\tau}$ and thus belongs to an eigenspace: its image in $\tilde{Z}/\hat{\tau}$ is resolved by a curve $\mathcal{C}_p$, that is intersected positively by either F_1 or F_2, and so by their sum. A similar argument can be applied for $n_i \in \{n_3, n_4, n_5, n_8\}$: if E_1 intersects two curves exchanged by $\hat{\tau}$, then $F_1 + F_2$ intersects with multeplicity 2 the curve which is their image in the resolved quotient; if E_1 does not intersect them, then neither $F_1 + F_2$ intersects their image.
Therefore, we've proved that how $F_1^{(i)} + F_2^{(i)}$ intersects the exceptional lattice of the resolution of $\tilde{Z}/\hat{\tau}$ depends on how E_i intersects that of $\tilde{Z}$.
Since the surfaces $\widetilde{\tilde{Z}/\hat{\tau}}$ and $\tilde{Y}$ are isomorphic (see Rem. 3.4.0.1), we have

$$H^0(X, L) = \bigoplus_{i=1,2} \pi_4^*H^0(\widetilde{\tilde{Z}/\hat{\tau}}, F_1^{(i)}) \oplus \pi_4^*H^0(\widetilde{\tilde{Z}/\hat{\tau}}, F_2^{(i)})$$

and a correspondence between each $H^0(\tilde{Y}, D_j)$ and one of the $H^0(\widetilde{\tilde{Z}/\hat{\tau}}, F_k^{(i)})$.
The fact that this correspondence is exactly as stated comes from a comparison between how the E_i and the D_j intersect the exceptional lattices of $\tilde{Z}$ and $\tilde{Y}$ respectively. □

2.6.3 Examples with $L^2 = 4$

There are three families of K3 surfaces X polarized with an ample class L such that $L^2 = 4$, corresponding to **no. 2**, **no. 3**, **no. 4** of Table 2.7: since it holds $\chi(D_i) = h^0(D_i)$, as we proved in Proposition 2.6.1.1, we can read from Table 2.7 the dimension of the eigenspaces of the action induced by τ. Moreover, by the correspondence between projective families of X and its quotients, and in particular by the degree of $\hat{L}$ and $\overline{L}$ (the pseudo-ample classes on $\tilde{Z}$, $\tilde{Y}$ respectively), we expect the dimension of the projective space in which the quotients are naturally embedded.

We proceed by firstly defining for each L an automorphism ψ on $\mathbb{P}^3$ with eigenspaces of the correct dimension according to Table 2.7, and then finding equations for the K3 surfaces which are invariant for ψ. Recall from Remark 1.4.0.4 that the projective dimension of each of the families of X is $5 = 20 - (rk(\Omega_4) + 1)$, so the equations for the general member of the family depend on 5 parameters.
We check the simplecticity of ψ a posteriori, as we find that the quotient surfaces are birational to K3 surfaces.

Remark 2.6.3.1. A symplectic automorphism τ of order four on a K3 surface X always fixes four points and exchanges two pairs of points; however, an automorphism α of order four with this property is not necessarily symplectic, as it can also hold $\alpha^*\omega_X = -\omega_X$ [2, Prop. 2]. Therefore, unlike involutions, to determine if an automorphism of order 4 on a K3 surface is symplectic it is not enough to check its fixed locus.

The projective models of X and its quotients are summarized in the following table:

no.	X	X/τ^2	X/τ
2	quartic in $\mathbb{P}^3$	complete intersection of 3 quadrics in $\mathbb{P}^5$	$(2,2) \cap (1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$
3	double cover of a quadric	complete intersection of 3 quadrics in $\mathbb{P}^5$	quartic in $\mathbb{P}^3$
4	quartic in $\mathbb{P}^3$	complete intersection of 3 quadrics in $\mathbb{P}^5$	double cover of a quadric

no. 2: The divisor $L_0(2)$ has square 4, and in $NS(X) = \Omega_4 \oplus \mathbb{Z}L$ there exists no class E such that $E^2 = 0$ and $EL_0(2) = 2$: therefore by [88, Thm. 5.2] $\phi_{|L_0(2)|} : X \hookrightarrow \mathbb{P}^3$ is an embedding of X in $\mathbb{P}^3$ as a quartic surface. Consider the automorphism of $\mathbb{P}^3$:

$$\psi_2 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ix_1 : -x_2 : -ix_3);$$

quartic surfaces of the form

$$Q_2 : \; x_0^3x_2 + x_0^2(\alpha x_1^2 + \beta x_3^2) + x_0(\gamma x_2^3 + \delta x_1x_2x_3) + x_2^2(\varepsilon x_1^2 + \zeta x_3^2) + \eta x_1^3x_3 + \theta x_1x_3^3 = 0$$

are invariant under the action of ψ_2, and they depend on 5 projective parameters up to projectivities of the form $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ax_1 : bx_2 : cx_3)$, which commute with ψ_2. Moreover, Q_2 contains exactly 8 points fixed by ψ_2^2, of wich 4 are fixed also by ψ_2; we have therefore

$$\phi_{|L_0(2)|} : X \xrightarrow{\simeq} Q_2 \subset \mathbb{P}^3.$$

To find models for the quotient surfaces, as in [32, §3.4] we consider the map given by the degree 2 invariants under the action of ψ_2^2

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_2 : x_1x_3) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$

then the surface Q_2 maps to the complete intersection of quadrics in $\mathbb{P}^5$

$$R_2 : \begin{cases} z_0z_2 - z_4^2 = 0 \\ z_5^2 - z_1z_3 = 0 \\ z_0z_4 + z_0(\alpha z_1 + \beta z_3) + z_4(\gamma z_2 + \delta z_5) + z_2(\varepsilon z_1 + \zeta z_3) + z_5(\eta z_1 + \theta z_3) = 0 \end{cases}$$

which is a projective model for Q_2/ψ_2^2. Since $\hat{L}_0(2) = \pi_{2*}L_0(2)$ has self-intersection 8, it holds

$$\phi_{|\hat{L}_0(2)|} : Z \xrightarrow{\simeq} R_2 \subset \mathbb{P}^5.$$

Now, the automorphism induced by ψ_2 on $\mathbb{P}^5$ is

$$\hat{\psi}_2 : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : -z_1 : z_2 : -z_3 : -z_4 : z_5) :$$

since the surface R_2 has the same form as in [32, §3.6], then the quotient of R_2 under the action of $\hat{\psi}_2$ is described by a complete intersection in $\mathbb{P}^2_{(z_0:z_2:z_5)} \times \mathbb{P}^2_{(z_1:z_3:z_4)}$ of two polynomials of bidegree respectively $(2,2)$, $(1,1)$, that is

$$S_2 : \begin{cases} z_0z_2z_1z_3 - z_4^2z_5^2 = 0 \\ z_0z_4 + z_0(\alpha z_1 + \beta z_3) + z_4(\gamma z_2 + \delta z_5) + z_2(\varepsilon z_1 + \zeta z_3) + z_5(\eta z_1 + \theta z_3) = 0. \end{cases}$$

no. 3: The divisor $L^{(1)}_{2,2}(0)$ is ample but not very ample: indeed (see [88, Thm. 5.2]) we have $L^{(1)}_{2,2}(0) = H_1 + H_2$ with

$$H_1 = \frac{L_0(0) + \rho + \sigma}{2}, \ H_2 = \frac{L_0(0) + \rho - \sigma}{2}; \quad \langle H_1, H_2 \rangle = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}; \quad \tau^*(H_1) = H_2.$$

Hence

$$\phi_{|L^{(1)}_{2,2}(0)|} = \phi_{|H_1+H_2|} : X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$$

is a double cover ramified along a curve $\mathcal{C}$ of bidegree $(4,4)$ invariant for the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$

$$\overline{\psi}_3 : (x_0 : x_1)(y_0 : y_1) \mapsto (y_0 : iy_1)(x_0 : ix_1);$$

which switches the two copies of $\mathbb{P}^1$ (as prescribed by $\tau^*H_1 = H_2$). The curve $\mathcal{C}$ depends on 5 projective parameters when taking into account the action of the group of projectivities of the form $(x_0 : x_1)(y_0 : y_1) \mapsto (x_0 : ax_1)(y_0 : ay_1)$, which are the only ones that commute with $\overline{\psi}_3$. We embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre map

$$(x_0 : x_1)(y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1) = (z_0 : z_1 : z_2 : z_3) :$$

now X is a double cover of the quadric surface $z_0z_3 = z_1z_2$ ramified along a curve of degree 4 invariant for the automorphism ψ_3 of $\mathbb{P}^3$ induced by $\overline{\psi}_3$ via the Segre map,

$$\psi_3 : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : iz_2 : iz_1 : -z_3);$$

notice that ψ_3 has eigenspaces of the same dimension, accordingly to Table 2.7. The surface X is therefore described in $\mathbb{P}(2,1,1,1,1)$ by

$$Q_3 : \begin{cases} z_0z_3 = z_1z_2 \\ w^2 = \alpha z_0^4 + \beta z_0^2 z_3^2 + \gamma z_3^4 + z_0z_3(\delta z_1^2 + \varepsilon z_1z_2 + \delta z_2^2) + \zeta z_1^4 + \eta z_1^2 z_2^2 + \zeta z_2^4. \end{cases}$$

The fixed locus of ψ_3 on $\mathbb{P}^3$ is $\{(1:0:0:0),(0:0:0:1),(0:1:1:0)\}$: only the first two of these points belong to the branch curve, so to have 4 points fixed by ψ_3 on Q_3, the action induced by ψ_3 on $\mathbb{P}(2,1,1,1,1)$ has to be

$$(w; z_0 : z_1 : z_2 : z_3) \mapsto (w; z_0 : iz_2 : iz_1 : -z_3).$$

To find a projective model of the quotient surface Z, we consider the degree 2 invariants for the action of ψ_3^2, that form a projective space of dimension 6:

$$(w; z_0 : z_1 : z_2 : z_3) \mapsto (w : z_0^2 : z_1^2 : z_2^2 : z_3^2 : z_0z_3 : z_1z_2) = (w : a_0 : a_1 : a_2 : a_3 : a_4 : a_5);$$

the surface Q_3 maps to

$$R_3 : \begin{cases} a_0a_3 = a_4^2 \\ a_1a_2 = a_5^2 \\ a_4 = a_5 \\ w^2 = \alpha a_0^2 + \beta a_0a_3 + \gamma a_3^2 + \delta a_4(a_1 + a_2) + \varepsilon a_4^2 + \zeta a_1^2 + \eta a_1a_2 + \zeta a_2^2, \end{cases}$$

the complete intersection of 3 quadrics in the hyperplane defined by $a_4 = a_5$ in $\mathbb{P}^6$. Eliminate a_5, and change coordinates to

$$b_0 = a_0, \quad b_1 = a_1 + a_2, \quad b_2 = a_1 - a_2, \quad b_3 = a_3, \quad b_4 = a_4;$$

then the automorphism induced by ψ_3 on $\mathbb{P}^5$ is

$$\hat{\psi}_3 : (w : b_0 : b_1 : b_2 : b_3 : b_4) \mapsto (w : b_0 : -b_1 : b_2 : b_3 : -b_4).$$

In the new coordinates we can write R_3 as follows: denote $\rho_1(w, b_0, b_2, b_3) = b_0b_3$, $\rho_2(w, b_0, b_2, b_3) = b_2^2 + 4b_0b_3$, and rewrite the last equation as $b_4^2 = \rho_3(w, b_0, b_2, b_3)$. Then

$$R_3 : \begin{cases} b_1^2 = \rho_1(w, b_0, b_2, b_3) \\ b_1b_4 = \rho_2(w, b_0, b_2, b_3) \\ b_4^2 = \rho_3(w, b_0, b_2, b_3), \end{cases}$$

The expression of R_3 is now similar to [32, §3.7], therefore the quotient surface $S_3 = R_3/\hat{\psi}_3$ is the quartic surface $\rho_1\rho_3 - \rho_2^2 = 0$ in $\mathbb{P}^3$.

no. 4: The eigenspaces associated to the action of τ on $H^0(X, L_{2,2}^{(2)}(1))$ have dimensions $2, 1, 0, 1$, and there is no $E \in NS(X)$ such that $E^2 = 0$ and $EL_{2,2}^{(2)}(1) = 2$, so $L_{2,2}^{(2)}(1)$ is very ample. Consider the automorphism of $\mathbb{P}^3$:

$$\psi_4 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : ix_2 : -ix_3);$$

quartic surfaces of the form

$$Q_4 : f_4(x_0, x_1) + x_2x_3f_2(x_0, x_1) + \alpha x_2^4 + \beta x_2^2x_3^2 + \gamma x_3^4 = 0,$$

where f_4, f_2 are respectively homogeneous quartic and quadric polynomials, are invariant under the action of ψ_4, and they form a family of projective dimension 5 when taking into account the action of the group of projectivities of the form $(x_0 : x_1 : x_2 : x_3) \mapsto (ax_0 + bx_1 : cx_0 + dx_1 : ex_2 : x_3)$, that commute with ψ_4. Moreover, Q_4 contains exactly 4 points fixed by ψ_4, and 4 more fixed only by ψ_4^2. Therefore

$$\phi_{|L_{2,2}^{(2)}(1)|} : X \xrightarrow{\simeq} Q_4 \subset \mathbb{P}^3.$$

Proceeding as in **no. 2**, we consider the map given by the degree 2 invariants under the action of ψ_4^2

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 : x_2x_3) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$

then the quotient $Q_4/\psi^2|_{Q_4}$ is the complete intersection of quadrics in $\mathbb{P}^5$

$$R_4 : \begin{cases} \rho_1 : z_0z_1 - z_4^2 = 0 \\ \rho_2 : z_5^2 - z_2z_3 = 0 \\ \rho_3 : \tilde{f}_4(z_0, z_1, z_4) + z_5\tilde{f}_2(z_0, z_1, z_4) + \alpha z_2^2 + \beta z_5^2 + \gamma z_3^2 = 0, \end{cases}$$

where $\tilde{f}_4, \tilde{f}_2$ are respectively homogeneous quadric and linear polynomials such that $\tilde{f}_4(x_0^2, x_1^2, x_0x_1) = f_4(x_0, x_1)$ (similarly $\tilde{f}_2$ and f_2). The automorphism induced by ψ_4 on $\mathbb{P}^5$ is

$$\hat{\psi}_4 : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : -z_2 : -z_3 : z_4 : z_5).$$

The surface R_4 is singular in 8 points, 4 obtained as $R_4 \cap \{z_0 = z_1 = z_4 = 0\}$, which are not fixed by $\hat{\psi}_4$, and 4 obtained as $R_4 \cap \{z_2 = z_3 = z_5 = 0\}$, which are fixed by $\hat{\psi}_4$.
To find the quotient $R_4/\hat{\psi}_4|_{R_4}$, we can consider the projection $\mathbb{P}^5 \to \mathbb{P}^3$

$$\pi : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : z_4 : z_5)$$

from the line $\ell = (0 : 0 : s : t : 0 : 0)$. Notice that $\pi(\rho_1) = \rho_1$, and that for every $(\overline{z}_0 : \overline{z}_1 : \overline{z}_4 : \overline{z}_5) \in \mathbb{P}^3$ we can compute its pre-image as

$$\begin{cases} \overline{z}_5^2 = st \\ \tilde{f}_4(\overline{z}_0, \overline{z}_1, \overline{z}_4) + \overline{z}_5\tilde{f}_2(\overline{z}_0, \overline{z}_1, \overline{z}_4) + \alpha s^2 + \beta st + \gamma t^2 = 0; \end{cases}$$

setting $B = \beta\overline{z}_5^2 + \overline{z}_5\tilde{f}_2 + \tilde{f}_4$, this gives

$$\begin{cases} s = \overline{z}_5^2/t \\ t^2 = \frac{-B \pm \sqrt{B^2 - 4\alpha\gamma\overline{z}_5^4}}{2\gamma}. \end{cases}$$

There are generally 4 solutions, pairwise identified by the action of $\hat{\psi}_4$: we can therefore define a surface S_4 that completes the diagram

$$\begin{array}{ccc} R_4 & \xrightarrow[\pi]{4:1} & \rho_1 \\ & {\scriptstyle 2:1}\searrow{\scriptstyle /\hat{\psi}_4} \qquad \nearrow{\scriptstyle 2:1} & \\ & S_4 & \end{array}$$

that is, the quotient $S_4 = R_4/\hat{\psi}_4|_{R_4}$ is a double cover of the quadric $\rho_1 \subset \mathbb{P}^3$ ramified over the curve defined by $B^2 - 4\alpha\gamma\overline{z}_5^4 = 0$, and thus is a K3 surface.

2.6.4 Other examples

Again, the following examples are numbered accordingly to the table presented in the proof of Prop. 2.6.1.1.

no. 1: the divisor $L_0(1)$ has self-intersection 2: it holds

$$\phi_{|L_0(1)|} : X \xrightarrow{2:1} \mathbb{P}^2,$$

ramified along a sextic curve $\mathcal{C}$, invariant for the automorphism of $\mathbb{P}^2$

$$\psi_1 : (x_0 : x_1 : x_2) \mapsto (x_0 : ix_1 : -x_2);$$

moreover, since ψ_1 fixes three points, $\mathcal{C}$ has to contain exactly two of them, to have four fixed points on the double cover; thus

$$\mathcal{C} : \alpha x_0^5 x_2 + \beta x_0^4 x_1^2 + \gamma x_0^3 x_2^3 + \delta x_0^2 x_1^2 x_2^2 + \varepsilon x_0 x_1^4 x_2 + \zeta x_0 x_2^5 + \eta x_1^6 + \theta x_1^2 x_2^4 = 0.$$

The moduli space of $\mathcal{C}$ has projective dimension 5, as we have to take into account the action of the projectivities of $\mathbb{P}^2$ that preserve the eigenspaces of ψ, that is, those of the form $(x_0 : x_1 : x_2) \mapsto (x_0 : ax_1 : bx_2)$.
Following [32, §3.2], to find a projective model of the quotient surface $\tilde{Z}$ we consider the embedding of $\mathbb{P}^2$ into the projective space defined by the degree 2 monomials that are invariant for ψ_1^2

$$\chi : (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_0 x_2) = (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$$

as the quadric cone $Q : y_0 y_2 - y_3^2 = 0$. Then, recalling $\hat{L}_0(1) = \pi_{2*} L_0(1)$, we have that

$$\phi_{|\hat{L}_0(1)|} : Z \xrightarrow{2:1} Q$$

is the double cover ramified along the union of the curve $\hat{\mathcal{C}} = \chi(\mathcal{C})$ and the hyperplane $y_1 = 0$; the branch curve of this double cover is therefore

$$\mathcal{C}_Z : y_1(\alpha y_0^2 y_3 + \beta y_0^2 y_1 + \gamma y_0 y_2 y_3 + \delta y_0 y_1 y_2 + \varepsilon y_1^2 y_3 + \zeta y_2^2 y_3 + \eta y_1^3 + \theta y_1 y_2^2) = 0.$$

Now, the automorphism $\hat{\psi}_1$ induced by ψ_1 on $\mathbb{P}^3$ is

$$\hat{\psi}_1 : (y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : -y_1 : y_2 : -y_3);$$

again, to find the quotient surface we consider the immersion of $\mathbb{P}^3$ in the space defined by the invariant monomials of degree 2

$$(y_0 : y_1 : y_2 : y_3) \mapsto (y_0^2 : y_1^2 : y_2^2 : y_3^2 : y_0 y_2 : y_1 y_3) = (v_0 : v_1 : v_2 : v_3 : v_4 : v_5)$$

as the complete intersection of quadrics

$$\begin{cases} v_0 v_2 = v_4^2 \\ v_1 v_3 = v_5^2; \end{cases}$$

the quotient surface Y is the image of Z via this map, that is

$$Y : \begin{cases} v_0 v_2 = v_4^2 \\ v_1 v_3 = v_5^2 \\ v_3 = v_4 \\ w^2 = \alpha v_0 v_5 + \beta v_0 v_1 + \gamma v_4 v_5 + \delta v_1 v_4 + \varepsilon v_1 v_5 + \zeta v_2 v_5 + \eta v_1^2 + \theta v_1 v_2 : \end{cases}$$

this is the intersection of three quadrics in the projective space cut by $v_3 = v_4$ in $\mathbb{P}^6$ with coordinates $(w : v_0 : v_1 : v_2 : v_3 : v_4 : v_5)$, so it is indeed a K3 surface.

no. 8: Consider the automorphism of $\mathbb{P}^5$

$$\psi_8 : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : ix_2 : -x_3 : -x_4 : -ix_5),$$

and the invariant surface

$$Q_8 : \begin{cases} x_2 x_5 = q_1(x_0, x_1) + q_2(x_3, x_4) \\ x_2^2 = \ell_1(x_0, x_1) m_1(x_3, x_4) \\ x_5^2 = \ell_2(x_0, x_1) m_2(x_3, x_4); \end{cases}$$

the fixed locus of ψ_8 comprises 3 isolated points (that are not contained in Q_8) and two lines, $(s : t : 0 : 0 : 0 : 0)$ and $(0 : 0 : 0 : u : v : 0)$, each intersecting Q_8 in two points; the fixed locus of ψ_8^2 comprises the line $(0 : 0 : z : 0 : 0 : w)$, which doesn't intersect Q_8, and the $\mathbb{P}^3$ space $\pi : x_2 = x_5 = 0$, which intersected with Q_8 gives 8 points, of which the 4 already considered. Moreover, the family of projectivities that preserve the eigenspaces of ψ_8 has projective dimension 9, so that the quotient of the moduli space of Q_8 by its action has dimension 5.
The automorphism ψ_8^2 on Q_8 is the same as the one described in [32, §3.7], so the quotient $R_8 = Q_8/\psi_8^2|_{Q_8}$ is the quartic surface

$$R_8 : \ell_1 \ell_2 m_1 m_2 - (q_1 + q_2)^2 = 0$$

in the projective space $\mathbb{P}^3_{(x_0:x_1:x_3:x_4)}$.
The automorphism induced by ψ_8 on this space is

$$\hat{\psi}_8 : (x_0 : x_1 : x_3 : x_4) \mapsto (x_0 : x_1 : -x_3 : -x_4)$$

which is the same as [32, §3.4]: thus, to find the quotient surface $S_8 = R_8/\hat{\psi}_8|_{R_8}$ we consider the space of invariants of degree 2 for $\hat{\psi}_8$:

$$(x_0 : x_1 : x_3 : x_4) \mapsto (x_0^2 : x_1^2 : x_3^2 : x_4^2 : x_0x_1 : x_3x_4) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$

write the equation that defines R_8 in the new coordinates: the product $\ell_1\ell_2$ is a quadric in x_0, x_1, and therefore it becomes a linear expression $\ell(z_0, z_1, z_4)$; similarly, m_1m_2 becomes $m(z_2, z_3, z_5)$, and $q_1 + q_2$ becomes $\lambda(z_0, \dots, z_5)$, both m and λ linear. Therefore, S_8 is described in $\mathbb{P}^5$ by the complete intersection of three quadrics:

$$S_8 : \begin{cases} \ell m = L^2 \\ z_0z_1 = z_4^2 \\ z_2z_3 = z_5^2. \end{cases}$$

no. 9: Consider the automorphism of $\mathbb{P}^5$

$$\psi_9 : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : ix_3 : -x_4 : -ix_5),$$

and the invariant surface

$$Q_9 : \begin{cases} x_3x_5 + q_1(x_0, x_1, x_2) = 0 \\ x_4^2 + q_2(x_0, x_1, x_2) = 0 \\ x_3^2 + x_5^2 + x_4\ell(x_0, x_1, x_2) = 0, \end{cases}$$

where q_1, q_2 are quadrics, and ℓ is linear. The fixed locus of ψ_9 comprises three isolated points that are not contained in Q_9, and the plane $x_3 = x_4 = x_5 = 0$, which intersected with Q_9 gives $q_1 \cap q_2 = \{4 \text{ points}\}$; the fixed locus of ψ_9^2 gives 4 more points on Q_9. Moreover, considering the action of the projectivities that preserve the eigenspaces, that have the form $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (ax_0 + bx_1 + cx_2 : dx_0 + ex_1 + fx_2 : gx_0 + hx_1 + kx_2 : mx_3 : nx_4 : mx_5)$, we find that surfaces of this type form a moduli space of projective dimension 5, as expected. We have

$$\phi_{|L_{4,4}(0)|} : X \xrightarrow{\simeq} Q_9.$$

The divisor $\hat{L}_{4,4}(0)$ on $\tilde{Z}$ can be written as $\hat{L}_{4,4}(0) = E_1 + E_2$, with (using the notation of Def. 2.4.2.3)

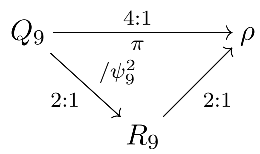
$$E_1 = \frac{\hat{L}_{4,4}(0) + \hat{e}_1 - \hat{f}_1 + \hat{e}_4 - \hat{f}_4 + n_3 + n_4}{2} - x_2 = \frac{\hat{L}_{4,4}(0)}{2} + \frac{(\hat{a}_1 - \hat{a}_2 + \hat{\sigma})}{4},$$

$$E_2 = \frac{\hat{L}_{4,4}(0)}{2} - \frac{(\hat{a}_1 - \hat{a}_2 + \hat{\sigma})}{4}; \qquad \langle E_1, E_2 \rangle = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Thus $\phi_{|\hat{L}_{4,4}(0)|}(\tilde{Z})$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ (see [88, Thm. 5.2]). To find $R_9 = Q_9/\psi_9^2|_{Q_9}$, we proceed as in **no. 4**: consider the projection

$$\pi : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : x_4)$$

from the line $r = (0 : 0 : 0 : s : 0 : t)$; then we have



and R_9 can be described as a double cover of $\rho : x_4^2 + q_2(x_0, x_1, x_2) = 0$ ramified along the quartic curve $x_4^2\ell^2(x_0, x_1, x_2) - 4q_1^2(x_0, x_1, x_2) = 0$. To prove R_9 is a K3 surface, we show that the branch curve intersects in 4 points each of the rulings of ρ: choose $q_2 = x_0x_1 - x_2^2$, in order to write the two lines through the point $(0 : 1 : 0 : 0)$,

$$\begin{cases} x_4 + x_2 = 0 \\ x_0 = 0, \end{cases} \qquad \begin{cases} x_4 - x_2 = 0 \\ x_0 = 0; \end{cases}$$

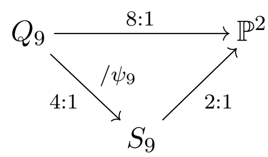
intersecting each of these lines with the branch curve, we obtain a polynomial of degree 4 in one variable.
Now, since $\overline{L}_{4,4}(0)^2 = 2$, $\phi_{|\overline{L}_{4,4}(0)|}(\tilde{Y})$ is a double cover of $\mathbb{P}^2$. This is coherent with the fact that, as the double cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is induced by an involution that exchanges the two copies of $\mathbb{P}^1$, the automorphism induced by ψ_9 on $\mathbb{P}^3$

$$\hat{\psi}_9 : (x_0 : x_1 : x_2 : x_4) \mapsto (x_0 : x_1 : x_2 : -x_4)$$

exchanges the two rulings of ρ, giving a double cover $\rho \to \mathbb{P}^2$; this can also be seen by looking at the divisors on $\tilde{Z}$, as $\hat{\tau}^*(E_1) = E_2$.
The quotient Q_9/ψ_9 is the double cover of $\mathbb{P}^2_{(x_0:x_1:x_2)}$ ramified along the sextic curve $q_2^2\ell^2 + 4q_1^2q_2$: indeed, projecting from the space $\mathbb{P}^2_{(x_3:x_4:x_5)}$ onto $\mathbb{P}^2_{(x_0:x_1:x_2)}$ we get generically 8 points on Q_9, that satisfy

$$x_5^4 = \frac{-(2q_1^2 + q_2\ell^2) \pm \ell\sqrt{(q_2^2\ell^2 + 4q_1^2q_2)}}{2};$$

we can distinguish them in two orbits of four points permuted cyclically by the action of ψ_9: the identification of the points in the same orbit gives the quotient surface S_9, that completes the diagram.



Chapter 3

Action of $(\mathbb{Z}/2\mathbb{Z})^2$ on a K3 surface

3.1 Introduction

In this chapter, proceeding similarly to the previous one, we study the symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \tau, \varphi \rangle$ on a K3 surface.
There are two main differences that set apart this action from that of $\mathbb{Z}/4\mathbb{Z}$: firstly, the image of the maps $\pi_{2,2*}, \pi_{2,2}^*$ induced in cohomology by the rational quotient map $\pi_{2,2} : X \dashrightarrow X/(\mathbb{Z}/2\mathbb{Z})^2$ is not primitive in Λ_{K3} (see Remark 3.4.3.2 and Corollary 3.4.8.2). Moreover, if X is projective then the surfaces $\tilde{Z}_\tau,\ \tilde{Z}_\varphi$, which are the resolution of the singularities of the quotient surfaces $X/\tau,\ X/\varphi$ respectively, may not belong to the same deformation families. Indeed, the involutions τ^*, φ^* may act differently on the polarization L of X, because they act differently on (some of) the orbits for the action of $O(\Omega_{2,2})$ on the discriminant group of the co-invariant lattice $\Omega_{2,2}$. The involution $\rho^* := \tau^* \circ \varphi^*$ however acts similarly to τ^*, and so the surface $\tilde{Z}_\rho = \widetilde{X/\rho}$ always belongs to the same deformation family as $\tilde{Z}_\tau$. We also remark that, in any case, the projective family of the surface $\tilde{Y}$, the resolution of the singularities of the quotient $X/(\mathbb{Z}/2\mathbb{Z})^2$, is completely determined by that of X.

The main results of this chapter are the following: the lattice-theoretic characterization of the intermediate quotient surface $\tilde{Z}_\iota$, that is the resolution of the singularities of the surface X/ι, where ι is any of the involutions in $(\mathbb{Z}/2\mathbb{Z})^2$; the comparison between its moduli space and those of $X, \tilde{Y}$.
The negative definite, rank 12 lattice $\Gamma_{2,2}$ that characterizes the intermediate quotient surface is introduced in Definition 3.4.1.5. Similarly to the lattice Γ_4 for the cyclic case, also $\Gamma_{2,2}$ contains primitively a copy of the Nikulin lattice N and of the co-invariant lattice for a symplectic involution Ω_2: the latter embedding is described in Remark 3.4.3.3.

Theorem 3.1.0.1 (see Thm. 3.4.7.1)**.** *A K3 surface $\tilde{Z}$ is the resolution of the singularities of X/ι, for some K3 surface X with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ and $\iota \in (\mathbb{Z}/2\mathbb{Z})^2$*

an involution, if and only if $\tilde{Z}$ is $\Gamma_{2,2}$-polarized.

Theorem 3.1.0.2 (see Thm. 3.5.2.3, 3.5.3.3)**.** *Let X be a general projective K3 surface with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, let $\tilde{Z}$ and $\tilde{Y}$ be the resolution of the singularities of X/ι and $X/(\mathbb{Z}/2\mathbb{Z})^2$ respectively. The following table describes the correspondence between the Néron-Severi lattices of X and its quotients. The lattices $NS(X)$ are as in Theorem 3.5.1.2, $NS(\tilde{Z})$ are as in Theorem 3.5.3.2 and $NS(\tilde{Y})$ are as in Theorem 3.5.2.2. Be aware that the lattices $(S \oplus \langle k\rangle)'^{(i)}$, $i = 1, 2$ are not isometric.*

$NS(X)$		$NS(\tilde{Z})$	$NS(\tilde{Y})$
$d =_2 1$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$(\Gamma_{2,2} \oplus \langle 4d\rangle)'$	$(M_{2,2} \oplus \langle 2d\rangle)'$
$d =_4 2$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(1)}$	$(M_{2,2} \oplus \langle 8d\rangle)'$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'$	$(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(2)}$ $\Gamma_{2,2} \oplus \langle d\rangle$ $(\Gamma_{2,2} \oplus \langle 4d\rangle)^\star$	$(M_{2,2} \oplus \langle 2d\rangle)'$
$d =_8 0$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(1)}$	$(M_{2,2} \oplus \langle 8d\rangle)'$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(1)}$	$\Gamma_{2,2} \oplus \langle d\rangle$ $(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(2)}$	$(M_{2,2} \oplus \langle 2d\rangle)'$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(2)}$	$(\Gamma_{2,2} \oplus \langle d\rangle)'$	$M_{2,2} \oplus \langle d/2\rangle$
$d =_8 4$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(1)}$	$(M_{2,2} \oplus \langle 8d\rangle)'$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(1)}$	$\Gamma_{2,2} \oplus \langle d\rangle$ $(\Gamma_{2,2} \oplus \langle 4d\rangle)'^{(2)}$	$(M_{2,2} \oplus \langle 2d\rangle)'^{(1)}$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(2)}$ $(\Omega_{2,2} \oplus \langle 2d\rangle)^\star$	$(\Gamma_{2,2} \oplus \langle d\rangle)'$	$(M_{2,2} \oplus \langle 2d\rangle)'^{(2)}$ $M_{2,2} \oplus \langle d/2\rangle$

3.2 A symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ on the surface X_ω

By Nikulin's uniqueness result [71, Thm. 4.7] we can describe the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on Λ_{K3} using a projective model. Again, as we did for the symplectic action of the cyclic group of order four, we choose an elliptic K3 surface $X \xrightarrow{\pi} \mathbb{P}^1$ such that $MW(\pi) = (\mathbb{Z}/2\mathbb{Z})^2$.

3.2.1 The surface X_ω

Let $\omega = e^{i\pi/3}$, consider the elliptic curve $E_\omega = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\omega)$ and define the K3 surface $X_\omega = Kum(E_\omega \times E_\omega)$: its transcendental lattice is

$$T(X_\omega) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix},$$

and its Néron-Severi has therefore $rk = 20$ and $d = 12$.

A description of all the possible Jacobian fibrations on X_ω is provided by Nishiyama [73, Table 1.3]: in particular there exists a fibration

$$\pi : X_\omega \to \mathbb{P}^1 \quad \text{s.t.} \quad MW(\pi) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

which provides a symplectic action of the group $(\mathbb{Z}/2\mathbb{Z})^2$ on X_ω by means of sections t, r that generate $MW(\pi)$.
The reducible fibers of π are one of type I_6^*, one of type I_6 and three of type I_2 that we'll denote I_2^j, $j = 1, 2, 3$. Call C_0 (respectively D_0, E_0^j) the component of I_6^* (resp. I_6, I_2^j) intersected by the curve s, and number the other components so that, for every $k \in \mathbb{Z}/6\mathbb{Z}$, D_i intersects only $D_{(k+1)}$ and $D_{(k-1)}$; C_0, C_1 intersect only C_2; C_9, C_{10} intersect only C_8 and, for $i = 2, \ldots, 8$, C_i intersects both $C_{(i+1)}$ and $C_{(i-1)}$; moreover, it holds $E_0^j E_1^j = 2$, $j = 1, 2, 3$. Thus the trivial lattice $\mathcal{T}(\pi)$ is generated by the classes of the generic fiber F of π, of the curve s and of the components C_i, D_k, E_1^j $i = 1 \ldots, 10$, $k = 1, \ldots, 5$, $j = 1, 2, 3$ of the reducible fibers: these curves are rational, so they have self-intersection -2, the only exception being the class of the generic fiber F, which satisfies $F^2 = 0$. The curves $D_k, k = 1, \ldots, 5$ generate the lattice A_5, and $C_1, \ldots, C_{10}$ generate the lattice D_{10}.
Using the height pairing formula [89, §11.8] we can determine the components of the reducible fibers C_i, D_k, E_m^j that have non-trivial intersection with the elements of $MW(\pi)$. We will choose the following notation for the elements of $MW(\pi)$: the zero section s intersects the components C_0, D_0 and E_0^j; the section t intersects the components C_1, D_3 and E_1^j; the section r intersects the components C_{10}, D_0 and E_1^j; the section $q = t + r$ (where $+$ is the sum in $MW(\pi)$) intersects the components C_9, D_3 and E_0^j. We can write t, r, q in function of the basis of the trivial lattice $\mathcal{T}(\pi)$ using the information about their intersections:

$$t = 2F + s - (\sum_{i=1}^{8} C_i + D_2 + D_3 + D_4) - (C_9 + C_{10} + D_1 + D_3 + D_5 + \sum_{j=1}^{3} E_1^j)/2;$$

$$r = 2F + s - (\sum_{i=1}^{8} iC_i + 4C_9 + 5C_{10} + \sum_{j=1}^{3} E_1^j)/2;$$

$$q = 2F + s - (\sum_{i=1}^{8} iC_i + 5C_9 + 4C_{10} + D_1 + 2D_2 + 3D_3 + 2D_4 + D_5)/2.$$

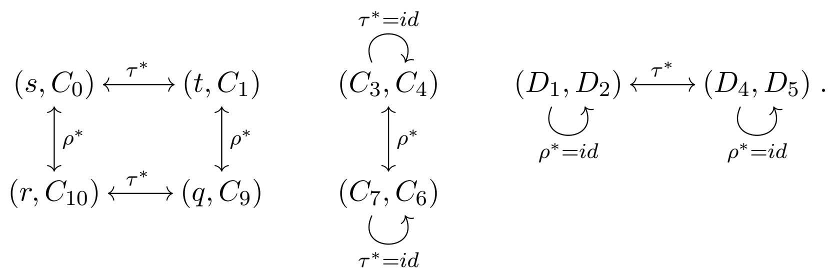
From (2.2.1.1) and the equations above it can be readily seen that $NS(X_\omega)$ admits as a $\mathbb{Z}$-basis $\mathcal{B} = \{F, s, t, r, C_2, \ldots, C_{10}, D_1, \ldots, D_5, E_1^1, E_1^2\}$: indeed the intersection form on the sublattice $L \subset NS(X)$ generated by $\mathcal{B}$ has discriminant -12, which is the opposite to that of $T(X_\omega)$, so $L = NS(X)$.

3.2.2 The action of $\langle \tau^*, \rho^* \rangle$ on the second cohomology of X_ω

The symplectic involutions τ, ρ and $\varphi = \rho \circ \tau$ on X_ω (corresponding respectively to the translation by the sections $t, r, q \in MW(\pi)$) induce isometries τ^*, ρ^* and φ^*: since they are symplectic, they act as the identity on the transcendental lattice $T(X_\omega)$; on $NS(X_\omega)$ we have

$$\tau^* : C_0 \leftrightarrow C_1, \quad C_9 \leftrightarrow C_{10}, \quad D_{[k]_6} \leftrightarrow D_{[k+3]_6}, \quad E_0^j \leftrightarrow E_1^j, \quad s \leftrightarrow t, \quad r \leftrightarrow q,$$
$$\rho^* : C_i \leftrightarrow C_{10-i}, \quad E_0^j \leftrightarrow E_1^j, \quad s \leftrightarrow r, \quad t \leftrightarrow q,$$

and τ^* acts as the identity on $F, C_2, \ldots, C_8$, while ρ^* on $F, D_0, \ldots, D_5$. Therefore, we can identify 8 orthogonal copies of A_2 in $NS(X_\omega)$ that are either fixed or exchanged in pairs by τ^*, ρ^*, as follows:



The orthogonal complement in $NS(X_\omega)$ of the direct sum $A_2^{\oplus 8}$ is generated over $\mathbb{Q}$ by the vectors

$$\begin{aligned}
S_1 &= C_3 + 2C_4 + 3C_5 + 2C_6 + C_7, \\
S_2 &= 4F + 2t + 2s - \Big(\sum_{i=2}^{8} iC_i + 4C_9 + 4C_{10}\Big), \\
S_3 &= E_1^1 - E_1^2, \\
S_4 &= -3(E_1^1 + E_1^2) - 4\Big(2r - 2F - t - s + \sum_{i=2}^{8}(i-1)C_i\Big) + \\
&\quad + 2(-7C_9 - 9C_{10} + D_1 + 2D_2 + 3D_3 + 2D_4 + D_5),
\end{aligned}$$

that are pairwise orthogonal, and whose self-intersection is as follows: $S_1^2 = -6$, $S_2^2 = 6$, $S_3^2 = -4$, $S_4^2 = -12$. It can be also verified that both τ^* and ρ^* act as the identity id on S_1, S_2 and as $-id$ on S_3, S_4.

The lattice $A_2^{\oplus 8} \oplus \langle -6\rangle \oplus \langle 6\rangle \oplus \langle -4\rangle \oplus \langle -12\rangle$ has discriminant group

$$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^{11},$$

so it has index $2^2 3^5$ in $NS(X_\omega)$ (see Remark 1.2.1.2); the latter can be obtained by adding the following generators:

$$\begin{aligned}
u_1 &= (S_1 + S_2)/2,\\
u_2 &= (S_3 + S_4)/2,\\
u_3 &= (q - s + C_0 - C_3 + C_4 - C_6 + C_7 - C_9 - D_1 + D_2 + D_4 - D_5)/3,\\
u_4 &= (r - t + C_1 - C_3 + C_4 - C_6 + C_7 - C_{10} + D_1 - D_2 - D_4 + D_5)/3,\\
u_5 &= (S_1 - C_3 + C_4 + C_6 - C_7)/3,\\
u_6 &= (u_2 - q - r - C_3 + C_4 + C_9 + C_{10})/3,\\
u_7 &= (u_1 + u_2 + r - C_3 + C_4 - C_{10} - D_1 + D_2 + D_4 - D_5 + S_3)/3.
\end{aligned} \tag{3.2.2.1}$$

Remark 3.2.2.1. Notice that the lattice $U(3) = \langle u_1, S_2 - u_1\rangle$ is an overlattice of index 2 of $\langle -6\rangle \oplus \langle 6\rangle$ on which $\tau^*,\ \rho^*$ act as the identity; similarly the lattice $A_2(2) = \langle u_2, S_3 - u_2\rangle$ is an overlattice of index 2 of $\langle -4\rangle \oplus \langle -12\rangle$ on which $\tau^*,\ \rho^*$ act as $-id$.

Now, $H^2(X_\omega, \mathbb{Z})$ is an overlattice of index 12 of the lattice $NS(X_\omega) \oplus T(X_\omega)$ (see Remark 1.2.1.2). Denoting $\{\omega_1, \omega_2\}$ the $\mathbb{Z}$-basis of $T(X_\omega)$ for which the intersection matrix is $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the discriminant form of $A_{NS(X_\omega)\oplus T(X_\omega)}$, in the basis given by

$$\begin{aligned}
n_1 &= (r + C_{10} + C_3 + C_4 + D_1 + D_2 + D_4 + D_5 + u_7 + u_1 + u_2)/2,\\
n_2 &= (r + C_{10} + C_3 + C_4 + u_7 + u_1)/2 + (D_1 - D_2 + D_4 - D_5)/6,\\
t_1 &= \omega_2/2,\\
t_2 &= \omega_1/6 - \omega_2/3,
\end{aligned} \tag{3.2.2.2}$$

is

$$\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1/3 \end{bmatrix}.$$

Therefore, in order to generate $H^2(X_\omega, \mathbb{Z})$ as an overlattice of $NS(X_\omega) \oplus T(X_\omega)$, following [72, Prop. 1.4.1.a] we can add to a $\mathbb{Z}$-basis of the latter the isotropic elements $n_1 + t_1, n_2 + t_2$.

3.3 The action of $(\mathbb{Z}/2\mathbb{Z})^2$ on the K3 lattice

3.3.1 A convenient description of the K3 lattice

We can now describe abstractly the isometries $\tau^*,\ \rho^*,\ \varphi^*$ induced on Λ_{K3} by the symplectic involutions τ, ρ, φ in $(\mathbb{Z}/2\mathbb{Z})^2 \subset Aut(X)$. We fix a marking $\Lambda_{\mathrm{K3}} \simeq H^2(X, \mathbb{Z})$ as follows.

Proposition 3.3.1.1. *The isometries τ^*, ρ^* and φ^* act on the sublattice of finite index of Λ_{K3} $W := A_2^{\oplus 8} \oplus A_2(2) \oplus \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix} \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ as follows:*

$$A_2 \overset{\tau^*}{\oplus} A_2 \underset{\rho^*\ \ \rho^*}{\oplus} A_2 \overset{\tau^*}{\oplus} A_2 \oplus \overbrace{A_2 \underset{\rho^*}{\oplus} A_2}^{\tau^*=id} \oplus \underbrace{A_2 \overset{\tau^*}{\oplus} A_2}_{\rho^*=id} \oplus \underbrace{\overbrace{A_2(2)}^{\tau^*=-id}}_{\rho^*=-id} \oplus \underbrace{U(3) \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}}_{\tau^*=\rho^*=id}$$

The isometry $\varphi^* := (\rho \circ \tau)^*$ exchanges pairwise the eight copies of A_2, and acts as the identity on $A_2(2) \oplus U(3) \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$.

Remark 3.3.1.2. Notice that τ^* and ρ^* act in similar ways on W up to reordering the A_2 components, while the action of φ^* is different. The consequences become apparent only when dealing with projective surfaces: indeed, there are some projective models $X \subset \mathbb{P}^n$ such that the resolution of the singularities of X/τ, X/φ belong to different deformation families (see Corollary 3.5.3.3): this is due to the different action of τ^*, φ^* on the polarization of X; on the other hand, the the resolution of the singularities of X/τ, X/ρ always belong to the same deformation family.
Therefore, from now on we will use τ and φ as generators of $(\mathbb{Z}/2\mathbb{Z})^2$.

For $i = 1, 2$, denote $a_i,\ b_i,\ c_i,\ d_i,\ e_i,\ f_i,\ g_i,\ h_i$ the generators of the eight copies of A_2, such that

$$\begin{aligned} &\tau^* : a_i \leftrightarrow b_i, \quad c_i \leftrightarrow d_i, \quad g_i \leftrightarrow h_i; \\ &\varphi^* : a_i \leftrightarrow d_i, \quad b_i \leftrightarrow c_i, \quad e_i \leftrightarrow f_i; \quad g_i \leftrightarrow h_i; \end{aligned}$$

denote w, z the generators of $A_2(2)$ (on which τ^* acts as $-id$ and φ^* as id), x, y the generators of $U(3)$, and v_1, v_2 the generators of $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. Then the lattice Λ_{K3} is isomorphic to the overlattice $H^2(X, \mathbb{Z})$ of W obtained by adding the following generators (cf. (3.2.2.1),(3.2.2.2)):

$$\begin{aligned} \alpha &= (-a_1 + a_2 + d_1 - d_2 - e_1 + e_2 + f_1 - f_2 - g_1 + g_2 + h_1 - h_2)/3, \\ \beta &= (-b_1 + b_2 + c_1 - c_2 - e_1 + e_2 + f_1 - f_2 + g_1 - g_2 - h_1 + h_2)/3, \\ \gamma &= (x - y - e_1 + e_2 - f_1 + f_2)/3, \\ \delta &= (x - c_1 + c_2 - d_1 + d_2 - e_1 + e_2)/3, \\ \varepsilon &= (x - z + w + c_1 - c_2 - e_1 + e_2 - g_1 + g_2 + h_1 - h_2)/3, \\ \zeta &= (x + z + c_1 + c_2 + e_1 + e_2 + g_1 + g_2 + h_1 + h_2 + \varepsilon)/2 + v_2/2, \\ \eta &= (x + c_1 + c_2 + e_1 + e_2 + \varepsilon)/2 + (g_1 - g_2 + h_1 - h_2)/6 + v_1/6 - v_2/3; \end{aligned} \tag{3.3.1.1}$$

The action of τ^* and φ^* on these elements is deduced by the one on the sublattice W described above by $\mathbb{Q}$-linear extension:

3.3.2 Invariant and co-invariant lattices

The group $(\mathbb{Z}/2\mathbb{Z})^2$ acts symplectically in a unique way on the second integral cohomology lattice of a K3 surface [71, Thm. 4.7]: the invariant and co-invariant lattices for this action can be found in [30, Prop. 4.3]. Here we want to provide an explicit embedding of them in $H^2(X,\mathbb{Z})$ with the description of it given in Section 3.3.1.

The invariant lattice $\Lambda_{\mathrm{K3}}^{\langle\tau,\varphi\rangle}$ is an overlattice of the lattice $I = A_2(4)\oplus A_2(2)^{\oplus 2}\oplus U(3)\oplus \begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = \langle a_1+b_1+c_1+d_1,\ a_2+b_2+c_2+d_2,\ e_1+f_1,\ e_2+f_2,\ g_1+h_1,\ g_2+h_2,\ x,\ y,\ v_1,\ v_2\rangle$ obtained by adding as generators the elements

$$\begin{aligned}
\iota_1 &= (v_1+v_2+g_1+h_1-g_2-h_2)/3,\\
\iota_2 &= (a_1+b_1+c_1+d_1-(a_2+b_2+c_2+d_2)+e_1+f_1-e_2-f_2+x)/3,\\
\iota_3 &= \gamma;
\end{aligned}$$

the co-invariant lattice $\Omega_{2,2}$ is an overlattice of the lattice

$$\Delta = \left[\begin{array}{c|c|c} A_2(2)^{\oplus 3} & 0 & 0\\ \hline 0 & A_3(2) & A_3\\ \hline 0 & A_3 & A_3(2)\end{array}\right]$$

spanned over $\mathbb{Z}$ by $\{z,\ w,\ f_1-e_1,\ f_2-e_2,\ h_1-g_1,\ h_2-g_2,\ b_1-a_1,\ a_1-c_1,\ c_1-d_1,\ a_2-b_2,\ c_2-a_2,\ d_2-c_2\}$ obtained by adding as generators the elements

$$\begin{aligned}
\omega_1 &= (a_1-b_1-c_1+d_1-a_2+b_2+c_2-d_2+z-w)/3,\\
\omega_2 &= (-a_1+d_1+a_2-d_2-e_1+f_1+e_2-f_2-g_1+h_1+g_2-h_2)/3,\\
\omega_3 &= (a_1+b_1-c_1-d_1-a_2-b_2+c_2+d_2-e_1+f_1+e_2-f_2)/3.
\end{aligned}$$

The discriminant group of $\Omega_{2,2}$ is $(\mathbb{Z}/2\mathbb{Z})^6\times(\mathbb{Z}/4\mathbb{Z})^2$. We can describe Λ_{K3} as overlattice of $\Lambda_{\mathrm{K3}}^{\langle\tau,\varphi\rangle}\oplus\Omega_{2,2}$ by adding the elements

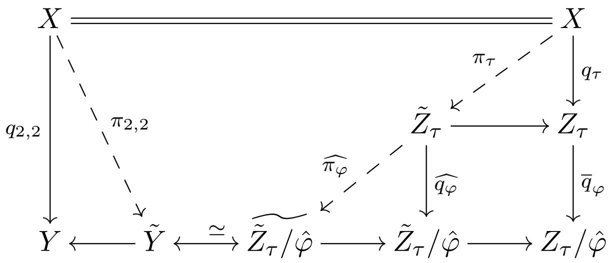
$$\begin{aligned}
r_1 =&(v_2+w)/2,\\
r_2 =&(v_1+e_2+f_2+f_1-e_1+\omega_1+\omega_3)/2,\\
r_3 =&(\iota_2+\iota_3+y+e_2+f_2+f_1-e_1+\omega_3)/2,\\
r_4 =&(e_1+f_1+f_1-e_1)/2,\\
r_5 =&(g_2+h_2+h_1-g_1+b_1+c_1-2a_1+b_2+c_2-2a_2+\omega_2+\omega_3)/2,\\
r_6 =&(\iota_1+v_1+g_2+h_2+w+h_1-g_1)/2,\\
r_7 =&(a_2+b_2+c_2+d_2)/4+3(b_2+c_2+d_2-3a_2)/4,\\
r_8 =&(\iota_2+3\iota_3+y-3a_1+b_1+c_1+d_1-a_2-b_2+3c_2+3d_2)/4+\\
&+(e_2+f_2+f_1-e_1)/2-a_2+\omega_3;
\end{aligned}$$

the elements $\iota_i,\ \omega_i,\ r_k$ (for $i=1,2,3,\ k=1,\dots,8$) are all integral in the lattice $H^2(X,\mathbb{Z})$ described in Section 3.3.1.

Remark 3.3.2.1. The lattice $\Omega_{2,2}$ contains three different copies of the lattice $\Omega_2 \simeq E_8(2)$, co-invarant for $\tau,\ \varphi$ and ρ.

3.4 Quotients

The symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2 = \{1,\tau,\varphi,\rho\}$ on a K3 surface X gives 24 isolated points with nontrivial stabilizer. Call $Fix_\tau = \{t_1,\dots,t_8\},\ Fix_\varphi = \{q_1,\dots,q_8\}, Fix_\rho = \{r_1,\dots,r_8\}$: then τ and φ act on Fix_ρ as the same permutation $(r_1,r_2)(r_3,r_4)(r_5,r_6)(r_7,r_8)$; τ and ρ act on Fix_φ as $(q_1,q_2)(q_3,q_4)(q_5,q_6)(q_7,q_8)$, ρ and φ act on Fix_τ as $(t_1,t_2)(t_3,t_4)(t_5,t_6)(t_7,t_8)$.
Consider the (singular) quotient surfaces $Y = X/(\mathbb{Z}/2\mathbb{Z})^2,\ Z_\tau = X/\tau,\ Z_\varphi = X/\varphi,\ Z_\rho = X/\rho$; resolve the singularities to obtain the K3 surfaces $\tilde{Y},\ \tilde{Z}_\tau,\ \tilde{Z}_\varphi,\ \tilde{Z}_\rho$: then φ induces an involution $\hat{\varphi}$ on Z_τ such that $Z_\tau/\hat{\varphi} \simeq Y$, and this involution can be extended to $\tilde{Z}_\tau$ (similarly for the other quotients). Denote the maps between these surfaces as in the following diagram (and similarly exchanging τ and φ, or considering other pairs including one of them and ρ):



(3.4.0.1)

Remark 3.4.0.1. The surfaces $\tilde{Y}$ and $\widetilde{\tilde{Z}_\tau/\hat{\varphi}}$ are isomorphic, because they are birationally equivalent K3 surfaces.

In the following sections, we are going to describe the maps

$$\begin{aligned}
\pi_{\tau *} &: \Lambda_{\mathrm{K3}} \simeq H^2(X,\mathbb{Z}) \xrightarrow{q_{\tau *}} q_{\tau *}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Z}_\tau,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \\
\widehat{\pi_\varphi}_* &: \Lambda_{\mathrm{K3}} \simeq H^2(\tilde{Z}_\tau,\mathbb{Z}) \xrightarrow{\widehat{\pi_\varphi}_*} \widehat{\pi_\varphi}_* H^2(\tilde{Z}_\tau,\mathbb{Z}) \hookrightarrow H^2(\tilde{Y},\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \\
\pi_{\varphi *} &: \Lambda_{\mathrm{K3}} \simeq H^2(X,\mathbb{Z}) \xrightarrow{q_{\varphi *}} q_{\varphi *}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Z}_\varphi,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \\
\widehat{\pi_\tau}_* &: \Lambda_{\mathrm{K3}} \simeq H^2(\tilde{Z}_\varphi,\mathbb{Z}) \xrightarrow{\widehat{\pi_\tau}_*} \widehat{\pi_\tau}_* H^2(\tilde{Z}_\varphi,\mathbb{Z}) \hookrightarrow H^2(\tilde{Y},\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \\
\pi_{2,2*} &: \Lambda_{\mathrm{K3}} \simeq H^2(X,\mathbb{Z}) \xrightarrow{q_{2,2*}} q_{2,2*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Y},\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}};
\end{aligned}$$

3.4.1 The map $\pi_{\tau *}$ and the surface $\tilde{Z}_\tau$

Proposition 3.4.1.1. *The map $\pi_{\tau *}$ acts in the following way on the sublattice W of $H^2(X,\mathbb{Z})$:*

$$\begin{array}{c}
\underset{a_1,a_2}{A_2} \oplus \underset{b_1,b_2}{A_2} \oplus \underset{c_1,c_2}{A_2} \oplus \underset{d_1,d_2}{A_2} \oplus \underset{e_1,e_2}{A_2} \oplus \underset{f_1,f_2}{A_2} \oplus \underset{g_1,g_2}{A_2} \oplus \underset{h_1,h_2}{A_2} \oplus \underset{z,w}{A_2(2)} \oplus \underset{x,y}{U(3)} \oplus \underset{v_1,v_2}{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}} \\
\downarrow \\
\underset{\hat{a}_1,\hat{a}_2}{A_2} \oplus \underset{\hat{c}_1,\hat{c}_2}{A_2} \oplus \underset{\hat{e}_1,\hat{e}_2}{A_2(2)} \oplus \underset{\hat{f}_1,\hat{f}_2}{A_2(2)} \oplus \underset{\hat{g}_1,\hat{g}_2}{A_2} \oplus 0 \oplus \underset{\hat{x},\hat{y}}{U(6)} \oplus \underset{\hat{v}_1,\hat{v}_2}{\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}}
\end{array}$$

Proof. The action of $\tau*$ on W is described in Proposition 3.3.1.1: we can use it to compute the intersection form of $\pi_{\tau*}W$ via the push-pull formula. Since π_τ is a finite morphism of degree 2, for any $x_1, x_2 \in W$ we get

$$\pi_{\tau*}x_1 \cdot \pi_{\tau*}x_2 = \frac{1}{2}(\pi_\tau^*\pi_{\tau*}x_1 \cdot \pi_\tau^*\pi_{\tau*}x_2)$$

where $\pi_\tau^*\pi_{\tau*}x_1 = x_1 + \tau^*x_1$. Therefore, if τ^* exchanges two copies of A_2, $\pi_{\tau*}(A_2 \oplus A_2) = A_2$; if τ^* acts as the identity on a lattice L, then $\pi_{\tau*}L = L(2)$; if τ^* acts as $-id$ on a lattice L, then $\pi_{\tau*}L = 0$. □

Corollary 3.4.1.2. *The embedding $\pi_{\tau*}H^2(X,\mathbb{Z}) \hookrightarrow H^2(\tilde{Z}_\tau,\mathbb{Z})$ is unique up to isometries of the latter; its orthogonal complement is the Nikulin lattice N.*

Proof. The lattice $\pi_{\tau*}H^2(X,\mathbb{Z})$ can be obtained by $\mathbb{Q}$-linear extension of $\pi_{\tau*}$ applied to the elements (3.3.1.1). The lattice $\pi_{\tau*}H^2(X,\mathbb{Z})$ is an even lattice of signature $(3, 11)$ and $\ell = 6$, so it satisfies the conditions of Theorem 1.2.1.10;. Since $\tilde{Z}_\tau$ is obtained as blow up of X/τ in its eight singular points, the exceptional lattice is the Nikulin lattice N. □

Remark 3.4.1.3. Calling $n_1, \ldots, n_8$ the (-2)-curves that generate N over $\mathbb{Q}$, the elements that glue together N and $\pi_{\tau*}H^2(X,\mathbb{Z})$ to Λ_{K3} can be then chosen to be:

$$\begin{aligned}
s_1 &= (\hat{c}_1 - \hat{c}_2 + \hat{e}_2 + \hat{f}_2 + \hat{\gamma} - \hat{\varepsilon} + n_5 - n_8 + n_3 + n_2)/2 - n_8, \\
s_2 &= (\hat{a}_1 - \hat{a}_2 - \hat{\alpha} + \hat{f}_1 + \hat{f}_2 - \hat{\varepsilon} + n_4 - n_8 + n_3 + n_2)/2 - n_8, \\
s_3 &= (\hat{e}_2 + \hat{f}_1 + n_7 + n_5 + n_4 + n_3)/2 - 2n_8, \\
s_4 &= (\hat{c}_1 - \hat{c}_2 + \hat{e}_2 + \hat{f}_1 - \hat{\varepsilon} + n_7 - n_8 + n_5 + n_4)/2 - n_8, \\
s_5 &= (\hat{a}_1 - \hat{a}_2 + \hat{c}_1 - \hat{c}_2 - \hat{\alpha} + n_6 + n_5 + n_4 + n_2)/2 - 2n_8, \\
s_6 &= (\hat{a}_1 - \hat{a}_2 + \hat{c}_1 - \hat{c}_2 - \hat{\alpha} + \hat{f}_1 + n_7 - n_8 + n_6 + n_3)/2 - n_8.
\end{aligned}$$

Remark 3.4.1.4. The lattice $\pi_{\tau*}\Omega_{2,2}$ is isomorphic to $D_4(2)$ with the following generators: $d_1 = (\hat{e}_2 - \hat{f}_2 + \hat{f}_1 - \hat{e}_1 + \hat{c}_1 - \hat{a}_1 - \hat{c}_2 + \hat{a}_2)/3 - \hat{f}_1 + \hat{e}_1$, $d_2 = (\hat{e}_2 - \hat{f}_2 + \hat{f}_1 - \hat{e}_1 + \hat{c}_1 - \hat{a}_1 - \hat{c}_2 + \hat{a}_2)/3$, $d_3 = \hat{a}_1 - \hat{c}_1$, $d_4 = \hat{c}_1 - \hat{a}_1 + \hat{c}_2 - \hat{a}_2$.

Definition 3.4.1.5. We define the lattice $\Gamma_{2,2}$ as the lattice of rank 12 obtained as over-lattice of $\pi_{\tau *}\Omega_{2,2} \oplus N$ by adding as generators the elements

$$\begin{aligned} x_1 =& (d_4 - d_2 + n_2 + n_4 + n_5 + n_6)/2; \\ x_2 =& (d_1 - d_2 + n_3 + n_7 + n_2 + n_5)/2. \end{aligned}$$

The lattice $\Gamma_{2,2}$ is primitively embedded in $H^2(\tilde{Z}_\tau, \mathbb{Z})$; an integral basis of its orthogonal complement is given by

$$\begin{aligned} s'_1 =& \hat{a}_1 + \hat{c}_1, \\ s'_2 =& \hat{a}_2 + 2(\hat{e}_2 - s_5 + s_6 + s_2) + 4(s_4 - s_3) + \hat{\zeta} + n_5 - n_7, \\ s'_3 =& 2(\hat{c}_1 + \hat{e}_2) - \hat{f}_1 - \hat{a}_2 - 3\hat{\zeta}, \\ s'_4 =& \hat{c}_2 - 8\hat{\alpha} + \hat{e}_2 + 4(\hat{f}_1 + s_1 - s_6) + 2(s_4 - s_5 - s_2 + s_3) - \hat{\zeta} + 3(n_6 - n_5), \\ s'_5 =& \hat{\alpha} - \hat{a}_2 - \hat{f}_1 - \hat{\zeta}, \\ s'_6 =& \hat{a}_2 + \hat{c}_2, \quad s'_7 = \hat{g}_1, \quad s'_8 = \hat{g}_2, \quad s'_9 = \hat{\gamma}, \quad s'_{10} = \hat{\eta} - \hat{\zeta}. \end{aligned}$$

3.4.2 The map $\pi_{\varphi *}$ and the surface $\tilde{Z}_\varphi$

The action of φ^* on the sublattice W of $H^2(X, \mathbb{Z})$ is different that that of τ^*, so the quotient map $\pi_{\varphi *}$ will be different as well: in particular, φ^* does not preserve any of the orthogonal copies of $A_2 \subset W$. The analogue to Proposition 3.4.1.1 is as follows, and the proof is similar.

Proposition 3.4.2.1. *The map $\pi_{\varphi *}$ acts in the following way on the sublattice W of $H^2(X, \mathbb{Z})$:*

$$\begin{array}{ccccccccccccccccccccccc}
\underset{a_1,a_2}{A_2} & \oplus & \underset{d_1,d_2}{A_2} & \oplus & \underset{b_1,b_2}{A_2} & \oplus & \underset{c_1,c_2}{A_2} & \oplus & \underset{e_1,e_2}{A_2} & \oplus & \underset{f_1,f_2}{A_2} & \oplus & \underset{g_1,g_2}{A_2} & \oplus & \underset{h_1,h_2}{A_2} & \oplus & \underset{z,w}{A_2(2)} & \oplus & \underset{x,y}{U(3)} & \oplus & \underset{v_1,v_2}{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}} \\
\searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \downarrow & & \downarrow & & \downarrow \\
& \underset{\tilde{a}_1,\tilde{a}_2}{A_2} & & \oplus & & \underset{\tilde{b}_1,\tilde{b}_2}{A_2} & & \oplus & & \underset{\tilde{e}_1,\tilde{e}_2}{A_2} & & \oplus & & \underset{\tilde{g}_1,\tilde{g}_2}{A_2} & & \oplus & \underset{\tilde{z},\tilde{w}}{A_2(4)} & \oplus & \underset{\tilde{x},\tilde{y}}{U(6)} & \oplus & \underset{\tilde{v}_1,\tilde{v}_2}{\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}}
\end{array}$$

Remark 3.4.2.2. The elements that glue together N and $\pi_{\varphi *}H^2(X, \mathbb{Z})$ to form Λ_{K3} can be then chosen as:

$$\begin{aligned} t_1 &= (\tilde{b}_1 + \tilde{e}_2 + \tilde{g}_1 + \tilde{g}_2 - \tilde{\varepsilon} + \tilde{\gamma} + \tilde{y} + \tilde{\eta})/2 + (n_2 + n_3 + n_5 + n_8)/2; \\ t_2 &= (\tilde{a}_1 + \tilde{a}_2 + \tilde{\delta} + \tilde{z} - \tilde{\varepsilon})/2 + (n_2 + n_3 + n_4 + n_8)/2; \\ t_3 &= (\tilde{\gamma} + \tilde{y})/2 + (n_3 + n_4 + n_5 + n_7)/2; \\ t_4 &= (\tilde{a}_1 + \tilde{a}_2 + \tilde{\delta} + \tilde{\varepsilon} + \tilde{y})/2 + (n_4 + n_5 + n_7 + n_8)/2; \\ t_5 &= (\tilde{b}_1 + \tilde{e}_2 + \tilde{g}_1 + \tilde{g}_2 + \tilde{\varepsilon} + \tilde{\zeta})/2 + (n_2 + n_4 + n_5 + n_6)/2; \\ t_6 &= (\tilde{a}_1 + \tilde{a}_2 + \tilde{\delta} + \tilde{\varepsilon})/2 + (n_3 + n_6 + n_7 + n_8)/2. \end{aligned}$$

Remark 3.4.2.3. The lattice $\pi_{\varphi *}\Omega_{2,2}$ is isomorphic to $D_4(2)$ with the following generators: $d_1' = (2\tilde{b}_2 - 2\tilde{a}_2 + \tilde{z} - \tilde{w} + \tilde{b}_1 - \tilde{a}_1)/3,\ d_2' = \tilde{w} + (2\tilde{b}_2 - 2\tilde{a}_2 + \tilde{z} - \tilde{w} + \tilde{b}_1 - \tilde{a}_1)/3,\ d_3' = \tilde{a}_2 - \tilde{b}_2,\ d_4' = \tilde{a}_1 - \tilde{b}_1$.

Definition 3.4.2.4. Define the lattice $\tilde{\Gamma}$ as the overlattice of $N \oplus \pi_{\varphi *}\Omega_{2,2}$ obtained by adding to the set of generators the elements

$$x_1' = (d_2' - d_1' + n_2 + n_3 + n_4 + n_8)/2, \quad x_2' = (d_2' + d_4' + n_3 + n_6 + n_7 + n_8)/2;$$

Proposition 3.4.2.5. *The lattices $\Gamma_{2,2}$ and $\tilde{\Gamma}$ are isomorphic.*

Proof. The lattices $\pi_{\tau *}\Omega_{2,2}$ and $\pi_{\varphi *}\Omega_{2,2}$ are both isomorphic to $D_4(2)$. Moreover, the gluings that realizes $\Gamma_{2,2}$ as an overlattice of $\pi_{\tau *}\Omega_{2,2} \oplus N$, and $\tilde{\Gamma}$ as an overlattice of $\pi_{\varphi *}\Omega_{2,2} \oplus N$, are isomorphic: indeed, one can easily check that the orbits for the action of $O(D_4(2))$ on $A_{D_4(2)}$, and of $O(N)$ on A_N, are determined by the order and square of their elements. □

3.4.3 The surface $\tilde{Y}$ as quotient of $\tilde{Z}_\tau$

We conclude with the description of the K3 surface $\tilde{Y}$, which is the minimal resolution of the quotient $X/(\mathbb{Z}/2\mathbb{Z})^2$. We can obtain $\tilde{Y}$ as quotient of either $\tilde{Z}_\tau, \tilde{Z}_\varphi$ by the residual symplectic involution $\hat{\varphi}, \hat{\tau}$ respectively.
The residual involution $\hat{\varphi}$ fixes eight isolated points, and it acts on the curves represented by the classes $n_1, \dots, n_8$ of the second integral cohomology of $\tilde{Z}_\tau$ (which are the exceptional curves introduced by the resolution $\tilde{Z}_\tau \to Z_\tau$) by exchanging them pairwise ($\hat{\tau}$ acts similarly on $\tilde{Z}_\varphi$).

Proposition 3.4.3.1. *Consider the sublattice $\pi_{\tau *}W \oplus A_1^{\oplus 8}$ of finite index of $H^2(\tilde{Z}_\tau, \mathbb{Z})$: the map $\widehat{\pi_\varphi}_*$ acts in the following way on it:*

$$\begin{array}{ccccccccccccccccccccc}
\underset{\hat{a}_1,\hat{a}_2}{A_2} & \oplus & \underset{\hat{c}_1,\hat{c}_2}{A_2} & \oplus & \underset{\hat{e}_1,\hat{e}_2}{A_2(2)} & \oplus & \underset{\hat{f}_1,\hat{f}_2}{A_2(2)} & \oplus & \underset{\hat{g}_1,\hat{g}_2}{A_2} & \oplus & \underset{\hat{x},\hat{y}}{U(6)} & \oplus & \underset{\hat{v}_1,\hat{v}_2}{\begin{bmatrix}8 & 4\\ 4 & 8\end{bmatrix}} & \oplus & \underset{n_1,n_8}{A_1^{\oplus 2}} & \oplus & \underset{n_2,n_5}{A_1^{\oplus 2}} & \oplus & \underset{n_3,n_7}{A_1^{\oplus 2}} & \oplus & \underset{n_4,n_6}{A_1^{\oplus 2}} \\
\searrow & & \swarrow & & \searrow & & \swarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \underset{\overline{a}_1,\overline{a}_2}{A_2} & & \oplus & & \underset{\overline{e}_1,\overline{e}_2}{A_2(2)} & & \oplus & \underset{\overline{g}_1,\overline{g}_2}{A_2(2)} & \oplus & \underset{\overline{x},\overline{y}}{U(12)} & \oplus & \underset{\overline{v}_1,\overline{v}_2}{\begin{bmatrix}16 & 8\\ 8 & 16\end{bmatrix}} & \oplus & \underset{\overline{n}_1}{A_1} & \oplus & \underset{\overline{n}_2}{A_1} & \oplus & \underset{\overline{n}_3}{A_1} & \oplus & \underset{\overline{n}_4}{A_1}
\end{array}$$

The lattice $\widehat{\pi_\varphi}_* H^2(\tilde{Z}_\tau, \mathbb{Z})$ can be obtained by $\mathbb{Q}$-linear extension to the elements $\hat{\alpha}, \hat{\gamma}, \hat{\varepsilon}, \hat{\zeta}, \hat{\eta}$ which are the image via $\pi_{\tau *}$ of the elements in (3.3.1.1), and $\nu, s_1, \dots, s_6$ defined in Remark 3.4.1.3. The symbol $\overline{\star}$ denotes the image of $\star$ in $\widehat{\pi_\varphi}_* H^2(\tilde{Z}_\tau, \mathbb{Z})$.

Proof. The only difficult thing is to determine which are the pairs of classes exchanged by $\widehat{\pi_{\varphi *}}|_{A_1^{\oplus 8}}$, and $\widehat{\pi_{\tau *}}|_{A_1^{\oplus 8}}$. To do this, we need to ensure that the the intersection form of the images of $s_1, \dots, s_6$ via $\widehat{\pi_\varphi}_*$ (respectively, of $\tilde{s}_1, \dots, \tilde{s}_6$ via $\widehat{\pi_{\tau *}}$), computed with the push-pull formula, is that of an integral even lattice: the only valid choice is the one in the statement. □

Remark 3.4.3.2. The element $\overline{\gamma}/2$ is integral in $\widehat{\pi_{\varphi}}_* H^2(\tilde{Z}_\tau, \mathbb{Z})$: in fact, it holds

$$\widehat{\pi_{\varphi}}_*(s_1 + s_3 + s_4) = \overline{\gamma}/2 + (\overline{a}_1 - \overline{a}_2 + \overline{e}_1 + 2\overline{e}_2 - \overline{\varepsilon} + 2\overline{n}_2 + \overline{n}_4 + 2\overline{n}_3 - 5\overline{n}_1).$$

A similar result holds for $\widehat{\pi_{\tau}}_* H^2(\tilde{Z}_\varphi, \mathbb{Z})$. A more geometric explanation is given in Remark 3.4.6.1, using the surface X_ω and its quotients.

Remark 3.4.3.3. The co-invariant lattice for $\hat{\varphi}^*$ (which is a copy of $\Omega_2 = E_8(2)$, since $\hat{\varphi}$ is a symplectic involution on $\tilde{Z}_\tau$), is entirely contained in $\Gamma_{2,2}$: indeed, it is the orthogonal complement in $\Gamma_{2,2}$ of $\langle n_1+n_8, n_2+n_5, n_3+n_7, n_4+n_6, \hat{e}_1+\hat{f}_1, \hat{e}_2+\hat{f}_2, \hat{a}_1+\hat{c}_1, \hat{a}_2+\hat{c}_2\rangle$.

The resolution of the singularities $\tilde{Y} \to \tilde{Z}_\tau/\hat{\varphi}$ introduces in cohomology another copy of the lattice N: calling $m_1, \dots m_8$ the (-2)-classes that generate N over $\mathbb{Q}$, we can choose as elements that glue $\widehat{\pi_{\varphi}}_* H^2(\tilde{Z}_\tau, \mathbb{Z}) \oplus N$ to $H^2(\tilde{Y}, \mathbb{Z})$ the following:

$$\begin{aligned}
k_1 &= (\overline{a}_2 + \overline{e}_1 + \overline{g}_2 + \overline{\eta})/2 + (m_2 + m_3 + m_5 + m_8)/2,\\
k_2 &= (\overline{g}_1 + \overline{\eta} + \overline{\zeta})/2 + (m_2 + m_3 + m_4 + m_8)/2,\\
k_3 &= (\overline{a}_1 + \overline{a}_2 + \overline{g}_1 + \overline{s}_1 + \overline{\varepsilon} + \overline{s}_3 + \overline{s}_4 + \overline{n}_8)/2 + (m_3 + m_4 + m_5 + m_7)/2,\\
k_4 &= (\overline{a}_2 + \overline{e}_1 + \overline{\zeta})/2 + (m_4 + m_5 + m_7 + m_8)/2,\\
k_5 &= (\overline{a}_2 + \overline{e}_2 + \overline{s}_1 + \overline{\zeta} + \overline{s}_3 + \overline{s}_4 + \overline{n}_8)/2 + (m_2 + m_4 + m_5 + m_6)/2,\\
k_6 &= (\overline{a}_1 + \overline{e}_2 + \overline{\varepsilon} + \overline{\zeta})/2 + (m_3 + m_6 + m_7 + m_8)/2.
\end{aligned} \tag{3.4.3.1}$$

3.4.4 The exceptional lattice $M_{2,2}$ and the map $\pi_{2,2*}$

The lattice $M_{2,2}$, as described in [71, §6, case 2a], is an overlattice of $A_1^{\oplus 12} = \langle v_1, \dots, v_{12}\rangle$ obtained by adding as generator the elements $(v_1 + \dots + v_8)/2$ and $(v_5 + \dots v_{12})/2$. We want to find an explicit embedding of $M_{2,2}$ in $H^2(\tilde{Y}, \mathbb{Z})$: actually, we can obtain it as overlattice of $N \oplus \widehat{\pi_{\rho}}_* \Gamma_{2,2} = N \oplus \langle \overline{n}_1, \overline{n}_2, \overline{n}_3, \overline{n}_4\rangle$, as follows.

Proposition 3.4.4.1. *The lattice $M_{2,2}$ is generated over $\mathbb{Q}$ by the elements $\overline{n}_1, \dots, \overline{n}_4$, $m_1, \dots, m_8$. To get a set of $\mathbb{Z}$-generators, add the elements*

$$\mu_1 = \frac{m_1 + \dots + m_8}{2}, \quad \mu_2 = \frac{\overline{n}_1 + \dots + \overline{n}_4 + m_1 + m_2 + m_7 + m_8}{2}.$$

Proof. A $\mathbb{Q}$-basis of $M_{2,2}$ is obviously $\{\overline{n}_1, \dots, \overline{n}_4, m_1, \dots, m_8\}$, as these are the classes that come from resolution of the singularities in our construction. Notice moreover that it holds $\mu_1 \in N$, while μ_2 is the one of the only two linear combinations of the form $(\overline{n}_1 + \dots + \overline{n}_4 + m_i + m_j + m_h + m_k)/2$ which are integral in $H^2(\tilde{Y}, \mathbb{Z})$, the other being of course $(\overline{n}_1 + \dots + \overline{n}_4 + m_3 + m_4 + m_5 + m_6)/2$. □

Remark 3.4.4.2. Consider the map $(\pi_{2,2})_*$, defined as the composition $\widehat{\pi_{\varphi}}_* \circ \pi_{\tau *}$: then $(\pi_{2,2})_* H^2(X, \mathbb{Z})$ is a sublattice of index 2 of $\widehat{\pi_{\varphi}}_* H^2(\tilde{Z}_\tau, \mathbb{Z})$. Indeed, it does not contain the element $\overline{\gamma}/2$ (see Remark 3.4.3.2).

3.4.5 The surface $\tilde{Y}$ as quotient of $\tilde{Z}_\varphi$

Proposition 3.4.5.1. *Consider the sublattice $\pi_{\varphi*}W \oplus A_1^{\oplus 8}$ of finite index of $H^2(\tilde{Z}_\varphi, \mathbb{Z})$: the map $\widehat{\pi_{\tau*}}$ acts in the following way on it:*

$$
\begin{array}{ccccccccccccccccccccccc}
\underset{\tilde{a}_1,\tilde{a}_2}{A_2} & \oplus & \underset{\tilde{b}_1,\tilde{b}_2}{A_2} & \oplus & \underset{\tilde{e}_1,\tilde{e}_2}{A_2} & \oplus & \underset{\tilde{g}_1,\tilde{g}_2}{A_2} & \oplus & \underset{\tilde{w},\tilde{z}}{A_2(2)} & \oplus & \underset{\hat{x},\hat{y}}{U(6)} & \oplus & \underset{\hat{v}_1,\hat{v}_2}{\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}} & \oplus & \underset{n_1,n_5}{A_1^{\oplus 2}} & \oplus & \underset{n_2,n_4}{A_1^{\oplus 2}} & \oplus & \underset{n_3,n_8}{A_1^{\oplus 2}} & \oplus & \underset{n_6,n_7}{A_1^{\oplus 2}} \\
\searrow & & \swarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \underset{\overline{a}_1,\overline{a}_2}{A_2} & & \oplus & \underset{\overline{e}_1,\overline{e}_2}{A_2(2)} & \oplus & \underset{\overline{g}_1,\overline{g}_2}{A_2(2)} & \oplus & 0 & \oplus & \underset{\overline{x},\overline{y}}{U(12)} & \oplus & \underset{\overline{v}_1,\overline{v}_2}{\begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}} & \oplus & \underset{\tilde{n}_1}{A_1} & \oplus & \underset{\tilde{n}_2}{A_1} & \oplus & \underset{\tilde{n}_3}{A_1} & \oplus & \underset{\tilde{n}_6}{A_1}
\end{array}
$$

The lattice $\widehat{\pi_{\tau*}}H^2(\tilde{Z}_\varphi, \mathbb{Z})$ can be obtained by $\mathbb{Q}$-linear extension applied to the elements $\tilde{\alpha}, \tilde{\gamma}, \tilde{\varepsilon}, \tilde{\zeta}, \tilde{\eta}$ which are the image via $\pi_{\varphi*}$ of the elements (3.3.1.1)*, and $\nu, t_1, \dots, t_6$ defined in Remark 3.4.2.2. We denote $\tilde{\star} = \widehat{\pi_{\tau*}}\star$; if $\star = \pi_{\varphi*}\bullet$ for $\bullet \in H^2(X, \mathbb{Z})$, then $\overline{\bullet} = \tilde{\star}$.*

The resolution of the singularities $\tilde{Y} \to \tilde{Z}_\varphi/\hat{\tau}$ introduces in cohomology another copy of the lattice N: calling $m_1, \dots m_8$ the (-2)-classes that generate N over $\mathbb{Q}$, we can choose as elements that glue $\widehat{\pi_{\tau*}}H^2(\tilde{Z}_\varphi, \mathbb{Z}) \oplus N$ to $H^2(\tilde{Y}, \mathbb{Z})$ the following:

$$
\begin{aligned}
h_1 &= (\overline{e}_2 + \overline{a}_1 - \tilde{t}_4 - \overline{\varepsilon} - \overline{\zeta} - \tilde{n}_1 - \tilde{t}_3)/2 + (m_5 + m_3 + m_2 + m_8)/2; \\
h_2 &= (\overline{a}_2 - \tilde{t}_4 - \overline{\zeta} - \tilde{n}_1 - \tilde{t}_3)/2 + (m_4 + m_3 + m_2 + m_8)/2; \\
h_3 &= (\overline{g}_2 + \overline{e}_2 + \overline{e}_1 + \overline{a}_2 - \tilde{t}_4 - \overline{\zeta} - \tilde{n}_1 - \tilde{t}_3)/2 + (m_7 + m_5 + m_4 + m_3)/2; \\
h_4 &= (\overline{g}_2 + \overline{e}_2 + \overline{a}_2 - \tilde{t}_4 - \overline{\zeta} - \tilde{n}_1 - \tilde{t}_3)/2 + (m_7 + m_5 + m_4 + m_8)/2; \\
h_5 &= (\overline{g}_1 + \overline{e}_2 + \overline{a}_1 - \overline{\varepsilon} + \overline{a}_2)/2 + (m_6 + m_5 + m_4 + m_2)/2; \\
h_6 &= (\overline{g}_1 + \overline{g}_2 + \overline{e}_1)/2 + (m_7 + m_6 + m_3 + m_8)/2.
\end{aligned}
$$

Proposition 3.4.5.2. *The lattice $M_{2,2}$ is generated over $\mathbb{Q}$ by the elements $\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_6$ $m_1, \dots, m_8$. To get a set of $\mathbb{Z}$-generators, add the elements*

$$
\mu_1 = \frac{m_1 + \cdots + m_8}{2}, \quad \mu_2' = \frac{\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 + \tilde{n}_6 + m_3 + m_4 + m_5 + m_8}{2}.
$$

Remark 3.4.5.3. We won't give the explicit change of basis of $H^2(\tilde{Y}, \mathbb{Z})$ between $\tilde{Y}$ obtained as quotient of $\tilde{Z}_\tau$ or $\tilde{Z}_\varphi$. Notice however that the lattice $M_{2,2}$ is preserved: indeed, it is generated over $\mathbb{Q}$ by the exceptional curves introduced in the resolution of $X/(\mathbb{Z}/2\mathbb{Z})^2$, which do not depend on the intermediate quotient.

3.4.6 Quotients of the surface X_ω

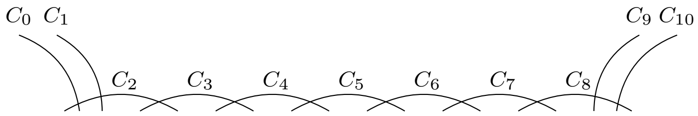
In this section, we give a geometric description of how the element described in Remark 3.4.3.2 appears: to this end, we will consider the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on the elliptic K3 surface X_ω described in Section 3.2.1, looking in particular at the reducible fiber I_6^*.

We are going to rewrite the curves $C_0, \dots, C_{10}$ in terms of the basis of $NS(X_\omega)$ described in Section 3.2.2: starting from its sublattice

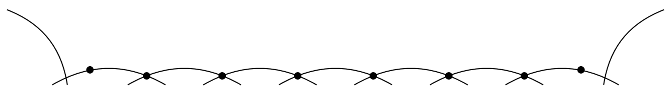
$$A_2^{\oplus 6} \oplus U(3) = \langle s, C_0, t, C_1, r, C_{10}, q, C_9, C_3, C_4, C_7, C_6, u_1, S_2 - u_1 \rangle$$

(which in Section 3.3.1 correspond, in the same order, to $\langle a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, x, y \rangle$) we recover the missing curves of the fiber I_6^* as

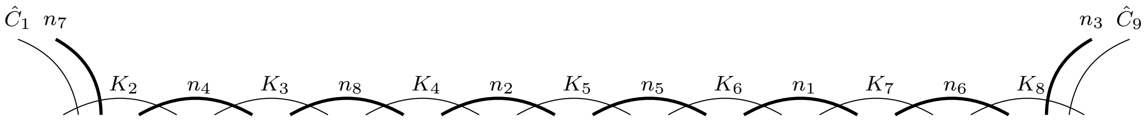
$$\begin{aligned}
&C_5 = (2u_1 - S_2 - C_3 - 2C_4 - 2C_6 - C_7)/3 = \gamma - e_2 - f_2,\\
&C_8 = (S_2 - u_1 - r - q - C_6 - 2C_7 - 2C_9 - 2C_{10})/3,\\
&C_2 = C_8 - (2C_0 + 2C_3 + C_4 - C_6 - 2C_7 - 2C_9 + 2C_1 - 2C_{10} - q + t + s - r)/3.
\end{aligned}$$



The action of τ exchanges C_0 with C_1, C_9 with C_{10}; on each of the remaining curves τ fixes two points (as it's a symplectic involution acting on a rational curve), giving the eight singular points of the quotient surface $Z_\omega := X_\omega/\tau$. Since $\hat{C}_i := \pi_{\tau *}C = d\pi(C)$, where d is the degree of $\pi|_C$, it holds $\hat{C}_1^2 = -2 = \hat{C}_9^2$, $\hat{C}_i^2 = -4$ for $i = 2, \dots 8$, $\hat{C}_i \hat{C}_{i+1} = 2$ for $i = 1, \dots 8$.



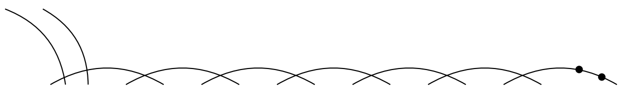
The resolution introduces eight rational curves, giving a fiber of type I_{12}^* on $\tilde{Z}_\omega$. Numbering the exceptional curves $n_1, \dots, n_8$ as in Section 3.4.1.1, the fiber of type I_{12}^* is spanned as follows:



with the curves K_i defined as:

$$\begin{gathered}
K_2 = (\hat{C}_2 - n_4 - n_7)/2; \quad K_3 = (\hat{C}_3 - n_4 - n_8)/2;\\
K_4 = (\hat{C}_4 - n_2 - n_8)/2; \quad K_5 = (\hat{C}_5 - n_2 - n_5)/2;\\
K_6 = (\hat{C}_6 - n_1 - n_5)/2; \quad K_7 = (\hat{C}_7 - n_1 - n_6)/2; \quad K_8 = (\hat{C}_8 - n_3 - n_6)/2.
\end{gathered}$$

The action of the involution $\hat{\varphi}$ on $\tilde{Z}_\omega$ is described in Section 3.4.3: the curves of the fiber I_{12}^* are identified symmetrically with respect to K_5, on which $\hat{\varphi}$ fixes two points (that become two of the eight isolated singularities of the quotient surface $Y_\omega := \tilde{Z}_\omega/\hat{\varphi}$). Again, by definition of the map $\widehat{\pi_{\varphi *}}$, calling $\overline{C}_i = \widehat{\pi_{\varphi *}}\hat{C}_i$ $(i = 1, \dots, 10)$, $\overline{K}_j = \widehat{\pi_{\varphi *}}K_j$ $(j = 2, \dots, 8)$ and $\overline{n}_k = \widehat{\pi_{\varphi *}}n_k$ $(k = 1, \dots, 8)$ all the curves $\overline{C}_1, \overline{C}_9, \overline{K}_j, \overline{n}_k$ have self-intersection -2, with the exception of $\overline{K}_5^2 = -4$.

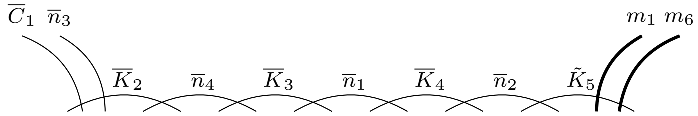


Remark 3.4.6.1. Since $\overline{n}_2 = \widehat{\pi_{\varphi*}} n_2 = \widehat{\pi_{\varphi*}} n_5$, and $\overline{C}_5 = \overline{\gamma} - 2\overline{e}_2$, it holds

$$\overline{K}_5 = (\overline{C}_5 - 2\overline{n}_2)/2 = \overline{\gamma}/2 - \overline{e}_2 - \overline{n}_2.$$

Therefore, since $\overline{e}_2, \overline{n}_2$ are integral elements, also $\overline{\gamma}/2$ is integral.

To conclude, we compute the resolution: we have to introduce two new rational curves, giving a fiber of type I_6^* on the K3 surface $\tilde{Y}_\omega$: numbering the exceptional curves $m_1, \dots, m_8$ as in Section 3.4.3, the I_6^*-type fiber is spanned as follows, with $\tilde{K}_5 := (\overline{K}_5 - m_1 - m_6)/2$:



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3.4.7 A lattice-theoretic characterization of the intermediate quotient surface

Similarly to theorem 2.4.5.1 for the action of $\mathbb{Z}/4\mathbb{Z}$, we give a lattice-theoretic characterization of K3 surfaces which admit a symplectic involution, and are themselves resolution of the singularities of a quotient of a K3 surface by a symplectic involution which is one of the generators of $Aut(X) = (\mathbb{Z}/2\mathbb{Z})^2$, acting symplectically.

Theorem 3.4.7.1. *Let $\tilde{Z}$ be a K3 surface such that $rk(NS(\tilde{Z})) = 12$. There exists a K3 surface X with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ such that $\tilde{Z}$ is birationally equivalent to the quotient X/ι, where ι is one of the generators of $(\mathbb{Z}/2\mathbb{Z})^2$ if and only if $NS(\tilde{Z}) = \Gamma_{2,2}$ (see Def. 3.4.1.5).*

Proof. The "only if" is true by construction (see Sections 3.4.1, 3.4.2). Conversely, suppose $NS(\tilde{Z}) = \Gamma_{2,2}$: the embedding $\Omega_2 \subset \Gamma_{2,2}$ described in Remark 3.4.3.3 defines a symplectic involution $\hat{\rho}$ on $\tilde{Z}$, and the Néron-Severi lattice of the resolution $\tilde{Y}$ of $Y = \tilde{Z}/\hat{\rho}$ is a copy of $M_{2,2}$, as proved in Proposition 3.4.4.1; therefore, by Nikulin's results in [71] the surface $\tilde{Y}$ is the resolution of the quotient of a K3 surface X by the symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, and it holds $NS(X) = \Omega_{2,2}$. The action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\Omega_{2,2}$ defines three copies of $\Omega_2 \subset \Omega_{2,2}$, as described in Section 3.3.2; choose one of them, and define ι as the involution for which it is the co-invariant lattice (this is always possible by the Torelli theorem). Taking the quotient map $\pi_\iota : X \to X/\iota$ and the resolution $\widetilde{X/\iota}$, it then holds $NS(\widetilde{X/\iota}) \simeq NS(\tilde{Z})$. □

3.4.8 The dual maps

We're now going to define the dual maps

$$\pi_\tau^* : H^2(\tilde{Z}_\tau, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

$$\pi_{\varphi}^{*}: H^2(\tilde{Z}_{\varphi}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$
$$\pi_{2,2}^{*}: H^2(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

using the descriptions of $H^2(\tilde{Z}_\tau, \mathbb{Z}), H^2(\tilde{Z}_\varphi, \mathbb{Z}), H^2(\tilde{Y}, \mathbb{Z})$ provided in Sections 3.4.1, 3.4.2, 3.4.3 respectively. The proof of the following proposition is similar to that of Proposition 2.4.4.1.

Proposition 3.4.8.1. *1. The map $\pi_\tau^*: H^2(\tilde{Z}_\tau, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ annihilates N, and acts on $\pi_{\tau*}W \subset \pi_{\tau*}\Lambda_{\mathrm{K3}}$ as follows*

$$\pi_\tau{}^*: \quad A_2^{\oplus 3} \oplus A_2(2)^{\oplus 2} \oplus U(6) \oplus \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \longrightarrow A_2^{\oplus 6} \oplus A_2^{\oplus 2} \oplus U(3) \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$
$$\begin{pmatrix} \hat{a}_1, \hat{a}_2 & & & \\ \hat{c}_1, \hat{c}_2 & \begin{matrix}\hat{e}_1, \hat{e}_2 \\ \hat{f}_1, \hat{f}_2\end{matrix} & \hat{x}, \hat{y} & \hat{v}_1, \ \hat{v}_2 \\ \hat{g}_1, \hat{g}_2 & & & \end{pmatrix} \longmapsto \begin{pmatrix} a_1+b_1, a_2+b_2 & & & \\ c_1+d_1, c_2+d_2 & \begin{matrix}2e_1, 2e_2 \\ 2f_1, 2f_2\end{matrix} & 2x, 2y & 2v_1, 2v_2 \\ g_1+h_1, g_2+h_2 & & & \end{pmatrix}$$

Its action can be extended to $\pi_{\tau*}\Lambda_{\mathrm{K3}}$ adding the following elements (and their respective image to the image lattice): $\hat{\alpha} = (-\hat{a}_1 + \hat{a}_2 + \hat{c}_1 - \hat{c}_2 - \hat{e}_1 + \hat{e}_2 + \hat{f}_1 - \hat{f}_2)/3$, $\hat{\gamma} = (\hat{x} - \hat{y} - \hat{e}_1 + \hat{e}_2 - \hat{f}_1 + \hat{f}_2)/3$, $\hat{\delta} = (\hat{x} - 2\hat{c}_1 + 2\hat{c}_2 - \hat{e}_1 + \hat{e}_2)/3$, $\hat{\varepsilon} = (\hat{x} + \hat{c}_1 - \hat{c}_2 - \hat{e}_1 + \hat{e}_2)/3$, $\hat{\zeta} = (\hat{x} + \hat{c}_1 + \hat{c}_2 + \hat{e}_1 + \hat{e}_2 + \hat{\varepsilon})/2 + \hat{v}_2/2 + \hat{g}_1 + \hat{g}_2$, $\hat{\eta} = (\hat{x} + \hat{c}_1 + \hat{c}_2 + \hat{e}_1 + \hat{e}_2 + \hat{\varepsilon})/2 + (\hat{g}_1 - \hat{g}_2 - \hat{v}_2)/3 + \hat{v}_1/6$;

to extend the action to $H^2(\tilde{Z}_\tau, \mathbb{Z})$, add also $s_1, \ldots, s_6$ (see Remark 3.4.1.3).

2. The map $\pi_\varphi^*: H^2(\tilde{Z}_\varphi, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ annihilates N, and acts on $\pi_{\varphi*}W \subset \pi_{\varphi*}\Lambda_{\mathrm{K3}}$ as follows

$$\pi_\varphi{}^*: \quad A_2^{\oplus 4} \oplus A_2(4) \oplus U(6) \oplus \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \longrightarrow A_2^{\oplus 6} \oplus A_2^{\oplus 2} \oplus U(3) \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$
$$\begin{pmatrix} \tilde{a}_1, \tilde{a}_2 & & & \\ \tilde{b}_1, \tilde{b}_2 & \tilde{z}, \tilde{w} & \tilde{x}, \tilde{y} & \tilde{v}_1, \ \tilde{v}_2 \\ \tilde{e}_1, \tilde{e}_2 & & & \\ \tilde{g}_1, \tilde{g}_2 & & & \end{pmatrix} \longmapsto \begin{pmatrix} a_1+d_1, a_2+d_2 & & & \\ b_1+c_1, b_2+c_2 & 2z, 2w & 2x, 2y & 2v_1, 2v_2 \\ e_1+f_1, e_2+f_2 & & & \\ g_1+h_1, g_2+h_2 & & & \end{pmatrix}$$

Its action can be extended to $\pi_{\varphi*}\Lambda_{\mathrm{K3}}$ adding the following elements (and their respective image to the image lattice): $\tilde{\gamma} = (\tilde{x} - \tilde{y} - 2\tilde{e}_1 + 2\tilde{e}_2)/3$, $\tilde{\delta} = (\tilde{x} - \tilde{b}_1 + \tilde{b}_2 - \tilde{a}_1 + \tilde{a}_2 - \tilde{e}_1 + \tilde{e}_2)/3$, $\tilde{\varepsilon} = (\tilde{x} - \tilde{z} + \tilde{w} + \tilde{b}_1 - \tilde{b}_2 - \tilde{e}_1 + \tilde{e}_2)/3$, $\tilde{\zeta} = (\tilde{x} + \tilde{z} + \tilde{b}_1 + \tilde{b}_2 + \tilde{e}_1 + \tilde{e}_2 + \tilde{\varepsilon})/2 + \tilde{v}_2/2 + \tilde{g}_1 + \tilde{g}_2$, $\tilde{\eta} = (\tilde{x} + \tilde{b}_1 + \tilde{b}_2 + \tilde{e}_1 + \tilde{e}_2 + \tilde{\varepsilon})/2 + (\tilde{g}_1 - \tilde{g}_2 - \tilde{v}_2)/3 + \tilde{v}_1/6$;

to extend the action to $H^2(\tilde{Z}_\varphi, \mathbb{Z})$, add also $t_1, \ldots, t_6$ (see Remark 3.4.2.2).

3. The map $\pi_{2,2}^* H^2(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ annihilates $M_{2,2}$, and acts on $\pi_{2,2*}W \subset \pi_{2,2*}\Lambda_{\mathrm{K3}}$ as follows

$$\pi_{2,2}{}^*: \quad A_2 \oplus A_2(2)^{\oplus 2} \oplus U(12) \oplus \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix} \longrightarrow A_2^{\oplus 8} \oplus U(3) \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{pmatrix} \overline{a}_1, \overline{a}_2 & \begin{matrix} \overline{e}_1, \overline{e}_2 \\ \overline{g}_1, \overline{g}_2 \end{matrix} & \overline{x}, \overline{y} & \overline{v}_1,\ \overline{v}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \begin{matrix} a_1+b_1+c_1+d_1 \\ a_2+b_2+c_2+d_2 \\ 2e_1+2f_1, 2e_2+2f_2 \\ 2g_1+2h_1, 2g_2+2h_2 \end{matrix} & 4x, 4y & 4v_1, 4v_2 \end{pmatrix}$$

Its action can be extended to $\pi_{\varphi *}\Lambda_{\mathrm{K3}}$ adding the following elements (and their respective image to the image lattice): $\overline{\gamma} = (\overline{x} - \overline{y} - 2\overline{e}_1 + 2\overline{e}_2)/3$, $\overline{\varepsilon} = (\overline{x} + \overline{a}_1 - \overline{a}_2 - \overline{e}_1 + \overline{e}_2)/3$, $\overline{\zeta} = (\overline{x} + \overline{a}_1 + \overline{a}_2 + \overline{e}_1 + \overline{e}_2 + \overline{v}_2 + \overline{\varepsilon})/2 + \overline{g}_1 + \overline{g}_2$, $\overline{\eta} = (x + \overline{a}_1 + \overline{a}_2 + \overline{e}_1 + \overline{e}_2 + \overline{\varepsilon})/2 + (\overline{g}_1 - \overline{g}_2 - \overline{v}_2)/3 + \overline{v}_1/6$;

to extend the action to $H^2(\tilde{Y}, \mathbb{Z})$, add also $\overline{\gamma}/2$ and $k_1, \dots, k_6$ (see (3.4.3.1)).

Corollary 3.4.8.2. *The image of the map $\pi_{2,2}^* : H^2(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is not primitive in $H^2(X, \mathbb{Z})$: indeed, it is a sublattice of index 2^3 of the invariant lattice for the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on X.*

Proof. One finds the elements $\pi_{2,2}^*(k_4 + k_6)/2, \pi_{2,2}^*(\overline{a}_1 + \overline{\varepsilon} + \overline{\eta})/2, \pi_{2,2}^*\overline{\gamma}/4$ are integral in $H^2(X, \mathbb{Z})$. However, they do not belong to $\pi_{2,2}^* H^2(\tilde{Y}, \mathbb{Z})$: indeed, a $\mathbb{Z}$-basis of the latter is given by the image via $\pi_{2,2}^*$ of $\{\overline{a}_1, \overline{\gamma}/2, \overline{\varepsilon}, \overline{\eta}, k_1, \dots, k_6\}$. □

3.5 Projective families of K3 surfaces with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ and their quotients

Projective families of K3 surfaces with a symplectic action of a finite group G are classified by the Néron-Severi lattice of their general member, which is always some overlattice of finite index of $\Omega_G \oplus \langle 2d \rangle$ (see Remark 1.4.0.5). The notation for the overlattices is explained in Remark 2.5.0.5.
As is the case with the action of $\mathbb{Z}/4\mathbb{Z}$, the family of X uniquely determines the family of $\tilde{Y}$, the resolution of $X/(\mathbb{Z}/2\mathbb{Z})^2$. However, unlike the cyclic case, when $G = (\mathbb{Z}/2\mathbb{Z})^2$, knowing the family which X belongs to is not always enough to determine the family the intermediate quotient surface $\tilde{Z}$ belongs to: indeed, two different phenomena can happen. When d is even, the two generators τ, φ of $(\mathbb{Z}/2\mathbb{Z})^2$ may act differently on the polarization L of X (the third involution $\rho = \varphi \circ \tau$ always acts as τ acts): if so, the two quotient surfaces $\tilde{Z}_\tau, \tilde{Z}_\varphi$ belong to different projective families. Moreover, when $d =_4 2$ the projective family polarized with the lattice $(\Omega_{2,2} \oplus \langle 2d \rangle)'$ admits two distinct actions of $(\mathbb{Z}/2\mathbb{Z})^2$, corresponding to two different primitive embeddings of $\Omega_{2,2}$: for one of them, the involutions τ^*, φ^* act differently on the ample class $L = \Omega_{2,2}^\perp$, for the other they act similarly. Therefore, if X belongs to this family, there are *three* different families $\tilde{Z}$ can belong to.

We proceed similarly to Section 2.5: we are going to skip repeating the explanation of the methods, and the proofs which are readily adapted from those presented there.

3.5.1 Projective families of K3 surfaces with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$

Proposition 3.5.1.1. *The relation $\sim_{\Omega_{2,2}}$ (see Def. 2.5.1.1) divides $A_{\Omega_{2,2}}$ in 4 non-trivial equivalence classes (plus the trivial one $\{0\}$):*

$k \backslash g$	0	1/2	1	3/2
2	111	0	144	0
4	0	384	0	384

The equivalence classes for $\approx_{\Omega_{2,2}}$ are given in the table below: for each one we give a representative element $x_{(k,g,n)}$ in terms of the generators of $\Omega_{2,2}$.

class (k,g,n)	representative $x_{(k,g,n)}$
$(2,0,108)$	$\frac{f_1-e_1+h_1-g_1}{2}$
$(2,0,3)$	$\frac{b_1+c_1+d_1-3a_1}{2}$
$(2,1,108)$	$\frac{w+f_1-e_1+h_1-g_1}{2}$
$(2,1,36)$	$w/2$
$(4,1/2,384)$	$\frac{b_1+c_1+d_1-3a_1}{4}$
$(4,3/2,384)$	$\frac{b_2+c_2+d_2-3a_2}{4}+\frac{w}{2}$

Theorem 3.5.1.2. *Let X be a projective K3 surface that admits a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, such that $rk(NS(X)) = 13$. Then, $NS(X)$ is one of the following lattices:*

1. *for every $d \in \mathbb{N}$, $NS(X) = \Omega_{2,2} \oplus \langle 2d\rangle$;*
2. *for any $d =_4 0$ there are two possible overlattices of index 2 of $\Omega_{2,2}\oplus\langle 2d\rangle$, $NS(X) = (\Omega_{2,2} \oplus \langle 2d\rangle)'^{(i)}$, $i = 1, 2$: these are not isometric lattices;*
3. *for $d =_4 2$, $NS(X) = (\Omega_{2,2} \oplus \langle 2d\rangle)'$: this lattice is uniquely determined by d and the index, but it admits two non isomorphic embeddings $\iota_1, \iota_2 : \Omega_{2,2} \hookrightarrow NS(X)$, i.e. no isometry $\psi \in O(\Omega_{2,2})$ exists such that $\iota_1 = \iota_2 \circ \psi$;*
4. *For $d =_{16} 4$ or $d =_{16} -4$, $NS(X) = (\Omega_{2,2} \oplus \langle 2d\rangle)^\star$ overlattice of index 4 of $\Omega_{2,2} \oplus \langle 2d\rangle$, uniquely determined by d and the index of the overlattice.*

Example 3.5.1.3. We are going to exhibit a primitive embedding in Λ_{K3} of each of the lattices presented in 3.5.1.2, having fixed the primitive embedding of $\Omega_{2,2}$ in $H^2(X,\mathbb{Z})$ described in Section 3.3.2, by providing examples of primitive classes $L \in \Omega_{2,2}^{\perp_{H^2(X,\mathbb{Z})}}$ such that $L^2 = 2d$ (we may assume that L is ample by Lemma 2.5.0.2). We use the notation of Section 3.3.1, and we construct the overlattices using the elements $x_{(k,g,n)}$ in Proposition 3.5.1.1. We remark that using $x_{(2,1,108)}$ and $x_{(2,1,36)}$ we obtain isomorphic lattices.

1. For every $d \in \mathbb{N} \setminus \{0\}$, the class

$$L_0 = L_0(d) = (x + 2y - e_1 - f_1 + e_2 + f_2)/3 + dy$$

generates the lattice $\langle 2d \rangle$ such that $\Omega_{2,2} \oplus \langle 2d \rangle$ is primitively embedded in $H^2(X, \mathbb{Z})$.

2. For $d = 4h,\ h \in \mathbb{N} \setminus \{0\}$ the class

$$L_{2,0}^{(1)}(h) = 2L_0(h) + e_1 + f_1 + g_1 + h_1$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{2,0}^{(1)}/2 + x_{(2,0,108)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

3. For $d = 4(h-1),\ h \in \mathbb{N} \setminus \{0, 1\}$ the class

$$L_{2,0}^{(2)}(h) = 2L_0(h) + a_1 + b_1 + c_1 + d_1$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{2,0}^{(2)}/2 + x_{(2,0,3)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

4. For $d = 4h + 2,\ h \in \mathbb{N}$, the class

$$L_{2,2}^{(a)}(h) = 2L_0(h) + v_2 + f_1 + e_1 + h_1 + g_1$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{2,2}^{(a)}/2 + x_{(2,1,108)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

5. For $d = 4h + 2,\ h \in \mathbb{N}$, the class

$$L_{2,2}^{(b)}(h) = 2L_0(h) + v_2$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)'$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{2,2}^{(b)}/2 + x_{(2,1,36)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

6. For $d = 16h - 4,\ h \in \mathbb{N}$, the class

$$L_{4,-4}(h) = 4L_0(h) + a_1 + b_1 + c_1 + d_1$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{4,-4}/4 + x_{(4,1/2,384)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

7. For $d = 16h + 4,\ h \in \mathbb{N}$, the class

$$L_{4,4}(h) = 4L_0(h) + 2v_2 + a_2 + b_2 + c_2 + d_2$$

generates the lattice $\langle 2d \rangle$ such that $(\Omega_{2,2} \oplus \langle 2d \rangle)^\star$ is primitively embedded in $H^2(X, \mathbb{Z})$; $L_{4,4}/4 + x_{(4,3/2,384)}$ is in fact integral in $H^2(X, \mathbb{Z})$.

3.5.2 Projective families of K3 surfaces that arise as resolution of the singularities of $X/(\mathbb{Z}/2\mathbb{Z})^2$

Projective surfaces $\tilde{Y}$ that are the resolution of $X/(\mathbb{Z}/2\mathbb{Z})^2$ have to primitively contain in their Néron-Severi both the exceptional lattice $M_{2,2}$ described in Section 3.4.4 and a positive class L of square $2e$ that generates $M_{2,2}^{\perp_{NS(\tilde{Y})}}$: therefore, $\tilde{Y}$ is polarized with the lattice $M_{2,2} \oplus \langle 2e \rangle$ or one of its cyclic overlattices.

Theorem 3.5.2.1. *The relation $\sim_{M_{2,2}}$ (see Def. 2.5.1.1) divides $A_{M_{2,2}}$ in 4 non-trivial equivalence classes (plus the trivial one $\{0\}$):*

$k \backslash g$	0	1/2	1	3/2
2	55	64	72	64

We give a representative element $x_{(k,g,n)}$ for each non-trivial equivalence class (k, g, n) for $\approx_{M_{2,2}}$ in terms of the generators of $M_{2,2}$ introduced in Proposition 3.4.4.1.

class (k, g, n)	representative $x_{(k,g,n)}$
$(2, 0, 54)$	$(\overline{n}_1 + \overline{n}_4 + m_1 + m_7)/2$
$(2, 0, 1)$	$(m_3 + m_4 + m_5 + m_6)/2$
$(2, 1/2, 64)$	$(\overline{n}_1 + m_1 + m_6)/2$
$(2, 1, 54)$	$(\overline{n}_1 + \overline{n}_4 + m_1 + m_4 + m_5 + m_7)/2$
$(2, 1, 18)$	$(m_3 + m_6)/2$
$(2, 3/2, 64)$	$(\overline{n}_1 + m_1 + m_3 + m_4 + m_5)/2$

Theorem 3.5.2.2. *Let $\tilde{Y}$ be a projective K3 surface such that $rk(NS(\tilde{Y})) = 13$ and $NS(\tilde{Y})$ contains primitively $M_{2,2}$ and $\langle 2e \rangle$, $e \in \mathbb{N} \setminus \{0\}$. Then, $NS(\tilde{Y})$ is one of the following:*

1. *for every e, $NS(\tilde{Y}) = M_{2,2} \oplus \langle 2e \rangle$;*
2. *for every e, $NS(\tilde{Y})$ is an overlattice of index 2 of $M_{2,2} \oplus \langle 2e \rangle$. If $e =_4 0$ there are two non isomorphic possibilities for $NS(\tilde{Y})$: $(M_{2,2} \oplus \langle 2e \rangle)'^{(i)}$, $i = 1, 2$; if $e =_4 2$ $NS(\tilde{Y}) = (M_{2,2} \oplus \langle 2e \rangle)'$ is unique, but there are two non isomorphic embeddings of $M_{2,2}$ in $NS(\tilde{Y})$. If e is odd, this overlattice uniquely determined by e and the index.*

Each of these lattices admits a unique primitive embedding in Λ_{K3}.

Proof. The overlattices of $M_{2,2} \oplus \langle 2e \rangle$ are in bijection with the equivalence classes for

$\approx_{M_{2,2}}$. Fix the primitive embedding $M_{2,2} \hookrightarrow \Lambda_{\mathrm{K3}}$ as in Proposition 3.4.4.1: the orthogonal complement of $M_{2,2}$ is the overlattice of index 2 of the lattice $(\pi_{2,2})_* H^2(X,\mathbb{Z})$ obtained by the addition of $\overline{\gamma}/2$ as generator.

We can use as generators of the lattice $\langle 2e\rangle = M_{2,2}^{\perp_{NS(\tilde{Y})}}$ the primitive classes $\overline{L}$ in $H^2(\tilde{Y},\mathbb{Z})$ obtained from $(\pi_{2,2})_* L$ (with L as in Example 3.5.1.3) as follows:

1. $NS(\tilde{Y}) = M_{2,2} \oplus \langle 2e\rangle$ is realized as follows:
   - if $e =_2 0$ by $\overline{L} = (\pi_{2,2})_* L^{(2)}_{2,0}(h)/4, h = e+1$;
   - if $e =_4 1$ by $\overline{L} = (\pi_{2,2})_* L_{4,4}(h)/4, h = (e-1)/4$;
   - if $e =_4 3$ by $\overline{L} = (\pi_{2,2})_* L_{4,-4}(h)/4, h = (e+1)/4$;
2. $NS(\tilde{Y}) = (M_{2,2} \oplus \langle 2e\rangle)'$ is realized as follows:
   - if $e =_2 1$, $\overline{L} = (\pi_{2,2})_* L_0(e)/2$; if $e =_4 1$ the element $\overline{L}/2 + x_{(2,3/2,64)}$ is integral in $H^2(\tilde{Y},\mathbb{Z})$, if $e =_4 3$ the element $\overline{L}/2 + x_{(2,1/2,64)}$ is;
   - if $e =_4 2$, $\overline{L} = (\pi_{2,2})_* L^{(k)}_{2,2}(h)/2$; if $k = a$ $\overline{L}$ glues to an element of the class $(2,1,54)$, that is $x_{(2,1,54)}$ if h is even, $(\overline{n}_1+\overline{n}_4+m_1+m_3+m_6+m_7)/2$ otherwise; if $k = b$, $\overline{L}$ glues to an element of the class $(2,1,18)$, that is $x_{(2,1,18)}$ if h is even, $(m_4+m_5)/2$ otherwise; the lattices obtained for $k = a, b$ are isomorphic, but the isomorphism does not restrict to an isometry of $M_{2,2}$;
   - if $e =_8 4$, either $\overline{L} = (\pi_{2,2})_* L^{(1)}_{2,0}(h)/2$, $h = e/4$, and $\overline{L}$ glues to $(\overline{n}_1 + \overline{n}_4 + m_1 + m_3 + m_4 + m_5 + m_6 + m_7)/2$, which belongs to the class $(2,0,54)$; or $\overline{L} = (\pi_{2,2})_* L^{(2)}_{2,0}(h)/2$, $h = e/4+1$, and $\overline{L}/2 + x_{(2,0,1)}$ is integral; the lattices obtained are not isomorphic.
   - if $e =_8 0$, either $\overline{L} = (\pi_{2,2})_* L^{(1)}_{2,0}(h)/2$, $h = e/4$, and $\overline{L}/2 + x_{(2,0,54)}$ is integral; or $\overline{L} = (\pi_{2,2})_* L_0(d)$, $d = e/4$, and $\overline{L}/2 + x_{(2,0,1)}$ is integral; again, the lattices obtained are not isomorphic.

Since all the equivalence classes for the relation $\approx_{M_{2,2}}$ have been used, we've exhausted all the possibile overlattices of $M_{2,2} \oplus \langle 2e\rangle$. Each non-isomorphic $NS(\tilde{Y})$ admits a unique primitive embedding in Λ_{K3}, by Proposition 1.2.1.11. □

Corollary 3.5.2.3. *In the following table we give the correspondence between families of K3 surfaces X with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, and $\tilde{Y}$ which is the minimal resolution of the quotient $X/(\mathbb{Z}/2\mathbb{Z})^2$, with the notation of Remark 2.5.0.5. The primitive classes $\overline{L} \in NS(\tilde{Y})$ that generate the sublattices $\langle nd\rangle$ as stated are indicated in curly brackets.*

$NS(X)$		$NS(\tilde{Y})$	
$d =_2 1$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(M_{2,2} \oplus \langle 2d \rangle)'$	$\{\overline{L} = \pi_{2,2*}L_0/2\}$
$d =_4 2$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(M_{2,2} \oplus \langle 8d \rangle)'$	$\{\overline{L} = \pi_{2,2*}L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'$	$(M_{2,2} \oplus \langle 2d \rangle)'$	$\{\overline{L} = \frac{\pi_{2,2*}L_{2,2}^{(k)}}{2}, k = a, b\}$
$d =_8 0$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(M_{2,2} \oplus \langle 8d \rangle)'$	$\{\overline{L} = \pi_{2,2*}L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$(M_{2,2} \oplus \langle 2d \rangle)'$	$\{\overline{L} = \frac{\pi_{2,2*}L_{2,0}^{(1)}}{2}\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$M_{2,2} \oplus \langle d/2 \rangle$	$\{\overline{L} = \frac{\pi_{2,2*}L_{2,0}^{(2)}}{4}\}$
$d =_8 4$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(M_{2,2} \oplus \langle 8d \rangle)'$	$\{\overline{L} = \pi_{2,2*}L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$(M_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$\{\overline{L} = \frac{\pi_{2,2*}L_{2,0}^{(1)}}{2}\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$(M_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$\{\overline{L} = \frac{\pi_{2,2*}L_{2,0}^{(2)}}{2}\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)^{\star}$	$M_{2,2} \oplus \langle d/2 \rangle$	$\{\overline{L} = \frac{\pi_{2,2*}L_{4,-4}}{4}, \frac{\pi_{2,2*}L_{4,4}}{4}\}$

3.5.3 Projective families of K3 surfaces that are intermediate quotient for the symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$

Projective surfaces $\tilde{Z}$ that are the resolution of $X/\langle\iota\rangle$, where ι is one of the generators of $(\mathbb{Z}/2\mathbb{Z})^2$ have to primitively contain in their Néron-Severi both the lattice $\Gamma_{2,2}$ described in Definition 3.4.1.5, and a positive class L of square $2x$ that generates $\Gamma_{2,2}^{\perp_{NS(\tilde{Z})}}$: therefore, $\tilde{Z}$ is polarized with the lattice $\Gamma_{2,2} \oplus \langle 2x \rangle$ or one of its cyclic overlattices.

Remark 3.5.3.1. Since the orthogonal complement of $\Gamma_{2,2}$ in $H^2(\tilde{Z}, \mathbb{Z})$ is the image of $H^2(X, \mathbb{Z})$ via the map $\pi_{\iota*}$, where ι is either τ, ρ or φ, one expects that any overlattice S of $\Gamma_{2,2} \oplus \langle 2x \rangle$ be realized with $\Gamma_{2,2}^{\perp_S} = \pi_{\iota*}L$, L chosen among those in Example 3.5.1.3, divided by an appropriate integer if not already primitive. This is, however, not entirely true: indeed, for $L \in \{L_{2,0}^{(1)}, L_{2,2}^{(b)}\}$ the classes $\pi_{\tau*}L$ and $\pi_{\varphi*}L$ realize different overlattices S; therefore, to get the full picture both involutions τ, φ must be considered simultaneously (while ρ gives the same results as τ).

There is an alternative path we could have taken: find classes $L_{2,0}^{(3)}, L_{2,2}^{(c)}$ which glue to the same orbit as $L_{2,0}^{(1)}, L_{2,2}^{(b)}$ respectively for the action induced by $O(\Omega_{2,2})$ on the discriminant group, but on which τ^* acts as φ^* acts on $L_{2,0}^{(1)}, L_{2,2}^{(b)}$ respectively. For instance, in the same orbit for the action induced by $O(\Omega_{2,2})$ on the discriminant group we find the elements $w/2$, which is killed by $\pi_{\tau*}$, but not by $\pi_{\varphi*}$, and $(e_1 - f_1)/2$, which is killed by $\pi_{\varphi *}$, but not by $\pi_{\tau *}$.

Theorem 3.5.3.2. *Let $\tilde{Z}$ be a K3 surface such that $rk(NS(\tilde{Z})) = 13$; suppose $NS(\tilde{Z})$ admits a primitive embedding of both $\Gamma_{2,2}$ and a class of positive square $2x$ that generates $\Gamma_{2,2}^{\perp_{NS(\tilde{Z})}}$. Then $NS(\tilde{Z})$ is one of the following:*

1. *for any x, $\Gamma_{2,2} \oplus \langle 2x \rangle$;*
2. *for any $x =_4 0$ there are two non-isomorphic overlattices of index 2: $(\Gamma_{2,2} \oplus \langle 2x \rangle)'^{(i)}$, $i = 1, 2$;*
3. *for $x =_4 2$, $(\Gamma_{2,2} \oplus \langle 2x \rangle)'$, uniquely determined by x and the index of the overlattice;*
4. *for $x =_8 4$, $(\Gamma_{2,2} \oplus \langle 2x \rangle)^{\star}$, uniquely determined by x and the index of the overlattice.*

Proof. Consider the table of non-trivial equivalence classes for $\sim_{\Gamma_{2,2}}$:

k \ g	0	1/2	1	3/2
2	23	0	40	0
4	0	96	0	96

An element of the form $(E + \alpha)/2$, with $E^2 = 2x$ and $\alpha \in \Gamma_{2,2}$, has integer, even self-intersection only if x is even, and an element of the form $(E + \alpha)/4$ only if $x =_8 4$. The equivalence classes for $\approx_{\Gamma_{2,2}}$ are presented in the following table:

class (k, g, n)	repr. $x_{(k,g,n)} \in \pi_{\tau *} H^2(X, \mathbb{Z})$	repr. $y_{(k,g,n)} \in \pi_{\varphi *} H^2(X, \mathbb{Z})$
$(2, 0, 3)$	$\frac{\hat{f}_1 - \hat{e}_1}{2}$	$\frac{n_2+n_3+n_4+n_8}{2}$
$(2, 0, 8)$	$\frac{n_3+n_5+n_6+n_8}{2}$	$\frac{n_3+n_4+n_5+n_7}{2}$
$(2, 0, 12)$	$\frac{\hat{f}_1-\hat{e}_1+n_4+n_6+\hat{c}_1-\hat{a}_1}{2}$	$\frac{\tilde{b}_1-\tilde{a}_1+n_6+n_7}{2}$
$(2, 1, 4)$	$\frac{\hat{f}_1-\hat{e}_1+n_4+n_6}{2}$	$\frac{n_6+n_7}{2}$
$(2, 1, 12)$	$\frac{\hat{c}_1-\hat{a}_1}{2}$	$\frac{\tilde{b}_1-\tilde{a}_1}{2}$
$(2, 1, 24)$	$\frac{n_3+n_6+n_5+n_8+\hat{c}_1-\hat{a}_1}{2}$	$\frac{\tilde{b}_1-\tilde{a}_1+n_3+n_4+n_5+n_7}{2}$
$(4, 1/2, 96)$	$\frac{3(\hat{f}_1-\hat{e}_1)}{4} + \frac{n_6+n_8}{2}$	$\frac{n_2+n_3+3n_4+3n_8}{4} + \frac{x'_1}{2}$
$(4, 3/2, 96)$	$\frac{\hat{f}_1-\hat{e}_1}{4} + \frac{n_3+n_4+n_5+n_6}{2}$	$\frac{n_2+3n_3+n_4+3n_8}{4} + \frac{x'_1+n_5+n_6}{2}$

The corresponding overlattice of $\Gamma_{2,2} \oplus \langle 2x \rangle$ can be realized having fixed either the embedding $\Gamma_{2,2} \in H^2(\tilde{Z}_\tau, \mathbb{Z})$ as in Definition 3.4.1.5, or $\tilde{\Gamma} \in H^2(\tilde{Z}_\varphi, \mathbb{Z})$ as in Proposition 3.4.2.5 (recall that $\Gamma_{2,2} \simeq \tilde{\Gamma}$, and it admits a unique primitive embedding in Λ_{K3}). We write $\pi_{\iota*}L$ where both $\pi_{\tau*}L, \pi_{\varphi*}L$ give the same projective family: this can be checked by direct computation of the discriminant group.

1. The lattice $\Gamma_{2,2} \oplus \langle 2x \rangle$ is realized as follows:
   - if $x = 2h+1$, $\Gamma_{2,2}^{\perp} = \pi_{\tau*}L_{2,2}^{(b)}(h)/2$;
   - if $x = 2h$, $\Gamma_{2,2}^{\perp} = \pi_{\varphi*}L_{2,0}^{(1)}(h)/2$;
2. The overlattices of index 2 of $\Gamma_{2,2} \oplus \langle 2x \rangle$ are realized as follows:
   - if $x = 4h+2$, $\Gamma_{2,2}^{\perp}$ can be $\pi_{\iota*}L_0(2h+1), \pi_{\iota*}L_{2,0}^{(2)}(2h+2)/2$ and, depending on the parity of h, one between $\pi_{\iota*}L_{4,-4}(h/2+1/2)/2, \pi_{\iota*}L_{4,4}(h/2)/2$, (respectively for h odd or even). The three overlattices thus realized are isometric, but the isometry does not restrict to $\Gamma_{2,2}$: indeed, the selected positive classes glue to elements belonging respectively to the orbits $(2,1,4), (2,1,24), (2,1,12)$.
   - if $x = 8h+4$, $\Gamma_{2,2}^{\perp}$ can be $\pi_{\iota*}L_0(4h+2), \pi_{\iota*}L_{2,0}^{(2)}(4h+3)/2$ or $\pi_{\iota*}L_{2,2}^{(a)}(h)/2$; the first two choices give isometric overlattices (the isometry does not restrict to $\Gamma_{2,2}$, since they glue to elements belonging respectively to the orbit $(2,0,8), (2,0,12)$), the last gives a different one (it glues to an element in $(2,0,3)$).
   - if $x = 8h$, $\Gamma_{2,2}^{\perp}$ can be $\pi_{\iota*}L_0(4h), \pi_{\iota*}L_{2,0}^{(2)}(4h+1)/2$ or $\pi_{\tau*}L_{2,0}^{(1)}(h)$; the first two choices give isometric overlattices (the isometry does not restrict to $\Gamma_{2,2}$, since they glue to elements belonging respectively to the orbit $(2,0,8), (2,0,12)$), the last gives a different one (it glues to an element in $(2,0,3)$).
3. The overlattice of index 4 of $\Gamma_{2,2} \oplus \langle 2x \rangle$ for $x = 8h+4$ is realized by $\Gamma_{2,2}^{\perp} = \pi_{\varphi*}L_{2,2}^{(b)}(h)$: it glues to an element in $(4, 3/2, 96)$ if h is even, in $(4, 1/2, 96)$ if h is odd.

□

Corollary 3.5.3.3. *We give the correspondence between families of projective K3 surfaces X with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ and $\tilde{Z}$, the resolution of the singularities of the quotient X/ι, with ι one of the generators of $(\mathbb{Z}/2\mathbb{Z})^2$, by describing the relation between the Néron-Severi lattices of their general member. When choosing different involutions gives different results, they are denoted τ, φ accordingly to the previous sections. The notation is explained in Remark 2.5.0.5.*

	$NS(X)$	$NS(\tilde{Z})$	
$d =_2 1$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(\Gamma_{2,2} \oplus \langle 4d \rangle)'$	$\{\pi_{\iota *} L_0\}$
$d =_4 2$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(1)}$	$\{\pi_{\iota *} L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'$	$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(2)}$	$\{\pi_{\iota *} L_{2,2}^{(a)}\}$
		$\Gamma_{2,2} \oplus \langle d \rangle$	$\{\pi_{\tau *} L_{2,2}^{(b)}/2\}$
		$(\Gamma_{2,2} \oplus \langle 4d \rangle)^\star$	$\{\pi_{\varphi *} L_{2,2}^{(b)}\}$
$d =_8 0$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(1)}$	$\{\pi_{\iota *} L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$\Gamma_{2,2} \oplus \langle d \rangle$	$\{\pi_{\varphi *} L_{2,0}^{(1)}/2\}$
		$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(2)}$	$\{\pi_{\tau *} L_{2,0}^{(1)}\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$(\Gamma_{2,2} \oplus \langle d \rangle)'$	$\{\pi_{\iota *} L_{2,0}^{(2)}/2\}$
$d =_8 4$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(1)}$	$\{\pi_{\iota *} L_0\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$\Gamma_{2,2} \oplus \langle d \rangle$	$\{\pi_{\varphi *} L_{2,0}^{(1)}/2\}$
		$(\Gamma_{2,2} \oplus \langle 4d \rangle)'^{(2)}$	$\{\pi_{\tau *} L_{2,0}^{(1)}\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$(\Gamma_{2,2} \oplus \langle d \rangle)'$	$\{\pi_{\iota *} L_{2,0}^{(2)}/2\}$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)^\star$		$\left\{\frac{\pi_{\iota *} L_{4,-4}}{2}, \frac{\pi_{\iota *} L_{4,4}}{2}\right\}$

3.6 Projective models

Given a nef and big divisor L on X, there is a natural map $\phi_{|L|} : X \to \mathbb{P}(H^0(X, L)^*) \simeq \mathbb{P}^n$, with $n = L^2/2+1$. Any automorphism σ of X that preserves L induces an action on $\mathbb{P}(H^0(X, L)^*)$: in particular, if σ is finite of order m, we can split $H^0(X, L)$ in eigenspaces corresponding to the m-roots of unity.

Remark 3.6.0.1. As we already discussed in Remark 2.6.0.1, the action of σ on $H^0(X, L)$ could have order km for some integer $k > 1$, being such that

$$\sigma^m : (x_0, \dots, x_n) \mapsto \xi_k(x_0, \dots, x_n)$$

for ξ_k a root of unity. Consider however the action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \tau, \varphi \rangle$ on X: even if we tacitly take τ^k and φ^h on $H^0(X, L)$ instead of τ and φ (as we did in Remark 2.6.0.1), we don't have any control on the order of their composition ρ – we only know that it divides kh. Therefore, if the group acting on $H^0(X,L)$ is bigger than $(\mathbb{Z}/2\mathbb{Z})^2$, we can assume it is dihedral.

3.6.1 Eigenspaces of τ, φ

Let X be a K3 surface with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, let L be the ample class that generates $\Omega_{2,2}^{\perp_{NS(X)}}$. We consider the action of the symplectic involutions $\tau, \varphi \in (\mathbb{Z}/2\mathbb{Z})^2$ (recall that the third involution $\rho = \varphi \circ \tau$ acts on L as τ acts): if $\tilde{Z}_\tau, \tilde{Z}_\varphi$ are the minimal resolution of $X/\tau, X/\varphi$ respectively, we have

$$\begin{aligned} H^0(X,L) &= \pi_\tau^* H^0(\tilde{Z}_\tau, E_1) \oplus \pi_\tau^* H^0(\tilde{Z}_\tau, E_2) \\ &= \pi_\varphi^* H^0(\tilde{Z}_\varphi, F_1) \oplus \pi_\varphi^* H^0(\tilde{Z}_\varphi, F_2); \end{aligned}$$

the nef divisors $E_1, E_2 \in NS(\tilde{Z}_\tau), F_1, F_2 \in NS(\tilde{Z}_\varphi)$ that satisfy these equalities for the choices of ample classes introduced in Example 3.5.1.3 are defined in the following tables with the exceptional curves numbered as in Sections 3.4.1, 3.4.2:

$L_0(d)$	$d =_2 0$	$d =_2 1$
E_1	$\pi_{\tau*}L_0/2 - (n_3+n_5+n_6+n_8)/2$	$\pi_{\tau*}L_0/2 - (n_1+n_8)/2$
E_2	$\pi_{\tau*}L_0/2 - (n_1+n_2+n_4+n_7)/2$	$\pi_{\tau*}L_0/2 - (n_2+n_3+n_4+n_5+n_6+n_7)/2$
F_1	$\pi_{\varphi*}L_0/2 - (n_1+n_2+n_6+n_8)/2$	$\pi_{\varphi*}L_0/2 - (n_6+n_7)/2$
F_2	$\pi_{\varphi*}L_0/2 - (n_3+n_4+n_5+n_7)/2$	$\pi_{\varphi*}L_0/2 - (n_1+n_2+n_3+n_4+n_5+n_8)/2$

$L_{2,0}^{(1)}(h)$	any h
E_1	$\pi_{\tau*}L_{2,0}^{(1)}/2 - (n_1+n_4+n_6+n_8)/2$
E_2	$\pi_{\tau*}L_{2,0}^{(1)}/2 - (n_2+n_3+n_5+n_7)/2$
F_1	$\pi_{\varphi*}L_{2,0}^{(1)}/2$
F_2	$\pi_{\varphi*}L_{2,0}^{(1)}/2 - \sum_{i=1}^8 n_i/2$

$L_{2,0}^{(2)}(h)$	any h
E_1	$\pi_{\tau*}L_{2,0}^{(2)}/2$
E_2	$\pi_{\tau*}L_{2,0}^{(2)}/2 - \sum_{i=1}^8 n_i/2$
F_1	$\pi_{\varphi*}L_{2,0}^{(2)}/2$
F_2	$\pi_{\varphi*}L_{2,0}^{(2)}/2 - \sum_{i=1}^8 n_i/2$

$L_{2,2}^{(a)}(h)$	any h
E_1	$\pi_{\tau*}L_{2,2}^{(a)}/2 - (n_1+n_4+n_6+n_8)/2$
E_2	$\pi_{\tau*}L_{2,2}^{(a)}/2 - (n_2+n_3+n_5+n_7)/2$
F_1	$\pi_{\tau*}L_{2,2}^{(a)}/2 - (n_1+n_5+n_6+n_7)/2$
F_2	$\pi_{\tau*}L_{2,2}^{(a)}/2 - (n_2+n_3+n_4+n_8)/2$

$L_{4,\pm4}(h)$	any h
E_1	$\pi_{\tau*}L_{4,\pm4}/2$
E_2	$\pi_{\tau*}L_{4,\pm4}/2 - \sum_{i=1}^8 n_i/2$
F_1	$\pi_{\varphi*}L_{4,\pm4}/2$
F_2	$\pi_{\varphi*}L_{4,\pm4}/2 - \sum_{i=1}^8 n_i/2$

$L^{(b)}_{2,2}(h)$	any h
E_1	$\pi_{\tau*}L^{(b)}_{2,2}/2$
E_2	$\pi_{\tau*}L^{(b)}_{2,2}/2-\sum_{i=1}^{8}n_i/2$
F_1	$\pi_{\varphi*}L^{(b)}_{2,2}/2-(n_1+n_5+n_6+n_7)/2$
F_2	$\pi_{\varphi*}L^{(b)}_{2,2}/2-(n_2+n_3+n_4+n_8)/2$

3.6.2 Eigenspaces and classes in $NS(\tilde{Y})$

To determine the eigenspaces of the full action of $(\mathbb{Z}/2\mathbb{Z})^2$, we will have to consider the residual involutions $\hat{\varphi},\hat{\tau}$ on $\tilde{Z}_\tau,\tilde{Z}_\varphi$ respectively, and how they act on the divisors E_i,F_j defined in Section 3.6.1. In particular, recall from Propositions 3.4.3.1, 3.4.5.1 the action of the residual involutions on the Nikulin lattice:

$$\begin{aligned}\hat{\varphi}=&(n_1,n_8)(n_2,n_5)(n_3,n_7)(n_4,n_6)\\ \hat{\tau}=&(n_1,n_5)(n_2,n_4)(n_3,n_8)(n_6,n_7).\end{aligned}$$

Theorem 3.6.2.1 (see [32, Prop. 2.7] and [26, Thm. 5.6])**.** *Let X be a K3 surface that admits a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$, and let L be an ample divisor on X invariant for this action. We distinguish two cases:*

1. *Let $L^2=2d=_4 0$, $NS(X)=\Omega_{2,2}\oplus\mathbb{Z}L$: then the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}(H^0(X,L)^*)$ is induced by an action of $\mathcal{D}_4$, the dihedral group of order 8, on $H^0(X,L)$ as follows. Consider the presentation*

$$\mathcal{D}_4=\langle a,b\mid a^2=b^2=1,(ab)^4=1\rangle$$

then $\mathcal{D}_4$ acts as $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^{d+1}$, as

$$\begin{aligned}a:&(x_0:\cdots:x_{d/2+1}:x_{d/2+2}:\cdots:x_{d+2})\mapsto(x_0:\cdots:x_{d/2+1}:-x_{d/2+2}:\cdots:-x_{d+2})\\ b:&(x_0:\cdots:x_{d/2+1}:x_{d/2+2}:\cdots:x_{d+2})\mapsto(x_{d/2+2}:\cdots:x_{d+2}:x_0:\cdots:x_{d/2+1}).\end{aligned}$$

2. *For any other deformation family, there exist divisors $D_1,\dots,D_4\in NS(\tilde{Y})$ such that*

$$H^0(X,L)=\pi^*_{2,2}H^0(\tilde{Y},D_1)\oplus\pi^*_{2,2}H^0(\tilde{Y},D_2)\oplus\pi^*_{2,2}H^0(\tilde{Y},D_3)\oplus\pi^*_{2,2}H^0(\tilde{Y},D_4)$$

and every $\pi^*_{2,2}H^0(\tilde{Y},D_i)$ corresponds to one of the subspaces which are the intersection of eigenspaces for the action of the two generators of $(\mathbb{Z}/2\mathbb{Z})^2$ on $H^0(X,L)$:

$$H^0(X,L)=V_{++}\oplus V_{+-}\oplus V_{-+}\oplus V_{--}.$$

Proof. For each projective family, consider for the associated ample class L the divisors E_1, E_2 defined in section 3.6.1: the residual involution $\hat{\varphi}$ on $\tilde{Z}_\tau$ fixes E_1, E_2 in all cases, except for $L_0(d)$ and d even, when they are exchanged. The same holds for the action of $\hat{\tau}$ on $\tilde{Z}_\varphi$ and the divisors F_1, F_2.
If the divisors are fixed by the residual involution, we can split $H^0(X, L)$ in four subspaces $V_{+,+}(L), V_{+,-}(L), V_{-,+}(L), V_{-,-}(L)$, each spanned by $\pi_{2,2}^* H^0(\tilde{Y}, D_i)$ for some nef divisors of the quotient surface: the proof follows the same argument of the cyclic case (see Proposition 2.6.1.1), using the divisors D_i defined below.

– Consider $L_0(d)$, d odd; depending on the value of d mod 4, we define $D_1, \ldots, D_4$ as follows:

$L_0(d)$	$d =_4 1$	$d =_4 3$
D_1	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_1+m_1+m_3+m_4+m_5}{2}$	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_1+m_1+m_6}{2}$
D_2	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_1+m_2+m_6+m_7+m_8}{2}$	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_1+m_2+m_3+m_4+m_5+m_7+m_8}{2}$
D_3	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_2+\overline{n}_3+\overline{n}_4+m_2+m_3+m_4+m_5+m_7+m_8}{2}$	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_2+\overline{n}_3+\overline{n}_4+m_2+m_6+m_7+m_8}{2}$
D_4	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_2+\overline{n}_3+\overline{n}_4+m_1+m_6}{2}$	$\frac{\pi_{2,2*}L_0}{4} - \frac{\overline{n}_2+\overline{n}_3+\overline{n}_4+m_1+m_3+m_4+m_5}{2}$

– Consider $L^{(1)}_{2,0}(h)$, whose square is $2d = 8h$; depending on the value of h mod 2, we define $D_1, \ldots, D_4$ as follows.

$L^{(1)}_{2,0}(h)$	$h =_2 0$
D_1	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_1+m_7}{2}$
D_2	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_2+m_3+m_4+m_5+m_6+m_8}{2}$
D_3	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_2+m_8}{2}$
D_4	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_1+m_3+m_4+m_5+m_6+m_7}{2}$

$L^{(1)}_{2,0}(h)$	$h =_2 1$
D_1	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_1+m_3+m_4+m_5+m_6+m_7}{2}$
D_2	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_2+m_8}{2}$
D_3	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_2+m_3+m_4+m_5+m_6+m_8}{2}$
D_4	$\frac{\pi_{2,2*}L^{(1)}_{2,0}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_1+m_7}{2}$

– Consider $L^{(2)}_{2,0}(h)$, whose square is $2d = 8(h-1)$; depending on the value of h mod 2, we define $D_1, \ldots, D_4$ as follows (the elements μ_1, μ_2 are defined in Proposition 3.4.4.1).

$L^{(2)}_{2,0}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \frac{m_3+m_4+m_5+m_6}{2}$	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4}$
D_2	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \frac{m_1+m_2+m_7+m_8}{2}$	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \mu_1$
D_3	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_2+\overline{n}_3+\overline{n}_4+\sum_i m_i}{2}$	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \mu_2$
D_4	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \frac{\overline{n}_1+\overline{n}_2+\overline{n}_3+\overline{n}_4}{2}$	$\frac{\pi_{2,2*}L^{(2)}_{2,0}}{4} - \mu_1 - \mu_2$

– Consider $L^{(k)}_{2,2}(h)$, $k = a, b$, whose square is $8h + 4$; depending on the value of h mod 2, we define $D_1, \ldots, D_4$ as follows:

$L^{(a)}_{2,2}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_1+m_4+m_5+m_7}{2}$	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_1+m_3+m_6+m_7}{2}$
D_2	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_2+m_3+m_6+m_8}{2}$	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_1+\overline{n}_4+m_2+m_4+m_5+m_8}{2}$
D_3	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_2+m_4+m_5+m_8}{2}$	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_2+m_3+m_6+m_8}{2}$
D_4	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_1+m_3+m_6+m_7}{2}$	$\frac{\pi_{2,2*}L^{(a)}_{2,2}}{4} - \frac{\overline{n}_2+\overline{n}_3+m_1+m_4+m_5+m_7}{2}$

$L^{(b)}_{2,2}(h)$	$h =_2 0$	$h =_2 1$
D_1	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{m_3+m_6}{2}$	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{m_4+m_5}{2}$
D_2	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{m_1+m_2+m_4+m_5+m_7+m_8}{2}$	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{m_1+m_2+m_3+m_6+m_7+m_8}{2}$
D_3	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{\sum_j \overline{n}_j+m_1+m_2+m_3+m_6+m_7+m_8}{2}$	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{\sum_j \overline{n}_j+m_1+m_2+m_4+m_5+m_7+m_8}{2}$
D_4	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{\sum_j \overline{n}_j+m_4+m_5}{2}$	$\frac{\pi_{2,2*}L^{(b)}_{2,2}}{4} - \frac{\sum_j \overline{n}_j+m_3+m_6}{2}$

– Consider $L_{4,-4}(h)$ and $L_{4,4}(h)$, whose square is $32h - 8, 32h + 8$ respectively; since $\pi_{2,2*}L_{4,\pm 4}/4$ is primitive in $NS(\tilde{Y})$, we define $D_1, \ldots, D_4$ as follows (the elements μ_1, μ_2 are defined in Proposition 3.4.4.1):

$L_{4,\pm 4}(h)$	any h
D_1	$\frac{\pi_{2,2*}L_{\pm 4,4}}{4}$
D_2	$\frac{\pi_{2,2*}L_{\pm 4,4}}{4} - \mu_1$
D_3	$\frac{\pi_{2,2*}L_{\pm 4,4}}{4} - \mu_2$
D_4	$\frac{\pi_{2,2*}L_{\pm 4,4}}{4} - \mu_1 - \mu_2$

Table 3.4: Euler characteristics

	no.	L	$\chi(D_1)$	$\chi(D_2)$	$\chi(D_3)$	$\chi(D_4)$
$d =_4 1$	1	L_0	$(d+3)/4$	$(d+3)/4$	$(d-1)/4$	$(d+3)/4$
$d =_4 3$	2	L_0	$(d+5)/4$	$(d+1)/4$	$(d+1)/4$	$(d+1)/4$
$d =_4 2$	3	$L_{2,2}^{(a)}$	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$	$(d+2)/4$
	4	$L_{2,2}^{(b)}$	$(d+6)/4$	$(d+2)/4$	$(d-2)/4$	$(d+2)/4$
$d =_8 0$	5	$L_{2,0}^{(1)}$	$d/4+1$	$d/4$	$d/4+1$	$d/4$
	6	$L_{2,0}^{(2)}$	$d/4+2$	$d/4$	$d/4$	$d/4$
$d =_8 4$	7	$L_{2,0}^{(1)}$	$d/4$	$d/4+1$	$d/4$	$d/4+1$
	8	$L_{2,0}^{(2)}$	$d/4+1$	$d/4+1$	$d/4-1$	$d/4+1$
	9	$L_{\pm4,4}$	$d/4+2$	$d/4$	$d/4$	$d/4$

Consider now the projective family with ample class $L = L_0(d)$, d even: then we cannot split $H^0(X, L)$ in four subspaces, but rather we find

$$H^0(X, L) = V_+ \oplus V_-$$

where V_+, V_- are the eigenspaces for one of the generators of $(\mathbb{Z}/2\mathbb{Z})^2$ (say τ), and the other generator (say φ) acts exchanging the two.
By Remark 3.6.0.1, on $H^0(X, L)$ we have an action as follows: choose a basis $\{x_0, \dots, x_{d+2}\}$ of $H^0(X, L)$ such that

$$\tau : (x_0, \dots, x_{d/2+1}, x_{d/2+2}, \dots, x_{d+2}) \mapsto \xi_k(x_0, \dots, x_{d/2+1}, -x_{d/2+2}, \dots, -x_{d+2})$$

with ξ_k some root of unity, so that τ^2 is the multiplication by ξ_k^2; then it holds

$$\begin{aligned}\varphi(x_i) =& \xi_m f_i(x_{d/2+2}, \dots, x_{d+2}) \text{ for every } i = 0, \dots d/2+1,\\ \varphi(x_j) =& \xi_m f_j(x_0, \dots, x_{d/2+1}) \text{ for every } j = d/2+2, \dots d+2,\end{aligned}$$

with ξ_m another root of unity and f_i linear such that φ^2 is the multiplication by ξ_m^2; composing them, we get

$$\begin{aligned}\varphi(\tau(x_i)) =& \xi_k \xi_m f_i(x_{d/2+2}, \dots, x_{d+2}) \text{ for every } i = 0, \dots, d/2+1\\ \varphi(\tau(x_j)) =& -\xi_k \xi_m f_j(x_0, \dots, x_{d/2+1}) \text{ for every } j = d/2+2, \dots, d+2,\end{aligned}$$

while

$$\tau(\varphi(x_i)) = -\xi_k \xi_m f_i(x_{d/2+2}, \dots, x_{d+2}) \text{ for every } i = 0, \dots, d/2+1$$

$$\tau(\varphi(x_j)) = \xi_k\xi_m f_j(x_0, \ldots, x_{d/2+1}) \text{ for every } j = d/2+2, \ldots, d+2,$$

so it holds $\tau\varphi = -\varphi\tau$. Therefore $(\tau\varphi)^2$ is the multiplication by $-\xi_k^2\xi_m^2$. Substituting $\tilde{\tau} = \tau^k$, and $\tilde{\varphi} = \varphi^m$, we still get $(\tilde{\tau}\tilde{\varphi})^2 = -id$, so $\tilde{\tau}\tilde{\varphi}$ has order 4 and $\tilde{\tau}, \tilde{\varphi}$ span the dihedral group $\mathcal{D}_4$ (as anticipated in Remark 3.6.0.1); by projectivizing, the action of $\mathcal{D}_4$ loses faithfulness, and we see on $\mathbb{P}(H^0(X, L)^*)$ an action of $(\mathbb{Z}/2\mathbb{Z})^2$ via the maps described in the statement. □

Proposition 3.6.2.2. *In case 2 of Theorem 3.6.2.1 it holds*

$$\pi_\tau^* H^0(\tilde{Z}_\tau, E_1) = \pi_{2,2}^* H^0(\tilde{Y}, D_1) \oplus \pi_{2,2}^* H^0(\tilde{Y}, D_2),$$
$$\pi_\tau^* H^0(\tilde{Z}_\tau, E_2) = \pi_{2,2}^* H^0(\tilde{Y}, D_3) \oplus \pi_{2,2}^* H^0(\tilde{Y}, D_4).$$

Proof. See the proof of Proposition 2.6.2.2. □

Remark 3.6.2.3. To define $D_1, \ldots, D_4$ we chose to use the description of $\tilde{Y}$ as resolution of the quotient $\tilde{Z}_\tau/\hat{\varphi}$. The same results can be obtained using $\tilde{Z}_\varphi/\hat{\tau}$ instead.

3.6.3 Projective models with $L^2 = 4$

There are three families of K3 surfaces X polarized with an ample class L such that $L^2 = 4$: for one of them $L = L_0(2)$, so the action of $(\mathbb{Z}/2\mathbb{Z})^2$ is as described in case 1 of Theorem 3.6.2.1; the other two correspond to **no. 3**, **no. 4** of Table 3.4, and we can read from there the dimension of the eigenspaces for the action of $(\mathbb{Z}/2\mathbb{Z})^2$. Moreover, by the correspondence between projective families of X and its quotients, and in particular by the degree of $\hat{L}_\iota$ (the pseudo-ample class on the intermediate quotient surface $\tilde{Z}_\iota$, $\iota \in \{\tau, \varphi\}$) and of $\overline{L}$ on $\tilde{Y}$, we expect the dimension of the projective space in which the quotients are naturally embedded.

We proceed by firstly defining an action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle\tau, \varphi\rangle$ on the correct projective space ($\mathbb{P}^3$ for $L = L_0(2), L_{2,2}^{(a)}(0)$, $\mathbb{P}^1 \times \mathbb{P}^1$ if $L = L_{2,2}^{(b)}(0)$) with eigenspaces of the expected dimension, and finding a family of K3 surfaces which are invariant for it. Recall from Remark 1.4.0.4 that each of the projective families of X has dimension $7 = 20 - (rk(\Omega_{2,2}) + 1)$. To check the simplecticity of the action of $(\mathbb{Z}/2\mathbb{Z})^2$, it is sufficient to check that each of the two generators is a symplectic involution, i.e. that each fixes 8 points on X.

Let $L = L_0(2)$: consider the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^3$ given by

$$(z_0 : z_1 : z_2 : z_3) \overset{\tau}{\mapsto} (-z_0 : -z_1 : z_2 : z_3)$$
$$\overset{\varphi}{\mapsto} (z_2 : z_3 : z_0 : z_1)$$

then φ exchanges the eigenspaces of τ, which is the action described in Section 3.6.1 for $L_0(2)$. Quartic surfaces invariant for this action are of the form

$$Q_3 : q(z_0, z_1) + q(z_2, z_3) + \alpha z_0^2 z_2^2 + \beta z_0 z_1 z_2 z_3 + \gamma z_1^2 z_3^2 + \delta(z_0^2 z_2 z_3 + z_0 z_1 z_2^2) +$$

$$+\varepsilon(z_0^2z_3^2+z_1^2z_2^2)+\zeta(z_0z_1z_3^2+z_1^2z_2z_3)=0;$$

they depend on 11 parameters, but taking into account projectivities of the form $(z_0 : z_1 : z_2 : z_3) \mapsto (az_0+bz_1 : cz_0+dz_1 : az_2+bz_3 : cz_2+dz_3)$ which commute with the given action of $(\mathbb{Z}/2\mathbb{Z})^2$ we find a moduli space of dimension 7. This is therefore a complete family of K3 surfaces with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$. The quotient surfaces Z_τ, Z_φ admit projective models as complete intersection of 3 quadrics in $\mathbb{P}^5$, as in [32, §3.4]. Since $\overline{L}^2 = 16$, we expect $\tilde{Y} \subset P^9$, so it doesn't admit a natural model as complete intersection of hypersurfaces.

The following two models are numbered according to Table 3.4.
no. 3: Consider the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^3$ given by

$$(x_0 : x_1 : x_2 : x_3) \overset{\tau}{\mapsto} (-x_0 : -x_1 : x_2 : x_3)$$
$$\overset{\varphi}{\mapsto} (-x_0 : x_1 : -x_2 : x_3)$$

then the eigenspaces are all of the same dimension. The family of quartic surfaces

$$Q_4 : \sum_{i=0}^{3} a_i x_i^4 + \sum_{\substack{i,j=0\dots3\\ j>i}} b_{ij}x_i^2x_j^2 + x_0x_1x_2x_3,$$

whose general member is smooth, is invariant for the action above, and it depends on 7 projective parameters up to the action of projectivities that commute with τ, φ.

Since the action of τ, φ is the same up to a change of coordinates, the quotient surfaces Z_τ, Z_φ will be described by similar equations. As in [32, §3.4], we consider the map given by the degree 2 invariants under the action of τ

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_3^2 : x_0x_1 : x_2x_3) = (z_0 : z_1 : z_2 : z_3 : z_4 : z_5);$$

then the surface Q_4 maps to the complete intersection of quadrics in $\mathbb{P}^5$

$$R_4 : \begin{cases} z_4^2 = z_0z_1 \\ z_5^2 = z_1z_2 \\ z_4z_5 = -\sum_{i=0}^{3} a_iz_i^2 - \sum_{\substack{i,j=0\dots3\\ j>i}} b_{ij}z_iz_j \end{cases}$$

which is a projective model for Z_τ. Now, the automorphism $\hat{\rho}$ on $\mathbb{P}^5$ is

$$\hat{\rho} : (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : z_2 : z_3 : -z_4 : -z_5) :$$

the surface R_4 has the same form as in [32, §3.7], so its quotient under the action of $\hat{\rho}$, which is a projective model for Y, is the quartic surface in $\mathbb{P}^3 = (z_0 : z_1 : z_2 : z_3)$

$$S_4 : z_0z_1z_2z_3 + (\sum_{i=0}^{3} a_iz_i^2 + \sum_{\substack{i,j=0\dots3\\ j>i}} b_{ij}z_iz_j)^2 = 0.$$

no. 4: we have $L^{(b)}_{2,2}(0) = H_1 + H_2$ with

$$H_1 = \frac{L_0(0) + v_2 + w}{2},\ H_2 = \frac{L_0(0) + v_2 - w}{2}; \quad \langle H_1, H_2 \rangle = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

and

$$\tau^*(H_1) = H_2, \quad \varphi^*(H_1) = H_1, \quad \varphi^*(H_2) = H_2.$$

Hence

$$\phi_{|L^{(b)}_{2,2}(0)|} = \phi_{|H_1+H_2|} : X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$$

is a double cover ramified along a curve $\mathcal{B}$ of bidegree $(4,4)$ invariant for the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\begin{aligned}(x_0 : x_1)(y_0 : y_1) &\overset{\tau}{\mapsto} (y_0 : y_1)(x_0 : x_1) \\ &\overset{\varphi}{\mapsto} (x_0 : -x_1)(y_0 : -y_1);\end{aligned}$$

curves of this type depend on 7 projective parameters when taking into account the action of the group of projectivities of the form $(x_0 : x_1)(y_0 : y_1) \mapsto (x_0 : ax_1)(y_0 : ay_1)$, which are the only ones that commute with the action above. We take the quotient of X by the action of τ as described in [32, §3.5]: the surface Z_τ is a double cover of $\mathbb{P}^2 = (x_0y_0 : x_0y_1 + x_1y_0 : x_1y_1) = (w_0 : w_1 : w_2)$ ramified along a sextic curve $\mathcal{C}$, the union of the image $\mathcal{B}_\tau$ of $\mathcal{B}$, which is a quartic curve, and the conic curve invariant for the action of $\hat{\varphi}$ induced on $\mathbb{P}^2$,

$$\hat{\varphi} : (w_0 : w_1 : w_2) \mapsto (w_0 : -w_1 : w_2).$$

To find a projective model of Y, we map Z_τ to the space of invariants of degree two of $\hat{\varphi}$, $\mathbb{P}^3 = (w_0^2 : w_1^2 : w_2^2 : w_0w_2) = (z_0 : z_1 : z_2 : z_3)$: then Y is a double cover of the surface $z_0z_2 = z_3^2$ ramified along $\overline{\mathcal{C}}$ (the image of the sextic curve $\mathcal{C}$) which is a cubic curve.

Now, let's go back and describe Z_φ: the action of φ on $\mathbb{P}^1 \times \mathbb{P}^1$ fixes 4 points, which do not belong to the branch curve: therefore, if we write $X : t^2 = b$, where b is the polynomial of bidegree (4,4) such that $\mathcal{B} : b = 0$, to have 8 fixed points on X we find that φ acts as the identity on t.
Proceeding as in case no. 3 of Section 2.6.3 we embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre map

$$(x_0 : x_1)(y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1) = (z_0 : z_1 : z_2 : z_3) :$$

now X is a double cover of the quadric surface $z_0z_3 = z_1z_2$ ramified along the image of $\mathcal{B}$.
We consider the induced action of φ on the weighted projective space $\mathbb{P}(2,1,1,1,1)$,

$$\varphi : (t; z_0 : z_1 : z_2 : z_3) \mapsto (t; z_0 : -z_1 : -z_2 : z_3);$$

the space of invariants of degree 2 for φ is $\mathbb{P}^6 = (t : z_0^2 : z_1^2 : z_2^2 : z_3^2 : z_0z_3 : z_1z_2) = (t : a_0 : a_1 : a_2 : a_3 : a_4 : a_5)$, and the quotient surface Z_φ is described by the complete intersection

$$\begin{cases} a_4 = a_5 \\ a_0a_3 = a_4^2 \\ a_1a_2 = a_5^2 \\ t^2 = \overline{b} \end{cases}$$

where $\overline{b}$ is now a quadric: this is therefore a projective model of Z_φ as the complete intersection of 3 quadrics in $\mathbb{P}^5$, as we expected since $(\pi_{\varphi*}L_{2,2}^{(b)}(0))^2 = 8$.
The action of $\hat{\tau}$ on $\mathbb{P}^5$ changes sign to t and exchanges a_1 with a_2, fixing the other coordinates. Let

$$\mathbb{P}^5 = (t : c_0 : c_1 : c_2 : c_3 : c_4) = (t : a_0 : a_1 + a_2 : a_1 - a_2 : a_3 : a_4) :$$

similarly to the surface S_4 of case no. 4 of Section 2.6.3, to compute the quotient surface we use the projection from the line $\ell = (\lambda : 0 : 0 : \mu : 0 : 0)$ on the invariant space for the action of $\hat{\tau}$:

$$\pi : \mathbb{P}^5 \to \mathbb{P}^3 = (c_0 : c_1 : c_3 : c_4).$$

Then, Z_φ covers 4:1 the surface $c_0^2 = c_3c_4$, and $\hat{\tau}$ exchanges pairwise the points on each regular fiber: therefore we get again a model of Y as double cover of a quadric surface in $\mathbb{P}^3$, as expected.

Chapter 4

Generalizing Shioda-Inose structures to the order 4

Introduction

Given any abelian surface A, if X is a K3 surface such that there exists a Hodge isometry $T(A) \simeq T(X)$, then X admits a symplectic involution ι such that X/ι is birational to the Kummer surface $Kum(A)$ [63, Thm. 6.3]: the triple (A, X, ι) is called a Shioda-Inose structure. In [27] it is proved that if A admits a symplectic automorphism σ_A of order 3 (which is not a condition satisfied by the general abelian surface), and X is a K3 surface such that there exists a Hodge isometry $T(A) \simeq T(X)$, then X admits a symplectic automorphism σ_X of order 3 such that X/σ_X is birational to the generalized Kummer surface $Kum_3(A)$, the resolution of the singularities of A/σ_A. The quadruple $(A, \sigma_A, X, \sigma_X)$ is called a generalized Shioda-Inose structure.

Some abelian surfaces A admit a symplectic automorphism α of order 4: however, in this chapter we prove that, if a K3 surface X satisfies the condition $T(X) \simeq T(A)$, it is not even guaranteed that X admit an action of a group of order 4 at all. Unlike the order 3 case, it is therefore impossible to give a full generalization of Shioda-Inose structures to the order 4, so we propose two different partial generalizations.

Definition 1 (see Def. 4.2.5.1). Let X be a projective K3 surface. We say that X admits a *strong* order 4 Shioda-Inose structure if there is a symplectic automorphism τ of order 4 on X and a pair (A, α) as above such that the resolution of the singularities of X/τ^2 and X/τ are isomorphic to $Kum(A)$ and $Kum_4(A)$ respectively.

The quadruple (A, α, X, τ) is a strong structure if and only if (A, X, τ^2) is a classical Shioda-Inose structure, so it holds $T(X) \simeq T(A)$; however, strong structures do not exist for any pair (A, α), but only if

$$T(A) \simeq \begin{bmatrix} -4k & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

To give a structure that exists for any pair (A, α), we turn to Morrison-Nikulin involutions: these are special symplectic involutions on a K3 surface X, that exchange two algebraic copies of the lattice E_8. For a projective K3 surface, the existence of a Morrison-Nikulin involution and of a Shioda-Inose structure are equivalent properties (see Theorem 4.2.1.4).

Definition 2 (see Def. 4.2.5.4). Let X be a projective K3 surface. We say that X admits a *weak* order 4 Shioda-Inose structure if there is a symplectic automorphism τ of order 4 on X such that τ^* cyclically permutes four orthogonal algebraic copies of D_4.

If a K3 surface X has a symplectic automorphism τ that satisfies this definition, then there exists a pair (A, α) such that the resolution of the singularities of X/τ is isomorphic to $Kum_4(A)$ (see Theorem 4.2.4.12); conversely, given any pair (A, α) we can find a K3 surface X with a symplectic automorphism τ as above, such that X/τ is birational to $Kum_4(A)$ (see Theorem 4.2.4.14).

If the quadruple (A, α, X, τ) is a weak structure which is not strong, then $T(A) \not\simeq T(X)$; even worse, $T(A)$ does not always uniquely determine $T(X)$. We remark that this is a crucial flaw of this construction: indeed, by Theorems 1.3.0.14, 1.3.0.8, in classical Shioda-Inose structures to each abelian surface A corresponds a unique K3 surface X, and to each X with a Morrison-Nikulin involution at most two abelian surfaces (A and $A^\vee$) (see Remark 4.2.5.2). Without a Hodge isometry of the transcendental lattices, we can only give a correspondence between families of A and X that form a weak order 4 Shioda-Inose structures.

Theorem 1 (see Cor. 4.2.4.16)**.** *In the following table we compare the transcendental lattices of A, $Kum_4(A)$ and X, where the quadruple (A, α, X, τ) is a weak order 4 Shioda-Inose structure.*

	$T(A)$	$T(Kum_4(A))$	$T(X)$
$\forall d$	$\langle 2\rangle^{\oplus 2} \oplus \langle -2d\rangle$	$\langle 2\rangle^{\oplus 2} \oplus \langle -8d\rangle$	$\langle 4\rangle^{\oplus 2} \oplus \langle -2d\rangle$ *or* $\begin{bmatrix} -2(d-1) & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{bmatrix}$
$d = 4h+1$	$\langle 2\rangle \oplus \begin{bmatrix} 2 & 1 \\ 1 & -2h \end{bmatrix}$	$\langle 2\rangle \oplus \begin{bmatrix} 8 & 4 \\ 4 & -8h \end{bmatrix}$	$\begin{bmatrix} 4-2h & 3 & 3 \\ 3 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

$d = 8k+6$	$\begin{bmatrix} -4k-2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4 \\ 4 & -8(2k+1) \end{bmatrix}$	$\langle 2 \rangle^{\oplus 2} \oplus \langle -2d \rangle$
$d = 8k+2$	$\begin{bmatrix} -4k & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4 \\ 4 & -16k \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4k+2 \\ 4k+2 & 4k^2 \end{bmatrix}$ *or* $\begin{bmatrix} -4k & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

4.1 The Kummer surfaces $Kum(A), Kum_4(A)$

Let A be an abelian surface, with local coordinates (z_1, z_2). Assume A admits the automorphism α defined by

$$\alpha(z_1, z_2) = (z_2, -z_1) :$$

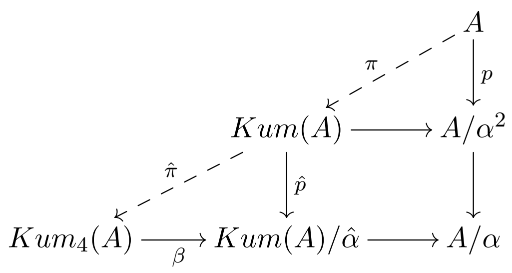
this is a symplectic automorphism of order 4, because $\alpha^*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2$. The wedge product

$$\wedge : H^2(A, \mathbb{Z}) \times H^2(A, \mathbb{Z}) \to H^4(A, \mathbb{Z}) \simeq \mathbb{Z}$$

gives $H^2(A, \mathbb{Z})$ a lattice structure $(H^2(A, \mathbb{Z}), \wedge) = U^{\oplus 3}$. Computing the action of α^* on $H^2(A, \mathbb{Z})$, we see that it acts on two of the copies of U exchanging the generators of each copy, and as the identity on the remaining one.

Remark 4.1.0.1. The co-invariant lattice of the action of α^* is spanned by the classes $dz_2 \wedge d\bar{z}_2 - dz_1 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1$: it is therefore the lattice $\Omega := \langle -2 \rangle^{\oplus 2}$. A complex 2-torus T admits α as an automorphism if and only if Ω is primitively embedded in $NS(T)$.

We want to describe explicitly the maps induced in cohomology by the rational (dashed) arrows in the following diagram:



(4.1.0.1)

where $\hat{\alpha}$ is the involution induced on the Kummer surface $Kum(A)$ by α on A.

4.1.1 The surface $Kum(A)$

The construction of the Kummer surface $Kum(A)$ is classical: take an abelian surface A, and the involution defined by $\iota(a) = -a$ for every $a \in A$; ι fixes exactly 16 points on A, and acts trivially on the second integral cohomology lattice $H^2(A, \mathbb{Z}) \simeq U^{\oplus 3}$: the minimal resolution of the quotient A/ι is the K3 surface $Kum(A)$, which is characterized by the existence of a primitive embedding of the *Kummer lattice* K_2 in its Néron-Severi lattice [70, §.3, Thm. 3]. We recall here the construction, assuming $A = E \times E$ with E an elliptic curve, $\alpha((e, f)) = (f, -e)$ symplectic of order 4, $\iota = \alpha^2$: by deformation, this gives us the description of the map

$$\pi_* : H^2(A, \mathbb{Z}) \to H^2(Kum(A), \mathbb{Z})$$

for any abelian surface A with a symplectic automorphism of order 4.
The involution $e \mapsto -e$ on an elliptic curve E fixes exactly 4 points, that can be identified with a copy of $(\mathbb{Z}/2\mathbb{Z})^2 = \{(0,0),\ (1,0),\ (0,1),\ (1,1)\}$ under the group law of E: thus, the fixed locus of α^2 on $A = E \times E$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, and the quotient surface A/α^2 has 16 A_1 singularities.

Definition 4.1.1.1 ([70, §3, Def. 1], see also [31, Rem. 2.3]). Call k_{abcd} the class of the rational curve that resolves the point $(a, b, c, d) \in (\mathbb{Z}/2\mathbb{Z})^4$ in the Kummer surface. The lattice K_2 is the overlattice of $\langle k_{abcd} \mid (a, b, c, d) \in (\mathbb{Z}/2\mathbb{Z})^4 \rangle$ obtained adding as generators the classes

$$\begin{aligned}
w_0 &= \sum_{(a,b,c,d)\in(\mathbb{Z}/2\mathbb{Z})^4} k_{abcd}/2,\\
w_1 &= (k_{0000} + k_{0001} + k_{0010} + k_{0011} + k_{0100} + k_{0101} + k_{0110} + k_{0111})/2,\\
w_2 &= (k_{0000} + k_{0001} + k_{0010} + k_{0011} + k_{1000} + k_{1001} + k_{1010} + k_{1011})/2,\\
w_3 &= (k_{0000} + k_{0001} + k_{0100} + k_{0101} + k_{1000} + k_{1001} + k_{1100} + k_{1101})/2,\\
w_4 &= (k_{0000} + k_{0010} + k_{0100} + k_{0110} + k_{1000} + k_{1010} + k_{1100} + k_{1110})/2.
\end{aligned}$$

Since α^2 acts trivially on $H^2(A, \mathbb{Z})$, we can apply the push-pull formula to conclude that $\pi_*(U \oplus U \oplus U) = U(2) \oplus U(2) \oplus U(2)$.

Theorem 4.1.1.2 ([31, Rem. 2.8]). *Call $\{v_1, \ldots, v_6\}$ the generators of $U(2)^{\oplus 3}$ such that $v_{2k+1}v_{2k+2} = 2,\ k = 0, 1, 2$ and the other intersections are trivial: then the lattice $H^2(Kum(A), \mathbb{Z})$ is the overlattice of $K_2 \oplus U(2)^3$ obtained by adding as generators the classes*

$$\begin{aligned}
\beta_1 &= (v_1 + k_{0000} + k_{0001} + k_{0010} + k_{0011})/2,\\
\beta_2 &= (v_2 + k_{0000} + k_{0100} + k_{1000} + k_{1100})/2,\\
\beta_3 &= (v_3 + k_{0000} + k_{0010} + k_{0100} + k_{0110})/2,\\
\beta_4 &= (v_4 + k_{0000} + k_{0001} + k_{1000} + k_{1001})/2,\\
\beta_5 &= (v_5 + k_{0000} + k_{0001} + k_{0100} + k_{0101})/2,\\
\beta_6 &= (v_6 + k_{0000} + k_{0010} + k_{1000} + k_{1010})/2.
\end{aligned}$$

4.1.2 The involution $\hat{\alpha}$ and the surface $Kum_4(A)$

Among the 16 points fixed by α^2, α fixes the four points $\{(a,b,a,b) \mid a,b \in \mathbb{Z}/2\mathbb{Z}\}$, and exchanges the remaining ones as $(a,b,c,d) \mapsto (c,d,a,b)$: therefore, there are 10 singular points on the quotient surface A/α, of which 4 of type A_3, and 6 of type A_1 [9, Prop. 2.1]. The K3 surface $Kum_4(A)$, that arises as minimal resolution of the quotient A/α, is characterized by the existence of a primitive embedding of the lattice K_4 in its Néron-Severi lattice.

Definition 4.1.2.1 ([9, §4, cas 4.2]). The lattice K_4 is the overlattice of $A_3^{\oplus 4} \oplus A_1^{\oplus 6} = \langle m_{\gamma,1}, m_{\gamma,2}, m_{\gamma,3} \mid \gamma = 1,\dots,4\rangle \oplus \langle m_j \mid j = 1,\dots,6\rangle$ obtained by adding as generator the class

$$\mu := \frac{\sum_\gamma 3m_{\gamma,1} + 2m_{\gamma,2} + m_{\gamma,3}}{4} + \frac{\sum_j m_j}{2}. \tag{4.1.2.1}$$

It is the smallest primitive sublattice of Λ_{K3} which contains the exceptional curves of $Kum_4(A)$.

We want to describe the surface $Kum_4(A)$ as minimal resolution of $Kum(A)/\hat{\alpha}$, where $\hat{\alpha}$ is the symplectic involution induced by the automorphism α of A.
Define the sublattice of $H^2(Kum(A),\mathbb{Z})$

$$W = \langle k_{abcd} \mid (a,b,c,d) \in (\mathbb{Z}/2\mathbb{Z})^4\rangle \oplus \langle v_j \mid j = 1,\dots,6\rangle,$$

isomorphic to $\langle -2\rangle^{\oplus 16} \oplus U(2)^{\oplus 3}$.

Proposition 4.1.2.2. *1. The isometry $\hat{\alpha}^*$ induced by $\hat{\alpha}$ acts on W as the permutation*

$$(k_{0010}, k_{1000})(k_{0001}, k_{0100})(k_{0011}, k_{1100})(k_{0110}, k_{1001})(k_{1011}, k_{1110})(k_{0111}, k_{1101})\\ (v_1, v_2)(v_3, v_4),$$

and as the identity on the remainig generators of W;

2. *Consider the map $\hat{\pi}_* : H^2(Kum(A)) \to H^2(Kum_4(A))$: then $\hat{\pi}_*(W) = \langle -4\rangle^{\oplus 4} \oplus\langle -2\rangle^{\oplus 6} \oplus \langle 2\rangle^{\oplus 2} \oplus U(4)$.*

3. *Denoting $\hat{\star} = \hat{\pi}_*\star$, a $\mathbb{Z}$-basis for $\hat{\pi}_*(H^2(Kum(A),\mathbb{Z}))$ is* $\{\hat{k}_{0000}, \hat{k}_{0001}, \hat{k}_{0010}, \hat{k}_{0011}, \hat{k}_{0101}, \hat{k}_{0110}, \hat{k}_{1010}, \hat{w}_0, \hat{w}_1, \hat{w}_2, \hat{\beta}_1, \hat{\beta}_3, \hat{\beta}_5, \hat{\beta}_6\}$.

Proof. The action of $\hat{\alpha}^*$ on W is deduced by the action of α^* on $H^2(A,\mathbb{Z})$, and by that of $\hat{\alpha}$ on the singular points of A/α^2; we extend $\hat{\alpha}^*$ to $H^2(Kum(A),\mathbb{Z})$ by $\mathbb{Q}$-linearity using the elements w_i, $i = 0,\dots,4$ (see Def. 4.1.1.1) and β_j, $j = 1,\dots,6$ (see Thm. 4.1.1.2). Then, applying the push-pull formula to the generators of W, we get $\hat{\pi}_*(W)$, and by $\mathbb{Q}$-linear extension we get the whole $\hat{\pi}_*(H^2(Kum(A),\mathbb{Z}))$. □

As any symplectic involution on a K3 surface, $\hat{\alpha}$ fixes 8 points on $Kum(A)$ [71, §5]: to resolve the singularities of $Kum(A)/\hat{\alpha}$, we have to introduce 8 rational curves, whose classes in cohomology generate a copy of the Nikulin lattice (see Definition 2.4.2.1).

Remark 4.1.2.3. The lattice $\hat{\pi}_*(H^2(Kum(A),\mathbb{Z}))$ has the same discriminant group of N, that is, $(\mathbb{Z}/2\mathbb{Z})^6$ with the same discriminant form of $U(2)^{\oplus 3}$. Thus, applying [72, Prop. 1.6.1] we find that $\hat{\pi}_*(H^2(Kum(A),\mathbb{Z}))\oplus N$ is a sublattice of finite index of $H^2(Kum_4(A),\mathbb{Z})$. In particular, we can choose as gluing classes the following:

$$\begin{aligned}
\gamma_1 &= (\hat{k}_{0000} + \hat{k}_{0101} + \hat{w}_0 + \hat{\beta}_6 + n_2 + n_3 + n_5 + n_8)/2;\\
\gamma_2 &= (\hat{k}_{0000} + \hat{k}_{0101} + n_2 + n_3 + n_4 + n_8)/2;\\
\gamma_3 &= (\hat{k}_{0000} + \hat{w}_0 + \hat{\beta}_5 + n_3 + n_4 + n_5 + n_7)/2;\\
\gamma_4 &= (\hat{k}_{0000} + \hat{w}_0 + n_4 + n_5 + n_7 + n_8)/2;\\
\gamma_5 &= (\hat{k}_{1010} + \hat{\beta}_5 + n_2 + n_4 + n_5 + n_6)/2;\\
\gamma_6 &= (\hat{k}_{1010} + \hat{w}_0 + n_3 + n_6 + n_7 + n_8)/2.
\end{aligned}$$

We now want to describe the lattice K_4 (see Def. 4.1.2.1) as embedded by our construction in $H^2(Kum_4(A),\mathbb{Z})$: since K_4 is the exceptional lattice for the quotient A/α, it is generated by the Nikulin lattice N, and the image via $\hat{\pi}_*$ of the Kummer lattice K_2.

Proposition 4.1.2.4. *The sublattice $A_3^{\oplus 4}\oplus A_1^{\oplus 6}$ of K_4 is spanned by the following classes of $\hat{\pi}_*K_2\oplus N$:*

$$\begin{aligned}
&m_{1,1} = n_2;\ m_{1,2} = (\hat{k}_{0000} - n_2 - n_8)/2;\ m_{1,3} = n_8;\\
&m_{2,1} = n_3;\ m_{2,2} = (\hat{k}_{0101} - n_3 - n_4)/2;\ m_{2,3} = n_4;\\
&m_{3,1} = n_1;\ m_{3,2} = (\hat{k}_{1010} - n_1 - n_7)/2;\ m_{3,3} = n_7;\\
&m_{4,1} = n_5;\ m_{4,2} = (\hat{k}_{1111} - n_5 - n_6)/2;\ m_{4,3} = n_6;\\
&m_1 = \hat{k}_{0010} = \hat{k}_{1000};\ m_2 = \hat{k}_{0011} = \hat{k}_{1100};\ m_3 = \hat{k}_{0100} = \hat{k}_{0001};\\
&m_4 = \hat{k}_{0110} = \hat{k}_{1001};\ m_5 = \hat{k}_{0111} = \hat{k}_{1101};\ m_6 = \hat{k}_{1110} = \hat{k}_{1011}.
\end{aligned}$$

To get the whole K_4, add to the generators the classes:

$$\begin{aligned}
\mu &= \frac{\sum_{\gamma=1,4} 3m_{\gamma,1} + 2m_{\gamma,2} + m_{\gamma,3}}{4} + \frac{\sum_{\gamma=2,3} m_{\gamma,1} + 2m_{\gamma,2} + 3m_{\gamma,3}}{4} + \frac{\sum_j m_j}{2};\\
\nu &= \frac{m_1 + m_2 + m_4 + m_5 + m_{3,1} + m_{3,3} + m_{4,1} + m_{4,3}}{2};\\
\xi &= \frac{m_2 + m_3 + m_4 + m_6 + m_{2,1} + m_{2,3} + m_{4,1} + m_{4,3}}{2}.
\end{aligned}$$

Proof. Consider the diagram (4.1.0.1). The automorphism $\hat{\alpha}$ acts on the exceptional curves of $Kum(A)$ fixing two points on each of the 4 curves whose class is k_{abab}, and exchanging the other curves in pairs: therefore the surface $Kum(A)/\hat{\alpha}$ is singular in eight points. Blowing up the two singular points on the image of each curve k_{abab} we get an A_3 lattice, and the eight exceptional curves introduced with the blow-up β span the Nikulin lattice $N = \langle m_1^i,\ m_3^i\rangle_{i=1}^4$. Denote $\hat{k}_{abcd}$ the pullback $\beta^*\hat{p}_*k_{abcd}$: then by push-pull we have $\hat{k}_{abcd}^2 = \hat{k}_{cdab}^2 = -2$ for every $(c,d) \neq (a,b)$, while $\hat{k}_{abab}^2 = -4$; moreover, the class $\hat{k}_{abab}$ is by definition orthogonal to the exceptional curves, therefore $\hat{k}_{abab} = m_{\gamma,1} + 2m_{\gamma,2} + m_{\gamma,3}$ for some γ. To determine which copy of $A_3,\ A_1$ in K_4 each $\hat{k}_{abab}, \hat{k}_{abcd}, n_i$ corresponds to, we still have to require that the image of the elements γ_i defined in Remark 4.1.2.3 be integral in $H^2(Kum_4(A), \mathbb{Z})$; this forces the definition of K_4 as stated. □

Remark 4.1.2.5. The orthogonal complement to K_4 is the lattice $\langle 2 \rangle^{\oplus 2} \oplus U(4)$, spanned by the classes $\hat{v}_1 = \hat{\pi}_* v_1 = \hat{\pi}_* v_2,\ \hat{v}_3 = \hat{\pi}_* v_3 = \hat{\pi}_* v_4,\ \hat{v}_5, \hat{v}_6$. This corrects a mistake in [61, Ex. 4.3].

4.2 Generalizing Shioda-Inose structures

4.2.1 Results on Shioda-Inose structures

Definition 4.2.1.1 ([63, Def. 6.1]). A K3 surface X admits a Shioda-Inose structure if there is a symplectic involution ι on X with rational quotient map $\pi : X \dashrightarrow Y$ such that Y is a Kummer surface, and π_* induces a Hodge isometry $T(X)(2) \simeq T(Y)$.

Recall that a K3 surface Y is the minimal resolution of X/ι if and only if there exists a primitive embedding of the Nikulin lattice $N \hookrightarrow NS(Y)$ [71, §5] (see also Definition 2.4.2.1). If we want Y to be a Kummer surface, then we're asking that both N and K_2 be primitively embedded (obviously non orthogonally because of their rank) in $NS(Y)$. The following proposition suggests how to do it:

Proposition 4.2.1.2 ([63, Thm. 5.7])**.** *The lattice $E_8 \oplus N$ has the same discriminant form as $U(2)^{\oplus 3}$: indeed, it belongs to the same genus as the lattice K_2.*

The lattices $E_8 \oplus N$ and K_2 are negative definite: they are in the same genus, but not isomorphic, as it can be proved by counting the number of elements of square -2.
Suppose however that we have a projective surface Y lattice-polarized with $E_8 \oplus N$ (so that $NS(Y)$ has signature $(1, 16)$). Then, Theorem 1.2.1.14 guarantees the existence of a primitive embedding of K_2 in $NS(Y)$, so that Y is a Kummer surface.

Theorem 4.2.1.3 ([63, Thm. 5.7])**.** *Let X be a K3 surface such that $E_8^{\oplus 2} \hookrightarrow NS(X)$: then there is a symplectic involution ι on X such that, if $\pi : X \dashrightarrow Y$ is the rational quotient map,*

1. *there is a primitive embedding $E_8 \oplus N \hookrightarrow NS(Y)$,*
2. *π_* induces a Hodge isometry $T(X)(2) \simeq T(Y)$.*

A sympletic involution ι that exchanges two copies of E_8 orthogonally embedded in $NS(X)$ is called a *Morrison-Nikulin involution* on X. As it turns out, for X projective admitting a Morrison-Nikulin involution is equivalent to admitting a Shioda-Inose structure.

Theorem 4.2.1.4 ([63, Thm. 6.3])**.** *Let X be a projective K3 surface. Then the following are equivalent:*

1. *X admits a Shioda-Inose structure.*
2. *There exists an abelian surface A and a Hodge isometry $T(X) \simeq T(A)$.*
3. *There is a primitive embedding $T(X) \hookrightarrow U^{\oplus 3}$.*
4. *There is a primitive embedding $E_8^{\oplus 2} \hookrightarrow NS(X)$.*

The following theorems gives the relation between $NS(X)$ and $NS(Y)$, with the notation of Remark 2.5.0.5; the lattice $(N \oplus \langle 4d \rangle)'$ appears also Theorem 2.5.3.1.

Theorem 4.2.1.5 ([31, Thm. 2.18])**.** *Let X be a projective K3 surface admitting a Morrison-Nikulin involution ι: then $\rho(X) \geq 17$ and $NS(X) \simeq R \oplus E_8^{\oplus 2}$, where R is an even lattice with signature $(1, \rho(X) - 17)$.*
Let Y be the resolution of the singularities of X/ι: then $NS(Y)$ is an overlattice of index $2^{rk(R)}$ of $R(2) \oplus N \oplus E_8$. In particular, if $\rho(X) = 17$ then $NS(X) \simeq \langle 2d \rangle \oplus E_8^{\oplus 2}$, the surface Y is the Kummer surface of a $(1, d)$-polarized abelian surface, and $NS(Y) \simeq (N \oplus \langle 4d \rangle)' \oplus E_8$.

Theorem 4.2.1.6 ([29, Thm. 2.7])**.** *Let $Kum(A)$ be a Kummer surface with Picard number 17, let H be the generator of $K_2^{\perp} \subset NS(Kum(A))$, $H^2 = 4d > 0$. Then $NS(Kum(A)) = (K_2 \oplus \langle 4d \rangle)'$, uniquely determined. More precisely, if $v_{4d} \in K_2$ is such that $(v_{4d} + H)/2$ is integral in $NS(Kum(A))$, one can assume that:*

- *if $H^2 =_8 0$, then $v_{4d} = k_{0000} + k_{1000} + k_{0100} + k_{1100}$;*
- *if $H^2 =_8 4$, then $v_{4d} = k_{0001} + k_{0010} + k_{0011} + k_{1000} + k_{0100} + k_{1100}$.*

From a lattice-theoretic perspective, the lattices $(K_2 \oplus \langle 4d \rangle)'$ and $(N \oplus \langle 4d \rangle)' \oplus E_8$ are abstractly isomorphic (one can apply for instance [54, Cor. VIII.4.2]). In the case $d = 1$, Naruki [69] provides an explicit description of this isomorphism, with the property that the (-2)-curves that generate the Kummer lattice and the lattice $E_8 \oplus N$ are all effective.

In the classical case, the condition $T(X) \simeq T(A)$ is equivalent to the existence on X of a symplectic involution ι (a Morrison-Nikulin involution) such that X/ι is birational to $Kum(A)$. This cannot be generalized to the order 4: indeed, in Section 4.2.3 we prove that if A admits a symplectic automorphism of order 4, the condition $T(X) \simeq T(A)$ does not even guarantee that X admit a symplectic action of a group of order 4, be it $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. However, in Section 4.2.4 we find a generalization of Morrison-Nikulin involutions: if a projective K3 surface X has a symplectic automorphism of order 4 τ that cyclically permutes four copies of D_4 contained in $NS(X)$, then there exists an abelian surface A with a symplectic automorphism of order 4 such that X/τ is birational to $Kum_4(A)$ (and the converse is also true).

4.2.2 Families of abelian surfaces A with an automorphism of order 4, and of K3 surfaces $Kum_4(A)$

Complex 2-tori T with a symplectic automorphism of order 4 are characterized by the fact that $NS(T)$ contains primitively the lattice $\Omega := \langle -2 \rangle^{\oplus 2}$. The moduli space of abelian surfaces (i.e.projective tori) with an automorphism of order 4 splits in irreducible components, each corresponding to a deformation family. The Néron-Severi lattice of general member A of each deformation family is a cyclic overlattice of finite index (possibly 1) of $\Omega \oplus \langle 2d \rangle$ (where $\Omega^\perp$ is generated by the ample class H of A) that can be primitively embedded in $U^{\oplus 3}$.

Remark 4.2.2.1. There exists a unique primitive embedding of Ω in $U^{\oplus 3} = \langle u_1, \dots, u_6 \rangle$: up to isometries of the latter, $\Omega = \langle u_1 - u_2, u_3 - u_4 \rangle$ (see also Section 4.1). From now on, call $\omega_1 = u_1 - u_2, \omega_2 = u_3 - u_4$, and $b_1 = u_1 + u_2, b_2 = u_3 + u_4 \in \Omega^\perp$.

Theorem 4.2.2.2. *Let A be an abelian surface with a symplectic automorphism of order 4, and let H with $H^2 = 2d$ the generator of $\Omega^{\perp_{NS(A)}}$. Then $NS(A)$ is one of the following:*

1. *For every d, $NS(A) = \Omega \oplus \langle 2d \rangle$.*
2. *For $d =_4 1$ or $d =_4 2$, $NS(A) = (\Omega \oplus \langle 2d \rangle)'$ (see Remark 2.5.0.5 for the notation), uniquely determined.*

Proof. Cyclic overlattices of $\Omega \oplus \langle 2d \rangle$ correspond to isotropic elements of the form $(H + w)/2 \in A_{\Omega \oplus \langle 2d \rangle}$, with $w \in A_\Omega$; non-isomorphic overlattices are in bijection with the equivalence classes of the action of $O(\Omega)$ induced on the discriminant group A_Ω, by Theorem 1.2.1.3. The action of $O(\Omega)$ splits A_Ω in 3 classes: $\{0\}$, $\{\omega_1/2, \omega_2/2\}$ and $\{(\omega_1 + \omega_2)/2\}$. Each of the corresponding overlattices admits a unique primitive embedding in $H^2(A, \mathbb{Z})$ by Theorem 1.2.1.5. □

Example 4.2.2.3. We give examples of $H \in \Omega^{\perp_{U^{\oplus 3}}}$ that realize each possible $NS(A)$, and the corresponding transcendental $T(A)$:

1. For every d, $H_0(d) = u_5 + d u_6$ generates the lattice $\langle 2d \rangle$ such that $NS(A) = \Omega \oplus \langle 2d \rangle$ is primitively embedded in $U^{\oplus 3}$;

$$T_0^{Ab}\{d\} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2d \rangle.$$

2. For $d = 4h+1$, $H_1(h) = 2H_0(h) + b_1$ generates the lattice $\langle 2d \rangle$ such that $NS(A) = (\Omega \oplus \langle 2d \rangle)'$ is primitively embedded in $U^{\oplus 3}$ (the element $(H_1(h) + \omega_1)/2$ is integral);

$$T_1^{Ab}\{h\} = \langle 2 \rangle \oplus \begin{bmatrix} 2 & 1 \\ 1 & -2h \end{bmatrix}.$$

3. For $d = 4h + 2$, $H_2(h) = 2H_0(h) + b_1 + b_2$ generates the lattice $\langle 2d \rangle$ such that $NS(A) = (\Omega \oplus \langle 2d \rangle)'$ is primitively embedded in $U^{\oplus 3}$ (the element $(H_2(h) + \omega_1 +$

$\omega_2)/2$ is integral);

$$T_2^{Ab}\{h\} = \begin{bmatrix} -2h & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Example 4.2.2.4. The abelian surface $A = E \times E$, with E elliptic curve, belongs to the second class of examples. Indeed, we can take as $\mathbb{Z}$-basis of $NS(E \times E)$ $\{e_1, e_2, \delta\}$, the classes of the curves $E_1 = \{(z_1, 0) \mid z_1 \in E\}$, $E_2 = \{(0, z_2) \mid z_2 \in E\}, \Delta = \{(z, z) \mid z \in E\}$. Each of these curves is isomorphic to an elliptic curve, and intersects the other two in one point: therefore, as a lattice we have

$$NS(A) = \langle e_1, e_2, \delta \rangle = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which can be embedded in $U^{\oplus 3}$ with orthogonal complement isomorphic to $T_1^{Ab}\{0\}$.

The surface $Y = Kum_4(A)$ is a projective K3 surface such that there exists a primitive embedding $K_4 \hookrightarrow NS(Y)$. Therefore, if Y is general $NS(Y) = K_4 \oplus \langle 2d \rangle$ or some cyclic overlattice of the latter.

Theorem 4.2.2.5. *We classify the overlattices of $K_4 \oplus \langle 2d \rangle$ that are admissible as Néron-Severi of a K3 surface, up to isomorphism:*

1. *for every h, $(K_4 \oplus \langle 8h \rangle)^*$, uniquely determined;*
2. *for every $d =_4 1, 2$, $(K_4 \oplus \langle 2d \rangle)'$, uniquely determined.*

Proof. The induced action of $O(K_4)$ splits A_{K_4} into 9 orbits: in the following table, the cardinality of each orbit (except $\{0\}$) is displayed, denoting k the order, and g the square of the elements cointained. Notice that there are two different orbits containing elements with $(k, g) = (2, 1)$.

k \ g	0	1/2	1	3/2
2	3	0	3+1	8
4	12	12	12	12

The lattice $K_4 \oplus \langle 2d \rangle$ is not admissible as Néron-Severi lattice of a K3 surface: indeed, it has rank 19 and length 5, so the corresponding transcendental lattice would have rank 3 and length 5, which is impossible.
Let L be the generator of $K_4^{\perp}$, $L^2 = 2d$. Let $v/4$ be a representative of one of the classes $(4, g)$. Then it holds

$$\left(\frac{L + v}{4}\right)^2 = \frac{2d + v^2}{16} = \frac{d}{8} + g,$$

which is an even integer only if $d = 4h$: more precisely, if $g = 0, 1/2, 1, 3/2$ then $h =_4 0, 3, 2, 1$ respectively. Each of the lattices $(K_4 \oplus \langle 8h \rangle)^*$ thus realized admits a unique primitive embedding in Λ_{K3} by Theorem 1.2.1.5.
Now let $w/2$ be a representative of one of the classes $(2, g)$: Then it holds

$$\left(\frac{L+w}{2}\right)^2 = \frac{2d + w^2}{4} = \frac{d}{2} + g;$$

since the orbit $(2, 1/2)$ is empty, $d \neq_4 3$. If $d = 4h$, the overlattice $(K_4 \oplus \langle 8h \rangle)'$ thus generated is not admissible as Néron-Severi lattice of a K3 surface: indeed, it has again rank 19 and length 5. For $d =_4 1, 2$ the overlattices $(K_4 \oplus \langle 2d \rangle)'$ are uniquely determined, and each of them admits a unique primitive embedding in Λ_{K3}.
The overlattices of index 2 are uniquely determined because the two orbits of elements of order 2 and square 1 glue to elements of different square in $K_4^\perp = U(4) \oplus \langle 2 \rangle^{\oplus 2} = \langle u_1, u_2, a_1, a_2 \rangle$. Indeed the discriminant form of A_{K_4} is

$$q = \begin{bmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 \end{bmatrix} = \langle x_1, x_2, x_3, x_4 \rangle,$$

and the two orbits are generated by the elements $v_1 = x_1 + x_2$ and $v_2 = x_1 + x_2 + 2(x_3 + x_4)$ respectively: v_1 glues to elements of the form $2F + a_1 + a_2 + u_1$ (any F in $K_4^\perp$), while v_2 glues to $2F + a_1 + a_2 + u_1 + u_2$; elements of the first form have square $16h + 4$, elements of the second form have square $16h + 12$. □

The following theorem generalizes Theorem 4.2.1.6 to abelian surfaces A with a symplectic automorphism of order 4, and surfaces $Kum_4(A)$. For the notation, see Remark 2.5.0.5.

Theorem 4.2.2.6. *If $NS(A) = \Omega \oplus \langle 2d \rangle$, then $NS(Kum_4(A)) = (K_4 \oplus \langle 8d \rangle)^*$.*
If $NS(A) = (\Omega \oplus \langle 2d \rangle)'$, then $NS(Kum_4(A)) = (K_4 \oplus \langle 2d \rangle)'$.

Proof. Take the image of H_j in Example 4.2.2.3 via the composite map $(\hat{\pi} \circ \pi)_*$: if necessary, divide it to get a primitive class $\hat{H}_j$, which by construction belongs to $R := K_4^\perp$ in $H^2(Kum_4(A), \mathbb{Z})$. Then, find the maximum integer k such that $\hat{H}_j/k$ belongs to A_R; since $H^2(Kum_4(A), \mathbb{Z})$ is unimodular, if $k > 1$ then $K_4 \oplus \hat{H}_j$ is not primitive in $H^2(Kum_4(A), \mathbb{Z})$, but an overlattice of index k of it is: indeed, there exists an element $\theta_j \in K_4$ such that $(\hat{H}_j + \theta_j)/k$ is integral in $H^2(Kum_4(A), \mathbb{Z})$ (see Remark 1.2.1.7).
More precisely, the class $\hat{H}_0(d)$ glues to one of the orbits $(4, g)$ depending on the value of d modulo 4; $\hat{H}_1(h)$ glues to an element in the orbit $(k, g) = (2, 3/2)$; $\hat{H}_2(h)$ glues to an element in one of the two orbits with $(k, g) = (2, 1)$ depending on the parity of h. □

4.2.3 K3 surfaces with transcendental lattice $T(A)$

This section provides a negative answer to the following question:

Question 1. *Given a general abelian surface A with a symplectic automorphism α of order 4, consider a K3 surface X_A such that $T(X_A) \simeq T(A)$: does X_A admit a symplectic action of a group of order 4?*

Remark 4.2.3.1. In [27] it is proved that the answer is affirmative if, instead of symplectic automorphisms of order 4, we consider those of order 3; it is also proved that it's negative for symplectic automorphisms of order 6. Moreover, for the order 3 the automorphism γ on X is such that X/γ is birational to $Kum_3(A)$: therefore, Shioda-Inose structures can be fully generalized to symplectic automorphisms of order 3.

We can rephrase this question as a lattice-theoretic problem: indeed, X_A admits a symplectic action of $G \in \{\mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2\}$ if and only if there exists a primitive embedding of the lattice $T(X_A)$, that is isomorphic to one of the lattices T_i^{Ab} of Example 4.2.2.3, in the invariant lattice for the action of G on Λ_{K3}.

Proposition 4.2.3.2. *Let A be a general abelian surface with a symplectic automorphism of order 4, let $\langle H\rangle = \Omega^{\perp_{NS(A)}}$, $H^2 = 2d$: then a K3 surface X_A such that $T(X_A) \simeq T(A)$ has a symplectic automorphism τ of order 4 if and only if d is even. More precisely, referring to Example 4.2.2.3 for the notation, the valid cases are:*

1. *$T(X_A) = T_0^{Ab}\{d\}$, for any $d > 0$ even;*
2. *$T(X_A) = T_2^{Ab}\{h\}$, for $d = 4h + 2$, $h \geq 0$.*

Proof. The invariant lattice for the action of $\mathbb{Z}/4\mathbb{Z}$ on Λ_{K3} is

$$I_4 \coloneqq U \oplus \langle -2\rangle^{\oplus 2} \oplus U(4)^{\oplus 2} = \langle u_1, u_2\rangle \oplus \langle a, b\rangle \oplus \langle v_1, v_2\rangle \oplus \langle w_1, w_2\rangle$$

(see [37, Table 10.3], and Proposition 5.1.1.1 for the isometry between I_4 and the invariant lattice as described in Section 2.3.2). Consider $\langle 2\rangle^{\oplus 2}$, that is a primitive sublattice of T_i^{Ab} for all the examples: since $\langle -2\rangle^{\oplus 2} \oplus U(4)^{\oplus 2}$ does not represent 2, the only primitive embedding of $\langle 2\rangle^{\oplus 2}$ in I_4 up to isometries is $\langle u_1 + u_2, u_2 - u_1 - a - b + w_1 + w_2\rangle \coloneqq \langle t_1, t_2\rangle$, its orthogonal complement being $D_4(2) \oplus U(4) = \langle w_2 - a - b, b - a, a - u_1 + u_2, w_1 - a - b\rangle \oplus \langle v_1, v_2\rangle$. Call $\langle d_1, \dots, d_4\rangle = D_4(2)$ (numbered as in Example 1.2.0.2): then, since $D_4(2) \oplus U(4)$ does not represent -2, $T_0^{Ab}\{d\}$ can be the transcendental of a K3 with a symplectic automorphism of order 4 only for d even. To do so, choose as the class of square $-2d$ that generates $(\langle 2\rangle^{\oplus 2})^{\perp_{T(A)}}$ as $v_1 - nv_2$ if $d = 4n$, as $d_1 + v_1 - nv_2$ if $d = 4n + 2$.
The lattice $T_1^{Ab}\{h\}$ cannot be realized: indeed, I_4 is obtained as overlattice of $\langle 2\rangle^{\oplus 2} \oplus D_4(2) \oplus U(4)$ by adding as generator the class

$$\alpha \coloneqq \frac{t_1 + t_2 + d_1 + d_2 + d_4}{2},$$

which has intersection 1 with both t_1 and t_2. The lattice $T_2^{Ab}\{h\}$ can be the transcendental of a K3 with a symplectic automorphism of order 4 for any h: the class that has square $-2h$ and intersection 1 with both t_1 and t_2 is as in the following table:

$h = 4n$	$\alpha + d_3 + v_1 - nv_2$
$h = 4n + 1$	$\alpha + v_1 - nv_2$
$h = 4n + 2$	$\alpha + d_2 + d_3 + v_1 - nv_2$
$h = 4n + 3$	$\alpha + d_2 + 2d_3 + v_1 - nv_2$

□

Proposition 4.2.3.3. *Let A be a general abelian surface with a symplectic automorphism of order 4, let $\langle H \rangle = \Omega^{\perp_{NS(A)}}$, $H^2 = 2d$: then a K3 surface X_A such that $T(X_A) \simeq T(A)$ has a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ if and only if d is even. More precisely, referring to Example 4.2.2.3 for the notation, the valid cases are:*

1. *$T(X_A) = T_0^{Ab}\{d\}$, for any $d > 0$;*
2. *$T(X_A) = T_2^{Ab}\{h\}$, for $d = 4h + 2$, $h \geq 0$ even.*

Proof. The invariant lattice for the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on Λ_{K3} is

$$I_{2,2} := U \oplus U(2)^{\oplus 2} \oplus D_4(2) = \langle u_1, u_2 \rangle \oplus \langle v_1, v_2 \rangle \oplus \langle w_1, w_2 \rangle \oplus \langle d_1, \ldots, d_4 \rangle$$

(see [37, Table 10.3], and Proposition 5.1.2.1 for the isometry between $I_{2,2}$ and the invariant lattice as described in Section 3.3.2). Up to isometries, $\langle 2 \rangle^{\oplus 2}$ admits a unique primitive embedding in $I_{2,2}$: choosing as its generators $\langle u_1 + u_2, v_1 + v_2 - u_1 + u_2 \rangle$, its orthogonal complement in $I_{2,2}$ is

$$U(2) \oplus D_4(2) \oplus \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix} = \langle w_1, w_2 \rangle \oplus \langle d_1, \ldots, d_4 \rangle \oplus \langle u_1 - u_2 - v_1, v_1 - v_2 \rangle,$$

and $I_{2,2}$ is obtained as overlattice by adding as generator the class $\beta = ((u_1 + u_2) + (v_1 + v_2 - u_1 + u_2) + (v_1 - v_2))/2 = u_2 + v_1$, that has intersection 1 with both generators of $\langle 2 \rangle^{\oplus 2}$.
Therefore, we can primitively embed $T_0^{Ab}\{d\}$ in $I_{2,2}$ for any d, choosing as generator of $\langle 2 \rangle^{\oplus 2 \perp}$ the class $w_1 - kw_2$ for $d = 2k$, $u_1 - u_2 - v_1 + w_1 - kw_2$ for $d = 2k + 1$; similarly, we can primitively embed $T_2^{Ab}\{h\}$ for h even (that is, for $d = 4h + 2 =_8 2$) by choosing as the class with square $-2h$ $u_2 + v_1 + w_1 - kw_2$; again, the other possible transcendental lattices are not admissible. □

4.2.4 Families of covering K3 surfaces

Since given an abelian surface A with a symplectic automorphism of order 4, the condition $T(X) \simeq T(A)$ does not imply the existence of a symplectic action of a group G of order 4 on the K3 surface X, in this section we try a different approach. An equivalent condition for the existence of a classical Shioda-Inose structure is the existence of

a Morrison-Nikulin involution ι on the K3 surface (see Theorem 4.2.1.4): in this case, the resolution of the singularities Z of the quotient X/i is such that $NS(Z)$ contains primitively the lattice $E_8 \oplus N$, which is in the same genus of the Kummer lattice K_2 (see Theorem 4.2.1.3 and Proposition 4.2.1.2), so Z is a Kummer surface.
The generalization of Morrison-Nikulin involutions we propose is that of a symplectic automorphism τ of order 4 that permutes cyclically four algebraic copies of the lattice D_4: indeed, in this case the surface $\tilde{Y}$ that is the minimal resolution of $Y = X/\tau$ is such that $NS(\tilde{Y})$ contains primitively the lattice Δ (see Definition 4.2.4.1), which is in the same genus of K_4 and admits a primitive embedding of M_4.

Recall from Section 2.4.3 that a K3 surface $\tilde{Y}$ is the minimal resolution of $Y = X/\tau$, with X a K3 surface and τ symplectic automorphism of order 4, if and only if there is a primitive embedding of M_4 in $NS(\tilde{Y})$. Refer to section 2.4.3 for the notations.

Definition 4.2.4.1. Consider the lattice $D_4 = \langle \overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4 \rangle$ (the intersection form is described in Example 1.2.0.2); define the lattice Δ as the overlattice of $M_4 \oplus D_4$ obtained by adding to the set of generators the class

$$\delta = \frac{m_1^2 + m_3^2 + m_1^3 + m_3^3 + \tilde{m}^1 + \tilde{m}^2 + \overline{e}_2 + \overline{e}_4}{2}. \qquad (4.2.4.1)$$

Remark 4.2.4.2. The lattice Δ has the same signature and discriminant form of the lattice K_4: it is not isomorphic to it, as it can be seen by comparing the number of classes of square -2 in each lattice.

Remark 4.2.4.3. One could consider the lattice $M_{2,2}$ in place of M_4, and study primitive embeddings of $M_{2,2}$ in K_4, to explore the possibility that $Kum_4(A)$ be covered by a K3 surface admitting a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$. We remark that the lattice Δ characterizes K3 surfaces covered by a K3 surface X with a symplectic automorphism τ whose action can be seen on algebraic classes (see Theorem 4.2.4.12 for a more precise statement); however, a similar approach applied to a K3 surface X with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ does not give a lattice that can play the role of Δ.

Proposition 4.2.4.4. *The lattice $H^2(Kum_4(A), \mathbb{Z})$ is an overlattice of $\Delta \oplus \langle -4 \rangle \oplus \langle 4 \rangle \oplus \langle 8 \rangle^{\oplus 2}$: calling $\overline{a}$ the generator of $\langle -4 \rangle$, $\overline{\chi}$ the generator of $\langle 4 \rangle$ and $\overline{\omega}_1, \overline{\omega}_2$ the generators of $\langle 8 \rangle^{\oplus 2}$, the classes we have to add to the set of generators are:*

$$\begin{aligned}
\phi &= (\overline{\omega}_1 + \overline{\omega}_2)/2, \\
\psi &= (\overline{a} + \overline{\chi} + \overline{\omega}_1)/2, \\
x_1 &= (m_1^4 + m_3^4 + \tilde{m}^1 + \overline{e}_1 + \overline{e}_4 + \psi + \overline{a})/2, \\
x_2 &= (m_1^3 + m_3^3 + \tilde{m}^2 + \overline{e}_1 + \overline{e}_4 + \overline{\chi} + \phi + \psi)/2, \\
x_3 &= \frac{\delta + m_2^2 + m_3^2 + m_1^3 + m_2^3 + m_1^4 + m_3^4 + \overline{e}_1}{2} + \frac{\tilde{m}^1 - \tilde{m}^2 + \overline{e}_2 - \overline{e}_4 + 3\overline{\chi} + \overline{a}}{4}, \\
x_4 &= \frac{\delta + m_2^2 + m_3^2 + m_2^4}{2} + \frac{m_1^3 + m_3^3 - m_1^4 + m_3^4 - \tilde{m}^1 + \tilde{m}^2 + \overline{e}_2 + \overline{e}_4 + 3(\overline{a} - \phi)}{4}.
\end{aligned}$$

Proof. By direct computation it can be proved that the lattice generated as above is unimodular of signature (3,19). Compare also to Section 2.4.3 (the notation is the same). □

Remark 4.2.4.5. This choice of generators of $H^2(Kum_4(A), \mathbb{Z})$ gives a primitive embedding of the lattice Δ in it.

Theorem 4.2.4.6. *We find the following isomorphisms of the lattices introduced in Theorem 4.2.2.5:*

1. *The lattice* $(K_4 \oplus \langle 8d \rangle)^\star$, *where* $K_4^\perp = \mathbb{Z}\hat{H}_0(d)$, *is isomorphic to* $(\Delta \oplus \langle 8d \rangle)^*$, *where* $\Delta^\perp$ *is generated by either* $F_0(d) = \overline{a} + \phi - d(\overline{a} + \phi - \overline{\omega}_1)$, *or* $G_0(d) = \overline{a} + \overline{\chi} - d(\overline{a} + \phi)$. *The isomorphism between the two overlattices associated to* $F_0(d)$ *and* $G_0(d)$ *does not preserve* Δ.
2. *Let* $d = 4h + 1$. *The lattice* $(K_4 \oplus \langle 2d \rangle)'$, *where* $K_4^\perp = \mathbb{Z}\hat{H}_1(h)$, *is isomorphic to* $(\Delta \oplus \langle 2d \rangle)'$, *where* $\Delta^\perp$ *is generated by* $F_1(h) = \overline{a} + \overline{\chi} + \psi - h(\overline{a} + \phi)$.
3. *Let* $d = 4h + 2$. *The lattice* $(K_4 \oplus \langle 2d \rangle)'$, *where* $K_4^\perp = \mathbb{Z}\hat{H}_2(h)$, *is isomorphic to* $(\Delta \oplus \langle 2d \rangle)'$, *where* $\Delta^\perp$ *is generated by either* $F_2(h) = \overline{a} + \overline{\chi} + \phi - h(\overline{a} + \phi - \overline{\omega}_1)$ *or* $G_2(h) = \overline{\chi} + h(\overline{a} + \overline{\chi})$. *If* h *is even, the isomorphism between the two overlattices associated to* $F_2(h)$ *and* $G_2(h)$ *does not preserve* Δ *(but it does for* h *odd).*

Proof. The induced action of $O(\Delta)$ splits A_Δ into 15 orbits: in the following table, the cardinality of each orbit (except $\{0\}$) is displayed, denoting k the order, and g the square of the elements cointained.

k \ g	0	1/2	1	3/2
2	2+1	0	2+1+1	8
4	8+4	8+4	8+4	8+4

This gives us more non-isomorphic overlattices of $\Delta \oplus \langle 2e \rangle$ than there are for $K_4 \oplus \langle 2e \rangle$, but the classes containing elements of the same order and square, when glued to the appropriate positive class, give rise to lattices in the same genus (this can be checked directly); by Proposition 1.2.1.11 we can conclude that they are actually isomorphic as lattices. □

Remark 4.2.4.7. The isomorphisms between overlattices of $\Delta \oplus \langle 2e \rangle$ obtained by different isometry classes of A_Δ do not preserve Δ (see Cor. 1.2.1.7). This will give more than one deformation families of K3 surfaces covering the same generalized Kummer (see Theorem 4.2.4.14 and Corollary 4.2.4.16).

We are now going to pull back the Néron-Severi lattices we found for $\tilde{Y} = Kum_4(A)$ in Theorem 4.2.4.6 through the map $\pi_4 : X \dashrightarrow \tilde{Y}$ induced by a symplectic automorphism of order 4 on a K3 surface X. Recall that $\tilde{Y}$ is polarized with $\Delta \oplus \langle 2d \rangle$ or one of its

overlattices (Theorem 4.2.4.6), that Δ itself is an overlattice of $D_4 \oplus M_4$ (Definition 4.2.4.1), and that $\pi_4^* D_4 = D_4^{\oplus 4}$ (see Proposition 2.4.4.1).
This process will allow us to find the Néron-Severi lattice of the K3 surfaces X with a symplectic automorphism τ of order 4 that complete the following diagram

$$\begin{array}{ccc} & & X \\ & \overset{\pi_4}{\swarrow} & \downarrow p_4 \\ Kum_4(A) & \longrightarrow & X/\tau \end{array} \tag{4.2.4.2}$$

and such that τ permutes cyclically four algebraic copies of D_4.

Recall from [71, Thm. 4.15] that a K3 surface X admits a symplectic automorphism of order 4 if and only if there exists a primitive embedding in $NS(X)$ of the co-invariant lattice Ω_4 (see Section 2.3.2). The following proposition is a re-statement of Proposition 2.4.4.1: recall from Definition 4.2.4.1 and Proposition 4.2.4.4 that $H^2(Kum_4(A), \mathbb{Z})$ is an overlattice of $M_4 \oplus D_4 \oplus \langle -4 \rangle \oplus \langle 4 \rangle \oplus \langle 8 \rangle^{\oplus 2}$.

Proposition 4.2.4.8. *The map π_4^* annihilates M_4, and acts on $D_4 \oplus \langle -4 \rangle \oplus \langle 4 \rangle \oplus \langle 8 \rangle^{\oplus 2}$ as*

$$\pi_4{}^*: \quad D_4 \oplus \langle -4 \rangle \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \longrightarrow D_4^{\oplus 4} \oplus A_1^{\oplus 2} \oplus \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{pmatrix} \begin{matrix} \overline{e}_1 \\ \overline{e}_2 \\ \overline{e}_3 \\ \overline{e}_4 \end{matrix}, & \overline{a}, & \overline{\chi}, \ \overline{\omega}_1, \ \overline{\omega}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \begin{matrix} e_1 + f_1 + g_1 + h_1 \\ e_2 + f_2 + g_2 + h_2 \\ e_3 + f_3 + g_3 + h_3 \\ e_4 + f_4 + g_4 + h_4 \end{matrix}, & 2a_1 + 2a_2, & 2\rho, 4\omega_1, 4\omega_2 \end{pmatrix}$$

where $e_1, \ldots, e_4$, $f_1, \ldots, f_4$, $g_1, \ldots, g_4$, $h_1, \ldots, h_4$ are the generators of the four copies of D_4, a_1 and a_2 are the generators of the two copies of A_1, ρ is the generator of $\langle 4 \rangle$, ω_1 and ω_2 are the generators of $\langle 2 \rangle^{\oplus 2}$.
The map π_4^* can be extended to $H^2(Kum_4(A), \mathbb{Z})$ adding the elements defined in (4.2.4.1) *and Proposition 4.2.4.4 (and their respective images to the image lattice).*
The lattice $\pi_4^* H^2(Kum_4(A), \mathbb{Z})$ is primitively embedded in $H^2(X, \mathbb{Z})$ with the lattice Ω_4 as orthogonal complement.

Definition 4.2.4.9. Consider the lattice $D_4^{\oplus 4} \oplus \langle -4 \rangle^{\oplus 2}$ spanned by the elements e_i, f_i, g_i, h_i for $i = 1, \ldots, 4$, $a_1 - a_2$ and σ. Define the lattice Π as its overlattice obtained by adding to this set of generators the following:

$$\begin{aligned} \zeta_1 &= (\sigma + e_1 - g_1 + e_2 - f_2 + f_4 - g_4)/2, \\ \zeta_2 &= (e_1 - g_1 + f_1 - h_1 + e_2 - g_2 + f_4 - h_4)/2, \\ \zeta_3 &= (\sigma + f_1 - h_1 + e_2 - h_2 + f_4 - e_4)/2 \\ \zeta_4 &= (e_2 - g_2 + e_4 - g_4 + a_1 - a_2 + \sigma)/2. \end{aligned}$$

Then Π is a negative definite lattice of rank 18 with the same discriminant form as $\langle -4\rangle^{\oplus 2}$.
The orthogonal complement of Π in $H^2(X,\mathbb{Z})$ is

$$\Pi^\perp = U \oplus \langle 4\rangle^{\oplus 2} = \langle \frac{\omega_1+\omega_2+a_1+a_2}{2}, \frac{\omega_1-\omega_2-a_1-a_2}{2}\rangle \oplus \langle \rho, 2\omega_2+a_1+a_2\rangle.$$

Remark 4.2.4.10. The lattice Π is the primitive saturation (see Definition 1.2.1.1) of $\pi_4^*\Delta \oplus \Omega_4$ in $H^2(X,\mathbb{Z})$.

Theorem 4.2.4.11. *Let X be a K3 surface. Then X admits a symplectic automorphism τ of order 4 such that τ^* permutes cyclically four copies of D_4 in $NS(X)$ if and only if there exists a primitive embedding $\Pi \hookrightarrow NS(X)$.*

Proof. Suppose that τ^* permutes cyclically four copies of D_4 in $NS(X)$, with generators $e_1,\dots,e_4,\ f_1,\dots,f_4,\ g_1,\dots,g_4,\ h_1,\dots,h_4$. Then the co-invariant lattice Ω_4 associated to its action contains the elements $e_i-f_i,\ e_i-g_i,\ e_i-h_i$ for $i=1,\dots,4$; these elements generate a lattice of rank 12, while Ω_4 has rank 14. Moreover, we also have to preserve the embedding $\Omega_2 \hookrightarrow \Omega_4$ of the co-invariant lattice for the action of τ^2: since by hypothesis the classes $e_i-g_i,\ f_i-h_i$ are co-invariant classes for the action of τ^2, we deduce that $\Omega_2 = E_8(2)$ is an overlattice of finite index of $\langle e_i-g_i,\ f_i-h_i\rangle_{i=1,\dots,4} = D_4(2)^{\oplus 2}$. There is only one way to construct this overlattice: using the same notation as in Section 2.3.2, we have to add the classes $\alpha-\gamma, \beta-\delta$, and therefore these classes belong to Ω_4 too; in the definition of Π, these are $\zeta_2, \zeta_1+\zeta_3$. We still have to add two generators to get a lattice of rank 14, that should be invariant for τ^2, but not for τ: by uniqueness of the action of $\mathbb{Z}/4\mathbb{Z}$ on Λ_{K3}, we can refer to Section 2.3.2 and conclude that they are two orthogonal classes of square -4; in the notation of that section, these are σ and a_1-a_2, and they glue to some of the other generators; in the definition of Π, the same gluing is given by ζ_1, ζ_4.
Therefore, Π is by construction the smallest primitive sublattice of $H^2(X,\mathbb{Z})$ containing Ω_4 and $D_4^{\oplus 4}$ such that the four copies of D_4 are cyclically permuted by the action of τ on Ω_4. □

The following theorem is a generalization of Theorem 4.2.1.3.

Theorem 4.2.4.12. *Let X be a K3 surface such that $D_4^{\oplus 4}$ is primitively embedded in $NS(X)$, and suppose there exists a symplectic automorphism τ of order 4 on X such that τ^* acts on the four copies of D_4 as the permutation $(1,2,3,4)$; let $\pi_4 : X \dashrightarrow \tilde{Y}$ be the rational quotient map, then:*

1. *there is a primitive embedding $\Delta \hookrightarrow NS(\tilde{Y})$, where Δ is the minimal primitive overlattice of $M_4 \oplus D_4$ in Λ_{K3};*
2. *if X is projective, then there exists an abelian surface A such that $\tilde{Y} = Kum_4(A)$.*

Remark 4.2.4.13. Notice that, unlike the original theorem, we have to suppose that τ exists, because the condition $D_4^{\oplus 4} \subset NS(X)$ is not enough by Theorem 4.2.4.11.

Moreover, $T(\tilde{Y})$ is not a multiple of $T(X)$: the relation between them will be furtherly explored in Corollary 4.2.4.16.

Theorem 4.2.4.14. *Let A be a general abelian surface with a symplectic automorphism of order 4, let X be a K3 surface with a symplectic automorphism τ of order 4 that cyclically permutes four copies of D_4 in $NS(X)$. Then $Kum_4(A)$ is a minimal resolution of X/τ, and $T(X)$ is as follows (see Remark 2.5.0.5 for the notation):*

1. *if $NS(Kum_4(A)) = (K_4 \oplus \langle 8d\rangle)^*$, then $NS(X) = \Pi \oplus \langle 2d\rangle$, and $T(X)$ is isomorphic to*
$$T_{0,F}^{\mathrm{K3}}\{d\} = \langle 4\rangle^{\oplus 2} \oplus \langle -2d\rangle$$
or $NS(X) = (\Pi \oplus \langle 8d\rangle)'$, and $T(X)$ is isomorphic to
$$T_{0,G}^{\mathrm{K3}}\{d\} = \begin{bmatrix} -2(d-1) & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

2. *if $d = 4h+1$ and $NS(Kum_4(A)) = (K_4 \oplus \langle 2d\rangle)'$, then $NS(X) = (\Pi \oplus \langle 8d\rangle)^*$, and $T(X)$ is isomorphic to*
$$T_1^{\mathrm{K3}}\{h\} = \begin{bmatrix} 4-2h & 3 & 3 \\ 3 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$$

3. *if $d = 4h+2$ and $NS(Kum_4(A)) = (K_4 \oplus \langle 2d\rangle)'$, then*
   - *if $h =_2 1$, $NS(X) = (\Pi \oplus \langle 2d\rangle)'$, and $T(X)$ is isomorphic to*
   $$T_2^{\mathrm{K3}}\{h\} = \langle 2\rangle^{\oplus 2} \oplus \langle -2d\rangle;$$
   - *if $h =_2 0$, then either $NS(X) = (\Pi \oplus \langle 2d\rangle)'$, and $T(X)$ is isomorphic to*
   $$T_{2,F}^{\mathrm{K3}}\{h\} = \langle 4\rangle \oplus \begin{bmatrix} 4 & 2h+2 \\ 2h+2 & h^2 \end{bmatrix},$$
   or $NS(X) = (\Pi \oplus \langle 2d\rangle)^*$, and $T(X)$ is isomorphic to
   $$T_{2,G}^{\mathrm{K3}}\{h\} = \begin{bmatrix} -2h & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Proof. We compute $\pi_4^* F_i, \pi_4^* G_i$ for each of the classes F_i, G_i of Theorem 4.2.4.6: if they are not primitive, we divide them accordingly to get primitive classes $\tilde{F}_i, \tilde{G}_i$. The lattice $NS(X)$ is the primitive saturation in Λ_{K3} of $\Pi \oplus \mathbb{Z}\tilde{F}_i$, or $\Pi \oplus \mathbb{Z}\tilde{G}_i$ respectively.
The action of $O(\Pi)$ induced on A_Π gives six orbits: $\{0\}, (2,0), (2,1), (4,3/4), (4,3/2), (4,7/4)$, where the orbit denoted (k,g) contains all the elements of A_Π of order k and square g.

More precisely, we find:

1. $\tilde{F}_0(d) = \pi_4^* F_0(d)/4 = (\omega_1+\omega_2+a_1+a_2)/2 + d(\omega_1-\omega_2-a_1-a_2)/2$, $\tilde{F}_0(d)^2 = 2d$, $NS(X) = \Pi \oplus \langle 2d \rangle$;
   $\tilde{G}_0(d) = \pi_4^* G_0(d)/2 = a_1+a_2+\rho - d(\omega_1+\omega_2+a_1+a_2)$, $\tilde{G}_0(d)^2 = 8d$, $NS(X)$ is an overlattice of index 2 of $\Pi \oplus \langle 8d \rangle$, associated to the class $(2,0)$ of A_Π.

2. $\tilde{F}_1(h) = \pi_4^* F_1(h) = 3(a_1+a_2+\rho)+2\omega_1 - 2h(\omega_1+\omega_2+a_1+a_2)$, $\tilde{F}_1(h)^2 = 32h+8$, $NS(X)$ is an overlattice of index 4 of $\Pi \oplus \langle 8d \rangle$, associated to the class $(4, 3/2)$ of A_Π.

3. $\tilde{F}_2(h) = \pi_4^* F_2(h)/2 = 2\tilde{F}_0(h)+\rho$, $\tilde{F}_2(h)^2 = 2d$, $NS(X)$ is an overlattice of index 2 of $\Pi \oplus \langle 2d \rangle$, associated to the class $(2,1)$ of A_Π.
   $\tilde{G}_2(h) = \pi_4^* G_2(h)/2 = \rho + h(a_1+a_2+\rho)$: if $h =_2 0$ $NS(X)$ is an overlattice of index 4 of $\Pi \oplus \langle 2d \rangle$, associated to the class $(4, 3/4)$ of A_Π; if $h =_2 1$ $NS(X)$ is an overlattice of index 2 of $\Pi \oplus \langle 2d \rangle$: in this case, there is a Π-preserving isometry between this lattice, and the one defined by $\tilde{F}_2$.

□

Remark 4.2.4.15. There is one orbit for the action of $O(\Pi)$ on A_Π we did not use, namely $(4, 7/4)$. Consider the positive class $\tilde{G}_3 = 4\tilde{F}_0+\rho$: it holds $\tilde{G}_3^2 = 32k+4$, and the primitive saturation of the lattice $\Pi \oplus \mathbb{Z}\tilde{G}_3(k)$, which is the overlattice associated to the orbit $(4, 7/4)$, is isomorphic to that of $\Pi \oplus \mathbb{Z}\tilde{G}_2(4k)$; the isometry doesn't preserve Π, but still these lattices, being isometric, give the same projective family of K3 surfaces polarized with the lattice Π. Now, having two different embeddings of a co-invariant lattice for some symplectic action in the same $NS(X)$ usually gives two different projective families of the quotient surfaces. However, in this case it holds $\tilde{G}_3 = \pi_4^* G_3/2$, with $G_3 = 2F_0(k)+\overline{\chi}$, $G_3^2 = 32k+4$ and there is a (Δ-preserving) isometry between the primitive saturation of $\Delta \oplus \mathbb{Z}G_3(k)$ and that of $\Delta \oplus \mathbb{Z}G_2(4k)$: therefore, also the Kummer surfaces belong to the same projective family.

Corollary 4.2.4.16. *We give a table comparing the transcendental lattices of A, X and $Kum_4(A)$. We remark that requesting that Π be primitively embedded into $NS(X)$ is not enough to give a bijection between deformation families of abelian and K3 surface which cover the same generalized Kummer surface.*

	$T(A)$	$T(Kum_4(A))$	$T(X)$
$\forall d$	$\langle 2 \rangle^{\oplus 2} \oplus \langle -2d \rangle$	$\langle 2 \rangle^{\oplus 2} \oplus \langle -8d \rangle$	$\langle 4 \rangle^{\oplus 2} \oplus \langle -2d \rangle$ *or* $\begin{bmatrix} -2(d-1) & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{bmatrix}$
$d = 4h+1$	$\langle 2 \rangle \oplus \begin{bmatrix} 2 & 1 \\ 1 & -2h \end{bmatrix}$	$\langle 2 \rangle \oplus \begin{bmatrix} 8 & 4 \\ 4 & -8h \end{bmatrix}$	$\begin{bmatrix} 4-2h & 3 & 3 \\ 3 & 4 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

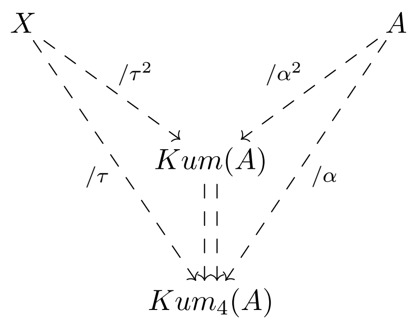
$d = 8k+6$	$\begin{bmatrix} -4k-2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4 \\ 4 & -8(2k+1) \end{bmatrix}$	$\langle 2 \rangle^{\oplus 2} \oplus \langle -2d \rangle$
$d = 8k+2$	$\begin{bmatrix} -4k & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4 \\ 4 & -16k \end{bmatrix}$	$\langle 4 \rangle \oplus \begin{bmatrix} 4 & 4k+2 \\ 4k+2 & 4k^2 \end{bmatrix}$ *or* $\begin{bmatrix} -4k & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

4.2.5 What is a generalized Shioda-Inose structure?

As it can be seen from the table above, it is indeed possible to find generalized Shioda-Inose structures in a strict sense, as in the following definition, but only for certain projective families.

Definition 4.2.5.1. A K3 surface X admits a *strong* order 4 Shioda-Inose structure if there is a quadruple (A, α, X, τ) such that:

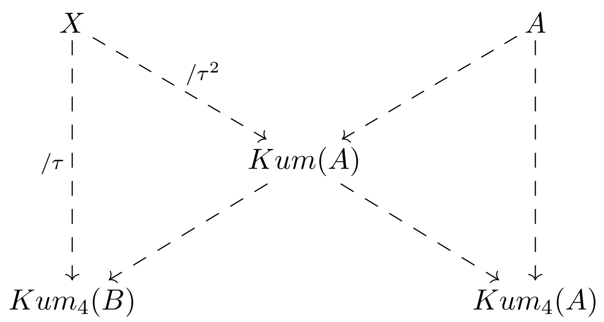
1. τ is a symplectic automorphism of order 4 on X;
2. A is an abelian surface with a symplectic automorphism of order 4 α such that the resolution of the singularities Y of X/τ is isomorphic to the surface $Kum_4(A)$;
3. the triple (A, X, τ^2) is a classical Shioda-Inose structure, that is, $T(X) \simeq T(A)$ and the resolution of the singularities of X/τ^2 is isomorphic to $Kum(A)$.



Remark 4.2.5.2. For classical Shioda-Inose structures, given any abelian surface A, the existence of a Hodge isometry $T(X) \simeq T(A)$ allows to single out exactly one K3 surface X that completes the structure: indeed, the image of the period of A through the isometry gives a period of a (projective) K3 surface, that by Theorem 1.3.0.8 corresponds to exactly one K3 surface. On the other hand, given (X, ι) such that X/ι is birational to a Kummer surface, one usually finds two abelian surfaces that complete the structure, one being the dual of the other (see Theorem 1.3.0.14).

Remark 4.2.5.3. If the surface X admits a strong structure, then there exists a Hodge isometry $T(X) \simeq T(A)$. However, with this definition it is not possible to generalize Morrison's theorem 4.2.1.4. Indeed, from Corollary 4.2.4.16 we derive the following facts:

1. if A, X are such that $T(A) \simeq T_0^{Ab}\{8k+6\}$ and $T(X) \simeq T_2^{\mathrm{K3}}\{2k+1\}$, then $T(A) \simeq T(X)$, so they form a Shioda-Inose structure and they admit a symplectic automorphism of order 4, but they do not form a strong order 4 Shioda-Inose structure: indeed the quotient X/τ is birational to a generalized Kummer surface $Kum_4(B)$, with $T(B) \simeq T_2^{Ab}\{2k+1\}$, and not to $Kum_4(A)$. We get a diagram like this instead:



2. an abelian surface A can belong to a strong order 4 Shioda-Inose structure only if $T(A) \simeq T_2^{Ab}\{2k\}$, because then we can take $T(X) \simeq T_{2,G}^{\mathrm{K3}}\{2k\} \simeq T(A)$;

3. the existence of a primitive embedding $\Pi \hookrightarrow NS(X)$ is not sufficient to conclude that X admits a strong order 4 Shioda-Inose structure: indeed we have a strong order 4 Shioda-Inose structure only if $T(X) \simeq T_{2,G}^{\mathrm{K3}}\{2k\}$, while the embedding exists for any of the transcendental lattices in Theorem 4.2.4.14.

We can therefore propose another definition:

Definition 4.2.5.4. A K3 surface X admits a *weak* order 4 Shioda-Inose structure if there exists a symplectic automorphism τ on X of order 4 permuting cyclically four copies of D_4 in $NS(X)$.

With this definition, we can better generalize Morrison's theorem:

Theorem 4.2.5.5. *Let X be a projective K3 surface. Then the following are equivalent:*

1. *X admits a weak order 4 Shioda-Inose structure;*

2. *there is a primitive embedding $\Pi \hookrightarrow NS(X)$;*

3. *There exists an abelian surface A with a symplectic automorphism of order 4 such that $Kum_4(A)$ is isomorphic to the resolution of the singularities of X/τ, and the projective families of A and X correspond accordingly to Corollary 4.2.4.16.*

Proof. The equivalence between 1) and 2) is proven in Theorem 4.2.4.11. Moreover, 2) is equivalent to 3) because if $T(X)$ is one of those in Corollary 4.2.4.16, the correspondent $NS(X)$ is one of those in Theorem 4.2.4.14, so it's a cyclic overlattice of $\Pi \oplus \langle 2d \rangle$. □

Remark 4.2.5.6. If X admits a strong order 4 Shioda-Inose structure (A, α, X, τ), then the same quadruple forms also weak structure.

The absence of a Hodge isometry between $T(A)$ and $T(X)$ in weak structures allows only, given X, to identify only the projective family to which A belongs; conversely, given A, we have also the problem that depending on $T(A)$ there can be up to two different families of K3 surfaces X that form a weak structure with A.
Therefore, we conclude that it is not possible to give a full generalization of Shioda-Inose structures to the order 4: one has to give up either the generality of the existence, or the (almost) uniqueness of the abelian and K3 surface that interact with each other.

4.2.6 Examples

We're going to provide two examples of K3 surfaces admitting an order 4 Shioda-Inose structures, both of them admitting a Jacobian fibration (see Section 2.2.1 for the related theory).
The first one is the surface X_4 of Section 2.2.2, that admits a strong order 4 Shioda-Inose structure. We already know from Section 2.2.3 that the symplectic automorphism τ induced by the generator of $MW(\pi)$ exchanges four copies of D_4 in $NS(X_4)$, and in Remark 2.3.1.2 also the two copies of E_8 exchanged by τ^2 are provided. We can prove that $\tilde{Z}_4$, the resolution of the singularities of X_4/τ^2, is a Kummer surface. By Theorem 4.2.1.3 we only need to check that $\pi_{2*}T(X_4) \simeq T(X_4)(2)$: since $T(X_4) = \langle \omega_1, \omega_2 \rangle$, we can see this holds by Proposition 2.4.1.1.

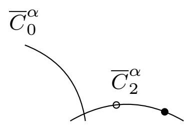
An example of weak order 4 Shioda-Inose structure can be found in Shimada's catalogue ([90], see also the tables in the preprint version). Consider $p : X \to \mathbb{P}^1$ (no. 2711) the Jacobian fibration whose trivial lattice is $\mathcal{T}(p) = U \oplus 2D_5 \oplus A_7$, and $MW(p) \simeq \mathbb{Z}/4\mathbb{Z}$: since $NS(X)$ is a cyclic overlattice of $\mathcal{T}(p)$ of index 4, the discriminant group of the Néron-Severi of X is $A_{NS(X)} = \mathbb{Z}/8\mathbb{Z}$: the transcendental lattice of X is $T_1^{\mathrm{K3}}\{0\}$ of Theorem 4.2.4.14.
The Kodaira type of X is $2I_1^* + I_8 + 2I_1$. Call $\{C_k^\alpha,\ k = 0, \dots, 5\}$ the irreducible components of the two copies of I_1^* $(\alpha = 1, 2)$, such that C_0^α, C_1^α intersect only C_2^α, C_4^α, C_5^α intersect only C_3^α, and C_2^α intersects C_3^α; call $\{B_i,\ i = 0, \dots, 7\}$ the irreducible components of I_8 such that, for every $i \in \mathbb{Z}/8\mathbb{Z}$, B_i intersects only $B_{(i+1)}$.
The trivial section s intersects C_0^1, C_0^2, B_0, and the torsion sections of p intersect B_i, C_j^1, C_k^2 only if i, j, k satisfy the height formula

$$0 = 4 - i(8-i)/8 - \delta_j - \delta_k, \quad \text{with } \delta_j, \delta_k = \begin{cases} 1 & j, k = 1 \\ 3/4 & j, k = 4, 5; \end{cases}$$

we choose as generator t of $MW(p)$ the section that intersects B_2, C_5^1, C_5^2. Then the action of the symplectic automorphism τ (induced by t) on the singular fibers is as follows:

$$C_0^\alpha \stackrel{\tau^*}{\longmapsto} C_5^\alpha \stackrel{\tau^*}{\longmapsto} C_1^\alpha \stackrel{\tau^*}{\longmapsto} C_4^\alpha \stackrel{\tau^*}{\longmapsto} C_0^\alpha \quad \text{for } \alpha = 1, 2, \qquad \tau^*(B_{[i]_8}) = B_{[i+2]_8}.$$

Call $u = \tau^* t, v = \tau^* u$: the four orthogonal copies of D_4 on which τ^* acts as a cycle of order 4 are $\{s, C_0^1, C_0^2, B_0\}$, $\{t, C_5^1, C_5^2, B_2\}$, $\{u, C_1^1, C_1^2, B_4\}$, $\{v, C_4^1, C_4^2, B_6\}$. Taking the quotient X/τ, we get an I_2 fiber in place of the I_8; the action on the two I_1 fibers of X fixes the nodal points, which become A_3 singularities in the quotient X/τ; in place of the I_1^* fibers we get a configuration of curves as follows ($\overline{C}_0^\alpha$ is the image of C_0^α via the quotient map, similarly $\overline{C}_2^\alpha$), where the black point is an A_3 singularity, the white one an A_1:



Resolving the singularities, we get another elliptic K3 surface $q : Y \to \mathbb{P}^1$ (no. 2717 in Shimada's catalogue): its Kodaira type is $2I_1^* + 2I_4 + I_2$, which alongside the information that $MW(q) \simeq \mathbb{Z}/4\mathbb{Z}$ gives $A_{NS(Y)} = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2$, which is the one expected by the correspondence of Corollary 4.2.4.16. The surface Y is a Kummer-4 surface $Kum_4(A)$, with $T(A) \simeq \langle 2 \rangle \oplus U$.

Remark 4.2.6.1. We can also find two orthogonal copies of E_8 in $NS(X)$: $\langle B_1, B_5, B_6, B_7, B_0, s, C_0^1, C_2^1 \rangle$ and $\langle C_4^2, u, C_1^2, C_2^2, C_3^2, C_5^2, t, C_5^1 \rangle$. Call ι the involution that exchanges them: then, ι is *not* τ^2, and the resolution of the singularities of X/ι is a Kummer surface $Kum(B)$ for an abelian surface B such that $T(B) \simeq T(X)$.

Chapter 5

Action of a group of order 4 on a $K3^{[2]}$ type manifold and involutions on Nikulin-type orbifolds

In this chapter, we apply our knowledge of symplectic actions of groups of order 4 G on K3 surfaces to study the symplectic action of G on a $K3^{[2]}$-type manifold X. Indeed, it is proven in [39] that this action is always standard, meaning that a pair (X, G) can be always deformed to a natural pair $(S^{[2]}, G)$. We classify the irreducible components of the moduli space of projective $K3^{[2]}$-type manifolds with a symplectic action of G (see Theorems 5.1.1.2 and 5.1.2.2). For some of them, we find the general member: either as Fano manifold over a cubic fourfold, or as Hilbert scheme of two points of a quartic surface with a mixed (partially non-symplectic) action of G, or as double cover of a cone over $\mathbb{P}^2 \times \mathbb{P}^2$.

We then turn our attention to Nikulin orbifolds and their deformation class (see Example 1.5.1.2): if X admits a symplectic action of G, and $i \in G$ is an element of order 2, then the Nikulin orbifold Y obtained as terminalization of X/i admits a symplectic involution induced by the quotient G/i. The two groups of order four induce two very different involutions on Y: indeed, we can see from the action on X that the one induced by $\mathbb{Z}/4\mathbb{Z}$ fixes only points on Y, while the locus of the one induced by $(\mathbb{Z}/2\mathbb{Z})^2$ has codimension 2. We describe the action of these involutions on $H^2(Y, \mathbb{Z})$, using the same quotient maps we introduced for K3 surfaces in Chapters 2 and 3. We then prove that these induced involutions extend to any deformations of Y that satisfy a given lattice-theoretic condition: thus, we can define *standard* symplectic involutions on Nikulin-type orbifolds.

Theorem 2 (Thm. 5.4.2.10)**.** *Let Y be a Nikulin-type orbifold such that $NS(Y)$ contains primitively either $D_4(2)$ or $D_6(2)$, and $T(Y)$ satisfies similar lattice theoretic condistions (see Lemma 5.4.2.8). Then Y admits a standard symplectic involution ι.*

We remark that, differently than what happens on the known IHS manifolds, it is not enough that a Nikulin-type orbifold Y be polarized with the correct anti-invariant lattice for it to admit a standard involution: there is also a gluing datum between invariant and co-invariant lattices, i.e. a specific embedding of the co-invariant lattice in $H^2(Y,\mathbb{Z})$, that has to be respected. The correct gluing is described in Lemma 5.4.2.8.

We conclude with the lattice-theoretic classification of projective Nikulin orbifolds that are terminalization of X/i, where X is a $\mathrm{K3}^{[2]}$-type manifold with a symplectic action of a group of order 4 G, and $i \in G$ is an element of order 2. After noticing that standard involutions on Nikulin-type orbifolds commute with the non-standard involution described in [52], we classify also projective Nikulin-type orbifolds that admit a mixed action of $(\mathbb{Z}/2\mathbb{Z})^2$, where one of the generators is standard, and the other is not.

5.1 Projective families of $\mathrm{K3}^{[2]}$-type manifolds with a symplectic action of a group of order 4

If X is a $\mathrm{K3}^{[2]}$-type manifold, then $H^2(X,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}^{[2]}} = E_8^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle -2 \rangle$. The symplectic action of a group of order 4 G on a $\mathrm{K3}^{[2]}$-type manifold X is always standard [39], so in particular the co-invariant lattice Ω_G will be the same as the one for the action of G on a K3 surface S: hence, if X is projective, it will be polarized with the lattice $\Omega_G \oplus \langle 2d \rangle$ or one of its overlattices [16, Prop. 2.1]: indeed, an ample class on X has necessarily positive self-intersection, and Ω_G is negative definite. These lattices have been already classified as Néron-Severi group of general projective K3 surfaces S with a symplectic action of G in Theorems 2.5.1.4, 3.5.1.2, so if X is also general there exists an S such that $NS(X) \simeq NS(S)$; however, since $\Lambda_{\mathrm{K3}^{[2]}}$ is not unimodular, unlike what happens for K3 surfaces $NS(X)$ does not necessarily determine $T(X)$, because there can be more than one primitive embedding $NS(X) \hookrightarrow \Lambda_{\mathrm{K3}^{[2]}}$. Each pair $(NS(X), T(X))$ gives a different projective family (see Remark 1.4.0.5).

Remark 5.1.0.1. Consider the isometry

$$\Lambda_{\mathrm{K3}^{[2]}} \simeq \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle :$$

For either group of order 4 G, Ω_G admits a unique primitive embedding in $\Lambda_{\mathrm{K3}^{[2]}}$. Indeed, we have to apply Theorem 1.2.1.14: the discriminant group of Ω_G does not contain any element of order 2 and square 3/2 (see Propositions 2.5.1.2, 3.5.1.1), so the only primitive embedding we get is the one such that $\Omega_G^{\perp} = \Lambda_{\mathrm{K3}}^G \oplus \langle -2 \rangle$.

As a consequence, we have that each projective family of X is determined by the embedding in $\Lambda_{\mathrm{K3}^{[2]}}$ of the class of positive square that generates $\Omega_G^{\perp_{NS(X)}}$.
We're going to distinguish between families polarized with a class $L = (L_S, 0) \in \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle$, i.e. those families such that $NS(X) \simeq NS(S)$, $T(X) \simeq T(S) \oplus \langle -2 \rangle$ for some general projective K3 surface S admitting a symplectic action of G, and families polarized with a class $M = (L_S, n)$ with $n > 0$, for which if $NS(X) \simeq NS(S)$, then $T(X) \not\simeq T(S) \oplus \langle -2 \rangle$.

5.1.1 Families of projective K3$^{[2]}$-type manifolds with a symplectic action of $\mathbb{Z}/4\mathbb{Z}$

Let $G = \mathbb{Z}/4\mathbb{Z} = \langle \tau \rangle$. Since the action of G on a K3$^{[2]}$-type manifold X is standard, it holds

$$\Lambda^{\tau}_{\mathrm{K3}^{[2]}} = \Lambda^{\tau}_{\mathrm{K3}} \oplus \langle -2 \rangle.$$

For convenience, we give a new description of the lattice $\Lambda^{\tau}_{\mathrm{K3}}$, introduced in Section 2.3.2, as direct sum of elementary lattices.

Proposition 5.1.1.1. *The lattice $\Lambda^{\tau}_{\mathrm{K3}}$ as described in Section 2.3.2 is isometric to the lattice*

$$\underset{s_1, s_2}{U} \oplus \underset{w_1, w_2}{\langle -2 \rangle^{\oplus 2}} \oplus \underset{w_3, \dots, w_6}{U(4)^{\oplus 2}}$$

via

$$\begin{aligned}
s_1 &= (a_1 + a_2 + \omega_1 + \omega_2)/2;\\
s_2 &= (-a_1 - a_2 + \omega_1 - \omega_2)/2;\\
w_1 &= \omega_2 + (e_1 + f_1 + g_1 + h_1 + e_4 + f_4 + g_4 + h_4 + a_1 + a_2 + \rho)/2;\\
w_2 &= \omega_2 + e_3 + f_3 + g_3 + h_3 + e_4 + f_4 + g_4 + h_4 +\\
&\qquad + (e_1 + f_1 + g_1 + h_1 + e_2 + f_2 + g_2 + h_2 + a_1 + a_2 + \rho)/2;\\
w_3 &= \rho + (e_2 + f_2 + g_2 + h_2 + e_4 + f_4 + g_4 + h_4)/2;\\
w_4 &= \rho - (e_2 + f_2 + g_2 + h_2)/2 + (e_4 + f_4 + g_4 + h_4)/2;\\
w_5 &= 2\omega_2 + e_1 + f_1 + g_1 + h_1 + e_3 + f_3 + g_3 + h_3 + a_1 + a_2 +\\
&\qquad + (e_2 + f_2 + g_2 + h_2 + e_4 + f_4 + g_4 + h_4)/2;\\
w_6 &= 2\omega_2 + e_4 + f_4 + g_4 + h_4 + a_1 + a_2 + \rho.
\end{aligned}$$

Theorem 5.1.1.2. *The deformation families of projective K3$^{[2]}$-type manifolds X with a symplectic automorphism τ of order 4 are determined by the pairs $(NS(X), T(X))$ appearing in the following table, where the class L indicated is the generator of $\langle 2d \rangle = \Omega_4^{\perp_{NS(X)}}$: the classes $L_0, L_{i,j}$ are defined in Example 2.5.1.6, while the M_i and $\tilde{M}_i$ are defined in the proof below.*

	$NS(X)$	$T(X)$	L	L^2
$d =_4 1$	$\Omega_4 \oplus \langle 2d \rangle$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 3} \oplus \langle 2d \rangle$	$L_0(d)$	$2d$
		$U(4)^{\oplus 2} \oplus K_m$	$M_1(m)$	$2(4m-3)$
		$U(4) \oplus D_m$	$\tilde{M}_1(m)$	$2(4m+1)$
$d =_4 2$	$\Omega_4 \oplus \langle 2d \rangle$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 3} \oplus \langle 2d \rangle$	$L_0(d)$	$2d$
		$U(4)^{\oplus 2} \oplus \langle -2 \rangle \oplus H_m$	$M_2(m)$	$2(4m-2)$
	$(\Omega_4 \oplus \langle 2d \rangle)'^{(1)}$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle \oplus H_m$	$L^{(1)}_{2,2}(m)$	$2(4m+2)$
	$(\Omega_4 \oplus \langle 2d \rangle)'^{(2)}$		$L^{(2)}_{2,2}(m)$	$2(4m-2)$

$d =_4 3$	$\Omega_4 \oplus \langle 2d \rangle$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 3} \oplus \langle 2d \rangle$	$L_0(d)$	$2d$
		$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus G_m$	$M_3(m)$	$2(4m-1)$
	$(\Omega_4 \oplus \langle 2d \rangle)'$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus G_m$	$L_{2,3}(m)$	$2(4m+3)$
$d =_4 0$	$\Omega_4 \oplus \langle 2d \rangle$	$U(4)^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 3} \oplus \langle 2d \rangle$	$L_0(d)$	$2d$
	$(\Omega_4 \oplus \langle 2d \rangle)'$	$U(4) \oplus \langle -2 \rangle^{\oplus 3} \oplus F_m$	$L_{2,0}(m)$	$2(4(m-1))$
		$U \oplus U(4) \oplus \langle -2 \rangle^{\oplus 2} \oplus E_m$	$M_4(m)$	$2(4(m-1))$
	$(\Omega_4 \oplus \langle 2d \rangle)^\star$	$U \oplus U(4) \oplus \langle -2 \rangle^{\oplus 2} \oplus E_m$	$L_{4,j}(h)$	$2(4(m-1))$, *see table 5.2*

Table 5.2: Relation between m, j, h

$m \pmod 4$	0	1	2	3
j	12	0	4	8
h	$(m-4)/4$	$(m+3)/4$	$(m-2)/4$	$(m+13)/4$

$$G_m = \begin{bmatrix} -2m & 1 \\ 1 & -2 \end{bmatrix} \quad H_m = \begin{bmatrix} -2m & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad K_m = \begin{bmatrix} -2m & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 2 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$D_m = \begin{bmatrix} -2m & 1 & 1 & 1 & 2 & 2 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \quad E_m = \begin{bmatrix} -8m & 4 \\ 4 & -2 \end{bmatrix} \quad F_m = \begin{bmatrix} -2(m-1) & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

Proof. We use Nikulin's theorem 1.2.1.14 to determine all primitive embeddings of each of the Néron-Severi lattices described for K3 surfaces in Theorem 2.5.1.4 in the ambient lattice $\Lambda_{\mathrm{K3}^{[2]}}$, whose discriminant group is $\mathbb{Z}/2\mathbb{Z}$ with discriminant form $q = [3/2]$. By Remark 5.1.0.1, we can start by fixing the embedding induced by $\Omega_4 \hookrightarrow \Lambda_{\mathrm{K3}}$. Now, for each Néron-Severi lattice we have the primitive embedding induced by $\Lambda_{\mathrm{K3}} \hookrightarrow \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle$, $x \mapsto (x, 0)$ (see Example 2.5.1.6); then, applying Theorem 1.2.1.14 we find as necessary condition to have alternative embeddings of the same Néron-Severi lattice that its discriminant form contain an element of order 2 and square $3/2$: if there is more than one such subgroup, we can then check if they give rise to different embeddings. If this is the case, according to Theorem 1.2.1.14 we find a different orthogonal complement $T(X)$ (in our case, each $T(X)$ is unique in its genus).
The embeddings we find through this process are unique up to isometries of $\Lambda_{\mathrm{K3}^{[2]}}$, so we can choose for each one a representative: we give a class M of square $2d$ in $\Lambda^{\tau}_{\mathrm{K3}^{[2]}} = \Omega_4^{\perp}$ as generator of $\Omega_4^{\perp_{NS(X)}}$, such that $T(X)$ has the correct discriminant form.

The lattice $\Omega_4 \oplus \langle 2d \rangle$ has discriminant form

$$q_{\Omega_4 \oplus \langle 2d \rangle} := \begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix}^{\oplus 2} \oplus \left[1/2\right]^{\oplus 2} \oplus \left[1/2d\right] :$$

let γ be the generator of the subgroup $[1/2d]$, α_1, α_2 those of $[1/2]^{\oplus 2}$, x_1, x_2 those of one of the $\begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix}$ blocks. For $d =_4 3$, $d\gamma$ has order 2 and square $3/2$; for $d =_4 1$, $d\gamma$ has order 2 and square $1/2$, so $\alpha_1 + \alpha_2 + d\gamma$ has order 2 and square $3/2$; for $d =_4 2$, $d\gamma$ has order 2 and square 1, so $\alpha_1 + d\gamma$ has order 2 and square $3/2$; for $d =_4 0$, $d\gamma$ has order 2 and square 0, so we have no alternative embeddings. These alternative embeddings of $\Omega_4 \oplus \langle 2d \rangle$ are realized by the following classes of square $2d$ in $\Lambda^\tau_{\mathrm{K3}^{[2]}} = \Lambda^\tau_{\mathrm{K3}} \oplus \mathbb{Z}\mu$, $\mu^2 = -2$ (see Proposition 5.1.1.1 for the notation):

- for $d = 4m - 3$, $M_1(m) = 2(s_1 + ms_2) + w_2 - w_1 + \mu$,
- for $d = 4m - 2$, $M_2(m) = 2(s_1 + ms_2) + w_1 + \mu$;
- for $d = 4m - 1$, $M_3(m) = 2(s_1 + ms_2) + \mu$.

For $d =_4 1$, the class

$$\tilde{M}_1(m) = 2(s_1 + ms_2) + w_3 + w_4 + w_2 - w_1 + \mu$$

provides a third different primitive embedding of $\Omega_4 \oplus \langle 2d \rangle$ in the ambient lattice: the associated subgroup of $q_{\Omega_4 \oplus \langle 2d \rangle}$ is generated by $\alpha_1 + \alpha_2 + d\gamma + x_1 + x_2$. For $d =_4 0$ the lattice $(\Omega_4 \oplus \langle 2d \rangle)'$ admits another primitive embedding, realized by the following classes:

- for $d = 4(m - 1)$, $M_4(m) = w_3 + mw_4 + 2\mu$.

These are the only cases in which there exist alternative embeddings, as the discriminant group of the other Néron-Severi lattices does not contain any element of order 2 and square $3/2$. □

Remark 5.1.1.3. Notice that there are pairs of projective families with general member X_1, X_2 such that $NS(X_1) \simeq NS(X_2)$ but $T(X_1) \not\simeq T(X_2)$, and others such that the converse holds.

5.1.2 Families of projective $\mathrm{K3}^{[2]}$-type manifolds with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$

Let $G = (\mathbb{Z}/2\mathbb{Z})^2 = \langle \tau, \varphi \rangle$. As above, since the action of G on a $\mathrm{K3}^{[2]}$-type manifold X is standard, it holds

$$\Lambda^G_{\mathrm{K3}^{[2]}} = \Lambda^G_{\mathrm{K3}} \oplus \langle -2 \rangle,$$

and we can give a description of Λ^G_{K3} as direct sum of elementary lattices.

Proposition 5.1.2.1. *The lattice* $\Lambda_{\mathrm{K3}}^{(\mathbb{Z}/2\mathbb{Z})^2}$ *as described in Section 3.3.2 is isometric to the lattice*

$$\begin{array}{ccccc} U & \oplus & U(2)^{\oplus 2} & \oplus & D_4(2) \\ s_1, s_2 & & u_1, \dots, u_4 & & m_1, \dots, m_4 \end{array}$$

via

$$\begin{aligned}
s_1 &= (x + 2y - e_1 - f_1 + e_2 + f_2)/3; \\
s_2 &= y; \\
u_1 &= (v_1 + v_2 + g_1 + h_1 - g_2 - h_2)/3; \\
u_2 &= (v_1 + v_2 + g_1 + h_1 + 2g_2 + 2h_2)/3; \\
u_3 &= (-v_1 + 2v_2 + 2g_1 + 2h_1 + g_2 + h_2)/3; \\
u_4 &= (-2v_1 + v_2 - 2g_1 - 2h_1 - g_2 - h_2)/3; \\
m_1 &= (a_1 + b_1 + c_1 + d_1 - (a_2 + b_2 + c_2 + d_2) + 2e_1 + 2f_1 + e_2 + f_2 - 2y)/3; \\
m_2 &= (a_1 + b_1 + c_1 + d_1 + 2(a_2 + b_2 + c_2 + d_2) + 2e_1 + 2f_1 + e_2 + f_2 - 2y)/3; \\
m_3 &= \; - e_1 - f_1 - e_2 - f_2; \\
m_4 &= e_2 + f_2 + 2y.
\end{aligned}$$

Theorem 5.1.2.2. *The deformation families of projective* $K3^{[2]}$*-type manifolds* X *with a symplectic action of* $(\mathbb{Z}/2\mathbb{Z})^2$ *are determined by the pairs* $(NS(X), T(X))$ *appearing in the following table, where the class* L *indicated is the generator of* $\langle 2d\rangle = \Omega_{2,2}^{\perp_{NS(X)}}$*: the classes* $L_0, L_{i,j}^{(h)}$ *are defined in Example 3.5.1.3, while the* M_i *are defined in the proof below.*

	$NS(X)$	$T(X)$	L	L^2
$d =_4 1$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$\langle -2d\rangle \oplus \langle -2\rangle \oplus U(2)^{\oplus 2} \oplus D_4(2)$	$L_0(d)$	$2d$
		$U(2)^{\oplus 2} \oplus B_m$	$M_1(m)$	$2(4m-3)$
$d =_4 3$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$\langle -2d\rangle \oplus \langle -2\rangle \oplus U(2)^{\oplus 2} \oplus D_4(2)$	$L_0(d)$	$2d$
		$U(2)^{\oplus 2} \oplus D_4(2) \oplus G_m$	$M_3(m)$	$2(4m-1)$
$d =_4 2$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$\langle -2d\rangle \oplus \langle -2\rangle \oplus U(2)^{\oplus 2} \oplus D_4(2)$	$L_0(d)$	$2d$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'$	$\langle -2\rangle \oplus D_4(2) \oplus P_h$	$L_{2,2}^{(a,b)}(h)$	$2(4h+2)$
$d =_8 0$	$\Omega_{2,2} \oplus \langle 2d\rangle$	$\langle -2d\rangle \oplus \langle -2\rangle \oplus U(2)^{\oplus 2} \oplus D_4(2)$	$L_0(d)$	$2d$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(1)}$	$\langle -2\rangle \oplus U \oplus R_h$	$L_{2,0}^{(1)}(h)$	$2(4h)$
	$(\Omega_{2,2} \oplus \langle 2d\rangle)'^{(2)}$	$\langle -2\rangle \oplus U(2)^{\oplus 2} \oplus Q_h$	$L_{2,0}^{(2)}(h)$	$2(4h-4)$
		$U \oplus U(2) \oplus C_m$	$M_8(m)$	$2(8m-8)$

$d =_8 4$	$\Omega_{2,2} \oplus \langle 2d \rangle$	$\langle -2d \rangle \oplus \langle -2 \rangle \oplus U(2)^{\oplus 2} \oplus D_4(2)$	$L_0(d)$	$2d$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(1)}$	$\langle -2 \rangle \oplus U \oplus R_h$	$L_{2,0}^{(1)}(h)$	$2(4h)$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$	$\langle -2 \rangle \oplus U(2)^{\oplus 2} \oplus Q_h$	$L_{2,0}^{(2)}(h)$	$2(4h-4)$
	$(\Omega_{2,2} \oplus \langle 2d \rangle)^{\star}$	$\langle -2 \rangle \oplus T_h^{(\pm 4)}$	$L_{4,\pm 4}(h)$	$2(16h \pm 4)$

$$G_m = \begin{bmatrix} -2m & 1 \\ 1 & -2 \end{bmatrix} \quad B_m = \left[\begin{array}{cc|cccc} \multicolumn{2}{c|}{\multirow{2}{*}{G_m}} & 1 & 1 & -2 & 1 \\ & & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & & & & \\ 1 & 0 & \multicolumn{4}{c}{D_4(2)} \\ -2 & 0 & & & & \\ 1 & 0 & & & & \end{array}\right] \quad C_m = \left[\begin{array}{cc|cccc} -4m & 2 & -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ \hline -2 & 0 & & & & \\ 2 & 0 & \multicolumn{4}{c}{D_4(2)} \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{array}\right]$$

$$P_h = \left[\begin{array}{c|cccc} -2h & 1 & 1 & 1 & 0 \\ \hline 1 & & & & \\ 1 & \multicolumn{4}{c}{U(2)^{\oplus 2}} \\ 1 & & & & \\ 0 & & & & \end{array}\right] \quad Q_h = \left[\begin{array}{c|cccc} -2h & 2 & 0 & 0 & 0 \\ \hline 2 & & & & \\ 0 & \multicolumn{4}{c}{D_4(2)} \\ 0 & & & & \\ 0 & & & & \end{array}\right] \quad T_h^{(-4)} = U(2)^{\oplus 2} \oplus \left[\begin{array}{c|cccc} -2h & 1 & 0 & 0 & 0 \\ \hline 1 & & & & \\ 0 & \multicolumn{4}{c}{D_4(2)} \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$$T_h^{(4)} = U \oplus \begin{bmatrix} -8h & -2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & -4 & 2 & 0 \\ 0 & 0 & 0 & -4 & -8 & 4 & 0 \\ 0 & 0 & 0 & 2 & 4 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix} \quad R_h = \begin{bmatrix} -8(h+1) & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & -8 \end{bmatrix}$$

Proof. The proof is analogous to that of Theorem 5.1.1.2. We now use the Néron-Severi lattices described for K3 surfaces S in Theorem 3.5.1.2; again, by Remark 5.1.0.1 $\Omega_{2,2}$ admits a unique primitive embedding in $\Lambda_{\mathrm{K3}^{[2]}}$ up to isometries, so we start by fixing the embedding induced by $\Omega_{2,2} \hookrightarrow \Lambda_{\mathrm{K3}}$. For each $NS(X) \simeq NS(S)$ we have at least the embedding such that $T(X) \simeq T(S) \oplus \langle -2 \rangle$ (see Example 3.5.1.3). The additional choices are as follows: for d odd, the lattice $\Omega_{2,2} \oplus \langle 2d \rangle$ admits another primitive embedding, realized by the following classes of square $2d$ in $\Lambda^G_{\mathrm{K3}^{[2]}} = \Lambda^G_{\mathrm{K3}} \oplus \mathbb{Z}\mu$, $\mu^2 = -2$ (see Proposition 5.1.2.1 for the notation):

- for $d = 4m-3$, $M_1(m) = 2(s_1 + ms_2) + m_3 + \mu$;
- for $d = 4m-1$, $M_3(m) = 2(s_1 + ms_2) + \mu$.

For $d =_8 0$ the lattice $(\Omega_{2,2} \oplus \langle 2d \rangle)'^{(2)}$ admits another primitive embedding, realized by

- $M_8(m) = 2(\mu + u_3 + mu_4) + m_2 - m_1$.

These are the only cases in which there exist alternative embeddings, for the discriminant group of the other Néron-Severi lattices does not contain any element of order 2 and square 3/2. □

5.2 Fixed loci

Given a pair (X, G) with X a K3$^{[2]}$-type manifold and G group of order four acting symplectically, by Remark 1.4.2.4 we know that the locus of points of X with nontrivial stabilizer is topologically the same of that of the natural pair $(S^{[2]}, G)$.

Proposition 5.2.0.1. *Let $G = \mathbb{Z}/4\mathbb{Z}$ act symplectically on a K3$^{[2]}$-type manifold X, let τ be a generator of G. Then τ fixes 16 points, of which exactly 8 lie on the K3 surface Σ fixed by τ^2.*

Proof. The action of $G = \mathbb{Z}/4\mathbb{Z} = \langle\tau\rangle$ on a K3 surface S fixes 4 points $\{p_1, \dots p_4\}$ and exchanges two pairs of points, $q_1 \mapsto q_2,\ r_1 \mapsto r_2$ (these are fixed by τ^2).
Let $[s_1, s_2]$ with $s_1 \neq s_2$ be the class in $S^{[2]}$ of the unordered pair of points $\{s_1, s_2\} \subset S$. The natural action of τ^2 on $S^{[2]}$ fixes 28 isolated points: 6 of the form $[p_i, p_j]$ with $i \neq j \in \{1, \dots, 4\}$, 16 of the form $[p_i, q_j]$ or $[p_i, r_j]$ with $i \in \{1, \dots, 4\}, j \in \{1, 2\}$, 4 of the form $[q_i, r_j],\ i, j \in \{1, 2\}$, and the points $[q_1, q_2], [r_1, r_2]$; moreover, it fixes the K3 surface $\Sigma = [s, \tau^2(s)],\ s \in S$. Let Δ be the blow-up of the singular locus of $S^{(2)} = (S \times S)/\sigma$, where σ is the exchange of the two copies of S. Then, $\Sigma \cap \Delta$ consists of 8 lines, over the points $[p_i, p_i], [q_j, q_j], [r_j, r_j], i \in \{1, \dots, 4\}, j \in \{1, 2\}$. The K3 surface Σ fixed by the natural symplectic involution ι on $S^{[2]}$ is isomorphic to the resolution of the singularities of S/ι: if $\iota = \tau^2$, we get a surface isomorphic to $\tilde{Z}$ (see Section 2.4.2); the involution induced by the action of G/τ^2 on Σ exchanges pairwise the four lines over $[q_j, q_j], [r_j, r_j]$, and fixes two points on each of the remaining four lines. □

Proposition 5.2.0.2. *Let $G = (\mathbb{Z}/2\mathbb{Z})^2$ act symplectically on a K3$^{[2]}$-type manifold X, let τ, φ, ρ be the three involutions in G. The action of G stabilizes (with order 2) 72 isolated points and three K3 surfaces $\Sigma_\tau, \Sigma_\varphi, \Sigma_\rho$, each fixed by the respective involution. The three surfaces intersect pairwise in 4 points as follows: the points in $\Sigma_\tau, \Sigma_\varphi$ belong to the set of 28 isolated points fixed by ρ, and similarly the other pairs. The fixed locus of G consists therefore of 12 points, lying in the intersection of the three K3 surfaces.*

Proof. Call $\{t_1, \dots t_8\}, \{q_1, \dots q_8\}, \{r_1, \dots r_8\}$ the points of the K3 surface S fixed respectively by τ, φ, ρ: the involutions φ, ρ act on the set $\{t_1, \dots t_8\}$ exchanging them pairwise in the same way, say $t_{2i-1} \leftrightarrow t_{2i}$. Similarly, each two involutions act in the same way on the set of points fixed by the third one.
The involution τ on $S^{[2]}$ fixes a K3 surface $\Sigma_\tau \simeq \tilde{Z}_\tau$ (see Section 3.4.1) given by the points $[s, \tau(s)]$, and the 28 isolated points $[t_i, t_j]$ for $i \neq j \in \{1, \dots 8\}$ (similarly the other two involution); moreover, it holds

$$\Sigma_\tau \cap \Sigma_\varphi = \{[p, \tau(p)] = [q, \varphi(q)]\} = \{[r_i, \tau(r_i)]\},$$

which are four of the isolated points fixed by ρ. The other pairs of fixed surfaces intersect similarly. Notice that each involution acts on each of the surfaces fixed by the other involutions (for instance, τ acts on Σ_ρ and Σ_φ). □

5.3 Examples

To find a projective model of a general member X in the moduli space of projective K3$^{[2]}$-type manifolds with a symplectic action of some group H, the action of H on X should not be natural, i.e. it cannot be $X = S^{[2]}$ with H acting symplectically on S, as it is explained in the following Remark 5.3.0.1. Therefore, one has to resort to alternative constructions that give IHS manifolds of K3$^{[2]}$-type.

Remark 5.3.0.1. Let S be a general member in the moduli space of projective K3 surfaces with a symplectic action of some group H. The Hilbert square $S^{[2]}$ is not a general member in the moduli space of K3$^{[2]}$-type manifolds with a symplectic action of H: indeed, $NS(S^{[2]}) = NS(S) \oplus \langle -2 \rangle$ is an overlattice of finite index of $\Omega_H \oplus \langle 2d \rangle \oplus \langle -2 \rangle$, so it has a bigger rank than that of a general member X, for which $NS(X)$ is an overlattice of finite index of $\Omega_H \oplus \langle 2d \rangle$. Notice also that the moduli space of S and of $S^{[2]}$ have the same dimension (see Remark 1.4.0.4).

Projective models of K3$^{[2]}$-type manifolds with a symplectic involution ι have been constructed in [15] and [16]. We are going to adapt some of these constructions, supposing that $\iota \in G$ a group of order 4.

Actions on the Fano variety of lines of a cubic fourfold.
Let $\mathcal{C}$ be a cubic fourfold in $\mathbb{P}^5$: if $\mathcal{C}$ is smooth, the Fano variety of lines $F(\mathcal{C})$ is an IHS manifold of K3$^{[2]}$-type. We recall here two important results about these manifolds:

Proposition 5.3.0.2 ([8, Prop. 2, Prop. 4, Prop. 6.ii]). *1. There exists a Hodge isometry $H^4(\mathcal{C},\mathbb{Z}) \simeq H^2(F(\mathcal{C}), Z)$, that maps $H^{i,j}(\mathcal{C})$ to $H^{i-1,j-1}F(\mathcal{C})$.*

2. The Fano variety of lines of a Pfaffian cubic fourfold $\mathcal{C}_V$ is isomorphic to the Hilbert scheme of two points over a K3 surface S_V of degree 14: this gives a Hodge isometry $H^2(F(\mathcal{C}_V),\mathbb{Z}) \simeq H^2(S_V^{[2]}, Z) \simeq H^2(S_V, Z) \oplus \langle -2 \rangle$; in particular, the image of the Plücker polarization g of $F(\mathcal{C}_V)$ is $2L - 5\delta$, where L is the polarization of degree 14 of S_V and δ is half the class of the exceptional divisor of $S_V^{[2]} \to Sym^2(S_V)$.

Remark 5.3.0.3. As a consequence of the first statement we get that an automorphism ϕ of $\mathcal{C}$ lifts to a symplectic automorphism of $F(\mathcal{C})$ if and only if ϕ acts as the identity on $H^{3,1}(\mathcal{C})$. As a consequence of the second we get that $F(\mathcal{C})$ is naturally polarized with a class of square 6 and divisibility 2, as this properties hold for $2L - 5\delta$.

In [15] it is shown that the only case in which an involution ι of a cubic fourfold $\mathcal{C}$ lifts to a symplectic involution of $F(\mathcal{C})$ is when ι is induced by the automorphism of $\mathbb{P}^5$ $\phi : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (-x_0 : -x_1 : x_2 : x_3 : x_4 : x_5)$ and the cubic fourfold has equation

$$\mathcal{C}_2 : x_0^2\lambda_1(x_2, x_3, x_4, x_5)+x_1^2\lambda_2(x_2, x_3, x_4, x_5)+x_0x_1\lambda_3(x_2, x_3, x_4, x_5)+\Gamma(x_2, x_3, x_4, x_5) = 0$$

where the λ_i are linear, while Γ is cubic. Moreover, in the same paper there is a description of the fixed locus of the symplectic involution on $F(\mathcal{C}_2)$: it consists of 28 points, given by the line $x_2 = x_3 = x_4 = x_5 = 0$ and the 27 lines on the cubic threefold $\Gamma(x_2, x_3, x_4, x_5) = 0$, and the K3 surface $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^3$ described by the complete intersection

$$\Sigma : \begin{cases} \Gamma(x_2, x_3, x_4, x_5) = 0 \\ x_0^2\lambda_1(x_2, x_3, x_4, x_5) + x_1^2\lambda_2(x_2, x_3, x_4, x_5) + x_0x_1\lambda_3(x_2, x_3, x_4, x_5) = 0. \end{cases} \tag{5.3.0.1}$$

1: Action of $\mathbb{Z}/4\mathbb{Z}$. Of all the automorphisms ψ of $\mathbb{P}^5$ of order four such that $\psi^2 = \phi$, the only one that acts on a smooth cubic hypersurface $\mathcal{C}_4$ such that $\psi^*|_{H^{3,1}(\mathcal{C}_4)} = id$ is $\psi : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (ix_0 : -ix_1 : x_2 : x_3 : -x_4 : -x_5)$, with

$$\mathcal{C}_4 : C(x_2, x_3)+x_2Q_1(x_4, x_5)+x_3Q_2(x_4, x_5)+x_0^2\ell_1(x_4, x_5)+x_1^2\ell_2(x_4, x_5)+x_0x_1\ell_3(x_2, x_3) = 0,$$

where C is a cubic polynomial, Q_i are quadric and ℓ_i are linear. Notice that $\mathcal{C}_4$ is a specialization of $\mathcal{C}_2$, as the general cubic $\Gamma(x_2, x_3, x_4, x_5)$ is replaced by $C(x_2, x_3) + x_2Q_1(x_4, x_5) + x_3Q_2(x_4, x_5)$, and the ℓ_i depend on less variables than the corresponding λ_i. This family has 6 moduli (the equation depends on 16 projective parameters, but the space of projectivities of $\mathbb{P}^5$ that commute with ϕ has dimension 10), so $F(\mathcal{C}_4)$ is a general member of a family of projective K3$^{[2]}$-type manifolds with a symplectic automorphism of order 4: in Theorem 5.1.1.2, this is the family associated to $M_3(1)$, because this polarization is the only one with square 6 and divisibility 2.
Following [15, §7], we give a model of the K3 surface fixed by τ^2, $\Sigma_4 \subset \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^3_{(x_2:\cdots:x_5)}$ as:

$$\Sigma_4 : \begin{cases} C(x_2, x_3) + x_2Q_1(x_4, x_5) + x_3Q_2(x_4, x_5) = 0 \\ x_0^2\ell_0(x_4, x_5) + x_1^2\ell_1(x_4, x_5) + x_0x_1\ell_2(x_2, x_3) = 0; \end{cases}$$

notice that this is a specialization of the surface Σ in (5.3.0.1), with the additional property that it admits a symplectic involution: indeed the residual involution induced by ψ fixes 8 points on Σ_4, so it is symplectic.
This family of Fano varieties was already described in [23], in which also the fixed locus of the automorphism of order 4 is partially computed (there is one line missing). There are 16 lines on $\mathcal{C}_4$ which are fixed by $\tau := \psi|_{\mathcal{C}_4}$: calling $P_0 = (1 : 0 : 0 : 0 : 0 : 0)$, $P_1 = (0 : 1 : 0 : 0 : 0 : 0)$, we have the six lines that join one of the points P_0, P_1 with one of the three solutions $\tilde{P}_j = (0 : 0 : s_j : t_j : 0 : 0)$ of the system $\{C(x_2, x_3) = 0,\ x_0 = x_1 = x_4 = x_5 = 0\}$, and the two lines that join P_0 with the solution of $\{\ell_1(x_4, x_5) = 0,\ x_0 = x_1 = x_2 = x_3 = 0\}$, P_1 with the solution of $\{\ell_2(x_4, x_5) = 0,\ x_0 = x_1 = x_2 = x_3 = 0\}$: these give the 8 fixed points of $F(\mathcal{C}_4)$ belonging to the K3 surface fixed by τ^2.
Moreover, we have the lines $(0 : 0 : 0 : 0 : x_4 : x_5)$ and $(x_0 : x_1 : 0 : 0 : 0 : 0)$, and the six lines that join each of the $\tilde{P}_j$ with each of the two solutions of $\{s_jQ_1(x_4, x_5) + t_jQ_2(x_4, x_5) = 0,\ x_0 = x_1 = 0\}$: these give the 8 points fixed by τ among the 28 isolated points fixed by τ^2.

2: Action of $(\mathbb{Z}/2\mathbb{Z})^2$. Consider the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^5$ given by

$$(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \overset{\tau}{\mapsto} (-x_0 : -x_1 : x_2 : x_3 : x_4 : x_5)$$

$$\overset{\varphi}{\mapsto} (-x_0 : x_1 : -x_2 : x_3 : x_4 : x_5)$$

and the invariant cubic fourfold

$$\mathcal{C}_{2,2} : x_0x_1x_2 + x_0^2\ell_0'(x_3, x_4, x_5) + x_1^2\ell_1'(x_3, x_4, x_5) + x_2^2\ell_2'(x_3, x_4, x_5) + C'(x_3, x_4, x_5) = 0,$$

where ℓ_i' are linear and C' is cubic. Again, $\mathcal{C}_{2,2}$ is a specialization of $\mathcal{C}_2$; this family has 8 moduli (the equation depends on 20 projective parameters, but the space of projectivities of $\mathbb{P}^5$ that commute with ϕ has dimension 12), so $F(\mathcal{C}_{2,2})$ is a general member of a family of projective $K3^{[2]}$-type manifolds with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ – the one in Theorem 5.1.2.2 associated to $M_3(1)$.
Again, the K3 surface fixed by τ can be described in $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^3_{(x_2:\cdots:x_5)}$ as:

$$\Sigma_{2,2} : \begin{cases} x_2^2\ell_2'(x_3, x_4, x_5) + C'(x_3, x_4, x_5) = 0 \\ x_0^2\ell_0'(x_3, x_4, x_5) + x_1^2\ell_1'(x_3, x_4, x_5) + x_0x_1x_2 = 0. \end{cases}$$

The surfaces fixed by the other involutions in $(\mathbb{Z}/2\mathbb{Z})^2$ are described by similar equations, obtained by permutation of the coordinates.

If $F(\mathcal{C})$ admits a symplectic action of a group G of order 4 and $i \in G$ has order 2, then there is an involution on the surface fixed by i: the surface Σ defined in (5.3.0.1) specializes to $\Sigma_4, \Sigma_{2,2}$ if $G = \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2$ respectively. We show that these are the most general specializations of Σ that admit a symplectic involution.

Proposition 5.3.0.4. *There are exactly two families of K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ with a symplectic involution. The surfaces Σ_4 and $\Sigma_{2,2}$ are general members of these families.*

Proof. A K3 surface in $\mathbb{P}^1 \times \mathbb{P}^3$ is a smooth complete intersection of divisors of bidegree $(2, 1)$ and $(0, 3)$ respectively. We exclude involutions of $\mathbb{P}^1 \times \mathbb{P}^3$ that act as the identity on either component, because they fix lines on the invariant surfaces (or these are singular). The only suitable involutions are therefore:

$$\begin{aligned} \iota_1 :&(x_0 : x_1)(x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : -x_1)(x_2 : x_3 : x_4 : -x_5), \\ \iota_2 :&(x_0 : x_1)(x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : -x_1)(x_2 : x_3 : -x_4 : -x_5). \end{aligned}$$

Invariant K3 surfaces for ι_1 are given by taking both divisors in the positive eigenspace (otherwise the resulting surfaces are singular), so they satisfy the same equations as $\Sigma_{2,2}$. Invariant K3 surfaces for ι_2 are given either by taking both divisors in the same eigenspace (positive or negative), or by taking one in the positive and one in the negative eigenspaces. However, in the former case we find that ι_2 fixes the resulting surface, so we have to exclude it; the latter case has Σ_4 as a general member. □

Non natural actions on the Hilbert square of a K3 surface. We start by recalling the construction of Beauville's involution [7], the first example of non-natural automorphism on an IHS manifold: it is a non-symplectic, non-natural involution defined on

$S^{[2]}$ for a smooth quartic surface $S \subset \mathbb{P}^3$ without lines (this condition is satisfied by the general smooth quartic surface).
The Hilbert square $S^{[2]}$ parametrizes non-ordinate pairs of points $[p, q]$ of S: take the line $\ell \subset \mathbb{P}^3$ through p and q: then $\ell \cap S = \{p, q, P, Q\}$ (not necessarily distinct). Define Beauville's involution β on $S^{[2]}$ generically as $[p, q] \mapsto [P, Q]$; the isometry β^* has invariant lattice $\langle 2 \rangle$, generated by the class $H - \mu$, where $\langle H \rangle \oplus \langle \mu \rangle = \langle 4 \rangle \oplus \langle -2 \rangle$: the latter is the Néron-Severi lattice of $S^{[2]}$ for S general, as $NS(S)$ is spanned by the hyperplane class, that has self-intersection 4.

Proposition 5.3.0.5. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface without lines, that has an automorphism α that preserves the polarization (that is, α is induced by an automorphism of $\mathbb{P}^3$): then, the induced automorphism α on $S^{[2]}$ commutes with Beauville's involution β.*

Proof. The automorphism of $\mathbb{P}^3$ maps lines into lines, so $\beta([\alpha(p), \alpha(q)]) = [\alpha(P), \alpha(Q)]$. From a lattice-theoretic perspective, we can show that $\alpha^* \circ \beta^* = \beta^* \circ \alpha^*$ on $H^2(S^{[2]}, \mathbb{Z})$, which implies that α and β commute (see Remark 1.3.0.13): firstly, notice that $\beta^* H = 3H - 4\mu$, $\beta^* \mu = 2H - 3\mu$, and there is a basis of the orthogonal complement K of $\langle H, \mu \rangle$ in $H^2(S^{[2]}, \mathbb{Z})$ such that β^* acts as $-id$ on each class. On the other hand, the induced automorphism α of $S^{[2]}$, being natural, is such that α^* acts as the identity on $\langle H, \mu \rangle$: its co-invariant lattice is therefore contained in K. Take a class of the form $u = aH + b\mu + cv$, with $v \in K$: then $\beta^* \alpha^* u = (3a + 2b)H - (4a + 3b)\mu - c\alpha^* v = \alpha^* \beta^* u$. □

If α acts as $-id$ on ω_S, then $\alpha \circ \beta$ is a non-natural symplectic automorphism of $S^{[2]}$. This construction can therefore be used to describe the general member of a projective family of K3$^{[2]}$-type manifolds with a finite symplectic action.
When α is an involution, this has been done in [16, Rem. 2.13]. To obtain a non-natural symplectic action of a group of order 4 G, we are going to start from K3 surfaces with a mixed action, i.e. an action of G such that only a proper subgroup $K \subset G$ acts symplectically. Mixed actions of finite groups on K3 surfaces have been classified in [12].

1: Non natural action of $\mathbb{Z}/4\mathbb{Z}$. Quartic surfaces with equation

$$S_4 : a_1 x_0^4 + x_0^2(a_2 x_1^2 + a_3 x_2 x_3) + x_0 x_1 (a_4 x_2^2 + a_5 x_3^2) + \\ + x_1^2(a_6 x_1^2 + a_7 x_2 x_3) + x_2^2(a_8 x_2^2 + a_9 x_3^2) + a_{10} x_3^4 = 0$$

are invariant for the automorphism $\gamma : (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, ix_2, -ix_3)$, which acts as $-id$ on its symplectic form [2, Ex. 1.2]. The general member of this family is smooth and contains no line. Since this family of surfaces has 6 moduli, taking the Hilbert square we obtain a complete family of projective IHS manifolds of type K3$^{[2]}$ with a symplectic automorphism of order 4 $\tau := \beta \circ \gamma$ (see Remark 5.3.0.1).

Proposition 5.3.0.6. *Let X be the general member of the projective family of K3$^{[2]}$-type manifolds associated in Theorem 5.1.1.2 to $\tilde{M}_1(1)$: then X is the Hilbert scheme of two points on S_4.*

Proof. There are 5 deformation families of K3 surfaces with an automorphism of order 4 that acts as $-id$ on the symplectic form; each of them has a different fixed locus, and the corresponding invariant lattices have different ranks [2, Prop. 2]. The automorphism γ on S_4 has empty fixed locus, and its invariant lattice is

$$L^\gamma = U(4) \oplus \begin{bmatrix} -8 & 4 & -4 & 0 \\ 4 & -8 & 4 & -4 \\ -4 & 4 & -4 & 2 \\ 0 & -4 & 2 & -4 \end{bmatrix}$$

(see [12] and the attached database [13], entry (1.2.7.55)): therefore $NS(S_4)$ (which has rank 14 and is recorded in the database) is an overlattice of $\Omega_2 \oplus L^\gamma$, where $\Omega_2 = E_8(2)$ is the coinvariant lattice for the symplectic involution γ^2. Since by Proposition 5.3.0.5 $X = S_4^{[2]}$ admits the symplectic automorphism $\tau = \beta \circ \gamma$ of order 4, the lattice $NS(X) = NS(S_4) \oplus \langle -2 \rangle$ is isomorphic to $\Omega_4 \oplus \langle 2d \rangle$ or one of its overlattices; moreover, the involution τ^2 on X is induced by γ^2 on S_4, so Ω_2 is embedded in Ω_4 according to Section 2.3.2, and it holds $NS(X)^{\tau^2} = NS(S)^{\gamma^2} \oplus \langle -2 \rangle$.
We find $L^\gamma \oplus \langle -2 \rangle$ as $\Omega_2^\perp$ in the lattice $\Omega_4 \oplus \langle 2 \rangle$, so $NS(X) = \Omega_4 \oplus \langle 2 \rangle$: the (-2)-class δ that generates $(L^\gamma \oplus \Omega_2)^{\perp_{NS(X)}}$ cannot glue to Ω_2, so $NS(S_4) = \delta^\perp$ in $\Omega_4 \oplus \langle 2 \rangle$. Finally, knowing $NS(S_4)$ we can compute (the genus of) $T(S_4)$, and since $T(X) \simeq T(S_4)$ we can completely determine the deformation family of $S_4^{[2]}$: this is the one associated in Theorem 5.1.1.2 to $\tilde{M}_1(1)$. ☐

2: Non natural action of $(\mathbb{Z}/2\mathbb{Z})^2$. The family of quartics

$$S_{2,2} : f_4(x_0, x_1) + x_2^2 f_2(x_0, x_1) + x_3^2 g_2(x_0, x_1) + \alpha x_2^2 + \beta x_2 x_3 + \gamma x_3^2 = 0$$

admits a mixed action of $(\mathbb{Z}/2\mathbb{Z})^2$, with generators $\sigma : (x_0, x_1, x_2, x_3) \mapsto (-x_0, -x_1, x_2, x_3)$ and $\iota : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, -x_2, x_3)$: here σ is symplectic, while ι and $\sigma \circ \iota$ are non-symplectic and each of them fixes a curve of genus 3 on $S_{2,2}$. The general member of this family is smooth and contains no line. This family has 8 moduli, so taking the Hilbert square we obtain a complete family of projective IHS manifolds of type $K3^{[2]}$ with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \sigma, \iota \circ \beta \rangle$.

Proposition 5.3.0.7. *1. The invariant lattice for the mixed action of* $(\mathbb{Z}/2\mathbb{Z})^2$ *on the quartic surface* $S_{2,2}$*, such that both non-symplectic involutions fix a curve of genus 3, is* $U(4) \oplus \langle -2 \rangle^{\oplus 2}$.

2. The Hilbert square $X = S_{2,2}^{[2]}$ *is a general member of a family of projective IHS of* $K3^{[2]}$*-type with a symplectic action of* $(\mathbb{Z}/2\mathbb{Z})^2$*: in Theorem 5.1.2.2, this is the family associated to the class* $M_1(1)$.

Proof. In the database [13] the collection of possible mixed actions of $(\mathbb{Z}/2\mathbb{Z})^2$ on a K3 surface S consists of 354 elements: there are two of them for which both non-symplectic involutions fix a curve of genus 3, the entries (1.2.9.13) and (1.2.9.21).

Since we know that one of the symplectic involutions on $X = S_{2,2}^{[2]}$ is natural, we can compare the orthogonal complement of Ω_2 appropriately embedded in $NS(X)$ with the invariant lattices of the two possible actions: thus we can exclude (1.2.9.13), and find that $NS(X) = \Omega_{2,2} \oplus \langle 2 \rangle = \langle -2 \rangle \oplus K$, where K is the Néron-Severi lattice of the K3 surface (1.2.9.21): this is therefore our $S_{2,2}$. Since $T(X) \simeq T(S_{2,2})$, we can then find the deformation family which X belongs to – the one associated in Theorem 5.1.2.2 to $M_1(1)$. □

Another example for $(\mathbb{Z}/2\mathbb{Z})^2$. Following [16, §5.1], we can search for a projective model of a K3$^{[2]}$-type manifold X as a double cover of a cone $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$, with a symplectic involution ι that acts exchanging the two copies of $\mathbb{P}^2$.
From a lattice-theoretic point of view, this model is given by a big and nef divisor $H = F_1 + F_2 \in NS(X)$ such that $\langle F_1, F_2 \rangle = U(2)$ and $\iota^* F_1 = F_2$.

Proposition 5.3.0.8. *The involution ι can never be the square of a symplectic automorphism of order 4 on X, but it can be one of the generators for a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. The same lattice-theoretic condition gives for K3 surfaces a projective model as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with a symplectic involution ι that exchanges the two copies of $\mathbb{P}^1$: if ι is the square of a symplectic automorphism of order 4 it always holds $\iota^* F_1 = F_1$ (see §2.6.3 no.3) so this model cannot be realized; if ι is one of the generators of $(\mathbb{Z}/2\mathbb{Z})^2$ however this model exists (see §3.6.3, no.5). □

5.4 The action of G in cohomology, and the induced involutions

Let X be a K3$^{[2]}$-type manifold with a symplectic involution i, and $\pi : X \to X/i$ be the quotient map. Since (X, i) is always a standard pair, if we denote i_S a symplectic involution on a K3 surface, and π_S the associated quotient map, then it holds:

$$i^* : \; H^2(X, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle \to \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle, \quad (x, \mu) \mapsto (i_S^* x, \mu)$$

and therefore

$$\pi_* : \; H^2(X, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle \to \pi_{S*}\Lambda_{\mathrm{K3}} \oplus \langle -4 \rangle, \quad (x, \mu) \mapsto (\pi_{S*} x, \tilde{\mu}). \tag{5.4.0.1}$$

Recall from Example 1.5.1.2 that if Y is the Nikulin orbifold obtained as terminalization of the quotient X/i, then

$$H^2(Y, \mathbb{Z}) \simeq \Lambda_{\mathrm{N}} := U(2)^{\oplus 3} \oplus E_8 \oplus \langle -2 \rangle^{\oplus 2}. \tag{5.4.0.2}$$

Remark 5.4.0.1. Notice that $U(2)^{\oplus 3} \oplus E_8 \simeq \pi_{S*}(H^2(S, \mathbb{Z}))$. Its orthogonal complement in $H^2(Y, \mathbb{Z})$, $\langle -2 \rangle^{\oplus 2}$ is generated by $(\tilde{\mu} \pm \tilde{\Sigma})/2$, where $\tilde{\Sigma}$ is the exceptional class introduced in the terminalization $Y \to X/i$.

If X admits a symplectic action of a group H with $\langle i\rangle$ a normal subgroup, then on Y there is a residual symplectic action of H/i. In this section, we're going to describe the residual involution ι induced on Y by a group G of order 4 acting symplectically on X. By the standardness of the action of G on K3$^{[2]}$-type manifolds, we can take as π_S in (5.4.0.1) one of the quotient maps π_{2*} (if G is cyclic, see Section 2.4.1), or $\pi_{\tau*},\pi_{\varphi*}$ (otherwise, see Sections 3.4.1, 3.4.2 respectively); then, ι acts on $U(2)^{\oplus 3}\oplus E_8\subset H^2(Y,\mathbb{Z})$ as the induced involutions on $\pi_{S*}\Lambda_{\mathrm{K3}}$, and as the identity on $\langle -2\rangle^{\oplus 2}=\langle(\tilde{\mu}\pm\tilde{\Sigma})/2\rangle$.

5.4.1 Action of G on the K3 lattice revised, and results on the cohomology of the terminalization of X/G

To describe the induced involution ι on $H^2(Y,\mathbb{Z})$ in a convenient way, we start by providing an explicit isometry between the invariant lattice Λ_{K3}^i and the abstract lattice $U^{\oplus 3}\oplus E_8(2)$: this will allow us to then compute the action of ι on the standard basis for Λ_{N}, the one that gives as intersection matrix exactly (5.4.0.2).

Using our knowledge from Chapters 2 and 3 of the action of G on Λ_{K3}, we can also give some partial results about the cohomology of the terminalization W of the quotient X/G, by computing the image via the quotient map of the invariant lattice Λ_{K3}^G.

Remark 5.4.1.1. What's left unknown is whether the image of $H^2(Y,\mathbb{Z})$ via the quotient map is a primitive sublattice of $H^2(W,\mathbb{Z})$. Moreover, in the case $G=(\mathbb{Z}/2\mathbb{Z})^2$ the terminalization $W\to X/G$ introduces two new divisors, and their contribution to the cohomology of W is yet to be determined.

The case $G=\mathbb{Z}/4\mathbb{Z}$

Recall from Section 2.3.2 that the co-invariant lattice $\Omega_2\simeq E_8(2)\subseteq\Omega_4$ for the involution τ^2 is generated by the elements $\alpha-\gamma,\ \beta-\delta,\ e_1-g_1,\ e_3-g_3,\ f_1-h_1,\ f_2-h_2,\ f_3-h_3,\ f_4-h_4$.
The lattice $R=\Omega_2^{\perp_{\Omega_4}}$ is generated by the elements $r_1=e_1-f_1+g_1-h_1,\ r_2=\alpha-\beta+\gamma-\delta,\ r_3=e_3-f_3+g_3-h_3,\ r_4=g_1-f_1+e_4-f_4+\alpha-\gamma-(e_2-g_2+e_4-g_4+a_1-a_2+\sigma)/2,\ r_5=a_1-a_2,\ r_6=\sigma$.

Proposition 5.4.1.2. *The lattice $\Lambda_{\mathrm{K3}}^{\tau^2}$ invariant for τ^{2*} is an overlattice of $\Lambda_{\mathrm{K3}}^{\tau}\oplus R$ isometric to the lattice $U^{\oplus 3}\oplus E_8(2)$ with the following generators, using the elements s_i,w_j introduced in Proposition 5.1.1.1, and r_j as above:*

$$\begin{aligned}U^{\oplus 3}=&\langle s_1,\ s_2\rangle\oplus\langle\frac{w_3+w_4+w_6+r_5+r_6}{2},\ \frac{w_1+w_3+w_4+w_6+r_4+r_5+r_6}{2}\rangle\oplus\\ &\oplus\langle\frac{w_1-3w_2+w_3+w_4-3w_5-w_6+r_1-r_6}{2}-4w_6-3r_3-4r_4-r_5-3r_6,\\ &\frac{w_1-w_2+w_3+w_4-w_5-r_6}{2}-w_6-r_3-r_4\rangle;\\ E_8(2)=&\langle\frac{w_3+r_1-r_2}{2}+w_4+w_6+r_5+r_6,\ \frac{w_3-r_1+r_2}{2}+w_2+r_4+r_5,\end{aligned}$$

$$\frac{3w_6 - w_3 + r_2}{2} + r_3 + r_4,\ -\frac{w_3 + w_4 + 3w_6 + 3r_5 + r_6}{2} - r_2 - r_3 - 2r_4,$$
$$\frac{w_3 + r_1 + r_2}{2},\ \frac{-w_3 + w_5 + w_6 + 3r_3 + 3r_5 + r_6}{2} - r_1 + r_2 + 3r_4,$$
$$\frac{-w_4 - r_2 - r_5 + r_6}{2} - w_1 + w_6,\ \frac{w_3 + r_1 - r_2}{2} + w_1 - r_4\rangle.$$

Lemma 5.4.1.3. *Denote* $\hat{\star} \coloneqq \pi_{2*}\star$, $\overline{\star} \coloneqq \pi_{4*}\star$.
The (primitive) image of

$$\Lambda^{\tau}_{\mathrm{K3}} = U \oplus \langle -2\rangle^{\oplus 2} \oplus U(4)^{\oplus 2} = \langle s_1, s_2\rangle \oplus \langle w_1, w_2\rangle \oplus \langle w_3, \ldots, w_6\rangle$$

via the maps π_{2*} *and* π_{4*} *introduced in Section 2.4.1 is as follows:*

$$\pi_{2*}(U \oplus \langle -2\rangle^{\oplus 2} \oplus U(4)^{\oplus 2}) = U(2) \oplus \langle -4\rangle^{\oplus 2} \oplus U(2)^{\oplus 2} = \langle \hat{s}_1, \hat{s}_2\rangle \oplus \langle \hat{w}_1, \hat{w}_2\rangle \oplus \langle \hat{w}_3/2, \ldots, \hat{w}_6/2\rangle,$$

$$\pi_{4*}(U \oplus \langle -2\rangle^{\oplus 2} \oplus U(4)^{\oplus 2}) = U(4) \oplus \langle -2\rangle^{\oplus 2} \oplus U^{\oplus 2} = \langle \overline{s}_1, \overline{s}_2\rangle \oplus \langle \overline{w}_1/2, \overline{w}_2/2\rangle \oplus \langle \overline{w}_3/4, \ldots, \overline{w}_6/4\rangle.$$

Proposition 5.4.1.4. *Let* X *be a* $K3^{[2]}$*-type manifold with a symplectic action of* $G = \mathbb{Z}/4\mathbb{Z}$. *The terminalization* W *of the quotient* X/G *is a primitive symplectic orbifold,* $\pi_1(W_{reg}) = \mathbb{Z}/2\mathbb{Z}$, *and* $H^2(W,\mathbb{Z})$ *is an overlattice of finite index (possibly 1) of* $U^{\oplus 2} \oplus U(4) \oplus \langle -2\rangle^{\oplus 2} \oplus \langle -4\rangle^{\oplus 2}$.

Proof. By Proposition 5.2.0.1 we know that the generator τ of G fixes points on X, so the terminalization of the quotient is a primitive symplectic orbifold, of which we can compute the fundamental group of the regular locus (see Proposition 1.5.1.1). Let Y be the Nikulin orbifold that arises as terminalization of X/τ^2: since the involution ι induced by G/τ^2 on the Nikulin orbifold Y does not fix any surface, the terminalization $W \to Y/\iota$ does not introduce any new divisor. The quotient map $\pi_{\iota*}$ acts on $H^2(Y,\mathbb{Z})$ as follows: on $\langle(\tilde{\mu} \pm \tilde{\Sigma})/2\rangle$ it doubles the intersection form; on its orthogonal complement, that is an overlattice of $\pi_{2*}(R \oplus \Lambda^{\tau}_{\mathrm{K3}})$, it annihilates $\pi_{2*}R$ and acts on $\pi_{2*}\Lambda^{\tau}_{\mathrm{K3}}$ as described by Lemma 5.4.1.3. Therefore, $\pi_{\iota*}H^2(Y,\mathbb{Z}) = U^{\oplus 2} \oplus U(4) \oplus \langle -2\rangle^{\oplus 2} \oplus \langle -4\rangle^{\oplus 2}$. □

The case $G = (\mathbb{Z}/2\mathbb{Z})^2$

Consider now $G = (\mathbb{Z}/2\mathbb{Z})^2 = \langle \tau, \varphi\rangle$. Since, having fixed a basis of $\Lambda^{G}_{\mathrm{K3}}$, the actions of τ^*, φ^* are differently described, we are going to give both quotient maps.

Proposition 5.4.1.5. *With the notation of Section 3.3.2, consider the elements:*

$$\omega_1 = (d_2 - a_2 + c_2 - b_2 + f_1 - e_1 - f_2 + e_2 - d_1 + a_1 + b_1 - c_1)/3;$$
$$\iota_1 = (m_2 + m_4 - b_1 + c_1 + d_1 - a_1)/2;$$
$$\iota_2 = (m_1 + m_2 - b_2 + c_2 + d_2 - a_2)/2;$$

$$
\begin{aligned}
\iota_3 =&(m_1 + f_1 - e_1 + \omega_1)/2;\\
\iota_4 =&(m_3 + m_4 + f_1 - e_1)/2.
\end{aligned}
$$

Then the lattice $\Lambda^{\tau}_{\mathrm{K3}}$ is isometric to the lattice $U^{\oplus 3} \oplus E_8(2)$ with the following generators, using the same notation as Proposition 5.1.2.1:

$$
\begin{aligned}
U^{\oplus 3} =&\langle s_1, s_2\rangle \oplus \langle \iota_4 + u_3 + u_4 + m_2, u_3 + u_4 + m_2\rangle \oplus\\
&\oplus \langle \iota_3 + u_3 + u_4 + m_2 + 2(\iota_1 + u_1 + u_2 + u_4), \iota_1 + u_1 + u_2 + u_4\rangle;\\
E_8(2) =&\langle m_3 + m_4 - 2(\iota_4 + u_3 + m_2) - 5u_4 - 3(\iota_1 + u_1 + u_2),\\
&\iota_2 - 2\iota_1 - 4\iota_3 + 6\iota_4 - u_1 + 2m_1 + m_2 - 2m_3 - 2m_4,\\
&u_1 - u_2,\ 2\iota_1 - \iota_2 + 2\iota_3 - 2\iota_4 + u_1 + 2u_2 + 2u_4 - m_1,\\
&\iota_1 + \iota_2 + u_3 + m_3,\ u_1 + u_2 + u_4 - \iota_2 + m_1 + m_2 + m_4,\\
&\iota_2 + u_4 - m_1,\ u_3 + m_2 - \iota_2 - 2(\iota_1 + u_4 + u_1 + u_2)\rangle.
\end{aligned}
$$

Proposition 5.4.1.6. *Consider the elements:*

$$
\begin{aligned}
\omega_2 =&(c_2 - a_2 - d_2 + b_2 + z - w + d_1 - b_1 - c_1 + a_1)/3;\\
\kappa_1 =&(m_1 + m_2 + b_2 - a_2 + c_2 - d_2)/2;\\
\kappa_2 =&(u_1 + u_2 + u_4 + w)/2;\\
\kappa_3 =&(m_2 + m_4 + b_1 - a_1 + c_1 - d_1)/2;\\
\kappa_4 =&(u_1 + u_2 + u_3 + m_1 + m_4 + \omega_2)/2;
\end{aligned}
$$

Then the lattice $\Lambda^{\varphi}_{\mathrm{K3}}$ is isometric to the lattice $U^{\oplus 3} \oplus E_8(2)$ with the following generators:

$$
\begin{aligned}
U^{\oplus 3} =&\langle s_1, s_2\rangle \oplus \langle u_1, \kappa_2\rangle \oplus \langle u_4, \kappa_4 - 2u_1 - \kappa_2 + 3u_4\rangle;\\
E_8(2) =&\langle m_2 - 2u_4 - \kappa_1,\ m_1 - \kappa_1,\ -(m_1 + m_3 + \kappa_3),\ \kappa_1 + \kappa_3 - m_2 - m_4,\\
&m_2 + m_3 - u_1 - 2u_2 - u_3 - u_4 - \kappa_1 + 2\kappa_2 + 2\kappa_4,\ u_2 - u_1 + 4u_4 - 2\kappa_2,\\
&m_4 - m_2 - m_3 + 2u_1 + u_2 + u_3 - u_4 + \kappa_1 - 2\kappa_4,\ m_3 - 2u_4\rangle.
\end{aligned}
$$

Lemma 5.4.1.7. *Recall that*

$$
\Lambda^{(\mathbb{Z}/2\mathbb{Z})^2}_{\mathrm{K3}} = U \oplus U(2)^{\oplus 2} \oplus D_4(2) = \langle s_1, s_2\rangle \oplus \langle u_1, \ldots, u_4\rangle \oplus \langle m_1, \ldots, m_4\rangle.
$$

Denote $\hat{\star} := \pi_{\tau *}\star$, $\tilde{\star} := \pi_{\varphi *}\star$ (see Sections 3.4.1, 3.4.2). Define

$$
\mu_1 = (\hat{m}_4 + \hat{m}_2)/2,\ \ \mu_2 = \hat{m}_3 + (\hat{m}_4 + \hat{m}_2)/2,\ \ \mu_3 = (\hat{m}_1 + \hat{m}_2)/2,\ \ \mu_4 = (\hat{m}_4 - \hat{m}_2)/2
$$

$$
\beta_1 = \tilde{u}_1,\ \ \beta_2 = (\tilde{u}_1 + \tilde{u}_2)/2,\ \ \beta_3 = (\tilde{u}_1 + \tilde{u}_3 + \tilde{u}_4)/2,\ \ \beta_4 = \tilde{u}_3.
$$

Then the intersection matrix of $\{\beta_1, \ldots, \beta_4\}$ is

$$
B = \begin{bmatrix} 0 & 2 & 0 & 0\\ 2 & 2 & 1 & 0\\ 0 & 1 & 2 & 2\\ 0 & 0 & 2 & 0 \end{bmatrix},
$$

and the (primitive) image of $\Lambda_{\mathrm{K3}}^{(\mathbb{Z}/2\mathbb{Z})^2}$ *via the maps* $\pi_{\tau*}, \pi_{\varphi*}$ *is*

$$\begin{aligned}\pi_{\tau*}(U \oplus U(2)^{\oplus 2} \oplus D_4(2)) = U(2) \oplus U^{\oplus 2} \oplus D_4(2) = \\ \langle \hat{s}_1, \hat{s}_2 \rangle \oplus \langle \hat{u}_1/2, \dots, \hat{u}_4/2 \rangle \oplus \langle \mu_1, \dots \mu_4 \rangle, \\ \pi_{\varphi*}(U \oplus U(2)^{\oplus 2} \oplus D_4(2)) = U(2) \oplus B \oplus D_4 = \\ \langle \tilde{s}_1, \tilde{s}_2 \rangle \oplus \langle \beta_1, \dots \beta_4 \rangle \oplus \langle \tilde{m}_1/2, \dots, \tilde{m}_4/2 \rangle;\end{aligned}$$

Denote $\overline{\star} := \pi_{2,2*}\star$ *(see Section 3.4.4): then* $\pi_{2,2*}\Lambda_{\mathrm{K3}}^{(\mathbb{Z}/2\mathbb{Z})^2}$ *is not a primitive sublattice of* Λ_{K3}*, but an overlattice of it of index 2 is. Define*

$$\nu_1 = (-\overline{m}_1 - \overline{m}_2)/4, \;\; \nu_2 = (\overline{m}_1 - \overline{m}_2)/4, \;\; \nu_3 = (\overline{m}_2 + \overline{m}_4)/4, \;\; \nu_4 = (\overline{m}_3)/2 + (\overline{m}_1 + \overline{m}_2)/4;$$

then we consider:

$$\begin{aligned}(\pi_{2,2*}(U \oplus U(2)^{\oplus 2} \oplus D_4(2)))' = \langle 2 \rangle \oplus \langle -2 \rangle \oplus U(2)^{\oplus 2} \oplus D_4 = \\ \langle (\overline{s}_1 + \overline{s}_2)/2, \; (\overline{s}_1 - \overline{s}_2)/2 \rangle \oplus \langle \overline{u}_1/2, \dots, \overline{u}_4/2 \rangle \oplus \langle \nu_1, \dots \nu_4 \rangle.\end{aligned}$$

Remark 5.4.1.8. The lattices $U^{\oplus 2} \oplus U(2) \oplus D_4(2)$ and $U(2) \oplus B \oplus D_4$ are isomorphic.

Proposition 5.4.1.9. *Let* X *be a K3*$^{[2]}$*-type manifold with a symplectic action of* $G = (\mathbb{Z}/2\mathbb{Z})^2$*. The terminalization* W *of the quotient* X/G *is an IHSO, and* $H^2(W, \mathbb{Z})$ *is a lattice of rank 14 containing the lattice* $U(2)^{\oplus 2} \oplus D_4 \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -4 \rangle^{\oplus 2}$.

Proof. Since G is generated by symplectic involutions, each fixing a surface on X, the terminalization of the quotient is an IHSO (see Proposition 1.5.1.1). Let Y be the Nikulin orbifold that arises as terminalization of the quotient of X by any involution of G, suppose τ. The involution ι induced on the Nikulin orbifold Y fixes two surface, the image in Y of the surfaces $\Sigma_\rho, \Sigma_\varphi$: indeed, τ acts on each of them, so they are not identified in the quotient. Therefore, the terminalization $W \to Y/\iota$ introduces two new divisors. The image of $H^2(X, \mathbb{Z})$ via the quotient map $\pi_{2,2*}$ is computed similarly to Proposition 5.4.1.4. ☐

5.4.2 Induced involutions on Nikulin orbifolds

In this section, we are going to describe the action of the induced involutions on Λ_{N} (5.4.0.2). At first, we take Y a Nikulin orbifold arising as terminalization of X/i, where X is a K3$^{[2]}$-type manifold and $i \in G$, a group of order 4 acting symplectically on X. We then prove that (under appropriate conditions) if $\tilde{Y}$ is a deformation of Y, it can carry a symplectic involution that acts in cohomology as ι. Therefore, in analogy to standard actions on K3$^{[2]}$-type manifolds, we introduce the notion of *standard* involution on a Nikulin-type orbifold, and find lattice-theoretic conditions for its existence using the Torelli theorem for IHSOs [51, Thm. 1.1].

Remark 5.4.2.1.

1. When giving the matrices for the induced involutions, we follow the convention of matrices acting on the left on column vectors: notice that this is the opposite of what Sage [87] uses as default.
2. We provide two different representations of the involution induced by the action of $G = (\mathbb{Z}/2\mathbb{Z})^2$ on X, corresponding respectively to the actions of $(\mathbb{Z}/2\mathbb{Z})^2/\tau$ and $(\mathbb{Z}/2\mathbb{Z})^2/\varphi$. There exists an isometry of Λ_{N} sending one to the other.
3. The involution induced by $\mathbb{Z}/4\mathbb{Z}$, and that induced by $(\mathbb{Z}/2\mathbb{Z})^2$ are different, as one can see by comparing the multiplicity of the eigenvalues: the invariant lattices for the two actions have different rank.

Proposition 5.4.2.2. *The involution induced by $\mathbb{Z}/4\mathbb{Z}$ on $H^2(Y,\mathbb{Z}) = \langle -2\rangle^{\oplus 2} \oplus U(2)^{\oplus 3} \oplus E_8$ is represented by the following matrix:*

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 3 & 3 & -3 & 1 & 3 & 1 & -1 & -2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 4 & 3 & -2 & 2 & 3 & 1 & -1 & -2 & 1 & 0 & -1 & 1\\
0 & 0 & 0 & 0 & 2 & 1 & 29 & 10 & 1 & 2 & -4 & 1 & 1 & -4 & -2 & 0\\
0 & 0 & 0 & 0 & -2 & -3 & 102 & 29 & -3 & 4 & -10 & 7 & 1 & -13 & -6 & 0\\
0 & 0 & 0 & 0 & 4 & -2 & 280 & 86 & -2 & 15 & -32 & 16 & 5 & -36 & -18 & -1\\
0 & 0 & 0 & 0 & 4 & 0 & 186 & 58 & 0 & 11 & -22 & 10 & 4 & -25 & -12 & 0\\
0 & 0 & 0 & 0 & 10 & 2 & 380 & 120 & 2 & 22 & -45 & 19 & 8 & -50 & -25 & 0\\
0 & 0 & 0 & 0 & 14 & 2 & 554 & 174 & 2 & 32 & -66 & 28 & 12 & -72 & -37 & 0\\
0 & 0 & 0 & 0 & 10 & 0 & 462 & 144 & 0 & 26 & -55 & 25 & 9 & -60 & -30 & 0\\
0 & 0 & 0 & 0 & 8 & 0 & 372 & 116 & 0 & 20 & -44 & 20 & 8 & -49 & -24 & 0\\
0 & 0 & 0 & 0 & 6 & 0 & 256 & 80 & 0 & 14 & -30 & 13 & 6 & -34 & -16 & 0\\
0 & 0 & 0 & 0 & 2 & 0 & 128 & 40 & -1 & 7 & -15 & 7 & 3 & -17 & -8 & 0
\end{bmatrix}$$

Proof. By Remark 5.4.0.1, the induced involution acts as the identity on $\langle -2\rangle^{\oplus 2}$, and it acts as $\hat{\tau}^*$ on $\pi_{2*}\Lambda_{\mathrm{K3}} \simeq U(2)^{\oplus 3} \oplus E_8$ (see Section 2.4.1). Using the description of $\Lambda_{\mathrm{K3}}^{\tau^2}$ provided in Prop. 5.4.1.2, we compute its image via the map π_{2*} (see Proposition 2.4.1.1): for every $x \in E_8(2)$, $\pi_{2*}x/2$ is integral and primitive, so that the primitive completion of $\pi_{2*}\Lambda_{\mathrm{K3}}^{\tau^2}$ is $U(2)^{\oplus 3} \oplus E_8$, as was already proved in [63, Thm. 5.7]. Since the elements $s_1, s_2, w_1, \dots, w_6$ are invariant for τ^*, while $r_1, \dots, r_6$ are anti-invariant, we can determine the action of $\hat{\tau}^*$ on $U(2)^{\oplus 3} \oplus E_8$. □

Corollary 5.4.2.3. *The invariant and co-invariant lattices for the involution ι induced on Y by $G = \mathbb{Z}/4\mathbb{Z}$ are:*

$$H^2(Y,\mathbb{Z})^{\iota} = U(2)^{\oplus 3} \oplus \langle -4\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}; \quad \left(H^2(Y,\mathbb{Z})^{\iota}\right)^{\perp} = D_6(2).$$

Proposition 5.4.2.4. *The involution ι induced by $(\mathbb{Z}/2\mathbb{Z})^2$ on $H^2(Y,\mathbb{Z}) = \langle -2\rangle^{\oplus 2} \oplus U(2)^{\oplus 3} \oplus E_8$ is represented by the following matrices, ι being the generator of $(\mathbb{Z}/2\mathbb{Z})^2/\tau$ in the first case, of $(\mathbb{Z}/2\mathbb{Z})^2/\varphi$ in the second one:*

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&1&3&0&-2&0&0&1&0&0&0&0\\
0&0&0&0&0&0&7&4&-6&-1&0&3&1&1&-1&-3\\
0&0&0&0&3&0&14&7&-12&-2&0&6&2&1&-1&-5\\
0&0&0&0&2&0&8&4&-7&2&0&2&2&0&0&-4\\
0&0&0&0&0&0&0&0&0&2&0&-1&1&-1&1&-1\\
0&0&0&0&0&0&-4&-2&3&4&1&-4&1&-2&2&0\\
0&0&0&0&0&0&-8&-4&6&7&0&-6&1&-3&3&1\\
0&0&0&0&0&0&-8&-4&6&6&0&-6&1&-2&2&2\\
0&0&0&0&0&0&-4&-2&3&4&0&-4&1&-1&2&0\\
0&0&0&0&0&0&2&2&-3&2&0&0&1&0&1&-2\\
0&0&0&0&0&0&6&4&-6&0&0&2&2&0&0&-3
\end{bmatrix}$$

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&2&0&-3&0&0&0&0&1&-2&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-3&1&6&-1&-1&1&0&-1&3&-2&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&2&-1&-1&0&1&0\\
0&0&0&0&0&0&0&0&-1&0&1&0&-1&0&1&0\\
0&0&0&0&0&0&0&-2&0&0&2&-1&-2&0&2&0\\
0&0&0&0&0&0&0&-2&0&0&3&-2&-2&0&2&0\\
0&0&0&0&0&0&0&-2&0&0&2&-2&-1&0&2&0\\
0&0&0&0&0&2&0&-4&0&0&2&-2&0&-1&2&0\\
0&0&0&0&0&0&0&0&0&0&2&-2&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&1
\end{bmatrix}$$

Proof. The proof is similar to that of Proposition 5.4.2.2. □

Corollary 5.4.2.5. *The invariant and co-invariant lattices for the involution ι induced on Y by $G = (\mathbb{Z}/2\mathbb{Z})^2$ are:*

$$H^2(Y,\mathbb{Z})^\iota = U^{\oplus 2} \oplus U(2) \oplus D_4(2) \oplus \langle -2\rangle^{\oplus 2}; \quad (H^2(Y,\mathbb{Z})^\iota)^\perp = D_4(2).$$

Definition 5.4.2.6. Let Y be a Nikulin-type orbifold. We call an involution ι *standard* if the pair (Y,ι) can be deformed to a pair $(\tilde{Y},\tilde{\iota})$, where $\tilde{Y}$ is the terminalization of $S^{[2]}/i$ for some K3 S with a symplectic action of a group G of order 4, $i \in G$ is a symplectic involution, and $G/i = \langle\tilde{\iota}\rangle$.

Remark 5.4.2.7. Let φ be an isometry of $H^2(Y,\mathbb{Z})$ with invariant and co-invariant lattices as those given in Corollary 5.4.2.3 or 5.4.2.5: this does not guarantee the existence of a standard involution f of Y such that $f^* = \varphi$.
Indeed, $H^2(Y,\mathbb{Z})$ can be obtained as overlattice of $D_6(2) \oplus (U(2)^{\oplus 3} \oplus \langle -4\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2})$ in more than one way: the correct way to have the induced involution is obtained by gluing along a certain subgroup of the discriminant form of $D_6(2)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$, but there is at least another gluing along $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$.
Similarly, $H^2(Y,\mathbb{Z})$ can be obtained as overlattice of $D_4(2) \oplus (U(2) \oplus U^{\oplus 2} \oplus D_4(2))$ by gluing along $(\mathbb{Z}/2\mathbb{Z})^4$, but also along $(\mathbb{Z}/4\mathbb{Z})^2$. Therefore, the lattice-theoretic data needed to characterize the induced involutions consist of invariant lattice, co-invariant lattice and their gluing: the latter is given in Lemma 5.4.2.8 below.

Lemma 5.4.2.8. *Let Y be a Nikulin-type orbifold which admits a standard symplectic involution ι. Then one of the following holds:*

1. *the gluing between $H^2(Y,\mathbb{Z})^\iota \simeq U(2)^{\oplus 3} \oplus \langle -4\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$ and $(H^2(Y,\mathbb{Z})^\iota)^\perp \simeq D_6(2)$ which gives $H^2(Y,\mathbb{Z})$ as overlattice is obtained by adding as generators the following elements: $(d_5+d_6+u_1+u_2+v_2)/2$, $(d_5+a_1+u_1+v_2)/2$, $(d_1+d_4+d_6+a_1)/2$, $(d_3+d_6+v_1)/2$, $(d_2+d_4+d_6+a_2)/2$, $(d_1+d_3+a_2+v_2)/2$; here $\{d_1,\dots,d_6\}$ are the generators of $D_6(2)$ (numbered as in Example 1.2.0.2), $\{u_1,u_2\}$ and $\{v_1,v_2\}$ those of two of the copies of $U(2)$, $\{a_1,a_2\}$ of $\langle -4\rangle^{\oplus 2}$; this corresponds to the case where ι is induced by $\mathbb{Z}/4\mathbb{Z}$;*

2. *the gluing between $H^2(Y,\mathbb{Z})^\iota \simeq U^{\oplus 2} \oplus U(2) \oplus D_4(2) \oplus \langle -2\rangle^{\oplus 2}$ and $(H^2(Y,\mathbb{Z})^\iota)^\perp \simeq D_4(2)$ which gives $H^2(Y,\mathbb{Z})$ as overlattice is obtained by adding as generators the following elements: $(d_1+d_2+e_2+e_4)/2$, $(d_2+d_3+e_4)/2$, $(d_2+e_1+e_3+e_4)/2$, $(d_4+e_2+e_3+e_4)/2$; here $\{d_1,\dots,d_4\}$ and $\{e_1,\dots,e_4\}$ are the generators of the two copies of $D_4(2)$ (numbered as in Example 1.2.0.2); this corresponds to the case where ι is induced by $(\mathbb{Z}/2\mathbb{Z})^2$.*

Remark 5.4.2.9. We say that an embedding $\varphi : D_6(2) \hookrightarrow \Lambda_{\mathrm{N}}$ satisfies the condition $(\star)$ if the gluing between $\varphi(D_6(2))$ and its orthogonal complement is as in Lemma 5.4.2.8.1. Similarly, we say that $\psi : D_4(2) \hookrightarrow \Lambda_{\mathrm{N}}$ satisfies the condition $(\star)$ if the gluing between $\psi(D_4(2))$ and its orthogonal complement is as in Lemma 5.4.2.8.2.

Theorem 5.4.2.10. *Let Y be a Nikulin-type orbifold such that there is an embedding of $D_k(2)$ in $NS(Y)$, with k either 4 or 6, such that the induced embedding $\varphi : D_k(2) \hookrightarrow H^2(Y,\mathbb{Z})$ satisfies the condition $(\star)$; then Y admits a standard symplectic involution.*

Proof. We define the isometry α on $H^2(Y,\mathbb{Z})$ that acts as $-id$ on $\varphi(D_k(2))$, and as the identity on its orthogonal complement. To apply the Torelli theorem for IHSOs

[51, Thm. 1.1] we need to show that α is 1) an integral Hodge isometry which 2) is a monodromy operator and 3) preserves the Kähler cone. The first condition is satisfied by construction, because $T(Y)$ is contained in the orthogonal complement to $\varphi(D_k(2))$, so α acts on it as the identity; the second is satisfied because there exists a Nikulin orbifold Y' with an induced involution ι such that ι^* acts as α, and monodromy is invariant under deformations. To prove 3), refer to the description of the walls of the Kähler cone of Y in [52]. The lattice $\varphi(D_6(2))$ does not contain any wall-divisor: indeed it contains no (-2)-classes and, if we assume $E_8 \oplus U(2)^{\oplus 3} = \langle e_1, \dots, e_8 \rangle \oplus \langle u_{i,1}, u_{i,2} \rangle_{i=1,2,3}$ (numbered as in example 1.2.0.2), we get as generators of $\varphi(D_6(2))$

$$\begin{aligned} d_1 &= 2e_1 + 2e_2 + 5e_3 + 7e_4 + 5e_5 + 4e_6 + 3e_7 + e_8 + u_{2,1} + 2u_{2,2} + u_{3,1} - u_{3,2}, \\ d_2 &= -12e_1 - 8e_2 - 17e_3 - 25e_4 - 21e_5 - 16e_6 - 11e_7 - 5e_8 - u_{3,1} - 5u_{3,2}, \\ d_3 &= -6e_1 - 4e_2 - 8e_3 - 12e_4 - 9e_5 - 8e_6 - 6e_7 - 3e_8 - u_{2,2} - u_{3,1} - u_{3,2}, \\ d_4 &= 3e_1 + 2e_2 + 3e_3 + 5e_4 + 4e_5 + 4e_6 + 3e_7 + 1e_8 - u_{2,1} + 2u_{3,2}, \\ d_5 &= 16e_1 + 11e_2 + 23e_3 + 33e_4 + 27e_5 + 22e_6 + 15e_7 + 8e_8 + u_{2,1} + 2u_{3,1} + 5u_{3,2}, \\ d_6 &= e_1 + e_5 - u_{2,1}, \end{aligned}$$

so no elements of square $-4, -6, -12$ in $\varphi(D_6(2))$ have divisibility 2; a similar result holds for $\varphi(D_4(2))$).
If $NS(Y) = D_k(2) \oplus \langle 2d \rangle$ ($k = 4, 6, d \in \mathbb{Z}_{>0}$) or one of its overlattices, then wall classes in its Néron-Severi will have the form $aL + bv$, with $a, b \in \mathbb{Q} \setminus \{0\}$, $v \in D_k(2)$ and L the generator of $D_k(2)^{\perp_{NS(Y)}}$: since α does not reflect any of these classes, the Kähler cone of the general projective Y is preserved. ☐

5.5 Classifying Nikulin-type Orbifolds with a standard symplectic involution

In the first part of this section, we classify projective Nikulin orbifolds with an induced symplectic involution: these are obtained as quotients of projective K3$^{[2]}$-type manifolds with a symplectic action of a group of order four G, whose deformation families are classified in Sections 5.1.1, 5.1.2. We establish a correspondence between the moduli space of projective K3$^{[2]}$-type manifolds with a symplectic action of a group G of order 4, and the Nikulin orbifolds obtained as terminalization of their partial quotient.

We then classify projective Nikulin-type orbifolds Y with an action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \iota, \kappa \rangle$, where ι is standard (see Definition 5.4.2.6), and κ is the non-standard involution defined in [52] (see Theorem 5.5.2.1): therefore, $NS(Y)$ admits a primitive embedding of either $D_6(2) \oplus \langle -2 \rangle$, or $D_4(2) \oplus \langle -2 \rangle$, with the additional property that the embedding $D_k(2) \hookrightarrow H^2(Y, \mathbb{Z})$ satisfies condition $(\star)$ (see Remark 5.4.2.9).

5.5.1 Families of Nikulin orbifolds with an induced symplectic involution

The general (non-projective) Nikulin orbifold with an induced involution has Néron-Severi lattice $\Omega_\iota \oplus \langle -4 \rangle$, where $\Omega_\iota = D_6(2)$ if ι is induced by $G = \mathbb{Z}/4\mathbb{Z}$, $\Omega_\iota = D_4(2)$ if ι

is induced by $G = (\mathbb{Z}/2\mathbb{Z})^2$; the class that generates $\Omega_\iota^\perp$ is the exceptional class $\tilde{\Sigma}$ that is introduced with the terminalization $Y \to X/i$.

The case $G = \mathbb{Z}/4\mathbb{Z}$

In the following table we classify the projective families of Nikulin orbifolds Y admitting a natural involution induced by an action of $\mathbb{Z}/4\mathbb{Z}$ on a $\mathrm{K3}^{[2]}$-type manifold X, by giving the possible pairs $(NS(Y), T(Y))$ in the second and third column. Projective families of X are classified in Theorem 5.1.1.2; we consider the map π_* introduced in (5.4.0.1), with $\pi_{S*} = \pi_{2*}$ described in Section 2.4.1. If X is polarized with a class L of square $2d$, then on Y we consider the pseudo-ample class $H = \pi_* L$, or $H = \pi_* L/2$ if the former is not primitive: therefore, it holds $H^2 = 4d$ or $H^2 = d$ respectively. The relation between the number d in the first column and m appearing in the last columns is given in the proof of Theorem 5.1.1.2, where the classes $M_i(m)$ are constructed; since in many cases the same lattice $T(Y)$ obtained using $H = \pi_* M_i(m)$, therefore depending on m, is obtained also using $L_i(h)$, we write $L_i(m)$ accordingly (see Example 2.5.1.6). The relation between m, j, h appearing in the last line of the table is explained in Table 5.5.

Remark 5.5.1.1. Notice that for the families with a polarization $L = (L_S, 0) \in \Lambda_{\mathrm{K3}} \oplus \langle -2 \rangle$, i.e. those families such that $T(X) = T(S) \oplus \langle -2 \rangle$ for some general projective K3 surface S admitting a symplectic action of G, the class $\tilde{\Sigma}$ does not glue to any element in $NS(Y)$; for the families with a polarization $M = (L_S, n)$ instead, $\tilde{\Sigma}$ glues to $\pi_* M$ (see Remark 1.5.1.3).

	$NS(Y)$	$T(Y)$	H	H^2
$d =_4 1$	$D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$\langle -4d \rangle \oplus \langle -4 \rangle^{\oplus 3} \oplus U(2)^{\oplus 2}$	$\pi_* L_0(d)$	$4d$
	$(D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle)'$	$C_m \oplus U(2)^{\oplus 2}$	$\pi_* \overset{(\sim)}{M}_1(m)$	$4(4m-3)$
$d =_4 2$	$D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$\langle -4d \rangle \oplus \langle -4 \rangle^{\oplus 3} \oplus U(2)^{\oplus 2}$	$\pi_* L_0(d)$	$4d$
	$(D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle)'$	$B_m \oplus \langle -4 \rangle \oplus U(2)^{\oplus 2}$	$\pi_* M_2(m)$	$4(4m-2)$
	$(D_6(2) \oplus \langle 4d \rangle)' \oplus \langle -4 \rangle$		$\pi_* L_{2,2}^{(1,2)}(m)$	$4(4m \pm 2)$
$d =_4 3$	$D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$\langle -4d \rangle \oplus \langle -4 \rangle^{\oplus 3} \oplus U(2)^{\oplus 2}$	$\pi_* L_0(d)$	$4d$
	$D_6(2) \oplus (\langle 4d \rangle \oplus \langle -4 \rangle)'$	$F_m \oplus \langle -4 \rangle^{\oplus 2} \oplus U(2)^{\oplus 2}$	$\pi_* M_3(m)$	$4(4m-1)$
	$(D_6(2) \oplus \langle 4d \rangle)' \oplus \langle -4 \rangle$		$\pi_* L_{2,3}(m)$	$4(4m+3)$
$d =_4 0$	$D_6(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$\langle -4d \rangle \oplus \langle -4 \rangle^{\oplus 3} \oplus U(2)^{\oplus 2}$	$\pi_* L_0(d)$	$4d$
	$D_6(2) \oplus \langle d \rangle \oplus \langle -4 \rangle$	$G_m \oplus \langle -4 \rangle^{\oplus 2} \oplus U(2)^{\oplus 2}$	$\pi_* L_{2,0}(m)/2$	$4(m-1)$
	$(D_6(2) \oplus \langle d \rangle \oplus \langle -4 \rangle)'$		$\pi_* M_4(m)/2$	$4(m-1)$
	$(D_6(2) \oplus \langle d \rangle)' \oplus \langle -4 \rangle$		$\pi_* L_{4,j}(h)/2$	$4(m-1)$, see Table 5.5

Table 5.5: Relation between m, j, h

$m \pmod 4$	0	1	2	3
j	12	0	4	8
h	$(m-4)/4$	$(m+3)/4$	$(m-2)/4$	$(m+13)/4$

$$B_m = \begin{bmatrix} -4m & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \quad C_m = \begin{bmatrix} -4m & 2 & 2 & 2 \\ 2 & -4 & 0 & 0 \\ 2 & 0 & -4 & 0 \\ 2 & 0 & 0 & -4 \end{bmatrix} \quad F_m = \begin{bmatrix} -4m & 2 \\ 2 & -4 \end{bmatrix} \quad G_m = \begin{bmatrix} -4m & 4 \\ 4 & -4 \end{bmatrix}$$

Remark 5.5.1.2. The projective families of $K3^{[2]}$-type manifolds associated to the polarizations M_1 and $\tilde{M}_1$ give Nikulin orbifolds that belong to the same projective family. A similar statement holds for $L_{2,2}^{(1)}$ and $L_{2,2}^{(2)}$. These are the only families for which this phenomenon happens.

The case $G = (\mathbb{Z}/2\mathbb{Z})^2$

In the following table we classify the projective families of Nikulin orbifolds Y admitting a natural involution induced by an action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \tau, \varphi \rangle$ on a $K3^{[2]}$-type manifold X, by giving the possible pairs $(NS(Y), T(Y))$ in the second and third column. Projective families of X are classified in Theorem 5.1.2.2; we consider the map π_* introduced in (5.4.0.1), with $\pi_{S*} = \pi_{\tau*}, \pi_{\varphi*}$: these maps are described in Sections 3.4.1, 3.4.2 respectively. They may act differently on the polarization of X, as given a class L of positive square, $L \in \Omega_{2,2}^{\perp_{H^2(X,\mathbb{Z})}}$, the two involutions τ^*, φ^* may act differently on L (see Lemma 5.4.1.7): in this case, the same family of X can give rise to more than one family of Y, as it happens for projective K3 surfaces (see Corollary 3.5.3.3). When there is no difference between $\pi_{\tau*}$ and $\pi_{\varphi*}$, we use the notation $\pi_{\iota*}$. On Y we consider the pseudo-ample class $H = \pi_* L$, or $H = \pi_* L/2$ if the former is not primitive: therefore, it holds $H^2 = 4d$ or $H^2 = d$ respectively. The same considerations as in Remark 5.5.1.1 can be applied here.

	$NS(Y)$	$T(Y)$	H	H^2
$d =_2 1$	$D_4(2) \oplus \langle 2d \rangle \oplus \langle -4 \rangle$	$U \oplus \langle -4 \rangle \oplus D_4(2) \oplus Q_h$	$\pi_{\tau*} L_{2,2}^{(b)}(h)/2$	$2(2h+1)$
$d =_4 0$	$D_4(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$U \oplus \langle -4 \rangle \oplus T_h$	$\pi_{\varphi*} L_{2,0}^{(1)}(h)/2$	$4h$
		$\langle -4d \rangle \oplus U^2 \oplus D_4(2) \oplus \langle -4 \rangle$	$\pi_{\iota*} L_0(d)$	$4d$
	$(D_4(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle)'$	$\langle -4d \rangle \oplus U^2 \oplus D_4(2) \oplus \langle -4 \rangle$	$\pi_{\iota*} M_8(m)/2$	$8(m-1)$
	$(D_4(2) \oplus \langle 4d \rangle)' \oplus \langle -4 \rangle$	$U \oplus \langle -4 \rangle \oplus \langle 2 \rangle \oplus P_h$	$\pi_{\tau*} L_{2,0}^{(1)}(h)$	$16h$
$d =_4 1$	$D_4(2) \oplus \langle 4d \rangle \oplus \langle -4 \rangle$	$U \oplus \langle -4 \rangle \oplus T_h$	$\pi_{\varphi*} L_{2,0}^{(1)}(h)/2$	$4h$
		$\langle -4d \rangle \oplus U^2 \oplus D_4(2) \oplus \langle -4 \rangle$	$\pi_{\iota*} L_0(d)$	$4d$

	$(D_4(2)\oplus\langle 4d\rangle)'\oplus\langle -4\rangle$	$\langle -4d\rangle\oplus U^2\oplus D_4(2)\oplus\langle -4\rangle$	$\pi_{\iota *}L_{4,4}(h)/2$	$16h+4$
	$(D_4(2)\oplus\langle 4d\rangle\oplus\langle -4\rangle)'$	$U^2\oplus R_m$	$\pi_{\tau *}M_1(m)$	$4(4m-3)$
	$D_4(2)\oplus(\langle 4d\rangle\oplus\langle -4\rangle)'$	$B\oplus F_m$	$\pi_{\varphi *}M_1(m)$	$4(4m-3)$
$d =_4 2$	$D_4(2)\oplus\langle 4d\rangle\oplus\langle -4\rangle$	$U\oplus\langle -4\rangle\oplus T_h$	$\pi_{\varphi *}L^{(1)}_{2,0}(h)/2$	$4h$
		$\langle -4d\rangle\oplus U^2\oplus D_4(2)\oplus\langle -4\rangle$	$\pi_{\iota *}L_0(d)$	$4d$
	$(D_4(2)\oplus\langle 4d\rangle\oplus\langle -4\rangle)'$	$\langle -4d\rangle\oplus U^2\oplus D_4(2)\oplus\langle -4\rangle$	$\pi_{\iota *}M_8(m)/2$	$8(m-1)$
	$(D_4(2)\oplus\langle 4d\rangle)'\oplus\langle -4\rangle$	$U\oplus\langle -4\rangle\oplus W_h$	$\pi_{\iota *}L^{(a)}_{2,2}(h)$	$16h+8$
		$D_4\oplus\langle -4\rangle\oplus V_h$	$\pi_{\varphi *}L^{(b)}_{2,2}(h)$	$16h+8$
$d =_4 3$	$D_4(2)\oplus\langle 4d\rangle\oplus\langle -4\rangle$	$U\oplus\langle -4\rangle\oplus T_h$	$\pi_{\varphi *}L^{(1)}_{2,0}(h)/2$	$4h$
		$\langle -4d\rangle\oplus U^2\oplus D_4(2)\oplus\langle -4\rangle$	$\pi_{\iota *}L_0(d)$	$4d$
	$(D_4(2)\oplus\langle 4d\rangle)'\oplus\langle -4\rangle$	$\langle -4d\rangle\oplus U^2\oplus D_4(2)\oplus\langle -4\rangle$	$\pi_{\iota *}L_{4,-4}(h)/2$	$16h-4$
	$D_4(2)\oplus(\langle 4d\rangle\oplus\langle -4\rangle)'$	$U^2\oplus D_4(2)\oplus S_m$	$\pi_{\iota *}M_3(m)$	$4(4m-1)$

$$P_h = \begin{bmatrix} -4-4h & 2 & 2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 & 2 & -2 \\ 0 & 0 & 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & -2 & 2 & 0 & -4 \end{bmatrix} \quad Q_h = \begin{bmatrix} -4h & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -2 \end{bmatrix} \quad S_m = \begin{bmatrix} -4m & 2 \\ 2 & -4 \end{bmatrix}$$

$$R_m = \begin{bmatrix} -4m & 2 & 2 & 2 & -2 & 0 \\ 2 & -4 & 0 & 0 & 2 & 0 \\ 2 & 0 & -4 & 0 & 2 & 0 \\ 2 & 0 & 0 & -4 & 0 & 0 \\ -2 & 2 & 2 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix} \quad T_h = \begin{bmatrix} -4-4h & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -8 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 4 & -2 & 0 \\ 0 & 2 & 4 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix}$$

$$V_h = \begin{bmatrix} -4h & 2 & 0 & 0 & 0 \\ 2 & 8 & 4 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \quad W_h = \begin{bmatrix} -4-4h & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & -4 & 2 & -2 \\ 0 & 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & -4 \end{bmatrix}$$

$$F_m = \begin{bmatrix} -4m & 2 & 0 & 0 & 0 & 0 \\ 2 & -8 & 4 & 0 & -2 & 2 \\ 0 & 4 & -4 & 2 & 4 & 0 \\ 0 & 0 & 2 & -4 & -4 & 0 \\ 0 & -2 & 4 & -4 & -6 & 0 \\ 0 & 2 & 0 & 0 & 0 & -6 \end{bmatrix}$$

Remark 5.5.1.3. The classes $\pi_{\iota*}L_0(e)$ and $\pi_{\iota*}L^{(2)}_{2,0}(e+1)/2$ give the same projective family of Nikulin orbifolds; in other words, the projective families of K3$^{[2]}$-type manifolds associated to the polarizations $L_0(e)$ and $L^{(2)}_{2,0}(e+1)$ give Nikulin orbifolds that belong to the same projective family, independently of the involution in $(\mathbb{Z}/2\mathbb{Z})^2$ we choose when taking the quotient. These are the only families for which this phenomenon happens.

5.5.2 Nikulin-type orbifolds with a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$

We now describe Nikulin-type orbifolds with a particular symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$. The two natural involutions induced respectively by the action of $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ on a K3$^{[2]}$-type manifold X can be extended to Nikulin-type orbifolds by deformation (see Theorem 5.4.2.10). Moreover, we can consider the non-standard involution described in [52], which consists in the reflection on a class of square -2 and divisibility 2.

Theorem 5.5.2.1 ([52, Thm. 8.5])**.** *Let Y be a Nikulin-type orbifold such that there exists $D \in NS(Y)$ with $D^2 = -2$ and $div(D) = 2$. Then there exists an irreducible symplectic orbifold Z bimeromophic to Y and a non-standard symplectic involution κ on Z such that:*

$$H^2(Z,\mathbb{Z})^\kappa \simeq U(2)^{\oplus 3} \oplus E_8 \oplus \langle -2 \rangle, \quad \Omega_\kappa := (H^2(Z,\mathbb{Z})^\kappa)^\perp \simeq \langle -2 \rangle.$$

Remark 5.5.2.2. If Y is a Nikulin orbifold obtained as terminalization of a natural pair $(S^{[2]},\iota)$, then $NS(Y) = \langle -2 \rangle^2$: therefore κ exists on Y, and it acts exchanging the exceptional classes $\tilde{\Sigma}$ and $\pi_*\delta$ [60, Prop. 4.5].

Corollary 5.5.2.3. *Let Y be a Nikulin-type orbifold. Then Y admits a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \iota, \kappa \rangle$, where ι is standard and κ is the non-standard involution described in Theorem 5.5.2.1, if and only if it satisfies the following conditions:*

1. *there exists a primitive embedding of $D_6(2) \oplus \langle -2 \rangle$ or $D_4(2) \oplus \langle -2 \rangle$ in $NS(Y)$;*
2. *the primitive embedding of $NS(Y)$ in $H^2(Y,\mathbb{Z})$ satisfies condition ($\star$) (see Remark 5.4.2.9).*

Proof. The co-invariant lattice Ω_κ of the non-standard involution κ can always be embedded in the invariant lattice of the standard involutions in a way compatible with Lemma 5.4.2.8: use one of the orthogonal $\langle -2 \rangle$ components. With this choice, κ commutes with the standard involution on Y, whose existence is guaranteed by Theorem 5.4.2.10: therefore, we get a symplectic action of $(\mathbb{Z}/2\mathbb{Z})^2$ on Y. Moreover, this is the only valid choice: indeed, κ exists if and only if Ω_κ is embedded with divisibility 2 in $H^2(Y,\mathbb{Z})$, and all such embeddings are equivalent; if ι is either standard involution, we can always embed Ω_ι in $\Omega_\kappa^\perp$ such that Lemma 5.4.2.8 is satisfied. □

Theorem 5.5.2.4. *Let Y be a general projective Nikulin-type orbifolds with a symplectic action of $\mathfrak{K} = \langle \iota, \kappa \rangle$, where κ is the non-standard involution described in Theorem 5.5.2.1* *and (Y,ι) is a deformation of the natural pair $(\tilde{Y},\tilde{\iota})$, where $\tilde{Y}$ is the terminalization of $S^{[2]}/i$, S is a K3 surface with a symplectic action of $G=\mathbb{Z}/4\mathbb{Z}$ and $i\in G$ of order 2. Then Y belongs to one of the deformation families described in the following table.*

$NS(Y)$	e	$T(Y)$	k
$D_6(2)\oplus\langle 2e\rangle\oplus\langle -2\rangle$	$e=_2 0$	$\langle -2k\rangle\oplus U(2)^{\oplus 2}\oplus\langle -4\rangle^{\oplus 2}\oplus\langle -2\rangle$	e
	$e=_8 1$	$G_k\oplus U(2)^{\oplus 2}\oplus\langle -4\rangle^{\oplus 2}$ $R'_k\oplus\langle 4\rangle$	$(e+1)/2$ $(e-1)/8$
	$e=_8 3$	$G_k\oplus U(2)^{\oplus 2}\oplus\langle -4\rangle^{\oplus 2}$ $M'_k\oplus U(2)$ $Q'_k\oplus\langle -4\rangle\oplus\langle 4\rangle$	$(e+1)/2$ $(e+5)/8$ $(e+29)/8$
	$e=_8 5$	$G_k\oplus U(2)^{\oplus 2}\oplus\langle -4\rangle^{\oplus 2}$ $N'_k\oplus\langle -4\rangle$ $P'_k\oplus U(2)\oplus\langle -4\rangle$	$(e+1)/2$ $(e-5)/8$ $(e-5)/8$
	$e=_8 7$	$G_k\oplus U(2)^{\oplus 2}\oplus\langle -4\rangle^{\oplus 2}$ $S'_k\oplus U(2)\oplus\langle -4\rangle$	$(e+1)/2$ $(e+9)/8$
$(D_6(2)\oplus\langle 4e\rangle)'^{(1)}\oplus\langle -2\rangle$	$e=_4 0$ $e=_4 1$ $e=_4 2$ $e=_4 3$	$S_k\oplus U(2)\oplus\langle -4\rangle\oplus\langle -2\rangle$ $R_k\oplus\langle 4\rangle\oplus\langle -2\rangle$ $Q_k\oplus\langle -4\rangle\oplus\langle 4\rangle\oplus\langle -2\rangle$ $P_k\oplus U(2)\oplus\langle -4\rangle\oplus\langle -2\rangle$	$e/4+1$ $(e-1)/4$ $(e+14)/4$ $(e-3)/4$
$(D_6(2)\oplus\langle 4e\rangle)'^{(2)}\oplus\langle -2\rangle$	$e=_4 0$ $e=_4 1$ $e=_4 2$ $e=_4 3$	$M''_k\oplus U(2)$ $N''_k\oplus\langle -4\rangle$ $M_k\oplus U(2)\oplus\langle -2\rangle$ $N_k\oplus\langle -4\rangle\oplus\langle -2\rangle$	$e/4+1$ $(e-1)/4$ $(e+2)/4$ $(e-3)/4$

$$G_k=\begin{bmatrix}-4k & 2\\ 2 & -2\end{bmatrix}\quad S_k=\begin{bmatrix}-4k & 2 & 0 & 4\\ 2 & 0 & 4 & 0\\ 0 & 4 & 0 & 0\\ 4 & 0 & 0 & -4\end{bmatrix}\quad S'_k=\left[\begin{array}{c|cccc}-8 & 2 & 0 & 0 & 0\\ \hline 2 & & & & \\ 0 & & S_k & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$$R_k=\begin{bmatrix}-4k+4 & 2 & 0 & 0 & 2 & 0\\ 2 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & -4 & 0 & 2 & 0\\ 0 & 0 & 0 & -4 & 2 & 0\\ 2 & 2 & 2 & 2 & -4 & 2\\ 0 & 0 & 0 & 0 & 2 & -4\end{bmatrix}\quad R'_k=\left[\begin{array}{c|cccc}-8 & 2 & 0 & \dots & 0\\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & R_k & & \\ 0 & & & & \end{array}\right]$$

$$
Q_k = \begin{bmatrix} 8-4k & 2 & 2 & 2 & -2 \\ 2 & -4 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 & 0 \end{bmatrix} \quad Q'_k = \left[\begin{array}{c|c} -8 & 2 \quad 0 \quad \dots \quad 0 \\ \hline 2 & \\ 0 & \\ \vdots & Q_k \\ 0 & \end{array}\right]
$$

$$
P_k = \begin{bmatrix} -4-4k & 2 & 2 & 0 \\ 2 & -4 & 0 & 0 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad P'_k = \left[\begin{array}{c|c} -8 & 2 \quad 0 \quad 0 \quad 0 \\ \hline 2 & \\ 0 & \\ 0 & P_k \\ 0 & \end{array}\right]
$$

$$
M_k = \begin{bmatrix} 8-4k & 2 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 2 & 2 & -8 \end{bmatrix} \quad M'_k = \left[\begin{array}{c|c} -8 & 2 \quad 0 \quad \dots \quad 0 \\ \hline 2 & \\ 0 & \\ \vdots & M_k \\ 0 & \end{array}\right] \quad M''_k = \left[\begin{array}{c|c} -2 & 2 \quad 0 \quad \dots \quad 0 \\ \hline 2 & \\ 0 & \\ \vdots & M_k \\ 0 & \end{array}\right]
$$

$$
N_k = \begin{bmatrix} -4k & 2 & 2 & 2 & 2 & 0 \\ 2 & -4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{bmatrix} \quad N'_k = \left[\begin{array}{c|c} -8 & 2 \quad 0 \quad \dots \quad 0 \\ \hline 2 & \\ 0 & \\ \vdots & N_k \\ 0 & \end{array}\right] \quad N''_k = \left[\begin{array}{c|c} -2 & 2 \quad 0 \quad \dots \quad 0 \\ \hline 2 & \\ 0 & \\ \vdots & N_k \\ 0 & \end{array}\right]
$$

Proof. The embedding $\Omega_\kappa \hookrightarrow \Lambda_{\mathrm{N}}$ is always such that $\Omega_\kappa^\perp \simeq \pi_{S*}H^2(S,\mathbb{Z}) \oplus \langle -2 \rangle$, where S is a K3 surface with a symplectic involution i, and $\pi_S : S \to S/i$. Call α the generator of $(\pi_{S*}H^2(S,\mathbb{Z}) \oplus \Omega_\kappa)^\perp$, and embed Ω_ι in $\Omega_\kappa^\perp$ such that the condition $(\star)$ is satisfied. Denote $\Omega = \Omega_\iota \oplus \Omega_\kappa$.
Suppose now that $i = \tau^2 \in \langle \tau \rangle = \mathbb{Z}/4\mathbb{Z}$, that acts symplectically on S. We remark that the gluings in Lemma 5.4.2.8.1 are exactly the image via π_{S*} of the ones that define $H^2(S,\mathbb{Z})$ as overlattice of finite index of $\Omega_4 \oplus \Lambda_{\mathrm{K3}}^\tau$. Therefore, we can obtain all the projective families of Y using as generator of $\Omega^{\perp_{NS(Y)}}$ an element of the form $\pi_{S*}L + n\alpha$, with L an ample class on S and $n = 0,1,2$. This bound on n is given by the condition $(\star)$, that allows overlattices of $\Omega \oplus \langle 2d \rangle$ of index at most 2: indeed, the classes of isomorphic overlattices of index 2 vary with the value of d (mod 4).
If $n = 0$ we find all the projective families with $NS(Y) = \pi_{S*}NS(S) \oplus \langle -2 \rangle$, $T(S) = \pi_{S*}T(S) \oplus \langle -2 \rangle$; if $n = 2$, since $(\pi_{S*}L + 2\alpha)^2 = (\pi_{S*}L)^2 - 8$, but $\pi_{S*}L + 2\alpha$ glues to the same isometry class of A_Ω as $\pi_{S*}L$, we can find new projective families; lastly, if $n = 1$ we find all the projective families with $NS(Y) = \Omega \oplus \langle 2e \rangle$ for e odd. □

Theorem 5.5.2.5. *Let Y be a general projective Nikulin-type orbifolds with a symplectic action of $\mathfrak{K} = \langle \iota, \kappa \rangle$, where κ is the non-standard involution described in Theorem 5.5.2.1 and (Y,ι) is a deformation of the natural pair $(\tilde{Y}, \tilde{\iota})$, where $\tilde{Y}$ is the terminalization of $S^{[2]}/i$, S is a K3 surface with a symplectic action of $G = (\mathbb{Z}/2\mathbb{Z})^2$ and $i \in G$ of order 2. Then Y belongs to one of the deformation families described in the following table.*

$NS(Y)$	e	$T(Y)$	k
$D_4(2)\oplus\langle 2e\rangle\oplus\langle -2\rangle$	$e =_2 0$	$\langle -4k\rangle\oplus\langle -2\rangle\oplus U^{\oplus 2}\oplus D_4(2)$	$e/2$
		$M_k''\oplus U\oplus D_4(2)$	$e/2$
	$e =_8 1$	$G_k\oplus U^{\oplus 2}\oplus D_4(2)$	$(e+1)/2$
		$M_k\oplus\langle -2\rangle\oplus U\oplus D_4(2)$	$(e-1)/2$
		$R_k'\oplus U$	$(e-1)/8$
	$e =_8 3$	$G_k\oplus U^{\oplus 2}\oplus D_4(2)$	$(e+1)/2$
		$M_k\oplus\langle -2\rangle\oplus U\oplus D_4(2)$	$(e-1)/2$
		$N_k'\oplus U\oplus U(2)$	$(e-3)/8$
		$S_k'\oplus D_4$	$(e-3)/8$
	$e =_8 5$	$G_k\oplus U^{\oplus 2}\oplus D_4(2)$	$(e+1)/2$
		$M_k\oplus\langle -2\rangle\oplus U\oplus D_4(2)$	$(e-1)/2$
		$Q_k'\oplus U^{\oplus 2}$	$(e+3)/8$
	$e =_8 7$	$G_k\oplus U^{\oplus 2}\oplus D_4(2)$	$(e+1)/2$
		$M_k\oplus\langle -2\rangle\oplus U\oplus D_4(2)$	$(e-1)/2$
$(D_4(2)\oplus\langle 4e\rangle)'\oplus\langle -2\rangle$	$e =_4 0$	$P_k\oplus\langle -2\rangle\oplus U$	$e/4$
		$S_k''\oplus D_4$	$e/4$
	$e =_4 1$	$R_k\oplus\langle -2\rangle\oplus U$	$(e-1)/4$
	$e =_4 2$	$N_k\oplus\langle -2\rangle\oplus U\oplus U(2)$	$(e-2)/4$
		$S_k\oplus\langle -2\rangle\oplus D_4$	$(e-2)/4$
	$e =_4 3$	$Q_k\oplus\langle -2\rangle\oplus U^{\oplus 2}$	$(e+1)/4$

$$G_k = \begin{bmatrix} -4k & 2 \\ 2 & -2 \end{bmatrix} \quad M_k = \begin{bmatrix} -4k & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -2 \end{bmatrix} \quad M_k'' = \left[\begin{array}{c|ccc} -2 & 2 & 0 & 0 \\ \hline 2 & & & \\ 0 & & M_k & \\ 0 & & & \end{array}\right]$$

$$N_k = \begin{bmatrix} -4-4k & 2 & 0 & 0 & 0 \\ 2 & -4 & 0 & 2 & -2 \\ 0 & 0 & -4 & 2 & 2 \\ 0 & 2 & 2 & -4 & 0 \\ 0 & -2 & 2 & 0 & -4 \end{bmatrix} \quad N_k' = \left[\begin{array}{c|cccc} -8 & 2 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & N_k & & \\ 0 & & & & \end{array}\right]$$

$$P_k = \begin{bmatrix} -4-4k & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & -4 & 2 & -2 \\ 0 & 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & -4 \end{bmatrix}$$

$$Q_k = \begin{bmatrix} -4k & 2 & 0 & 0 & 0 \\ 2 & -16 & 4 & 4 & 4 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & 4 & 0 & 0 & -4 \end{bmatrix} \quad Q'_k = \left[\begin{array}{c|cccc} -8 & 2 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & Q_k & & \\ 0 & & & & \end{array}\right]$$

$$R_k = \begin{bmatrix} -4k & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & -4 & 4 \\ 0 & 0 & 0 & 0 & -4 & -8 & 4 \\ 0 & 0 & 0 & 0 & 4 & 4 & -8 \end{bmatrix} \quad R'_k = \left[\begin{array}{c|cccc} -8 & 2 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & R_k & & \\ 0 & & & & \end{array}\right]$$

$$S_k = \begin{bmatrix} -4k & 2 & 0 & 0 & 0 \\ 2 & 8 & 4 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \quad S'_k = \left[\begin{array}{c|cccc} -8 & 2 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & S_k & & \\ 0 & & & & \end{array}\right] \quad S''_k = \left[\begin{array}{c|cccc} -2 & 2 & 0 & \dots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & S_k & & \\ 0 & & & & \end{array}\right]$$

Proof. The proof is similar to that of Theorem 5.5.2.4. □

Chapter 6

Transcendental lattices of the known IHS manifolds

6.1 Introduction

The contents in this chapter come from a paper written jointly with Ángel David Ríos Ortiz [83]. Our aim is to study the connection between complex algebraic manifolds sharing the same transcendental lattice. Indeed, it was already known by Beauville [6] that $S^{[n]}$, the Hilbert scheme of n points on a K3 surface S, shares the same transcendental lattice with S, and the same happens for $\mathrm{Km}_n(A)$ and the abelian surface A itself. With a few exceptions, all known constructions of IHS manifolds are actually obtained as some moduli space on either a symplectic surface (abelian or K3) or a cubic fourfold: hence, the transcendental part of their Hodge structure will be completely determined by the symplectic surface or cubic fourfold.

The philosophy behind this work is that two (projective) IHS manifolds that share the same transcendental lattices are tightly related to each other, even if they are *not* of the same deformation type, and sometimes even if one of them is not a IHS manifold but has a Hodge structure that *resembles* one.
With this philosophy, in this work we will investigate the following question: given an IHS manifold belonging to one of the known deformation types for which the transcendental lattice is that of a symplectic surface (we call them *induced*, see Definition 6.2.0.1), is this manifold birational to a moduli space over said surface?

In Section 3 we positively answer such a question for Beauville's examples [6], the first discovered examples of IHS manifolds. We prove that a IHS manifold of Km_n-type X is induced by an abelian surface A if and only if X is birational to a moduli space on A, and an analogous result holds if X is a $\mathrm{K3}^{[n]}$-type manifold and is induced by a projective K3 surface S (the latter case was already essentially proved by Markman in [46]).
In Section 4 we study O'Grady's examples [76, 75], where we can actually see new

phenomena appearing: transcendental data are not enough to determine whether a manifold will be a (desingularized) moduli space, but we have to include *algebraic* data; this is manifestly shown in the cohomology of these manifolds, and to explore these cases lattice theory is needed. We give lattice-theoretic criteria and construct many examples of Hodge structures of manifolds in O'Grady's families which are induced by symplectic surfaces but are *not* moduli spaces. We highlight the very different behavior of induced IHS manifolds which are or aren't moduli spaces, and we point out connections with the non-modular construction of O'Grady's 10 dimensional example due to Laza-Saccà-Voisin.

6.2 Induced IHS manifolds

Definition 6.2.0.1. Let X be a projective IHS manifold and (T, q) be a Hodge structure of K3-type with a Beauville-Bogomolov form (see Definition 1.3.0.1). We say that X is *induced* by T if there exists a Hodge isometry $T(X) \cong_{\mathrm{Hdg}} T$.

Definition 6.2.0.2. Let X be a projective IHS manifold belonging to one of the known deformation families: we say that X is induced by an abelian or K3 surface S if there exists a Hodge isometry $T(X) \cong_{\mathrm{Hdg}} T(S)$.

By results due to Orlov [78, Thm. 3.3] a projective K3 surface which is induced by another K3 surface is derived equivalent to the latter (that is, their derived categories are isomorphic). Mukai showed that a K3 surface which is induced by another K3 surface is a moduli space over the latter [65]. This gives a very precise interpretation of induced IHS manifolds in dimension 2.

Theorem 6.2.0.3 (Orlov, Mukai)**.** *Let S and S' be two projective K3 surfaces. The following are equivalent:*

1. *S is induced by S'.*
2. *S and S' are derived equivalent.*
3. *S is isomorphic to a fine two-dimensional moduli space over S'.*

Remark 6.2.0.4. In the literature, two projectice K3 surfaces S and S' are said to be *Fourier-Mukai partners* if any of the equivalent statements of Theorem 6.2.0.3 hold for S and S'.

Let S be a K3 surface, then the set of K3 surfaces induced by S modulo isomorphism is finite [41, Prop. 16.3.10]. We can generalize this result to the case of induced IHS manifolds.

Proposition 6.2.0.5. *Let (T, q) be a fixed Hodge structure of K3-type, and fix a lattice Λ; then the set*

$$\{X \text{ IHS with } H^2(X,\mathbb{Z}) = \Lambda,\ X \text{ induced by } (T,q)\}/\sim_{bir}$$

is finite.

Proof. Let X be a IHS manifold induced by (T, q) and fix a marking $H^2(X, \mathbb{Z}) \simeq \Lambda$, where Λ is the abstract lattice of the deformation type of X (see Example 1.2.0.3). Denote by $\mathrm{Mon}(X)$ the monodromy group of X. By [93, Thm. 3.4] this is a finite index subgroup of $O(\Lambda)$. Consider the set $\mathrm{Emb}(T(X), \Lambda)$ of primitive embeddings of abstract lattices. The set of orbits of $\mathrm{Emb}(T(X), \Lambda)$ under the action of $\mathrm{Mon}(X)$ is finite. Under a fixed orbit, the lattice $T(X) \oplus NS(X)$ is a sublattice of $H^2(X, \mathbb{Z})$ of finite index: this implies that there exists finitely many ways of extending the Hodge structure on $T(X)$ to Λ, and any such determines X up to bimeromorphisms by Theorem 1.3.0.11. $\square$

Corollary 6.2.0.6. *Let (T, q) be a fixed Hodge structure of K3-type, and fix a lattice Λ of rank at least 5. Then the set*

$$\{X \text{ IHS with } H^2(X, \mathbb{Z}) \cong \Lambda \text{ and induced by } (T, q)\}/\cong_{iso}$$

is finite.

Proof. With the assumption on the rank of Λ, the positive solution by Amerik and Verbitsky to the Kawamata-Morrison Cone Conjecture for IHS manifolds proves that there are a finite number of bimeromorphic models of a given X (see [1, Thm. 1.9]): this, combined with Proposition 6.2.0.5, gives the stated result. $\square$

A result analogous to Theorem 6.2.0.3 for IHS manifolds is at the moment out of reach: however, if we restrict to those belonging to the known deformation types, and which are moreover induced by a symplectic (abelian or K3) surface, some result can be indeed obtained. This will be done in the following sections.

6.3 Induced IHSs in Beauville's deformation families

6.3.1 Moduli of sheaves on symplectic surfaces

In the following, S will be a projective K3 or abelian surface. Let $\tilde{H}(S)$ denote the even cohomology ring, i.e.

$$\tilde{H}(S) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}). \tag{6.3.1.1}$$

We define a pure weight-two Hodge structure on $\tilde{H}(S) \otimes \mathbb{C}$ by requiring the degree 0 and 4 parts to be algebraic:

$$\tilde{H}^{0,2}(S) := H^{0,2}(S), \quad \tilde{H}^{2,0}(S) := H^{2,0}(S), \quad \tilde{H}^{1,1}(S) := H^0(S) \oplus H^{1,1}(S) \oplus H^4(S).$$

For any $v = (v_0, v_2, v_4) \in \tilde{H}(S)$, with degree q component given by v_q, we set $v^\vee := (v_0, -v_2, v_4)$. On $\tilde{H}(S)$ we define *Mukai's bilinear symmetric form* by

$$\langle u, w\rangle := -\int_S u \wedge w^\vee = \int_S u_2 \wedge w_2 - \int_S (u_0 \wedge w_4 + u_4 \wedge w_0). \tag{6.3.1.2}$$

Definition 6.3.1.1. The *Mukai lattice* of S is the free module $\tilde{H}(S)$ with Mukai's bilinear symmetric form $\langle\cdot,\cdot\rangle$. An element $v = (v_0, v_2, v_4) \in \tilde{H}(S)$ is called a *Mukai vector* if $v_0 \geq 0$ and $v_2 \in NS(S)$.

As a lattice, $\tilde{H}(S)$ is isometric to the *abstract Mukai lattice*

$$\widetilde{\Lambda_S} := \begin{cases} \widetilde{\Lambda_{\mathrm{K3}}} = \Lambda_{\mathrm{K3}} \oplus U & \text{if } S \text{ is a K3 surface} \\ \widetilde{\Lambda_{\mathrm{Ab}}} = \Lambda_{\mathrm{Ab}} \oplus U & \text{if } S \text{ is an abelian surface} \end{cases} \tag{6.3.1.3}$$

where $\Lambda_{\mathrm{K3}} = E_8^{\oplus 2} \oplus U^{\oplus 3}$ and $\Lambda_{\mathrm{Ab}} := U^{\oplus 3}$; notice that $\tilde{H}(S)$ is a Hodge structure of K3-type.

Let $\mathscr{F}$ be a coherent sheaf on S. Define the *Mukai vector* of $\mathscr{F}$ to be

$$v(\mathscr{F}) := \mathrm{ch}(\mathscr{F})\sqrt{\mathrm{td}(S)} = (rk(\mathscr{F}), c_1(\mathscr{F}), \frac{1}{2}(c_1(\mathscr{F})^2 - 2c_2(\mathscr{F})) + \varepsilon rk(\mathscr{F})), \tag{6.3.1.4}$$

where the last equality is Hirzebruch-Riemann-Roch Theorem, with $\varepsilon = 1$ if S is K3, and $\varepsilon = 0$ if S is abelian. Notice that the Mukai vector of a coherent sheaf is indeed a Mukai vector in the sense of Definition 6.3.1.1.

Let H be a polarization and v a Mukai vector on S. We write $M_v(S, H)$ (resp. $M_v^s(S, H)$) for the moduli space of H-semistable (resp. H-stable) sheaves on S with Mukai vector v. If S is abelian, a further construction is necessary: we define $K_v(S, H) := \mathrm{Alb}^{-1}(0)$, where Alb is the Albanese morphism, cf. [99].

Example 6.3.1.2. Let H be an ample divisor on S and let $v := (1, 0, 1 - n)$. Then $M_v(S, H) = S^{[n]}$. If S is abelian then $K_v(S, H) = \mathrm{Km}_n(S)$. This construction induces an embedding $\iota : H^2(S, \mathbb{Z}) \to \tilde{H}(S)$ with $v^\perp = H^2(X, \mathbb{Z})$, where X is either $S^{[n]}$ (when S is a K3 surface) or $\mathrm{Km}_n(S)$ (when S is an abelian surface).

The moduli spaces constructed above can be singular for two reasons: either the Mukai vector is non-primitive, or the polarization is not *v-generic*.

Definition 6.3.1.3. Fix a Mukai vector $v \in \tilde{H}(S)$ and let $\mathrm{Amp}_{\mathbb{R}}(S)$ be the ample cone of S. A *v-wall* is a hyperplane defined by $W_D := D^\perp \cap \mathrm{Amp}_{\mathbb{R}}(S)$ where $D \in NS(S)$ satisfies

$$\frac{v_0^2}{4}(2v_0v_4 - (v_0 - 1)v_2 \cdot v_2) < D \cdot D < 0. \tag{6.3.1.5}$$

A polarization H is called *v-generic* if H is not contained in any v-wall.

Remark 6.3.1.4. For any chosen v there exists a locally finite union of hyperplanes in $NS(S) \otimes \mathbb{R}$, outside of which any polarization is v-generic.

The following theorem is the final result of works by several authors: at first, Mukai proved that the moduli space of simple sheaves on an abelian or K3 surface, of which $M_v(S, H)$ is a compact subscheme (assuming that v is primitive and H is v-generic), is smooth and admits a non-degenerate holomorphic 2-form [64]. O'Grady [74] proved the relation between $M_v(S,H)$ and $S^{[n]}$ when S is a K3 surface, and Yoshioka [99] did the same for $K_v(S,H)$ and $\mathrm{Km}_n(S)$ if S is abelian.

Theorem 6.3.1.5. *Let S be an abelian or projective K3 surface, v a primitive Mukai vector and H a v-generic polarization. Then $M_v(S,H) = M_v^s(S,H)$, and we have the following results:*

1. *if S is a K3 surface and $\langle v,v\rangle \geq 2$, then $M_v(S,H)$ is an IHS manifold of dimension $2n = \langle v,v\rangle + 2$, which is deformation equivalent to $S^{[n]}$, the Hilbert scheme of n points on S. Moreover, there is a Hodge isometry between $v^\perp$ and $H^2(M_v(S,H),\mathbb{Z})$, where the latter has a lattice structure given by the Beauville-Bogomolov form;*

2. *if S is abelian and $\langle v,v\rangle \geq 6$, then $K_v(S,H)$ is an IHS manifold of dimension $2n = \langle v,v\rangle - 2$, which is deformation equivalent to $Km_n(S)$, the generalized Kummer manifold on S. Moreover, there is a Hodge isometry between $v^\perp$ and $H^2(K_v(S,H),\mathbb{Z})$.*

In particular, $T((M_v(S,H)) \cong_{\mathrm{Hdg}} T(S)$ and $T((K_v(S,H)) \cong_{\mathrm{Hdg}} T(S)$ as pure Hodge structures of weight 2.

6.3.2 Induced IHSs of K3$^{[n]}$-type

Let X be a projective IHS manifold of K3$^{[n]}$-type and let $\Lambda_{\mathrm{K3}^{[n]}}$ be the abstract K3$^{[n]}$ lattice (see Example 1.2.0.3). Markman constructed in [46] a natural $O(\widetilde{\Lambda_{\mathrm{K3}}})$-orbit i_X of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda_{\mathrm{K3}}}$. This allowed him to prove the following.

Theorem 6.3.2.1 ([47, Cor. 9.9])**.** *Let X and Y be two manifolds of K3$^{[n]}$-type. Then X and Y are bimeromorphic if and only if there exists a Hodge-isometry $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, satisfying $i_X = i_Y \circ f$.*

In particular, this gives a criterion to check if a IHS manifold of K3$^{[n]}$-type is bimeromorphic to a moduli space of sheaves on a K3 surface, which we will apply to the IHSs of K3$^{[n]}$-type that are induced by a K3 surface.

Denote by $\mathrm{Emb}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda_{\mathrm{K3}}})$ the set of isometric embeddings of the lattice $\Lambda_{\mathrm{K3}^{[n]}}$ in $\widetilde{\Lambda_{\mathrm{K3}}}$. In [46, Lemma 4.3.(i)] Markman establishes a bijective correspondence between the set

$$P_n = \{(r,s) \in \mathbb{Z}^2 \text{ coprime such that } -s \geq r > 0 \text{ and } -rs = n-1\}$$

and the set of $O(\widetilde{\Lambda_{\mathrm{K3}}})$-orbits in $\mathrm{Emb}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda_{\mathrm{K3}}})$. This correspondence is given by assigning to each pair (r,s) the embedding $(r,0,s)^\perp$ in $\widetilde{\Lambda_{\mathrm{K3}}}$. The following theorem is essentially proved in [46], for convenience we give the proof.

Theorem 6.3.2.2. *Let X be a projective IHS of $K3^{[n]}$-type and S a projective K3 surface, then X is induced by S if and only if X is bimeromorphic to a moduli space $M_v(S,H)$ for some $v \in \tilde{H}(S)$ and a v-generic polarization H.*

Proof. To prove the implication from right to left notice that by Theorem 6.3.1.5 for any Mukai vector v and v-generic polarization H on S we have that $T(M_v(S,H)) \cong_{\mathrm{Hdg}} T(S)$. By Lemma 1.3.0.9, if X is bimeromorphic to $M_v(S,H)$, then $T(X) \cong_{\mathrm{Hdg}} T(M_v(S,H))$ and we are done.
For the other implication, suppose that $T(X) \cong T(S)$ and consider the embedding

$$\iota : H^2(S^{[n]}, \mathbb{Z}) \to \tilde{H}(S)$$

given as $(1,0,1-n)^{\perp}$ as in Example 6.3.1.2. By construction we have that $T(S)^{\perp}$ in $\tilde{H}(S)$ contains a copy of U, that is $H^0(S) \oplus H^4(S)$, so by Theorem 1.2.1.10 the embedding $\iota|_{T(S)}$ is unique up to isometries; therefore, if we consider also $\iota_X : H^2(X,\mathbb{Z}) \to \widetilde{\Lambda}$, then there exists an isometry $\phi : \widetilde{\Lambda} \to \tilde{H}(S)$ such that $\phi(\iota_X(T(X))) = T(S)$.
Set $\Lambda = \phi(H^2(X,\mathbb{Z}))$ and let $v := \Lambda^{\perp}$: this is a primitive vector that takes values in $\tilde{H}(S)_{\mathrm{alg}}$. Possibly by changing sign, we can assume that v is a Mukai vector, hence by Theorem 6.3.1.5 for any v-generic ample class H in S we have $v^{\perp} \cong H^2(M_v(S,H),\mathbb{Z})$ as Hodge structures: this means that ϕ restricted to $H^2(X,\mathbb{Z})$ is in fact an isometry of Hodge structures that, by construction, extends to the Mukai lattice. By Theorem 6.3.2.1 we conclude that X and $M_v(S,H)$ are bimeromorphic. □

Remark 6.3.2.3. In the previous section we claimed that X *is* a moduli space and not just bimeromorphic to one. Using [5, Thm. 1.2], when X is bimeromorphic to a moduli space of semistable sheaves, then there exists a Bridgeland stability condition σ such that X is *isomorphic* to the moduli space of σ-stable sheaves.

Corollary 6.3.2.4. *Every IHS manifold of $K3^{[n]}$-type X with Picard rank $\rho \geq 13$ is induced by a unique K3 surface.*

Proof. By [63, Cor. 2.10] the assumption on the Picard rank guarantees that for every lattice T of signature $(2, 20-\rho)$ there exists a unique primitive embedding $T \hookrightarrow \Lambda_{\mathrm{K3}}$. If T has a Hodge structure of K3-type, we can extend it uniquely to a Hodge structure on Λ_{K3}, imposing $T^{\perp} \subset \Lambda^{1,1}_{\mathrm{K3}}$. By Theorem 1.3.0.8, this allows us (having fixed the marking $H^2(S,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$) to find a unique K3 S such that $T(X) \cong_{\mathrm{Hdg}} T(S)$: the result then follows by Theorem 6.3.2.2. □

6.3.3 Induced IHSs of Km_n-type

The strategy of proof given in the previous section has an analogue for the Km_n deformation type. Let X be a projective IHS manifold of Km_n-type and let Λ_{Km_n} be the abstract Km_n lattice (see Example 1.2.0.3).

Theorem 6.3.3.1 ([59, Thm. 4.3]). *Let $\mathcal{W}(\Lambda_{\mathrm{Km}_n})$ denote the subgroup of $O^+(\Lambda_{\mathrm{Km}_n})$ consisting of orientation preserving isometries acting as ± 1 on the discriminant group $\Lambda^*_{\mathrm{Km}_n}/\Lambda_{\mathrm{Km}_n}$. Denote by*

$$\chi : \mathcal{W}(\Lambda_{\mathrm{Km}_n}) \to \{1, -1\}$$

the associated character. Then $\mathrm{Mon}^2(X)$ consists precisely of orientation preserving isometries $g \in \mathcal{W}(\Lambda_{\mathrm{Km}_n})$ such that $\chi(g) \cdot \det(g) = 1$.

We need also an analogue of the monodromy invariant embedding due to Markman. Recall the following result by Wieneck.

Theorem 6.3.3.2 ([97, Thm. 4.9]). *Let X be a IHS manifold of Km_n-type, $n \geq 2$. Then there exists a canonical monodromy invariant $O(\widetilde{\Lambda_{\mathrm{Ab}}})$-orbit ι_X of primitive isometric embeddings of $\Lambda = H^2(X, \mathbb{Z})$ in the Mukai lattice $\widetilde{\Lambda_{\mathrm{Ab}}}$.*

Lemma 6.3.3.3. *There exists a bijective correspondence between the set*

$$Q_n = \{(r, s) \text{ coprime such that } -s \geq r > 0 \text{ and } rs = n + 1\}$$

and the set of $O(\widetilde{\Lambda_{\mathrm{Ab}}})$ orbits in $O(\Lambda_{\mathrm{Km}_n}, \widetilde{\Lambda_{\mathrm{Ab}}})$.

Proof. Recall that $\Lambda_{\mathrm{Km}_n} \simeq \Lambda_{\mathrm{Ab}} \oplus \langle -2(n+1)\rangle$. For every $(r, s) \in Q_n$ define the embedding $\iota_{r,s} : \Lambda_{\mathrm{Km}_n} \to \widetilde{\Lambda_{\mathrm{Ab}}}$ by sending Λ_{Ab} to $\Lambda_{\mathrm{Ab}} \oplus 0$ and the generator of $\langle -2(n + 1)\rangle = (\Lambda_{\mathrm{Ab}})^{\perp_{\Lambda_{\mathrm{Km}_n}}}$ to $(r, 0, -s)$. Notice that $\iota_{r,s}(\Lambda_{\mathrm{Km}_n}) = (r, 0, s)^{\perp}$. Suppose that there exists an isometry $g \in O(\widetilde{\Lambda_{\mathrm{Ab}}})$ such that $\iota_{r,s} = \iota_{r',s'} \circ g$: then g must fix $\Lambda_{\mathrm{Ab}} \oplus 0$ and send $(r, 0, -s)$ to $(r', 0, -s')$. Since both vectors are contained in $0 \oplus U$, then g has to be given by an isometry of $U = \langle u, v\rangle$, but $O(U) \cong (\mathbb{Z}/2\mathbb{Z})^2$, generated by the isometries $(u, v) \mapsto (-u, -v)$ and $(u, v) \mapsto (v, u)$: therefore g must be the identity (otherwise the conditions on r, s are not met).
Conversely, let $\phi : \Lambda_{\mathrm{Km}_n} \to \widetilde{\Lambda_{\mathrm{Ab}}}$ be an embedding. By [72, Thm. 1.14.4] there is a unique $O(\widetilde{\Lambda_{\mathrm{Ab}}})$-orbit of embeddings of $U^{\oplus 3}$ into $\widetilde{\Lambda_{\mathrm{Ab}}}$, hence there exists an isometry $g' \in O(\widetilde{\Lambda_{\mathrm{Ab}}})$ such that $\phi(U^{\oplus 3})$ is sent to $U^{\oplus 3} \oplus 0$. Let $v \in \Lambda_{\mathrm{Km}_n}$ be a vector generating the sublattice $\langle -2(n + 1)\rangle = (U^{\oplus 3})^{\perp}$; since $\widetilde{\Lambda_{\mathrm{Ab}}} = U^{\oplus 3} \oplus U$, we reduce the problem to classify the possible embeddings of v into U; since $O(U) \cong (\mathbb{Z}/2\mathbb{Z})^2$ we have that these embeddings are classified as pairs in the set Q_n. □

Lemma 6.3.3.4 ([91, Lemma 3]). *Let A be an abelian surface. There exists a canonical isomorphism of Hodge structures $g : H^2(A^\vee, \mathbb{Z}) \to H^2(A, \mathbb{Z})$ such that $\det(g) = -1$.*

Remark 6.3.3.5. By definition of the Mukai lattice, Lemma 6.3.3.4 implies in particular that there is a Hodge isometry $\tilde{g} : \tilde{H}(A) \cong_{\mathrm{Hdg}} \tilde{H}(A^\vee)$.

Lemma 6.3.3.6. *Let A be an abelian surface and fix a Mukai vector $v \in \tilde{H}(A)$; let $\tilde{v} = \tilde{g}(v) \in \tilde{H}(A^\vee)$; then there exists a Hodge isometry $\phi : H^2(K_v(A^\vee), \mathbb{Z}) \to H^2(K_v(A), \mathbb{Z})$ for v-generic polarizations. Moreover, with respect to the Mukai embedding we have $\det(\phi) = -1$ and $\chi(\phi) = 1$.*

Proof. By Lemma 6.3.3.3 we can assume that $v = (r, 0, s)$ for some r, s. Then, by Theorem 6.3.1.5 we know that

$$H^2(K_v(A^\vee), \mathbb{Z}) \cong_{\mathrm{Hdg}} v^\perp \cong_{\mathrm{Hdg}} H^2(K_v(A), \mathbb{Z})$$

as Hodge structures; observe that due to the specific choice of v we can assume the previous isomorphism to be given as g on $H^2(A, \mathbb{Z})$ and fixing $(r, 0, -s)$. This is a Hodge isometry by construction and $\det(\phi) = -1$; observe that since this restricts to the identity on $(r, 0, -s)$, then $\chi(\phi) = 1$ as claimed. □

Theorem 6.3.3.7. *Let X be a IHS manifold of Km_n-type, then X is induced by A if and only if X is bimeromorphic to $K_v(A, H)$ or $K_{\tilde{v}}(A^\vee, H')$ for some $v \in \tilde{H}(A)$, $\tilde{v} = \tilde{g}(v)$ (see Remark 6.3.3.5), H a v-generic polarization, H' a $\tilde{v}$-generic polarization.*

Proof. By Lemma 1.3.0.9 we just need to prove the implication from left to right. Suppose that $T(X) \cong T(A)$ and consider the standard embedding $\iota : H^2(\mathrm{Km}_n(A), \mathbb{Z}) \to \tilde{H}(A)$ given as $(1, 0, 1-n)^\perp$. By construction, we have that $T(A)^\perp$ in $\tilde{H}(A)$ contains a copy of U, that is $H^0(A) \oplus H^4(A)$, so by Theorem 1.2.1.10 the embedding $\iota|_T(A)$ is unique; hence there exists an isometry $\varphi : \widetilde{\Lambda} \to \tilde{H}(A)$ such that $\varphi(\iota_X(T(X))) = T(A)$. Set $\Lambda = \varphi(H^2(X, \mathbb{Z}))$ and $v = \Lambda^\perp$: this is a primitive vector that takes values in $\widetilde{H}(S)_{\mathrm{alg}}$. Possibly by changing sign, we can assume that v is a Mukai vector; by Theorem 6.3.1.5 for any v-generic ample class H in S we have $v^\perp \cong_{\mathrm{Hdg}} H^2(M_v(S, H), \mathbb{Z})$. This means that φ restricted to $H^2(X, \mathbb{Z})$ is in fact an isometry of Hodge structures that, by construction, extends to the Mukai lattice. We already know that $\varphi \in \mathcal{W}(\Lambda)$ (cf. the previous Remark), by Theorem 6.3.3.1 what we are left to show is that $\chi(\varphi)\det(\varphi) = 1$. If not, by Lemma 6.3.3.6 there exists a Hodge isometry $\psi : H^2(K_v(A^\vee), \mathbb{Z}) \to H^2(K_v(A), \mathbb{Z})$ such that $\det(\psi) = -1$ and $\chi(\psi) = 1$, and composing with ψ we obtain the required equality. □

We provide a result analogous to Corollary 6.3.2.4 for this deformation type.

Corollary 6.3.3.8. *Every IHS manifold of Km_n-type with Picard rank ≥ 4 is induced by a unique abelian surface or its dual.*

Proof. Let X be a IHS manifold of Km_n-type and of Picard rank at least 4. Then its transcendental lattice $T(X)$ is of rank at most 3, therefore it occurs as the transcendental lattice of an abelian surface A by [63, Cor. 2.6]; moreover, the embedding $T(X) \subseteq U^{\oplus 3}$ is unique, therefore it induces a unique Hodge structure on $T(A) \simeq T(X)$: by Theorem 1.3.0.14, the same Hodge structure is shared by two abelian surfaces, A or $A^\vee$. By Lemma 6.3.3.4 we get that X is induced by either of them. □

Remark 6.3.3.9. An abelian surface A is isomorphic to its dual if and only if it is principally polarized, i.e. it is polarized with a class $H \in NS(A)$ such that $H^2 = 2$. This happens when A is isomorphic either to the Jacobian of a genus 2 curve or the product of two elliptic curves [11, Cor. 11.8.2].

6.4 Induced IHSs in O'Grady's deformation families

6.4.1 Singular moduli of sheaves on symplectic surfaces

Let S be a projective K3 or abelian surface. We will be using the same notation as in Section 6.3.1. Moduli spaces of stable sheaves with *non-primitive* Mukai vector are singular. Through the works of O'Grady [74, 76] Lehn and Sorger [45] and its final form by Perego and Rapagnetta in [80, 81], we have a clear understanding on when these spaces are resolvable symplectic varieties (see Definition 1.5.0.8); in all the other cases, they have terminal singularities.

Theorem 6.4.1.1. *Let S be an abelian or projective K3 surface, v a primitive Mukai vector such that $\langle v, v\rangle = 2$ and H a v-generic polarization. Then $M_{2v}(S,H)$ (resp. $K_{2v}(S,H)$ if S is abelian) is a resolvable symplectic variety and we have the following results:*

1. *if S is a K3 surface, then the symplectic resolution $\pi : \widetilde{M_{2v}}(S,H) \to M_{2v}(S,H)$ is a IHS manifold of OG10-type. There is a Hodge isometry between $v^\perp$ and $H^2(M_v(S,H),\mathbb{Z})$, where the latter has a lattice structure given by Lemma 1.5.0.10. Moreover, if $\alpha \in v^\perp$ has divisibility 2, then $\frac{\alpha\pm\sigma}{2} \in H^2(\widetilde{M_{2v}}(S,H),\mathbb{Z})$, where σ is the exceptional divisor of π.*
2. *if S is abelian, then the symplectic resolution $\pi : \widetilde{K_{2v}}(S,H) \to K_{2v}(S,H)$ is a IHS manifold of OG6-type, and there is a Hodge isometry between $v^\perp$ and $H^2(K_{2v}(S,H),\mathbb{Z})$. Moreover, $H^2(\widetilde{K}_{2v}(S,H),\mathbb{Z}) \cong_{\mathrm{Hdg}} H^2(K_{2v}(S,H),\mathbb{Z}) \oplus_\perp \mathbb{Z}\sigma$, where σ is the exceptional divisor of π.*

In particular, there are isomorphisms $T(\widetilde{M_{2v}}(S,H)) \cong_{\mathrm{Hdg}} T(S)$ and $T(\widetilde{K_{2v}}(S,H)) \cong_{\mathrm{Hdg}} T(S)$.

6.4.2 Induced IHS's of O'Grady 6 type

Let X be a projective IHS manifold of OG6-type. In this case the monodromy group is *maximal* [60]. Hence the bimeromorphic class of a IHS manifold of OG6-type is fully determined by the Hodge structure of its Beauville-Bogomolov lattice.

Theorem 6.4.2.1 ([60, Thm. 1.1])**.** *Let X, Y be two IHS manifolds of OG6-type. Then X and Y are bimeromorphic if and only if $H^2(X,\mathbb{Z}) \cong_{\mathrm{Hdg}} H^2(Y,\mathbb{Z})$.*

We want to study whether induced OG6-type manifolds are symplectic resolution of a singular moduli space over an abelian surface or not, similarly to Beauville's examples in the previous sections. The following theorem follows a similar strategy as in [35, Thm. 1.1].

Theorem 6.4.2.2. *Let X be a projective IHS manifold of OG6-type induced by an abelian surface A. Then the following are equivalent:*

1. *There exists an algebraic class $\sigma \in NS(X)$ of square -2 and divisibility 2.*

2. *X is bimeromorphic to a* $\widetilde{K_{2v}}(S,H)$ *for some* $v \in \tilde{H}(A)$ *and some v-generic polarization* H.

Proof. The second item implies the first by Theorem 6.4.1.1. For the other implication, notice that by Theorem 1.2.1.14 there exist exactly two non-isomorphic primitive embeddings of the lattice $\langle -2\rangle$ in $\Lambda_{\mathrm{OG6}} = U^{\oplus 3} \oplus \langle -2\rangle^{\oplus 2} = \langle u_1^i, u_2^i\rangle_{i=1,2,3} \oplus \langle a_1, a_2\rangle$: up to isometries of the latter, to realize the first embedding we can choose as generator $\rho = u_1^1 - u_2^1$: then ρ has divisibility 1, so we exclude this case; to realize the other embedding, we choose as generator $\sigma = a_1$, the generator of one of the orthogonal components $\langle -2\rangle$: then σ has divisibility 2.
In the latter case $NS(X)$ has $\langle\sigma\rangle$ as an orthogonal summand. Let $\Lambda' := \langle\sigma\rangle^{\perp}$ in Λ_{OG6}, then we have

$$\Lambda' = \Lambda_{\mathrm{Ab}} \oplus \langle -2\rangle \cong H^2(K_v(A,H),\mathbb{Z}). \tag{6.4.2.1}$$

where the last isomorphism is as abstract lattices. If we let $\pi : \Lambda_{\mathrm{OG6}} \to \Lambda'$ be the projection map, then $T(X)$ is isomorphic to its image under π. Since $T(X) \cong T(A)$, this defines a level 2 Hodge structure on $H^2(K_v(A,H),\mathbb{Z})$ that by construction lifts to an isomorphism of level 2 Hodge structures

$$H^2(X,\mathbb{Z}) \cong H^2(\widetilde{K}_v(A,H),\mathbb{Z}). \tag{6.4.2.2}$$

By Theorem 6.4.2.1 we have that X is bimeromorphic to $\widetilde{K}_v(A,H)$. □

In the rest of this section we will study several examples of induced IHS of OG6-type which are not resolution of the singularities of moduli spaces. We first notice that there are strong conditions a lattice has to satisfy to be the transcendental lattice of an abelian surface.

Theorem 6.4.2.3 ([63, Thm. 1.6 and Cor. 2.6])**.** *Let T be an even lattice of signature $(2,k)$.*

1. *If $k = 0, 1$ there is a primitive embedding T in Λ_{Ab}.*
2. *If $k = 2$ there is a primitive embedding T in Λ_{Ab} if and only if $T \cong U \oplus T'$.*
3. *If $k = 3$ there is a primitive embedding T in Λ_{Ab} if and only if $T \cong U^{\oplus 2} \oplus T'$.*

Moreover there exists a (not necessarily unique) abelian surface A such that $T(A) \cong_{\mathrm{Hdg}} T$.

We are going to say that an induced OG6-type manifold X *arises from a moduli space* to mean that there exist an abelian surface A, a primitive Mukai vector v and a v-generic polarization H such that $X = \widetilde{K_{2v}}(A,H)$.
As a consequence of Theorems 6.4.1.1 and 6.4.2.2, given a Hodge structure on a lattice T and a primitive embedding $\phi : T \hookrightarrow \Lambda_{\mathrm{OG6}}$, the condition

$$\text{there exists } \sigma \in \phi(T)^{\perp} \text{ such that } \sigma^2 = -2,\ \sigma H^2(X,\mathbb{Z}) = 2\mathbb{Z} \tag{6.4.2.3}$$

is equivalent to the fact that the OG6-type manifold X such that $T(X) = \phi(T)$, $NS(X) = \phi(T)^\perp$ arises from a moduli space.

We are going to give examples of induced IHS manifold X of OG6-type that do not arise from a moduli space for each of the transcendental lattices $T(X) \simeq T$ allowed as transcendental lattices of abelian surfaces by Theorem 6.4.2.3. In each of the following examples the orthogonal complement $T^\perp$ of T in Λ_{OG6} does not admit any primitive embedding of $\langle -2\rangle$ with divisibility 2, so condition (6.4.2.3) is not satisfied.

Example 6.4.2.4. Fix generators $u_1, u_2, v_1, v_2, w_1, w_2, a, b$ of the OG6-lattice Λ_{OG6} where $u_1, ..., w_2$ are generators of the hyperbolic planes and a, b are the two classes of square -2 and divisibility 2.

1. $rk(T) = 5$: Consider the lattice $T = U \oplus U \oplus \langle -4\rangle$: embed it in Λ_{OG6} as $\langle u_1, u_2, v_1, v_2, a+b\rangle$. Then $T^\perp = U \oplus \langle -4\rangle$, so condition (6.4.2.3) is not satisfied. Indeed, a primitive class of even divisibility is of the type $(2k+1)(a-b)+2(hw_1+jw_2)$ with k, h, j integers, but none of these classes has square -2 (to this end, odd multiples of w_1, w_2 are needed).
2. $rk(T) = 4$: Consider $T = U \oplus \langle 6\rangle \oplus \langle -10\rangle$: embed it in Λ_{OG6} as $\langle u_1, u_2, 2(v_1 + v_2) + a, 2(w_1 - w_2) + b\rangle$; then $T^\perp = A_2 \oplus \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$.
3. $rk(T) = 3$: Consider $T = U \oplus \langle 4\rangle$: embed it in Λ_{OG6} as $\langle u_1, u_2, 2(v_1 + v_2) + a + b\rangle$; then $T^\perp = U \oplus A_3$.
4. $rk(T) = 2$: Consider $T = \langle 6\rangle^{\oplus 2}$: embed it in Λ_{OG6} as $\langle 2(u_1+u_2)+a, 2(v_1+v_2)+b\rangle$: then $T^\perp = U \oplus A_2^{\oplus 2}$.

The following theorem provides a characterization of induced IHSs with transcendental $T_d = U \oplus U \oplus \langle -2d\rangle$, $d \in \mathbb{N}_{\neq 0}$, as in case (3) of Theorem 6.4.2.3.

Theorem 6.4.2.5. *Let X be an induced IHS of OG6-type such that $rk(NS(X)) = 3$. Then X does not arise from a moduli space if and only if it holds*

$$d = 4k+2, \qquad T(X) \simeq U \oplus U \oplus \langle -2d\rangle, \qquad NS(X) \simeq \begin{bmatrix} 2k & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}. \quad (6.4.2.4)$$

Proof. The proof relies heavily on Theorem 1.2.1.14. Since X is induced by an abelian surface, its transcendental lattice is of the form $T_d = U \oplus U \oplus \langle -2d\rangle$. If $d = 0, 3 \pmod 4$ there exists only one possible embedding of T_d in Λ_{OG6} up to isometries of the latter, given by $T_d \hookrightarrow U^{\oplus 3} \hookrightarrow \Lambda_{\mathrm{OG6}}$: the corresponding $NS(X)$ is $S_d := \langle 2d\rangle \oplus \langle -2\rangle^{\oplus 2}$, which obviously satisfies condition (6.4.2.3). Moreover, S_d is unique in its genus (see Theorem 1.2.1.10).
If $d = 1 \pmod 4$, there are two possible embeddings of T_d in Λ_{OG6}: one as above with orthogonal complement S_d, the other described by $\langle u_1, u_2, v_1, v_2, \frac{d-1}{2}w_1 + \frac{d-1}{2}w_2 + \frac{d+1}{2}a\rangle$, which gives as $NS(X)$ a lattice of the form $S' \oplus \langle -2\rangle$, with $sign(S') = (1, 1)$, so it satisfies

condition (6.4.2.3). The lattice $NS(X)$ is again unique in its genus, since its discriminant form is $q_S = [-1/2d]$ so $rkNS(X) = 2 + \ell(NS(X))$ (see Theorem 1.2.1.10).
If $d = 2 \pmod 4$ there are again two possible embeddings, one of which gives $NS(X) = S_d$, and the other described by $\langle u_1, u_2, v_1, v_2, 2kw_1 - 2kw_2 + a + b\rangle$, where $k = (d-2)/4$: in this case it holds

$$NS(X) = \begin{bmatrix} 2k & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \tag{6.4.2.5}$$

which is unique in its genus (its discriminant form is $[(1-d)/2d]$). This lattice does not contain any classes σ that satisfy the requirements of condition (6.4.2.3): indeed, calling $\{b_1, b_2, b_3\}$ a basis of $NS(X)$, to have even divisibility it has to be $\sigma = \sum x_i b_i$ with x_1 even, and $x_2 =_2 x_3$: none of these classes however have square -2. □

Remark 6.4.2.6. Taking $k = 0$ in (6.4.2.5), choosing the basis $\{b_1, b_2 + b_1, b_3 - b_2 - 2b_1\}$ we get $NS(X) = U \oplus \langle -4\rangle$ as in Example 6.4.2.4.

Remark 6.4.2.7. Let $T = U \oplus U \oplus \langle -4\rangle$, and consider X an OG6-type manifold induced by T. If $NS(X) = U \oplus \langle -4\rangle = \langle u_1, u_2, x\rangle$, then by Theorem 6.4.2.5 X does not arise from a moduli space; the movable cone coincides with the positive cone, because there are no classes of divisibility 2; the walls of the Kähler cone are $\alpha^\perp$, for any α of the form $\alpha = au_1 + bu_2 + cx$ such that $c^2 - ab = 1$; on the other hand, if $NS(X) = \langle 4\rangle \oplus \langle -2\rangle^{\oplus 2} = \langle y, \alpha_1, \alpha_2\rangle$ (i.e. X arises from a moduli space), then the walls are of the form $ay + b\alpha_1 + c\alpha_2$ such that either $b^2 + c^2 = 2a^2 + 1$, or the following conditions hold: b, c are odd, and $b^2 + c^2 = 2a^2 + 2$. The movable cone coincides with the ample cone (all classes of square -2 have divisibility 2).

Proposition 6.4.2.8. *If X is an IHS manifold of OG6-type such that $T(X) = U \oplus U \oplus \langle -4\rangle$ and $NS(X) = U \oplus \langle -4\rangle$, then $Bir(X)$ is infinite.*

Proof. By the solution of the Kawamata-Morrison Cone Conjecture [1] we need to prove that there exist an infinite number of walls inside the movable cone. By the previous remark it is therefore sufficient to show that for any class α of the form $\alpha = au_1 + bu_2 + cx$ such that $c^2 - ab = 1$ there exists a positive class $\beta = du_1 + eu_2 + fx$ such that $\alpha\beta = 0$: to this end, we need to prove that the system

$$\Sigma : \begin{cases} af + be - 4cg = 0 \\ 2ef - 4g > 0 \end{cases}$$

admits a solution for any choice of a, b, c such that $c^2 - ab = 1$.
Notice firstly that we can always suppose $a \geq b$. If $c = 0$, then $(a, b) = (1, -1)$: then any choice of e, f, g that satisfies $e = f, g < e^2/2$ is a solution to Σ. If $c = \pm 1$, then $a = b = 0$, so $g = 0$ and any choice of e, f such that $ef > 0$ is a solution to Σ. If $c \neq 0, \pm 1$, then a, b have the same sign, and we write

$$\Sigma : \begin{cases} g = (af + be)/4c \\ 2ef - (af + be)/c > 0; \end{cases}$$

if $c > 0$, choose e, f such that $ef > c > 0$: then $2efc - (af + be) > 2 + 2ab - af - be$, which is positive for any choice of e, f with opposite sign to a, b. Therefore, a solution to Σ is given by any pair e, f with opposite sign to a, b, such that $4c$ divides both e and f. If $c < 0$, take instead any pair e, f with the same sign as a, b, such that $4c$ divides both e and f. □

We will now give some more details in case $rk(T) = 4$: it always holds $T = U \oplus Q$, where $Q = \begin{bmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{bmatrix}$ such that $d := \beta^2 - 4\alpha\gamma > 0$.

Remark 6.4.2.9. If $det(Q)$ is odd, then any IHS of OG6-type with transcendental lattice $T = U \oplus Q$ arises from a moduli space, because there exists only one primitive embedding of T in Λ_{OG6}.

Example 6.4.2.10. If $Q = \begin{bmatrix} 2\alpha & 0 \\ 0 & -2\gamma \end{bmatrix}$, then $T = U \oplus Q$ can be the transcendental lattice of an induced IHS of OG6-type that does not arise from a moduli space if and only if either $\gamma = 1, \alpha = 3 \pmod 4$, or $\gamma = 2, \alpha = 2 \pmod 4$: in the first case, the embedding can be realized as $Q = \langle 2(v_1 + kv_2) + a, 2(w_1 - hw_2) + b \rangle$, giving $\alpha = 4k - 1, \gamma = 4h + 1$; in the second case, $Q = \langle 2(v_1 + kv_2) + a + b, 2(w_1 - hw_2) + a - b \rangle$, giving $\alpha = 4k - 2, \gamma = 4h + 2$. The corresponding Néron-Severi lattice will be, respectively,

$$\begin{bmatrix} -2k^2 & 1-2k & 0 & 0 \\ 1-2k & -2 & 0 & 0 \\ 0 & 0 & -2h^2 & 1+2h \\ 0 & 0 & 1+2h & -2 \end{bmatrix}, \qquad \begin{bmatrix} -2k & 0 & 1 & 1 \\ 0 & 2h & 1 & -1 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix}.$$

Neither contains classes of square -2 and divisibility 2. All the other combinations for the values of α, γ modulo 4 give only one possible primitive embedding of $U \oplus Q$ in Λ_{OG6}.

6.4.3 Induced IHS's of O'Grady 10 type

Let X be a projective IHS manifold of OG10-type. As in the OG6-type case the monodromy group is *maximal*, therefore the bimeromorphic class of a IHS manifold of OG10-type is fully determined by the Hodge structure of its Beauville-Bogomolov lattice.

Theorem 6.4.3.1 ([77, Thm. 5.4])**.** *Let X, Y be two IHS manifolds of OG10-type. Then X and Y are bimeromorphic if and only if $H^2(X, \mathbb{Z}) \cong_{\mathrm{Hdg}} H^2(Y, \mathbb{Z})$.*

We will again follow the strategy of [35, Thm. 1.1] in order to find a necessary and sufficient criterion to decide whether a IHS of OG10-type induced by a K3 surface is the resolution of the singularities of a moduli space. See also [21] for a similar result.

Theorem 6.4.3.2. *Let X be a projective IHS manifold of OG10-type which is induced by a K3 surface S. Then the following are equivalent:*

1. *There exists a class $\sigma \in NS(X)$ such that $q_X(\sigma) = -6$ and $div(\sigma) = 3$.*

2. *X is bimeromorphic to a* $\widetilde{M_{2v}}(S, H)$ *for some* $v \in \tilde{H}(S)$ *and some v-generic polarization* H*.*

Proof. By Theorem 1.2.1.14 there exist exactly two non-isomorphic primitive embeddings of the lattice $\langle -6 \rangle$ in Λ_{OG10} up to isometries of the latter: to realize the first embedding we can choose as generating class $u_1 - 3u_2$ (where $\langle u_1, u_2 \rangle = U$), which has divisibility 1; for the other case, take instead as generating class $a_1 + 2a_2$ (where $\langle a_1, a_2 \rangle = A_2$), which has divisibility 3.
Let σ be a class of square -6 and divisibility 3 in Λ_{OG10} and let $\Lambda' := \sigma^\perp$ be its orthogonal complement in Λ_{OG10}: then we have $\Lambda' = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$. The lattice Λ' embeds in the Mukai lattice $\tilde{H}(S)$ as $\lambda^\perp$, with $\lambda^2 = 2$, so $\Lambda' \simeq_{\mathrm{Hdg}} H^2(M_{2\lambda}(S, H), \mathbb{Z})$, for any λ-general polarization H.
Now, if we let $\pi : \Lambda_{\mathrm{OG10}} \to \Lambda'$ be the projection map, then $T(X)$ is isomorphic to its image under π. Since $T(X) \cong T(S)$, this defines a level 2 Hodge structure on $H^2(M_{2\lambda}(S, H), \mathbb{Z})$. We can extend this to an isomorphism of level 2 Hodge structures

$$\Lambda_{\mathrm{OG10}} \cong H^2(\widetilde{M_{2\lambda}}(S, H), \mathbb{Z}), \tag{6.4.3.1}$$

via the construction of the lattice A_2 as overlattice of $\langle -2 \rangle \oplus \langle -6 \rangle$, which is the only way to embed Λ' in Λ_{OG10} primitively: calling α (respectively β) the generator of $\langle -2 \rangle$ (resp. $\langle -6 \rangle$), it holds $A_2 = \langle \alpha, (\alpha + \beta)/2 \rangle$. By the strong form of Torelli's theorem we have therefore that X is bimeromorphic to $\widetilde{M_{2\lambda}}(S, H)$. □

We are going to say that an induced OG10-type manifold X *arises from a moduli space* to mean that there exist a K3 surface S, a primitive Mukai vector v and a v-generic polarization H such that $X = \widetilde{M_{2v}}(S, H)$.
As a consequence of Theorems 6.4.1.1 and 6.4.3.2, given a Hodge structure on a lattice T and a primitive embedding $\phi : T \hookrightarrow \Lambda_{\mathrm{OG10}}$, the condition

$$\text{there exists } \sigma \in \phi(T)^\perp \text{ such that } \sigma^2 = -6,\ \sigma H^2(X, \mathbb{Z}) = 3\mathbb{Z} \tag{6.4.3.2}$$

is equivalent to the fact that the OG10-type manifold X such that $T(X) = \phi(T)$, $NS(X) = \phi(T)^\perp$ arises from a moduli space..

Example 6.4.3.3. Let $T = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$: there exists a unique K3 surface S such that $T(S) \simeq T$, which is the surface X_ω described in Section 3.2. Embed T in Λ_{OG10} as follows: call $\langle u_1, u_2, v_1, v_2, a_1, a_2 \rangle$ the standard basis of $U \oplus U \oplus A_2$, then $T = \langle u_1 + 2u_2 - v_1 - v_2 - a_1,\ u_2 - u_1 - 2v_1 - 2v_2 - a_1 - a_2 \rangle$; its orthogonal complement in Λ_{OG10} is the lattice $N = U \oplus E_8^{\oplus 2} \oplus D_4$. If X is such that $(NS(X), T(X)) = (N, T)$, then X is induced, but it does not arise from a moduli space: moreover, the discriminant of the Néron-Severi lattice of the latter is bigger.

The induced OG10-type manifolds in the example above contain a copy if U in their Neron-Severi lattice. One important family of OG10-type manifolds which also has this property was produced by Laza, Saccà and Voisin in [44] via a compactification of the intermediate Jacobians of hyperplane sections on a general cubic fourfold; this construction was further extended by Saccà in [86] to include every smooth cubic fourfold. More precisely, if $Y \subseteq \mathbb{P}^5$ is a smooth cubic fourfold, and $J_V \to V \subset (\mathbb{P}^5)^\vee$ the fibration in intermediate Jacobians of smooth hyperplane sections, in [86, Thm. 1.6] it is proven that there exists a smooth IHS compactification of J_V.

Definition 6.4.3.4. A IHS manifold of OG10-type bimeromorphic to a smooth compactification of the fibration $J_V \to V \subset (\mathbb{P}^5)^\vee$ for some smooth cubic fourfold $Y \subset \mathbb{P}^5$ as considered above is called an *LSV manifold*.

Remark 6.4.3.5. The IHS compactification of the intermediate Jacobian fibration associated with a smooth cubic fourfold Y is not unique, but its bimeromorphic class it is. We will use J_Y to denote any compactification of $J_V \to V$.

Remark 6.4.3.6. The algebraic copy of U in LSV manifolds comes from the construction as follows. There are always two distinguished algebraic classes, an isotropic class F coming from the naturally associated fibration, and a rigid class θ given by the compactification of the Theta divisor on the fibers: hence, it holds

$$\langle F, \theta \rangle = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

Therefore $U = \langle F, \theta + F \rangle$.

Proposition 6.4.3.7. *Let X be a IHS manifold of OG10-type. If X is an LSV manifold then $U \subseteq NS(X)$. Conversely, if $U \subseteq NS(X)$ and X is very general, then there exists a cubic fourfold Y such that X is bimeromorphic to J_Y.*

Proof. If X is an LSV manifold, then is bimeromorphic to J_Y, where Y is a cubic fourfold. By [86, Lemma 3.5] it holds $U \subseteq NS(J_Y) = NS(X)$, the last equality by Lemma 1.3.0.9.
IHS manifolds of OG10-type which contain a copy of U in the Neron-Severi form an irreducible moduli space $\mathcal{M}_U$ of dimension 20 (see Remark 1.4.0.4). Hence, by counting dimensions, in order to prove the claim it suffices to show that LSV manifolds are dense inside $\mathcal{M}_U$. Let Y be a smooth cubic fourfold, then by [86, Lemma 3.2] there exists an isomorphism

$$T(J_Y) \otimes \mathbb{Q} \cong_{\mathrm{Hdg}} H^4(Y, \mathbb{Q})_{tr} \tag{6.4.3.3}$$

of rational Hodge structures.
Let $\mathcal{M}$ be the moduli space of smooth cubic fourfolds (which has again dimension 20) and define the subspace

$$\mathcal{V} = \{Y \in \mathcal{M} \text{ such that } H^{2,2}(Y) \cap H^4_{prim}(Y, \mathbb{Z}) = \langle h^2 \rangle\}$$

of very general cubic fourfolds (see Example 1.3.0.2 for the notation): this is an open subset in $\mathcal{M}$. The Torelli Theorem for cubic fourfolds 1.3.0.15 states that two cubic fourfolds Y, Y' are isomorphic if and only if there exists a Hodge isometry $H^4(Y,\mathbb{Z}) \cong_{\mathrm{Hdg}} H^4(Y',\mathbb{Z})$ preserving the square of the hyperplane class. If Y is in $\mathcal{V}$, then $(h^2)^{\perp} = H^4(Y,\mathbb{Z})_{tr} = H^4(Y,\mathbb{Z})_{prim}$: therefore the only distinction between cubic fourfolds in $\mathcal{V}$ is given by the Hodge structure we put on the abstract primitive lattice $H^4_{prim} = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$. By (6.4.3.3) this shows the claim. □

Remark 6.4.3.8. Although Proposition 6.4.3.7 implies that an open dense set of U-polarized manifolds of OG10-type X is actually in the LSV family, it is not *explicit* in any way: this prevents us from finding a cubic fourfold Y such that J_Y is bimeromorphic to a given X.

Proposition 6.4.3.7 prompts a question: if X is induced by a K3 surface and it does not arise from a moduli space, does it always belong to the LSV family? We negatively answer the question in the case of Picard rank 3, which is the most generic case.

Proposition 6.4.3.9. *Let S be a projective K3 surface such that $NS(S) = \langle 2d\rangle$, $d \in \mathbb{Z}_{>0}$: then there exists an OG10-type manifold X which is induced by S, but is doesc not arise from a moduli space, if and only if $d = 3(3h+1)$. Moreover, X belongs to the LSV family if and only if d is odd and no prime of the form $6n+5$ divides it.*

Proof. The condition $d = 3(3h+1)$ ensures by Theorem 1.2.1.14 that there exist two different embeddings of the transcendental lattice $T = T(S)$ in Λ_{OG10}: indeed if γ is a generator of A_T, then the group $G = \langle \frac{2d}{3}\gamma\rangle$ is isomorphic to the discriminant group of A_2. We then compute the orthogonal complement to G in A_T, which is $H = \langle 3\gamma\rangle$: following Theorem 1.2.1.14, if X is not a moduli space, then the discriminant form q of $NS(X)$ is the opposite to that of H, that is, $q = [3/2(3h+1)]$. The lattice $U \oplus \langle -2(3h+1)\rangle$ is unique in its genus [72, Prop. 1.13.4], and its discriminant form is $\tilde{q} = [-1/2(3h+1)]$: therefore we only need to find conditions under which $\tilde{q}$ is equivalent to q.
Recall that two quadratic forms defined on a finite abelian group G are equivalent if and only if they are p-equivalent for every prime p (see Definition 1.2.0.12). In our case, since $G = \mathbb{Z}/2(3h+1)\mathbb{Z}$ is cyclic, we can use [54, Lemma IV.1.4]. Let $|A_2| = 2^n$: then, it holds $q|_{A_2} = 3m/2^n$, and $\tilde{q}|_{A_2} = -m/2^n$ (for some m odd), which are equivalent if and only if $n = 1$. Therefore $3h+1$ should be odd. Suppose now $p \neq 2$: then $q|_{A_p} = \tilde{q}|_{A_p}$ if and only if $-2, 6$ are both square, or both non-square numbers modulo p. Since -2 is a square if and only if $p =_8 1, 3$, and -6 is a square if and only if $p =_{24} 1, 5, 19, 23$, it holds $q|_{A_p} = \tilde{q}|_{A_p}$ if and only if $p =_{24} 1, 7, 13, 15, 19, 21$. Taking the complementary (and excluding $p = 3$, that does not divide $3h+1$), we get the statement. □

Example 6.4.3.10. Let $\{e_1, \dots e_8\}$ be a $\mathbb{Z}$-basis of E_8 and let $\{u_1, u_2\}$ be a basis of U as in Example 1.2.0.2. Let S_0 be a K3 surface such that $NS(S_0) = \langle 6\rangle$: let X_0 be an OG10-type manifold such that $NS(X_0)$ is generated as a sublattice of Λ_{OG10} by $\langle e_6, u_1, u_2\rangle$: then X_0 is induced by S_0 and it does not arise from a moduli space, but it is a member of the LSV family. Indeed, it holds $NS(X_0) = U \oplus \langle -2\rangle$.
Let S_1 be a K3 surface such that $NS(S_1) = \langle 24\rangle$; let X_1 be a IHS of OG10-type such

that $NS(X_1)$ is generated as a sublattice of Λ_{OG10} by $\langle e_6, e_7, -e_5 + e_8 + u_1 + 3u_2\rangle$: then X_1 is induced by S_1, but it's neither a resolution of a moduli space, nor a member of the LSV family. Indeed, it holds

$$NS(X_1) = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 2 \end{bmatrix}:$$

this lattice has discriminant form $q = [3/8]$, so there exists no primitive embedding of U in it. Indeed, if by contradiction such an embedding existed, then there would exist an even lattice K of rank 1 such that $NS(X_1) = U \oplus K$, and $q_K = [3/8]$ [72, Cor. 1.13.4]; but comparing signatures, it should also hold $K = \langle -2k\rangle, k \in \mathbb{Z}_{>0}$, which has discriminant form $[-1/2k]$, that is never equivalent to q_K.

6.5 The most algebraic IHS manifolds

K3 surfaces whose Néron-Severi lattice has maximum rank 20 are called *singular* K3 surfaces: their moduli space is reduced to a point (see Remark 1.4.0.4), and they have interesting geometric properties. Some notable examples of singular K3 surfaces are Fermat's quartic, and also the surfaces X_4 and X_ω from Sections 2.2.2, 3.2.1.

Theorem 6.5.0.1 ([92, Thm. 4.4]). *Let $\mathcal{Q}$ be the set of matrices with integral entries of the form $Q = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$ such that $a, c > 0, b^2 - 4ac < 0$: there is a bijection between the set of singular K3 surfaces and $\mathcal{Q}/SL_2(\mathbb{Z})$, given by $S \mapsto T(S)$.*

Remark 6.5.0.2. A similar result holds for abelian surfaces: indeed the set $\mathcal{Q}/SL_2(\mathbb{Z})$ also parametrizes the minimal rank transcendental lattices of abelian surfaces (see Theorem 6.4.2.3) up to isometries, and for each lattice T_Q with intersection matrix Q the corresponding Néron-Severi lattice satisfies Theorem 1.2.1.10; the only difference is that, by the Torelli theorem for abelian surfaces 1.3.0.14, one finds for each $T(S)$ two abelian surfaces S and $S^\vee$ instead of the unique K3 surface.

A natural question is how to extend this result to IHS manifolds, at least for the known deformation types. In the case of Beauville's deformation families we get the following result.

Corollary 6.5.0.3. *Let X be a $K3^{[n]}$-type manifold such that $rk(NS(X)) = 21$: then X is bimeromorphic to a moduli space of sheaves on the unique K3 surface with transcendental lattice $T(X)$.*
Let X be a Km_n-type manifold such that $rk(NS(X))5$: then X is bimeromorphic to a moduli space of sheaves on the abelian surface A or its dual $A^\vee$, with transcendental lattice $T(A) \simeq T(X)$.

Proof. If X is a $\mathrm{K3}^{[n]}$-type manifold of Picard rank 21, the transcendental lattice $T(X)$ is a positive definite lattice of rank 2, hence it is the transcendental lattice of a unique

K3 surface S by Theorem 6.5.0.1. By Theorem 6.3.2.2 we get that X is bimeromorphic to a moduli space of sheaves over S.
If X is a Km_n-type manifold, an analogue result holds by Remark 6.5.0.2 and Theorem 6.3.3.7. □

For O'Grady's deformation families, due to the classification results of induced OG6-type and OG10-type manifolds given in Section 6.4, there is no analogous result as Corollary 6.5.0.3. Instead, we get the following.

Corollary 6.5.0.4. *The transcendental lattice of rank* 2 *and smallest discriminant group for which an induced IHS manifold in one of the O'Grady's deformation families does not arise from a moduli space of sheaves over a K3 surface or an abelian surface is:*

1. $\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$ *for OG6-type;*
2. $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ *for OG10-type (see Example 6.4.3.3).*

Proof. It follows by the classification of reduced positive definite binary forms of small determinant [19, Table 15.1, pp. 360], via a direct computation using Theorems 6.4.2.2 and 6.4.3.2. □

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