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Radon transforms: Unitarization, Inversion and Wavefront sets

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A Filippo ed Ernesto. Sono stata fortunata ad avervi incontrato.

Introduction

The Radon transform has its origin in the problem of determining a function defined on $\mathbb{R}^d$ from its integrals over hyperplanes. In 1917 Radon proved the reconstruction formula for two and three-dimensional signals. In $\mathbb{R}^3$ it reads

$$f(x) = -\frac{1}{8\pi^2}\Delta \int_{S^2} g(\theta, x \cdot \theta) \mathrm{d}\theta, \tag{1}$$

where Δ is the Laplacian acting on the variable x, S^2 is the sphere in $\mathbb{R}^3$ and for every $\theta \in S^2$ and $t \in \mathbb{R}$ we denote with $g(\theta, t)$ the integral of f over the hyperplane $x \cdot \theta = t$. Formula (1) suggests to define two dual transforms $f \mapsto \mathcal{R}f$, $g \mapsto \mathcal{R}^{\#}g$, known as Radon transform and dual Radon transform or back-projection. The Radon transform $\mathcal{R}$ maps a function on $\mathbb{R}^d$ into the set of integrals over the hyperplanes, while the dual Radon transform $\mathcal{R}^{\#}$ maps a function defined on the set of hyperplanes of $\mathbb{R}^d$ into its integrals over the sheafs of hyperplanes through a point. Then, formula (1) can be rewritten

$$f = -\frac{1}{2}\Delta\mathcal{R}^{\#}\mathcal{R}f.$$

This classical inverse problem is a particular case of the more general issue of recovering an unknown function on a manifold by means of its integrals over a family of submanifolds, already investigated by Gelfand in the 1950s. A natural framework for such general inverse problems was considered by Helgason [38] and is motivated by the group structure hidden in the classical Radon transform setting. He considers two homogeneous spaces X and Ξ of the same locally compact group G. The space X is meant to describe the ambient in which the functions to be analysed live, for example the Euclidean plane, or the sphere S^2 or the hyperbolic plane H^2. The second space Ξ is meant to parametrise the set of submanifolds of X over which one wants to integrate functions, for instance lines in the Euclidean plane, great circles in S^2, geodesics or horocycles in H^2. The spaces X and Ξ are both homogeneous spaces and thus $X = G/K$ and $\Xi = G/H$, with K and H closed subgroups of G. Any element $\xi \in \Xi$ defines a subset $\hat{\xi}$ of X by taking all the points of X whose coset intersects the coset of ξ in G. The map $\xi \mapsto \hat{\xi}$ is required to be injective in order to avoid an overlapping parametrisation of the submanifolds of X over which we integrate. The Radon transform $\mathcal{R}$ takes functions on X into functions on Ξ and is abstractly defined by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x),$$

provided that, for all $\xi \in \Xi$, m_ξ is a suitable measure on the manifold $\hat{\xi}$ and the right hand side is meaningful, possibly in some weak sense. As for the right space of functions $f\colon X \to \mathbb{C}$ for which the Radon transform makes sense, a natural choice is the L^2 setting. Indeed, both X and Ξ are transitive spaces, so that there exist quasi-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$. We stress that in Helgason's approach, it is assumed that m_ξ, $\mathrm{d}x$ and $\mathrm{d}\xi$ are all invariant measures. In this context, two classical issues are to generalize formula (1) and to prove that the Radon transform, up to composition with a suitable pseudo-differential operator, can be extended to a unitary map $\mathcal{Q}$ from $L^2(X, \mathrm{d}x)$ to $L^2(\Xi, \mathrm{d}\xi)$.

The thesis is devoted to investigate these two classical problems and the role of the Radon transform in different issues in harmonic analysis and is centered on the intertwining properties of the Radon transform with irreducible quasi-regular representations of G. Precisely, we investigate the link of the shearlet transform and the wavelet transform with the affine Radon transform and we study the consequences of this connection. We focus on the role of the Radon transform in microlocal analysis, especially in the resolution of the wavefront set in shearlet analysis and finally, we present the extension of the shearlet transform to distributions based on the intimate connection between the shearlet transform with the Radon and the wavelet transforms.

In the following four sections, these main topics are illustrated more precisely, and full details are given in Chapters 2, 3, 4 and 5, respectively. Chapter 1 is devoted to an introduction to the theory of locally compact groups and their homogeneous spaces, the theory of group representations and the classical Radon transform.

Unitarization and Inversion Formulae for the Radon Transforms

Chapter 2 of this thesis deals with a new approach based on the theory of group representations in order to solve in a general and unified way the unitarization and the inversion problems for generalized Radon transforms. Our treatment differs from the Helgason's framework since we make weaker assumptions on the measures carried by the basic spaces X and Ξ and the submanifolds $\hat{\xi} \subset X$, namely their relative invariance instead of invariance. This allows to consider a wider variety of cases of interest in applications, such as the similitude group studied by Murenzi [5], and the generalized shearlet dilation groups introduced by Führ in [29, 32] for the purpose of generalizing the standard shearlet group introduced in [54, 20]. We suppose that there exists a non-trivial π-invariant subspace $\mathcal{A}$ of $L^2(X, \mathrm{d}x)$ such that $\mathcal{R}$ is well defined for all $f \in \mathcal{A}$ and that the adjoint of the operator $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ has non-trivial domain. Then, we prove that the Radon transform $\mathcal{R}$ is a closable operator from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$ and that its closure $\overline{\mathcal{R}}$ is independent of the choice of $\mathcal{A}$ and is the unique closed extension of $\mathcal{R}$. Our main result states that if the quasi regular representations π of G on $L^2(X, \mathrm{d}x)$ and $\hat{\pi}$ of G on $L^2(\Xi, \mathrm{d}\xi)$ are irreducible, then the Radon transform $\mathcal{R}$, up to composition with a suitable pseudo-differential operator, can be extended to a unitary operator $\mathcal{Q} : L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ which intertwines them, namely

$$\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, \qquad g \in G.$$

The proof is based on the orthogonality relations for non-unimodular groups introduced by Duflo and Moore [26]. Such unitarization problem for the Radon transform was already addressed and essentially solved by Helgason in the context of symmetric spaces [39] which, however, does not fully fit in the framework considered in our work. If, in addition, we require that π is square integrable, under a suitable choice of the admissible vector ψ, we derive a general inversion formula for $\mathcal{R}$ given by the weakly convergent Haar integral

$$f = \int_G \chi(g)\langle \mathcal{R}f, \hat{\pi}(g)\Psi\rangle \, \pi(g)\psi \; \mathrm{d}\mu(g), \tag{2}$$

where χ is a character of G and where $\psi \in L^2(X, \mathrm{d}x)$ and $\Psi \in L^2(\Xi, \mathrm{d}\xi)$ are related suitable mother wavelets. We stress that the coefficients $\langle \mathcal{R}f, \hat{\pi}(g)\Psi\rangle$ depend on f only through its Radon transform $\mathcal{R}f$, so that the above equation allows to reconstruct an unknown signal from its Radon transform by computing the family of coefficients $\{\langle \mathcal{R}f, \hat{\pi}(g)\Psi\rangle\}_{g\in G}$, thereby showing alternative approaches to the generalized Radon inversion problem. The results of Chapter 2 are based on the content of [2].

We believe that our contribution opens the way to very interesting research directions. In particular, it would be interesting to relax the assumption of irreducibility of the representations. This would allow to extend the findings of Chapter 2 to groups and representations of interest in applications when the hypothesis of irreducibility does not hold, such as the class of groups studied in [1, 4] and to the context of homogeneous trees.

The affine Radon transform intertwines wavelets and shearlets

In Chapter 3 we prove that the unitary Radon transform in affine coordinates intertwines the shearlet representation with the tensor product of two wavelet representations. This intertwining result yields a formula for the shearlet coefficients that involves only integral transforms applied to the affine Radon transform of the signal, namely a one-dimensional wavelet transform followed by a convolution with a scale-dependent filter, and it opens new perspectives for the inversion of the Radon transform.

In order to formulate this result precisely, we recall the definition of the three main ingredients, namely wavelets, shearlets and the affine Radon transform. The wavelet group is $\mathbb{R} \rtimes \mathbb{R}^\times$ with law $(b, a)(b', a') = (b + ab', aa')$. The square integrable wavelet representation W acts on $L^2(\mathbb{R})$ by

$$W_{b,a}\psi(x) = |a|^{-1/2}\psi(\frac{x-b}{a})$$

and the wavelet transform, defined by $\mathcal{W}_\psi f(b, a) = \langle f, \psi_{b,a}\rangle$, is a multiple of an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R} \rtimes \mathbb{R}^\times)$ provided that ψ satisfies the Calderón condition, see (1.9) below. Next, denote by $\mathbb{S}$ the (parabolic) shearlet group, namely $\mathbb{R}^2 \rtimes (\mathbb{R} \rtimes \mathbb{R}^\times)$ with multiplication

$$(b, s, a)(b', s', a') = (b + S_s A_a b', s + |a|^{1/2}s', aa')$$

where

$$A_a = a\begin{bmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{bmatrix}, \qquad S_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$$

and where the vectors are understood as column vectors. The group $\mathbb{S}$ acts on $L^2(\mathbb{R}^2)$ via the shearlet representation, namely

$$S_{b,s,a}f\,(x) = |a|^{-3/4} f(A_a^{-1}S_s^{-1}(x - b)).$$

The shearlet transform is then $\mathcal{S}_\psi f(b, s, a) = \langle f, S_{b,s,a}\psi\rangle$, and is a multiple of an isometry provided that an admissibility condition on ψ is satisfied [19, 51], see (3.12) below.

Finally, the Radon transform in affine coordinates of a signal $f \in L^1(\mathbb{R}^2)$ is the function $\mathcal{R}^{\mathrm{aff}} f : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\mathcal{R}^{\mathrm{aff}} f(v,t) = \int_{\mathbb{R}} f(t - vy, y)\,\mathrm{d}y, \qquad (v,t) \in \mathbb{R}^2.$$

Our main result shows that $\mathcal{R}^{\mathrm{aff}}$ extends, up to composition with a pseudo-differential operator, to a unitary operator $\mathcal{Q}$ from $L^2(\mathbb{R}^2)$ onto itself such that

$$\mathcal{Q}\, S_{b,s,a} f = \left(W_{s,|a|^{1/2}} \otimes \mathrm{I}\right) W_{(1,\mathbf{v})\cdot b,a}\, \mathcal{Q} f \tag{3}$$

where the meaning of the dummy variable $\mathbf{v}$ is

$$W_{(1,\mathbf{v})\cdot b,a} g(v,t) = |a|^{-\frac{1}{2}}\, g\left(v, \frac{t - (1,v)\cdot b}{a}\right).$$

Our second most important result is the formula

$$\mathcal{S}_\psi f(x,y,s,a) = |a|^{-\frac{3}{4}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(x+vy,a) \overline{\phi_2\Big(\frac{v-s}{|a|^{1/2}}\Big)}\,\mathrm{d}v, \tag{4}$$

provided that $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and ψ is of the form

$$\mathcal{F}\psi(\xi_1,\xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2\Big(\frac{\xi_2}{\xi_1}\Big).$$

The one-dimensional wavelet χ_1 and the one-dimensional filter ϕ_2 are related to the shearlet admissible vector ψ by the following relations

$$\mathcal{F}\chi_1(\xi_1) = |\xi_1|\mathcal{F}\psi_1(\xi_1),$$

$$\phi_2(\xi_2/\xi_1) = \mathcal{F}\psi_2(\xi_2/\xi_1).$$

Equation (4) shows that for any signal f in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ the shearlet coefficients can be computed by means of three classical transforms. Indeed, in order to obtain $\mathcal{S}_\psi f(x,y,s,a)$ one can:

1. compute the Radon transform $\mathcal{R}^{\mathrm{aff}} f(v,t)$ of the original signal f;
2. apply the wavelet transform with respect to the variable t, that is

$$G(v,b,a) = \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(b,a); \tag{5}$$

3. convolve the result with the scale-dependent filter

$$\Phi_a(v) = \overline{\phi_2\left(-\frac{v}{|a|^{1/2}}\right)},$$

   where the convolution is computed with respect to the variable v, that is

$$\mathcal{S}_\psi f(x,y,s,a) = \left(G(\cdot, x + \cdot\, y, a) * \Phi_a\right)(s).$$

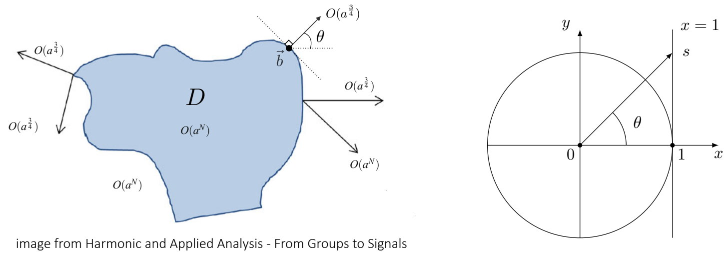

Figure 1: The wavefront set of a region D in $\mathbb{R}^2$ is the set of pairs $(x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, where x is a point on the boundary of D and ξ is the direction perpendicular to the boundary at x. The continuous shearlet transform $\mathcal{S}_\psi \chi_D(b, s, a)$ of χ_D has rapid asymptotic decay as $a \to 0$ except when b belongs to the boundary of D and s identifies the normal direction to the boundary at b (see the figure on the right hand side); in this case $\mathcal{S}_\psi \chi_D(b, s, a) \sim a^{3/4}$, as $a \to 0$.

Finally, since S is a square-integrable representation, there is a reconstruction formula, namely

$$f = \int_{\mathbb{S}} \mathcal{S}_\psi f(x, y, s, a) \; S_{x,y,s,a}\psi \; \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{6}$$

where the integral converges in the weak sense. Note that $\mathcal{S}_\psi f$ depends on f only through its Radon transform $\mathcal{R}^{\mathrm{aff}} f$, see (4). The above equation allows to reconstruct an unknown signal f from its Radon transform by computing the shearlet coefficients by means of (4). Formula (6) is a continuous version of the reconstruction formula presented in Theorem 3.3 in [16]. It is worth observing that while we work with the full shearlet group, the authors of [16] use the cone-adapted shearlet system introduced in [51]. We show that a cone-adapted version also works in our setting. Moreover, we generalize the above findings to the class of semidirect products introduced by Führ in [29, 32] for the purpose of generalizing the standard shearlet group, known as shearlet dilation groups. The contents of Chapter 3 are essentially contained in [8, 6].

The Radon Transform in Microlocal Analysis

Among the large reservoir of directional multiscale representations which has been introduced over the years, the shearlet representation has gained considerable attention for its capability to resolve the wavefront set of distributions, providing both the location and the geometry of the singularity set of signals. Indeed, when we shift from one-dimensional to multidimensional signals, it is not just of interest to locate singularities in space but also to describe how they are distributed. This additional information is expressed by the notion of wavefront set introduced by Hörmander in [43].

In [51] the authors show that the decay rate of the shearlet coefficients of a signal f with respect to suitable shearlets characterises the wavefront set of f. Precisely, they show that for any signal $f \in L^2(\mathbb{R}^2)$ the shearlet coefficients $\mathcal{S}_\psi f(b, s, a)$ exhibit fast asymptotic decay as $a \to 0$ except when the pair $(b, (\xi_1, \xi_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$, with $\xi_2/\xi_1 = s$, belongs to the wavefront set of f (see Figure 1). Later this result has been generalised in [34] where it is shown that the same result holds under much weaker assumptions on the continuous shearlets by means of a new approach based on an adaptation of the Radon transform to the shearlet structure, the affine Radon transform. Our research falls within this context and this construction has in part inspired our work.

In Chapter 4 we show that formula (4) provides a new geometric insight on the ability of the shearlet transform to resolve the wavefront set of signals. Precisely, we introduce a new approach based on the wavelet transform and on the affine Radon transform which clarifies how the ability of the shearlet transform to characterize the wavefront set of signals follows directly by the combination of the microlocal properties inhereted by the one-dimensional wavelet transform with a sensitivity for directions inhereted by the Radon transform.

We think that our approach opens the way to new future directions. It would be interesting to generalize this kind of arguments to arbitrary distributions and to generalized shearlet dilation groups [29, 32], which provide a large reservoir of families of natural transformations in dimension greater than 2, where there is no canonical shearlet group, and exhibit different behaviors in terms of wavefront set resolution properties, as investigated in [27, 3]. The idea would be to exploit the geometric construction based on the Legendre transform that relates the wavefront sets of $\mathcal{R}f$ and that of f (see [59, 49, 61] – to name a few). This investigation could lead to a new approach based on the Radon transform to investigate if a directional multiscale representation is able to resolve the wavefront set of distributions in more general cases. Part of the contents of Chapter 4 is contained in [7].

The shearlet transform of distributions

The last chapter of this thesis is devoted to present the extension of the shearlet transform to distributions. This work arises from the lack of a complete distributional framework for the shearlet transform in the literature and from the link between the shearlet transform with the Radon and the wavelet transforms, whose distribution theory is a deeply investigated and well known subject in applied mathematics. We refer respectively to [42] and to [38, 40, 48] for the extension of the wavelet transform and the Radon transform to various spaces of distributions. We prove that the shearlet transform and its transpose, called the shearlet synthesis operator, are continuous operators on various test function spaces. Then, we use these continuity results to develop a distributional framework for the shearlet transform via a duality approach. Precisely, we show that the shearlet transform can be extended as a continuous map from $\mathcal{S}_0'(\mathbb{R}^2)$ into $\mathcal{S}'(\mathbb{S})$, where $\mathcal{S}_0'(\mathbb{R}^2)$ is the Lizorkin distribution space and $\mathcal{S}'(\mathbb{S})$ is the space of distributions of slow growth on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$. The Lizorkin distribution space plays a crucial role in the extension of both the Radon and the wavelet transforms to distributions and it turns out to be a natural domain for the shearlet transform, too. Finally, we show that the shearlet transform of Lizorkin distributions extends the one considered in [51, 34] on the space of tempered distributions and allows to write the action of any Lizorkin distribution on any test function as an absolutely convergent integral over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$. All these findings are presented in [10], which is still in preparation.

Most of the content of this thesis is contained in a series of journal articles, proceedings, chapters and preprints:

Journal Articles

(A3) G. S. Alberti, F. Bartolucci, F. De Mari, E. De Vito, *Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs*, SIAM J. Math. Anal., 51(6):4356–4381, 2019.

(A2) F. Bartolucci, F. De Mari, E. De Vito, F. Odone, *The Radon Transform Intertwines Wavelets and Shearlets*, Appl. Comput. Harmon. Anal., 47(3):822-847, 2019.

(A1) F. Bartolucci, F. De Mari, E. De Vito, F. Odone, *Shearlets as Multi-scale Radon Transforms*, Sampl. Theory Signal Image Process., Special Issue – SampTA 2017, 17(1):1-15, 2018.

Chapters

(B1) F. Bartolucci, F. De Mari, E. De Vito, *Cone-Adapted Shearlets and Radon Transforms.* In Advances in Microlocal and Time-Frequency Analysis, Appl. Numer. Harmon. Anal., Birkhäuser, to appear, arXiv:1910.10219, 2019.

Proceedings

(C2) F. Bartolucci, M. Monti, *Unitarization and Inversion Formula for the Radon Transform for Hyperbolic Motions*, 2019 International Conference on Sampling Theory and Applications (SampTA), to appear, 2019.

(C1) F. Bartolucci, F. De Mari, E. De Vito, F. Odone, *Shearlets as Multi-scale Radon Transforms*, 2017 International Conference on Sampling Theory and Applications (SampTA), pp. 625-629, 2017.

Preprints

(P1) F. Bartolucci, S. Pilipović, N. Teofanov, *Shearlet Transform of Distributions*, in preparation, 2019.

Contents

Chapter 1

Preliminaries

1.1 Locally Compact Groups and Homogeneous Spaces

This subsection is devoted to a brief introduction to the measure theory on locally compact groups and their homogeneous spaces. Some of the results are used in the chapters that follow, others are presented for completeness. The contents of this section are contained in the classical reference [28].

1.1.1 Locally compact second countable groups

We start with some fundamental definitions and results.

Definition 1.1. A topological group is a group equipped with a topology with respect to which the group operations are continuous. Precisely, $(x, y) \mapsto xy$ is continuous from $G \times G$ to G and $x \mapsto x^{-1}$ is continuous from G to G.

Let G be a topological group. We denote the identity element of G by e. For every $A, B \subseteq G$ and $x \in G$, we define

$$Ax = \{yx : y \in A\}, \quad xA = \{xy : y \in A\}, \quad A^{-1} = \{y^{-1} : y \in A\},$$

$$AB = \{xy : x \in A, y \in B\}.$$

We say that $A \subseteq G$ is symmetric if $A = A^{-1}$. If H is a subgroup of G we write $H < G$. If in addition H is normal, we write $H \lhd G$. The next Proposition lists several basic properties of topological groups.

Proposition 1.2 ([28, Proposition 2.1])**.** *Let G be a topological group. Then*

i) the topology of G is invariant under translations and inversion, that is, if U is open, then xU , Ux and U^{-1} are open for every $x \in G$. Moreover, if U is open and $W \subseteq G$, then UW and WU are open;

ii) for every neighborhood U of e there exists a symmetric neighborhood V of e such that $VV \subseteq U$;

iii) if $H < G$, then $\overline{H} < G$;

iv) every open subgroup of G is closed;

v) if A and B are compact sets in G, then AB is also compact.

Let G be a topological group. Consider $H < G$ and the space G/H of left cosets of H. We denote by p the canonical projection of G onto G/H and we endow G/H with the quotient topology. We recall that $U \subseteq G/H$ is open if and only if $p^{-1}(U)$ is open in G.

Proposition 1.3 ([28, Proposition 2.2])**.** *Let $H < G$. Then*

i) p is an open map;

ii) if H is closed, then G/H is Hausdorff;

iii) if G is locally compact, then G/H is locally compact;

iv) if $H \triangleleft G$, then G/H is a topological group.

Corollary 1.4 ([28, Corollary 2.3])**.** *If G is T_1, then G is Hausdorff. If G is not T_1, then $\overline{\{e\}} \triangleleft G$ and $G/\overline{\{e\}}$ is Hausdorff.*

By Corollary 1.4, without loss of generality, we always assume that a topological group G is Hausdorff. By a locally compact group we mean a topological group whose topology is locally compact and Hausdorff. We shall also assume our groups to be second-countable. As stated in Theorem 1.5, this is equivalent to require that G is first-countable and σ-compact. In case G is not σ-compact, some technical complications in the measure theory arise, as pointed out and explained in subsection 2.3 in [28].

Theorem 1.5 ([67])**.** *Let G be a locally compact group. Then, the following conditions are equivalent*

i) G is second-countable;

ii) G is first-countable and σ-compact;

iii) G admits a left invariant metric which induces the topology of G.

Example 1.6. The affine group "$ax+b$" is $\mathbb{R} \times \mathbb{R}_+$, with $\mathbb{R}_+ = (0, +\infty)$, endowed with the product rule

$$(b', a')(b, a) = (a'b + b', a'a) \tag{1.1}$$

and it is a locally compact second countable group.

1.1.2 Haar Measure

Let G be a locally compact second countable (lcsc) group. We denote by $\mathcal{B}(G)$ the Borel σ-algebra of G and we consider a Borel measure μ on G. We recall that μ is a Radon measure if it satisfies the following three conditions

(i) μ is finite on compact sets;

(ii) μ is outer regular on the Borel sets, that is, for every Borel set E

$$\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\};$$

(iii) μ is inner regular on the open sets, that is, for every open set U

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}.$$

Definition 1.7. A left Haar measure on the topological group G is a nonzero Radon measure μ such that for every Borel set $E \subseteq G$ and every $x \in G$

$$\mu(xE) = \mu(E).$$

Similarly for right Haar measures.

The following theorem states the existence and uniqueness of Haar measures and is one of the fundamental results in the theory of locally compact groups. It is worth observing that its proof is constructive.

Theorem 1.8 ([28, Theorem 2.10])**.** *Every locally compact group G has a left Haar measure μ, which is essentially unique in the sense that if λ is any other left Haar measure, then there exists a positive constant c such that $\lambda = c\mu$.*

The next Proposition gives a simple and explicit formula to compute the Haar measure for a class of topological groups which covers most of the examples in which we are interested.

Proposition 1.9 ([28, Proposition 2.21])**.** *If G is an open set in $\mathbb{R}^d$ and if the left translations are given by affine maps, that is*

$$xy = A(x)y + b(x),$$

where $A(x)$ is a linear transformation and $b(x) \in \mathbb{R}^d$, then $|\det A(x)|^{-1}\mathrm{d}x$ is a Haar measure on G, with $\mathrm{d}x$ the Lebesgue measure of $\mathbb{R}^d$.

Example 1.2.1 continued**.** If we consider the affine group, the left translations are given by

$$(b, a)(\beta, \alpha) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} = A(b, a) \begin{bmatrix} \beta \\ \alpha \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix},$$

so that, by Proposition 1.9,

$$|\det A(b, a)|^{-1}\ \mathrm{d}b\mathrm{d}a = a^{-2}\mathrm{d}b\mathrm{d}a$$

is a left Haar measure on "$ax + b$".

1.1.3 Modular Function

Let G be a lcsc group. We fix a left Haar measure μ on G, then for any $x \in G$ the measure μ_x defined by

$$\mu_x(E) = \mu(Ex)$$

is again a left Haar measure. Therefore there must exist a positive real number $\Delta(x)$ such that

$$\mu_x = \Delta(x)\mu.$$

It is worth observing that $\Delta(x)$ is independent of the initial choice of μ. The function

$$\Delta : G \to \mathbb{R}_+$$

is called the modular function of G.

Proposition 1.10 ([28, Proposition 2.24]). *The modular function* Δ *is a continuous homomorphism from* G *into the multiplicative group* $\mathbb{R}_+$. *Furthermore, for every* $f \in L^1(G)$ *we have*

$$\int_G f(xy)\mathrm{d}\mu(x) = \Delta(y)^{-1}\int_G f(x)\mathrm{d}\mu(x).$$

We say that a lcsc group G is unimodular if its modular function is identically equal to one. Large classes of groups are unimodular, such as the Abelian and the compact groups (see Corollary 2.18 in [28]). Nevertheless, we are interested in non-unimodular groups, which play a prominent role in applied harmonic analysis, such as the affine group.

Example 1.2.1 continued. It is not difficult to show that the modular function of "$ax+b$" is given by $\Delta(b,a) = a^{-1}$.

1.1.4 Homogeneous Spaces

Definition 1.11. We say that a group G acts on a set X if there exists a map, called action, $G \times X \ni (g,x) \mapsto g[x] \in X$ such that

i) $h[g[x]] = hg[x]$ for every g, $h \in G$ and every $x \in X$;

ii) $e[x] = x$ for every $x \in X$.

We define the orbit of an element $x \in X$ as the set $\mathcal{O}_x = \{g[x] : g \in G\}$ and the isotropy H_x in $x \in X$ as the subgroup of G consisting of those elements of the group which leave x fixed, that is $H_x = \{g \in G : g[x] = x\}$. The set X is called a G-space. Furthermore, we say that X is a transitive G-space, or equivalently that the action is transitive, if for every x, $y \in X$ there exists $g \in G$ such that $g[x] = y$.

If X is a transitive G-space, $x_0 \in X$ is fixed and H denotes the isotropy at x_0, then it is not difficult to show that the map $\Phi : G/H \to X$ defined by $\Phi(gH) = g[x_0]$ is a bijection. In particular, if G is a lcsc group, then G/H is a lcsc topological space by Proposition 1.3. If in addition, X is a lcsc transitive G-space and the action of G on X is continuous with respect to the product topology of $G \times X$, then Φ is an homeomorphism by Proposition 2.44 in [28], so that X is homeomorphic to the quotient space G/H. In such a case, we say that X is a homogenous space.

1.1.5 Measure Theory on Homogeneous Spaces

Let G be a lcsc group and let $X \simeq G/H$ be a homogeneous space, where H is a closed subgroup of G. We consider a measure α on the Borel σ-algebra $\mathcal{B}$ of X. Then, for fixed $g \in G$, we can define a new measure $g(\alpha)$ on X by

$$g(\alpha)(E) = \alpha(g^{-1}[E]), \qquad E \in \mathcal{B}.$$

Definition 1.12. A Borel measure α on X is said to be G-invariant if $g(\alpha) = \alpha$ for every $g \in G$, which may be expressed equivalently by the equality

$$\int_X f(g^{-1}[x])\mathrm{d}\alpha(x) = \int_X f(x)\mathrm{d}\alpha(x), \qquad f \in C_c(X),\, g \in G.$$

Definition 1.13. A Borel measure α on X is said to be quasi-invariant if $g(\alpha)$ and α are equivalent for every $g \in G$, that is $\alpha(E) = 0$ if and only if $g(\alpha)(E) = 0$ for every $g \in G$ and $E \in \mathcal{B}$.

Theorem 1.14 ([28, Theorem 2.56])**.** *Every homogeneous space admits a non-trivial quasi-invariant measure.*

A character χ of the group G is a group homomorphism from G into the multiplicative group $\mathbb{C}^{\times}$. We say that χ is a positive character if $\chi(g) > 0$ for every $g \in G$. Finally, we call χ a unitary character if $|\chi(g)| = 1$ for every $g \in G$.

Definition 1.15. A Borel measure α on X is said to be relatively invariant with positive character χ if $g(\alpha) = \chi(g)\alpha$ for every $g \in G$, or equivalently

$$\int_X f(g^{-1}[x])\mathrm{d}\alpha(x) = \chi(g) \int_X f(x)\mathrm{d}\alpha(x), \qquad f \in C_c(X),\, g \in G.$$

Observe that if a Borel measure α on X is relatively invariant, then it is automatically quasi-invariant. The viceversa is not true.

Finally, we recall that a Borel section is a measurable map $s\colon X \to G$ satisfying $s(x)[x_0] = x$ for every $x \in X$ and $s(x_0) = e$; a Borel section always exists since G is second countable [71, Theorem 5.11].

1.2 Representation Theory

Our manuscript mainly deals with a new approach based on the theory of representations for the inversion problem for generalized Radon transforms. This section is devoted to recalling the fundamental notions and results in representation theory that we use throughout our manuscript. In subsection 1.2.1 we start with a basic introduction to the theory of unitary representations and we state the Schur's Lemma in its classical version. Subsection 1.2.2 is devoted to a generalization of the Schur's Lemma due to Duflo and Moore [26] and finally in subsection 1.2.3 we recall the theory of square-integrable representations. Most of the content of this section is contained in the classical references [28] and [26].

1.2.1 Unitary Representations

In this subsection we recall some fundamental definitions and results in representation theory and we illustrate them with the example of the wavelet representation which plays a crucial role throughout our manuscript. We refer the reader to [28] for details.

If $\mathcal{H}$ is an Hilbert space, we denote its scalar product and the associated norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively. Sometimes, in order to simplify the notation, we omit $\mathcal{H}$ in the scalar product and in the norm, but the Hilbert space is clear from the context. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of bounded linear operators of $\mathcal{H}_1$ into $\mathcal{H}_2$. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ we denote $\mathcal{B}(\mathcal{H}, \mathcal{H})$ by $\mathcal{B}(\mathcal{H})$. We recall that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an isometry if it satisfies $\|Tu\|_{\mathcal{H}_2} = \|u\|_{\mathcal{H}_1}$ for every $u \in \mathcal{H}_1$. It is immediate to see that if T is an isometry, then $\operatorname{Ker} T$ is trivial. Thus, every isometry is injective, but not necessarily surjective. A surjective isometry is called a unitary operator. Observe that, since $\|Tu\|^2_{\mathcal{H}_2} = \langle Tu, Tu \rangle_{\mathcal{H}_2} = \langle u, T^*Tu \rangle_{\mathcal{H}_1}$

for every $u \in \mathcal{H}_1$, the polarization identity implies that T is an isometry if and only if $T^*T = \mathrm{id}_{\mathcal{H}_1}$. Hence, if T is a unitary operator its inverse coincides with its adjoint. Finally, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, the set

$$\mathcal{U}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is unitary}\}$$

is a group with respect to the composition. Let G be a locally compact second countable Hausdorff topological group and let $\mathcal{H}$ be a separable Hilbert space.

Definition 1.16. A unitary representation of G on the Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ continuous in the strong operator topology, that is for every g, $h \in G$:

i) $\pi(gh) = \pi(g)\pi(h)$,

ii) $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$,

iii) $g \mapsto \pi(g)u$ is continuous for every $u \in \mathcal{H}$.

The Hilbert space $\mathcal{H}$ is called the representation space of π and its dimension is called the dimension or degree of π.

It is immediate to see that from the equality $\|\pi(g)u - \pi(h)u\|_{\mathcal{H}} = \|\pi(h^{-1}g)u - u\|_{\mathcal{H}}$ it is enough to check the continuity of the maps $g \mapsto \pi(g)u$, $u \in \mathcal{H}$, at the identity $e \in G$. Furthermore, since $\pi(g)$ is a unitary operator for every $g \in G$, we have that $\|\pi(g)u - u\|^2_{\mathcal{H}} = 2\|u\|^2_{\mathcal{H}} - 2\mathrm{Re}(\langle \pi(g)u, u\rangle_{\mathcal{H}})$. Therefore, the strong continuity is implied by requiring that the maps $g \mapsto \langle \pi(g)u, u\rangle_{\mathcal{H}}$ are continuous for every $u \in \mathcal{H}$. This is because the weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{H})$.

Example 1.17. Let G be a lcsc group and let $X \simeq G/H$ be a homogeneous space, where H is a closed subgroup of G. We denote by $g[\cdot]$ the transitive action of G on X and we suppose that X admits a G-invariant measure μ. Then G acts on $L^2(X, \mathrm{d}\mu)$ by the unitary representation $\pi : G \to \mathcal{U}(L^2(X, \mathrm{d}\mu))$ defined by

$$\pi(g)f(x) = f(g^{-1}[x]).$$

In general, if μ is a relatively-invariant measure on X with positive character χ, then G acts on $L^2(X, \mathrm{d}\mu)$ by the unitary representation $\pi : G \to \mathcal{U}(L^2(X, \mathrm{d}\mu))$ defined by

$$\pi(g)f(x) = \chi(g)^{-\frac{1}{2}} f(g^{-1}[x]),$$

which is called the quasi-regular representation of G on $L^2(X, \mathrm{d}\mu)$. We refer to Chapter 3 in [28] for details.

Example 1.2.1 continued**.** The affine group acts on $L^2(\mathbb{R})$ by means of the wavelet representation π : "$ax + b$" $\to \mathcal{U}(L^2(\mathbb{R}))$ defined for every $(b, a) \in$ "$ax + b$" by

$$\pi(b, a)f(x) = a^{-\frac{1}{2}} f\left(\frac{x - b}{a}\right).$$

We observe that the operator $\pi(b, a)$ is just the composition of the translation and dilation operators $T_b \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $D_a \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined respectively for every $b \in \mathbb{R}$, $a \in \mathbb{R}_+$ as

$$T_b f(x) = f(x - b), \quad D_a f(x) = a^{-\frac{1}{2}} f\left(\frac{x}{a}\right).$$

Precisely,

$$\pi(b,a) = T_b D_a. \tag{1.2}$$

It is not difficult to check that π is a unitary representation.

Let π be a unitary representation of G on the Hilbert space $\mathcal{H}$.

Definition 1.18. Let $\mathcal{M}$ be a subspace of $\mathcal{H}$. We say that $\mathcal{M}$ is a π-invariant subspace if $\pi(g)\mathcal{M} \subseteq \mathcal{M}$ for every $g \in G$.

Definition 1.19. We say that π is irreducible if the only closed π-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.

Definition 1.20. For every $u, v \in \mathcal{H}$, the function $G \ni g \mapsto \langle u, \pi(g)v\rangle_{\mathcal{H}} \in \mathbb{C}$ is called the coefficient of π relative to u and v. If $u = v$ this function is called a diagonal coefficient.

It is worth observing that every coefficient is a continuous function of G into $\mathbb{C}$ by (iii) in Definition 1.16 and it is bounded since $|\langle u, \pi(g)v\rangle_{\mathcal{H}}| \leq \|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$ for every $g \in G$.

Proposition 1.21 ([22, Proposition 2.47])**.** *The following conditions are equivalent:*

i) π *is irreducible,*

ii) if u *and* v *are non-zero in* $\mathcal{H}$*, then the coefficient of* π *relative to* u *and* v *is not the zero function.*

Example 1.2.1 continued**.** We show that the wavelet representation is not irreducible. The Fourier transform is denoted by $\mathcal{F}$ both on $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$, where it is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\mathrm{d}x, \quad f \in L^1(\mathbb{R})$$

and its inverse is denoted by $\mathcal{F}^{-1}$. A direct calculation gives that

$$\mathcal{F}(\pi(b,a)f)(\xi) = a^{\frac{1}{2}}e^{-2\pi i b\xi}\mathcal{F}f(a\xi). \tag{1.3}$$

Consider two nonzero functions $f, g \in L^2(\mathbb{R})$. By Plancherel formula we have that

$$\begin{aligned}\int_{\text{“}ax+b\text{”}} |\langle \pi(b,a)f, g\rangle|^2 \frac{\mathrm{d}b\mathrm{d}a}{a^2} &= \int_{\text{“}ax+b\text{”}} |\langle \mathcal{F}(\pi(b,a)f), \mathcal{F}g\rangle|^2 \frac{\mathrm{d}b\mathrm{d}a}{a^2} \\ &= \int_{\text{“}ax+b\text{”}} \left|\int_{\mathbb{R}} \mathcal{F}f(a\xi)\overline{\mathcal{F}g(\xi)}e^{-2\pi i b\xi}\mathrm{d}\xi\right|^2 \frac{\mathrm{d}b\mathrm{d}a}{a} \\ &= \int_{\text{“}ax+b\text{”}} |\mathcal{F}^{-1}\omega_a(-b)|^2 \mathrm{d}b\frac{\mathrm{d}a}{a},\end{aligned}$$

where $\omega_a(\xi) = \mathcal{F}f(a\xi)\overline{\mathcal{F}g(\xi)}$. Hence, by applying again Plancherel formula,

$$\begin{aligned}\int_{\text{“}ax+b\text{”}} |\langle \pi(b,a)f, g\rangle|^2 \frac{\mathrm{d}b\mathrm{d}a}{a^2} &= \int_{\mathbb{R}_+} ||\mathcal{F}^{-1}\omega_a||_2^2 \frac{\mathrm{d}a}{a} = \int_{\mathbb{R}_+} ||\omega_a||_2^2 \frac{\mathrm{d}a}{a} \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} |\mathcal{F}f(a\xi)\overline{\mathcal{F}g(\xi)}|^2 \mathrm{d}\xi\frac{\mathrm{d}a}{a} \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} |\mathcal{F}f(a\xi)|^2\frac{\mathrm{d}a}{a}\right) |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi. \end{aligned} \tag{1.4}$$

Now, if we consider the Hardy spaces

$$\mathcal{H}_+ = \{f \in L^2(\mathbb{R}) : \mathcal{F}f(\xi) = 0 \text{ if } \xi < 0\}, \quad \mathcal{H}_- = \{f \in L^2(\mathbb{R}) : \mathcal{F}f(\xi) = 0 \text{ if } \xi > 0\}$$

and we take two nonzero functions $f \in \mathcal{H}_+$, $g \in \mathcal{H}_-$, then $\operatorname{supp}(\mathcal{F}g) \subseteq (-\infty, 0]$ and $\mathcal{F}f(a\xi) = 0$ for all $\xi \in \operatorname{supp}(\mathcal{F}g)$ and $a \in \mathbb{R}_+$. Hence, by (1.4), the coefficient $(b,a) \mapsto \langle \pi(b,a)f, g\rangle$ is identically zero and the wavelet representation is not irreducible by Proposition 1.21. The Hardy space $\mathcal{H}_+$ is a closed π-invariant subspace of $L^2(\mathbb{R})$ and the wavelet representation π is irreducible if restricted to $\mathcal{H}_+$. Indeed, by the change of variables $a \mapsto a/\xi$ in the inner integral, (1.4) becomes

$$\int_{\text{“}ax+b\text{”}} |\langle \pi(b,a)f, g\rangle|^2 \frac{\mathrm{d}b\mathrm{d}a}{a^2} = \|g\|^2 \int_{\mathbb{R}_+} |\mathcal{F}f(a)|^2 \frac{\mathrm{d}a}{a}, \tag{1.5}$$

which is not zero if both $f, g \in \mathcal{H}_+$ are not zero and we conclude by Proposition 1.21. The same holds true for $\mathcal{H}_-$.

Let π_1 and π_2 be two unitary representations of G on the separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Definition 1.22. We call $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ an intertwining operator for π_1 and π_2 if it satisfies

$$\pi_2(g)T = T\pi_1(g)$$

for every $g \in G$.

We denote by $C(\pi_1, \pi_2)$ the set of all the operators that intertwine π_1 and π_2. If $\pi_1 = \pi_2 = \pi$, we write $C(\pi, \pi) = C(\pi)$.

Definition 1.23. The representations π_1 and π_2 are called unitarily equivalent if there exists a unitary operator $U \in C(\pi_1, \pi_2)$ and we write $\pi_1 \sim \pi_2$.

The next result is one of the fundamental theorems in representation theory and it is known as Schur's lemma. It gives a necessary and sufficient condition for a representation to be irreducible and it describes the set $C(\pi_1, \pi_2)$ if both the representations are irreducible.

Theorem 1.24 (Schur's Lemma [28, Lemma 3.5])**.** *Let π_1 and π_2 be two unitary representations of G on the separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and let $T \in C(\pi_1, \pi_2)$.*

i) Suppose that $\pi_1 = \pi_2$. Then, π_1 is irreducible if and only if $C(\pi_1)$ contains only scalar multiples of the identity.

ii) Suppose that π_1 is irreducible. Then $T = \lambda S$ for some $\lambda \geq 0$ and some isometry S. If in addition π_2 is irreducible, then

$$\dim C(\pi_1, \pi_2) = \begin{cases} 1 & \textit{if} \quad \pi_1 \sim \pi_2 \\ 0 & \textit{if} \quad \pi_1 \not\sim \pi_2. \end{cases}$$

Corollary 1.25 ([28, Corollary 3.6])**.** *If G is Abelian, then every irreducible representation of G is one dimensional.*

We are interested in a generalization of Schur's Lemma for unbounded operators and for this reason we start extending the definition of intertwining operator.

Definition 1.26. Let T be a densely defined closed operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. We call T an intertwining operator for π_1 and π_2 if $\mathrm{dom}(T)$ is π_1-invariant and it satisfies for every $g \in G$

$$\pi_2(g)T = T\pi_1(g).$$

Theorem 1.27 ([23, Proposition 12.2.2]). *Let T be a densely defined closed intertwining operator for π_1 and π_2 and suppose that π_1 is irreducible. Then $\mathrm{dom}(T) = \mathcal{H}_1$ and $T = \lambda S$ for some $\lambda \geq 0$ and some isometry S. If in addition π_2 is irreducible, then S may be taken as a unitary operator. Furthermore, given another densely defined closed intertwining operator T' for π_1 and π_2, then T' is proportional to T.*

1.2.2 An Extension of Schur's Lemma

This subsection is devoted to the extension of Schur's lemma due to Duflo and Moore [26] on which our main result of Chapter 2 is based. First, we state this result in its classical version [26], then we reformulate it under weaker assumptions in view of Chapter 2.

Let G be a locally compact second countable group and let π_1 and π_2 be two unitary representations of G on the separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Finally, let ζ be a character of G.

Definition 1.28. A densely defined closed operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is called semi-invariant with weight ζ if it satisfies for every $g \in G$

$$\pi_2(g)T\pi_1(g)^{-1} = \zeta(g)T.$$

If the character ζ is unitary, then T is an intertwining operator for the unitary representations $\zeta \otimes \pi_1$ and π_2, where $\zeta \otimes \pi_1$ is defined on $\mathcal{H}_1$ for every $g \in G$ by

$$\zeta \otimes \pi_1(g)u = \zeta(g)\pi_1(g)u.$$

It is worth observing that if π_1 is irreducible, then such is $\zeta \otimes \pi_1$. Therefore, assuming that π_1 is irreducible, T is a scalar multiple of an isometry by Theorem 1.27. In [26] Duflo and Moore extend such a result to the case when ζ is not unitary. Before stating this generalization, we need the polar decomposition for unbounded operators. Recall that a bounded operator U between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is called a partial isometry if U is an isometry when restricted to the closed subspace $\mathrm{Ker}\, U^{\perp}$.

Theorem 1.29 ([62, Theorem VIII.32]). *Let T be a closed operator on a Hilbert space $\mathcal{H}$. Then, there exists a positive self-adjoint operator $|T|$, with $\mathrm{dom}\,|T| = \mathrm{dom}\, T$ and a partial isometry U, with $\mathrm{dom}\, U = \mathrm{Ker}\, T^{\perp}$ and $\mathrm{Ran}\, U = \overline{\mathrm{Ran}\, T}$, so that $T = U|T|$.*

We are finally ready to state a generalization of Schur's lemma for semi-invariant operators.

Theorem 1.30 ([26, Theorem 1]). *With the above notation, assume that π_1 is irreducible. Let T be a densely defined closed nonzero operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, semi-invariant with weight ζ.*

i) Suppose that $\pi_1 = \pi_2$. If T' is another densely defined closed operator from $\mathcal{H}_1$ to $\mathcal{H}_1$, semi-invariant with weight ζ, then T' is proportional to T.

ii) Let $T = \mathcal{Q}|T|$ be the polar decomposition of T. Then $|T|$ is a positive selfadjoint operator in $\mathcal{H}_1$ semi-invariant with weight $|\zeta|$, and $\mathcal{Q}$ is a partial isometry of $\mathcal{H}_1$ into $\mathcal{H}_2$, semi-invariant with weight $\zeta/|\zeta|$.

As shown by our next corollary, the assumption that T is closed can be removed assuming that π_2 is also irreducible and the domain of T^* is non-trivial. The following version of Theorem 1.30 will perfectly suit our hypotheses in Chapter 2.

Corollary 1.31 ([2, Corollary 3.4])**.** *With the above notation, assume that π_1 and π_2 are irreducible and let ζ be a character of G. Suppose that T is an operator of $\mathcal{H}_1$ into $\mathcal{H}_2$ such that* $\mathrm{dom}(T)$ *is non-trivial and π_1-invariant and $\pi_2(g)T\pi_1(g)^{-1} = \zeta(g)T$ for all $g \in G$. Then,* $\mathrm{dom}(T)$ *is dense and if* $\mathrm{dom}(T^*)$ *is non-trivial, then we have that*

i) $\mathrm{dom}(T^*)$ *is π_2-invariant and dense;*

ii) T is closable, the closure $\overline{T}$ of T is semi-invariant with weight ζ and $\overline{T}$ is the unique closed extension of T;

iii) $T^* = \overline{T}^*$ is a densely defined closed operator, and is semi-invariant with weight $\overline{\zeta}$.

Proof. Since $\mathrm{dom}(T)$ is non-trivial and π_1-invariant, the irreducibility of π_1 implies that $\mathrm{dom}(T)$ is dense and the adjoint operator T^* is well-defined. We suppose that $\mathrm{dom}(T^*)$ is non-trivial and we prove the first claim. Given $h \in \mathrm{dom}(T^*)$ and $g \in G$, for all $f \in \mathrm{dom}(T)$

$$\begin{aligned}|\langle \pi_2(g)h, Tf\rangle| &= |\overline{\zeta(g^{-1})}\langle h, T\pi_1(g^{-1})f\rangle| = |\zeta(g^{-1})||\langle T^*h, \pi_1(g^{-1})f\rangle| \\ &\leq |\zeta(g^{-1})|\|T^*h\|_{\mathcal{H}_1}\|f\|_{\mathcal{H}_1},\end{aligned}$$

which implies that $\pi_2(g)h \in \mathrm{dom}(T^*)$. Then, the irreducibility of π_2 implies that $\mathrm{dom}(T^*)$ is dense and, hence, T is closable, see [62, Theorem VIII.1].

Let T' be a closed extension of T. We claim that T' is semi-invariant with weight ζ. Given $f \in \mathrm{dom}(T')$, then there exists a sequence (f_n) in $\mathrm{dom}(T)$ such that it converges to f and (Tf_n) convergences to $T'f$. Given $g \in G$, clearly $(\pi_1(g)f_n)$ converges to $\pi_1(g)f$ and

$$\lim_{n\to+\infty} T\pi_1(g)f_n = \lim_{n\to+\infty} \zeta(g^{-1})\pi_2(g)Tf_n = \zeta(g^{-1})\pi_2(g)T'f.$$

Since T' is closed, then $\pi_1(g)f \in \mathrm{dom}(T')$ and

$$\zeta(g^{-1})\pi_2(g)T'f = T'\pi_1(g)f,$$

so that T' is semi-invariant with weight ζ. It follows, in particular, that $\overline{T}$ is a densely defined closed operator semi-invariant with weight ζ. By item ii) of Theorem 1.30, $|T'|$ and $|\overline{T}|$ are semi-invariant operators with weight $|\zeta|$, and by item i) of the same theorem, $|T'| = c|\overline{T}|$ for a constant $c > 0$, hence $\mathrm{dom}(T') = \mathrm{dom}(\overline{T})$, so that $T' = \overline{T}$.

Finally, $T^* = \overline{T}^*$ is a densely defined closed operator by [62, Theorem VIII.1] and the semi-invariance of T^* follows straightforwardly. □

1.2.3 Square-Integrable Representations

In this subsection we focus on square-integrable representations. This class of irreducible representations is of great interest for our purposes since they ensure the existence of reproducing formulae. Such reproducing formulae allow to reconstruct any element f in the representation space $\mathcal{H}$ of π by gluing the projections of f along the "directions" in $\mathcal{H}$ given by the action of G on a fixed vector $\psi \in \mathcal{H}$, which is said to be admissible. We will see in Chapter 2 how the reproducing formulae given by the theory of representations allow to obtain inversion formulae for generalized Radon transforms starting from the group structure which defines these transforms.

Let G be a locally compact second countable group and let π be a unitary representation of G on the separable Hilbert space $\mathcal{H}$.

Definition 1.32. Given a vector $\psi \in \mathcal{H}$, the voice transform associated to ψ is the function $\mathcal{V}_\psi : \mathcal{H} \to L^\infty(G) \cap C(G)$ defined for any $u \in \mathcal{H}$ by

$$\mathcal{V}_\psi u(g) = \langle u, \pi(g)\psi \rangle_{\mathcal{H}}.$$

Observe that the continuity of $\mathcal{V}_\psi u$ follows by (iii) in Definition 1.16 and the boundedness follows from $|\mathcal{V}_\psi u(g)| = |\langle u, \pi(g)\psi\rangle_{\mathcal{H}}| \leq \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$ for every $g \in G$. Thus, for every fixed $\psi \in \mathcal{H}$, the voice transform maps elements in $\mathcal{H}$ to functions on G that are bounded and continuous.

Definition 1.33. A vector $\psi \in \mathcal{H}$ is called an admissible vector for the representation π if the corresponding voice transform $\mathcal{V}_\psi$ is an isometry from $\mathcal{H}$ into $L^2(G)$, the Hilbert space of square-integrable functions with respect to the left Haar measure on G.

We refer to an admissible vector for a representation π also in the case when the corresponding voice transform $\mathcal{V}_\psi$ is a multiple of an isometry from $\mathcal{H}$ into $L^2(G)$.

Definition 1.34. If π is irreducible and admits an admissible vector, then we say that π is square-integrable.

Example 1.2.1 continued**.** Fix $\psi \in L^2(\mathbb{R})$. The voice transform $\mathcal{V}_\psi$ defined by

$$\mathcal{V}_\psi f(b,a) = \langle f, \pi(b,a)\psi \rangle = a^{-\frac{1}{2}} \int_{\mathbb{R}} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} \mathrm{d}x$$

is called the continuous wavelet transform. By (1.4), if $\psi \in \mathcal{H}_+$ is such that

$$\int_{\mathbb{R}_+} |\mathcal{F}\psi(a)|^2 \frac{\mathrm{d}a}{a} = 1, \tag{1.6}$$

then for every $f \in \mathcal{H}_+$

$$||\mathcal{V}_\psi f||_2^2 = \int_{\text{“}ax+b\text{”}} |\langle f, \pi(b,a)\psi\rangle|^2 \frac{\mathrm{d}b\mathrm{d}a}{a^2} = \|f\|^2 \int_{\mathbb{R}_+} |\mathcal{F}\psi(a)|^2 \frac{\mathrm{d}a}{a} = \|f\|^2,$$

which shows that the voice transform $\mathcal{V}_\psi$ is an isometry of $\mathcal{H}_+$ into $L^2(G)$ and that ψ is an admissible vector. Condition (1.6) is called Calderón equation and characterizes the admissible vectors for the wavelet representation restricted to the Hardy space $\mathcal{H}_+$, the so-called wavelets. Therefore, the wavelet representation is square-integrable if restricted to the Hardy space $\mathcal{H}_+$. The same holds true for $\mathcal{H}_-$.

Now, we state a classical result due to Duflo and Moore [26].

Theorem 1.35 ([26, Theorem 3])**.** *Let π be a square-integrable representation of G on $\mathcal{H}$. Then there exists a unique positive self-adjoint operator*

$$C\colon \operatorname{dom} C \subseteq \mathcal{H} \to \mathcal{H},$$

semi-invariant with weight $\Delta^{\frac{1}{2}}$, where Δ is the modular function of G, satisfying the following conditions:

i) let ψ, $u \in \mathcal{H}$, $u \neq 0$, then $\mathcal{V}_\psi u \in L^2(G)$ if and only if $\psi \in \operatorname{dom} C$;

ii) let ψ_1, $\psi_2 \in \operatorname{dom} C$ and u_1, $u_2 \in \mathcal{H}$, then

$$\int_G \mathcal{V}_{\psi_1} u_1(g)\overline{\mathcal{V}_{\psi_2} u_2(g)}\mathrm{d}g = \langle C\psi_2, C\psi_1\rangle\langle u_1, u_2\rangle. \tag{1.7}$$

By the above theorem, $\psi \in \mathcal{H}$ is an admissible vector for π if and only if $\psi \in \operatorname{dom} C$ and $\|C\psi\|_{\mathcal{H}} = 1$. Furthermore, it is worth observing that if G is unimodular, then by Theorem 1.27 the operator C is a scalar multiple of the identity. In the nonunimodular case, instead, the operator C cannot be bounded. Indeed, if it was bounded, then by the semi-invariance property

$$\pi(g)C\pi(g)^{-1} = \Delta(g)^{\frac{1}{2}}C,$$

and it would follow that $\|C\| = \Delta(g)^{\frac{1}{2}}\|C\|$ for every $g \in G$, where here $\|\cdot\|$ denotes the operator norm. This would imply $\Delta(g) = 1$ for every $g \in G$, a contradiction. The operator C is known as the Duflo-Moore operator of π.

Example 1.2.1 continued**.** Consider the subspace

$$\mathcal{D} = \{\psi \in \mathcal{H}_+ : \int_{\mathbb{R}_+} \xi^{-1}|\mathcal{F}\psi(\xi)|^2\mathrm{d}\xi < +\infty\}$$

of $\mathcal{H}_+$ and define the Fourier multiplier $C\colon \mathcal{D} \to \mathcal{H}_+$ by

$$\mathcal{F}C\psi(\xi) = \xi^{-\frac{1}{2}}\mathcal{F}\psi(\xi).$$

Then, the spectral theorem for unbounded operators, see Theorem VIII.6 of [62], shows that $\mathcal{D}$ is dense and that C is a positive, densely defined self-adjoint operator. Furthermore, by (1.3) and by definition of C, it satisfies the semi-invariance property

$$\begin{aligned}
\mathcal{F}(\pi(b,a)C\pi(b,a)^{-1}\psi)(\xi) &= a^{\frac{1}{2}}e^{-2\pi ib\xi}\mathcal{F}(C\pi(b,a)^{-1}\psi)(a\xi)\\
&= \xi^{-\frac{1}{2}}e^{-2\pi ib\xi}\mathcal{F}(\pi(b,a)^{-1}\psi)(a\xi)\\
&= a^{-\frac{1}{2}}\xi^{-\frac{1}{2}}e^{-2\pi ib\xi}e^{2\pi i\frac{b}{a}a\xi}\mathcal{F}\psi(\frac{a\xi}{a})\\
&= a^{-\frac{1}{2}}\xi^{-\frac{1}{2}}\mathcal{F}\psi(\xi) = \Delta^{\frac{1}{2}}\mathcal{F}C\psi(\xi).
\end{aligned}$$

Hence, since $\Delta(b,a) = a^{-1}$ and by the injectivity of the Fourier transform, it follows that C is a semi-invariant operator with weight $\Delta^{\frac{1}{2}}$. Furthermore, by (1.5), the relation (1.7) is satisfied and for every $f \in \mathcal{H}_+$ the wavelet transform $\mathcal{V}_\psi f$ is an isometry from $\mathcal{H}_+$ into $L^2(G)$ if and only if $\psi \in \mathcal{D}$ and $\|C\psi\|_{L^2(\mathbb{R}_+)} = 1$.

In order to state the last result of this subsection we need to introduce the concept of weak integral.

Definition 1.36. If $\Psi\colon G\to\mathcal{H}$ is a map such that

$$u\mapsto\int_G\langle\Psi(g),u\rangle_{\mathcal{H}}\mathrm{d}g$$

is a continuous linear functional on $\mathcal{H}$, we denote by

$$\int_G\Psi(g)\mathrm{d}g$$

the unique element in $\mathcal{H}$ for which

$$\langle\int_G\Psi(g)\mathrm{d}g,u\rangle_{\mathcal{H}}=\int_G\langle\Psi(g),u\rangle_{\mathcal{H}}\mathrm{d}g,$$

for every $u\in\mathcal{H}$. We call such an element the weak integral of Ψ.

The existence and uniqueness of the weak integral of Ψ follows by the Riesz lemma [62, Theorem II.4].

Theorem 1.37 ([30, Theorem 2.25])**.** *Suppose that π admits an admissible vector $\psi\in\mathcal{H}$. Then, for any $u\in\mathcal{H}$ we have the reproducing formula*

$$u=\int_G\mathcal{V}_\psi u(g)\pi(g)\psi\ \mathrm{d}\mu(g),\tag{1.8}$$

where μ is the Haar measure and the right-hand side is interpreted as weak integral, and

$$\|u\|^2_{\mathcal{H}}=\int_G|\mathcal{V}_\psi u(g)|^2\ \mathrm{d}\mu(g).$$

For every $u\in\mathcal{H}$, the right-hand side in (1.8) is the weak integral of the map $\Psi_u(g)=\mathcal{V}_\psi u(g)\pi(g)\psi$, which is therefore defined by the relation

$$\langle\int_G\mathcal{V}_\psi u(g)\pi(g)\psi\ \mathrm{d}\mu(g),v\rangle_{\mathcal{H}}=\int_G\mathcal{V}_\psi u(g)\langle\pi(g)\psi,v\rangle_{\mathcal{H}}\ \mathrm{d}\mu(g),\qquad v\in\mathcal{H}.$$

Theorem 1.37 states that if a unitary representation π admits an admissible vector ψ, then (1.8) allows to reconstruct any element $u\in\mathcal{H}$ by computing the family of coefficients $\{\langle u,\pi(g)\psi\rangle_{\mathcal{H}}\}_{g\in G}$. This result will have a central role in Chapter 2, where we obtain inversion formulae for generalized Radon transforms based on square-integrable representations which arise from the group structure which defines these transforms.

Example 1.2.1 continued**.** Consider $\psi\in\mathcal{H}_+$ which satisfies equation (1.6). Then, for every $f\in\mathcal{H}_+$ we have the reproducing formula

$$f=\int_{\mathbb{R}\times\mathbb{R}_+}\mathcal{V}_\psi f(b,a)\pi(b,a)\psi\frac{\mathrm{d}b\mathrm{d}a}{a^2}=a^{-\frac{1}{2}}\int_{\mathbb{R}\times\mathbb{R}_+}\mathcal{V}_\psi f(b,a)\psi\left(\frac{x-b}{a}\right)\frac{\mathrm{d}b\mathrm{d}a}{a^2},$$

where the right-hand side is interpreted as weak integral.

If we allow the scale parameter a to vary over $\mathbb{R}^\times=\mathbb{R}\setminus\{0\}$ instead of $\mathbb{R}_+$, we obtain a non-connected version of the affine group which is called the full affine group, denoted by $\mathbb{W}$. This is $\mathbb{R} \times \mathbb{R}^\times$ with the group operation (1.1) and left Haar measure $|a|^{-2}\mathrm{d}b\mathrm{d}a$. It acts on $L^2(\mathbb{R})$ by means of the square-integrable representation

$$W_{b,a}f(x) = |a|^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right).$$

The wavelet transform is then $\mathcal{W}_\psi f(b,a) = \langle f, W_{b,a}\psi\rangle$, which is a multiple of an isometry provided that $\psi \in L^2(\mathbb{R})$ satisfies the admissibility condition, namely the Calderón equation,

$$0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi|} \mathrm{d}\xi < +\infty \tag{1.9}$$

and, in such a case, ψ is called a one-dimensional wavelet. The proof of these statements follows by analogues arguments to the example of the classical affine group "$ax + b$". See [22] for details.

1.3 The Radon Transform

In this Section we recall the basic definitions and the fundamental results in Radon transform theory. In subsection 1.3.1 we present the classical Radon transform focusing on the L^2 setting in view of Chapter 2. Indeed, although the classical Radon theory is well-known and deeply investigated, all the classical references (see for example [38, 57]) mostly deal with the function spaces $C_c^\infty(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$. Then in subsection 1.3.2 we present the theory of general Radon transform between homogeneous spaces in duality introduced by Helgason [38, Chapter 2]. We explain how this general theory is motivated starting from the group structure lying under the classical Radon transform, we recall the whole construction in the classical setting and finally we list some natural problems motivated by the classical Radon theory which we address in Chapter 2.

1.3.1 The Classical Radon Transform

We first define the Radon transform on $L^1(\mathbb{R}^d)$ by following the approach in [61], see also [38] as a classical reference. Then we introduce the particular restriction of the Radon transform in which we are interested, the so-called polar Radon transform, obtained by parametrizing the space of hyperplanes by polar coordinates.

Given $f \in L^1(\mathbb{R}^d)$ its Radon transform is the function $\mathcal{R}f : (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{R}f(n,t) = \frac{1}{|n|} \int_{n\cdot x = t} f(x)\, \mathrm{d}m(x), \tag{1.10}$$

where m is the Euclidean measure on the hyperplane

$$(n:t) = \{x \in \mathbb{R}^d : n \cdot x = t\} \tag{1.11}$$

and the equality (1.10) holds for almost all $(n,t) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}$. We add some comments. Definition (1.10) makes sense since, given $n \in \mathbb{R}^d \setminus \{0\}$ Fubini theorem gives that

$$\int_{\mathbb{R}^d} |f(x)|\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{n\cdot x=t} |f(x)|\mathrm{d}m(x)\right) \mathrm{d}t < +\infty,$$

so that for almost all $t \in \mathbb{R}$ the integral $\int_{n\cdot x=t} |f(x)|\mathrm{d}m(x)$ is finite and $\mathcal{R}(n,t)$ is well defined.

Furthermore, each pair $(n,t) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}$ defines the hyperplane $(n:t)$ by means of (1.11). Clearly, the correspondence between parameters (n,t) and hyperplanes is not bijective. Indeed (n',t') and (n,t) determine the same hyperplane if and only if there exists $\lambda \in \mathbb{R}^\times$ such that $n' = \lambda n$ and $t' = \lambda t$ and this equivalence relation motivates the notation $(n:t)$ for the hyperplane in (1.11). Because of the factor $1/|n|$ in (1.10), $\mathcal{R}f$ is a positively homogenous function of degree -1, i.e. for all $\lambda \in \mathbb{R}^\times$

$$\mathcal{R}f(\lambda n, \lambda t) = |\lambda|^{-1} \mathcal{R}f(n,t). \tag{1.12}$$

This means that $\mathcal{R}f$ is completely defined by choosing a representative (n,t) for each hyperplane $(n:t)$, i.e. by choosing a suitable system of coordinates on the affine Grassmannian

$$\{\text{hyperplanes of } \mathbb{R}^d\} \simeq \mathbb{P}^{d-1} \times \mathbb{R}.$$

The canonical choice [38] is given by parametrizing $\mathbb{P}^{d-1}$ with its two-fold covering S^{d-1}, where S^{d-1} is the unit sphere in $\mathbb{R}^d$ (see Figure 1.1 for $d=2$). The next proposition, whose proof can be found in [61], summarizes the behaviour of the Radon transform under affine linear actions. The translation and dilation operators act on a function $f : \mathbb{R}^d \to \mathbb{C}$ as

$$T_b f(x) = f(x-b), \qquad D_A f(x) = |\det A|^{-1} f\left(A^{-1}x\right),$$

respectively, for $b \in \mathbb{R}^d$ and $A \in \mathrm{GL}(d,\mathbb{R})$. Both operators map each $L^p(\mathbb{R}^d)$ onto itself and D_A is normalized to be an isometry on $L^1(\mathbb{R}^d)$.

Proposition 1.38 ([61])**.** *Given $f \in L^1(\mathbb{R}^d)$, the following properties hold true:*

(i) $\mathcal{R}T_b f(n,t) = \mathcal{R}f(n, t - n \cdot b)$, *for all* $b \in \mathbb{R}^d$*;*

(ii) $\mathcal{R}D_A f(n,t) = \mathcal{R}f({}^t\!An, t)$, *for all* $A \in GL(d,\mathbb{R})$.

We now state a crucial result in Radon transform theory in its standard version. Proposition 1.39 and its two variations taylored to our setting and presented in this subsection have a key role in several proofs throughout our manuscript and are also of some independent interest. We denote by I the identity operator.

Proposition 1.39 (Fourier slice theorem, 1 [38])**.** *For any* $f \in L^1(\mathbb{R}^d)$

$$(\mathrm{I} \otimes \mathcal{F})\mathcal{R}f(n,\tau) = \mathcal{F}f(\tau n).$$

for all $n \in \mathbb{R}^d \setminus \{0\}$ *and all* $\tau \in \mathbb{R}$.

Here the Fourier transform on the right-hand side is in $\mathbb{R}^d$, whereas the operator $\mathcal{F}$ on the left-hand side is one-dimensional and acts on the variabile t. We repeat this slight abuse of notation in other formulas below. Proposition 1.39 shows that the Radon transform is injective on $L^1(\mathbb{R}^d)$, indeed if $\mathcal{R}f$ is identically zero then $\mathcal{F}f$ is zero which implies that also f is zero by the injectivity of the Fourier transform. Proposition 1.39 explains how the Radon transform and the Fourier transform are related. More specifically, for any fixed $n \in \mathbb{R}^d \setminus \{0\}$, the Fourier transform of the one-dimensional function $\mathcal{R}f(n,\cdot)$ is equal to the Fourier transform of f restricted to the line passing through the origin with direction parallel to the vector n.

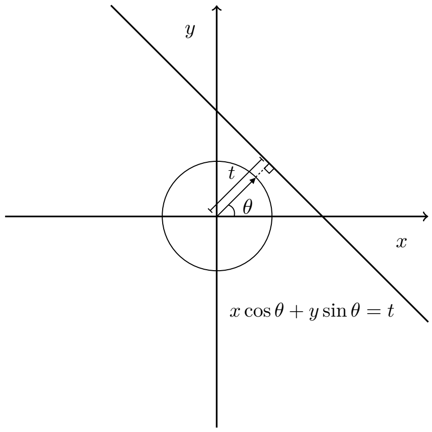

Figure 1.1: We parametrize lines in the plane by pairs $(\theta, t) \in [0, 2\pi) \times \mathbb{R}$ where θ identifies the direction perpendicular to the line and t identifies its distance from the origin.

As mentioned above, the natural restriction of the Radon transform is the polar Radon transform $\mathcal{R}^{\mathrm{pol}} f$, which is obtained by restricting $\mathcal{R}f$ to the closed subset $S^{d-1} \times \mathbb{R}$, where S^{d-1} is the unit sphere in $\mathbb{R}^d$. Define $\Theta^{d-1} = [0, \pi]^{d-2} \times [0, 2\pi)$. For all $\theta \in \Theta^{d-1}$ we write inductively

$$ {}^t\theta = (\theta_1, {}^t\hat{\theta}), \qquad \theta_1 \in [0, \pi], \ \hat{\theta} \in \Theta^{d-2} $$

and then we put

$$ {}^t n(\theta) = (\cos\theta_1, \sin\theta_1 {}^t n(\hat{\theta})), $$

where $n(\hat{\theta}) \in S^{d-2}$ corresponds to the previous inductive step. Clearly, the map $n : \Theta^{d-1} \to S^{d-1}$ induces a parametrization of the unit sphere in $\mathbb{R}^d$. Also, observe that the map $\Theta^{d-1} \times \mathbb{R} \to \mathbb{P}^{d-1}$ given by $(\theta, t) \mapsto (n(\theta) : t)$ is a two-fold covering of $\mathbb{P}^{d-1}$.

Definition 1.40. Take $f \in L^1(\mathbb{R}^d)$. The polar Radon transform of f is the function $\mathcal{R}^{\mathrm{pol}} f : \Theta^{d-1} \times \mathbb{R} \to \mathbb{C}$ defined by

$$ \mathcal{R}^{\mathrm{pol}} f(\theta, t) = \mathcal{R}f(n(\theta), t) = \int_{n(\theta)\cdot x = t} f(x) \, \mathrm{d}m(x). \tag{1.13} $$

For simplicity, from now on we fix $d = 2$ (see Figure 1.1), so $n(\theta) = (\cos\theta, \sin\theta)$ and the polar Radon transform $\mathcal{R}^{\mathrm{pol}} f : [0, 2\pi) \times \mathbb{R} \to \mathbb{C}$ has the following expression

$$ \mathcal{R}^{\mathrm{pol}} f(\theta, t) = \int_{\mathbb{R}} f(t\cos\theta - y\sin\theta, t\sin\theta + y\cos\theta) \mathrm{d}y, \tag{1.14} $$

where the equality (1.14) holds for almost every $(\theta, t) \in [0, 2\pi) \times \mathbb{R}$.

We have the following continuity results. We denote by $\mathcal{S}([0, 2\pi) \times \mathbb{R})$ the space of functions g which can be extended to be smooth and 2π-periodic in the variable θ and such that for any $k, l \in \mathbb{N}$

$$ \sup_{\theta \in [0,2\pi), t \in \mathbb{R}} |(1 + |t|^2)|^{\frac{k}{2}} |\frac{d^l}{dt^l} g(\theta, t)| < +\infty. $$

Theorem 1.41 ([37])**.** *The polar Radon transform is a continuous operator from $\mathcal{S}(\mathbb{R}^2)$ into $\mathcal{S}([0, 2\pi) \times \mathbb{R})$.*

Theorem 1.42 (Theorem 2.1 in [60]). *The polar Radon transform is a continuous operator from $L^1(\mathbb{R}^2)$ into $L^1([0, 2\pi) \times \mathbb{R})$.*

As mentioned above, the correspondence between parameters $(\theta, t) \in [0, 2\pi) \times \mathbb{R}$ and lines is not bijective. Indeed (θ, t) and $(\theta + \pi \bmod 2\pi, -t)$ determine the same line. Because of this fact the Radon transform satisfies the evenness property

$$\mathcal{R}^{\mathrm{pol}} f(\theta, t) = \mathcal{R}^{\mathrm{pol}} f(\theta + \pi \bmod 2\pi, -t). \tag{1.15}$$

We now prove a variation of Proposition 1.39 in the L^2 setting. This is a reformulation of [8, Proposition 6] adapted to the polar setting (see Figure 1.2).

Proposition 1.43 (Fourier slice theorem 2 [8, Proposition 6]). *Define $\psi : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^2$ by $\psi(\theta, \tau) = \tau n(\theta)$. For every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ there exists a negligible set $E \subseteq [0, 2\pi)$ such that for all $\theta \notin E$ the function $\mathcal{R}^{\mathrm{pol}} f(\theta, \cdot)$ is in $L^2(\mathbb{R})$ and satisfies*

$$\mathcal{R}^{\mathrm{pol}} f(\theta, \cdot) = \mathcal{F}^{-1}[\mathcal{F} f \circ \psi(\theta, \cdot)]. \tag{1.16}$$

Proof. By Proposition 1.39 we know that for all $\theta \in [0, 2\pi)$ the Radon transform $\mathcal{R}^{\mathrm{pol}} f(\theta, \cdot)$ is in $L^1(\mathbb{R})$ and satisfies

$$(\mathrm{I} \otimes \mathcal{F}) \mathcal{R}^{\mathrm{pol}} f(\theta, \tau) = \mathcal{F} f \circ \psi(\theta, \tau), \qquad \tau \in \mathbb{R}.$$

We start by proving that the function $\tau \mapsto \mathcal{F} f \circ \psi(\theta, \tau)$ is in $L^2(\mathbb{R})$, that is

$$\int_{\mathbb{R}} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 \mathrm{d}\tau < +\infty.$$

By hypothesis and by a change of variables in polar coordinates we have that

$$\|f\|_2^2 = \int_{\mathbb{R}^2} |\mathcal{F} f(\xi)|^2 \mathrm{d}\xi = \frac{1}{2} \int_0^{2\pi} \int_{\mathbb{R}} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 |\tau| \mathrm{d}\tau \mathrm{d}\theta < +\infty,$$

so that there exists a negligible set $E \subseteq [0, 2\pi)$ such that

$$C_f := \int_{\mathbb{R}} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 |\tau| \mathrm{d}\tau < +\infty$$

for all $\theta \notin E$. Therefore, for all $\theta \notin E$ it holds

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 \mathrm{d}\tau &= \int_{|\tau| \leq 1} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 \mathrm{d}\tau + \int_{|\tau| > 1} \frac{|\tau|}{|\tau|} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 \mathrm{d}\tau \\ &\leq 2 \|\mathcal{F} f\|_\infty^2 + \int_{\mathbb{R}} |\mathcal{F} f \circ \psi(\theta, \tau)|^2 |\tau| \mathrm{d}\tau \\ &\leq 2 \|f\|_1^2 + C_f < +\infty. \end{aligned}$$

Hence the function $t \mapsto \mathcal{R}^{\mathrm{pol}} f(\theta, t)$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and (1.16) follows by the Fourier inversion formula in $L^2(\mathbb{R})$. □

We now introduce the pseudo-differential operators $-\Delta \colon \mathrm{dom}(-\Delta) \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ and $-\Lambda_t \colon \mathrm{dom}(-\Lambda_t) \subseteq L^2([0, 2\pi) \times \mathbb{R}) \to L^2([0, 2\pi) \times \mathbb{R})$ defined by

$$\mathcal{F}(-\Delta f)(\xi) = |\xi|^2 \mathcal{F} f(\xi), \qquad (I \otimes \mathcal{F})(-\Lambda_t g)(\theta, \tau) = |\tau|^2 (I \otimes \mathcal{F}) g(\theta, \tau)$$

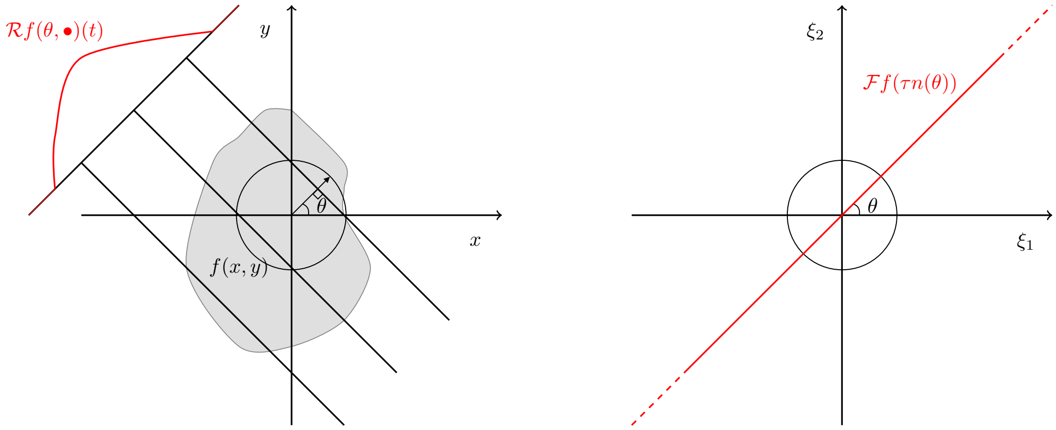

Figure 1.2: For every $\theta \in [0, 2\pi)$ the Fourier transform of the function $t \mapsto \mathcal{R}^{\mathrm{pol}} f(\theta, t)$ coincides with the Fourier transform of f restricted to the line passing through the origin with slope $\tan\theta$.

on the subspaces

$$\mathrm{dom}(-\Delta) = \{f \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\xi|^4 |\mathcal{F}(\xi)|^2 \mathrm{d}\xi < +\infty\},$$

$$\mathrm{dom}(-\Lambda_t) = \{g \in L^2([0, 2\pi) \times \mathbb{R}) : \int_{[0,2\pi)\times\mathbb{R}} |\tau|^4 |(I \otimes \mathcal{F})g(\theta, \tau)|^2 \mathrm{d}\theta\mathrm{d}\tau < +\infty\}.$$

The Radon transform satisfies the following intertwining property.

Proposition 1.44. *For any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$,*

$$\mathcal{R}^{\mathrm{pol}}(-\Delta f)(\theta, t) = -\Lambda_t \mathcal{R}^{\mathrm{pol}} f(\theta, t),$$

for almost every $(\theta, t) \in [0, 2\pi) \times \mathbb{R}$.

Proof. Given $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, applying Proposition 1.43 repeatedly, we have that

$$\begin{aligned}(\mathrm{I} \otimes \mathcal{F})\mathcal{R}^{\mathrm{pol}}(-\Delta f)(\theta, \tau) &= \mathcal{F}(-\Delta f)(\tau n(\theta)) = |\tau|^2 \mathcal{F} f(\tau n(\theta)) \\ &= |\tau|^2 (\mathrm{I} \otimes \mathcal{F})\mathcal{R}^{\mathrm{pol}} f(\theta, \tau) = (\mathrm{I} \otimes \mathcal{F})(-\Lambda_t \mathcal{R}^{\mathrm{pol}} f)(\theta, \tau)\end{aligned}$$

and we conclude by the injectivity of the unitary operator $(\mathrm{I} \otimes \mathcal{F})$ of $L^2([0, 2\pi) \times \mathbb{R})$ onto itself. □

The operators $-\Delta$ and $-\Lambda_t$, if restricted to the space of smooth functions with compact support $C_c^\infty(\mathbb{R}^2)$, coincide with minus the Laplacian operator on $\mathbb{R}^2$ and minus the partial derivative with respect to the variable t applied two times, respectively. In this case Proposition 1.44 reduces to Lemma 2.1 in [38] which follows immediately by Proposition 1.38, (i), as shown in [38].

It is possible to extend the Radon transform to an isometry of $L^2(\mathbb{R}^2)$ onto the closed subspace of functions

$$L^2_e([0, 2\pi) \times \mathbb{R}) = \{g \in L^2([0, 2\pi) \times \mathbb{R}) : g(\theta, t) = g(\theta + \pi \bmod 2\pi, -t)\}$$

of $L^2([0,2\pi)\times\mathbb{R})$, endowed with the measure $\mathrm{d}\theta\mathrm{d}t/2$, up to the composition with a pseudo-differential operator. Consider the subspace

$$\mathcal{D}=\{f\in L^2([0,2\pi)\times\mathbb{R}):\int_{[0,2\pi)\times\mathbb{R}}|\tau||(\mathrm{I}\otimes\mathcal{F})f(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau<+\infty\}$$

of $L^2([0,2\pi)\times\mathbb{R})$ and define the operator $\mathcal{I}\colon\mathcal{D}\to L^2([0,2\pi)\times\mathbb{R})$ by

$$(\mathrm{I}\otimes\mathcal{F})\mathcal{I}f(\theta,\tau)=|\tau|^{\frac{1}{2}}(\mathrm{I}\otimes\mathcal{F})f(\theta,\tau),$$

a Fourier multiplier with respect to the second variable. Since $\tau\mapsto|\tau|^{\frac{d-1}{2}}$ is a strictly positive (almost everywhere) Borel function on $\mathbb{R}$, the spectral theorem for unbounded operators, see Theorem VIII.6 of [62], shows that $\mathcal{D}$ is dense and that $\mathcal{I}$ is a positive self-adjoint injective operator. By Proposition 1.43, the Radon transform of every $f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ satisfies

$$\begin{aligned}
&\frac{1}{2}\int_{[0,2\pi)\times\mathbb{R}}|\mathcal{R}^{\mathrm{pol}}f(\theta,t)|^2\mathrm{d}\theta\mathrm{d}t=\frac{1}{2}\int_0^{2\pi}\int_{\mathbb{R}}|(\mathrm{I}\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}f(\theta,\tau)|^2\mathrm{d}\tau\mathrm{d}\theta\\
&=\frac{1}{2}\int_{[0,2\pi)\times\mathbb{R}}|\mathcal{F}f(\tau n(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau\\
&\leq\frac{1}{2}\int_0^{2\pi}\int_{|\tau|\leq1}|\mathcal{F}f(\tau n(\theta))|^2\mathrm{d}\tau\mathrm{d}\theta+\frac{1}{2}\int_0^{2\pi}\int_{|\tau|>1}|\tau||\mathcal{F}f(\tau n(\theta))|^2\mathrm{d}\tau\mathrm{d}\theta\\
&\leq 2\pi\|f\|_1^2+\frac{1}{2}\int_{[0,2\pi)\times\mathbb{R}}|\tau||\mathcal{F}f(\tau n(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau=2\pi\|f\|_1^2+\|f\|_2^2<+\infty,
\end{aligned}$$

which proves that $\mathcal{R}^{\mathrm{pol}}f\in L^2([0,2\pi)\times\mathbb{R})$. Furthermore, by Proposition 1.43 the Radon transform of $f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ always satisfies the condition

$$\begin{aligned}
\frac{1}{2}\int_{[0,2\pi)\times\mathbb{R}}|\tau||(\mathrm{I}\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}f(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau&=\frac{1}{2}\int_{[0,2\pi)\times\mathbb{R}}|\tau||\mathcal{F}f(\tau n(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau\\
&=\|f\|_2^2<+\infty.
\end{aligned}$$

Hence, $\mathcal{R}^{\mathrm{pol}}f\in\mathcal{D}$ for all $f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ and we can consider the composite operator $\mathcal{I}\mathcal{R}^{\mathrm{pol}}:L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)\to L^2([0,2\pi)\times\mathbb{R})$. Now, we are in a position to state one of the fundamental results in Radon transform theory, which will be of inspiration for our unitarization result in Chapter 2.

Theorem 1.45 (Theorem 4.1 in [38])**.** *The composite operator $\mathcal{I}\mathcal{R}^{\mathrm{pol}}$ from $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ into $L^2([0,2\pi)\times\mathbb{R})$ extends to a unique unitary operator $\mathcal{Q}$ from $L^2(\mathbb{R}^2)$ onto $L^2_e([0,2\pi)\times\mathbb{R})$.*

The image space is given by $L^2_e([0,2\pi)\times\mathbb{R})$ because of the evenness property (1.15) of the Radon transform. We give a third version of the Fourier slice theorem for the unitarized version of the Radon transform $\mathcal{Q}$. This is [8, Proposition 9] adapted to the polar setting. We think that it is perhaps known, but we could not locate it in the literature and for completeness we give the proof.

Proposition 1.46 ([8, Proposition 9])**.** *For all $f\in L^2(\mathbb{R}^2)$*

$$\mathcal{F}(\mathcal{Q}f(\theta,\cdot))(\tau)=|\tau|^{\frac{1}{2}}\mathcal{F}f(\tau n(\theta))\tag{1.17}$$

for almost every $(\theta,\tau)\in[0,2\pi)\times\mathbb{R}$.

Proof. We start by observing that (1.17) is true if $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ by the Fourier slice theorem and by the definition of $\mathcal{Q}$. Take now $f \in L^2(\mathbb{R}^2)$. By density there exists a sequence $(f_n)_n \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that $f_n \to f$ in $L^2(\mathbb{R}^2)$. Since $\mathcal{Q}$ is unitary from $L^2(\mathbb{R}^2)$ onto $L^2_e([0, 2\pi) \times \mathbb{R})$ and $\mathrm{I} \otimes \mathcal{F}$ is unitary from $L^2_e([0, 2\pi) \times \mathbb{R})$ into itself, where I is the identity operator, $(\mathrm{I} \otimes \mathcal{F})\mathcal{Q}f_n \to (\mathrm{I} \otimes \mathcal{F})\mathcal{Q}f$ in $L^2_e([0, 2\pi) \times \mathbb{R})$. Since $f_n \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, for almost every $(\theta, \tau) \in [0, 2\pi) \times \mathbb{R}$

$$
\begin{aligned}
(\mathrm{I} \otimes \mathcal{F})\mathcal{Q}f_n(\theta, \tau) &= (\mathrm{I} \otimes \mathcal{F})\mathcal{I}\mathcal{R}^{\mathrm{pol}} f_n(\theta, \tau) \\
&= |\tau|^{\frac{1}{2}} (\mathrm{I} \otimes \mathcal{F})\mathcal{R}^{\mathrm{pol}} f_n(\theta, \tau) \\
&= |\tau|^{\frac{1}{2}} \mathcal{F} f_n(\tau n(\theta)).
\end{aligned}
$$

So that, passing to a subsequence if necessary,

$$
|\tau|^{\frac{1}{2}} \mathcal{F} f_n(\tau n(\theta)) \to (\mathrm{I} \otimes \mathcal{F})\mathcal{Q}f(\theta, \tau)
$$

for almost every $(\theta, \tau) \in [0, 2\pi) \times \mathbb{R}$. Therefore for almost every $(\theta, \tau) \in [0, 2\pi) \times \mathbb{R}$,

$$
(\mathrm{I} \otimes \mathcal{F})\mathcal{Q}f(\theta, \tau) = \lim_{n \to +\infty} |\tau|^{\frac{1}{2}} \mathcal{F} f_n(\tau n(\theta)) = |\tau|^{\frac{1}{2}} \mathcal{F} f(\tau n(\theta)),
$$

where the last equality holds true using a subsequence if necessary. □

In order to state the most commonly used inversion formula for the Radon transform, known as Filtered Back Projection, we introduce the dual Radon transform. While the polar Radon transform is defined for any given line $(n(\theta) : t)$ as the integral over the set of points belonging to $(n(\theta) : t)$, the dual Radon transform is defined for any given point $x \in \mathbb{R}^2$ as the integral over the set of lines passing through x, that is $\{(n(\theta) : n(\theta) \cdot x) : \theta \in [0, 2\pi)\}$.

Definition 1.47. Given $g \in L^\infty([0, 2\pi) \times \mathbb{R})$, the dual Radon transform (or back-projection) of g is the L^∞ function $\mathcal{R}^\# g : \mathbb{R}^2 \to \mathbb{C}$ given by

$$
\mathcal{R}^\# g(x) = \int_0^{2\pi} g(\theta, n(\theta) \cdot x) \mathrm{d}\theta, \qquad x \in \mathbb{R}^2. \tag{1.18}
$$

Proposition 1.48 ([57])**.** *For any $f \in L^1(\mathbb{R}^2)$ and $g \in L^\infty([0, 2\pi) \times \mathbb{R})$,*

$$
\int_{\mathbb{R}^2} f(x)\mathcal{R}^\# g(x) \mathrm{d}x = \int_{[0,2\pi) \times \mathbb{R}} \mathcal{R}^{\mathrm{pol}} f(\theta, t) g(\theta, t) \mathrm{d}\theta \mathrm{d}t. \tag{1.19}
$$

The duality relation (1.19) can be exploited to extend the definition of the dual Radon transform as a continuous map between distribution spaces. We define the dual Radon transform $\mathcal{R}^\# : \mathcal{S}'([0, 2\pi) \times \mathbb{R}) \to \mathcal{S}'(\mathbb{R}^2)$ by

$$
\langle \mathcal{R}^\# g, f \rangle = \langle g, \mathcal{R}^{\mathrm{pol}} f \rangle,
$$

for any $g \in \mathcal{S}'([0, 2\pi) \times \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^2)$, which is well defined and weakly continuous by Theorem 1.41.

Theorem 1.49 (Theorem 1.4 in [57])**.** *For any $g \in \mathcal{S}([0, 2\pi) \times \mathbb{R})$, the distribution $\mathcal{F}\mathcal{R}^\# g$ is represented by the function*

$$
\mathcal{F}\mathcal{R}^\# g(\xi) = |\xi|^{-1} \left((I \otimes \mathcal{F}) g(\frac{\xi}{|\xi|}, |\xi|) + (I \otimes \mathcal{F}) g(-\frac{\xi}{|\xi|}, -|\xi|) \right).
$$

With slight abuse of notation, for every $\xi \in \mathbb{R}^2$, the unit vector $\xi/|\xi|$ stands for the angle $\theta_\xi \in [0, 2\pi)$ such that $n(\theta_\xi) = \xi/|\xi|$.

We introduce the space $\mathcal{S}_0(\mathbb{R}^d)$ of functions $f \in \mathcal{S}(\mathbb{R}^d)$ having all vanishing moments, i.e.

$$\int_{\mathbb{R}^d} x^m f(x)dx = 0,$$

for all $m \in \mathbb{N}^d$. We denote by $\mathcal{S}_0([0, 2\pi) \times \mathbb{R})$ the space of functions g in $\mathcal{S}([0, 2\pi) \times \mathbb{R})$ such that

$$\int_{\mathbb{R}} t^m g(\theta, t)dt = 0$$

for every $m \in \mathbb{N}$. The next Lemma is a reformulation of [56, Theorem 6.2].

Lemma 1.50 ([10])**.** *Let $f \in \mathcal{S}_0(\mathbb{R})$. Then, for any given $m \in \mathbb{N}$ there exists $g \in \mathcal{S}_0(\mathbb{R})$ such that*

$$\mathcal{F}f(\xi) = \xi^m \mathcal{F}g(\xi), \quad \xi \in \mathbb{R},$$

and vice versa.

Proof. We start proving the above statement for $m = 1$. Let $f \in \mathcal{S}_0(\mathbb{R})$ and consider

$$g(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = -\int_{x}^{+\infty} f(t)\mathrm{d}t,$$

where in the second equality we use the fact that $f \in \mathcal{S}_0(\mathbb{R})$. We denote $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. For every $k \in \mathbb{N}$ and $x > 0$

$$\begin{aligned}\langle x\rangle^k |g(x)| &= |\int_{x}^{+\infty} (1 + x^2)^{\frac{k}{2}} f(t)\mathrm{d}t| \leq \int_{x}^{+\infty} (1 + t^2)^{\frac{k}{2}} |f(t)|\mathrm{d}t \\ &\lesssim \int_{-\infty}^{+\infty} (1 + t^2)^{\frac{k}{2}} \frac{1}{(1 + t^2)^{k+2}}\mathrm{d}t < +\infty.\end{aligned}$$

Analogously, for $x < 0$ it holds

$$\begin{aligned}\langle x\rangle^k |g(x)| &= |\int_{-\infty}^{x} (1 + x^2)^{\frac{k}{2}} f(t)\mathrm{d}t| \leq \int_{-\infty}^{x} (1 + t^2)^{\frac{k}{2}} |f(t)|\mathrm{d}t \\ &\lesssim \int_{-\infty}^{+\infty} (1 + t^2)^{\frac{k}{2}} \frac{1}{(1 + t^2)^{k+2}}\mathrm{d}t < +\infty.\end{aligned}$$

Thus, g is a well defined function and $\sup_{x\in\mathbb{R}} \langle x\rangle^k |g(x)| < +\infty$ for every $k \in \mathbb{N}$. Moreover, $g'(x) = f(x)$, so that $\sup_{x\in\mathbb{R}} \langle x\rangle^k |g^{(l)}(x)| < +\infty$ for every $k \in \mathbb{N}$ and $l \geq 1$, since $f \in \mathcal{S}_0(\mathbb{R})$. Therefore, $g \in \mathcal{S}(\mathbb{R})$. Furthermore, for any $n \in \mathbb{N}$, we have that

$$\int_{-\infty}^{+\infty} x^n g(x)\mathrm{d}x = -\int_{-\infty}^{+\infty} x^{n+1} g'(x)\mathrm{d}x = -\int_{-\infty}^{+\infty} x^{n+1} f(x)\mathrm{d}x = 0,$$

since $f \in \mathcal{S}_0(\mathbb{R})$. Hence, $g \in \mathcal{S}_0(\mathbb{R})$ and by the definition of g we have

$$\mathcal{F}f(\xi) = \mathcal{F}g'(\xi) = (2\pi i)\xi\mathcal{F}g(\xi), \qquad \xi \in \mathbb{R}.$$

It is not difficult to see that the analogous statement for $m > 1$ follows by iterating the above proof m-times. The opposite direction is obviously true since the space $\mathcal{S}_0(\mathbb{R})$ is closed under multiplication by a polynomial. □

By Theorem 1.49 and Lemma 1.50, we see that if $g \in \mathcal{S}_0([0,2\pi)\times\mathbb{R})$ then $\mathcal{R}^{\#}g$ belongs to $L^2(\mathbb{R}^2)$. Indeed, by Lemma 1.50, for any $m \in \mathbb{N}$ there exists $h \in \mathcal{S}_0([0,2\pi)\times\mathbb{R})$ such that $(I\otimes\mathcal{F})g(\theta,\tau) = \tau^m(I\otimes\mathcal{F})h(\theta,\tau)$ and we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\mathcal{F}\mathcal{R}^{\#}g(\xi)|^2 \mathrm{d}\xi \leq \int_{\overline{B(0,1)}} |\xi|^{m-2}|(I\otimes\mathcal{F})h(\frac{\xi}{|\xi|},|\xi|) + (I\otimes\mathcal{F})h(-\frac{\xi}{|\xi|},-|\xi|)|^2\mathrm{d}\xi \\
&+ \int_{B(0,1)^c} |\xi|^{-2}|(I\otimes\mathcal{F})g(\frac{\xi}{|\xi|},|\xi|) + (I\otimes\mathcal{F})g(-\frac{\xi}{|\xi|},-|\xi|)|^2\mathrm{d}\xi \\
&\leq \int_0^{2\pi}\int_0^1 \tau^{m-1}|(I\otimes\mathcal{F})h(\theta,\tau) + (I\otimes\mathcal{F})h(-\theta,-\tau)|^2\mathrm{d}\tau\mathrm{d}\theta \\
&+ 4\int_{B(0,1)^c} |\xi|^{-2}(1+|\xi|^2)^{-k}\mathrm{d}\xi \\
&\leq \|(I\otimes\mathcal{F})(h+\tilde{h})\|^2_{L^2([0,2\pi)\times\mathbb{R})} + 4\int_{B(0,1)^c} |\xi|^{-2}(1+|\xi|^2)^{-k}\mathrm{d}\xi \\
&\leq \|h\|^2_{L^2([0,2\pi)\times\mathbb{R})} + \|\tilde{h}\|^2_{L^2([0,2\pi)\times\mathbb{R})} + 4\int_{B(0,1)^c} |\xi|^{-2}(1+|\xi|^2)^{-k}\mathrm{d}\xi < +\infty,
\end{aligned}
$$

where $m > 1$, $k \in \mathbb{N}$ and $\tilde{h}(\theta,t) = h(-\theta,-t)$. Therefore, for any $g \in \mathcal{S}_0([0,2\pi)\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^2)$, we can read equation (1.19) in the L^2 setting.

We shall now recall the most commonly used inversion formula for the Radon transform, known as Filtered Back Projection. We consider the operator $\sqrt{-\Delta}\colon \operatorname{dom}\sqrt{-\Delta} \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined as

$$
\mathcal{F}\sqrt{-\Delta}f(\xi) = |\xi|\mathcal{F}f(\xi),
$$

where

$$
\operatorname{dom}\sqrt{-\Delta} = \{f \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\xi|^2|\mathcal{F}f(\xi)|^2\mathrm{d}\xi < +\infty\}.
$$

Theorem 1.51 ([57, Theorem 2.1])**.** *For any* $f \in \mathcal{S}(\mathbb{R}^2)$,

$$
f = \frac{1}{2}\sqrt{-\Delta}\mathcal{R}^{\#}\mathcal{R}^{\mathrm{pol}}f. \tag{1.20}
$$

In Chapter 2 we will show a deeper understanding of inversion formula (1.20) as a result of intertwining properties of the operator $\mathcal{R}^{\#}\mathcal{R}^{\mathrm{pol}}$ with irreducible representations associated to the group structure that lies under the classical Radon transform. This will allow us to prove formula (1.20) more in general in the L^2 setting. Furthermore, Theorem 1.51 opens the problem whether it is possible to reconstruct an unknown signal f starting from the knowledge of its integral over a family of submanifolds. This question motivates the next section.

1.3.2 The Radon Transform between Dual Pairs

The inversion of the Radon transform consists in reconstructing an unknown signal f on $\mathbb{R}^2$ from its line integrals. This classical inverse problem raises the general problem of recovering an unknown function on a manifold by means of its integrals over a family of submanifolds. The natural framework for such general inverse problems was introduced by Helgason [38] and is motivated by the group structure hidden in the classical Radon transform setting. More precisely, $\mathbb{R}^2$, that is the space on which the signals live, and the space of parameters $[0, 2\pi) \times \mathbb{R}$ are both transitive spaces of the the same locally compact second countable group G. The group of rigid motions in the plane is $G = \mathbb{R}^2 \rtimes K$, with $K = \{R_\phi \in \mathrm{GL}(2, \mathbb{R}) : \phi \in [0, 2\pi)\}$ where

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

By the identification $K \simeq [0, 2\pi)$, we write (b, ϕ) for the elements in G and the group law becomes

$$(b, \phi)(b', \phi') = (b + R_\phi b', \phi + \phi' \bmod 2\pi).$$

The group G acts transitively on $\mathbb{R}^2$ by the action

$$(b, \phi)[x] = R_\phi x + b \tag{1.21}$$

and the isotropy at the origin $x_0 = (0, 0)$ is the Abelian subgroup

$$K = \{(0, \phi) : \phi \in [0, 2\pi)\} \simeq K.$$

Therefore $\mathbb{R}^2 \simeq G/K$ under the canonical isomorphism $gK \mapsto g[x_0]$. The group G is a subgroup of affine transformations of the plane and then maps lines into lines. Its action on lines is given by

$$(b, \phi).(\theta, t) = (\theta + \phi \bmod 2\pi, t + n(\theta) \cdot R_\phi^{-1} b)$$

and is easily seen to be transitive. The isotropy at the y-axis $\xi_0 = (0, 0)$ is

$$H = \{((0, b_2), \phi) : b_2 \in \mathbb{R}, \phi \in \{0, \pi\}\}.$$

Thus, $[0, 2\pi) \times \mathbb{R} \simeq G/H$ under the canonical isomorphism $gH \mapsto g.\xi_0$. From this group-theoretic point of view, the fact that a point $x \in \mathbb{R}^2$ belongs to the line $(\theta, t) \in [0, 2\pi) \times \mathbb{R}$ is equivalent to require that the left cosets $x = g_1 K$ and $(\theta, t) = g_2 H$ intersect. Indeed, $g_1[x_0]$ belongs to the line $g_2.\xi_0$ if and only if there exists $h \in H$ such that $g_1[x_0] = g_2 h[x_0]$, then $g_1(g_2 h)^{-1} \in K$ and $g_1 K \cap g_2 H \neq \emptyset$. This structure motivates the following general framework.

We consider two transitive G-spaces X and Ξ, where the actions on $x \in X$ and $\xi \in \Xi$ are

$$(g, x) \mapsto g[x], \quad (g, \xi) \mapsto g.\,\xi.$$

We fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote by K and H the corresponding stability subgroups, so that $X \simeq G/K$ and $\Xi \simeq G/H$ under the isomorphisms $gK \mapsto g[x_0]$ and $gH \mapsto g.\xi_0$, respectively. The space X is meant to describe the ambient in which the functions to be analyzed live, while the space Ξ is meant to parametrize the set of submanifolds of X over which one wants to integrate functions. Motivated by the group structure behind the classical Radon transform, the elements in Ξ can be realized as submanifolds of X introducing the concept of incidence. Two elements $x = g_1 K$ and $\xi = g_2 H$ are said to be incident if as cosets in G they intersect. The concept of incidence is meant to translate the fact that a point $x \in X$ belongs to the submanifold parametrized by $\xi \in \Xi$. In this way we realize any point $\xi \in \Xi$ as a submanifold $\hat{\xi} \subset X$ by taking all the points $x \in X$ that are incident to ξ. Precisely,

$$\hat{\xi} = \{x \in X : x \text{ and } \xi \text{ are incident}\} \subset X. \tag{1.22}$$

Conversely, one builds the "sheaf" of manifolds $\check{x}$ through the point $x \in X$ by taking all the points $\xi \in \Xi$ that are incident to x

$$\check{x} = \{\xi \in \Xi : \xi \text{ and } x \text{ are incident}\} \subset \Xi. \tag{1.23}$$

By (1.22) and (1.23) we have that

$$\check{x}_0 = K.\xi_0 \subset \Xi, \qquad \hat{\xi}_0 = H[x_0] \subset X.$$

By definition, $\check{x}_0$ and $\hat{\xi}_0$ are K and H transitive spaces, respectively, hence they carry quasi-invariant measures. By definition, for any $x = gK$ and $\xi = \gamma H$

$$\check{x} = g.\check{x}_0 \subset \Xi, \qquad \hat{\xi} = \gamma[\hat{\xi}_0] \subset X, \tag{1.24}$$

which are closed subsets by [38, Lemma 1.1]. If the maps $\xi \to \hat{\xi}$ and $x \to \check{x}$ are both injective the pair of homogeneous space (X, Ξ) is called a dual pair. This assumption is called transversality, see [38, Lemma 1.3] about an equivalent characterization. The transversality condition avoids a redundant parametrisation of the submanifolds of X over which one wants to integrate functions. The reader may consult [38] for numerous examples of dual pairs. It is worth observing that the leading example of the classical Radon transform does not satisfy the transversality condition. Indeed, the points (θ, t) and $(\theta + \pi \mod 2\pi, -t)$ in $[0, 2\pi) \times \mathbb{R}$ both parametrize the line given by the set of points

$$\widehat{(\theta, t)} = (\theta + \pi \widehat{\mod 2\pi, -t)} = \{x \in \mathbb{R}^2 : x \cdot n(\theta) = t\}.$$

In Helgason's approach the transitive spaces $\check{x}_0$ and $\hat{\xi}_0$ are supposed to carry K-invariant and H-invariant measures, respectively, that is

$$\int_{\check{x}_0} g(k^{-1}.\xi) \mathrm{d}\mu_0(\xi) = \int_{\check{x}_0} g(\xi) \mathrm{d}\mu_0(\xi), \qquad g \in L^1(\check{x}_0, \mathrm{d}\mu_0),\, k \in K,$$

$$\int_{\hat{\xi}_0} f(h^{-1}[x]) \mathrm{d}m_0(x) = \int_{\hat{\xi}_0} f(x) \mathrm{d}m_0(x), \qquad g \in L^1(\hat{\xi}_0, \mathrm{d}m_0),\, h \in H.$$

In order to define the Radon transform and its dual, we push-forward the measure dm_0 to $\check{x} = (gK)\check{}$ by the map $\check{x}_0 \ni \xi \mapsto g.\xi \in \check{x}$ and the measure $d\mu_0$ to $\hat{\xi} = (gH)\hat{}$ by the map $\hat{\xi}_0 \ni x \mapsto g[x] \in \hat{\xi}$. We denote by $\mathrm{d}\mu_x$ the measure on $\check{x}$ and by $\mathrm{d}m_\xi$ the measure on $\hat{\xi}$. Since the measures $\check{x}_0$ and $\hat{\xi}_0$ are supposed to be K-invariant and H-invariant, respectively, the measures $\mathrm{d}\mu_x$ and $\mathrm{d}m_\xi$ do not depend on the choice of the representatives of x and ξ. Furthermore, the transversality condition guarantees that we endow every $\check{x}$ and $\hat{\xi}$ with unique measures $\mathrm{d}\mu_x$ and $\mathrm{d}m_\xi$. It is worth observing that, even if the transversality condition is not satisfied in the classical Radon transform setting, $\widehat{(\theta, t)}$ and $(\theta + \pi \widehat{\mod 2\pi, -t)}$ are endowed with the same measure since the arc-lenght measure is invariant under translations and rotations. For this reason the classical Radon transform satisfies the evenness property (1.15).

Definition 1.52. We define the Radon transform of f as the map $\mathcal{R}f : \Xi \to \mathbb{C}$ given by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x) \mathrm{d}m_\xi(x), \tag{1.25}$$

and the dual Radon transform of g as the map $\mathcal{R}^{\#}g : X \to \mathbb{C}$ given by

$$\mathcal{R}^{\#}g(x) = \int_{\check{x}} g(\xi)\mathrm{d}\mu_x(\xi), \tag{1.26}$$

for any f and g for which the integrals converge.

We may think of $\mathcal{R}f(\xi)$ as the integral of f over the submanifold parametrized by $\xi \in \Xi$ and of $\mathcal{R}^{\#}g(x)$ as the integral of g over the sheaf of submanifolds passing through the point $x \in X$. It is worth observing that definitions (1.25) make sense even if the transversality condition is not satisfied, as in the case of the classical Radon transform.

We list some natural problems suggested by the classical Radon theory:

- Extend the generalized Radon transform between a pair of homogeneous spaces (X, Ξ) to a unitary operator from $L^2(X)$ onto $L^2(\Xi)$, up to the composition with a pseudo-differential operator;
- Invert the generalized Radon transform.

We address these two problems in the next chapter.

Chapter 2

Unitarization and inversion formulae for the Radon transform between dual pairs

In this Chapter we start presenting under weaker hypothesis the construction of the Radon transform between dual pairs recalled in § 1.3.2. Indeed, besides having a theoretical interest, the following weakened treatment allows us to consider a wider variety of cases of interest in applications, such as the similitude group, the standard shearlet group and the generalized shearlet dilation groups. One of the novelties of our treatment consists in requiring the relative invariance of the measures in X, Ξ, $\check{x}_0$ and $\hat{\xi}_0$ instead of invariance. We illustrate this construction with the example where G is the similitude group of the plane. Then, we shown how representation theory allows treating in a general and unified way the problem of inverting the Radon transform. Precisely, under some technical assumptions, we prove that if the quasi-regular representations of G acting on $L^2(X)$ and $L^2(\Xi)$ are irreducible, then the Radon transform, up to a composition with a suitable pseudo-differential operator, can be extended to a unitary operator intertwining the two representations, see Theorem 2.11. Such unitarization problem for the Radon transform was already addressed and essentially solved by Helgason in the context of symmetric spaces [39] which, however, does not fit in the framework considered in this chapter. If, in addition, the representations are square-integrable, we provide an inversion formula for the Radon transform based on the voice transform associated to these representations. Finally, in the last section we illustrate further examples. Most of the findings of this Chapter can be found in [2].

2.1 Dual homogeneous spaces

We keep the notation as in § 1.3.2. We fix a lcsc group G and we consider two transitive G-spaces X and Ξ, where the continuous actions on $x \in X$ and $\xi \in \Xi$ are denoted by

$$(g, x) \mapsto g[x], \quad (g, \xi) \mapsto g.\,\xi.$$

We also fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote the corresponding stability subgroups by K and H. In our weakened treatment we require the relative-invariance of the measures instead of invariance. Precisely, we assume that the G-transitive spaces X and Ξ admit relatively G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$ with positive characters $\alpha\colon G \to (0, +\infty)$ and $\beta\colon G \to (0, +\infty)$, respectively, which may be expressed by the equalities

$$\int_X f(g^{-1}[x])\,\mathrm{d}x = \alpha(g)\int_X f(x)\,\mathrm{d}x, \qquad f \in L^1(X, \mathrm{d}x),\ g \in G, \tag{2.1a}$$

$$\int_\Xi f(g^{-1}.\xi)\,\mathrm{d}\xi = \beta(g)\int_\Xi f(\xi)\,\mathrm{d}\xi, \qquad f \in L^1(\Xi, \mathrm{d}\xi),\ g \in G. \tag{2.1b}$$

Furthermore, we assume that $\check{x}_0$ and $\hat{\xi}_0$ carry relatively K-invariant and H-invariant measures μ_0 and m_0, respectively, that is

$$\int_{\check{x}_0} g(k^{-1}.\xi)\,\mathrm{d}\mu_0(\xi) = \iota(k)\int_{\check{x}_0} g(\xi)\,\mathrm{d}\mu_0(\xi), \qquad f \in L^1(\check{x}_0, \mathrm{d}\mu_0),\ k \in K, \tag{2.2a}$$

$$\int_{\hat{\xi}_0} f(h^{-1}[x])\,\mathrm{d}m_0(x) = \gamma(h)\int_{\hat{\xi}_0} f(x)\,\mathrm{d}m_0(x), \qquad f \in L^1(\hat{\xi}_0, \mathrm{d}m_0),\ h \in H. \tag{2.2b}$$

In order to define the Radon transform we need to endow each $\hat{\xi}$ with a suitable measure. Since the measure m_0 is H-relatively invariant, the choice of the representative of $\xi = gH$ is now crucial in the definition of $\hat{\xi}$. The same holds true for the definition of $\check{x}$ in order to define the dual Radon transform since the measure μ_0 is not K-invariant anymore but it is K-relatively invariant. For this reason we fix two Borel sections

$$s : X \to G, \qquad \sigma : \Xi \to G,$$

which always exist since G is second countable [71, Theorem 5.11]. By (1.24), for any $\xi \in \Xi$ and $x \in X$, the closed subsets $\hat{\xi}$ and $\check{x}$ are defined by

$$\hat{\xi} = \sigma(\xi)[\hat{\xi}_0] \subset X, \qquad \check{x} = s(x).\check{x}_0 \subset \Xi.$$

We then push-forward the measure m_0 to $\check{x}$ by the map $\check{x}_0 \ni \xi \mapsto s(x).\xi \in \check{x}$ and the measure μ_0 to $\hat{\xi}$ by the map $\hat{\xi}_0 \ni x \mapsto \sigma(\xi)[x] \in \hat{\xi}$. We denote by μ_x the so obtained measure on $\check{x}$ and by m_ξ that on $\hat{\xi}$.

Definition 2.1 ([2])**.** We define the Radon transform of f as the map $\mathcal{R}f : \Xi \to \mathbb{C}$ given by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x) = \int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x) \tag{2.3}$$

and we define the dual Radon transform of g as the map $\mathcal{R}^\# g : X \to \mathbb{C}$ given by

$$\mathcal{R}^\# g(x) = \int_{\check{x}} g(\xi)\mathrm{d}\mu_x(\xi) = \int_{\check{x}_0} g(s(x).\xi)\mathrm{d}\mu_0(\xi), \tag{2.4}$$

for any f and g for which the integrals converge.

It is worth observing that the definitions of $\mathcal{R}$ and $\mathcal{R}^\#$ depend intrinsically on the choices of the measures m_0 and μ_0 and of the Borel sections σ and s, and not only on the subsets of integration $\hat{\xi}$ and $\check{x}$. Indeed, if we choose a different section σ' and we define the Radon transform accordingly we obtain

$$\begin{aligned}
\mathcal{R}'f(\xi) &= \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x) = \int_{\hat{\xi}_0} f(\sigma'(\xi)[x])\mathrm{d}m_0(x) \\
&= \int_{\hat{\xi}_0} f(\sigma(\xi)\sigma(\xi)^{-1}\sigma'(\xi)[x])\mathrm{d}m_0(x) \\
&= \gamma(\sigma'(\xi)^{-1}\sigma(\xi))\int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x) \\
&= \gamma(\sigma'(\xi)^{-1}\sigma(\xi))\mathcal{R}f(\xi),
\end{aligned}$$

since $\sigma'(\xi)^{-1}\sigma(\xi) \in H$. Therefore, the dependence of the Radon transform on σ, that is on the choice of the representative of ξ, is only through a multiplicative factor depending on ξ and not on f. Analogous observations hold true for the dual Radon transform and the choice of the section s.

Now, we see how the construction of the measure $\mathrm{d}x$ and of the section s simplifies when G is the semidirect product of the Euclidean space $\mathbb{R}^d$ with a closed subgroup K of $\mathrm{GL}(d, \mathbb{R})$. This structure is enjoyed by several groups of interest in applications, such as the similitude group studied by Murenzi [5], and the generalized shearlet dilation groups introduced by Führ in [29, 32] for the purpose of generalizing the standard shearlet group introduced in [54, 20].

We recall that $G = \mathbb{R}^d \rtimes K$ is $\mathbb{R}^d \times K$ endowed with the group operation

$$(b_1, k_1)(b_2, k_2) = (b_1 + k_1 b_2, k_1 k_2), \qquad b_1, b_2 \in \mathbb{R}^d,\ k_1, k_2 \in K, \tag{2.5}$$

where kb is the natural linear action of the matrix k on the column vector b, so that G is a lcsc group. The inverse of an element in G is given by $(b,k)^{-1} = (-k^{-1}b, k^{-1})$. A left Haar measure of G is

$$\mathrm{d}\mu(b, k) = |\det k|^{-1}\mathrm{d}b\mathrm{d}k, \tag{2.6}$$

where $\mathrm{d}b$ is the Lebesgue measure of $\mathbb{R}^d$ and $\mathrm{d}k$ is a left Haar measure on K. The transitive space we consider is $X = \mathbb{R}^d$, regarded as G-space with respect to the canonical action

$$(b, k)[x] = b + kx, \qquad (b, k) \in G,\ x \in X.$$

The action is clearly transitive, the isotropy at the origin $x_0 = 0$ is the subgroup $\{(0, k) : k \in K\}$ which we identify with K, so that $X \simeq G/K$. Furthermore, the map

$$s\colon X \to G, \qquad s(x) = (x, \mathrm{I}_d), \tag{2.7}$$

is a Borel section and the Lebesgue measure $\mathrm{d}x$ on X is a relatively G-invariant measure. Indeed for any measurable set $E \subseteq \mathbb{R}^d$ we have $|(b,k)[E]| = |b+kE| = |kE| = |\det k||E|$ and so

$$\alpha(b, k) = |\det k|. \tag{2.8}$$

The following example shows that the classical Radon transform together with its dual transform can be obtained in this weakened framework starting from the similitude group of the plane [2, Example 2.2].

Example 2.2. The (connected component of the identity of the) similitude group $SIM(2)$ of the plane is $\mathbb{R}^2 \rtimes K$, with $K = \{R_\phi A_a \in \mathrm{GL}(2, \mathbb{R}) : \phi \in [0, 2\pi),\ a \in \mathbb{R}^+\}$ where

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \qquad A_a = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Under the identification $K \simeq [0, 2\pi) \times \mathbb{R}^+$, we write (b, ϕ, a) for the elements in $SIM(2)$. With this identification the group law becomes

$$(b, \phi, a)(b', \phi', a') = (b + R_\phi A_a b', \phi + \phi' \bmod 2\pi, aa'),$$

and the inverse of (b, ϕ, a) is given by

$$(b, \phi, a)^{-1} = (-A_a^{-1} R_\phi^{-1} b, -\phi \bmod 2\pi, a^{-1}). \tag{2.9}$$

By (2.6), a left Haar measure of $SIM(2)$ is

$$\mathrm{d}\mu(b,\phi,a)=a^{-3}\mathrm{d}b\mathrm{d}\phi\mathrm{d}a, \tag{2.10}$$

where $\mathrm{d}b$, $\mathrm{d}\phi$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $[0,2\pi)$ and $\mathbb{R}_+$, respectively.

It remains to choose the space Ξ and the corresponding subgroup H of $SIM(2)$. The group $SIM(2)$ acts transitively on $\Xi=[0,\pi)\times\mathbb{R}$ by

$$(b,\phi,a).(\theta,t)=(\theta+\phi \bmod \pi, n(\theta+\phi)\cdot n(\theta+\phi \bmod \pi)(at+n(\theta)\cdot R_\phi^{-1}b)),$$

where $n(\theta)={}^t(\cos\theta,\sin\theta)$, or equivalently

$$(b,\phi,a)^{-1}.(\theta,t)=\left(\theta-\phi \bmod \pi, n(\theta-\phi)\cdot n(\theta-\phi \bmod \pi)\frac{t-n(\theta)\cdot b}{a}\right).$$

The isotropy at $\xi_0=(0,0)$ is

$$H=\{((0,b_2),\phi,a):b_2\in\mathbb{R},\phi\in\{0,\pi\},a\in\mathbb{R}^+\}.$$

Thus, $[0,\pi)\times\mathbb{R}\simeq SIM(2)/H$. An immediate calculation gives

$$\int_\Xi f\left((b,\phi,a)^{-1}.(\theta,t)\right)\mathrm{d}\theta\mathrm{d}t=a\int_\Xi f(\theta,t)\,\mathrm{d}\theta\mathrm{d}t, \qquad f\in L^1(\Xi,\mathrm{d}\theta\mathrm{d}t),$$

namely, (2.1) is satisfied with the character $\beta(b,\phi,a)=a$. Thus, the Lebesgue measure $d\xi=\mathrm{d}\theta\mathrm{d}t$ is a relatively invariant measure on Ξ.

Consider now the sections $s\colon\mathbb{R}^2\to SIM(2)$ and $\sigma\colon[0,\pi)\times\mathbb{R}\to SIM(2)$ defined by

$$s(x)=(x,0,1), \qquad \sigma(\theta,t)=(t\,n(\theta),\theta,1).$$

It is easy to verify by direct computation that

$$\hat{\xi}_0=H[x_0]=\{(0,b_2):b_2\in\mathbb{R}\}\simeq\mathbb{R},$$
$$\check{x}_0=K.\xi_0=\{(\theta,0):\theta\in[0,\pi)\}\simeq[0,\pi).$$

It is immediate to see that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with character $\gamma((0,b_2),\phi,a)=a$ and the Lebesgue measure $\mathrm{d}\theta$ on $\check{x}_0$ is a K-invariant measure, thus $\iota(0,\phi,a)\equiv 1$. Further, we have that

$$\widehat{(\theta,t)}=\sigma(\theta,t)[\hat{\xi}_0]=\{x\in\mathbb{R}^2:x\cdot n(\theta)=t\},$$

which is the set of all points lying on the line of equation $x\cdot n(\theta)=t$ and

$$\check{x}=s(x).\check{x}_0=\{(\theta,t)\in[0,\pi)\times\mathbb{R}:t-n(\theta)\cdot x=0\},$$

which parametrizes the set of all lines passing through the point x.

It is easy to verify that $X=\mathbb{R}^2$ and $\Xi=[0,\pi)\times\mathbb{R}$ are homogeneous spaces in duality. Indeed, the map $x\mapsto\check{x}$ which sends a point to the set of all lines passing through that point and the map $(\theta,t)\mapsto\widehat{(\theta,t)}$ which sends a line to the set of points lying on that line are both injective. Finally, we compute by (2.3) the Radon transform between the homogeneous spaces in duality $\mathbb{R}^2$ and $[0,\pi)\times\mathbb{R}$ obtaining

$$\mathcal{R}^{\mathrm{pol}}f(\theta,t)=\int_\mathbb{R} f(t\cos\theta-y\sin\theta,t\sin\theta+y\cos\theta)\mathrm{d}y, \qquad (\theta,t)\in[0,\pi)\times\mathbb{R}, \tag{2.11}$$

which is the classical Radon transform (see Definition 1.40), and we get by (2.4) the dual Radon transform (see Definition 1.47) defined as

$$\mathcal{R}^{\#} f(x) = \int_0^{\pi} g(\theta, n(\theta) \cdot x) \mathrm{d}\theta, \qquad x \in \mathbb{R}^2. \tag{2.12}$$

Observe that, unlike the classical Radon transform presented in Section 1.3, the angle θ belongs to the smaller interval $[0, \pi)$. For this reason the map $(\theta, t) \mapsto \widehat{(\theta, t)}$ is injective and (X, Ξ) is a dual pair. Despite this difference between the two settings, all the results stated in section 1.3 still apply to (2.11) and (2.12). For our purposes, in what follows we will focus mainly on the Radon transform.

For clarity, we recall the main assumptions that we have made until now ((A1) and (A2)) and we list the further ones that we need along the way ((A3) – (A6)):

(A1) the spaces X and Ξ carry relatively G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$, respectively;

(A2) $\check{x}_0$ and $\hat{\xi}_0$ carry relatively K-invariant and H-invariant measures μ_0 and m_0 with characters ι and γ, respectively;

(A3) there exist a Borel section $\sigma \colon \Xi \to G$ and a character $\zeta \colon G \to (0, +\infty)$ such that

$$\gamma(\sigma(\xi)^{-1} g \sigma(g^{-1}.\xi)) = \zeta(g), \qquad g \in G,\ \xi \in \Xi;$$

(A4) the quasi-regular representation π of G acting on $L^2(X, \mathrm{d}x)$ is irreducible and square-integrable;

(A5) the quasi-regular representation $\hat{\pi}$ of G acting on $L^2(\Xi, \mathrm{d}\xi)$ is irreducible;

(A6) there exists a non-trivial π-invariant subspace $\mathcal{A} \subseteq L^2(X, \mathrm{d}x)$ such that for all $f \in \mathcal{A}$

$$f(\sigma(\xi)[\cdot]) \in L^1(\hat{\xi}_0, m_0) \quad \text{for almost all } \xi \in \Xi, \tag{2.13a}$$

$$\mathcal{R}f = \int_{\hat{\xi}_0} f(\sigma(\cdot)[x]) \mathrm{d}m_0(x) \in L^2(\Xi, \mathrm{d}\xi), \tag{2.13b}$$

and the adjoint of the operator $\mathcal{R} \colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ has non-trivial domain.

It is worth observing that the whole construction, and in particular the existence of the character ζ and the construction of the Radon transform $\mathcal{R}$ itself, depends on the choice of the section σ. We do not require the injectivity of the maps $\xi \mapsto \hat{\xi}$ and $x \mapsto \check{x}$, but our framework completely fits that of Helgason's dual pairs (X, Ξ). Apart from the cases considered below, the reader may consult [38] for numerous examples of dual pairs (X, Ξ). We add a few comments on assumption (A6). We will show that $\mathcal{R}$ is a semi-invariant densely defined operator, see Lemma 2.5, and as a consequence the assumption that the domain of $\mathcal{R}^*$ is non-trivial is equivalent to the closability of $\mathcal{R}$, see Corollary 1.31. By the irreducibility of π, the minimal choice for $\mathcal{A}$ is $\mathrm{span}\{\pi(g) f_0 : g \in G\}$, where f_0 is a non-zero function in $L^2(X, \mathrm{d}x)$ satisfying conditions (2.13). In general the closure of an operator depends on its domain, however if the Radon transform extends to a larger domain we will show that under some weak conditions also its extension is closable and the two closures coincide, see Corollary 2.8 and also the comment below Lemma 2.7. This delicate issue is further discussed in Example 2.10.

2.1.1 The representations

The group G acts unitarily on $L^2(X, \mathrm{d}x)$ via the quasi-regular representation defined by

$$\pi(g)f(x) = \alpha(g)^{-1/2} f(g^{-1}[x]),$$

with α given in (2.1a). By Assumption (A4), π is irreducible and square-integrable. We stress that this latter condition is needed only in Section 2.3.

The group G acts also on $L^2(\Xi, \mathrm{d}\xi)$ via the quasi-regular representation

$$\hat{\pi}(g)F(\xi) = \beta(g)^{-1/2} F(g^{-1}.\,\xi),$$

where β is defined in (2.1b). The representation $\hat{\pi}$ is irreducible by Assumption (A5).

Example 2.3 (Example 2.2 continued)**.** The group $SIM(2)$ acts on $L^2(\mathbb{R}^2)$ by means of the unitary irreducible representation π defined by

$$\pi(b, \phi, a)f(x) = a^{-1} f(A_a^{-1} R_\phi^{-1}(x - b)), \tag{2.14}$$

or, equivalently, in the frequency domain

$$\mathcal{F}[\pi(b, \phi, a)f](\omega) = a e^{-2\pi i b\cdot\omega} \mathcal{F}f(A_a R_\phi^{-1}\omega). \tag{2.15}$$

Furthermore, G acts on $L^2([0, \pi) \times \mathbb{R})$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b, \phi, a)F(\theta, t) = a^{-\frac{1}{2}} F\left(\theta - \phi \bmod \pi, n(\theta - \phi)\cdot n(\theta - \phi \bmod \pi)\frac{t - n(\theta)\cdot b}{a}\right), \tag{2.16}$$

which is irreducible, too.

2.2 The Unitarization Theorem

Assumption (A6) states that there exists a non-trivial π-invariant subspace $\mathcal{A}$ of $L^2(X, \mathrm{d}x)$ such that $\mathcal{R}f$ is well defined for all $f \in \mathcal{A}$ and the adjoint of the Radon transform $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ has non-trivial domain.

Example 2.4 (Example 2.2 continued)**.** We show that assumption (A6) holds true. We denote here by $\mathcal{R}_e^{\mathrm{pol}}$ the extension of the Radon transform as an even function on $[0, 2\pi) \times \mathbb{R}$, i.e.

$$\mathcal{R}_e^{\mathrm{pol}} f(\theta, t) = \mathcal{R}^{\mathrm{pol}} f(\theta', n(\theta)\cdot n(\theta')t),$$

where $\theta' \in [0, \pi)$ is such that $n(\theta') = \pm n(\theta)$. We recall that, for any $g \in \mathcal{S}_0([0, 2\pi) \times \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^2)$

$$\langle \mathcal{R}_e^{\#} g, f\rangle = \langle g, \mathcal{R}_e^{\mathrm{pol}} f\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$ and in $L^2([0, 2\pi)\times\mathbb{R})$, respectively, and where $\mathcal{R}_e^{\#}$ denotes here the dual Radon transform given by (1.18). We fix a non-zero $g \in \mathcal{S}_0([0, 2\pi)\times\mathbb{R})$ which satisfies the symmetry condition $g(\theta, t) = g(\theta + \pi \bmod 2\pi, -t)$ and we denote its restriction to $[0, \pi) \times \mathbb{R}$ by g_0. Then, there exists a positive constant C such that

$$|\langle g_0, \mathcal{R}^{\mathrm{pol}} f\rangle_{L^2([0,\pi)\times\mathbb{R})}| = \frac{1}{2}|\langle g, \mathcal{R}_e^{\mathrm{pol}} f\rangle_{L^2([0,2\pi)\times\mathbb{R})}| = \frac{1}{2}|\langle \mathcal{R}_e^{\#} g, f\rangle| \leq C\|f\|,$$

for any $f \in \mathcal{S}(\mathbb{R}^2)$. Therefore, if we take $f_0 \in \mathcal{S}(\mathbb{R}^2)$ and define the vector subspace $\mathcal{A} = \operatorname{span}\{\pi(g)f_0 : g \in G\} \subseteq \mathcal{S}(\mathbb{R}^2)$, then the domain of the adjoint of the restriction $\mathcal{R}$ of $\mathcal{R}^{\mathrm{pol}}$ to $\mathcal{A}$ is non-trivial since $g_0 \in \operatorname{dom}(\mathcal{R}^*)$ and we can conclude.

Our construction is based on the following lemma, which shows that the Radon transform intertwines the representations π and $\hat{\pi}$ up to a positive character of G.

Lemma 2.5 ([2, Lemma 3.1]). *The Radon transform $\mathcal{R}$ restricted to $\mathcal{A}$ is a densely defined operator from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$ satisfying*

$$\mathcal{R}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\mathcal{R}, \tag{2.17}$$

for all $g \in G$, where

$$\chi(g) = \alpha(g)^{1/2}\beta(g)^{-1/2}\gamma(g\sigma(g^{-1}.\xi_0))^{-1}. \tag{2.18}$$

For the classical Radon transform this result is a direct consequence of the behavior of $\mathcal{R}^{\mathrm{pol}}$ under linear actions [61, Chapter 2]. With a slight abuse of notation, $\mathcal{R}$ denotes both the Radon transform defined by (2.3) and its restriction to $\mathcal{A}$.

Proof. By Assumption (A6), $\mathcal{R}$ is a well-defined operator from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$. We now prove (2.17). By the π-invariance of $\mathcal{A}$, for $f \in \mathcal{A}$ and $g \in G$ we have

$$\begin{aligned}
(\mathcal{R}\pi(g)f)(\xi) &= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(g^{-1}\sigma(\xi)[x])\mathrm{d}m_0(x)\\
&= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\,\xi)\sigma(g^{-1}.\,\xi)^{-1}g^{-1}\sigma(\xi)[x])\mathrm{d}m_0(x)\\
&= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\,\xi)m(g,\xi)^{-1}[x])\mathrm{d}m_0(x),
\end{aligned}$$

where $m(g,\xi)^{-1} := \sigma(g^{-1}.\,\xi)^{-1}g^{-1}\sigma(\xi)$. It is known that for any $g \in G$ and any $\xi \in \Xi$

$$m(g,\xi) = \sigma(\xi)^{-1}g\sigma(g^{-1}.\,\xi) \in H. \tag{2.19}$$

We show this property for the reader's convenience. Indeed

$$\sigma(\xi)^{-1}g\sigma(g^{-1}.\,\xi).\xi_0 = \sigma(\xi)^{-1}g.(g^{-1}.\,\xi) = \sigma(\xi)^{-1}.\xi = \xi_0,$$

so that $m(g,\xi) \in H$. Thus, using (2.1b) we obtain

$$(\mathcal{R}\pi(g)f)(\xi) = \alpha(g)^{-1/2}\gamma(m(g,\xi))\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\xi)[x])\mathrm{d}m_0(x).$$

Then,

$$\begin{aligned}
(\mathcal{R}\pi(g)f)(\xi) &= \alpha(g)^{-1/2}\gamma(m(g,\xi))(\mathcal{R}f)(g^{-1}.\xi)\\
&= \alpha(g)^{-1/2}\beta(g)^{1/2}\gamma(m(g,\xi))\hat{\pi}(g)\mathcal{R}f(\xi).
\end{aligned}$$

Thanks to assumption (A3), there exists a character $\iota\colon G \to (0,+\infty)$ such that $\gamma(m(g,\xi)) = \iota(g)$ for every $g \in G$ and $\xi \in \Xi$. In particular, $\gamma(m(g,\xi)) = \gamma(m(g,\xi_0)) = \gamma(g\sigma(g^{-1}.\xi_0))$, and (2.17) follows.

We finally prove that $\mathcal{A}$ is dense. By assumption (A6), the domain of $\mathcal{R}$ is π invariant, so that $\pi(g)\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}}$ for every $g \in G$. Since $\mathcal{A} \neq \{0\}$ and π is irreducible, then $\overline{\mathcal{A}} = L^2(X, \mathrm{d}x)$. □

Observe that, if γ extends to a positive character of G, then

$$\gamma(m(g,\xi)) = \gamma(\sigma(\xi))^{-1}\gamma(g)\gamma(\sigma(g^{-1}.\,\xi))$$

and the independence of ξ is implied by the stronger condition

$$\gamma(\sigma(g^{-1}.\,\xi)) = \gamma(\sigma(\xi)),$$

that must be satisfied for all $g \in G$ and $\xi \in \Xi$. This is equivalent to requiring that $\gamma(\sigma(\xi)) = 1$ for all $\xi \in \Xi$, which is true in all our examples.

Lemma 2.5 is at the base of our construction and its validity strongly depends on the choice of the Borel section $\sigma\colon \Xi \to G$ (see (A3)). For instance, if we change the Borel section in Example 2.2 and we choose $\sigma'\colon \Xi \to G$ defined as $\sigma'(\theta,t) = (tn(\theta),\theta,e^t)$, then hypothesis (A3) does not hold true anymore and consequently Lemma 2.5 fails. This example underlines once again that our whole construction depends on the choice of σ in (A3).

Example 2.6 (Example 2.2 continued)**.** By (2.17) and (2.18) we have that

$$\mathcal{R}^{\mathrm{pol}}\pi(b,\phi,a) = \chi(b,\phi,a)^{-1}\hat{\pi}(b,\phi,a)\mathcal{R}^{\mathrm{pol}}, \tag{2.20}$$

where $\chi(b,\phi,a) = a^{-1/2}$ since $\alpha(b,\phi,a) = a^2$ and $\beta(b,\phi,a) = \gamma(b,\phi,a) = a$.

As a consequence of Corollary 1.31 and Assumption (A6), we get the following property.

Lemma 2.7 ([2, Lemma 3.5])**.** *The Radon transform $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure $\overline{\mathcal{R}}$ is a densely defined operator satisfying*

$$\overline{\mathcal{R}}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\overline{\mathcal{R}}, \tag{2.21}$$

for all $g \in G$, where χ is given by (2.18)*. Furthermore, $\overline{\mathcal{R}}$ is the unique closed extension of $\mathcal{R}$.*

We note that by formula (2.3) $\mathcal{R}$ naturally extends to a densely defined operator $\mathcal{R}^{\max} : \mathcal{A}^{\max} \to L^2(\Xi, \mathrm{d}\xi)$ where $\mathcal{A}^{\max}$ is the π-invariant domain

$$\mathcal{A}^{\max} = \{f \in L^2(X, \mathrm{d}x) : f(\sigma(\xi)[\cdot]) \in L^1(\hat{\xi}_0, m_0) \text{ a.e. } \xi \in \Xi, \\ \int_{\hat{\xi}_0} f(\sigma(\cdot)[x])\mathrm{d}m_0(x) \in L^2(\Xi, \mathrm{d}\xi)\}.$$

In general we are not able to show that $\mathcal{R}^{\max}$ is closable on $\mathcal{A}^{\max}$ and we need Assumption (A6) to ensure that the Radon transform is closable at least on a smaller domain $\mathcal{A} \subseteq \mathcal{A}^{\max}$. Observe that, if $\mathcal{A}'$ is another π-invariant vector space such that $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}^{\max}$ and the restriction $\mathcal{R}'$ of $\mathcal{R}^{\max}$ to $\mathcal{A}'$ is closable, then by Corollary 1.31, its closure $\overline{\mathcal{R}'}$ coincides with $\overline{\mathcal{R}}$. Hence, the choice of $\mathcal{A}$ in Assumption (A6) is not crucial. The most delicate issue is to prove that the Radon transform is closable and, by the irreducibility of π, it is natural to choose $\mathcal{A}$ as a "minimal" domain, for example $\mathcal{A} = \mathrm{span}\{\pi(g)f_0 : g \in G\}$, where $f_0 \in \mathcal{A}^{\max}$ is a suitable non-zero function. However, with this minimal choice, it would be nice to have a larger domain $\mathcal{A}'$ such that $\overline{\mathcal{R}}f = \mathcal{R}^{\max}f$ for all $f \in \mathcal{A}'$. The following result provides an equivalent characterization of this property, which is useful in the examples.

Corollary 2.8 ([2, Corollary 3.6])**.** *Let $\mathcal{A}'$ be a subspace of $L^2(X, \mathrm{d}x)$ such that*

$$\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}^{\max}$$

and $\mathcal{R}'$ denote the restriction of $\mathcal{R}^{\max}$ to $\mathcal{A}'$. Then $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and $\mathcal{A}$ is dense in $\mathcal{A}'$ with respect to the graph norm of $\mathcal{R}'$ if and only if $\mathcal{R}'\colon \mathcal{A}' \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure coincides with $\overline{\mathcal{R}}$. In particular, $\overline{\mathcal{R}}f = \mathcal{R}'f$ for any $f \in \mathcal{A}'$.

The result is a direct consequence of the following Lemma 2.9, whose proof is standard and we include it for completeness.

Lemma 2.9 ([2, Lemma 3.7])**.** *If $T_0\colon \mathrm{dom}(T_0) \subseteq \mathcal{H} \to \mathcal{K}$ and $T\colon \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{K}$ are two operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ such that* $\mathrm{dom}(T_0)$ *is dense and T extends T_0. Then T_0 is closable and* $\mathrm{dom}(T_0)$ *is dense in* $\mathrm{dom}(T)$ *with respect to the graph norm of T if and only if T is closable and $\overline{T} = \overline{T}_0$.*

Proof. We suppose that T_0 is closable and that $\mathrm{dom}(T_0)$ is dense in $\mathrm{dom}(T)$ with respect to the graph norm, namely

$$\|f\|_T^2 = \|f\|_{\mathcal{H}}^2 + \|Tf\|_{\mathcal{K}}^2, \qquad f \in \mathrm{dom}(T),$$

which gives to $\mathrm{dom}(T)$ a pre-Hilbertian structure. If we take $f \in \mathrm{dom}(T)$, then by hypothesis there exists a subsequence $(f_n)_n$ in $\mathrm{dom}(T_0)$ such that $f_n \to f$ in $\mathcal{H}$ and $Tf_n \to Tf$ in $\mathcal{K}$. Then, $f \in \mathrm{dom}(\overline{T}_0)$ and $\overline{T}_0 f = \lim_{n\to+\infty} T_0 f_n = Tf$. Therefore, $T \subseteq \overline{T}_0$ and so T is closable and $\overline{T} = \overline{T}_0$. Conversely, if we suppose that T is closable, then such is also T_0. Furthermore, by hypothesis $\overline{T_0} = \overline{T}$ and in particular $\mathrm{dom}(\overline{T}) = \mathrm{dom}(\overline{T}_0)$. This implies that, if we consider $f \in \mathrm{dom}(T)$, there exists a subsequence $(f_n)_n$ in $\mathrm{dom}(T_0)$ such that $f_n \to f$ in $\mathcal{H}$ and $Tf_n \to Tf$ in $\mathcal{K}$ and we can conclude that $\mathrm{dom}(T_0)$ is dense in $\mathrm{dom}(T)$ with respect to the graph norm. □

Example 2.10 (Example 2.2 continued)**.** Fix $f_0 \in \mathcal{S}(\mathbb{R}^2)$. We already know that the polar Radon transform $\mathcal{R}\colon \mathrm{span}\{\pi(g)f_0 : g \in G\} \to L^2([0,\pi) \times \mathbb{R})$ is closable and we denote its closure by $\overline{\mathcal{R}}$. We also know by Section 1.3 that the integral (2.11) is well defined for any $f \in L^1(\mathbb{R}^2)$ and that $\mathcal{R}^{\mathrm{pol}} f \in L^2([0,\pi) \times \mathbb{R})$ for every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Therefore, $\mathcal{A}' = L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ comes out as a natural domain for the polar Radon transform and, denoting the restriction of $\mathcal{R}^{\mathrm{pol}}$ to $\mathcal{A}'$ by $\mathcal{R}'$, the question whether $\overline{\mathcal{R}}f = \mathcal{R}^{\mathrm{pol}} f$ for every $f \in \mathcal{A}'$ naturally rises. By Corollary 2.8, we need to show that $\mathrm{span}\{\pi(g)f_0 : g \in G\}$ is dense in $\mathcal{A}'$ with respect to the graph norm, namely

$$\|f\|_{\mathcal{R}'}^2 = \|f\|^2 + \|\mathcal{R}'f\|^2, \qquad f \in \mathcal{A}'.$$

We denote by $\langle \cdot, \cdot \rangle_{\mathcal{R}'}$ the respective scalar product. For this it suffices to prove that if $f \in \mathcal{A}'$ and satisfies

$$\langle f, \pi(g)f_0 \rangle_{\mathcal{R}'} = 0$$

for every $g \in G$, then $f = 0$. We choose $f_0(x) = e^{-\pi|x|^2}$. By the Fourier slice theorem and equation (2.15), for any $g = (b, \phi, a) \in SIM(2)$ we have that

$$\begin{aligned} 0 = \langle f, \pi(g)f_0 \rangle_{\mathcal{R}} &= \langle f, \pi(g)f_0 \rangle + \langle \mathcal{R}f, \mathcal{R}\pi(g)f_0 \rangle_{L^2([0,\pi)\times\mathbb{R})} \\ &= a \int_{\mathbb{R}^2} \mathcal{F}f(\omega)(1 + \frac{1}{|\omega|})\overline{\mathcal{F}f_0(aR_\phi^{-1}\omega)} e^{2\pi i b\cdot\omega} \mathrm{d}\omega. \end{aligned}$$

If $(\phi, a) = (0, 1)$, then by the injectivity of the Fourier transform there exists a negligible set E such that

$$\mathcal{F}f(\omega)(1 + \frac{1}{|\omega|})\overline{\mathcal{F}f_0(\omega)} = 0 \tag{2.22}$$

for every $\omega \notin E$. Since $\mathcal{F}f_0(\omega) > 0$ for every $\omega \in \mathbb{R}^2$, this implies $\mathcal{F}f(\omega) = 0$ for any $\omega \notin E$ and we conclude that $f = 0$. Therefore, we have that $\overline{\mathcal{R}}f = \mathcal{R}^{\mathrm{pol}}f$ for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. It is also possible to show directly that $\mathcal{R}'$ is closable following the same arguments as in § 2.4.1 and $\overline{\mathcal{R}'} = \overline{\mathcal{R}}$ by Corollary 1.31.

We are finally in a position to state and prove our main result.

Theorem 2.11 ([2, Theorem 3.9])**.** *There exists a unique positive self-adjoint operator*

$$\mathcal{I} \colon \operatorname{dom}(\mathcal{I}) \supseteq \operatorname{Im} \overline{\mathcal{R}} \to L^2(\Xi, \mathrm{d}\xi),$$

semi-invariant with weight $\zeta = \chi^{-1}$ with the property that the composite operator $\mathcal{I}\overline{\mathcal{R}}$ extends to a unitary operator $\mathcal{Q} \colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ intertwining π and $\hat{\pi}$, namely

$$\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, \qquad g \in G. \tag{2.23}$$

Furthermore, π and $\hat{\pi}$ are equivalent representations.

The above result is a generalization of Helgason's theorem on the unitarization of the classical Radon transform, [38, Theorem 4.1], because by definition of extension it holds that

$$\mathcal{I}\mathcal{R}f = \mathcal{Q}f, \qquad f \in \mathcal{A}. \tag{2.24}$$

The isometric extension problem for the Radon transform was actually addressed and implicitly solved by Helgason in the general context of symmetric spaces, see [39, Corollary 3.11].

It is worth observing that the irreducibility of $\hat{\pi}$ enters only in the surjectivity of $\mathcal{Q}$. The first part of the statement holds true without this assumption but does need that $\mathcal{R}$ is closable. For this reason in the proof that follows we use Assumption (A5) only to show that $\mathcal{Q}$ is surjective.

Proof. The unitarization of $\mathcal{R}$ is based on the polar decomposition $\overline{\mathcal{R}} = \mathcal{Q}|\overline{\mathcal{R}}|$ of $\overline{\mathcal{R}}$. By Lemma 2.7 and Theorem 1.30, item (ii), $|\overline{\mathcal{R}}| \colon \operatorname{dom}(\overline{\mathcal{R}}) \to L^2(X, \mathrm{d}x)$ is a positive self-adjoint operator semi-invariant with weight $|\chi| = \chi$, where χ is defined by (2.18), i.e.

$$\pi(g)|\overline{\mathcal{R}}|\pi(g)^{-1} = \chi(g)|\overline{\mathcal{R}}|, \qquad g \in G, \tag{2.25}$$

and $\mathcal{Q} \colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ is a partial isometry with

$$\operatorname{Ker} \mathcal{Q} = \operatorname{Ker} \overline{\mathcal{R}}, \qquad \operatorname{Im} \mathcal{Q} = \overline{\operatorname{Im}(\overline{\mathcal{R}})},$$

and is semi-invariant with weight $\chi/|\chi| \equiv 1$, i.e. (2.23) is satisfied. Since π is irreducible, $\operatorname{Ker} \mathcal{Q} = \{0\}$ and it follows that $\mathcal{Q}$ is an isometry. Furthermore, since $\hat{\pi}$ is irreducible and $\operatorname{Im}(\mathcal{Q})$ is a $\hat{\pi}$-invariant closed subspace of $L^2(\Xi, \mathrm{d}\xi)$ by (2.23), it follows that $\mathcal{Q}$ is surjective, so that $\mathcal{Q}$ is unitary and π and $\hat{\pi}$ are equivalent.

Define $W = \mathcal{Q}|\overline{\mathcal{R}}|\mathcal{Q}^*$ with $\hat{\pi}$-invariant domain

$$\operatorname{dom} W = \{f \in L^2(\Xi, \mathrm{d}\xi) : \mathcal{Q}^* f \in \operatorname{dom} \overline{\mathcal{R}}\} = \mathcal{Q}(\operatorname{dom} \overline{\mathcal{R}}) \oplus \overline{\operatorname{Im}(\overline{\mathcal{R}})}^{\perp},$$

which is a densely defined positive operator in $L^2(\Xi, \mathrm{d}\xi)$, semi-invariant with weight χ. Indeed, $\mathcal{Q}(\mathrm{dom}\,\overline{\mathcal{R}})$ is dense in $\mathcal{Q}(L^2(X, \mathrm{d}x)) = \overline{\mathrm{Im}(\overline{\mathcal{R}})}$ since $\overline{\mathcal{R}}$ is densely defined by Lemma 2.7. Observe that the $\hat{\pi}$-invariance of $\mathrm{dom}\,W$ follows from the π-invariance of $\mathrm{dom}\,\overline{\mathcal{R}}$. Further, by (2.23) and (2.25) and using that $\pi(g)$ is a unitary operator we readily derive

$$\begin{aligned}\hat{\pi}(g)W\hat{\pi}(g)^{-1}f &= \hat{\pi}(g)\mathcal{Q}|\overline{\mathcal{R}}|\mathcal{Q}^*\hat{\pi}(g)^{-1}f\\ &= \left(\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1}\right)\left(\pi(g)|\overline{\mathcal{R}}|\pi(g)^{-1}\right)\left(\pi(g)\mathcal{Q}^*\hat{\pi}(g)^{-1}\right)f\\ &= \mathcal{Q}\left(\chi(g)|\overline{\mathcal{R}}|\right)\mathcal{Q}^*f\\ &= \chi(g)Wf,\end{aligned}$$

for every $f \in \mathrm{dom}\,W$.

Since $\mathcal{Q}^*\mathcal{Q} = \mathrm{Id}$, then $\overline{\mathcal{R}} = W\mathcal{Q}$ and $\mathrm{Im}\,\overline{\mathcal{R}} \subseteq \mathrm{Im}\,W$. We denote by $\mathcal{I}$ the Moore-Penrose inverse of W [11, Chapter 9, §3, Theorem 2] with densely defined domain given by

$$\mathrm{Im}\,W \oplus \mathrm{Im}\,W^{\perp} \supseteq \mathrm{Im}\,W\mathcal{Q} = \mathrm{Im}\,\overline{\mathcal{R}}.$$

Since W is a positive operator in $L^2(\Xi, \mathrm{d}\xi)$, then $\mathcal{I}$ is positive, too, and

$$\begin{aligned}\mathcal{I}Wf &= f, \qquad f \in \mathrm{dom}\,W \cap \mathrm{Ker}\,W^{\perp},\\ W\mathcal{I}f &= f, \qquad f \in \mathrm{Im}\,W.\end{aligned}$$

We claim that $\mathcal{I}$ is semi-invariant with weight χ^{-1} and

$$\mathcal{I}\overline{\mathcal{R}}f = \mathcal{Q}f, \qquad f \in \mathrm{dom}\,\overline{\mathcal{R}}.$$

Indeed, if $f \in \mathrm{Im}\,W$, by definition $\mathcal{I}f = h$ with $h \in \mathrm{dom}\,W \cap \mathrm{Ker}\,W^{\perp}$ and $Wh = f$. Thus, by the semi-invariance of W we have that

$$\begin{aligned}\hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1}f &= \hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1}Wh\\ &= \chi(g)^{-1}\hat{\pi}(g)\mathcal{I}W\hat{\pi}(g)^{-1}h\\ &= \chi(g)^{-1}\mathcal{I}f,\end{aligned}\tag{2.26}$$

where we used that $\hat{\pi}(g)^{-1}h \in \mathrm{Ker}\,W^{\perp}$, which follows from the $\hat{\pi}$-invariance of $\mathrm{Ker}\,W$. If $f \in \mathrm{Im}\,W^{\perp}$, by definition of $\mathcal{I}$ the semi-invariance property (2.26) is trivial.

Finally, since by (2.21) $\overline{\mathcal{R}}$ is an injective operator, we have that $\mathrm{Ker}\,W = \mathrm{Ker}\,\mathcal{Q}^*$ and hence $\mathrm{Ker}\,W^{\perp} = \overline{\mathrm{Im}\mathcal{Q}} \supseteq \mathrm{Im}\mathcal{Q}$, whence $\mathcal{Q}f \in \mathrm{dom}\,W \cap \mathrm{Ker}\,W^{\perp}$ for any $f \in \mathrm{dom}\,\overline{\mathcal{R}}$. Therefore $\mathcal{I}\overline{\mathcal{R}}f = \mathcal{I}W\mathcal{Q}f = \mathcal{Q}f$, as desired. □

It is worth observing that in the proof of Theorem 2.11 we exclusively use the fact that the Radon transform $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure $\overline{\mathcal{R}}$ is a semi-invariant operator between the irreducible representations π and $\hat{\pi}$. Therefore, we can formulate the same result for any semi-invariant operator between two irreducible representations.

Theorem 2.12. *Assume that π_1 and π_2 are two irreducible unitary representations and let T be an operator of $\mathcal{H}_1$ into $\mathcal{H}_2$ semi-invariant with weight χ. Then, there exists a unique positive self-adjoint operator*

$$\mathcal{I}\colon \mathrm{dom}(\mathcal{I}) \supseteq \mathrm{Im}\,T \to \mathcal{H}_2,$$

semi-invariant with weight $\zeta = \chi^{-1}$ with the property that the composite operator $\mathcal{I}T$ extends to a unitary operator $\mathcal{Q}\colon \mathcal{H}_1 \to \mathcal{H}_2$ intertwining π_1 and π_2.

Example 2.13 (Example 2.2 continued)**.** Applying Lemma 2.7 and Corollary 2.8 to $\mathcal{R}$, by (2.20) its closed extension $\overline{\mathcal{R}}$ is a semi-invariant operator from $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ to $L^2([0,\pi)\times\mathbb{R})$ with weight $\chi(b,\phi,a) = a^{-1/2}$. By Theorem 2.11 there exists a positive selfadjoint operator $\mathcal{I}\colon \operatorname{dom}(\mathcal{I}) \supseteq \operatorname{Im}(\overline{\mathcal{R}}) \to L^2([0,\pi)\times\mathbb{R})$, semi-invariant with weight $\chi(g)^{-1} = a^{1/2}$, such that $\mathcal{I}\overline{\mathcal{R}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2) \to L^2([0,\pi)\times\mathbb{R})$ intertwining the quasi-regular (irreducible) representations π and $\hat{\pi}$. Hence

$$\begin{aligned} &\mathcal{I}\mathcal{R}^{\mathrm{pol}} f = \mathcal{Q}f, && f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2), \\ &\mathcal{Q}^*\mathcal{Q}f = f, && f \in L^2(\mathbb{R}^2), \\ &\mathcal{Q}\mathcal{Q}^*F = F, && F \in L^2([0,\pi)\times\mathbb{R}), \\ &\hat{\pi}(g)\,\mathcal{Q}\,\pi(g)^{-1} = \mathcal{Q}, && g \in SIM(2). \end{aligned} \tag{2.27}$$

We can provide an explicit formula for $\mathcal{I}$. Consider the subspace

$$\mathcal{D} = \{f \in L^2([0,\pi)\times\mathbb{R}) : \int_{[0,\pi)\times\mathbb{R}} |\tau||(I\otimes\mathcal{F})f(\theta,\tau)|^2 \,\mathrm{d}\theta\mathrm{d}\tau < +\infty\}$$

and define the operator $\mathcal{J}\colon \mathcal{D} \to L^2([0,\pi)\times\mathbb{R})$ by

$$(I\otimes\mathcal{F})\mathcal{J}f(\theta,\tau) = |\tau|^{\frac{1}{2}}(I\otimes\mathcal{F})f(\theta,\tau), \tag{2.28}$$

a Fourier multiplier with respect to the last variable. A direct calculation shows that $\mathcal{J}$ is a densely defined positive self-adjoint injective operator and is semi-invariant with weight $\zeta(g) = \chi(g)^{-1} = a^{1/2}$. By Theorem 1.30, item (i), there exists $c > 0$ such that $\mathcal{I} = c\mathcal{J}$ and we now show that $c = 1$. Consider a non-zero function $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Then, by Plancherel theorem and the Fourier slice theorem (1.16) we have that

$$\begin{aligned} \|f\|^2 = \|\mathcal{I}\mathcal{R}^{\mathrm{pol}}f\|^2_{L^2([0,\pi)\times\mathbb{R})} &= c^2\|(I\otimes\mathcal{F})\mathcal{J}\mathcal{R}^{\mathrm{pol}}f\|^2_{L^2([0,\pi)\times\mathbb{R})} \\ &= c^2\int_{[0,\pi)\times\mathbb{R}} |\tau||(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}f(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau \\ &= c^2\int_{[0,\pi)\times\mathbb{R}} |\tau||\mathcal{F}f(\tau w(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau \\ &= c^2\|f\|^2. \end{aligned}$$

Thus, we obtain $c = 1$.

Analogous results can also be formulated for the dual Radon transform. We need two further assumptions

(A7) there exist a Borel section $s\colon X \to G$ and a character $\zeta\colon G \to (0,+\infty)$ such that

$$\iota(s(x)^{-1}gs(g^{-1}[x])) = \zeta(g), \qquad g \in G,\, x \in X;$$

(A8) there exists a non-trivial $\hat{\pi}$-invariant subspace $\mathcal{A}^{\#} \subseteq L^2(\Xi, \mathrm{d}\xi)$ such that for all $f \in \mathcal{A}^{\#}$

$$g(s(x).\cdot) \in L^1(\check{x}_0, \mu_0) \quad \text{for almost all } x \in X, \tag{2.29a}$$

$$\mathcal{R}^{\#} g = \int_{\check{x}_0} g(s(\cdot).\xi) \mathrm{d}\mu_0(\xi) \in L^2(X, \mathrm{d}x), \tag{2.29b}$$

and the adjoint of the operator $\mathcal{R}^{\#} \colon \mathcal{A}^{\#} \to L^2(X, \mathrm{d}x)$ has non-trivial domain.

Then, mimicking the proofs of Lemma 2.5 and Lemma 2.7, $\mathcal{R}^{\#} \colon \mathcal{A}^{\#} \to L^2(X, \mathrm{d}x)$ is closable and its unique closed extension $\overline{\mathcal{R}^{\#}}$ is a semi-invariant operator with weight $\chi^{\#}$ given by

$$\chi^{\#}(g) = \alpha(g)^{-1/2} \beta(g)^{1/2} \iota(gs(g^{-1}[x_0]))^{-1}. \tag{2.30}$$

Now, we show that assumption (A7) is always satisfied when $G = \mathbb{R}^d \rtimes K$. Indeed, set $g = (b, k)$ and considered the Borel section s given by (2.7), we have that

$$\begin{aligned} s(x)^{-1} g s(g^{-1}[x] &= (x, \mathrm{I}_d)^{-1}(b, k) s((b, k)^{-1}[x]) \\ &= (-x + b, k) s(k^{-1}(x - b)) \\ &= (-x + b, k)(k^{-1}(x - b), \mathrm{I}_d) = (0, k) \end{aligned}$$

and $\zeta(b, k) = \iota(0, k)$ is a character of G.

Furthermore, by Corollary 1.31, $\mathcal{R}^*$ is a semi-invariant operator with weight χ from $L^2(\Xi, \mathrm{d}\xi)$ into $L^2(X, \mathrm{d}x)$. Therefore, if $\chi = \chi^{\#}$, then by Theorem 1.30, (i), $\mathcal{R}^*$ is a constant multiple of $\overline{\mathcal{R}^{\#}}$. This is the case of the classical dual Radon transform.

Example 2.14 (Example 2.2 continued)**.** By Example 2.4, assumption (A8) is satisfied. We keep the notation as in Example 2.4. We fix $h \in \mathcal{S}_0([0, 2\pi) \times \mathbb{R})$ which satisfy the symmetry condition $h(\theta, t) = h(\theta + \pi \mod 2\pi, -t)$ and we denote its restriction on $[0, \pi) \times \mathbb{R}$ by h_0. Then, for any given $f \in \mathcal{S}(\mathbb{R}^2)$, there exists a positive constant C such that

$$\begin{aligned} |\langle \mathcal{R}^{\#} h_0, f \rangle| = \frac{1}{2} |\langle \mathcal{R}_e^{\#} h, f \rangle| = \frac{1}{2} |\langle h, \mathcal{R}_e^{\mathrm{pol}} f \rangle_{L^2([0,2\pi) \times \mathbb{R})}| &= |\langle h_0, \mathcal{R}^{\mathrm{pol}} f \rangle_{L^2([0,\pi) \times \mathbb{R})}| \\ &\le C \|h_0\|_{L^2([0,\pi) \times \mathbb{R})}. \end{aligned} \tag{2.31}$$

Therefore, if we define $\mathcal{A}^{\#} = \mathrm{span}\{\hat{\pi}(g) h_0 : g \in G\}$, then the domain of the adjoint of the restriction of $\mathcal{R}^{\#}$ to $\mathcal{A}^{\#}$ is non-trivial since contains $\mathcal{S}(\mathbb{R}^2)$ and assumption (A8) holds true. Then, by Corollary 1.31, $\mathcal{R}^{\#} : \mathcal{A}^{\#} \to L^2(X, \mathrm{d}x)$ is closable and, since $\alpha(b, \phi, a) = a^2$, $\beta(b, \phi, a) = a$ and $\iota \equiv 1$, its unique closed extension $\overline{\mathcal{R}^{\#}}$ is a semi-invariant operator with weight $\chi^{\#}(b, \phi, a) = a^{-1/2}$. By (2.20) $\chi = \chi^{\#}$, then by Theorem 1.30, (i), $\overline{\mathcal{R}^{\#}}$ is a constant multiple of $\mathcal{R}^*$. Moreover, $\mathcal{R}^*$ is a closed extension of $\mathcal{R}^{\#}$ and we conclude that $\mathcal{R}^* = \overline{\mathcal{R}^{\#}}$ by the uniqueness of the closure.

2.2.1 Generalized Filtered Back Projection Inversion Formulae

This subsection is devoted to show that it is possible to obtain Theorem 1.51 as a consequence of the intertwining properties of the classical Radon transform and its adjoint operator. Lemma 2.7 states that the Radon transform $\mathcal{R} \colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure $\overline{\mathcal{R}}$ is a semi-invariant operator with weight χ given by (2.18).

Furthermore, as a consequence of Corollary 1.31, $\mathcal{R}^* = \overline{\mathcal{R}}^*$ is a semi-invariant operator with weight χ. The composite operator $\mathcal{R}^*\overline{\mathcal{R}}$ is densely defined and self-adjoint. Furthermore it is immediate to show that it is semi-invariant with weight χ^2. Indeed, for any $g \in G$ we have that

$$\pi(g)\mathcal{R}^*\overline{\mathcal{R}}\pi(g)^{-1} = \chi(g)\mathcal{R}^*\pi(g)\overline{\mathcal{R}}\pi(g)^{-1} = \chi(g)^2\mathcal{R}^*\overline{\mathcal{R}}.$$

Hence, by Theorem 2.12, there exists a unique positive self-adjoint operator

$$\mathcal{J}\colon \operatorname{dom}(\mathcal{I}) \supseteq \operatorname{Im}(\mathcal{R}^*\overline{\mathcal{R}}) \to L^2(X, \mathrm{d}x),$$

semi-invariant with weight χ^{-2} with the property that the composite operator $\mathcal{J}\mathcal{R}^*\overline{\mathcal{R}}$ extends to the identity operator $\mathrm{I} : L^2(X, \mathrm{d}x) \to L^2(X, \mathrm{d}x)$ by the Schur's Lemma (see Theorem 1.24). Therefore, for any $f \in \operatorname{dom}(\mathcal{R}^*\overline{\mathcal{R}})$

$$\mathcal{J}\mathcal{R}^*\overline{\mathcal{R}}f = f. \tag{2.32}$$

The above result is a generalization of the inversion formula for the classical Radon transform. Indeed, we show that if $\mathcal{R}$ is the classical Radon transform, then (2.32) is the filtered back projection inversion formula given in Theorem 1.51.

Example 2.15 (Example 2.2 continued)**.** By equation (2.20), the composite operator $\mathcal{R}^*\overline{\mathcal{R}}$ is a semi-invariant operator with weight $\chi(b, \phi, a)^2 = a^{-1}$. Now, consider the subspace

$$\mathcal{D}_s = \{f \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\omega|^{2s}|\mathcal{F}f(\omega)|^2 \mathrm{d}\omega < +\infty\}$$

of $L^2(\mathbb{R}^2)$. It is not difficult to verify that the Fourier multiplier $A_s\colon \mathcal{D}_s \to L^2(\mathbb{R}^2)$ defined by

$$\mathcal{F}A_s f(\omega) = |\omega|^s \mathcal{F}f(\omega) \tag{2.33}$$

is a densely defined positive self-adjoint operator and is semi-invariant with weight $\chi_s(b, \phi, a) = \Delta(b, \phi, a)^{-s/2} = a^s$. Thus, by Theorem 1.30, (i), the operator $\mathcal{R}^*\overline{\mathcal{R}}$ is given, up to a constant, by (2.33) with $s = -1$. Furthermore, by Theorem 2.12, there exists a unique positive self-adjoint operator

$$\mathcal{J}\colon \operatorname{dom}(\mathcal{I}) \supseteq \operatorname{Im}(\mathcal{R}^*\overline{\mathcal{R}}) \to L^2(X, \mathrm{d}x),$$

semi-invariant with weight $\chi(b, \phi, a)^{-2} = a$ with the property that the composite operator $\mathcal{J}\mathcal{R}^*\overline{\mathcal{R}}$ extends to the identity operator and by (2.33) the operator $\mathcal{J}$ is given, up to a constant, by $\sqrt{-\Delta}$. Therefore, there exists a constant c such that for any $f \in \operatorname{dom}(\mathcal{R}^*\overline{\mathcal{R}})$

$$f = c\sqrt{-\Delta}\mathcal{R}^*\overline{\mathcal{R}}f. \tag{2.34}$$

Finally, for any $f \in \mathcal{S}(\mathbb{R}^2)$, $\overline{\mathcal{R}}f$ is the restriction on $[0, \pi)$ of a function in $\mathcal{S}([0, 2\pi) \times \mathbb{R})$ which satisfies the evenness property (1.15) and then, by equation (2.31) in Example 2.14, equation (2.34) becomes

$$f = c\sqrt{-\Delta}\mathcal{R}^{\#}\mathcal{R}^{\mathrm{pol}}f, \tag{2.35}$$

which is the well-known inversion formula presented in Theorem 1.51. We know show that $c = 1$. By Plancherel theorem, Theorem 1.49 and the Fourier slice theorem (1.39) we have that

$$
\begin{aligned}
\|f\|^2 &= c^2 \|\mathcal{F}\sqrt{-\Delta}\mathcal{R}^{\#}\mathcal{R}^{\mathrm{pol}} f\|^2 = c^2 \int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}\mathcal{R}^{\#}\mathcal{R}^{\mathrm{pol}} f(\xi)|^2 \mathrm{d}\xi \\
&= c^2 \int_{\mathbb{R}^2} \left| \left(\chi_{\{\xi_2>0\}}(\xi)(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}} f(\frac{\xi}{|\xi|}, |\xi|) + \chi_{\{\xi_2<0\}}(\xi)(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}} f(-\frac{\xi}{|\xi|}, -|\xi|) \right) \right|^2 \mathrm{d}\xi \\
&= c^2 \int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2 \mathrm{d}\xi = c^2 \|f\|_2^2.
\end{aligned}
$$

Thus, we obtain $c = 1$. Observe that in Theorem 1.51 the constant is equal to $1/2$. This follows from the fact that in the classical Radon transform the angle θ is allowed to vary in $[0, 2\pi)$, while in our case it is restricted to the interval $[0, \pi)$. Indeed, in this latter case, Theorem 1.49 becomes

$$
\mathcal{F}\mathcal{R}^{\#} g(\xi) = |\xi|^{-1} \left(\chi_{\{\xi_2>0\}}(\xi)(I\otimes\mathcal{F})g(\frac{\xi}{|\xi|}, |\xi|) + \chi_{\{\xi_2<0\}}(\xi)(I\otimes\mathcal{F})g(-\frac{\xi}{|\xi|}, -|\xi|) \right),
$$

with g the restriction on $[0, \pi)$ of a function in $\mathcal{S}([0, 2\pi) \times \mathbb{R})$ satisfying the evenness property (1.15) and where χ_D denotes the characteristic function of a region D in $\mathbb{R}^2$. This is a consequence of the fact that, when we make a change of variables in polar coordinates, if $\theta \in [0, 2\pi)$ we are recovering the plane two times.

Analogously, the composite operator $\overline{\mathcal{R}}\mathcal{R}^*$ is a densely defined self-adjoint operator from $L^2(\Xi, \mathrm{d}\xi)$ into itself and it is semi-invariant with weight χ^2. Hence, by Theorem 2.12, there exists a unique positive self-adjoint operator

$$
\mathcal{K}\colon \operatorname{dom}(\mathcal{K}) \supseteq \operatorname{Im}(\overline{\mathcal{R}}\mathcal{R}^*) \to L^2(\Xi, \mathrm{d}\xi),
$$

semi-invariant with weight χ^{-2} with the property that the composite operator $\mathcal{K}\overline{\mathcal{R}}\mathcal{R}^*$ extends to the identity operator on $L^2(\Xi, \mathrm{d}\xi)$. Therefore, for any $g \in \operatorname{dom}(\overline{\mathcal{R}}\mathcal{R}^*)$

$$
\mathcal{K}\overline{\mathcal{R}}\mathcal{R}^* g = g. \tag{2.36}
$$

This is a generalization of [38, Chapter I, Theorem 3.5]. In particular, since $\mathcal{I}^2$ is semi-invariant with weight χ^{-2}, by Theorem 1.30, (i), then $c\mathcal{K} = \mathcal{I}^2$ for some constant c and equation (2.36) becomes

$$
\mathcal{I}^2\overline{\mathcal{R}}\mathcal{R}^* g = c\, g, \tag{2.37}
$$

with $g \in \operatorname{dom}(\overline{\mathcal{R}}\mathcal{R}^*)$.

Example 2.16 (Example 2.2 continued)**.** By (2.20), the operator $\overline{\mathcal{R}}\mathcal{R}^*$ is semi-invariant with weight $\chi(b, \phi, a)^2 = a^{-1}$. Hence, by Theorem 2.12, there exists a unique positive self-adjoint operator

$$
\mathcal{K}\colon \operatorname{dom}(\mathcal{K}) \supseteq \operatorname{Im}(\overline{\mathcal{R}}\mathcal{R}^*) \to L^2(\Xi, \mathrm{d}\xi),
$$

semi-invariant with weight $\chi(b, \phi, a)^{-2} = a$ with the property that the composite operator $\mathcal{K}\overline{\mathcal{R}}\mathcal{R}^*$ extends to the identity operator. By Theorem 1.30, (i), $\mathcal{K}$ is given, up to a constant, by $\mathcal{I}^2$ where $\mathcal{I}$ is defined by (2.28). Therefore, for any $g \in \operatorname{dom}(\overline{\mathcal{R}}\mathcal{R}^*)$

$$
c\mathcal{I}^2\overline{\mathcal{R}}\mathcal{R}^* g = g, \tag{2.38}
$$

for some constant c. In particular, if we take g the restriction on $[0,\pi)$ of a function in $\mathcal{S}([0,2\pi)\times\mathbb{R})$ which satisfies the evenness property (1.15), then, by Example 2.14, $\mathcal{R}^*$ coincides with $\mathcal{R}^\#$ and $\mathcal{R}^\# g \in \mathcal{S}(\mathbb{R}^2)$ by Corollary 2.5 in [38]. Thus, formula (2.38) becomes

$$c\mathcal{I}^2\mathcal{R}^{\mathrm{pol}}\mathcal{R}^\# g = g. \tag{2.39}$$

We now show that $c=1$. By Plancherel theorem, Theorem 1.49 and the Fourier slice theorem (1.16) we have that

$$\begin{aligned}
\|g\|^2 &= c^2\|(I\otimes\mathcal{F})\mathcal{I}^2\mathcal{R}^{\mathrm{pol}}\mathcal{R}^\# g\|^2 = c^2\int_{[0,\pi)\times\mathbb{R}}|\tau|^2|(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}\mathcal{R}^\# g(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau \\
&= c^2\int_{[0,\pi)\times\mathbb{R}}|\tau|^2|\mathcal{F}\mathcal{R}^\# g(\tau n(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau \\
&= c^2\int_{\mathbb{R}^2}|\xi||\mathcal{F}\mathcal{R}^\# g(\xi)|^2\mathrm{d}\xi \\
&= c^2\int_{\mathbb{R}^2}|\xi|^{-1}\left|\Big(\chi_{\{\xi_2>0\}}(\xi)(I\otimes\mathcal{F})g(\frac{\xi}{|\xi|},|\xi|)+\chi_{\{\xi_2<0\}}(\xi)(I\otimes\mathcal{F})g(-\frac{\xi}{|\xi|},-|\xi|)\Big)\right|^2\mathrm{d}\xi \\
&= c^2\int_{[0,\pi)\times\mathbb{R}}|(I\otimes\mathcal{F})g(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau = c^2\|g\|^2,
\end{aligned}$$

thus $c=1$ and (2.39) is the inversion formula given in [38, Theorem 3.5].

2.3 Inversion of the Radon transform

In this section, we make explicit use of the assumption that π is square-integrable to invert the Radon transform. We recall that, under this assumption, there exists a self-adjoint operator

$$C\colon\ \mathrm{dom}\, C\subseteq L^2(X,\mathrm{d}x)\to L^2(X,\mathrm{d}x),$$

semi-invariant with weight $\Delta^{\frac{1}{2}}$, where Δ is the modular function of G, such that for all $\psi\in\mathrm{dom}\, C$ with $\|C\psi\|=1$, the voice transform $\mathcal{V}_\psi$

$$(\mathcal{V}_\psi f)(g) = \langle f,\pi(g)\psi\rangle, \qquad g\in G,$$

is an isometry from $L^2(X,\mathrm{d}x)$ into $L^2(G)$ and we have the weakly-convergent reproducing formula

$$f=\int_G(\mathcal{V}_\psi f)(g)\pi(g)\psi\ \mathrm{d}\mu(g) \tag{2.40}$$

where μ is the Haar measure. The vector ψ is called admissible vector.

As shown in the previous section, there exists a positive self-adjoint operator $\mathcal{I}$ semi-invariant with weight χ^{-1} such that $\mathcal{I}\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$, which intertwines the quasi-regular representations π and $\hat{\pi}$ of G on $L^2(X,\mathrm{d}x)$ and $L^2(\Xi,\mathrm{d}\xi)$ respectively.

Since $\mathcal{Q}$ is unitary and satisfies (2.23), the voice transform reads

$$\mathcal{V}_\psi f(g)=\langle f,\pi(g)\psi\rangle=\langle\mathcal{Q}f,\mathcal{Q}\pi(g)\psi\rangle=\langle\mathcal{Q}f,\hat{\pi}(g)\mathcal{Q}\psi\rangle, \qquad g\in G, \tag{2.41}$$

for all $f \in L^2(X, \mathrm{d}x)$. Furthermore, the assumption that π is square-integrable ensures that any $f \in L^2(X, \mathrm{d}x)$ can be reconstructed from its unitary Radon transform $\mathcal{Q}f$ by means of the reconstruction formula (2.40), which becomes

$$f = \int_G \langle \mathcal{Q}f, \hat{\pi}(g)\mathcal{Q}\psi \rangle \, \pi(g)\psi \; \mathrm{d}\mu(g).$$

Moreover, if we can choose ψ in such a way that $\mathcal{Q}\psi$ is in the domain of the operator $\mathcal{I}$, by (2.41), for all $f \in \operatorname{dom} \overline{\mathcal{R}}$, we have

$$\begin{aligned} \mathcal{V}_\psi f(g) &= \langle \mathcal{Q}f, \hat{\pi}(g)\mathcal{Q}\psi \rangle \\ &= \langle \mathcal{I}\overline{\mathcal{R}}f, \hat{\pi}(g)\mathcal{Q}\psi \rangle \\ &= \langle \overline{\mathcal{R}}f, \mathcal{I}\hat{\pi}(g)\mathcal{Q}\psi \rangle \\ &= \chi(g)\langle \overline{\mathcal{R}}f, \hat{\pi}(g)\mathcal{I}\mathcal{Q}\psi \rangle, \end{aligned} \tag{2.42}$$

where we use that $\mathcal{I}$ is a selfadjoint operator, semi-invariant with weight χ^{-1}.

By (2.42) the voice transform $\mathcal{V}_\psi f$ depends on f only through its Radon transform $\overline{\mathcal{R}}f$. Therefore, (2.42) together with (2.40) allow to reconstruct an unknown signal $f \in \operatorname{dom} \overline{\mathcal{R}}$ from its Radon transform. Explicitly, we have derived the following inversion formula for the Radon transform.

Theorem 2.17 ([2, Theorem 4.1]). *Let $\psi \in L^2(X, \mathrm{d}x)$ be an admissible vector for the representation π such that $\mathcal{Q}\psi \in \operatorname{dom} \mathcal{I}$, and set $\Psi = \mathcal{I}\mathcal{Q}\psi$. Then, for any $f \in \operatorname{dom} \overline{\mathcal{R}}$,*

$$f = \int_G \chi(g)\langle \overline{\mathcal{R}}f, \hat{\pi}(g)\Psi \rangle \, \pi(g)\psi \; \mathrm{d}\mu(g), \tag{2.43}$$

where the integral is weakly convergent,and

$$\|f\|^2 = \int_G \chi(g)^2 |\langle \overline{\mathcal{R}}f, \hat{\pi}(g)\Psi \rangle|^2 \mathrm{d}\mu(g). \tag{2.44}$$

If, in addition, $\psi \in \operatorname{dom} \overline{\mathcal{R}}$, then $\Psi = \mathcal{I}^2 \overline{\mathcal{R}}\psi$.

Note that the datum $\overline{\mathcal{R}}f$ is analyzed by the family $\{\hat{\pi}(g)\Psi\}_{g \in G}$ and the signal f is reconstructed by a different family, namely $\{\pi(g)\psi\}_{g \in G}$.

As we will show in the next example, sometimes it is useful to choose first the vector Ψ and then to determine the admissible vector ψ as a function of Ψ. Suppose that $\Psi \in L^2(\Xi, \mathrm{d}\xi)$ is such that $\Psi \in \operatorname{dom} \mathcal{R}^*$ and $\mathcal{R}^*\Psi \in \operatorname{dom}(\overline{\mathcal{R}})$ is an admissible vector for the representation π. Then, $\Psi \in \operatorname{dom}(\overline{\mathcal{R}}\mathcal{R}^*)$ and by equation (2.37) we have that

$$\mathcal{I}^2 \overline{\mathcal{R}}\mathcal{R}^*\Psi = c\Psi, \tag{2.45}$$

for some constant c. Therefore, $\psi = c^{-1}\,\mathcal{R}^*\Psi$ is an admissible vector such that $\mathcal{Q}\psi \in \operatorname{dom} \mathcal{I}$ and we have the inversion formula (2.43).

Example 2.18 (Example 2.2 continued). It is known that π is square-integrable and the corresponding voice transform gives rise to $2D$-directional wavelets [5]. An admissible vector is a function $\psi \in L^2(\mathbb{R}^2)$ satisfying the following admissibility condition [5]

$$\int_{[0,2\pi)\times\mathbb{R}^+} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi \frac{\mathrm{d}a}{a} = 1, \qquad \text{for all } \omega \in \mathbb{R}^2/\{0\}, \tag{2.46}$$

which is equivalent to

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\omega_1,\omega_2)|^2}{\omega_1^2+\omega_2^2} \mathrm{d}\omega_1 \mathrm{d}\omega_2 = 1. \tag{2.47}$$

Given $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, define $\mathcal{G}(b,\phi,a) = a^{\frac{1}{2}} \langle \mathcal{R}^{\mathrm{pol}} f, \hat{\pi}(b,\phi,a)\Psi\rangle$, i.e. by (2.16)

$$\mathcal{G}(b,\phi,a) = \int_{[0,\pi)\times\mathbb{R}} \mathcal{R}^{\mathrm{pol}} f(\theta,t)\, \overline{\Psi\left(\theta-\phi \bmod \pi, \frac{t-b\cdot n(\theta)}{a}\right)} \mathrm{d}\theta \mathrm{d}t. \tag{2.48}$$

Then, taking into account that $\chi(b,\phi,a) = a^{-\frac{1}{2}}$, (2.43) reads

$$f(x) = \int_{\mathbb{R}^2 \rtimes ([0,2\pi)\times\mathbb{R}^+)} \mathcal{G}(b,\phi,a)\, \psi(R_\phi^{-1}\frac{x-b}{a})\, \mathrm{d}b \mathrm{d}\phi \frac{\mathrm{d}a}{a^5}. \tag{2.49}$$

By (2.44), formula (2.49) is equivalent to

$$\|f\|^2 = \int_{\mathbb{R}^2 \rtimes ([0,2\pi)\times\mathbb{R}^+)} |\mathcal{G}(b,\phi,a)|^2 \mathrm{d}b \mathrm{d}\phi \frac{\mathrm{d}a}{a^5}. \tag{2.50}$$

The idea to exploit the theory of the continuous wavelet transform to derive inversion formulae for the Radon transform is not new, we refer to [41, 12, 55, 72, 58, 63, 74]–to name a few.

Moreover, if we choose Ψ the restriction on $[0,\pi)$ of a function in $\mathcal{S}_0([0,2\pi)\times\mathbb{R})$ such that $\Psi(\theta,t) = \Psi_2(\theta)\Psi_1(t)$, we obtain a formula for the coefficients $\mathcal{G}(b,\phi,a)$ which involves only integral transforms applied to the Radon transform, precisely a one-dimensional wavelet, followed by a convolution. We observe that it is enough to choose $\Psi_1 \in \mathcal{S}_0(\mathbb{R})$ and Ψ_2 bounded. In such a case Ψ_1 is a one-dimensional wavelet and equation (2.48) becomes

$$\mathcal{G}(b_1,b_2,\phi,a) = a^{\frac{1}{2}} \left(\mathcal{W}_{\Psi_1}\left(\mathcal{R}^{\mathrm{pol}} f(\theta,\cdot)\right)(a, b_1\cos\theta + b_2\sin\theta)\right) *_\theta \check{\Psi}_2(\phi), \tag{2.51}$$

where $\mathcal{W}_{\Psi_1}$ is the canonical wavelet transform and the convolution is with respect to the variable θ running on the Abelian compact group $[0,\pi)$. It remains to find the reconstruction vector ψ. By Corollary 2.5 in [38], $\mathcal{R}^{\#}\Psi \in \mathcal{S}_0(\mathbb{R}^2)$ and then it satisfies (2.47). Therefore, by (2.39), $\psi = \mathcal{R}^{\#}\Psi$ is an admissible vector such that $\mathcal{I}\mathcal{R}^{\mathrm{pol}}\psi \in \operatorname{dom}\mathcal{I}$ and inversion formula (2.49) holds true. Formula (2.51) is very useful in applications since it can give rise to an algorithm to compute the coefficients $\mathcal{G}(b,\phi,a)$ based on the efficient codes available for the wavelet transform.

Finally, we show that it is possible to obtain a version of (2.50) in which the scale parameter a varies only in a compact set. Consider a C^∞ function $\Phi \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ such that

$$|\mathcal{F}\Phi(\omega)|^2 + \int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi \frac{\mathrm{d}a}{a} = 1. \tag{2.52}$$

By Plancherel's theorem, we have that

$$\begin{aligned}\int_{\mathbb{R}^2} |\langle f, T_b\Phi\rangle|^2\, \mathrm{d}b &= \int_{\mathbb{R}^2} \left|\int_{\mathbb{R}^2} \mathcal{F}f(\omega)\overline{\mathcal{F}\Phi(\omega)} e^{2\pi i b\cdot\omega}\, \mathrm{d}\omega\right|^2 \mathrm{d}b \\ &= \int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f\, \overline{\mathcal{F}\Phi})(b)|^2\, \mathrm{d}b \\ &= \int_{\mathbb{R}^2} |\mathcal{F}f(\omega)|^2 |\mathcal{F}\Phi(\omega)|^2\, \mathrm{d}\omega. \end{aligned} \tag{2.53}$$

Using an analogous computation, by Plancherel theorem, formula (2.15) for the Fourier transform of $\pi(g)\psi$ and Fubini's theorem we have

$$\begin{aligned}
&\int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5} = \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\langle f,\pi(b,\phi,a)\psi\rangle|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^3} \\
&= \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} \left|\int_{\mathbb{R}^2} \mathcal{F}f(\omega)\overline{\mathcal{F}\psi(A_aR_\phi^{-1}\omega)}e^{2\pi i b\cdot\omega}\,\mathrm{d}\omega\right|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a} \\
&= \int_{[0,2\pi)\times(0,1)} \left(\int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f\overline{\mathcal{F}\psi(A_aR_\phi^{-1}\cdot)})(b)|^2\mathrm{d}b\right)\mathrm{d}\phi\frac{\mathrm{d}a}{a} \\
&= \int_{\mathbb{R}^2} |\mathcal{F}f(\omega)|^2 \left(\int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_aR_\phi^{-1}\omega)|^2\mathrm{d}\phi\frac{\mathrm{d}a}{a}\right)\mathrm{d}\omega. \qquad (2.54)
\end{aligned}$$

Thus, combining equations (2.52), (2.53) and (2.54) we obtain the reconstruction formula

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle f,T_b\Phi\rangle|^2\mathrm{d}b + \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2\mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5}. \qquad (2.55)$$

It is worth observing that there always exists a function Φ satisfying (2.52) provided that the admissible vector ψ has fast Fourier decay. Indeed, if we require $\mathcal{F}\psi$ to satisfy a decay estimate of the form

$$|\mathcal{F}\psi(\omega)| = O(|\omega|^{-L}), \qquad \text{for every } L>0,$$

then, by (2.46) we have that

$$\begin{aligned}
z(\omega) &:= 1 - \int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_aR_\phi^{-1}\omega)|^2\mathrm{d}\phi\frac{\mathrm{d}a}{a} \\
&= \int_{[0,2\pi)\times[1,+\infty)} |\mathcal{F}\psi(A_aR_\phi^{-1}\omega)|^2\mathrm{d}\phi\frac{\mathrm{d}a}{a} \\
&\lesssim \int_{[0,2\pi)\times[1,+\infty)} a^{-2L}|\omega|^{-2L}\frac{\mathrm{d}a}{a}\mathrm{d}\phi \\
&\lesssim |\omega|^{-2L}.
\end{aligned}$$

Therefore, there exists a C^∞ function Φ such that $\mathcal{F}\Phi(\omega) = \sqrt{z(\omega)}$, so that (2.52) holds true.

Finally, let us show that the first term in the right hand side of (2.55) may be expressed in terms of $\mathcal{R}^{\mathrm{pol}}f$ only. We readily derive

$$\begin{aligned}
\langle f,T_b\Phi\rangle = \langle f,\pi(b,0,1)\Phi\rangle &= \langle \mathcal{Q}f,\mathcal{Q}\pi(b,0,1)\Phi\rangle \\
&= \langle \mathcal{Q}f,\hat{\pi}(b,0,1)\mathcal{Q}\Phi\rangle \\
&= \langle \mathcal{I}\mathcal{R}^{\mathrm{pol}}f,\hat{\pi}(b,0,1)\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi\rangle \\
&= \langle \mathcal{R}^{\mathrm{pol}}f,\hat{\pi}(b,0,1)\mathcal{I}^2\mathcal{R}^{\mathrm{pol}}\Phi\rangle, \qquad (2.56)
\end{aligned}$$

where we observe that $\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi$ is always in the domain of the operator $\mathcal{I}$ since

$$\begin{aligned}
\int_{[0,\pi)\times\mathbb{R}} |\tau||(I\otimes\mathcal{F})\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau &= \int_{[0,\pi)\times\mathbb{R}} |\tau|^2|(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}\Phi(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau \\
&= \int_{[0,\pi)\times\mathbb{R}} |\tau|^2|\mathcal{F}\Phi(\tau w(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} |\omega||\mathcal{F}\Phi(\omega)|^2\mathrm{d}\omega < +\infty,
\end{aligned}$$

since by definition Φ is a C^∞ function. Therefore, formula (2.55) reads

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle \mathcal{R}^{\rm pol} f, \hat{\pi}(b,0,1)\mathcal{I}^2\mathcal{R}^{\rm pol}\Phi\rangle|^2 {\rm d}b + \int_{\mathbb{R}^2 \rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2 {\rm d}b{\rm d}\phi \frac{{\rm d}a}{a^5},$$

where all the coefficients depend on f only through its polar Radon transform.

It is worth observing that the domain of $\overline{\mathcal{R}}$ is related to the domain of C, which defines the admissible vectors of π. By Theorem 1.30, (ii), the operator $|\overline{\mathcal{R}}|$ is a positive self-adjoint operator semi-invariant with weight $\chi(b,\phi,a) = a^{-1/2}$, which is a power of the modular function $\Delta(b,\phi,a) = a^{-2}$, i.e. $\chi(b,\phi,a) = \Delta(b,\phi,a)^{1/4}$. On the other hand, C is a positive self-adjoint operator semi-invariant with weight $\Delta^{1/2}$ and is such that $\psi \in L^2(\mathbb{R}^2)$ is an admissible vector of the square-integrable representation π if and only if $\psi \in \operatorname{dom} C$ and $\|C\psi\| = 1$. Therefore, $|\overline{\mathcal{R}}|$ and C are both positive self-adjoint operators on $L^2(\mathbb{R}^2)$ semi-invariant with a power of the modular function of $SIM(2)$ as weight. Thus, by Theorem 1.30, (i), the operators $|\overline{\mathcal{R}}|$ and C are given, up to a constant, by (2.33) with $s = -1/2$ and $s = -1$, respectively. The above argument explains why the domain of $\overline{\mathcal{R}}$ and the domain of C, and thus the admissibility condition (2.46) of π, are strictly related. A similar result can be proved for the examples illustrated in Section 2.4.

2.4 Examples

In this section, we illustrate two additional examples. The first one is presented in subsection 5.1 in [2, Lemma 3.5] and the second one is studied in [9].

2.4.1 The spherical means Radon transform

Groups and spaces

Take the same group $G = SIM(2)$ as in Example 2.2, namely $G = \mathbb{R}^2 \rtimes K$, with $K = \{R_\phi A_a \in {\rm GL}(2,\mathbb{R}) : \phi \in [0,2\pi),\ a \in \mathbb{R}^+\}$. Firstly, we choose $X = \mathbb{R}^2$ and, for what concerns this space, we keep the notation as in Example 2.2. Then, we consider the space $\Xi = \mathbb{R}^2 \times \mathbb{R}^+$, which we think of as parametrizing centers and radii of circles in $\mathbb{R}^2$, with the action

$$(b,\phi,a).(c,r) = (b + aR_\phi c, ar). \tag{2.57}$$

An immediate calculation shows that the isotropy at $\xi_0 = ((1,0),1)$ is

$$H = \{\big((1-\cos\phi, -\sin\phi), \phi, 1\big) : \phi \in [0,2\pi)\}.$$

By direct computation, recalling that $x_0 = 0$,

$$\hat{\xi}_0 = H[x_0] = \{(1-\cos\phi, -\sin\phi) : \phi \in [0,2\pi)\}$$

is the circle with center $(1,0)$ and radius 1 and

$$\check{x}_0 = K.\xi_0 = \{((a\cos\phi, a\sin\phi), a) : \phi \in [0,2\pi),\ a \in \mathbb{R}^+\}$$

is the family of circles passing through the origin. The measure ${\rm d}m_0 = {\rm d}\phi$ is H-invariant on $\hat{\xi}_0$, since the action of H on $\hat{\xi}_0$ is given by a simple rotation of a fixed angle. This gives $\gamma(h) \equiv 1$.

We define the section $\sigma\colon \Xi \to SIM(2)$ by $\sigma(c,r) = (c-(r,0),0,r)$. Thus, for $\xi = (c,r) \in \Xi$ we have

$$\hat{\xi} = \sigma(c,r)[\hat{\xi}_0] = \{c - rn(\phi) : \phi \in [0,2\pi)\}, \tag{2.58}$$

namely, the circle with center c and radius r and, for $x \in \mathbb{R}^2$ we have

$$\check{x} = s(x).\check{x}_0 = \{(x + (a\cos\phi, a\sin\phi), a) : \phi \in [0,2\pi),\ a \in \mathbb{R}^+\},$$

that is, the family of circles passing through the point x. It is easy to see that the maps $x \mapsto \check{x}$ and $\xi \mapsto \hat{\xi}$ are both injective. Thus, $X = \mathbb{R}^2$ and $\Xi = \mathbb{R}^2 \times \mathbb{R}^+$ are homogeneous spaces in duality.

We now fix a relatively invariant measure on Ξ: as we will show, this requires some care. Given $\alpha \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} f\left((b,\phi,a)^{-1}.(c,r)\right) \mathrm{d}c\frac{\mathrm{d}r}{r^\alpha} = a^{3-\alpha}\int_{\mathbb{R}^2\times\mathbb{R}^+} f(c,r)\mathrm{d}c\,\frac{\mathrm{d}r}{r^\alpha},$$

so that the measure $\mathrm{d}\xi = \mathrm{d}c\frac{\mathrm{d}r}{r^\alpha}$ is a relatively invariant measure on Ξ with character $\beta(b,\phi,a) = a^{3-\alpha}$.

The representations

Since the group G is the same as in Example 2.2, the representation π is given by (2.14), whereas we have to compute the quasi-regular representation $\hat{\pi}$ acting on $L^2(\Xi,\mathrm{d}\xi)$. Since $\beta(b,\phi,a) = a^{3-\alpha}$, by (2.9) and (2.57) we have

$$\begin{aligned}\hat{\pi}(b,\phi,a)F(c,r) &= a^{\frac{\alpha-3}{2}}F((-A_a^{-1}R_\phi^{-1}b, -\phi \bmod 2\pi, a^{-1}).(c,r))\\ &= a^{\frac{\alpha-3}{2}}F(a^{-1}R_{-\phi}(c-b), a^{-1}r),\end{aligned} \tag{2.59}$$

which is irreducible by Mackey imprimitivity theorem [28]. The proof is based on classical arguments and we omit it.

The Radon transform

By (2.58) and (2.3), the Radon transform in this case is given by

$$\mathcal{R}^{\mathrm{cir}}f(c,r) = \int_0^{2\pi} f(c - rn(\phi))\mathrm{d}\phi,$$

namely, the integral of f over the circle of center c and radius r. This is the so-called spherical means Radon transform [50]. It is worth observing that more interesting problems arise when the available centers and radii are restricted to some hypersurface: this does not easily fit into our assumptions, and it is left for future investigation.

Let us now determine a suitable π-invariant subspace $\mathcal{A}$ of $L^2(\mathbb{R}^2)$ as in (A6). In order to do that, it is useful to derive a Fourier slice theorem for $\mathcal{R}^{\mathrm{cir}}$. For any $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, by Fubini's theorem and using [24, Eq. 10.9.1], we have

$$\begin{aligned}(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}}f(\tau,r) &= \int_0^{2\pi}\int_{\mathbb{R}^2} f(c - rn(\phi))e^{-2\pi i c\cdot\tau}\mathrm{d}c\mathrm{d}\phi\\ &= \int_0^{2\pi} e^{-2\pi i r n(\phi)\cdot\tau}\mathrm{d}\phi\int_{\mathbb{R}^2} f(c)e^{-2\pi i c\cdot\tau}\mathrm{d}c\\ &= 2\pi J_0(2\pi|\tau|r)\mathcal{F}f(\tau),\end{aligned} \tag{2.60}$$

where J_0 is the Bessel function of the first kind. As a consequence, by Plancherel theorem, recalling that $\mathrm{d}\xi = \mathrm{d}c \frac{\mathrm{d}r}{r^\alpha}$ we obtain

$$\left\|\mathcal{R}^{\mathrm{cir}} f\right\|^2_{L^2(\Xi,\mathrm{d}\xi)} = \left\|(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}} f\right\|^2_{L^2(\Xi,\mathrm{d}\tau\frac{\mathrm{d}r}{r^\alpha})} = c_\alpha \int_{\mathbb{R}^2} |\mathcal{F}f(\tau)|^2 |\tau|^{\alpha-1}\,\mathrm{d}\tau,$$

where

$$c_\alpha = (2\pi)^{\alpha+1} \int_{\mathbb{R}^+} \frac{|J_0(r)|^2}{r^\alpha} \mathrm{d}r. \tag{2.61}$$

Observe that c_α is finite if and only if $\alpha \in (0,1)$, so that from now on we assume that $\alpha \in (0,1)$ and we set

$$\mathcal{A}_\alpha = \{f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\mathcal{F}f(\tau)|^2 |\tau|^{\alpha-1}\,\mathrm{d}\tau < +\infty\},$$

which is π-invariant and is such that $\mathcal{R}^{\mathrm{cir}} f \in L^2(\Xi, \mathrm{d}\xi)$ for all $f \in \mathcal{A}_\alpha$. Next we show that $\mathcal{R}^{\mathrm{cir}}$, regarded as operator from $\mathcal{A}_\alpha$ to $L^2(\Xi, \mathrm{d}\xi)$, is closable.

Suppose that $(f_n)_n \subseteq \mathcal{A}_\alpha$ is a sequence such that $f_n \to f$ in $L^2(\mathbb{R}^2)$ and $\mathcal{R}^{\mathrm{cir}} f_n \to g$ in $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$. Since $\mathcal{F}\otimes I$ is unitary from $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$ onto $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$, we have that $(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}} f_n \to (\mathcal{F}\otimes I)g$ in $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$. Since $f_n \in \mathcal{A}_\alpha$, by the Fourier slice theorem adapted to this setting (2.60), for every $(\tau, r) \in \mathbb{R}^2 \times \mathbb{R}_+$

$$(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}} f_n(\tau, r) = 2\pi J_0(2\pi|\tau|r)\mathcal{F}f_n(\tau).$$

Hence, passing to a subsequence if necessary,

$$2\pi J_0(2\pi|\tau|r)\mathcal{F}f_n(\tau) \to (\mathcal{F}\otimes I)g(\tau, r)$$

for almost every $(\tau, r) \in \mathbb{R}^2 \times \mathbb{R}_+$. Therefore, for almost every $(\tau, r) \in \mathbb{R}^2 \times \mathbb{R}_+$

$$(\mathcal{F}\otimes I)g(\tau, r) = \lim_{n\to+\infty} 2\pi J_0(2\pi|\tau|r)\mathcal{F}f_n(\tau) = 2\pi J_0(2\pi|\tau|r)\mathcal{F}f(\tau),$$

where the last equality holds true using a subsequence if necessary. Therefore, if $(h_n)_n \in \mathcal{A}_\alpha$ is another sequence such that $h_n \to f$ in $L^2(\mathbb{R}^2)$ and $\mathcal{R}^{\mathrm{cir}} h_n \to h$ in $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$, then, for almost every $(\tau, r) \in \mathbb{R}^2 \times \mathbb{R}_+$

$$(\mathcal{F}\otimes I)h(\tau, r) = 2\pi J_0(2\pi|\tau|r)\mathcal{F}f(\tau).$$

Therefore,

$$(\mathcal{F}\otimes I)g(\tau, r) = (\mathcal{F}\otimes I)h(\tau, r)$$

for almost every $(\tau, r) \in \mathbb{R}^2 \times \mathbb{R}_+$. Then $\lim_{n\to+\infty} \mathcal{R}^{\mathrm{cir}} f_n = \lim_{n\to+\infty} \mathcal{R}^{\mathrm{cir}} h_n$, and $\mathcal{R}^{\mathrm{cir}}$ is closable and we denote its closure by $\overline{\mathcal{R}}$. However, it is also possible to prove the closability of the operator $\mathcal{R}^{\mathrm{cir}}$ reasoning as in § 2.10 by choosing a "minimal" domain of the form $\mathrm{span}\{\pi(g)f_0 : g \in G\}$.

We stress that, if $\alpha \notin (0,1)$, the set

$$\{f \in L^2(X, \mathrm{d}x) : \mathcal{R}^{\mathrm{cir}} f \in L^2(\Xi, \mathrm{d}\xi)\} = \{0\},$$

i.e. it is trivial. This motivates the role of Assumption (A6) in our construction.

The Unitarization theorem

By (2.17) and (2.18) we have that

$$\mathcal{R}^{\mathrm{cir}}\pi(b,\phi,a) = a^{\frac{1-\alpha}{2}}\hat{\pi}(b,\phi,a)\mathcal{R}^{\mathrm{cir}},$$

since $\alpha(b,\phi,a) = a^2$, $\beta(b,\phi,a) = a^{3-\alpha}$ and $\gamma(b,\phi,a) = 1$, and so $\chi(b,\phi,a) = a^{\frac{\alpha-1}{2}}$.

By Theorem 2.11, there exists a positive self-adjoint operator $\mathcal{I}$, semi-invariant with weight $a^{\frac{1-\alpha}{2}}$, such that $\mathcal{I}\mathcal{R}^{\mathrm{cir}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2) \to L^2(\Xi, \mathrm{d}\xi)$. Moreover, $\mathcal{Q}$ intertwines π and $\hat{\pi}$, namely

$$\hat{\pi}(b,\phi,a)\,\mathcal{Q}\,\pi(b,\phi,a)^{-1} = \mathcal{Q}, \qquad (b,\phi,a) \in SIM(2).$$

As in the other examples, by using Theorem 1.30, part (i), it is possible to show that there exists a constant $k_\alpha \in \mathbb{R}^+$ such that $\mathcal{I} = k_\alpha \mathcal{J}$ with

$$(\mathcal{F}\otimes I)\mathcal{J}f(\tau,r) = |\tau|^{\frac{1-\alpha}{2}}(\mathcal{F}\otimes I)f(\tau,r), \qquad f \in \mathcal{D},$$

where

$$\mathcal{D} = \{f \in L^2(\Xi,\mathrm{d}\xi) : \int_{\mathbb{R}^2\times\mathbb{R}^+} |\tau|^{1-\alpha}|(\mathcal{F}\otimes I)f(\tau,r)|^2\,\mathrm{d}\tau\frac{\mathrm{d}r}{r^\alpha} < +\infty\}.$$

Using the same argument as in Example 2.2, it is possible to determine the constant k_α. Take a function $f \in \mathcal{A}_\alpha \setminus \{0\}$. By Plancherel theorem and the Fourier slice theorem obtained for $\mathcal{R}^{\mathrm{cir}}$ we have that

$$\begin{aligned}\|f\|^2 = \|\mathcal{I}\mathcal{R}^{\mathrm{cir}}f\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^+)} &= k_\alpha^2\|(\mathcal{F}\otimes I)\mathcal{J}\mathcal{R}^{\mathrm{cir}}f\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^+)}\\ &= k_\alpha^2\int_{\mathbb{R}^2\times\mathbb{R}^+}|\tau|^{1-\alpha}|(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}}f(\tau,r)|^2\mathrm{d}\tau\frac{\mathrm{d}r}{r^\alpha}\\ &= k_\alpha^2 c_\alpha\|f\|^2,\end{aligned}$$

where c_α is given by (2.61). Thus, we obtain that $k_\alpha = c_\alpha^{-1/2}$.

The inversion formula

Applying Theorem 2.17 to $\mathcal{R}^{\mathrm{cir}}$ we obtain the inversion formula for $f \in \mathcal{A}_\alpha$

$$f = \int_{SIM(2)} a^{\frac{\alpha-9}{2}}\langle\mathcal{R}^{\mathrm{cir}}f, \hat{\pi}(b,\phi,a)\Psi\rangle_{L^2(\Xi,\mathrm{d}\xi)}\,\psi(R_{-\phi}\frac{x-b}{a})\,\mathrm{d}b\mathrm{d}\phi\mathrm{d}a,$$

where we used that $\chi(b,\phi,a) = a^{\frac{\alpha-1}{2}}$, the expression for the Haar measure of $SIM(2)$ given in (2.10) and the expression for the representation π given in (2.14).

2.4.2 The Radon transform for hyperbolic motions

We consider the semidirect product $G = \mathbb{R}^2 \rtimes K$, with $K = \{aA_s\Omega_\epsilon \in \mathrm{GL}(2,\mathbb{R}) : a \in \mathbb{R}^\times,\ s \in \mathbb{R},\ \epsilon \in \{-1,1\}\}$ where

$$A_s = \begin{bmatrix}\cosh s & \sinh s\\ \sinh s & \cosh s\end{bmatrix}, \qquad \Omega_{-1} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

and Ω_1 is the identity matrix. We denote by C_2 the multiplicative group $\{-1, 1\}$. Under the identification $K \simeq \mathbb{R} \times \mathbb{R}^* \times C_2$, we write (b, s, a, ϵ) for the elements in G, so that the group law becomes

$$(b, s, a, \epsilon)(b', s', a', \epsilon') = (b + aA_s\Omega_\epsilon b', s + s', aa', \epsilon\,\epsilon').$$

A left Haar measure of G is $\mathrm{d}\mu(b, s, a, \epsilon) = |a|^{-3}\mathrm{d}b\mathrm{d}s\mathrm{d}a\mathrm{d}\epsilon$, where $\mathrm{d}b$, $\mathrm{d}s$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^*$, respectively and $\mathrm{d}\epsilon$ is the counting measure on C_2.

The group G acts transitively on $X = \mathbb{R}^2$ by the canonical action

$$(b, s, a, \epsilon)[x] = b + aA_s\Omega_\epsilon x, \quad (b, s, a, \epsilon) \in G,\ x \in X. \tag{2.62}$$

The isotropy at the origin $x_0 = 0$ is the closed subgroup $\{(0, k) : k \in K\} \simeq K$, so that $X \simeq G/K$ and the Lebesgue measure $\mathrm{d}x$ on X is a relatively G-invariant measure with positive character $\alpha(b, s, a, \epsilon) = |a|^2$. It is possible to parametrize lines in the plane, except those with slope -1 or 1, by the space of parameters $\Xi = C_2 \times \mathbb{R} \times \mathbb{R}$ as in Figure 2.1. The group G is a subgroup of affine transformations of the plane and thus maps lines into lines. Its action on this set of lines is given by the formula

$$(b, s, a, \epsilon)^{-1}.(\eta, u, t) = \left(\epsilon\,\eta, u + s, \frac{t - \Omega_\eta w(u) \cdot b}{a}\right),$$

where $w(u) = {}^t(\cosh u, \sinh u)$, and is easily seen to be transitive. The isotropy at $\xi_0 = (1, 0, 0)$ is

$$H = \{((0, b_2), 0, a, 1) : b_2 \in \mathbb{R}, a \in \mathbb{R}^\times\}.$$

Thus, $\Xi \simeq G/H$. An immediate calculation gives that the measure $\mathrm{d}\xi = \mathrm{d}\eta\mathrm{d}u\mathrm{d}t$, where $\mathrm{d}u$ and $\mathrm{d}t$ are the Lebesgue measures on $\mathbb{R}$ and $\mathrm{d}\eta$ is the counting measure on C_2, is a G-relatively invariant measure on Ξ with positive character $\beta(b, s, a, \epsilon) = |a|$.

Consider now the section $\sigma \colon \Xi \to G$ defined by

$$\sigma(\eta, u, t) = (t\,\Omega_\eta w(-u), -u, 1, \eta).$$

By direct computation

$$\hat{\xi}_0 = H[x_0] = \{(0, b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R}.$$

It is immediate to see that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with character $\gamma((0, b_2), 0, a, 1) = |a|$ and that $\gamma(\sigma(\eta, u, t)) = 1$ for all $(\eta, u, t) \in \Xi$, so that $(g, \xi) \mapsto \gamma(\sigma(\xi)^{-1}g\sigma(g^{-1}.\,\xi))$ extends to a positive character of G independent of ξ. Further, we have that

$$\widehat{(\eta, u, t)} = \sigma(\eta, u, t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x \cdot \Omega_\eta n(u) = t\},$$

which is the set of all points laying on the line of equation $x \cdot \Omega_\eta n(u) = t$. Therefore, the submanifolds over which we integrate functions are lines in $\mathbb{R}^2$, except to the ones with slope -1 or 1, and are parametrized by Ξ through the injective map $(\eta, u, t) \mapsto \widehat{(\eta, u, t)}$.

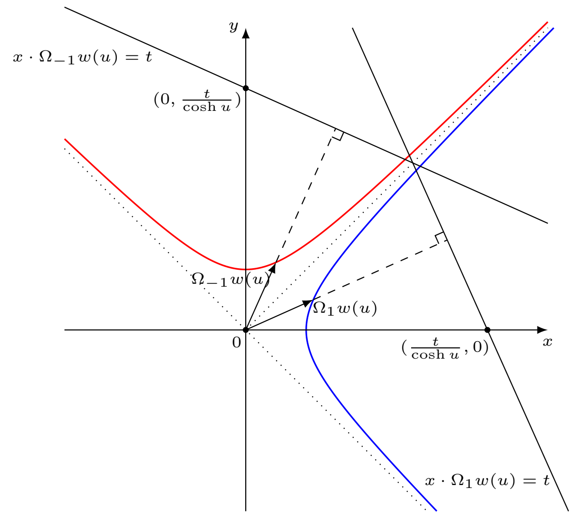

Figure 2.1: The lines in $\mathbb{R}^2$ except those with slope 1 or -1 are parametrized by triples $(\eta, u, t) \in \Xi = C_2 \times \mathbb{R} \times \mathbb{R}$. The vector u parametrizes the slope. The choice $\eta = 1$ $(\eta = -1)$ corresponds to the absolute value of the slope > 1 (< 1) and fixes as reference line the $x-$axis (y-axis). Then t parametrizes the intersection of the line with the reference axis.

The representations

The group G acts on $L^2(X)$ by means of the unitary representation π defined by

$$\pi(b, s, a, \epsilon)f(x) = |a|^{-1} f(a^{-1}\Omega_\epsilon^{-1} A_s^{-1}(x - b)).$$

The dual action $\mathbb{R}^2 \times K \ni (\eta, k) \mapsto {}^t k\eta$ has a single open orbit $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : |x| \neq |y|\}$ for ${}^t(1, 0) \in \mathbb{R}^2$ of full measure and the stabilizer $K_{(1,0)} = \{(0, 1, 1)\}$ is compact. Then, by a result due to Führ in [31], the representation π is square-integrable. Furthermore, G acts on $L^2(\Xi, \mathrm{d}\xi)$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b, s, a, \epsilon)F(\eta, u, t) = |a|^{-\frac{1}{2}} F\left(\epsilon\,\eta, u + s, \frac{t - \Omega_\eta w(u) \cdot b}{a}\right).$$

By Mackey imprimitivity theorem [28], one can show that also $\hat{\pi}$ is irreducible. The proof, although not trivial, is based on classical arguments and we omit it.

The Radon transform

We compute by (2.3) the Radon transform between the homogeneous spaces X and Ξ and we obtain

$$\mathcal{R}f(\eta, u, t) = \int_{\mathbb{R}} f(\Omega_\eta A_{-u}{}^t(t, y))\mathrm{d}y, \tag{2.63}$$

which maps any $(\eta, u, t) \in \Xi$ in the integral of f over the line parametrized by (η, u, t) through the map $(\eta, u, t) \mapsto \widehat{(\eta, u, t)}$, i.e. the line of equation $x \cdot \Omega_\eta w(u) = t$. Observe that, by Fubini's theorem, the integral (2.63) converges for any $f \in L^1(\mathbb{R}^2)$. Then, we define

$$\mathcal{A} = \{f \in L^1 \cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\omega_1, \omega_2)|^2}{\sqrt{|\omega_1{}^2 - \omega_2{}^2|}} \mathrm{d}\omega_1 \mathrm{d}\omega_2 < +\infty\},$$

which is π-invariant and is such that $\mathcal{R}f \in L^2(\Xi, \mathrm{d}\xi)$ for all $f \in \mathcal{A}$. Furthermore, as in subsection 2.4.1, it is possible to show that $\mathcal{R}$, regarded as operator from $\mathcal{A}$ to $L^2(\Xi, \mathrm{d}\xi)$, is closable. In order to prove that $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable, we derive the Fourier slice theorem adapted to this context, precisely

$$(I \otimes \mathcal{F})\mathcal{R}f(\eta, u, \tau) = \mathcal{F}f(\tau\, \Omega_\eta w(u)),$$

for every $f \in L^1(\mathbb{R}^2)$ and $(\eta, u, \tau) \in \Xi$, where I is the identity operator on $L^2(C_2 \times \mathbb{R}, \mathrm{d}\eta \mathrm{d}u)$.

It is worth observing that when we fix $\eta = 1$ ($\eta = -1$) in (2.63) we are restricting the integration of f over all lines with slope > 1 (< 1). Then, for $\eta = 1$ and $\eta = -1$ we have the limited angle horizontal and vertical Radon transforms, respectively. We will see in the next section how these two different contributions enter in the inversion formula when we reconstruct an unknown signal from its Radon transform.

Unitarization and Inversion formula

Applying Theorem 2.11, $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is a densely defined operator which intertwines the representations π and $\hat{\pi}$ up to the positive character $\chi(b, s, a, \epsilon) = |a|^{-1/2}$, namely

$$\hat{\pi}(b, s, a, \epsilon)\mathcal{R}\pi(b, s, a, \epsilon)^{-1} = |a|^{-1/2}\mathcal{R},$$

for all $(b, s, a, \epsilon) \in G$.

The composition of $\mathcal{R}$ with a positive selfadjoint operator $\mathcal{I}$ satisfying

$$\hat{\pi}(b, s, a, \epsilon)\mathcal{I}\hat{\pi}(b, s, a, \epsilon)^{-1} = |a|^{1/2}\mathcal{I}$$

can be extended to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2) \to L^2(\Xi, \mathrm{d}\xi)$ intertwining the irreducible representations π and $\hat{\pi}$.

We can provide an explicit formula for $\mathcal{I}$. We consider the subspace $\mathcal{D}$ of $L^2(\Xi, \mathrm{d}\xi)$ of the functions F such that

$$\int_{\mathbb{R}\times\mathbb{R}} |\tau||(I \otimes \mathcal{F})F(\eta, u, \tau)|^2 \, \mathrm{d}u\mathrm{d}\tau < +\infty, \qquad \eta = -1, 1,$$

and we define the operator $\mathcal{J}$ on $\mathcal{D}$ by

$$(I \otimes \mathcal{F})\mathcal{J}F(\eta, u, \tau) = |\tau|^{\frac{1}{2}}(I \otimes \mathcal{F})F(\eta, u, \tau),$$

a Fourier multiplier with respect to the last variable. A direct calculation shows that $\mathcal{J}$ is a densely defined positive selfadjoint operator with the property

$$\hat{\pi}(b, s, a, \epsilon)\mathcal{J}\hat{\pi}(b, s, a, \epsilon)^{-1} = |a|^{1/2}\mathcal{J}.$$

By Theorem 1.30, (i), there exists $c > 0$ such that $\mathcal{I} = c\mathcal{J}$ and it is easy to show that $c = 1$.

It is possible to prove that the admissible vectors for π are the functions $\psi \in L^2(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\omega_1, \omega_2)|^2}{|\omega_1{}^2 - \omega_2{}^2|} \mathrm{d}\omega_1 \mathrm{d}\omega_2 = 1. \tag{2.64}$$

The voice transform is $(\mathcal{V}_\psi f)(g) = \langle f, \pi(g)\psi\rangle$, and is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(G, \mathrm{d}\mu)$ provided that ψ satisfies the admissible condition (2.64). If $\mathcal{Q}\psi \in \operatorname{dom}\mathcal{I}$, by equation (2.42), we have that

$$(\mathcal{V}_\psi f)(b,s,a,\epsilon) = \int_{\mathbb{R}\times\mathbb{R}} \mathcal{R}f(1,u,t)\overline{\Psi(\epsilon, u+s, \frac{t-w(u)\cdot b}{a})}\frac{\mathrm{d}u\mathrm{d}t}{|a|} + \int_{\mathbb{R}\times\mathbb{R}} \mathcal{R}f(-1,u,t)\overline{\Psi(-\epsilon, u+s, \frac{t-\Omega_{-1}w(u)\cdot b}{a})}\frac{\mathrm{d}u\mathrm{d}t}{|a|}, \qquad (2.65)$$

for any $f \in \mathcal{A}$, where $\Psi = \mathcal{I}\mathcal{Q}\psi$. Note that the coefficients depend on f only through its Radon transform and do not involve the operator $\mathcal{I}$ as applied to the signal. Hence, equation (2.40) allows to reconstruct an unknown signal $f \in \mathcal{A}$ from its Radon transform by computing the coefficients $(\mathcal{V}_\psi f)(b,s,a,\epsilon)$ by means of (2.65). It is worth observing that the different contributions in (2.65) with $\eta = 1$ and $\eta = -1$ reconstruct the frequency projections of f onto the horizontal cone $\{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_2/\omega_1| < 1\}$ and onto the vertical cone $\{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_1/\omega_2| < 1\}$, respectively.

Chapter 3

The Affine Radon Transform Intertwines Wavelets and Shearlets

The purpose of this chapter is to investigate the connection between the shearlet transform and the wavelet transform. We show that this link has to be found in the unitary Radon transform in affine coordinates. Precisely, we prove that the affine Radon transform, up to a composition with a suitable pseudo-differential operator, can be extended to a unitary operator intertwining the shearlet representation with the tensor product of two wavelet representations. This intertwining result yields a formula for the shearlet coefficients that involves only integral transforms applied to the affine Radon transform of the signal, thereby opening new perspectives both for finding a new algorithm to compute the shearlet coefficients of a signal and for the inversion of the Radon transform. In this direction we have obtained formulae for the so-called cone-adapted shearlet coefficients [51] which we present in section 3.5.

Even if these results perfectly fit our weakened construction of generalized Radon transforms between dual homogeneous spaces (see § 3.4), we do not make use of the theory presented in Chapter 2 in order to obtain both an intertwining and a unitarization result, and consequently an inversion formula, for the affine Radon transform. This choice has been motivated by several reasons.

First of all, most of the results of this chapter are contained in [8], which is a work prior to [2] that has strongly inspired us for this latter work. Indeed, we realized later that the findings in [8] could be treated in a more general and unified way using the theory of irreducible representations and for this reason we used a different approach which does not make use of the hypothesis of irreducibility. The proof mimics the approach followed by Helgason to solve the unitarization problem for the Radon transform in the context of symmetric spaces [39] and can be adapted to other examples in which the hypothesis of irreducibility does not hold, for example to the context of homogeneous trees.

In this chapter we actually work with a large class of semidirect products introduced by Führ in [29, 32] for the purpose of generalizing the standard shearlet group, known as shearlet dilation groups. This family reduces to the standard shearlet group in dimension two, while it provides a large reservoir of families of natural transformations in higher dimensions, where there is no canonical shearlet group. Moreover it exhibits different behaviours in terms of wavefront set resolution properties, as investigated in [27].

We start presenting in full details all the various ingredients, namely the groups, the representations and the affine Radon transform. Then, we state and prove the main results and we show with the example of the standard shearlet group how these findings perfectly fit the general framework presented in Chapter 2.

3.1 The Groups ad their Representations

3.1.1 Shearlet dilation groups

In this section we introduce the groups in which we are interested. This family includes the groups introduced by Führ in [29, 32], and called generalized shearlet dilation groups, for the purpose of generalizing the standard shearlet group introduced in [54, 20]. It is worth observing that we endow the groups with a differentiable structure just to include in our analysis the generalized shearlet dilation groups but it does not play any role in the proofs of the main theorems. For this reason we do not recall the theory of Lie groups and we refer to [73, 70, 47] for classical references.

We denote the (real) general linear group of size $d \times d$ by $\mathrm{GL}(d,\mathbb{R})$ and by $\mathrm{T}(d,\mathbb{R})$ the closed subgroup of unipotent upper triangular matrices.

Definition 3.1 ([8, Definition 1]). A *shearlet dilation group* $H < \mathrm{GL}(d,\mathbb{R})$ is a subgroup of the form $H = SD$, where

(i) S is a Lie subgroup of $\mathrm{T}(d,\mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} 1 & -{}^t s \\ 0 & B(s) \end{bmatrix}$$

with $s \in \mathbb{R}^{d-1}$ and $B : \mathbb{R}^{d-1} \to \mathrm{T}(d-1,\mathbb{R})$ a smooth map;

(ii) D is the one-parameter subgroup of $\mathrm{GL}(d,\mathbb{R})$ consisting of the diagonal matrices

$$a\,\mathrm{diag}(1, |a|^{\lambda_1}, \ldots, |a|^{\lambda_{d-1}}) = a \begin{bmatrix} 1 & 0 \\ 0 & \Lambda(a) \end{bmatrix} \tag{3.1}$$

as a ranges in $\mathbb{R}^\times$. Here $(\lambda_1, \ldots, \lambda_{d-1})$ is a fixed vector in $\mathbb{R}^{d-1}$.

The group S is called the shearing subgroup of H and D is called the diagonal complement or scaling subgroup of H.

Several observations are in order. First of all, if one requires the shearing subgroup S to be Abelian, then one obtains the class introduced by Führ, with a slightly more general definition. This has inspired Definition 3.1.

Since the map B is continuous, S is automatically connected, and hence by Theorem 3.6.2 in [70], it is closed and simply connected. By construction the elements of H are of the form

$$h_{s,a} = h_{s,1}h_{0,a} = a \begin{bmatrix} 1 & -{}^t s\Lambda(a) \\ 0 & B(s)\Lambda(a) \end{bmatrix}. \tag{3.2}$$

Furthermore, since the diagonal matrices of $\mathrm{GL}(d,\mathbb{R})$ normalize $\mathrm{T}(d,\mathbb{R})$, then H is the semidirect product of S and D.

Finally, the assumption that S is a subgroup normalized by D forces the maps B and Λ to satisfy some equalities. Indeed, since

$$\begin{bmatrix} 1 & -{}^t u \\ 0 & B(u) \end{bmatrix} \begin{bmatrix} 1 & -{}^t v \\ 0 & B(v) \end{bmatrix} = \begin{bmatrix} 1 & -{}^t(v + {}^t B(v) u) \\ 0 & B(u)B(v) \end{bmatrix},$$

then S is a group if and only if

$$B(0) = \mathrm{I}_{d-1} \tag{3.3}$$

$$B(u)B(v) = B(v + {}^t B(v) u) \tag{3.4}$$

$$B(u)^{-1} = B(-{}^t B(u)^{-1} u) \tag{3.5}$$

for every $u, v \in \mathbb{R}^{d-1}$. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & \Lambda(a) \end{bmatrix} \begin{bmatrix} 1 & -{}^t s \\ 0 & B(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Lambda(a)^{-1} \end{bmatrix} = \begin{bmatrix} 1 & {}^t(\Lambda(a)^{-1} s) \\ 0 & \Lambda(a)B(s)\Lambda(a)^{-1} \end{bmatrix}$$

the compatibility of D with S is equivalent to asking for the following condition to hold for all $a \neq 0$ and all $s \in \mathbb{R}^{d-1}$:

$$\Lambda(a)B(s)\Lambda(a)^{-1} = B(\Lambda(a)^{-1} s). \tag{3.6}$$

It follows that H is diffeomorphic as a manifold to $\mathbb{R}^{d-1} \times \mathbb{R}^{\times}$, so that we can identify the element $h_{s,a}$ with the pair (s, a). With this identification the product law amounts to

$$(s, a)(s', a') = \left(\Lambda(a)^{-1} s' + {}^t B(\Lambda(a)^{-1} s') s, aa'\right). \tag{3.7}$$

We stress that, in general, S is not isomorphic as a Lie group to the additive Abelian group $\mathbb{R}^{d-1}$, unless S is the standard shearlet group introduced in [20], see the examples below.

It should be clear that a slightly larger class would be obtained by allowing for diagonal matrices of the form

$$\mathrm{sign}(a)\,\mathrm{diag}(|a|^{\mu_0}, |a|^{\mu_1}, \ldots, |a|^{\mu_{d-1}}), \qquad a \in \mathbb{R}^{\times}.$$

The case $\mu_0 = 0$, however, is uninteresting because any shearlet dilation group corresponding to this choice never admits admissible vectors [32, 3]. But then a simple change of variables permits to assume $\mu_0 = 1$, as we did, and to set $\lambda_j = \mu_j - 1$.

In [32] the authors introduce the notion of shearlet dilation group by means of structural properties and then prove that in the case when S is Abelian they can be parametrized as in Definition 3.1.

We now give three examples. If S is Abelian a full characterization is provided in [32], see also [3] for a connection with a suitable class of subgroups of the symplectic group.

Example 3.2 (The standard shearlet group [8])**.** A possible choice for B is the map $B(s) = \mathrm{I}_{d-1}$, which satisfies all the above properties. In this case, $s \mapsto h_{s,1}$ defines a group isomorphism between $\mathbb{R}^{d-1}$ and the Abelian group S. Furthermore, in this example, we use the standard notation S_s for the matrices $h_{s,1}$.

Clearly, any choice of the weights $\lambda_1, \dots, \lambda_{d-1}$ is compatible with (3.6). In particular, if we choose as D the group of matrices

$$A_a = a \begin{bmatrix} 1 & 0 \\ 0 & |a|^{\gamma-1}\,\mathrm{I}_{d-1} \end{bmatrix} \qquad \Longleftrightarrow \qquad \Lambda(a) = |a|^{\gamma-1}\,\mathrm{I}_{d-1} \qquad a \in \mathbb{R}^{\times},$$

where $\gamma \in \mathbb{R}$ is a fixed parameter, then we obtain the d-dimensional shearlet group, usually denoted $\mathbb{S}^{\gamma}$, and, often, the parameter γ is chosen to be $1/d$ [20, 18].

Example 3.3 (The Toeplitz shearlet group [8])**.** Another important example arises when $B(s)$ is the Toeplitz matrix

$$B(s) = T(\hat{s}) = \begin{bmatrix} 1 & -s_1 & -s_2 & \dots & -s_{d-2} \\ 0 & 1 & -s_1 & -s_2 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & -s_1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \tag{3.8}$$

where $\hat{s} = {}^t(s_1, \dots, s_{d-2})$. It is easy to see that $T(\hat{u})T(\hat{v}) = T(\hat{u}\sharp\hat{v})$ where

$$(\hat{u}\sharp\hat{v})_i := u_i + v_i + \sum_{j+k=i} v_j u_k, \quad i = 1, \dots d-2$$

and that consequently all the equalities in (3.4) hold. This case corresponds to Toeplitz shearlet groups (see [17]).

Not all dilation matrices as in (3.1) are compatible with (3.6). In [32] it is shown that

$$\lambda_k = k\lambda_1, \qquad k = 2, \dots, d-1 \tag{3.9}$$

for any fixed λ_1.

Example 3.4 (A non-Abelian shearlet dilation group [8])**.** The matrices

$$g(u_1, u_2, u_3) = \begin{bmatrix} 1 & -u_1 & -u_2 & -u_3 \\ 0 & 1 & -u_1 & -u_2 - \frac{1}{2}u_1^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as $u = (u_1, u_2, u_3)$ ranges in $\mathbb{R}^3$ give rise to a non-Abelian shearlet group S. Indeed, it is easily checked that

$$g(u_1, u_2, u_3)g(v_1, v_2, v_3) = g(u_1 + v_1, u_2 + v_2 - u_1v_1, u_3 + v_3 - u_1(v_2 + \frac{1}{2}v_1^2)),$$

a product which is not Abelian in the third coordinate. Evidently,

$$B(u) = \begin{bmatrix} 1 & -u_1 & -u_2 - \frac{1}{2}u_1^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a smooth function of u. The group S is isomorphic to the standard Heisenberg group, as is most clearly seen at the level of Lie algebra. Indeed, the Lie algebra of S is given by the matrices

$$X(q,p,t) = \begin{bmatrix} 0 & q & p & t \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

because $X^3(q,p,t) = 0$ and hence

$$\begin{aligned} \exp(X(q,p,t)) &= \mathrm{I}_4 + X(q,p,t) + \frac{1}{2}X^2(q,p,t) \\ &= \begin{bmatrix} 1 & q & p+\frac{1}{2}q^2 & t+\frac{1}{2}qp \\ 0 & 1 & q & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= g(-q, -(p+\frac{1}{2}q^2), -(t+\frac{1}{2}qp)). \end{aligned}$$

Further,

$$[X(q,p,t), X(q',p',t')] = X(0,0,qp'-pq')$$

exhibits the Lie algebra of S as the three dimensional Heisenberg Lie algebra. A straightforward calculation shows that for any choice of $\lambda \in \mathbb{R}$ the diagonal matrices

$$\Lambda(a) = \begin{bmatrix} |a|^{\lambda} & & \\ & |a|^{2\lambda} & \\ & & |a|^{3\lambda} \end{bmatrix}$$

normalize $B(u)$ because $\Lambda(a)B(u)\Lambda(a)^{-1} = B(\Lambda(a)^{-1}u)$. Conversely, these are easily seen to be the only rank-one dilations that normalize the matrices $B(u)$. In conclusion, the group D consisting of the matrices

$$a \begin{bmatrix} 1 & \\ & \Lambda(a) \end{bmatrix}$$

together with S give rise to the non-Abelian shearlet dilation group $H = SD$. It is worth observing that the dilations in D are not the standard dilations of the Heisenberg group. Indeed, the Lie algebra of D consists of the diagonal matrices $A_\lambda(\tau) = \mathrm{diag}(\tau, (\lambda+1)\tau, (2\lambda+1)\tau, (3\lambda+1)\tau)$ and

$$[A_\lambda(\tau), X(q,p,t)] = X(-\lambda\tau q, -2\lambda\tau p, -3\lambda\tau t)$$

shows that these homogeneous dilations are not the standard dilations of the Heisenberg Lie algebra (see [66], p. 620).

3.1.2 The shearlet representation and the admissible vectors

From now on we fix a group $G = \mathbb{R}^d \rtimes H$ where H is a shearlet dilation group as in Definition 3.1 and we parametrize its elements as (b, s, a). By Proposition 1.9 and equation (3.7), we get that a left Haar measure of H is

$$\mathrm{d}h = |a|^{\lambda_D - 1}\mathrm{d}s\mathrm{d}a$$

where $\lambda_D = \lambda_1 + \ldots + \lambda_{d-1}$ and $\mathrm{d}s$, $\mathrm{d}a$ are the Lebesgue measures of $\mathbb{R}^{d-1}$ and $\mathbb{R}^\times$. As a consequence, by (2.6) a left Haar measure on G is

$$\mathrm{d}g = \mathrm{d}b \, \frac{\mathrm{d}h}{|\det h_{s,a}|} = |a|^{-(d+1)} \mathrm{d}b \mathrm{d}s \mathrm{d}a, \tag{3.10}$$

where $\mathrm{d}b$ is the Lebesgue measure on $\mathbb{R}^d$ and the last equality holds true since

$$|\det h_{s,a}| = |a|^{d+\lambda_D}.$$

Finally, by (2.8) the Lebesgue measure on $\mathbb{R}^d$ is a relatively G-invariant measure with character $\chi(b, s, a) = |a|^{d+\lambda_D}$ and G acts on $L^2(\mathbb{R}^d)$ via the quasi-regular representation

$$S_{b,s,a} f(x) = |a|^{-\frac{d+\lambda_D}{2}} f(h_{s,a}^{-1}(x - b)). \tag{3.11}$$

The next result generalizes Theorem 4.12 in [32] to the case when S is not Abelian.

Theorem 3.5 ([8, Theorem 2])**.** *The representation S is square-integrable and its admissible vectors ψ are the elements of $L^2(\mathbb{R}^d)$ satisfying*

$$0 < C_\psi = \int_{\mathbb{R}^d} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi_1|^d} \mathrm{d}\xi < +\infty, \tag{3.12}$$

where $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$.

Proof. The proof is an immediate consequence of the following result due to Führ, see [31] and the references therein. The quasi-regular representation of $\mathbb{R}^d \rtimes H$ is square integrable if and only if there exists a vector $\xi_0 \in \mathbb{R}^d$ such that

(i) the dual orbit $\mathcal{O}_{\xi_0} = \{{}^t h \xi_0 \in \mathbb{R}^d : h \in H\}$ is open and it is of full measure,

(ii) the stabilizer $H_{\xi_0} = \{h \in H : {}^t h \xi_0 = \xi_0\}$ is compact,

where saying that $\mathcal{O}_{\xi_0}$ has full measure means that its complement has Lebsegue measure zero. In such case, a vector ψ is admissible if and only if

$$C_\psi = \int_H |\mathcal{F}\psi({}^t h \xi_0)|^2 \mathrm{d}h < +\infty. \tag{3.13}$$

In our setting, with the choice $\xi_0 = (1, 0, \ldots, 0)$ we have that

$${}^t h_{s,a} \xi_0 = {}^t h_{0,a} {}^t h_{s,1} \begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ \Lambda(a) s \end{bmatrix}$$

so that $\mathcal{O}_{\xi_0} = \mathbb{R}^\times \times \mathbb{R}^{d-1}$, which is of full measure, and H_{ξ_0} is trivial. Hence S is square-integrable.

To compute the admissible vectors, notice that by (3.13)

$$\begin{aligned} \int_H |\mathcal{F}\psi({}^t h \xi_0)|^2 \mathrm{d}h &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^\times} |\mathcal{F}\psi(a\Lambda(a)s, a)|^2 |a|^{\lambda_D - 1} \mathrm{d}s \mathrm{d}a \\ &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}^\times} \frac{|\mathcal{F}\psi(\xi_1, \xi')|^2}{|\xi_1|^d} \mathrm{d}\xi_1 \mathrm{d}\xi' \end{aligned}$$

with the change of variables $a = \xi_1$ and $s = \Lambda(\xi_1)^{-1} \xi' / \xi_1$. □

Theorem 3.5 states the surprising fact that the admissibility condition is the same for all generalized shearlet dilation groups. A canonical choice is to assume that

$$\mathcal{F}\psi(\xi_1, \xi') = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2(\xi'/\xi_1) \tag{3.14}$$

where $\psi_1 \in L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}^\times} \frac{|\mathcal{F}\psi_1(\xi_1)|^2}{|\xi_1|} \mathrm{d}\xi_1 < +\infty. \tag{3.15}$$

and $\psi_2 \in L^2(\mathbb{R}^{d-1})$. However, other choices are available and, in particular, it is possible to build shearlets with compact support in space [46]. Finally, since the representation S is square-integrable, the associated voice transform $\mathcal{S}_\psi f(b, s, a) = \langle f, S_{b,s,a}\psi\rangle$, known as shearlet transform, is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(G)$ and we have the weakly-convergent reproducing formula [30]

$$f = \frac{1}{C_\psi} \int_G \mathcal{S}_\psi f(b, s, a)\, S_{b,s,a}\psi \,\frac{\mathrm{d}b\,\mathrm{d}s\,\mathrm{d}a}{|a|^{d+1}}. \tag{3.16}$$

3.1.3 The quasi-regular representation of H

Consider now the shearlet dilation group H, with S and D its shearing and dilation subgroups, respectively (see Definition 3.1). As mentioned above, we identify H with $\mathbb{R}^{d-1} \times \mathbb{R}^\times$ as manifolds and sometimes denote by (s, a) the element $h_{s,a}$ of H. Recall that by (3.7) the product law is then

$$(s, a)(s', a') = (\Lambda(a)^{-1}s' + {}^tB(\Lambda(a)^{-1}s')s, aa').$$

Observe that H acts naturally on $\mathbb{R}^d$ and its (right) dual action is

$${}^th_{s,a} \begin{bmatrix} v_1 \\ v \end{bmatrix} = a \begin{bmatrix} v_1 \\ \Lambda(a)({}^tB(s)v - s) \end{bmatrix}.$$

This implies that H acts naturally on $\mathbb{P}^{d-1} = (\mathbb{R}^d \setminus \{0\})/\sim$ as well. By identifying $\mathbb{R}^{d-1}$ with $\{(1, v) : v \in \mathbb{R}^{d-1}\}$ we get that H acts on $\mathbb{R}^{d-1}$ as

$${}^th_{s,a}.v = \Lambda(a)({}^tB(s)v - s).$$

Hence we can define the quasi-regular representation of H acting on $L^2(\mathbb{R}^{d-1})$ by means of

$$V_{s,a}f(v) = |a|^{\frac{\lambda_D}{2}} f(\Lambda(a)({}^tB(s)v - s)),$$

where we recall that $\lambda_D = \lambda_1 + \cdots + \lambda_{d-1}$. In general, V is not irreducible, but we can always define the voice transform associated to a fixed vector $\psi \in L^2(\mathbb{R}^{d-1})$, namely the mapping $\mathcal{V}_\psi : L^2(\mathbb{R}^{d-1}) \longrightarrow C(H)$ defined by

$$\mathcal{V}_\psi f(s, a) = \langle f, V_{s,a}\psi\rangle,$$

where $C(H)$ is the space of continuous functions on H.

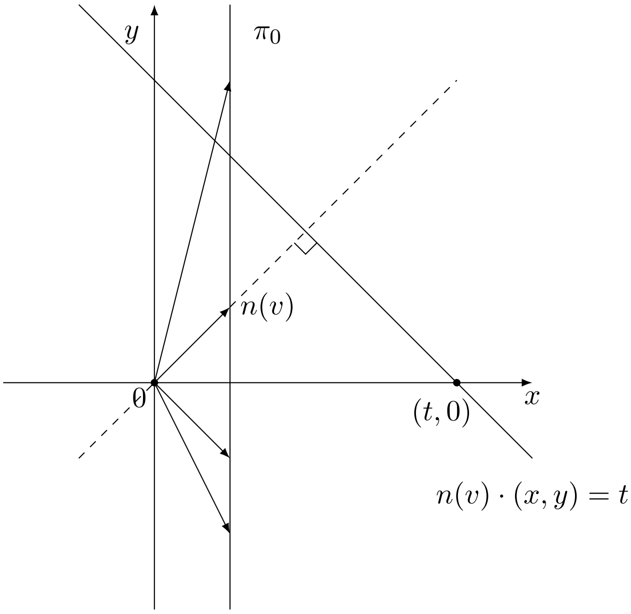

Figure 3.1: space of hyperplanes parametrized by affine coordinates (2-dimensional case)

Example 3.6 (The standard shearlet group, continued [8])**.** For the classical shearlet group $\mathbb{S}^\gamma$ the shearlet representation on $L^2(\mathbb{R}^d)$ becomes

$$S^\gamma_{b,s,a}f(x) = |a|^{-\frac{1+\gamma(d-1)}{2}} f(A_a^{-1}S_s^{-1}(x-b)), \tag{3.17}$$

whereas the group H is the semidirect product $\mathbb{R}^{d-1} \rtimes \mathbb{R}^\times$ and V is given by

$$V_{s,a}f(v) = |a|^{\frac{(d-1)(\gamma-1)}{2}} f\left(\frac{v-s}{|a|^{1-\gamma}}\right), \tag{3.18}$$

which is not irreducible unless $d = 2$. Furthermore, the voice transform can be written as convolution operator

$$\mathcal{V}_\psi f(s,a) = |a|^{\frac{(d-1)(\gamma-1)}{2}} \int_{\mathbb{R}^{d-1}} f(v)\overline{\psi\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\mathrm{d}v = f * \Psi_a(s)$$

where

$$\Psi_a(v) = |a|^{\frac{(d-1)(\gamma-1)}{2}} \overline{\psi\left(-\frac{v}{|a|^{1-\gamma}}\right)}.$$

3.2 The affine Radon Transform and its Unitarization

We start introducing the affine Radon transform. In what follows $\mathcal{R}$ denotes the classical Radon transform in $\mathbb{R}^d$ defined by (1.10) and we set ${}^t n(v) = (1, {}^t v)$ for all $v \in \mathbb{R}^{d-1}$.

Definition 3.7. Given $f \in L^1(\mathbb{R}^d)$, the affine Radon transform of f is the function $\mathcal{R}^{\mathrm{aff}}f : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{C}$ given by

$$\begin{aligned}\mathcal{R}^{\mathrm{aff}}f(v,t) &= \mathcal{R}f(n(v),t)\\ &= \frac{1}{\sqrt{1+|v|^2}} \int_{n(v)\cdot x = t} f(x)\mathrm{d}m(x) = \int_{\mathbb{R}^{d-1}} f(t - v\cdot y, y)\mathrm{d}y.\end{aligned} \tag{3.19}$$

The transform $\mathcal{R}^{\text{aff}}$ is obtained from $\mathcal{R}$ by parametrizing the projective space $\mathbb{P}^{d-1}$ with affine coordinates. Indeed, the map $(v,t) \mapsto (n(v) : t)$ is a diffeomorphism of $\mathbb{R}^{d-1} \times \mathbb{R}$ onto the open subset

$$U_0 = \{(n:t) : \exists \lambda \in \mathbb{R}^{\times} \text{ s.t. } \lambda n \in \pi_0\},$$

where $\pi_0 = \{n(v) : v \in \mathbb{R}^{d-1}\}$. The complement of U_0 is the set of horizontal hyperplanes, those for which the normal vector has the first component equal to zero (see Figure 3.1 for the 2-dimensional case). The set of pairs (v,t) such that $(n(v):t) \notin U_0$ is negligible, so that $\mathcal{R}^{\text{aff}}f$ completely defines $\mathcal{R}f$.

It is easy to find the relation between $\mathcal{R}^{\text{pol}}$ and $\mathcal{R}^{\text{aff}}$. Using the parametrization n of the unit sphere, we can write any vector $n(v)$ as $n(v) = \sqrt{1+|v|^2}\, n(\theta)$. More precisely, there exists ${}^t\theta = (\theta_1, {}^t\hat{\theta}) \in \Theta^{d-1}$ such that

$$(1, {}^tv) = \sqrt{1+|v|^2}(\cos\theta_1, \sin\theta_1 {}^tn(\hat{\theta})). \tag{3.20}$$

Equality (3.20) holds if and only if

$$\cos\theta_1 = \frac{1}{\sqrt{1+|v|^2}}, \qquad \sin\theta_1 n(\hat{\theta}) = \frac{v}{\sqrt{1+|v|^2}}.$$

It follows that

$$\theta_1 = \arccos\Big(\frac{1}{\sqrt{1+|v|^2}}\Big) \in [0, \frac{\pi}{2}), \qquad n(\hat{\theta}) = \frac{v}{|v|},$$

unless $v = 0$, in which case $n(\hat{\theta})$ can be any vector in S^{d-2}. Then, item (i) of Proposition 1.38 gives that

$$\mathcal{R}^{\text{aff}}f(v,t) = \frac{1}{\sqrt{1+|v|^2}}\mathcal{R}^{\text{pol}}f\Big(\theta, \frac{t}{\sqrt{1+|v|^2}}\Big) \tag{3.21}$$

and this is the relation that we need.

The next result is a formulation of the Fourier slice theorem written for the affine Radon transform. The function f to which $\mathcal{R}^{\text{aff}}$ is applied is taken in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. The proof is analogous to the one of Proposition 1.43.

Proposition 3.8 ([8, Proposition 6]). *Define $\psi : \mathbb{R}^{d-1} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^d$ by $\psi(v,\tau) = \tau n(v)$. For every $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ there exists a negligible set $E \subseteq \mathbb{R}^{d-1}$ such that for all $v \notin E$ the function $\mathcal{R}^{\text{aff}}f(v,\cdot)$ is in $L^2(\mathbb{R})$ and satisfies*

$$\mathcal{R}^{\text{aff}}f(v,\cdot) = \mathcal{F}^{-1}[\mathcal{F}f \circ \psi(v,\cdot)]. \tag{3.22}$$

The following Lemma shows that the affine Radon transform intertwines the shearlet representation with the tensor product of two unitary representations up to a positive character of G. The proof is a direct consequence of the behavior of the affine Radon transform under linear actions, see Proposition 1.38.

Lemma 3.9 ([8]). *The affine Radon transform $\mathcal{R}^{\text{aff}}$ satisfies the following intertwining relation between the shearlet representation and the tensor product of two unitary representations, precisely*

$$\mathcal{R}^{\text{aff}}S_{b,s,a}f(v,t) = |a|^{\frac{d-1}{2}}(V_{s,a} \otimes W_{n(v)\cdot b,a})\mathcal{R}^{\text{aff}}f(v,t) \tag{3.23}$$

for every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

Proof. Since for all $(b,s,a) \in G$ it holds $(b,s,a) = (b,0,1)(0,s,1)(0,0,a)$, it is sufficient to prove the equality for each of the three factors. For $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $b \in \mathbb{R}^d$ we have

$$\begin{aligned} \mathcal{R}^{\text{aff}} S_{b,0,1} f(v,t) &= \mathcal{R}^{\text{aff}} T_b f(v,t) \\ &= \mathcal{R} T_b f(n(v),t) \\ &= \mathcal{R} f(n(v), t - n(v)\cdot b) \\ &= \mathcal{R}^{\text{aff}} f(v, t - n(v)\cdot b) \\ &= (\mathrm{I} \otimes W_{n(v)\cdot b,1}) \mathcal{R}^{\text{aff}} f(v,t). \end{aligned} \tag{3.24}$$

For $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $a \in \mathbb{R}^\times$ we have

$$\begin{aligned} \mathcal{R}^{\text{aff}} S_{0,0,a} f(v,t) &= |a|^{\frac{d+\lambda_D}{2}} \mathcal{R}^{\text{aff}} D_{h_{0,a}} f(v,t) \\ &= |a|^{\frac{d+\lambda_D}{2}} \mathcal{R} D_{h_{0,a}} f(n(v),t) \\ &= |a|^{\frac{d+\lambda_D}{2}} \mathcal{R} f({}^t h_{0,a} n(v), t). \end{aligned}$$

A direct calculation gives

$${}^t h_{0,a} n(v) = a \begin{bmatrix} 1 & 0 \\ 0 & \Lambda(a) \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = a \begin{bmatrix} 1 \\ \Lambda(a) v \end{bmatrix} = an\left(\Lambda(a)v\right).$$

The behavior of the Radon transform under linear operations implies that

$$\begin{aligned} \mathcal{R}^{\text{aff}} S_{0,0,a} f(v,t) &= |a|^{\frac{d+\lambda_D}{2}} \mathcal{R} f\left(an\left(\Lambda(a)v\right), a\frac{t}{a} \right) \\ &= |a|^{\frac{d+\lambda_D}{2}-1} \mathcal{R}^{\text{aff}} f\left(\Lambda(a)v, \frac{t}{a} \right) \\ &= |a|^{\frac{d-1}{2}} (V_{0,a} \otimes W_{0,a}) \mathcal{R}^{\text{aff}} f\left(v,t\right). \end{aligned} \tag{3.25}$$

Finally, let $s = {}^t(s_1, \ldots, s_{d-1}) \in \mathbb{R}^{d-1}$. Then

$$\begin{aligned} \mathcal{R}^{\text{aff}} S_{0,s,1} f(v,t) &= \mathcal{R}^{\text{aff}} D_{h_{s,1}} f(v,t) \\ &= \mathcal{R} D_{h_{s,1}} f(n(v),t) = \mathcal{R} f({}^t h_{s,1} n(v), t). \end{aligned}$$

Since

$${}^t h_{s,1} n(v) = \begin{bmatrix} 1 & 0 \\ -s & {}^t B(s) \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ {}^t B(s) v - s \end{bmatrix} = n\left({}^t B(s) v - s \right),$$

by Proposition 1.38 we obtain the following string of equalities:

$$\begin{aligned} \mathcal{R}^{\text{aff}} S_{0,s,1} f(v,t) &= \mathcal{R} f\left(n\left({}^t B(s) v - s \right), t \right) \\ &= \mathcal{R}^{\text{aff}} f\left({}^t B(s) v - s, t \right) \\ &= (V_{s,1} \otimes \mathrm{I}) \mathcal{R}^{\text{aff}} f\left(v,t\right). \end{aligned} \tag{3.26}$$

Therefore, by

$$\mathcal{R}^{\text{aff}} S_{b,s,a} f = \mathcal{R}^{\text{aff}} S_{b,0,1} S_{0,s,1} S_{0,0,a} f,$$

equation (3.23) follows applying the relations obtained above. □

It is possible to extend the affine Radon transform $\mathcal{R}^{\text{aff}}$ to $L^2(\mathbb{R}^d)$ as a unitary map. However, this raises some technical issues, that are addressed in the next section.

3.2.1 The unitary extension

Consider the subspace

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}) : \int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1} |\mathcal{F}f(\xi,\tau)|^2 \, \mathrm{d}\xi\mathrm{d}\tau < +\infty\}$$

of $L^2(\mathbb{R}^{d-1} \times \mathbb{R})$ and define the operator $\mathcal{I} : \mathcal{D} \to L^2(\mathbb{R}^{d-1} \times \mathbb{R})$ by

$$\mathcal{F}\mathcal{I}f(\xi,\tau) = |\tau|^{\frac{d-1}{2}} \mathcal{F}f(\xi,\tau), \tag{3.27}$$

a Fourier multiplier with respect to the last variable. Since $\tau \mapsto |\tau|^{\frac{d-1}{2}}$ is a strictly positive (almost everywhere) Borel function on $\mathbb{R}$, the spectral theorem for unbounded operators, see Theorem VIII.6 of [62], shows that $\mathcal{D}$ is dense and that $\mathcal{I}$ is a positive self-adjoint injective operator.

The operator $\mathcal{I}$ is related to the inverse of the Riesz potential with exponent $(d-1)/2$ on $L^2(\mathbb{R})$. Indeed, if $\psi_2 \in L^2(\mathbb{R}^{d-1})$ and if $\psi_1 \in L^2(\mathbb{R})$ is such that

$$\int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau < +\infty,$$

then $\psi_2 \otimes \psi_1 \in \mathcal{D}$, because

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1} |\mathcal{F}(\psi_2 \otimes \psi_1)(\xi,\tau)|^2 \, \mathrm{d}\xi\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1}} |\mathcal{F}\psi_2(\xi)|^2 \, \mathrm{d}\xi \int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau < +\infty,
\end{aligned}$$

so that

$$\mathcal{I}(\psi_2 \otimes \psi_1) = \psi_2 \otimes \mathcal{I}_0\psi_1,$$

where $\mathcal{I}_0$ is the inverse of the standard Riesz potential defined by

$$\mathcal{F}\mathcal{I}_0\psi_1(\tau) = |\tau|^{\frac{d-1}{2}} \mathcal{F}\psi_1(\tau) \tag{3.28}$$

on the domain

$$\operatorname{dom} \mathcal{I}_0 = \{\varphi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\varphi(\tau)|^2 \mathrm{d}\tau < +\infty\}.$$

Furthermore, $\mathcal{D}$ is invariant under translations and dilations by matrices of the form

$$A = \begin{bmatrix} A_0 & 0 \\ v & a \end{bmatrix}, \tag{3.29}$$

where $A_0 \in \mathrm{GL}(d-1,\mathbb{R}),\ v \in \mathbb{R}^{d-1},\ a \in \mathbb{R}^\times$.

Lemma 3.10 ([8, Lemma 7])**.** *For all $b \in \mathbb{R}^d$ and A as in* (3.29) *it holds*

$$\mathcal{I}T_b = T_b\mathcal{I}, \qquad \mathcal{I}D_A = |a|^{-\frac{d-1}{2}} D_A\mathcal{I}. \tag{3.30}$$

Proof. The first of relations (3.30) is a consequence of the fact that $\mathcal{F}T_b f(\xi,\tau) = e^{-2\pi i b\cdot\xi}\mathcal{F}f(\xi,\tau)$ for all $f \in L^2(\mathbb{R}^{d-1}\times\mathbb{R})$. Precisely, for all $f \in \mathcal{D}$ we have that

$$\begin{aligned}
\mathcal{F}\mathcal{I}T_b f(\xi,\tau) &= |\tau|^{\frac{d-1}{2}}\mathcal{F}T_b f(\xi,\tau)\\
&= |\tau|^{\frac{d-1}{2}}e^{-2\pi i b\cdot\xi}\mathcal{F}f(\xi,\tau)\\
&= e^{-2\pi i b\cdot\xi}\mathcal{F}\mathcal{I}f(\xi,\tau)\\
&= \mathcal{F}T_b\mathcal{I}f(\xi,\tau),
\end{aligned}$$

whence $\mathcal{I}T_b = T_b\mathcal{I}$. The second follows from $\mathcal{F}D_A f(\xi,\tau) = \mathcal{F}f({}^tA(\xi,\tau))$. Indeed, for all $f \in \mathcal{D}$

$$\begin{aligned}
\mathcal{F}\mathcal{I}D_A f(\xi,\tau) &= |\tau|^{\frac{d-1}{2}}\mathcal{F}D_A f(\xi,\tau)\\
&= |\tau|^{\frac{d-1}{2}}\mathcal{F}f({}^tA(\xi,\tau))\\
&= |\tau|^{\frac{d-1}{2}}\mathcal{F}f({}^tA_0\xi+\tau v, a\tau)\\
&= |\tau|^{\frac{d-1}{2}}|a\tau|^{-\frac{d-1}{2}}\mathcal{F}\mathcal{I}f({}^tA_0\xi+\tau v, a\tau)\\
&= |a|^{-\frac{d-1}{2}}\mathcal{F}D_A\mathcal{I}f(\xi,\tau).
\end{aligned}$$

This proves (3.30). □

The space $\mathcal{D}$ becomes a pre-Hilbert space with respect to the scalar product

$$\begin{aligned}
\langle f,g\rangle_{\mathcal{D}} &= \langle \mathcal{I}f,\mathcal{I}g\rangle_2\\
&= \int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1}\mathcal{F}(f(v,\cdot))(\tau)\overline{\mathcal{F}(g(v,\cdot))(\tau)}\,\mathrm{d}v\mathrm{d}\tau.
\end{aligned} \tag{3.31}$$

Furthermore,

$$||f||^2_{\mathcal{D}} = \langle f,f\rangle_{\mathcal{D}} = \langle \mathcal{I}f,\mathcal{I}f\rangle_2 = ||\mathcal{I}f||_2^2,$$

for all $f \in \mathcal{D}$. Hence $\mathcal{I}$ is an isometric operator from $\mathcal{D}$, with the new scalar product (3.31), to $L^2(\mathbb{R}^{d-1}\times\mathbb{R})$. Since $\mathcal{I}$ is self-adjoint and injective, $\mathrm{Ran}(\mathcal{I})$ is dense in $L^2(\mathbb{R}^{d-1}\times\mathbb{R})$. Hence, by standard arguments, it extends uniquely to a unitary operator, denoted $\mathscr{I}$, from the completion $\mathcal{H}$ of $\mathcal{D}$ onto $L^2(\mathbb{R}^{d-1}\times\mathbb{R})$.

To extend $\mathcal{R}^{\mathrm{aff}}$ to $L^2(\mathbb{R}^d)$ as a unitary operator, note that, by Proposition 3.8, the affine Radon transform of $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ belongs to $L^2(\mathbb{R}^{d-1}\times\mathbb{R})$ if and only if is finite the integral

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\mathcal{R}^{\mathrm{aff}}f(v,t)|^2\mathrm{d}v\mathrm{d}t &= \int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} |\mathcal{F}(\mathcal{R}^{\mathrm{aff}}f(v,\cdot))(\tau)|^2\mathrm{d}\tau\mathrm{d}v\\
&= \int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\mathcal{F}f(\tau,\tau v)|^2\mathrm{d}\tau\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\frac{|\mathcal{F}f(\xi)|^2}{|\xi_1|^{d-1}}\mathrm{d}\xi,
\end{aligned}$$

where ξ_1 is the first component of the vector $\xi \in \mathbb{R}^d$. Therefore requiring that $\mathcal{R}^{\mathrm{aff}}f$ belongs to $L^2(\mathbb{R}^{d-1}\times\mathbb{R})$ is equivalent to

$$\int_{\mathbb{R}^d}\frac{|\mathcal{F}f(\xi)|^2}{|\xi_1|^{d-1}}\mathrm{d}\xi < +\infty.$$

We denote by

$$\mathcal{A} = \{f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{|\mathcal{F}f(\xi)|^2}{|\xi_1|^{d-1}} \mathrm{d}\xi < +\infty\},$$

which is dense in $L^2(\mathbb{R}^d)$ since it contains the functions whose Fourier transform is smooth and has compact support disjoint from the hyperplane $\xi_1 = 0$. By definition of $\mathcal{A}$, $\mathcal{R}^{\mathrm{aff}} f \in L^2(\mathbb{R}^{d-1} \times \mathbb{R})$ for all $f \in \mathcal{A}$.

We need a suitable formulation of the main result in Radon transform theory, namely the following version of Theorem 4.1 in [38]. For the sake of completeness and the reader's convenience we include the proof which is an adaptation of the proof of Theorem 4.1 in [38] to our context.

Theorem 3.11 ([8, Theorem 8]). *The affine Radon transform extends to a unique unitary operator from $L^2(\mathbb{R}^d)$ onto $\mathcal{H}$, denoted with $\mathscr{R}$ and, hence, $\mathcal{Q} = \mathscr{I}\mathscr{R}$ is a unitary operator from $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^{d-1} \times \mathbb{R})$.*

Proof. The map $\psi : \mathbb{R}^{d-1} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^d$, defined by $\psi(v, \tau) = \tau n(v)$, is a diffeomorphism onto the open set $V = \{\xi \in \mathbb{R}^d : \xi_1 \neq 0\}$ with Jacobian $J\Phi(v, \tau) = \tau^{d-1}$. Thus, the Plancherel theorem and the Fourier slice theorem give that for any $f \in \mathcal{A}$

$$\begin{aligned}
||f||_2^2 &= \int_V |\mathcal{F}f(\xi)|^2 \ \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{d-1} \times (\mathbb{R} \setminus \{0\})} |\mathcal{F}f(\tau n(v))|^2 |\tau|^{d-1} \ \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1} \times \mathbb{R}} |\mathcal{F}(\mathcal{R}^{\mathrm{aff}} f(v, \cdot))(\tau)|^2 |\tau|^{d-1} \ \mathrm{d}v\mathrm{d}\tau \\
&= ||\mathcal{R}^{\mathrm{aff}} f||_{\mathcal{D}}^2.
\end{aligned}$$

Thus, $\mathcal{R}^{\mathrm{aff}} f$ belongs to $\mathcal{D}$ for all $f \in \mathcal{A}$ and $\mathcal{R}^{\mathrm{aff}}$ is an isometric operator from $\mathcal{A}$ into $\mathcal{D}$. We want to prove that $\mathcal{R}^{\mathrm{aff}} : \mathcal{A} \to \mathcal{H}$ has dense image in $\mathcal{H}$. Since $\mathcal{H}$ is the completion of $\mathcal{D}$, it is enough to prove that $\mathcal{R}^{\mathrm{aff}}$ has dense image in $\mathcal{D}$, that is $(\mathrm{Ran}(\mathcal{R}^{\mathrm{aff}}))^\perp = \{0\}$ in $\mathcal{D}$. Take then $\varphi \in \mathcal{D}$ such that $\langle \varphi, \mathcal{R}^{\mathrm{aff}} f \rangle_{\mathcal{D}} = 0$ for all $f \in \mathcal{A}$. By the definition of the scalar product on $\mathcal{D}$ and the Fourier slice theorem we have that

$$\begin{aligned}
\langle \varphi, \mathcal{R}^{\mathrm{aff}} f \rangle_{\mathcal{D}} &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}} |\tau|^{d-1} \mathcal{F}(\varphi(v, \cdot))(\tau) \overline{\mathcal{F}(\mathcal{R}^{\mathrm{aff}} f(v, \cdot))(\tau)} \ \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1} \times \mathbb{R}} |\tau|^{d-1} \mathcal{F}(\varphi(v, \cdot))(\tau) \overline{\mathcal{F}f(\tau n(v))} \ \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^d} \mathcal{F}\Big(\varphi(\frac{\tilde{\xi}}{\xi_1}, \cdot)\Big)(\xi_1) \overline{\mathcal{F}f(\xi)} \mathrm{d}\xi,
\end{aligned}$$

where ${}^t\xi = (\xi_1, {}^t\tilde{\xi})$. Therefore, if $\langle \varphi, \mathcal{R}^{\mathrm{aff}} f \rangle_{\mathcal{D}} = 0$ for all $f \in \mathcal{A}$, then

$$\mathcal{F}\Big(\varphi(\frac{\tilde{\xi}}{\xi_1}, \cdot)\Big)(\xi_1) = 0$$

almost everywhere. However,

$$\|\varphi\|_{\mathcal{D}} = \int_{\mathbb{R}^{d-1} \times \mathbb{R}} |\tau|^{d-1} |\mathcal{F}(\varphi(v, \cdot))(\tau)|^2 \ \mathrm{d}v\mathrm{d}\tau = \int_{\mathbb{R}^d} |\mathcal{F}\Big(\varphi(\frac{\tilde{\xi}}{\xi_1}, \cdot)\Big)(\xi_1)|^2 \ \mathrm{d}\xi$$

and hence $\varphi = 0$ in $\mathcal{D}$. Therefore $\mathcal{R}^{\mathrm{aff}} : \mathcal{A} \to \mathcal{H}$ has dense image in $\mathcal{H}$ and we can extend it to a unique unitary operator $\mathscr{R}$ from $L^2(\mathbb{R}^d)$ onto $\mathcal{H}$. Hence, $\mathcal{Q} = \mathscr{I}\mathscr{R}$ is a unitary operator from $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^{d-1} \times \mathbb{R})$. □

We need yet another generalization of the Fourier slice theorem (Proposition 3.8). The proof is analogous to the one of Proposition 1.46 and can be found in [8].

Proposition 3.12 ([8, Proposition 9])**.** *For all* $f \in L^2(\mathbb{R}^d)$

$$\mathcal{F}(\mathcal{Q}f(v,\cdot))(\tau) = |\tau|^{\frac{d-1}{2}} \mathcal{F}f(\tau n(v)) \tag{3.32}$$

for almost every $(v, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

If $f \in \mathcal{A}$, (3.32) is an easy consequence of Proposition 3.12 and the definition of $\mathcal{I}$, and this is known (see [16], Section 3.2 and [57]). For arbitrary $f \in L^2(\mathbb{R}^d)$ the proof is not trivial because $\mathcal{Q}$ cannot be written as $\mathcal{I}\mathcal{R}^{\mathrm{aff}}$, and is based on the fact that $\mathcal{I}$ is a Fourier multiplier. The proof is analogous to the one of Proposition 1.46. However, it is worth observing that if we consider $\mathcal{Q}f(v,\cdot)$ as a function of the only second variable it always factorizes for every $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Proposition 3.13 ([8])**.** *For any* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*, for almost every* $v \in \mathbb{R}^{d-1}$ *the function* $\mathcal{R}^{\mathrm{aff}} f(v,\cdot)$ *is in* $\operatorname{dom} \mathcal{I}_0$ *and satisfies*

$$\mathcal{Q}f(v,\cdot) = \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}}(v,\cdot). \tag{3.33}$$

Proof. If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, Proposition 3.8 and Proposition 3.12 imply that for almost all $v \in \mathbb{R}^{d-1}$, $\mathcal{R}^{\mathrm{aff}} f(v,\cdot)$ is in $L^2(\mathbb{R})$ and

$$\mathcal{F}(\mathcal{Q}f(v,\cdot)) = |\cdot|^{\frac{d-1}{2}} \mathcal{F}\mathcal{R}^{\mathrm{aff}} f(v,\cdot).$$

Since by Fubini's theorem $\mathcal{F}(\mathcal{Q}f(v,\cdot)) \in L^2(\mathbb{R})$ for almost all $v \in \mathbb{R}^{d-1}$, the above equality implies that, for almost every $v \in \mathbb{R}^{d-1}$, $\mathcal{R}^{\mathrm{aff}} f(v,\cdot)$ is in the domain of $\mathcal{I}_0$ and, by definition of $\mathcal{I}_0$,

$$\mathcal{Q}f(v,\cdot) = \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v,\cdot).$$

□

Proposition 3.13 has a key role in several proofs in the following.

3.3 The Intertwining Theorem and its consequences

3.3.1 The main Theorem

We recall that the group G is the semidirect product $G = \mathbb{R}^d \rtimes H$ where $H = SD$ is the shearlet dilation group, S is the shearing subgroup and D the scaling subgroup of H, as in Definition 3.1. Each element in G is parametrized by a triple $(b, s, a) \in \mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^\times$ and $S_{b,s,a}$ is as in (3.11).

Theorem 3.14. *The unitary operator* $\mathcal{Q}$ *intertwines the shearlet representation with the tensor product of two unitary representations, precisely*

$$\mathcal{Q}S_{b,s,a}f(v,t) = (V_{s,a} \otimes W_{n(v)\cdot b,a})\mathcal{Q}f(v,t) \tag{3.34}$$

for every $f \in L^2(\mathbb{R}^d)$.

Proof. By density, it is enough to prove the equality on $\mathcal{A}$. We shall use throughout the fact that $\mathcal{R}^{\mathrm{aff}} f \in \mathcal{D}$ for every $f \in \mathcal{A}$ and that $\mathcal{D}$ is invariant under all translations and under the dilations described in (3.29). Since for all $(b,s,a) \in G$ it holds $(b,s,a) = (b,0,1)(0,s,1)(0,0,a)$, it is sufficient to prove the equality for each of the three factors. By (3.24), for $f \in \mathcal{A}$ and $b \in \mathbb{R}^d$ we have

$$\mathcal{R}^{\mathrm{aff}} S_{b,0,1} f(v,t) = (\mathrm{I} \otimes W_{n(v)\cdot b,1}) \mathcal{R}^{\mathrm{aff}} f(v,t).$$

Since $\mathcal{I}$ commutes with translations, $\mathrm{I} \otimes W_{n(v)\cdot b,1} = T_{(0,n(v)\cdot b)}$ implies

$$\mathcal{I}\mathcal{R}^{\mathrm{aff}} S_{b,0,1} f(v,t) = \mathcal{I}(\mathrm{I} \otimes W_{n(v)\cdot b,1}) \mathcal{R}^{\mathrm{aff}} f(v,t) = (\mathrm{I} \otimes W_{n(v)\cdot b,1}) \mathcal{I}\mathcal{R}^{\mathrm{aff}} f(v,t).$$

For $f \in \mathcal{A}$ and $a \in \mathbb{R}^\times$ equation (3.25) reads

$$\mathcal{R}^{\mathrm{aff}} S_{0,0,a} f(v,t) = |a|^{\frac{d-1}{2}} (V_{0,a} \otimes W_{0,a}) \mathcal{R}^{\mathrm{aff}} f\,(v,t)\,.$$

Since

$$(V_{0,a} \otimes W_{0,a}) = |a|^{\frac{3(1-\lambda_D)}{2}} D_A,$$

where the matrix A is of the form

$$A = \begin{bmatrix} \Lambda(a)^{-1} & 0 \\ 0 & a \end{bmatrix},$$

and because of the behavior of the operator $\mathcal{I}$ under dilations, we obtain

$$\begin{aligned} \mathcal{I}\mathcal{R}^{\mathrm{aff}} S_{0,0,a} f(v,t) &= \mathcal{I}|a|^{\frac{d-1}{2}} (V_{0,a} \otimes W_{0,a}) \mathcal{R}^{\mathrm{aff}} f(v,t) \\ &= (V_{0,a} \otimes W_{0,a}) \mathcal{I}\mathcal{R}^{\mathrm{aff}} f(v,t). \end{aligned}$$

Finally, let $f \in \mathcal{A}$ and $s = {}^t(s_1, \ldots, s_{d-1}) \in \mathbb{R}^{d-1}$. By (3.26) we have that

$$\mathcal{R}^{\mathrm{aff}} S_{0,s,1} f(v,t) = (V_{s,1} \otimes \mathrm{I}) \mathcal{R}^{\mathrm{aff}} f\,(v,t)$$

and since

$$(V_{s,1} \otimes \mathrm{I}) = |a|^{\frac{3(1-\lambda_D)}{2}} T_{(-({}^tB(s))^{-1}s,0)} D_A,$$

where

$$A = \begin{bmatrix} {}^tB(s)^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

the behavior of $\mathcal{I}$ under dilations implies

$$\begin{aligned} \mathcal{I}\mathcal{R}^{\mathrm{aff}} S_{0,s,1} f(v,t) &= \mathcal{I}(V_{s,1} \otimes \mathrm{I}) \mathcal{R}^{\mathrm{aff}} f(v,t) \\ &= (V_{s,1} \otimes \mathrm{I}) \mathcal{I}\mathcal{R}^{\mathrm{aff}} f(v,t). \end{aligned}$$

Therefore, by

$$\mathcal{I}\mathcal{R}^{\mathrm{aff}} S_{b,s,a} f = \mathcal{I}\mathcal{R}^{\mathrm{aff}} S_{b,0,1} S_{0,s,1} S_{0,0,a} f,$$

equation (3.34) follows straightforwardly. □

It is worth observing that the proof of Theorem 3.14 follows by the fact that the operator $\mathcal{I}$ satisfies the intertwining property

$$\mathcal{I}(V_{s,a} \otimes W_{n(v)\cdot b,a})f(v,t) = |a|^{-\frac{d-1}{2}}(V_{s,a} \otimes W_{n(v)\cdot b,a})\mathcal{I}f(v,t), \tag{3.35}$$

for every $f \in \mathcal{D}$. Indeed, the weight $|a|^{-\frac{d-1}{2}}$ deletes the weight $|a|^{\frac{d-1}{2}}$ which pops up in the intertwining lemma for the affine Radon transform (see Lemma 3.9) and then (3.34) follows since $\mathcal{Q} = \mathcal{I}\mathcal{R}^{\text{aff}}$ on $\mathcal{A}$.

Hence, Theorem 3.14 follows directly by the intertwining properties of the operators $\mathcal{R}^{\text{aff}}$ and $\mathcal{I}$ stated in Lemma 3.9 and 3.10, respectively. Moreover, the irreducibility of the shearlet representation and the closability of the affine Radon transform do not come into play. For this reason, this approach could be an alternative method to obtain unitarization and intertwining results when the hypothesis of irreducibility does not hold. The isometric extension problem for the Radon transform in the general context of symmetric spaces, see [39, Corollary 3.11], is a clear example. We will show in Section 3.4 that the family of examples treated in this Chapter satisfies all the assumptions (A1)-(A6) and consequently Theorem 3.14 can also be obtained as an application of Theorem 2.11.

3.3.2 The admissibility conditions

In this subsection we discuss the admissibility conditions and some of their consequences.

Our objective is to obtain an expression for the shearlet transform that makes use of formula (3.34). To this end, we start by looking for natural conditions that guarantee that $\psi \in L^2(\mathbb{R}^d)$ is an admissible vector for the shearlet representation S, namely that it satisfies (3.12).

Equation (3.34) suggests that a good choice for the admissible vector ψ is of the form

$$\mathcal{Q}\psi = \phi_2 \otimes \phi_1$$

where $\phi_1 \in L^2(\mathbb{R})$, $\phi_2 \in L^2(\mathbb{R}^{d-1})$. If this is the case, then by (3.32) it follows that

$$\phi_2(v)\mathcal{F}\phi_1(\tau) = |\tau|^{\frac{d-1}{2}}\mathcal{F}\psi(\tau n(v))$$

so that $\mathcal{F}\psi$ factorizes as

$$\mathcal{F}\psi(\tau, \tau v) = \mathcal{F}\psi_1(\tau)\mathcal{F}\psi_2(v), \tag{3.36}$$

where we assume that $\psi_2 \in L^2(\mathbb{R}^{d-1})$ and $\psi_1 \in L^2(\mathbb{R})$. Equation (3.36) is the canonical choice of admissible vectors given by (3.14). Furthermore,

$$\mathcal{F}\phi_1(\tau) = |\tau|^{\frac{d-1}{2}}\mathcal{F}\psi_1(\tau), \qquad \phi_2(v) = \mathcal{F}\psi_2(v),$$

so that the assumption that $\psi_2 \in L^2(\mathbb{R}^{d-1})$ is automatically satisfied. Since $\phi_1 \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} |\tau|^{d-1}|\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau < +\infty.$$

This, together with the fact that $\psi_1 \in L^2(\mathbb{R})$, implies that ψ_1 belongs to the domain of the differential operator $\mathcal{I}_0$ (see (3.28)). Therefore

$$\phi_1 = \mathcal{I}_0\psi_1. \tag{3.37}$$

With the choice (3.36) the admissibility condition (3.12) reduces to

$$0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi_1(\tau)|^2}{|\tau|} \mathrm{d}\tau < +\infty.$$

From now on we fix $\psi \in L^2(\mathbb{R}^d)$ of the form (3.36) with $\psi_1 \in L^2(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau < +\infty, \qquad 0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi_1(\tau)|^2}{|\tau|} \mathrm{d}\tau < +\infty, \tag{3.38}$$

and $\psi_2 \in L^2(\mathbb{R}^{d-1})$.

Corollary 3.15. *Under the assumptions* (3.38)*, for every* $L^2(\mathbb{R}^d)$

$$\mathcal{S}_\psi f(b,s,a) = \mathcal{V}_{\phi_2} \left(\mathcal{W}_{\phi_1}(\mathcal{Q}f(v,t))(n(v) \cdot b, a) \right)(s,a) \tag{3.39}$$

Proof. For all $f \in L^2(\mathbb{R}^d)$ and $(b,s,a) \in G$

$$\begin{aligned}
\mathcal{S}_\psi f(b,s,a) &= \langle f, S_{b,s,a}\psi \rangle_2 \\
&= \langle \mathcal{Q}f, \mathcal{Q}S_{b,s,a}\psi \rangle_2 \\
&= \langle \mathcal{Q}f, (V_{s,a} \otimes W_{n(\cdot)\cdot b,a})\mathcal{Q}\psi \rangle_2 \\
&= \langle \mathcal{Q}f, (V_{s,a} \otimes W_{n(\cdot)\cdot b,a})(\phi_2 \otimes \phi_1) \rangle_2 \\
&= \langle \mathcal{Q}f, V_{s,a}\phi_2 \otimes W_{n(\cdot)\cdot b,a}\phi_1 \rangle_2 \\
&= \int_{\mathbb{R}^{d-1}\times\mathbb{R}} \mathcal{Q}f(v,\tau) \overline{V_{s,a}\phi_2(v) W_{n(v)\cdot b,a}\phi_1(\tau)} \, \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \mathcal{Q}f(v,\tau) \overline{W_{n(v)\cdot b,a}\phi_1(\tau)} \, \mathrm{d}\tau \right) \overline{V_{s,a}\phi_2(v)} \, \mathrm{d}v \\
&= \int_{\mathbb{R}^{d-1}} \mathcal{W}_{\phi_1}(\mathcal{Q}f(v,\cdot))(n(v)\cdot b, a) \overline{V_{s,a}\phi_2(v)} \, \mathrm{d}v,
\end{aligned} \tag{3.40}$$

where in the last equality we have used the fact that ϕ_1 is an admissible wavelet. This is true because by (3.37)

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|\mathcal{F}\phi_1(\tau)|^2}{|\tau|} \, \mathrm{d}\tau &= \int_{\mathbb{R}} \frac{|\mathcal{F}\mathcal{I}_0\psi_1(\tau)|^2}{|\tau|} \, \mathrm{d}\tau \\
&\leq \int_{0<|\tau|<1} \frac{|\mathcal{F}\psi_1(\tau)|^2}{|\tau|} \, \mathrm{d}\tau + \int_{|\tau|\geq 1} |\tau|^{d-1} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau \\
&\leq \int_{\mathbb{R}} \frac{|\mathcal{F}\psi_1(\tau)|^2}{|\tau|} \, \mathrm{d}\tau + \int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau,
\end{aligned}$$

which are both finite. □

Equation (3.40) shows that the shearlet coefficients $\mathcal{S}_\psi f(b,s,a)$ can be computed in terms of the unitary Radon transform $\mathcal{Q}f$, which involves the pseudo-differential operator $\mathcal{I}$ and it is difficult to compute numerically. However, if $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, there is yet a different way to express the shearlet transform. To this end we need to choose ψ in such a way that $\mathcal{Q}\psi$ is in the domain of the operator $\mathcal{I}$, that is, in such a way that

$$\int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\mathcal{Q}\psi(v,\tau)|^2 \, \mathrm{d}v\mathrm{d}\tau < +\infty.$$

Assuming this and recalling that $\mathcal{Q}\psi = \mathcal{F}\psi_2 \otimes \mathcal{I}_0\psi_1$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\mathcal{Q}\psi(v,\tau)|^2 \, \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1}\times\mathbb{R}} |\tau|^{d-1} |\mathcal{F}(\mathcal{F}\psi_2 \otimes \mathcal{I}_0\psi_1)(v,\tau)|^2 \, \mathrm{d}v\mathrm{d}\tau \\
&= \int_{\mathbb{R}^{d-1}} |\mathcal{F}\psi_2(v)|^2 \, \mathrm{d}v \int_{\mathbb{R}} |\tau|^{d-1} |\mathcal{F}\mathcal{I}_0\psi_1(\tau)|^2 \, \mathrm{d}\tau \\
&= ||\psi_2||_2^2 \int_{\mathbb{R}} |\tau|^{2(d-1)} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau.
\end{aligned}$$

This shows that $\mathcal{Q}\psi$ is in the domain of $\mathcal{I}$ if and only if ψ_1 satisfies the additional condition

$$\int_{\mathbb{R}} |\tau|^{2(d-1)} |\mathcal{F}\psi_1(\tau)|^2 \, \mathrm{d}\tau < +\infty. \tag{3.41}$$

In this case, by (3.37)

$$\mathcal{I}Q\psi = \mathcal{I}(\phi_2 \otimes \phi_1) = \phi_2 \otimes \mathcal{I}_0\phi_1.$$

Corollary 3.16. *Under the assumptions* (3.38) *and* (3.41)*,*

$$\mathcal{S}_\psi f(b,s,a) = |a|^{-\frac{d-1}{2}} \mathcal{V}_{\phi_2}\left(\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,t))(n(v)\cdot b, a)\right)(s,a).$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Proof. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $(b,s,a) \in G$

$$\begin{aligned}
\mathcal{S}_\psi f(b,s,a) &= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \mathcal{Q}f(v,\tau) \overline{W_{n(v)\cdot b,a}\phi_1(\tau)} \, \mathrm{d}\tau \right) \overline{V_{s,a}\phi_2(v)} \, \mathrm{d}v \\
&= \int_{\mathbb{R}^{d-1}} \langle \mathcal{Q}f(v,\cdot), W_{n(v)\cdot b,a}\phi_1 \rangle_2 \overline{V_{s,a}\phi_2(v)} \, \mathrm{d}v.
\end{aligned} \tag{3.42}$$

Since $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, Proposition 3.13 implies that, for almost all $v \in \mathbb{R}^{d-1}$, $\mathcal{R}^{\mathrm{aff}} f(v,\cdot)$ is in the domain of $\mathcal{I}_0$ and

$$\mathcal{Q}f(v,\cdot)(\tau) = \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v,\cdot).$$

By assumption ϕ_1 is in the domain of $\mathcal{I}_0$ and the same property holds true for $W_{n(v)\cdot b,a}\phi_1$. Since $\mathcal{I}_0$ is self-adjoint, we get

$$\begin{aligned}
\langle \mathcal{Q}f(v,\cdot), W_{n(v)\cdot b,a}\phi_1 \rangle_2 &= \langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), \mathcal{I}_0 W_{n(v)\cdot b,a}\phi_1 \rangle_2 \\
&= |a|^{-\frac{d-1}{2}} \langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), W_{n(v)\cdot b,a} \mathcal{I}_0\phi_1 \rangle_2,
\end{aligned}$$

by taking into account that

$$\mathcal{I}_0 W_{n(\cdot)\cdot b,a} = |a|^{-\frac{d-1}{2}} W_{n(\cdot)\cdot b,a} \mathcal{I}_0.$$

Setting $\chi_1 = \mathcal{I}_0\phi_1 = \mathcal{I}_0^2\psi_1$, i.e.

$$\mathcal{F}\chi_1(\tau) = |\tau| \mathcal{F}\psi_1(\tau), \tag{3.43}$$

from (3.42) we finally get

$$\mathcal{S}_\psi f(b,s,a) = \tag{3.44}$$
$$|a|^{\frac{\lambda_D+1-d}{2}} \int_{\mathbb{R}^{d-1}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\Lambda(a)({}^tB(s)v-s)\right)}\,\mathrm{d}v$$

Observe that we have used the fact that χ_1 is an admissible wavelet, too, the proof is analogous to the proof that ψ_1 is such. We can rewrite the above formula by using the voice transform of H as follows

$$\mathcal{S}_\psi f(b,s,a) = |a|^{-\frac{d-1}{2}} \mathcal{V}_{\phi_2}\left(\mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,t))(n(v)\cdot b,a)\right)(s,a).$$

□

Notice that formula (3.44) cannot be obtained using the very same manipulations as those used in (3.40) because the factorization $\mathcal{Q}f = \mathcal{I}\mathcal{R}^{\text{aff}} f$ does not hold for all $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, but only if $f \in \mathcal{A}$.

Observe that formulas (3.40) and (3.44) can be also written in terms of the polar Radon transform using relation (3.21).

Equation (3.44) shows that for any signal $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ the shearlet coefficients can be computed by means of three classical transforms: first compute the affine Radon transform $\mathcal{R}^{\text{aff}} f$, then apply the wavelet transform to the last variable

$$G(v,b,a) = \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(n(v)\cdot b,a),$$

where χ_1 is given by (3.43), and, finally, "mock-convolve" with respect to the variable v

$$\mathcal{S}_\psi f(b,s,a) = \int_{\mathbb{R}^{d-1}} G(v,b,a)\Phi_a(s-{}^tB(s)v)\,\mathrm{d}v$$

with the scale-dependent filter

$$\Phi_a(v) = \overline{\phi_2\left(-\Lambda(a)v\right)}.$$

Note that the "mok-convolution" reduces to the standard convolution in $\mathbb{R}^{d-1}$ when $B(t) = \mathrm{I}_{d-1}$.

Notice that the shearlet coefficients $\mathcal{S}_\psi f(b,s,a)$ depend on f only through its affine Radon transform $\mathcal{R}^{\text{aff}} f$. Therefore Equation (3.16) allows to reconstruct any unknown signal $f \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ from its Radon transform by computing the shearlet coefficients by means of (3.44). Finally, it is worth observing that this reconstruction does not involve the differential operator $\mathcal{I}$ as applied to the signal. Hence, another interesting aspect of our result is that it could open the way to new methods for inverting the Radon transform, a very important issue in applications. Indeed, this result leads to an inversion formula for the Radon transform based on the shearlet and the wavelet transforms. A discrete version of our reconstruction formula is presented in [16] for $d = 2$.

Example 3.17 (The standard shearlet group, continued)**.** For the classical shearlet group $\mathbb{S}^\gamma$, (3.34) becomes

$$\mathcal{Q}S^\gamma_{b,s,a} f(v,t) = (V_{s,a}\otimes W_{n(v)\cdot b,a})\mathcal{Q}f(v,t), \tag{3.45}$$

where $S^{\gamma}_{b,s,a}$ is given by (3.17) and $V_{s,a}$ is the wavelet representation in dimension $d-1$ as in (3.18). Therefore in the case of the standard shearlet group Theorem 3.14 shows that the unitary operator $\mathcal{Q}$ intertwines the shearlet representation S^{γ} with the tensor product of two wavelet representations.

For a fixed admissible vector $\psi \in L^2(\mathbb{R}^d)$ of the form (3.36) with $\psi_1 \in L^2(\mathbb{R})$ satisfying (3.38) and $\psi_2 \in L^2(\mathbb{R}^{d-1})$, equation (3.40) becomes

$$\begin{aligned} &\mathcal{S}^{\gamma}_{\psi} f(b,s,a) \\ &= |a|^{\frac{(d-1)(\gamma-1)}{2}} \int_{\mathbb{R}^{d-1}} \mathcal{W}_{\phi_1}(\mathcal{Q}f(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v, \end{aligned} \tag{3.46}$$

for any $f \in L^2(\mathbb{R}^d)$ and $(b,s,a) \in G$. Assuming that ψ_1 satisfies the additional condition (3.41), for any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $(b,s,a) \in G$, equality (3.44) becomes

$$\begin{aligned} &\mathcal{S}^{\gamma}_{\psi} f(b,s,a) \\ &= |a|^{\frac{(d-1)(\gamma-2)}{2}} \int_{\mathbb{R}^{d-1}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v, \end{aligned} \tag{3.47}$$

where χ_1 is the admissible vector defined by (3.43).

For the sake of clarity we write the above equation for $d=2$ and in terms of the Radon transform in polar coordinates

$$\begin{aligned} &\mathcal{S}^{\gamma}_{\psi} f(x,y,s,a) \\ &= |a|^{\frac{\gamma-2}{2}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}\left(\mathcal{R}^{\mathrm{pol}} f(\arctan v, \frac{\cdot}{\sqrt{1+v^2}})\right)(x+vy,a)\,\overline{\phi_2\Big(\frac{v-s}{|a|^{1-\gamma}}\Big)}\,\frac{\mathrm{d}v}{\sqrt{1+v^2}}, \end{aligned}$$

where $x,y,s \in \mathbb{R}$, $a \in \mathbb{R}^{\times}$ and $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

3.4 The Radon Transform between Dual Pairs

In this section we show how the results stated in the previous sections perfectly fit in the general theory presented in Chapter 2. For simplicity, we show the all construction with the example where G is the standard shearlet group with $d=2$ and $\gamma=1/2$, see Example 3.2. The same construction can be adapted to any semidirect product $G=\mathbb{R}^d \rtimes K$, where K is a shearlet dilation group as in Definition 3.1. For clarity and to fix the notation, we recall the definition and the basic properties of the standard shearlet group in dimension two. For what concerns dual homogeneous spaces, we keep the notation as in Chapters 1 and 2.

Groups and spaces

We recall that the (parabolic) shearlet group $\mathbb{S}$, where the word "parabolic" refers to the choice $\gamma=1/2$, is the semidirect product of $\mathbb{R}^2$ with the closed subgroup $K=\{S_sA_a \in \mathrm{GL}(2,\mathbb{R}) : s \in \mathbb{R}, a \in \mathbb{R}^{\times}\}$ where

$$S_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}, \qquad A_a = a\begin{bmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{bmatrix}.$$

We can identify the element $S_s A_a$ with the pair (s, a) and we write (b, s, a) for the elements in $\mathbb{S}$. With this identification, by (2.5) and (3.7) the product law amounts to

$$(b, s, a)(b', s', a') = (b + S_s A_a b', s + |a|^{1/2} s', aa')$$

and the inverse of (b, s, a) is given by

$$(b, s, a)^{-1} = (-A_a^{-1} S_s^{-1} b, -|a|^{-1/2} s, a^{-1}).$$

By (3.10), a left Haar measure of $\mathbb{S}$ is

$$\mathrm{d}\mu(b, s, a) = |a|^{-3} \mathrm{d}b \mathrm{d}s \mathrm{d}a,$$

with $\mathrm{d}b$, $\mathrm{d}s$ and $\mathrm{d}a$ the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^\times$, respectively. We need to show that assumptions (A1)-(A6) are satisfied. We already know that the group $\mathbb{S}$ acts transitively on $X = \mathbb{R}^2$ by

$$(b, s, a)[x] = S_s A_a x + b$$

and by (2.8), we have $\alpha(b, s, a) = |a|^{3/2}$.

Furthermore, the shearlet group acts transitively on $\Xi = \mathbb{R} \times \mathbb{R}$ by the action

$$(b, s, a)^{-1}.(v, t) = \left(|a|^{-1/2}(v - s), \frac{t - n(v) \cdot b}{a} \right),$$

where $n(v) = {}^t(1, v)$. The isotropy at $\xi_0 = (0, 0)$ is

$$H = \{((0, b_2), 0, a) : b_2 \in \mathbb{R}, a \in \mathbb{R}^\times\},$$

so that $\Xi = \mathbb{S}/H$. It is immediate to verify that the Lebesgue measure $\mathrm{d}\xi = \mathrm{d}v\mathrm{d}t$ is a relatively invariant measure on Ξ with positive character $\beta(b, s, a) = |a|^{3/2}$. Now, we consider the sections $s \colon \mathbb{R}^2 \to \mathbb{S}$ and $\sigma \colon \mathbb{R} \times \mathbb{R} \to \mathbb{S}$ defined by

$$s(x) = (x, 0, 1), \qquad \sigma(v, t) = ((t, 0), v, 1).$$

Thus, we have that

$$\hat{\xi}_0 = H[x_0] = \{(0, b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R},$$
$$\check{x}_0 = K.\xi_0 = \{(s, 0) : s \in \mathbb{R}\} \simeq \mathbb{R}.$$

It is easy to check that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with $\gamma((0, b_2), 0, a) = |a|^{1/2}$ and that $\gamma(\sigma(v, t)) = 1$ for all $(v, t) \in \Xi$, so that $(g, \xi) \mapsto \gamma\big(\sigma(\xi)^{-1} g \sigma(g^{-1}.\,\xi)\big)$ extends to a positive character of G independent of ξ.

Further, we can compute

$$\widehat{(v, t)} = \sigma(v, t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x \cdot n(v) = t\},$$

which is the set of all points laying on the line of equation $x \cdot n(v) = t$ and

$$\check{x} = s(x).\check{x}_0 = \{(v, t) \in \mathbb{R} \times \mathbb{R} : t - n(v) \cdot x = 0\},$$

which parametrizes the set of all lines passing through the point x except the horizontal one. Thus, the maps $x \mapsto \check{x}$ and $(v, t) \mapsto \widehat{(v, t)}$ are both injective. Therefore, $X = \mathbb{R}^2$ and $\Xi = \mathbb{R} \times \mathbb{R}$ are homogeneous spaces in duality. Observe that, up to this point we have shown that assumptions (A1)-(A3), are satisfied.

The representations

The (parabolic) shearlet group $\mathbb{S}$ acts on $L^2(\mathbb{R}^2)$ via the shearlet representation given by (3.11), namely

$$\pi(b,s,a)f(x) = |a|^{-3/4} f(A_a^{-1}S_s^{-1}(x-b)), \tag{3.48}$$

which is square-integrable by Theorem 3.5.

Furthermore, since $\beta(b,s,a) = |a|^{3/2}$, the group $\mathbb{S}$ acts on $L^2(\mathbb{R}\times\mathbb{R})$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b,s,a)F(v,t) = |a|^{-\frac{3}{4}} F\left(|a|^{-1/2}(v-s), \frac{t-n(v)\cdot b}{a}\right). \tag{3.49}$$

By Mackey imprimitivity theorem [28], one can show that also $\hat{\pi}$ is irreducible. The proof, although not trivial, is based on classical arguments and we omit it. Thus, assumptions (A4) and (A5) hold true.

Observe that the unitary representation $\hat{\pi}$ is just the tensor product of two wavelet representations. Indeed, it acts on F, seen as a function of the only first variable, as the unitary representation V defined by (3.18), which reduces to the wavelet representation when $d=2$, namely

$$\hat{\pi}(b,s,a)F(v,t) = (V_{s,a}\otimes W_{n(v)\cdot b,a})F(v,t) = (W_{s,|a|^{1/2}}\otimes W_{n(v)\cdot b,a})F(v,t).$$

Now, it remains to check assumption (A6).

The Radon transform

By (2.3), the Radon transform between the homogeneous spaces in duality $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{R}$ is defined as

$$\mathcal{R}^{\mathrm{aff}} f(v,t) = \int_{\mathbb{R}} f(t-vy,y)\mathrm{d}y, \tag{3.50}$$

which is the affine Radon transform introduced in Definition 3.7.

Following the arguments in subsection 3.2.1, we define

$$\mathcal{A} = \{f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\omega)|^2}{|\omega_1|}\mathrm{d}\omega < +\infty\}, \tag{3.51}$$

where $\omega = (\omega_1,\omega_2)\in\mathbb{R}^2$, which is π-invariant and is such that $\mathcal{R}^{\mathrm{aff}}f\in L^2(\mathbb{R}\times\mathbb{R})$ for all $f\in\mathcal{A}$. Furthermore, as in §2.4.1, it is easy to show that the restriction $\mathcal{R}$ of $\mathcal{R}^{\mathrm{aff}}$ to $\mathcal{A}$ is closable and we denote its closure by $\overline{\mathcal{R}}$. However, it is also possible to prove the closability of the operator $\mathcal{R}$ reasoning as in § 2.10 by choosing a "minimal" domain of the form $\mathrm{span}\{\pi(g)f_0 : g\in G\}$.

The Unitarization theorem

Since $\alpha(b,s,a) = |a|^{3/2}$, $\beta(b,s,a) = |a|^{3/2}$ and $\gamma(b,s,a) = |a|^{1/2}$, by Lemma 2.5 the affine Radon transform satisfies the intertwining property

$$\mathcal{R}^{\mathrm{aff}}\pi(b,s,a) = \chi(b,s,a)^{-1}\hat{\pi}(b,s,a)\mathcal{R}^{\mathrm{aff}},$$

where $\chi(b,\phi,a)^{-1} = |a|^{1/2}$ and we recover Lemma 3.9.

By Lemma 2.7, the closure $\overline{\mathcal{R}}$ of the affine Radon transform is a semi-invariant operator with weight $\chi(b,s,a) = |a|^{-1/2}$. Therefore, by Theorem 2.11, there exists a positive self-adjoint operator $\mathcal{J}\colon \operatorname{dom}(\mathcal{I}) \subseteq L^2(\mathbb{R}\times\mathbb{R}) \to L^2(\mathbb{R}\times\mathbb{R})$ semi-invariant with weight $\zeta(g) = \chi(g)^{-1} = |a|^{1/2}$ such that $\mathcal{J}\mathcal{R}^{\mathrm{aff}}$ extends to a unitary operator $\mathcal{Q}$ from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}\times\mathbb{R})$, which intertwines the quasi-regular (irreducible) representations π and $\hat{\pi}$. The positive self-adjoint operator operator $\mathcal{I}$ defined by (3.27) is semi-invariant with weight $\zeta(g) = \chi(g)^{-1} = |a|^{1/2}$ by (3.35). By Theorem 1.30, item (i), there exists $c > 0$ such that $\mathcal{J} = c\mathcal{I}$ and we now show that $c = 1$. Consider a non-zero function $f \in \mathcal{A}$. Then, by Plancherel theorem and the Fourier slice theorem (3.22) we have that

$$\begin{aligned}\|f\|^2 = \|\mathcal{I}\mathcal{R}^{\mathrm{aff}} f\|^2_{L^2(\mathbb{R}\times\mathbb{R})} &= c^2 \|(I\otimes\mathcal{F})\mathcal{I}\mathcal{R}^{\mathrm{aff}} f\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \\ &= c^2 \int_{\mathbb{R}\times\mathbb{R}} |\tau| |(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{aff}} f(v,\tau)|^2 \mathrm{d}v\mathrm{d}\tau \\ &= c^2 \int_{\mathbb{R}\times\mathbb{R}} |\tau| |\mathcal{F}f(\tau n(v))|^2 \mathrm{d}v\mathrm{d}\tau \\ &= c^2\|f\|^2.\end{aligned}$$

Thus, we obtain $c = 1$.

The inversion formula

We already know that the shearlet representation π is square-integrable and by Theorem 3.5 its admissible vectors are the functions ψ in $L^2(\mathbb{R}^2)$ satisfying

$$0 < \int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\omega)|^2}{|\omega_1|^2} \mathrm{d}\omega < +\infty, \tag{3.52}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ [19]. We recall that the shearlet transform is $\mathcal{S}_\psi f(b,s,a) = \langle f, \pi(b,s,a)\psi\rangle$, and is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{S})$ provided that ψ satisfies the admissible condition (3.52). Let ψ be an admissible vector for the shearlet representation such that $\mathcal{Q}\psi \in \operatorname{dom}\mathcal{I}$. Then, by Theorem 2.17, for any $f \in \mathcal{A}$ we have the reconstruction formula

$$f = \int_{\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^\times} \mathcal{S}_\psi f(b,s,a)\, \pi(b,s,a)\psi\, \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{3.53}$$

where the coefficients $\mathcal{S}_\psi f(b,s,a)$ are given by

$$\mathcal{S}_\psi f(b_1,b_2,s,a) = |a|^{-5/4} \int_{\mathbb{R}\times\mathbb{R}} \mathcal{R}^{\mathrm{aff}} f(v,t) \overline{\Psi\left(\frac{v-s}{|a|^{1/2}}, \frac{t-n(v)\cdot b}{a}\right)} \mathrm{d}v\mathrm{d}t,$$

with $\Psi = \mathcal{I}\mathcal{Q}\psi$. Moreover, if we choose ψ such that $\Psi(v,t) = \Psi_2(v)\Psi_1(t)$, then

$$\mathcal{S}_\psi f(b_1,b_2,s,a) = |a|^{-3/4} \int_{\mathbb{R}} \mathcal{W}_{\Psi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(n(v)\cdot b, a) \overline{\Psi_2\left(\frac{v-s}{|a|^{1/2}}\right)} \mathrm{d}v, \tag{3.54}$$

provided that Ψ_1 is a 1D-wavelet. We have shown in subsection 3.3.2 that it is enough to choose $\psi \in L^2(\mathbb{R}^2)$ of the form (3.36) with $\psi_1 \in L^2(\mathbb{R})$ satisfying conditions (3.38) and (3.41) and $\psi_2 \in L^2(\mathbb{R})$. Then, ψ is an admissible vector for the shearlet representation such that $\mathcal{Q}\psi \in \operatorname{dom}\mathcal{I}$ and $\Psi = \mathcal{I}\mathcal{Q}\psi$ factorizes as $\Psi_2 \otimes \Psi_1$, with $\Psi_2 = \mathcal{F}\psi_2$ and Ψ_1 the one-dimensional wavelet χ_1 defined by (3.43).

This argument gives an alternative proof of Corollary 3.16, where it is also proved that formula (3.54) can actually be extended to the whole $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

3.5 Cone-Adapted Shearlets and Radon Transforms

In Section 3.3 we have shown that the classical shearlet transform can be realised by applying first the affine Radon transform, then by computing a one-dimensional wavelet transform and, finally, performing a one-dimensional convolution. This relation opens the possibility to recover a signal from its affine Radon transform by using the shearlet inversion formula (3.53), where the coefficients $\mathcal{S}_\psi f(b,s,a)$ depend on f only through its Radon transform. Thus, formula (3.53) allows to reconstruct an unknown signal f from its affine Radon transform $\mathcal{R}^{\mathrm{aff}} f$ by computing the family of coefficients $\{\mathcal{S}_\psi f(b,s,a)\}_{b\in\mathbb{R}^2, s\in\mathbb{R}, a\in\mathbb{R}^\times}$. Equation (3.53) has a disadvantage if one wants to use it in applications since the shearing parameter s is allowed to vary over a non-compact set. This gives rise to problems in the reconstruction of signals mostly concentrated on the x-axis since the energy of such signals is mostly concentrated in the coefficients $\mathcal{S}_\psi f(b,s,a)$ as $s \to \infty$. The standard way to address this problem is the so-called "shearlets on the cone" construction introduced by Kutyniok and Labate [51]. In this section, applying the "shearlets on the cone" construction to our results, we obtain for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ a reconstruction formula where both the scale parameter a and the shearing parameter s range over compact intervals, and where the coefficients depend on f only through its Radon transform.

3.5.1 Cone-Adapted Shearlets

Although the shearlet transform exhibits an elegant group structure and is based on the theory of square integrable representations, the reconstruction formula (3.53) has one disadvantage: the shearing parameter ranges over a non-compact set and this can constitute a limitation in applications. For example, if f is the delta distribution supported on the x-axis, a classical model for an edge in an image, the high amplitude shearlet coefficients, i.e. the shearlet coefficients in which the energy of the signal is mostly concentrated, correspond to the shearlet coefficients $\mathcal{S}_\psi f(b,s,a)$ as $s \to \infty$ [51]. In order to avoid this problem Kutyniok and Labate [51] proposed the "shearlets on the cone" construction which leads to a reconstruction formula where both the scale parameter a and the shearing parameter s are restricted over compact sets. We briefly recall this construction and we refer to [51] for more details.

Let $f \in L^2(\mathbb{R}^2)$. We consider the horizontal and vertical cones in the frequency plane

$$C = \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_2}{\xi_1}\right| \leq 1\right\}, \qquad C^{\mathrm{v}} = \left\{(\xi_1,\xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_1}{\xi_2}\right| \leq 1\right\}. \tag{3.55}$$

If D is a region in the plane, we denote by χ_D its characteristic function, i.e.

$$\chi_D(\xi) = \begin{cases} 1 & \text{if } \xi \in D \\ 0 & \text{if } \xi \notin D \end{cases}$$

and we define the frequency projections of f onto C and C^{v} by

$$\begin{aligned} \mathcal{F}(P_C f)(\xi_1,\xi_2) &= \mathcal{F}f(\xi_1,\xi_2)\chi_C(\xi_1,\xi_2) \\ \mathcal{F}(P_{C^{\mathrm{v}}} f)(\xi_1,\xi_2) &= \mathcal{F}f(\xi_1,\xi_2)\chi_{C^{\mathrm{v}}}(\xi_1,\xi_2) \end{aligned} \tag{3.56}$$

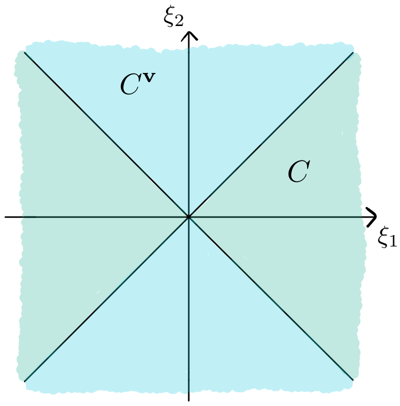

Figure 3.2: In the "shearlets on the cone" construction the frequency plane is divided in two cones C and C^{v}. In formula (3.57), $P_C f$ is reconstructed via the classical shearlet transform and $P_{C^{\mathrm{v}}} f$ via the so-called vertical shearlet transform.

respectively.

We need a modified version of the continuous shearlet transform obtained by switching the roles of the x-axis and the y-axis. We introduce the vertical shearlet representation

$$S^{\mathrm{v}}_{b,s,a} f(x) = |a|^{-3/4} f(\tilde{A}_a^{-1} \tilde{S}_s^{-1}(x-b))$$

where

$$\tilde{S}_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}, \qquad \tilde{A}_a = a \begin{bmatrix} |a|^{-1/2} & 0 \\ 0 & 1 \end{bmatrix},$$

and the associated vertical shearlet transform $\mathcal{S}^{\mathrm{v}}_{\psi} f(b,s,a) = \langle f, S^{\mathrm{v}}_{b,s,a}\psi\rangle$.

Then, chosen a suitable window function g, the following reconstruction formula holds true:

$$\begin{aligned} \|f\|^2 = &\int_{\mathbb{R}^2} |\langle f, T_b g\rangle|^2 \, \mathrm{d}b + \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_{\psi}[P_C f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s \frac{\mathrm{d}a}{|a|^3} \\ &+ \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathrm{v}}_{\psi^{\mathrm{v}}}[P_{C^{\mathrm{v}}} f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s \frac{\mathrm{d}a}{|a|^3}, \end{aligned} \tag{3.57}$$

where $\mathcal{F}\psi^{\mathrm{v}}(\xi_1,\xi_2) = \mathcal{F}\psi(\xi_2,\xi_1)$. In formula (3.57), $P_C f$ is reconstructed via the classical shearlet transform and $P_{C^{\mathrm{v}}} f$ via the vertical shearlet transform and this allows to restrict the shearing parameter s over a compact interval.

Reconstruction formulas of the form (3.57) were firstly proved by Labate and Kutinyok [51] for classical admissible shearlets ψ and then generalized by Grohs [34] requiring weaker conditions on ψ. We have chosen to present our results within the second approach. We fix $\psi \in L^2(\mathbb{R}^2)$ satisfying the admissibility condition (3.52). We require that ψ is a smooth function with all directional vanishing moments in the x_1-direction [34], that is

$$\int_{\mathbb{R}} x_1^N \psi(x_1,x_2)\mathrm{d}x_1 = 0, \qquad \text{for all}\, x_2 \in \mathbb{R},\ N \in \mathbb{N}.$$

Then, we have the following result.

Theorem 3.18 ([52, Chapter 2][34]). *For any $f \in L^2(\mathbb{R}^2)$, we have the reconstruction formula*

$$\begin{aligned}\|f\|^2 = &\int_{\mathbb{R}^2} |\langle f, T_b g\rangle|^2 \, \mathrm{d}b + \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_\psi[P_C f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3} \\ &+ \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}, \qquad (3.58)\end{aligned}$$

with $g \in C^\infty(\mathbb{R}^2)$ such that for all $\xi \in \mathbb{R}^2$

$$\begin{aligned}|\mathcal{F}g(\xi)|^2 + \chi_C(\xi)\int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^t S_s \xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} \\ + \chi_{C^{\mathbf{v}}}(\xi)\int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_a S_s \xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} = 1. \qquad (3.59)\end{aligned}$$

In [51] Labate and Kutinyok prove a reconstruction formula of the form (3.58) with $\psi \in L^2(\mathbb{R}^2)$ a classical admissible shearlet defined as in Definition 3.3 in [51].

3.5.2 Cone-Adapted Radon Transforms

We recall that the affine Radon transform is obtained labelling the normal vector to a line by affine coordinates and it is defined for any $f \in L^1(\mathbb{R}^2)$ as the function $\mathcal{R}^{\mathrm{aff}} f : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\mathcal{R}^{\mathrm{aff}} f(v,t) = \int_{\mathbb{R}} f(t - vy, y)\,\mathrm{d}y, \qquad \text{a.e. } (v,t) \in \mathbb{R}^2.$$

The choice of the affine parametrization is particularly well-adapted to the mathematical structure of the shearlet transform, see also [34]. By Theorem 3.11, the map

$$f \longmapsto \mathcal{I}\mathcal{R}^{\mathrm{aff}} f$$

from $\mathcal{A}$ to $L^2(\mathbb{R}^2)$ extends to a unitary map, denoted by $\mathcal{Q}$, from $L^2(\mathbb{R}^2)$ onto itself, where $\mathcal{I}$ is the self-adjoint unbounded operator defined by (3.27) and $\mathcal{A}$ is the dense subspace of $L^2(\mathbb{R}^2)$ given by (3.51).

We need a modified version of the affine Radon transform obtained by switching the roles of the x-axis and the y-axis. We parametrize lines in the plane, except the vertical ones, by pairs $(v,t) \in \mathbb{R} \times \mathbb{R}$ as follows

$$\Gamma_{v,t} = \{(x,y) \in \mathbb{R}^2 \mid vx + y = t\},$$

see Figure 3.3. The vertical (affine) Radon transform of any $f \in L^1(\mathbb{R}^2)$ is the function $\mathcal{R}^{\mathbf{v}} f : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\mathcal{R}^{\mathbf{v}} f(v,t) = \int_{\mathbb{R}} f(x, t - vx)\,\mathrm{d}x, \qquad \text{a.e. } (v,t) \in \mathbb{R}^2.$$

As for the affine Radon transform, we define the dense subspace $\mathcal{A}^{\mathbf{v}}$ of $L^2(\mathbb{R}^2)$ by

$$\mathcal{A}^{\mathbf{v}} = \{f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi_1,\xi_2)|^2}{|\xi_2|}\mathrm{d}\xi_1\mathrm{d}\xi_2 < +\infty\}.$$

Then, the composite operator $\mathcal{I}\mathcal{R}^{\mathbf{v}} : \mathcal{A}^{\mathbf{v}} \to L^2(\mathbb{R}^2)$ extends to a unitary map $\mathcal{Q}^{\mathbf{v}}$ from $L^2(\mathbb{R}^2)$ onto itself. The proof is analogous to the one of Theorem 3.11 and we omit it.

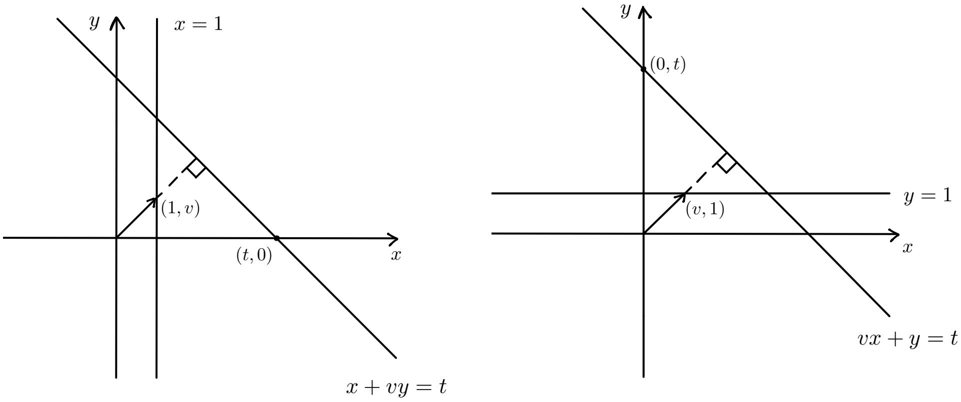

Figure 3.3: The affine Radon transform is defined by labeling the normal vector to a line, except the horizontal ones, by affine coordinates (figure on top): v parametrizes the slope of a line and t its intersection with the x-axis. The vertical Radon transform is obtained just switching the roles of the x-axis and the y-axis in the previous parametrization.

3.5.3 The Radon transform intertwines wavelets and shearlets

We fix $\psi \in L^2(\mathbb{R}^2)$ of the form

$$\mathcal{F}\psi(\xi_1, \xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2\left(\frac{\xi_2}{\xi_1}\right), \tag{3.60}$$

with $\psi_1 \in L^2(\mathbb{R})$ satisfying the conditions

$$0 < \int_{\mathbb{R}} \frac{|\mathcal{F}\psi_1(\tau)|^2}{|\tau|} \, d\tau < +\infty, \qquad \int_{\mathbb{R}} |\tau||\mathcal{F}\psi_1(\tau)|^2 \, d\tau < +\infty \tag{3.61}$$

and $\psi_2 \in L^2(\mathbb{R})$. Then, ψ satisfies the admissible condition (3.52) and the function $\phi_1 \in L^2(\mathbb{R})$ defined by

$$\mathcal{F}\phi_1(\tau) = |\tau|^{\frac{1}{2}}\mathcal{F}\psi_1(\tau) \tag{3.62}$$

is a one-dimensional wavelet. By Corollary 3.15, for any $f \in L^2(\mathbb{R}^2)$ and $(x, y, s, a) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times$,

$$\mathcal{S}_\psi f(x, y, s, a) = |a|^{-\frac{1}{4}} \int_{\mathbb{R}} \mathcal{W}_{\phi_1}(\mathcal{Q}f(v, \cdot))(x + vy, a)\overline{\phi_2\left(\frac{v - s}{|a|^{1/2}}\right)} \, dv \tag{3.63}$$

and analogously, for the vertical shearlet transform,

$$\mathcal{S}^{\mathrm{v}}_{\psi^{\mathrm{v}}} f(x, y, s, a) = |a|^{-\frac{1}{4}} \int_{\mathbb{R}} \mathcal{W}_{\phi_1}(\mathcal{Q}^{\mathrm{v}} f(v, \cdot))(vx + y, a)\overline{\phi_2\left(\frac{v - s}{|a|^{1/2}}\right)} \, dv, \tag{3.64}$$

where ϕ_1 is the one-dimensional wavelet defined by (3.62) and $\phi_2 = \mathcal{F}\psi_2$. The proof of formula (3.64) is analogous to the one of Corollary 3.15 where the roles of the x-axis and the y-axis are switched and the affine Radon transform is substituted by the vertical Radon transform.

3.5.4 Cone-adapted shearlets and Radon transforms

Equation (3.53) together with formula (3.63) allows to reconstruct an unknown signal f from its unitary Radon transform $\mathcal{Q}f$ but it is difficult to implement in applications since $\mathcal{Q}$ involves both a limit and the pseudo-differential operator $\mathcal{I}$. Furthermore, in the reconstruction formula (3.53) the shearing parameter s is allowed to range over $\mathbb{R}$ and this can give rise to the problems discussed above. The aim of this section is to obtain a reconstruction formula of the form (3.57), i.e. where both the scale and the shearing parameters belong to compact intervals, where the shearlet coefficients depend on f only through its Radon transform and do not involve the operator $\mathcal{I}$ applied to the signal.

We fix an admissible vector ψ of the form (3.60) satisfying conditions (3.61) and such that ψ_1 satisfies the further condition (3.41).

Proposition 3.19. *For any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $(x, y, s, a) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times$,*

$$\mathcal{S}_\psi[P_C f](x, y, s, a) = |a|^{-\frac{3}{4}} \int_{-1}^{1} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v, \cdot))(x + vy, a)\overline{\phi_2\left(\frac{v - s}{|a|^{1/2}}\right)}\, \mathrm{d}v,$$

$$\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](x, y, s, a) = |a|^{-\frac{3}{4}} \int_{-1}^{1} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathbf{v}} f(v, \cdot))(vx + y, a)\overline{\phi_2\left(\frac{v - s}{|a|^{1/2}}\right)}\, \mathrm{d}v,$$

where $\mathcal{F}\chi_1(\tau) = |\tau|\mathcal{F}\psi_1(\tau)$ and $\phi_2 = \mathcal{F}\psi_2$.

Proof. We take a function $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and we consider its frequency projection $P_C f$ on the horizontal cone C defined by (3.55) and (3.56). Since $P_C f$ belongs to $L^2(\mathbb{R}^2)$, we can apply formula (3.63) and we obtain

$$\mathcal{S}_\psi[P_C f](x, y, s, a) = |a|^{-\frac{1}{4}} \int_{\mathbb{R}} \mathcal{W}_{\phi_1}(\mathcal{Q}[P_C f](v, \cdot))(x + vy, a)\overline{\phi_2\left(\frac{v - s}{|a|^{1/2}}\right)}\, \mathrm{d}v, \quad (3.65)$$

where ϕ_1 is the admissible wavelet defined by (3.62) and $\phi_2 = \mathcal{F}\psi_2$. We consider the functions $t \mapsto \mathcal{Q}[P_C f](v, t)$ in equation (3.65). By Proposition 3.12 and the definition of $P_C f$, we have

$$\mathcal{F}(\mathcal{Q}[P_C f](v, \cdot))(\tau) = |\tau|^{\frac{1}{2}} \mathcal{F}(P_C f)(\tau, \tau v) = |\tau|^{\frac{1}{2}} \mathcal{F}f(\tau, \tau v)\chi_C(\tau, \tau v), \quad (3.66)$$

for almost every $(v, \tau) \in \mathbb{R}^2$. Furthermore, by the definition of the horizontal cone C, the function $\tau \mapsto \chi_C(\tau, \tau v)$ is identically one if $|v| \leq 1$ and zero otherwise. Thus, (3.66) becomes

$$\mathcal{F}(\mathcal{Q}[P_C f](v, \cdot))(\tau) = \begin{cases} |\tau|^{\frac{1}{2}} \mathcal{F}f(\tau, \tau v) & \text{if } |v| \leq 1 \\ 0 & \text{if } |v| > 1 \end{cases}. \quad (3.67)$$

From now on we consider the case $|v| \leq 1$. Since $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, Proposition 3.12 and equation (3.67) imply that for almost all $v \in \mathbb{R}$, $\mathcal{R}^{\mathrm{aff}} f(v, \cdot) \in L^2(\mathbb{R})$ and

$$\mathcal{F}(\mathcal{Q}[P_C f](v, \cdot))(\tau) = |\tau|^{\frac{1}{2}} \mathcal{F}f(\tau, \tau v) = |\tau|^{\frac{1}{2}} \mathcal{F}\mathcal{R}^{\mathrm{aff}} f(v, \cdot)(\tau). \quad (3.68)$$

Since $\tau \mapsto \mathcal{F}(\mathcal{Q}[P_C f](v, \cdot))(\tau) \in L^2(\mathbb{R})$ for almost all $v \in \mathbb{R}$, equality (3.68) implies that $\mathcal{R}^{\mathrm{aff}} f(v, \cdot)$ is in the domain of the differential operator $\mathcal{I}_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by (3.28) and, by the definition of $\mathcal{I}_0$,

$$\mathcal{Q}[P_C f](v, \cdot) = \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v, \cdot).$$

Since $\mathcal{I}_0$ is a self-adjoint operator, the wavelet coefficients in (3.65) become

$$\begin{aligned}\mathcal{W}_{\phi_1}(\mathcal{Q}[P_C f](v,\cdot))(x+vy,a) &= \langle \mathcal{Q}[P_C f](v,\cdot), W_{x+vy,a}\phi_1\rangle_2 \\ &= \langle \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v,\cdot), W_{x+vy,a}\phi_1\rangle_2 \\ &= \langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), \mathcal{I}_0 W_{x+vy,a}\phi_1\rangle_2 \\ &= |a|^{-\frac{1}{2}}\langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), W_{x+vy,a}\chi_1\rangle_2 \\ &= |a|^{-\frac{1}{2}}\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(x+vy,a),\end{aligned}$$

by taking into account that

$$\mathcal{I}_0 W_{b,a} = |a|^{-\frac{1}{2}} W_{b,a}\mathcal{I}_0,$$

for any $(b,a) \in \mathbb{R}\times\mathbb{R}^\times$ and where $\chi_1 = \mathcal{I}_0\phi_1 = \mathcal{I}_0^2\psi_1$. From the above calculations, we can conclude that for almost every $v \in \mathbb{R}$

$$\mathcal{W}_{\phi_1}(\mathcal{Q}[P_C f](v,\cdot))(x+vy,a) = \begin{cases} |a|^{-\frac{1}{2}}\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(x+vy,a) & |v| \le 1 \\ 0 & |v| > 1 \end{cases}$$

and formula (3.65) becomes

$$\mathcal{S}_\psi[P_C f](x,y,s,a) = |a|^{-\frac{3}{4}}\int_{-1}^{1}\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(x+vy,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)}\,\mathrm{d}v. \tag{3.69}$$

Using the same arguments as in the case of the horizontal cone, we obtain the following formula for the vertical shearlet transform

$$\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](x,y,s,a) = |a|^{-\frac{3}{4}}\int_{-1}^{1}\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathbf{v}} f(v,\cdot))(vx+y,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)}\,\mathrm{d}v. \tag{3.70}$$

This completes the proof. □

It is worth observing that formulas (3.69) and (3.70) turn the action of the frequency projections P_C and $P_{C^{\mathbf{v}}}$ on f into the restriction of the interval over which we integrate the directional variable v and so, (3.69) and (3.70) eliminate the need to perform a frequency projection on f prior to the analysis. Furthermore, as a consequence, the shearlet coefficients $\mathcal{S}_\psi[P_C f](b,s,a)$ and $\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](b,s,a)$ depend on f through its limited angle affine and vertical Radon transforms $\mathcal{R}^{\mathrm{aff}} f(v,t)$ and $\mathcal{R}^{\mathbf{v}} f(v,t)$, with $|v| \le 1$, respectively.

Finally, let us show that also the first integral in the right hand side of reconstruction formula (3.57) may be expressed in terms of $\mathcal{R}^{\mathrm{aff}} f$ only.

Proposition 3.20. *For any $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ and for any smooth function g in $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ we have that*

$$\langle f, T_b g\rangle = \int_{\mathbb{R}}\langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), T_{n(v)\cdot b}\zeta(v,\cdot)\rangle \mathrm{d}v,$$

for any $b \in \mathbb{R}^2$, where $\zeta = \mathcal{I}_0^2\mathcal{R}^{\mathrm{aff}} g$ and $n(v) = {}^t(1,v)$.

Proof. We take a function $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and we consider a smooth function $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. We readily derive

$$\langle f, T_b g\rangle = \langle \mathcal{Q}f, \mathcal{Q}T_b g\rangle = \int_{\mathbb{R}} \langle \mathcal{Q}f(v,\cdot), \mathcal{Q}T_b g(v,\cdot)\rangle \mathrm{d}v.$$

Since f and g are in $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, Proposition 3.13 implies that for almost all $v \in \mathbb{R}$, $\mathcal{R}^{\mathrm{aff}} f(v,\cdot)$ and $\mathcal{R}^{\mathrm{aff}} g(v,\cdot)$ are square-integrable functions and

$$\langle f, T_b g\rangle = \int_{\mathbb{R}} \langle \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v,\cdot), \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} T_b g(v,\cdot)\rangle \mathrm{d}v,$$

where we recall that $\mathcal{I}_0$ is the differential operator defined by (3.28). By the behavior of the affine Radon transform under translations and since the operator $\mathcal{I}_0$ commutes with translations, we have that

$$\mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} T_b g(v,t) = \mathcal{I}_0 (I \otimes T_{n(v)\cdot b}) \mathcal{R}^{\mathrm{aff}} g(v,t) = (I \otimes T_{n(v)\cdot b}) \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,t).$$

We need to choose g in such a way that $\mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)$ is in the domain of the operator $\mathcal{I}_0$ for almost every $v \in \mathbb{R}$. Assuming this, the same property holds true for $T_{n(v)\cdot b} \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)$ by the translation invariance of $\operatorname{dom} \mathcal{I}_0$ and we obtain

$$\begin{aligned}\langle f, T_b g\rangle &= \int_{\mathbb{R}} \langle \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} f(v,\cdot), T_{n(v)\cdot b} \mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)\rangle \mathrm{d}v \\ &= \int_{\mathbb{R}} \langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), T_{n(v)\cdot b} \mathcal{I}_0^2 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)\rangle \mathrm{d}v. \end{aligned} \tag{3.71}$$

It is worth observing that the extra assumption that $\mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)$ is in the domain of $\mathcal{I}_0$ for almost every $v \in \mathbb{R}$ is always satisfied. Indeed, by the definition of $\mathcal{I}_0$ and Proposition 3.8

$$\begin{aligned}\int_{\mathbb{R}} |\tau| |\mathcal{F}\mathcal{I}_0 \mathcal{R}^{\mathrm{aff}} g(v,\cdot)(\tau)|^2 \mathrm{d}\tau &= \int_{\mathbb{R}} |\tau|^2 |\mathcal{F}\mathcal{R}^{\mathrm{aff}} g(v,\cdot)(\tau)|^2 \mathrm{d}\tau \\ &= \int_{\mathbb{R}} |\tau|^2 |\mathcal{F}g(\tau,\tau v)|^2 \mathrm{d}\tau < +\infty \end{aligned}$$

since by definition g is a smooth function. We set $\zeta(v,\tau) = \mathcal{I}_0^2 \mathcal{R}^{\mathrm{aff}} g(v,\tau)$, that is

$$\mathcal{F}\zeta(v,\cdot)(\tau) = |\tau| \mathcal{F}g(\tau,\tau v),$$

so that (3.71) becomes

$$\langle f, T_b g\rangle = \int_{\mathbb{R}} \langle \mathcal{R}^{\mathrm{aff}} f(v,\cdot), T_{n(v)\cdot b} \zeta(v,\cdot)\rangle \mathrm{d}v. \tag{3.72}$$

Furthermore, if possible, we choose g of the form

$$\mathcal{F}g(\xi_1,\xi_2) = \mathcal{F}g_1(\xi_1) \mathcal{F}g_2\left(\frac{\xi_2}{\xi_1}\right),$$

with $g_1 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ satisfying the condition

$$\int_{\mathbb{R}} |\tau|^2 |\mathcal{F}g_1(\tau)|^2 \ \mathrm{d}\tau < +\infty \tag{3.73}$$

and $g_2 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Under these hypotheses, (3.72) becomes

$$\langle f, T_b g \rangle = \int_{\mathbb{R}} \langle \mathcal{R}^{\text{aff}} f(v, \cdot), T_{n(v)\cdot b}\zeta_1 \rangle \zeta_2(v) \mathrm{d}v,$$

where $\zeta_1 = \mathcal{I}_0^2 g_1$, which is well-defined by (3.73), and $\zeta_2 = \mathcal{F} g_2$. □

Theorem 3.18 and formulas (3.69), (3.70) and (3.72) give our main result of this section. We recall that ψ is an admissible vector of the form (3.60) satisfying conditions (3.61) and such that ψ_1 satisfies (3.41). Furthermore we require that ψ is smooth with all directional vanishing moments in the x_1-direction [34].

Theorem 3.21. *For any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we have the reconstruction formula*

$$\begin{aligned} \|f\|^2 = &\int_{\mathbb{R}^2} |\langle f, T_b g \rangle|^2 \, \mathrm{d}b + \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_\psi[P_C f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s \frac{\mathrm{d}a}{|a|^3} \\ &+ \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s \frac{\mathrm{d}a}{|a|^3}, \end{aligned} \tag{3.74}$$

where g is a smooth function in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that (3.59) holds true and for any $b = (x, y) \in \mathbb{R}^2$, $s \in \mathbb{R}$, $a \in \mathbb{R}^\times$

$$\mathcal{S}_\psi[P_C f](x,y,s,a) = |a|^{-\frac{3}{4}} \int_{-1}^{1} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(x+vy,a) \overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)} \, \mathrm{d}v,$$

$$\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](x,y,s,a) = |a|^{-\frac{3}{4}} \int_{-1}^{1} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathbf{v}} f(v,\cdot))(vx+y,a) \overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)} \, \mathrm{d}v,$$

$$\langle f, T_{(x,y)} g \rangle = \int_{\mathbb{R}} \langle \mathcal{R}^{\text{aff}} f(v,\cdot), T_{x+vy}\zeta(v,\cdot) \rangle \mathrm{d}v,$$

where $\zeta = \mathcal{I}_0^2 \mathcal{R}^{\text{aff}} g$, $\mathcal{F}\chi_1(\tau) = |\tau| \mathcal{F}\psi_1(\tau)$, $\phi_2 = \mathcal{F}\psi_2$.

Proof. The proof follows immediately by Theorem 3.18 and Propositions 3.19 and 3.20. □

This theorem gives an alternative reproducing formula for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ in which, by the "shearlets on the cone" construction, the scale and shearing parameters range over compact sets and, by Propositions 3.19 and 3.20, the coefficients depend on f only through its Radon transform. Therefore equation (3.74) allows to reconstruct an unknown signal f from its Radon transform by computing the family of coefficients $\{\langle f, T_b g \rangle, \mathcal{S}_\psi[P_C f](b,s,a), \mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[P_{C^{\mathbf{v}}} f](b,s,a)\}_{b \in \mathbb{R}^2, s \in \mathbb{R}, a \in \mathbb{R}^\times}$ by means of Theorem 3.21. It is worth observing that the different contributions in (3.74) with $\mathcal{R}^{\text{aff}} f(v,t)$ and $\mathcal{R}^{\mathbf{v}} f(v,t)$, $|v| \leq 1$, reconstruct the frequency projections $P_C f$ and $P_{C^{\mathbf{v}}} f$, respectively.

3.5.5 Generalizations

A disadvantage in formula (3.57), and therefore in formula (3.74), is that the frequency projections P_C and $P_{C^{\mathbf{v}}}$ performed on f can lead to artificially slow decaying shearlet coefficients. In order to avoid this problem we consider an open cover $\{U, U^{\mathbf{v}}\}$ of the unit circle in the plane $S^1 \simeq (-\pi, \pi]$, where

$$\begin{aligned} U =& (-\pi, -\frac{3}{4}\pi + \epsilon) \cup (-\frac{\pi}{4} - \epsilon, \frac{\pi}{4} + \epsilon) \cup (\frac{3}{4}\pi - \epsilon, \pi], \\ U^{\mathbf{v}} =& (-\frac{3}{4}\pi - \epsilon, -\frac{\pi}{4} + \epsilon) \cup (\frac{\pi}{4} - \epsilon, \frac{3}{4}\pi + \epsilon). \end{aligned}$$

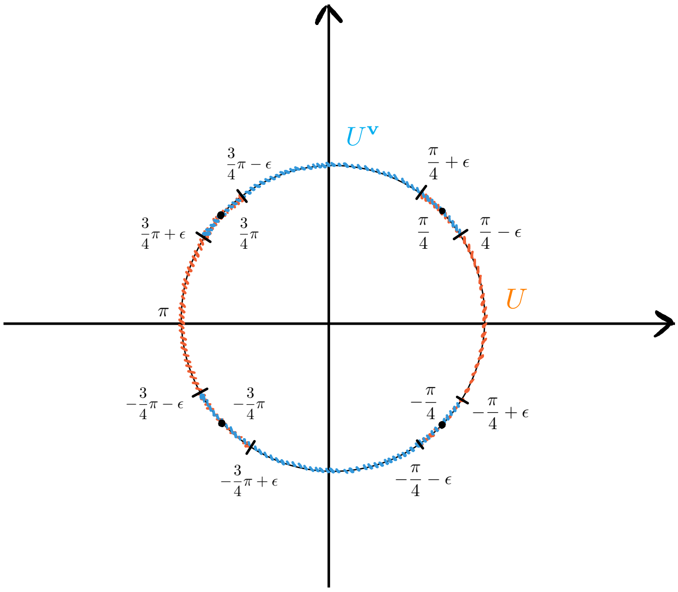

Figure 3.4: The open cover $\{U, U^{\vee}\}$ of the unit circle in the plane $S^1 \simeq (-\pi, \pi]$.

Then, there exist even functions $\varphi, \varphi^{\vee} \in C^{\infty}((-\pi,\pi])$ such that $\operatorname{supp}\varphi \subseteq U$, $\operatorname{supp}\varphi^{\vee} \subseteq U^{\vee}$ and $\varphi(\theta)^2 + \varphi^{\vee}(\theta)^2 = 1$ for all $\theta \in (-\pi,\pi]$, see [13]. For any $\xi \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\theta_\xi \in (-\pi,\pi]$ the angle corresponding to $\xi/|\xi| \in S^1$ by the canonical isomorphism $S^1 \simeq (-\pi,\pi]$. Then, we define the functions Φ, $\Phi^{\vee} \in C^{\infty}(\mathbb{R}^2 \setminus \{0\})$ by

$$\Phi(\xi) = \varphi(\theta_\xi), \qquad \Phi^{\vee}(\xi) = \varphi^{\vee}(\theta_\xi).$$

It is easy to verify that $\operatorname{supp}\Phi = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\xi_2/\xi_1| \leq \tan(\frac{\pi}{4}+\epsilon)\}$, $\operatorname{supp}\Phi^{\vee} = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\xi_1/\xi_2| \leq \cot(\frac{\pi}{4}-\epsilon)\}$ and $\Phi(\xi)^2 + \Phi^{\vee}(\xi)^2 = 1$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. We define the operators $L\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ and $L^{\vee}\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ as follows

$$\mathcal{F}(Lf)(\xi) = \mathcal{F}f(\xi)\Phi(\xi)$$

and

$$\mathcal{F}(L^{\vee}f)(\xi) = \mathcal{F}f(\xi)\Phi^{\vee}(\xi).$$

We recall that ψ is an admissible vector of the form (3.60) satisfying conditions (3.61) and such that ψ_1 satisfies (3.41). Using analogous computations as in Section 3.5.4, it is possible to show that for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $(x,y,s,a) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{\times}$

$$\begin{aligned}\mathcal{S}_\psi[Lf](x,y,s,a) & \\ &= |a|^{-\frac{3}{4}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(x+vy,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)}\varphi(\arctan v)\,\mathrm{d}v, \qquad (3.75)\end{aligned}$$

$$\begin{aligned}\mathcal{S}^{\vee}_{\psi^{\vee}}[L^{\vee}f](x,y,s,a) & \\ &= |a|^{-\frac{3}{4}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\vee} f(v,\cdot))(vx+y,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)}\varphi^{\vee}\left(\arctan\frac{1}{v}\right)\,\mathrm{d}v, \quad (3.76)\end{aligned}$$

where $\mathcal{F}\chi_1(\tau) = |\tau|\mathcal{F}\psi_1(\tau)$ and $\phi_2 = \mathcal{F}\psi_2$. Furthermore, following the proof of Theorem 3 in [52, Chapter 2], it is possible to derive a reconstruction formula of the form (3.74) in this new setup.

Theorem 3.22. *For any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we have the reconstruction formula*

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle f, T_b g\rangle|^2 \, \mathrm{d}b + \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_\psi[Lf](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}$$
$$+ \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[L^{\mathbf{v}}f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}, \tag{3.77}$$

where g is a smooth function in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that for all $\xi \in \mathbb{R}^2$

$$|\mathcal{F}g(\xi)|^2 + \Phi(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}$$
$$+ \Phi^{\mathbf{v}}(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_aS_s\xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} = 1 \tag{3.78}$$

and the coefficients in (3.77) *are given by* (3.75)*,* (3.76) *and* (3.72)*.*

Proof. Consider a smooth function $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that (3.78) holds true. By Plancherel theorem, we have that

$$\int_{\mathbb{R}^2} |\langle f, T_b g\rangle|^2 \, \mathrm{d}b = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}e^{2\pi i b\cdot\xi} \, \mathrm{d}\xi \right|^2 \, \mathrm{d}b$$
$$= \int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f \, \overline{\mathcal{F}g})(b)|^2 \, \mathrm{d}b$$
$$= \int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2 |\mathcal{F}g(\xi)|^2 \, \mathrm{d}\xi. \tag{3.79}$$

Using an analogous computation, by Plancherel theorem and Fubini's theorem we have

$$\int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_\psi[Lf](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}$$
$$= \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\langle Lf, S_{b,s,a}\psi\rangle|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}$$
$$= \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathcal{F}f(\xi)\Phi(\xi)\overline{\mathcal{F}\psi(A_a{}^tS_s\xi)}e^{2\pi i\xi b} \, \mathrm{d}\xi \right|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}$$
$$= \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f\Phi\overline{\mathcal{F}\psi(A_a{}^tS_s\cdot)})(b)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}$$
$$= \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2\Phi(\xi)^2|\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \, \mathrm{d}\xi\mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}$$
$$= \int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2\Phi(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}\mathrm{d}\xi. \tag{3.80}$$

Similarly, we have that

$$\int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[L^{\mathbf{v}}f](b,s,a)|^2 \, \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}$$
$$= \int_{\mathbb{R}^2} |\mathcal{F}f(\xi)|^2\Phi^{\mathbf{v}}(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_aS_s\xi)|^2 \, \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}\mathrm{d}\xi. \tag{3.81}$$

Thus, combining equations (3.79), (3.80) and (3.81) we obtain the reconstruction formula

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle f, T_b g\rangle|^2 \ \mathrm{d}b + \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}_\psi[Lf](b,s,a)|^2 \ \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}$$
$$+ \int_{-1}^{1}\int_{-2}^{2}\int_{\mathbb{R}^2} |\mathcal{S}^{\mathbf{v}}_{\psi^{\mathbf{v}}}[L^{\mathbf{v}}f](b,s,a)|^2 \ \mathrm{d}b\mathrm{d}s\frac{\mathrm{d}a}{|a|^3}, \tag{3.82}$$

for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where the shearlet coefficients are given by (3.75) and (3.76). It is worth observing that there always exists a function g satisfying (3.78) provided that the admissible vector ψ is smooth and possesses all vanishing moments in the x_1-direction [34]. Indeed, we have that

$$\begin{aligned} z(\xi) &:= 1 - \Phi(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} \\ &\quad - \Phi^{\mathbf{v}}(\xi)^2 \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_aS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} \\ &= \Phi(\xi)^2(1 - \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}) \\ &\quad + \Phi^{\mathbf{v}}(\xi)^2(1 - \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_aS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}}). \end{aligned}$$

Following the proof of Lemma 3 in [52, Chapter 2] it is possible to prove that

$$1 - \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi(A_a{}^tS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} = O(|\xi|^{-N}), \quad \left|\frac{\xi_2}{\xi_1}\right| \leq \tan\left(\frac{\pi}{4} + \epsilon\right),$$

for all $N \in \mathbb{N}$. Analogously,

$$1 - \int_{-1}^{1}\int_{-2}^{2} |\mathcal{F}\psi^{\mathbf{v}}(\tilde{A}_aS_s\xi)|^2 \ \mathrm{d}s\frac{\mathrm{d}a}{|a|^{3/2}} = O(|\xi|^{-N}), \quad \left|\frac{\xi_1}{\xi_2}\right| \leq \cot\left(\frac{\pi}{4} - \epsilon\right),$$

for all $N \in \mathbb{N}$. Therefore, there exists a smooth function $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that $\mathcal{F}g(\xi) = \sqrt{z(\xi)}$, so that (3.78) holds true. Finally, by Proposition 3.20, we can express the coefficients $\langle f, T_b g\rangle$ in reconstruction formula (3.82) in terms of $\mathcal{R}^{\mathrm{aff}} f$ only. □

Chapter 4

Wavefront Set Resolution in Shearlet Analysis and the Radon Transform

The use of wavelets in signal analysis and computer vision has proved almost optimal for one-dimensional signals in many ways, and the mathematics behind classical wavelets has reached a high degree of elaboration. One of the main reasons why the wavelet transform is widely exploited in signal analysis is its ability to describe pointwise smoothness properties of univariate functions in terms of the decay behaviour of the wavelet coefficients (we refer to [45, 56] as classical references).

However, when we shift from one-dimensional to multidimensional signals, the wavelet transform has proved not flexible enough to capture the geometry of the singularity set. Indeed, when we handle multidimensional signals, it is not just of interest to locate singularities in space but also to describe how they are distributed (see Figure 4.1). This additional information is expressed by the notion of wavefront set introduced by Hörmander in [43].

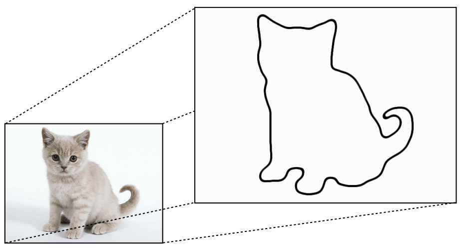

Figure 4.1: The edge detection is a clear and practical example of the interest to capture the geometry of the singularity set of multidimensional signals (see also the website `http://www.shearlab.org/` for further details and examples.)

For this reason a huge class of directional multiscale representations has been introduced over the years to handle high dimensional problems, such as directional wavelets [5], ridgelets [14], curvelets [15], wavelets with composite dilations [36], contourlets [25], shearlets [54], reproducing groups of the symplectic group [33], Gabor ridge functions [33] and mocklets [21] – to name a few. Among them the shearlet representation has gained considerable attention for their capability to resolve the wavefront set of distributions, providing both the location and the geometry of the singularity set. In [51] the authors show that the decay rate of the shearlet coefficients of a signal f with respect to suitable shearlets characterises the wavefront set of f. Precisely, they show that for every signal $f \in L^2(\mathbb{R}^2)$ the shearlet coefficients $\mathcal{S}_\psi f(b, s, a)$ exhibit fast asymptotic decay as $a \to 0$ except when the pair $(b, (\xi_1, \xi_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$, with $\xi_2/\xi_1 = s$, belongs to the wavefront set of f (see Figure 1).

Later this result has been generalised in [34], in which it is shown that the same result can be verified under much weaker assumptions on the continuous shearlets by means of a new approach based on the affine Radon transform. Further results in this line of research are given in [35, 53, 44]. In some sense, shearlets behave for two-dimensional signals as wavelets do for one-dimensional signals, and it is therefore natural to try to understand if the many strong connections are a consequence of some general mathematical principle.

The purpose of this chapter is to show that formula (3.47) provides a new geometric insight on the ability of the shearlet transform to resolve the wavefront set of signals. In particular, we introduce a new approach based on the wavelet transform and on the affine Radon transform to prove that the shearlet transform characterizes the wavefront set of signals.

This new approach clarifies how this ability of the shearlet transform follows directly by the combination of the microlocal properties inhereted by the one-dimensional wavelet transform with a sensitivity for directions inhereted by the Radon transform.

We start with the definition and the basic properties of the wavefront set and we show how it can be described through the affine Radon transform. Then, we state the main results and finally we show the example of the characteristic function of the unit disk.

4.1 The Wavefront Set and the Radon Transform

We recall the definition and the basic properties of the wavefront set. Then, we give an equivalent definition based on the affine Radon transform.

We say that a subset $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ is conic if $x \in \Gamma$ implies that $\lambda x \in \Gamma$ for every $\lambda \in \mathbb{R}^\times$. We denote by $\mathcal{D}'(\mathbb{R}^d)$ the set of all continuous linear functionals on the set of smooth functions of compact support.

Definition 4.1. Let $f \in \mathcal{D}'(\mathbb{R}^d)$. We say that a point $x_0 \in \mathbb{R}^d$ is a regular point of f if there exists a function $\phi \in C_c^\infty(U_{x_0})$, where U_{x_0} is a neighborhood of x_0 and $\phi(x_0) \neq 0$, such that $\phi f \in C_c^\infty(\mathbb{R}^d)$. The complement of the set of regular points of f is called the singular support of f.

Recall that if $f \in \mathcal{D}'(\mathbb{R}^d)$ and ϕ is a smooth cut-off function, then $\phi f \in \mathcal{E}'(\mathbb{R}^d)$, the space of compactly supported distributions. Furthermore, by the Paley-Wiener theorem [64, Chapter 7], the Fourier transform of every compactly supported distribution u is a smooth function and its decay behavior is related to the smoothness of u.

Theorem 4.2 ([64, Chapter 7]). *Let $u \in \mathcal{E}'(\mathbb{R}^d)$. Then $u \in C_c^\infty(\mathbb{R}^d)$ if and only if for every $N \in \mathbb{N}$ there exists a constant C_N such that*

$$|\mathcal{F}u(\xi)| \leq C_N(1 + |\xi|)^{-N} \tag{4.1}$$

for all $\xi \in \mathbb{R}^d$.

By Theorem 4.2, if $u \in \mathcal{E}'(\mathbb{R}^d)$ is not in C^∞, then there exists at least one direction $\xi \neq 0$ such that the Fourier transform of f does not satisfy (4.1) in any conic neighborhood Γ containing ξ. In some sense, we can say that the direction ξ is the reason way u fails to be smooth.

This leads to the concept of regular directed point, which allows to study at the same time the local behavior of a distribution f in a neighborhood of a given point $x_0 \in \mathbb{R}^d$ and the decay behavior of its Fourier transform in a certain direction $\xi_0 \in \mathbb{R}^d \setminus \{0\}$.

Definition 4.3. Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, a point $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ is a regular directed point of f if there exist a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\phi(x_0) \neq 0$ and a conic neighborhood Γ_{ξ_0} of ξ_0 such that, for every $N \in \mathbb{N}$, there exists a constant C_N such that

$$|\mathcal{F}(\phi f)(\xi)| \leq C_N(1 + |\xi|)^{-N} \tag{4.2}$$

for all $\xi \in \Gamma_{\xi_0}$. The wavefront set of $f \in \mathcal{D}'(\mathbb{R}^d)$, denoted by $WF(f)$, is the complement of the set of the regular directed points of f.

By definition, it is clear that $WF(f)$ is a closed subset of $\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ and it is conic in the ξ-variable, i.e., $(x, \xi) \in \mathrm{WF}(f)$ if and only if $(x, \lambda\xi) \in \mathrm{WF}(f)$ for every $\lambda \in \mathbb{R}^\times$. Furthermore, the projection $WF(f) \ni (x, \xi) \mapsto x \in \mathbb{R}^2$ gives the singular support of f. Therefore, through the notion of wavefront set we describe not only where the singularities of f are located but also which directions create the singularities. This additional information allows to capture the geometric features of the singularity set.

Our investigation is based on geometric observations and on the fact that the Fourier slice theorem indicates that the Radon transform is a useful tool in microlocal analysis. We show an equivalent definition of the set of regular directed points, and thus of the wavefront set, based on the affine Radon transform.

First of all, note that any conic subset $\Gamma \subseteq \mathbb{R}^d \setminus \{0\}$, except those which intersect the ξ_2-axis, is uniquely determined by its intersection with the line of equation $\xi_1 = 1$. Precisely, assume that $\xi_0 = (\xi_{0,1}, \xi_{0,2})$, $\xi_{0,1} \neq 0$, is such that $\xi_{0,2}/\xi_{0,1} = s_0$ and denote by Γ_{ξ_0} a conic neighborhood of ξ_0. Then, there exists a neighborhood V_{s_0} of s_0 such that $\xi \in \Gamma_{\xi_0}$ if and only if $\xi = (\lambda, \lambda v)$ for some $\lambda \in \mathbb{R}^\times$ and $v \in V_{s_0}$.

In the following, as we did for the affine Radon transform, we parametrize directions by using affine coordinates and, with slight abuse of notation, each $v \in \mathbb{R}$ denotes the set of directions $\{\xi = (\lambda, \lambda v) \,|\, \lambda \in \mathbb{R}^\times\}$. Thus, with this parametrization, we can reformulate the definition of the wavefront set as follows.

Definition 4.4. Let $f \in \mathcal{D}'(\mathbb{R}^2)$. A point $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point of f if there exist a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ a neighborhood V_{v_0} of v_0 such that, for every $N \in \mathbb{N}$, there exists a constant C_N such that

$$|\mathcal{F}(\phi f)(\xi)| \leq C_N(1 + |\xi|)^{-N} \tag{4.3}$$

for all $\xi = (\xi_1, \xi_2)$ with $\xi_2/\xi_1 \in V_{v_0}$.

The complement of the set of pairs $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ satisfying Definition 4.4 is the intersection of $WF(f)$ with the subset $\{(x, \lambda(1, v)) | x \in \mathbb{R}^2,\, \lambda \in \mathbb{R}^\times\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$, which we denote by $WF_0(f)$. As shown in the next lemma, Definition 4.4 is particularly well-adapted to the mathematical structure of the affine Radon transform.

Lemma 4.5. *Let $f \in \mathcal{D}'(\mathbb{R}^2)$. A point $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point of f if and only if there exist a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ and a neighborhood V_{v_0} of v_0 such that for every $N \in \mathbb{N}$ there exists a constant C_N such that*

$$|\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)| \leq C_N(1+|\tau|)^{-N} \tag{4.4}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$.

Proof. Let $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ be a regular directed point of f. Then, by definition, there exist a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ and a conic neighborhood V_{v_0} of v_0 such that, for every $N \in \mathbb{N}$, there exists a constant C_N with

$$|\mathcal{F}(\phi f)(\tau, \tau v)| \leq C_N(1+|(\tau,\tau v)|)^{-N}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$. By the Fourier Slice Theorem, for every $N \in \mathbb{N}$

$$|\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)| = |\mathcal{F}(\phi f)(\tau,\tau v)| \leq C_N(1+|\tau|\sqrt{1+v^2})^{-N} \leq C_N(1+|\tau|)^{-N}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$, where C_N is a constant independent of v.

Conversely, suppose that there exist a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ and a neighborhood V_{v_0} of v_0 such that, for every $N \in \mathbb{N}$, there exists a constant C_N such that

$$|\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)| \leq C_N(1+|\tau|)^{-N} \tag{4.5}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$. Then, for every $N \in \mathbb{N}$,

$$\begin{aligned} |\mathcal{F}(\phi f)(\tau,\tau v)| &= |\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)| \leq C_N(1+|\tau|)^{-N} \\ &\leq C_N(\sqrt{1+v^2})^N(1+|(\tau,\tau v)|)^{-N} \\ &\leq D_N(1+|(\tau,\tau v)|)^{-N} \end{aligned}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$, where $D_N = C_N \max\{(\sqrt{1+v^2})^N : v \in \overline{V}_{v_0}\}$ is a constant independent of v. Therefore, $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point of f. □

Lemma 4.5 follows by the simple observation that requiring the fast decay of the Fourier transform in an open conic subset Γ is equivalent to require a uniform fast decay of the Fourier transform on any line belonging to Γ. Then, by the Fourier slice theorem, this means to look at the decay behavior of the Fourier transform of the univariate functions $t \mapsto \mathcal{R}^{\mathrm{aff}}\phi f(v,t)$ for any fixed $v \in \Gamma \cap \{(\xi_1, \xi_2) : \xi_1 = 1\}$ (see Figure 4.2).

4.2 The role of the Radon transform in the Wavefront set Resolution in Shearlet Analysis

Lemma 4.5 shows that the wavefront set of a distribution can be characterized in terms of decay properties of the Fourier transform of the univariate functions $t \mapsto \mathcal{R}^{\mathrm{aff}}\phi f(v,t)$, as v varies in a bounded open interval. Therefore, since wavelets characterize pointwise smoothness of univariate functions, the decay behavior of the wavelet coefficients of

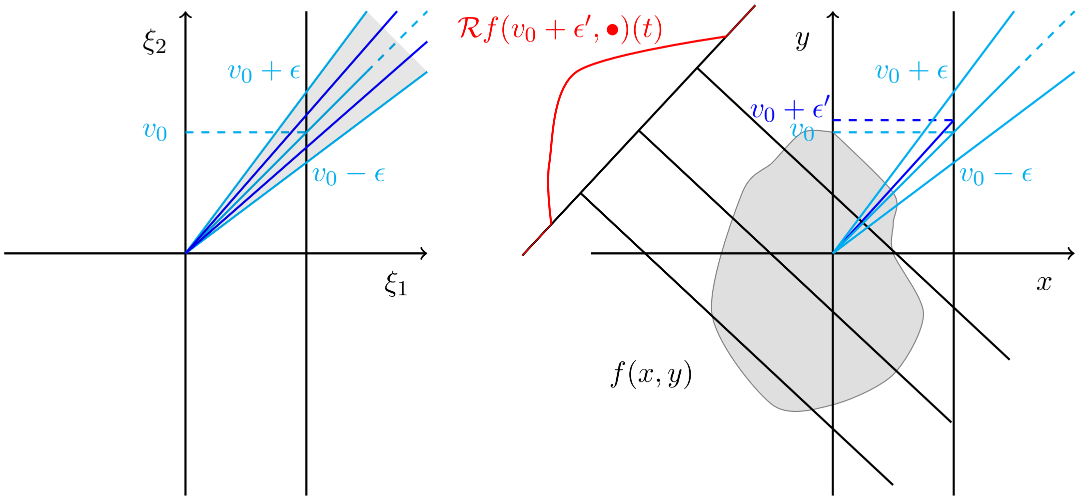

Figure 4.2: Looking at the decay behavior of the Fourier transform in a conic neighborhood V_{v_0} of v_0 is equivalent to look at the decay behavior of the Radon transform as a function of the second variable for any fixed $v \in V_{v_0}$.

the univariate functions $t \mapsto \mathcal{R}^{\text{aff}}\phi f(v,t)$ should give informations about microlocal properties of the signal f. This fact is stated and proved in the following lemma. From now on, we restrict ourselves to $f \in L^2(\mathbb{R}^2)$. The generalization of our following approach to a general distribution $f \in \mathcal{D}'(\mathbb{R}^2)$ is left to future investigation.

Lemma 4.6. *Let $f \in L^2(\mathbb{R}^2)$ and let (x_0, v_0) be a regular directed point of f. Let $\chi \in L^2(\mathbb{R})$ be an admissible wavelet with all vanishing moments and such that $\mathcal{F}\chi \in L^1(\mathbb{R})$. Then, there exist a neighborhood U_{x_0} of x_0, a neighborhood V_{v_0} of v_0 and a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ such that, for every $N > 0$, there exists a constant C_N such that*

$$|\mathcal{W}_\chi(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(b,a)| \leq C_N |a|^N \tag{4.6}$$

for all $v \in V_{v_0}$ and $b \in \mathbb{R}$.

Proof. Assume that $(x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point of $f \in L^2(\mathbb{R}^2)$. By Lemma 4.5, there exist a neighborhood U_{x_0} of x_0, a neighborhood V_{v_0} of v_0 and a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ such that, for every $N \in \mathbb{N}$, there exists a constant C_N with

$$|\mathcal{F}(\mathcal{R}^{\text{aff}}\phi f(v,t))(\tau)| \leq C_N(1+|\tau|)^{-N}$$

for all $v \in V_{v_0}$ and $\tau \in \mathbb{R}^\times$. For every $M > 0$ we have that for every $0 < \alpha < \frac{2M}{2M+1}$

$$\begin{aligned}
&|\mathcal{W}_\chi(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(b,a)| = |\int_{\mathbb{R}} \mathcal{F}(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(\tau)\overline{\mathcal{F}(W_{b,a}\chi)(\tau)}\mathrm{d}\tau| \\
&= |a|^{\frac{1}{2}}|\int_{\mathbb{R}} \mathcal{F}(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(\tau)e^{2\pi i b\tau}\overline{\mathcal{F}\chi(a\tau)}\mathrm{d}\tau| \\
&\leq |a|^{\frac{1}{2}}\int_{\mathbb{R}} |\mathcal{F}(\mathcal{R}^{\text{aff}}\phi f(v,t))(\tau)||\mathcal{F}\chi(a\tau)|\mathrm{d}\tau \\
&= \underbrace{|a|^{\frac{1}{2}}\int_{|\tau|<|a|^{-\alpha}} |\mathcal{F}(\mathcal{R}^{\text{aff}}\phi f(v,t))(\tau)||\mathcal{F}\chi(a\tau)|\mathrm{d}\tau}_{A} +
\end{aligned}$$

$$+\underbrace{|a|^{\frac{1}{2}}\int_{|\tau|>|a|^{-\alpha}}|\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)||\mathcal{F}\chi(a\tau)|\mathrm{d}\tau}_{B}\,.$$

Since χ possesses all vanishing moments, then by [56, Theorem 6.2] there exists $\theta \in L^2(\mathbb{R})$ such that $\mathcal{F}\chi(\tau) = \tau^M\mathcal{F}\theta(\tau)$ and we can estimate A as follows

$$\begin{aligned}
A &= |a|^{\frac{1}{2}}\int_{|\tau|<|a|^{-\alpha}}|\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)||\mathcal{F}\chi_1(a\tau)|\mathrm{d}\tau\\
&= |a|^{\frac{1}{2}}\int_{|\tau|<|a|^{-\alpha}}|a|^M|\tau|^M|\mathcal{F}\theta(a\tau)||\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)|\mathrm{d}\tau\\
&\leq |a|^{\frac{1}{2}}|a|^{M(1-\alpha)}\int_{|\tau|<|a|^{-\alpha}}|\mathcal{F}\theta(a\tau)||\mathcal{F}(\mathcal{R}^{\mathrm{aff}}\phi f(v,t))(\tau)|\mathrm{d}\tau\\
&= |a|^{\frac{1}{2}}|a|^{M(1-\alpha)}\int_{|\tau|<|a|^{-\alpha}}|\mathcal{F}\theta(a\tau)||\mathcal{F}\phi f(\tau(1,v))|\mathrm{d}\tau\\
&\leq |a|^{\frac{1}{2}}|a|^{M(1-\alpha)}\|\mathcal{F}\phi f\|_\infty\int_{|\tau|<|a|^{-\alpha}}|\mathcal{F}\theta(a\tau)|\mathrm{d}\tau\\
&\leq |a|^{M(1-\alpha)}\|\phi f\|_1\int_{|\tau|<|a|^{-\alpha}}|\mathcal{F}(W_{x+vy,a}\theta)(\tau)|\mathrm{d}\tau\\
&\leq \sqrt{2}|a|^{M(1-\alpha)-\frac{\alpha}{2}}\|\phi f\|_1\|\mathcal{F}(W_{x+vy,a}\theta)\|_2\\
&= \sqrt{2}|a|^{M(1-\alpha)-\frac{\alpha}{2}}\|\phi f\|_1\|\theta\|_2.
\end{aligned}$$

We can estimate B as

$$\begin{aligned}
B &\leq C_N|a|^{\frac{1}{2}}\int_{|\tau|>|a|^{-\alpha}}(1+|\tau|)^{-N}|\mathcal{F}\chi(a\tau)|\mathrm{d}\tau\\
&\leq C_N|a|^{\frac{1}{2}}\int_{|\tau|>|a|^{-\alpha}}|\tau|^{-N}|\mathcal{F}\chi(a\tau)|\mathrm{d}\tau\\
&\leq C_N|a|^{\alpha N+\frac{1}{2}}\int_{|\tau|>|a|^{-\alpha}}|\mathcal{F}\chi(a\tau)|\mathrm{d}\tau\\
&\leq C_N|a|^{\alpha N-\frac{1}{2}}\|\mathcal{F}\chi\|_1,
\end{aligned}$$

where we recall that C_N is a constant independent of v. Therefore, for every $M, N > 0$, we have the decay estimate

$$|\mathcal{W}_\chi(\mathcal{R}^{\mathrm{aff}}\phi f(v,\cdot))(b,a)| = O(|a|^{M(1-\alpha)-\frac{\alpha}{2}} + |a|^{\alpha N-\frac{1}{2}}),$$

with the implied constants uniform over $v \in V_{v_0}$ and $b \in \mathbb{R}$. Therefore, since χ has all vanishing moments, for every $N > 0$

$$|\mathcal{W}_\chi(\mathcal{R}^{\mathrm{aff}}\phi f(v,\cdot))(b,a)| = O(|a|^N),$$

with the implied constants uniform over $v \in V_{v_0}$ and $b \in \mathbb{R}$ and we conclude. □

From the above Theorem and formula (3.47) the decay behavior of the shearlet coefficients follows, see Theorem 4.7 below.

We consider an admissible wavelet χ_1 with all vanishing moments and a function $\phi_2 \in L^2(\mathbb{R})$. Then, the function $\psi_1 \in L^2(\mathbb{R})$ defined by

$$\mathcal{F}\psi_1(\tau) = |\tau|^{-1}\mathcal{F}\chi_1(\tau)$$

satisfies the conditions (3.38) and (3.41) and ψ defined by

$$\mathcal{F}\psi(\xi_1, \xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2\left(\frac{\xi_2}{\xi_1}\right), \tag{4.7}$$

where $\mathcal{F}\psi_2 = \phi_2$, satisfies the admissible condition (3.52). Then, by Corollary 3.16, for every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ the shearlet coefficients of f satisfy

$$\mathcal{S}^\gamma_\psi f(b, s, a) = |a|^{\frac{\gamma-2}{2}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v, \cdot))(n(v) \cdot b, a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)} \,\mathrm{d}v,$$

for every $(b, s, a) \in \mathbb{R}^2 \rtimes \mathbb{R} \rtimes \mathbb{R}^\times$.

From now on, we will assume that the admissible vector ψ is of the form (4.7) with $\chi_1 \in \mathcal{S}_0(\mathbb{R})$ and $\phi_2 \in C^\infty_c(\mathbb{R})$. Furthermore, we require that the shearlet ψ is a rapidly decreasing function, i.e.,

$$\psi(x) = O((1 + |x|)^{-N}), \qquad x \in \mathbb{R}^2,$$

for every $N \in \mathbb{N}$.

Theorems 4.7 and 4.10 give new proofs of Theorems 3.1 and 5.5 in [34]. Our approach shows the roles of the Radon transform and the wavelet transform in the resolution of the wavefront set in shearlet analysis.

Theorem 4.7. *Let f be a function in $L^2(\mathbb{R}^2)$ and let (b_0, s_0) be a regular directed point of f. Then there exist neighbourhoods U_{b_0} of b_0 and V_{s_0} of s_0 such that, for every $N > 0$, there exists a constant C_N such that*

$$|\mathcal{S}^\gamma_\psi f(b, s, a)| \le C_N |a|^N, \qquad \textit{as} \quad a \to 0,$$

for all $b \in U_{b_0}$ and $s \in V_{s_0}$.

Proof. Assume that $(b_0, s_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a regular directed point of $f \in L^2(\mathbb{R}^2)$. By Lemma 4.6, there exist a neighborhood U_{b_0} of b_0, a neighborhood V_{s_0} of s_0 and a function $\phi \in C^\infty_c(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ such that, for every $N > 0$, there exists a constant C_N such that

$$|\mathcal{W}_\chi(\mathcal{R}^{\text{aff}} \phi f(v, \cdot))(b, a)| \le C_N |a|^N \tag{4.8}$$

for all $v \in V_{s_0}$ and $b \in \mathbb{R}$.

First of all we show that, since ψ is assumed to be a rapidly decreasing function, for every $N > 0$ there exists a constant C_N such that

$$|\mathcal{S}^\gamma_\psi((1 - \phi)f)(b, s, a)| \le C_N |a|^{-N}, \qquad \text{as} \quad a \to 0,$$

for every $b \in U_{b_0}$ and $s \in V_{s_0}$ and it will be sufficient to study the behavior of the shearlet coefficients of the localized version ϕf of f. The proof is given in [34] but we show it for the reader's convenience.

By assumption, for every $N > 0$ there exists a constant C_N such that

$$|\psi(x)| \le C_N (1 + |x|)^{-N}, \qquad x \in \mathbb{R}^2,$$

and we estimate

$$|S^\gamma_{b,s,a}\psi(x)| = |a|^{-\frac{1+\gamma}{2}}|\psi(A_a^{-1}S_s^{-1}(x-b))| \leq C_N|a|^{-\frac{1+\gamma}{2}}(1+|A_a^{-1}S_s^{-1}(x-b)|)^{-N}.$$

Observe that, for every $x \in \mathbb{R}^d$ and $M \in GL(d,\mathbb{R})$, $|x| = |MM^{-1}x| \leq \|M\||M^{-1}x|$, where $\|M\|$ denotes the spectral norm of the matrix M. Thus,

$$|M^{-1}x| \geq \|M\|^{-1}|x|.$$

Since $\|A_a\| = |a|^\gamma$ for $|a| < 1$ and $\|S_s\| = (1+\frac{s^2}{2}+(s^2+\frac{s^2}{2})^{1/2})^{1/2}$ for all $s \in \mathbb{R}$, then

$$\begin{aligned}|S^\gamma_{b,s,a}\psi(x)| &\leq C_N|a|^{-\frac{1+\gamma}{2}}(1+\|S_s\|^{-1}|a|^{-\gamma}|x-b|)^{-N}\\ &\leq |a|^{-\frac{1+\gamma}{2}+\gamma N}\|S_s\|^N(\|S_s\||a|^\gamma+|x-b|)^{-N}\end{aligned}$$

and we can estimate the shearlet transform as follows

$$\begin{aligned}|\mathcal{S}^\gamma_\psi((1-\phi)f)(b,s,a)| &= \int_{\mathbb{R}^2}|1-\phi(x)||f(x)||S^\gamma_{b,s,a}\psi(x)|\mathrm{d}x\\ \leq C_N|a|^{-\frac{1+\gamma}{2}+\gamma N}\|S_s\|^N &\int_{\mathbb{R}^2}|1-\phi(x)||f(x)|(\|S_s\|a^\gamma+|x-b|)^{-N}\mathrm{d}x.\end{aligned}$$

Without loss of generality, we can assume that the cut-off function ϕ satisfies $\phi \equiv 1$ on U_{b_0}. Then, for any $b \in U_{b_0}$, we can assume that $|x-b| > \delta$ for some $\delta > 0$ and we have

$$\begin{aligned}|\mathcal{S}^\gamma_\psi((1-\phi)f)(b,s,a)| &\leq C_N|a|^{-\frac{1+\gamma}{2}+\gamma N}\|S_s\|^N\int_{|x-b|>\delta}|1-\phi(x)||f(x)||x-b|^{-N}\mathrm{d}x\\ &\leq C|a|^{-\frac{1+\gamma}{2}+\gamma N}\end{aligned}$$

for every $b \in U_{b_0}$ and $s \in V_{s_0}$, where C is a constant independent of b and v, and this concludes the first part of the proof.

Now, we go on to estimate the shearlet coefficients $|\mathcal{S}^\gamma_\psi\phi f(b,s,a)|$. By (3.47), we have that

$$|\mathcal{S}^\gamma_\psi\phi f(b,s,a)| \leq |a|^{\frac{\gamma-2}{2}}\int_{\mathbb{R}}|\mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(n(v)\cdot b,a)||\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)|\,\mathrm{d}v \tag{4.9}$$

for every $b \in U_{b_0}$ and $s \in V_{s_0}$. Without loss of generality, we can assume that $\text{supp}\phi_2 \subseteq [-1,1]$. Then, the integral in (4.9) reduces to the interval

$$I_s = (s-|a|^{1-\gamma}, s+|a|^{1-\gamma})$$

and for a sufficiently small $I_s \subseteq V_{s_0}$. Therefore, by (4.8) for every $N > 0$ there exists a constant C_N such that

$$\begin{aligned}|\mathcal{S}^\gamma_\psi\phi f(b,s,a)| &\leq |a|^{\frac{\gamma-2}{2}}\int_{V_{s_0}}|\mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}}\phi f(v,\cdot))(n(v)\cdot b,a)||\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)|\,\mathrm{d}v\\ &\leq C_N|a|^{N+\frac{\gamma-2}{2}}\int_{V_{s_0}}|\phi_2\left(\frac{v-s}{|a|^{1/2}}\right)|\,\mathrm{d}v \leq C_N\sqrt{2}|a|^{N-\frac{\gamma}{2}}\|\phi_2\|_2\end{aligned}$$

for any $b \in U_{b_0}$ and $s \in V_{s_0}$ and we conclude. □

The novelty of this proof is to show how the decay behavior of the shearlet transform around a regular directed point (b_0, s_0) follows directly by the decay properties of the wavelet coefficients of the univariate functions $t \mapsto \mathcal{R}^{\mathrm{aff}}\phi f(v,t)$, where ϕ is a cut-off function around b_0 and v is in a neighborhood of s_0 (see Lemma 4.6), or equivalently by the smoothness properties of such functions (see Lemma 4.5). This aspect of our approach will be further invastigated and clarified in the last section with the example of the unit disc.

In order to prove the converse of Theorem 4.7 we need an equivalent definition of directional regular point. We start giving an equivalent definition to Definition 4.3.

Definition 4.8. Let $f \in \mathcal{D}'(\mathbb{R}^2)$. A point $(x_0, \xi_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$ is a regular directed point of f if there exists a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ and a conic neighborhood Γ_{ξ_0} of ξ_0 such that, for every $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|\mathcal{F}(\phi f)(\xi)| \leq C_N |\xi|^{-N} \tag{4.10}$$

for all $\xi \in \Gamma^*_{\xi_0} = \Gamma_{\xi_0} \cap \overline{B(0,1)}^c$, with $B(0,1)$ the ball of centre 0 and radius 1.

Since $\mathcal{F}(\phi f)$ is continuous, equation (4.10) is equivalent to require that the function $\xi \mapsto |\xi|^N \mathcal{F}(\phi f)(\xi)$ belongs to $L^\infty(\Gamma^*_{\xi_0})$, which is equivalent to the fact that the functional

$$\varphi \mapsto \int_{\Gamma^*_{\xi_0}} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi$$

is continuous on $L^1(\Gamma^*_{\xi_0})$. We can thus rewrite the definition of regular directional point as follows.

Definition 4.9. Let $f \in \mathcal{D}'(\mathbb{R}^2)$. A point $(x_0, \xi_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$ is a regular directed point of f if there exists a neighborhood U_{x_0} of x_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ and a conic neighborhood Γ_{ξ_0} of ξ_0 such that, for every $N \in \mathbb{N}$, the functional

$$\varphi \mapsto \int_{\Gamma^*_{\xi_0}} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi$$

is continuous on $L^1(\Gamma^*_{\xi_0})$, i.e.,

$$|\int_{\Gamma^*_{\xi_0}} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi| \leq C_N \|\varphi\|_{L^1(\Gamma^*_{\xi_0})}, \tag{4.11}$$

for every $\varphi \in L^1(\Gamma^*_{\xi_0})$.

If $f \in L^2(\mathbb{R}^2)$ and $\varphi \in C_c(\Gamma^*_{\xi_0})$, we can read the integral in (4.11) as a scalar product in $L^2(\mathbb{R}^2)$ and we have that

$$\begin{aligned}\int_{\Gamma^*_{\xi_0}} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi &= \int_{\mathbb{R}^2} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi \\ &= \langle \mathcal{F}(\phi f), |\cdot|^N \overline{\varphi}\rangle \\ &= \langle \phi f, \mathcal{F}^{-1}|\cdot|^N \overline{\varphi}\rangle.\end{aligned}$$

Given an admissible vector ψ, since the shearlet transform is an isometry from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2 \rtimes \mathbb{R} \rtimes \mathbb{R}^\times)$, we can continue the previous chain of equalities as follows

$$\begin{aligned}\int_{\Gamma^*_{\xi_0}} |\xi|^N \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi &= \langle \mathcal{S}^\gamma_\psi \phi f, \mathcal{S}^\gamma_\psi(\mathcal{F}^{-1}|\cdot|^N\overline{\varphi})\rangle \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^\times} \mathcal{S}^\gamma_\psi \phi f(b,s,a)\overline{\mathcal{S}^\gamma_\psi(\mathcal{F}^{-1}|\cdot|^N\overline{\varphi})(b,s,a)}\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}.\end{aligned}$$

We now focus our attention on the shearlet transform of the function $\mathcal{F}^{-1}|\cdot|^N\overline{\varphi}$. We can assume without loss of generality that N is even. By formula (3.47) and by Proposition 1.44 together with relation (3.21), we have that

$$\begin{aligned}&\mathcal{S}^\gamma_\psi(\mathcal{F}^{-1}(\cdot)^N\overline{\varphi})(b,s,a) \\ &= |a|^{\frac{\gamma-2}{2}}\int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}}(\mathcal{F}^{-1}(\cdot)^N\overline{\varphi})(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v \\ &= |a|^{\frac{\gamma-2}{2}}\int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}}((-\Delta)^{\frac{N}{2}}\mathcal{F}^{-1}\overline{\varphi})(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v \\ &= |a|^{\frac{\gamma-2}{2}}\int_{\mathbb{R}} (1+v^2)^{\frac{N}{2}}\mathcal{W}_{\chi_1}((-\Lambda_t)^{\frac{N}{2}}\mathcal{R}^{\mathrm{aff}}(\mathcal{F}^{-1}\overline{\varphi})(v,\cdot))(n(v)\cdot b,a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v \\ &= |a|^{\frac{\gamma-2}{2}-N}\int_{\mathbb{R}^2} (1+v^2)^{\frac{N}{2}}\overline{\varphi(\tau n(v))}\overline{\mathcal{F}W_{n(v)\cdot b,a}(\mathcal{I}_0^{2N}\chi_1)(\tau)}\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}\tau\mathrm{d}v \\ &= |a|^{\frac{\gamma-1}{2}-N}\int_{\mathbb{R}^2} (1+v^2)^{\frac{N}{2}}\overline{\varphi(\tau n(v))}e^{2\pi i(n(v)\cdot b)\tau}\overline{\mathcal{F}(\tilde{\chi}_1)(a\tau)}\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}\tau\mathrm{d}v, \qquad (4.12)\end{aligned}$$

where $\tilde{\chi}_1 = \mathcal{I}_0^{2N}\chi_1$ with $\mathcal{I}_0$ defined by (3.28). Observe that if $\chi_1 \in \mathcal{S}_0(\mathbb{R})$, then also $\tilde{\chi}_1 \in \mathcal{S}_0(\mathbb{R})$.

By (4.12), we have the chain of inequalitis

$$\begin{aligned}&|\mathcal{S}^\gamma_\psi(\mathcal{F}^{-1}(\cdot)^N\overline{\varphi})(b,s,a)| \\ &\leq |a|^{\frac{\gamma-1}{2}-N}\int_{\mathbb{R}^2}(1+v^2)^{\frac{N}{2}}|\varphi(\tau n(v))||\mathcal{F}(\tilde{\chi}_1)(a\tau)||\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)|\,\mathrm{d}\tau\mathrm{d}v \\ &= |a|^{\frac{\gamma-1}{2}-N}\int_{\mathbb{R}^2}(1+(\frac{\xi_2}{\xi_1})^2)^{\frac{N}{2}}|\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)||\xi_1|^{-1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &\leq C|a|^{\frac{\gamma-1}{2}-N}\int_{\Gamma^*_{\xi_0}}|\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)|\,\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned}$$

with C a constant dependent only on N and $\Gamma^*_{\xi_0}$. Hence, we have that

$$\begin{aligned}|\int_{\Gamma^*_{\xi_0}} \xi^N\mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi| \leq C\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^\times}|\mathcal{S}^\gamma_\psi\phi f(b,s,a)||a|^{\frac{\gamma-1}{2}-N}\times \qquad (4.13)\\ \times\int_{\Gamma^*_{\xi_0}}|\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)|\,\mathrm{d}\xi_1\mathrm{d}\xi_2\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}.\end{aligned}$$

It is worth observing how the combination of the wavelet transform with the affine Radon transform makes appear a negative power of $|a|$. It is therefore clear that if we want to control (4.13), then we need some fast decaying hypothesis on the shearlet transform $\mathcal{S}_\psi^\gamma \phi f(b,s,a)$ when $|a| \to 0$. This is the idea under the proof of the next theorem. As well as in Theorem 4.7, the roles of the wavelet and the Radon transforms show up clearly.

Theorem 4.10. *Let f be a function in $L^2(\mathbb{R}^2)$. Suppose that there exist a neighborhood U_{b_0} of b_0 and a neighborhood V_{s_0} of s_0 such that, for every $N>0$, there exists a constant C_N such that*

$$|\mathcal{S}_\psi^\gamma f(b,s,a)| \leq C_N |a|^N$$

for every $b \in U_{b_0}$ and $s \in V_{s_0}$. Then (b_0, s_0) is a regular directed point for f.

The proof of Theorem 4.10 adapts several ideas from [51] and [34].

Proof. By Definition 4.9, we have to prove that there exists a neighborhood U'_{b_0} of b_0, a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$, a neighborhood V'_{s_0} of s_0 such that, for every $L \in \mathbb{N}$, the functional

$$\varphi \mapsto \int_{\Gamma^*_{s_0}} |\xi|^L \mathcal{F}(\phi f)(\xi)\varphi(\xi) \mathrm{d}\xi$$

is continuous on $L^1(\Gamma^*_{s_0})$, where $\Gamma^*_{s_0} = \Gamma_{s_0} \cap \overline{B(0,1)}^c$, with Γ_{s_0} the cone parametrized by the interval V'_{s_0}. This is equivalent to show that for every $L \in \mathbb{N}$

$$|\int_{\Gamma^*_{s_0}} (\xi)^{2L} \mathcal{F}(\phi f)(\xi)\varphi(\xi)\mathrm{d}\xi| \leq C_L \|\varphi\|_{L^1(\Gamma^*_{s_0})}, \tag{4.14}$$

for every $\varphi \in C_c(\Gamma^*_{s_0})$. We consider $V'_{s_0} = (s_0 - \epsilon', s_0 + \epsilon') \subset\subset V_{s_0}$, for some $\epsilon' > 0$, and a function $\phi \in C_c^\infty(\mathbb{R}^2)$ such that supp$\phi \subset\subset U'_{b_0} \subset\subset U_{b_0}$. With the notation $W \subset\subset W'$ we mean that there exists $R > 0$ such that a tube of diameter R around W is still contained in W'. Then, by Lemma 5.6 and 5.7 in [51], we have that for every $N > 0$ there exists a constant C_N such that

$$|\mathcal{S}_\psi^\gamma \phi f(b,s,a)| \leq C_N |a|^N$$

for every $b \in U'_{b_0}$ and $s \in V_{s_0}$. We refer the reader to the proof of Theorem 5.1 in [51]. Some comments are in order. Results in [51] are given under a very specific choice of the admissible vector ψ. Precisely, $\psi \in L^2(\mathbb{R}^2)$ is of the form (4.7) where $\psi_1 \in L^2(\mathbb{R})$ is a wavelet, $\mathcal{F}\psi_1 \in C_0^\infty(\mathbb{R})$ and supp $\mathcal{F}\psi_1 \subseteq [-2,-1/2] \cup [1/2, 2]$. Futhermore, $\|\psi_2\|_2 = 1$, $\mathcal{F}\psi_2 \in C_0^\infty(\mathbb{R})$ with supp $\mathcal{F}\psi_2 \subseteq [-1,1]$ and $\mathcal{F}\psi_2 > 0$ on (-1,1). This particular choice of ψ is covered by our hypothesis on the admissible vector. It is worth observing that these stronger assumptions on ψ enter only in this first part of the proof. The rest of the proof holds true without these assumptions and for this reason in the following we will never use them. The generalization of Lemma 5.6 and 5.7 in [51] is left to future investigation.

By equation (4.13), it is enough to prove that for every $L > 0$ there exists $C_L > 0$ such that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^\times} |\mathcal{S}_\psi^\gamma \phi f(b,s,a)||a|^{\frac{\gamma-1}{2}-L} \times$$
$$\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}\xi_1\mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \leq C_L \|\varphi\|_{L^1(\Gamma^*_{s_0})},$$

for every $\varphi \in C_c(\Gamma^*_{s_0})$. We show the case $|a| < 1$, the proof of the case $|a| > 1$ is very similar. We divide the above integral in four parts. For simplicity, in the following proof C is a generic positive constant and may vary from expression to expression. By the uniform fast decay of the shearlet transform for $b \in U'_{b_0}$ and $s \in V_{s_0} = (s_0 - \epsilon, s_0 + \epsilon)$, with $\epsilon > \epsilon'$, we have that

$$\int_{U'_{b_0}} \int_{s_0-\epsilon}^{s_0+\epsilon} \int_{|a|<1} |\mathcal{S}^\gamma_\psi \phi f(b,s,a)||a|^{\frac{\gamma-1}{2}-L} \times$$

$$\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}$$

$$\leq C \int_{U'_{b_0}} \int_{s_0-\epsilon}^{s_0+\epsilon} \int_{|a|<1} |a|^{N+\frac{\gamma-1}{2}-L} \times$$

$$\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}$$

$$\leq C \int_{U'_{b_0}} \int_{s_0-\epsilon}^{s_0+\epsilon} \int_{|a|<1} |a|^{N+\frac{\gamma-1}{2}-L-3} \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)| \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}b\mathrm{d}s\mathrm{d}a \leq C\|\varphi\|_{L^1(\Gamma^*_{s_0})},$$

where we have used the boundedness of $\tilde{\chi}_1$ and ϕ_2. The last inequality follows unless we choose N such that $N > -\frac{\gamma-1}{2} + L + 2$, which is always possible since we can choose N arbitrarily large. We now treat the case $b \in U'_{b_0}$, $|s - s_0| > \epsilon$ and $|a| < 1$ and we use the fact that $\tilde{\chi_1}$ possesses all vanishing moments and $\phi_2 \in C^\infty_c(\mathbb{R})$. Without loss of generality, we assume that $\mathrm{supp}\phi_2 \subseteq [-1,1]$. By formula (3.47), we estimate the shearlet transform as follows

$$|\mathcal{S}^\gamma_\psi \phi f(b,s,a)| \leq |a|^{\frac{\gamma-2}{2}} \int_{\mathbb{R}} |\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}}\phi f(v,\cdot))(n(v)\cdot b,a)||\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}v \tag{4.15}$$

$$\leq |a|^{\frac{\gamma-3}{2}} \|\mathcal{F}\phi f\|_\infty \|\mathcal{F}\chi_1\|_1 \int_{\mathbb{R}} |\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}v \tag{4.16}$$

$$\leq |a|^{\frac{\gamma-3}{2}} \|\mathcal{F}\phi f\|_\infty \|\mathcal{F}\chi_1\|_1 \int_{s-|a|^{1-\gamma}}^{s+|a|^{1-\gamma}} |\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}v \tag{4.17}$$

$$\leq \sqrt{2}|a|^{-\frac{\gamma+1}{2}} \|\mathcal{F}\phi f\|_\infty \|\mathcal{F}\chi_1\|_1 \|\phi_2\|_2. \tag{4.18}$$

Then, we have the following chain of inequalities

$$\int_{U'_{b_0}} \int_{|s-s_0|>\epsilon} \int_{|a|<1} |\mathcal{S}^\gamma_\psi \phi f(b,s,a)||a|^{\frac{\gamma-1}{2}-L} \times$$

$$\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1}-s}{|a|^{1-\gamma}}\right)| \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}$$

$$\leq C \int_{U'_{b_0}} \int_{|s-s_0|>\epsilon} \int_{|a|<1} |a|^{M-1-L-L_2(1-\gamma)} \times$$

$$\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}\theta(a\xi_1)||\xi_1|^M |\frac{\xi_2}{\xi_1} - s|^{-L_2} \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}$$

$$\leq C \int_{|s-s_0|>\epsilon} ||s-s_0| - \epsilon'|^{-L_2} \int_{|a|<1} |a|^{M-4-L-L_2(1-\gamma)} \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)| \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}b\mathrm{d}s\mathrm{d}a, \tag{4.19}$$

where, by Lemma 1.50, $\theta \in \mathcal{S}_0(\mathbb{R})$ is such that $\mathcal{F}\tilde{\chi_1}(\xi_1) = \xi_1^M \mathcal{F}\theta(\xi_1)$, $\xi_1 \in \mathbb{R}$. The last inequality follows since $|\xi_2/\xi_1 - s|^{-L_2} \geq ||\xi_2/\xi_1 - s_0| - |s - s_0||^{-L_2} \geq ||s - s_0| - \epsilon'|^{-L_2}$ due to the fact that $|\xi_2/\xi_1 - s_0| < \epsilon' < \epsilon$. It is now easy to verify that (4.19) can be estimated, up to a constant, by $\|\varphi\|_{L^1(\Gamma^*_{s_0})}$ unless we choose $L_2 > 1$ and $M > 1 + L + L_2(1 - \gamma)$, which is always possible since we can choose M arbitrarily large. Finally, we treat the case $b \in {U'_{b_0}}^c$, $|s - s_0| < \epsilon$ and $|a| < 1$. The case $b \in {U'_{b_0}}^c$, $|s - s_0| > \epsilon$ and $|a| < 1$ can be treated exploiting estimations analogues to the previous cases and for this reason we omit it. We have that

$$\begin{aligned}
&\int_{{U'_{b_0}}^c} \int_{|s-s_0|<\epsilon} \int_{|a|<1} |\mathcal{S}^\gamma_\psi \phi f(b,s,a)||a|^{\frac{\gamma-1}{2}-L} \times \\
&\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1} - s}{|a|^{1-\gamma}}\right)| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \\
&\leq \int_{{U'_{b_0}}^c} \int_{|s-s_0|<\epsilon} \int_{|a|<1} |a|^{\frac{\gamma-1}{2}-L} \int_{\mathrm{supp}\phi} |\phi f(x)||S^\gamma_{b,s,a}\psi(x)| \mathrm{d}x \times \\
&\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1} - s}{|a|^{1-\gamma}}\right)| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \\
&\leq C \int_{{U'_{b_0}}^c} \int_{|s-s_0|<\epsilon} \int_{|a|<1} |a|^{-1-L+\gamma N} \int_{\mathrm{supp}\phi} |\phi f(x)||x-b|^{-N} \mathrm{d}x \times \\
&\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1} - s}{|a|^{1-\gamma}}\right)| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3},
\end{aligned}$$

since for every $N > 0$ we can estimate $|S_{b,s,a}\psi(x)| \leq C|a|^{-\frac{1+\gamma}{2}+\gamma N}|x - b|^{-N}$ as in Theorem 4.7. Then, since $b \in {U'_{b_0}}^c$ and $x \in \mathrm{supp}\phi \subset\subset U'_{b_0}$, we have that $|x - b| \geq C|b_0 - b|$ and we continue the chain of inequalities as follows

$$\begin{aligned}
&\int_{{U'_{b_0}}^c} \int_{|s-s_0|<\epsilon} \int_{|a|<1} |a|^{-1-L+\gamma N} \int_{\mathrm{supp}\phi} |\phi f(x)||x-b|^{-N} \mathrm{d}x \times \\
&\times \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)||\mathcal{F}(\tilde{\chi}_1)(a\xi_1)||\phi_2\left(\frac{\frac{\xi_2}{\xi_1} - s}{|a|^{1-\gamma}}\right)| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3} \\
&\leq C \int_{{U'_{b_0}}^c} \int_{|s-s_0|<\epsilon} \int_{|a|<1} |a|^{-4-L+\gamma N} \|\phi f\|_1 |b_0 - b|^{-N} \int_{\Gamma^*_{\xi_0}} |\varphi(\xi_1,\xi_2)| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}b\mathrm{d}s\mathrm{d}a,
\end{aligned}$$

which can be estimated, up to a constant, by $\|\varphi\|_{L^1(\Gamma^*_{s_0})}$ unless we choose N sufficiently large and this concludes our proof. □

4.3 A Leading Example: the Unit Disk

In this section we present the example of the characteristic function of the unit disc, namely

$$f(x,y) = \frac{1}{2} \begin{cases} 1 & x^2 + y^2 \leq 1 \\ 0 & x^2 + y^2 > 1 \end{cases},$$

depicted in Fig. 4.3 (the factor 1/2 is introduced to simplify subsequent computations).

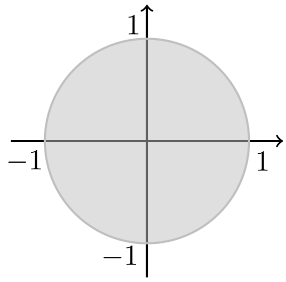

Figure 4.3: Characteristic function on the unit disc.

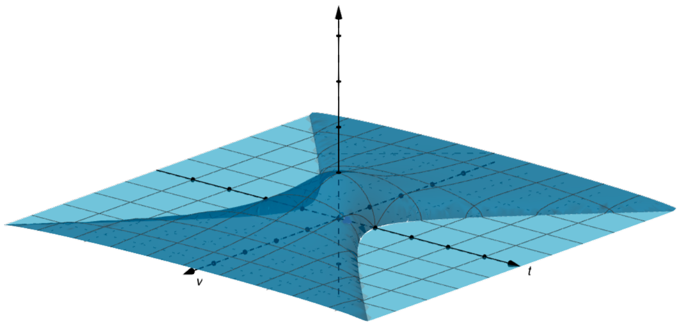

Figure 4.4: The affine Radon transform of the characteristic function of the unit disc.

The next Proposition describes $WF(g)$ when g is the characteristic function of a region in the plane with smooth boundary.

Proposition 4.11 ([43])**.** *Let D be a region in $\mathbb{R}^2$ whose boundary ∂D is smooth and let g be the characteristic function of D. Then*

$$WF(g) = \{(x, \xi) \mid x \in \partial D,\, \xi \text{ perpendicular to } \partial D \text{ at } x\}.$$

By Proposition 4.11, the wavefront set of f is

$$WF(f) = \{\big(\cos\theta, \sin\theta, \lambda\cos\theta, \lambda\sin\theta\big) \mid \theta \in (-\pi, \pi],\, \lambda \in \mathbb{R}^\times\}. \tag{4.20}$$

By a direct computation, the affine Radon transform of f is

$$\mathcal{R}^{\text{aff}} f(v, t) = \begin{cases} \frac{\sqrt{1+v^2-t^2}}{1+v^2} & t^2 - v^2 \le 1 \\ 0 & t^2 - v^2 > 1, \end{cases} \tag{4.21}$$

depicted in Figure 4.4. As a consequence of the rotational invariance of f and of relation (3.21), it holds that

$$\begin{aligned} \mathcal{R}^{\text{aff}} f(v, t) &= \frac{1}{\sqrt{1+v^2}} \mathcal{R}^{\text{pol}} f\Big(\theta_v, \frac{t}{\sqrt{1+v^2}}\Big) \\ &= \frac{1}{\sqrt{1+v^2}} \mathcal{R}^{\text{pol}} f\Big(0, \frac{t}{\sqrt{1+v^2}}\Big) \\ &= \frac{1}{\sqrt[4]{1+v^2}} (W_{0,\sqrt{1+v^2}}\varphi)(t), \end{aligned} \tag{4.22}$$

where $\theta_v = \arctan v$ and

$$\varphi(t) = \begin{cases} \sqrt{1-t^2} & |t| \leq 1 \\ 0 & |t| > 1 \end{cases},$$

which is depicted in Fig. 4.5.

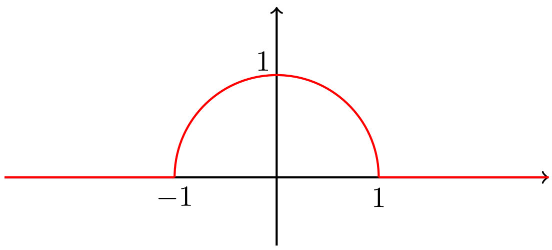

Figure 4.5: $\mathcal{R}f(0,t) = \varphi(t)$.

By formula (3.47) and (4.22) and by the fact that the wavelet transform commutes with dilations, we obtain

$$\mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(b,a) = \frac{1}{\sqrt[4]{1+v^2}}(\mathcal{W}_{\chi_1}\varphi)\,(\frac{b}{\sqrt{1+v^2}}, \frac{a}{\sqrt{1+v^2}}). \tag{4.23}$$

The wavelet transform of φ is depicted in Fig. 4.6 and, as expected, the wavelet coeffi-

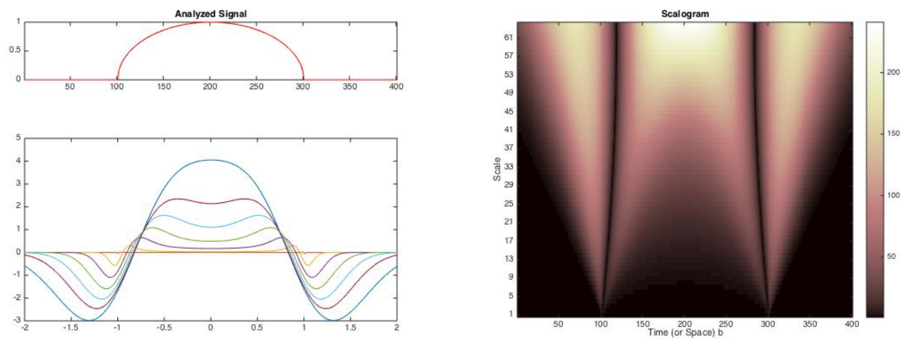

Figure 4.6: The wavelet coefficients at different scales.

cients $\mathcal{W}_{\chi_1}\varphi(b,a)$ of φ slowly decrease if and only if $b = \pm 1$ when a goes to zero. Recall that, if χ_1 has compact support equal to $[-1,1]$, the singularities in $t = \pm 1$ create cones of influence in the scale-space plane defined by

$$|b \mp 1| \leq a,$$

to which the high amplitude wavelet coefficients belong, i.e. the wavelet coefficients which don't exhibit rapid asymptotic decay at $b = \pm 1$, as $a \to 0$. The cones of influence created by the singular points $t = \pm 1$ are clearly visible in Fig. 4.6 (see Chapter 6 in [56] for further details). We now show that this behaviour implies the ability of the shearlet transform to correctly detect the wavefront set of the distribution f.

We now parametrise the directions by using affine coordinates, so that (4.20) reads

$$WF(f)_0 = \{(\cos\theta, \sin\theta, \tan\theta) \mid \theta \in (-\pi,\pi], \theta \neq \pm\frac{\pi}{2}\}.$$

We recall that $WF(f)_0$ is the wavefront set of f whose singular support does not intersect the vertical axis, i.e. the points $(0, \pm 1, 0, \lambda)$ and, with slight abuse of notation, each $s \in \mathbb{R}$ denotes the set of directions $\{\lambda(1, s) \mid \lambda \in \mathbb{R}^\times\}$.

We start by considering a point $(b_0, s_0) \notin WF(f)_0$, i.e. a regular directed point. By Lemma 4.6, if $\chi \in L^2(\mathbb{R})$ is an admissible wavelet with all vanishing moments and such that $\mathcal{F}\chi \in L^1(\mathbb{R})$, then there exist a neighborhood U_{b_0} of b_0, a neighborhood V_{s_0} of s_0 and a function $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\phi(x_0) \neq 0$ such that

$$\mathcal{W}_\chi(\mathcal{R}^{\mathrm{aff}} \phi f(v, \cdot))(b, a) = O(|a|^N),$$

for all $N > 0$, and for all $b \in \mathbb{R}$ and $v \in V_{s_0}$, with the implied constants uniform over $\mathbb{R}$ and V_{s_0}.

Therefore, by Theorem 4.7, we obtain the decay estimate

$$\mathcal{S}_\psi f(b, s, a) = O(|a|^N), \quad a \to 0,$$

for all $N > 0$, uniformly over V_{s_0} and U_{b_0}.

Now, take a point in the wavefront set $(\cos\theta_s, \sin\theta_s, s)$, with $s = \tan\theta_s$. For simplicity and without loss of generality, we can consider the point $(1, 0, 0) \in WF_0(f)$, that is the point on the boundary $(1, 0)$ together with a direction $\lambda(1, 0)$, $\lambda > 0$, perpendicular to the boundary at $(1, 0)$. We fix ϕ_2 to be the characteristic function of the interval $[-1, 1]$ and we assume that $\mathrm{supp}\chi_1 \subseteq [-1, 1]$. From (4.23), we obtain

$$\begin{aligned}\left|\mathcal{S}_\psi^\gamma f(1, 0, 0, a)\right| &= |a|^{\frac{\gamma-2}{2}} \Big| \int_{-|a|^{1-\gamma}}^{|a|^{1-\gamma}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v, \cdot))(1, a)\, \mathrm{d}v \Big| \\ &= |a|^{\frac{\gamma-2}{2}} \Big| \int_{-|a|^{1-\gamma}}^{|a|^{1-\gamma}} \mathcal{W}_{\chi_1}\varphi\Big(\frac{1}{\sqrt{1+v^2}}, \frac{a}{\sqrt{1+v^2}}\Big)\, \frac{\mathrm{d}v}{\sqrt[4]{1+v^2}} \Big|. \end{aligned} \tag{4.24}$$

By the mean value theorem, there exists $\overline{v} \in [-|a|^{1-\gamma}, |a|^{1-\gamma}]$ such that equation (4.24) is equal to

$$2|a|^{-\frac{\gamma}{2}} \Big| \frac{1}{\sqrt[4]{1+\overline{v}^2}} \mathcal{W}_{\chi_1}\varphi\Big(\frac{1}{\sqrt{1+\overline{v}^2}}, \frac{a}{\sqrt{1+\overline{v}^2}}\Big) \Big| \geq \frac{1}{\sqrt[4]{1+\overline{v}^2}} \Big| \mathcal{W}_{\chi_1}\varphi\Big(\frac{1}{\sqrt{1+\overline{v}^2}}, \frac{a}{\sqrt{1+\overline{v}^2}}\Big) \Big|, \tag{4.25}$$

where we are assuming $|a| \leq 1$. The last assumption is not restrictive since we are interested in the behaviour of (4.25) when a goes to zero. Observe that $1/\sqrt{1+\overline{v}^2} \to 1$ as $a \to 0$ and that $t = 1$ belongs to the singular support of φ. Therefore, in order to prove that the wavelet coefficients in (4.25) don't exhibit fast asymptotic decay as $a \to 0$, we have to verify that if $\overline{v} \in [-|a|^{1-\gamma}, |a|^{1-\gamma}]$ then $1/\sqrt{1+\overline{v}^2}$ belongs to the cone of influence created by the singularity $t = 1$, i.e., the inequality

$$\Big|1 - \frac{1}{\sqrt{1+\overline{v}^2}}\Big| \leq \frac{|a|}{\sqrt{1+\overline{v}^2}} \tag{4.26}$$

holds true when $a \to 0$. By direct computation, (4.26) is equivalent to

$$-\sqrt{2|a| + a^2} \leq \overline{v} \leq \sqrt{2|a| + a^2}.$$

Therefore, inequality (4.26) is satisfied if and only if the cone of directions $-|a|^{1-\gamma} \leq v \leq |a|^{1-\gamma}$ falls within the cone $-\sqrt{2|a| + a^2} \leq v \leq \sqrt{2|a| + a^2}$. It is immediate to see

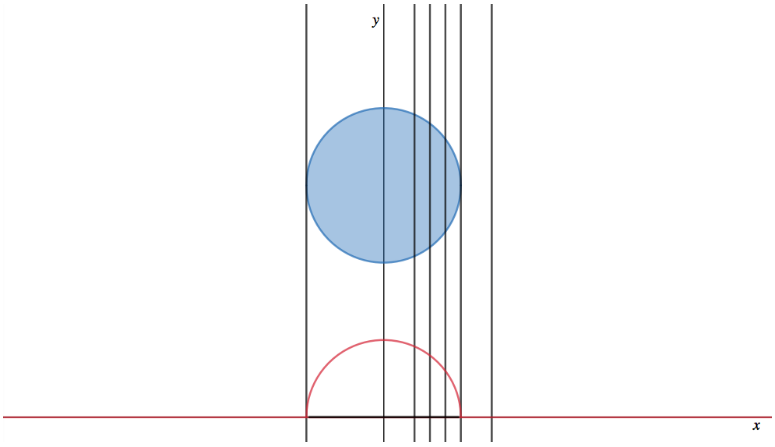

Figure 4.7: The function $\varphi(t) = \mathcal{R}^{\mathrm{aff}} f(0,t)$ is smooth except at $t = \pm 1$ corresponding to the lines parametrized by the pairs $(0, \pm 1)$. These are the lines tangent to the boundaries of the characteristic function f.

that this happens when $|a|^{1-\gamma} \leq \sqrt{2|a| + a^2}$, which is satisfied for every $0 < \gamma \leq 1/2$. Indeed, if $\gamma = 1/2$ the above condition becomes $a^2 + |a| \geq 0$ which is always satisfied and for $\gamma < 1/2$ we can conclude analyzing the infinitesimal order as $a \to 0$ of the right and the left hand side.

Finally, if $1/2 < \gamma < 1$, the inequality (4.26) is not satisfied at fine scales but the two cones of directions intersect because they are two cones centered in the x axis (see Figure 4.8). Therefore we can conclude splitting the shearlet coefficient as in what follows

$$\begin{aligned}
&|\mathcal{S}_\psi f(1,0,0,a)| \\
&= |a|^{\frac{\gamma-2}{2}} | \underbrace{\int_{-\sqrt{2|a|+a^2}}^{\sqrt{2|a|+a^2}} \mathcal{W}_{\chi_1}\varphi(\frac{1}{\sqrt{1+v^2}}, \frac{a}{\sqrt{1+v^2}}) \frac{\mathrm{d}v}{\sqrt[4]{1+v^2}}}_{\text{(i)}} \\
&+ \underbrace{\int_{[-|a|^{1-\gamma}, -\sqrt{2|a|+a^2}] \cup [\sqrt{2|a|+a^2}, |a|^{1-\gamma}]} \mathcal{W}_{\chi_1}\varphi(\frac{1}{\sqrt{1+v^2}}, \frac{a}{\sqrt{1+v^2}}) \frac{\mathrm{d}v}{\sqrt[4]{1+v^2}}}_{\text{(ii)}} |.
\end{aligned}$$

As for (i), we know that inequality (4.26) holds true for every v with $|v| \leq \sqrt{2a + a^2}$ and then $1/\sqrt{1+v^2}$ belongs to the cone of influence created by the singularity $t = 1$ as $a \to 0$. Finally, as for (ii), inequality (4.26) is not satisfied at fine scales if $v \in [-|a|^{1-\gamma}, -\sqrt{2a + a^2}] \cup [\sqrt{2a + a^2}, |a|^{1-\gamma}]$ and then the wavelet coefficients in (ii) have fast decay behavior. Therefore, the behavior of the shearlet coefficient as a goes to zero follows directly from the behavior of part (i) and we conclude.

This example shows that the the ability of the shearlet transform to resolve the wavefront set of signals is a direct consequence of two facts, first that singularities of f produce singularities of $\mathcal{R}^{\mathrm{aff}}$ and second that the wavelet transform is able to describe the smoothness of univariate functions. This is seen via formula (3.47), in which the Radon transform plays a crucial role.

It is worth observing that an explicit expression of the Radon transform of f is not needed, but what really matters is the location of its singularities. If we look again at our formula

$$\left|\mathcal{S}^{\gamma}_{\psi} f(1,0,0,a)\right| = |a|^{\frac{\gamma-2}{2}} \Big| \int_{-|a|^{1-\gamma}}^{|a|^{1-\gamma}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(1,a)\ \mathrm{d}v \Big|$$

it is clear that the decay behavior of the shearlet coefficients at fine scales follows directly by the decay behavior of the wavelet coefficients within the integral, which is determined by the location of the singularities of the affine Radon transform.

By equation 4.21, for every fixed $v \in [-|a|^{1-\gamma}, |a|^{1-\gamma}]$, the function

$$t \mapsto \mathcal{R}^{\text{aff}} f(v,t)$$

has two singularities in $t = \pm\sqrt{1+v^2}$ corresponding to the lines tangent to the unit circle with slope parametrized by v. In particular $\varphi(t) = \mathcal{R}^{\text{aff}} f(0,t)$ has two singularities in $t = \pm 1$ which correspond to the vertical lines $x = -1$ and $x = 1$ (see Figure 4.7). This is a consequence of a more general result about the propagation of singularities for $\mathcal{R}^{\text{pol}}$.

Theorem 4.12 ([49])**.** *Let $f \in \mathcal{E}'(\mathbb{R}^2)$ and $l_{(\theta_0,t_0)}$ be a line in the plane parametrized by the pair (θ_0, t_0) as in Definition 1.40. Let $(x_0, \xi_0) \in WF(f)$ such that $x_0 \in l_{(\theta_0,t_0)}$ and $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$ is a normal vector to $l_{(\theta_0,t_0)}$. Then the following holds.*

(i) The singularity of f in (x_0, ξ_0) causes a unique singularity in $WF(\mathcal{R}f)$ above (θ_0, t_0).

(ii) Singularities of f not tangent to $l_{(\theta_0,t_0)}$ do not cause singularities in $\mathcal{R}f$ above (θ_0, t_0).

This result follows by the fact that $\mathcal{R}^{\text{pol}}$ is an elliptic Fourier integral transform and by a stronger version of the Hörmander-Sato Lemma for elliptic operators [69]. We refer to [49] for the proof. Furthermore, by relation (3.21), Theorem 4.12 can be stated for the affine Radon transform. Theorem 4.12 has been widely exploited in limited angle data tomography to explain visible and invisible singularities and artifacts in the reconstructed signal (see [59, 49] – to name a few). The content of Theorem 4.12 can be summarized in the following general principle [49]: "$\mathcal{R}^{\text{pol}}$ detects singularities of f perpendicular to the line of integration ("visible" directions) but not in other ("invisible" directions)". Hence, the information on a singularity $(x_0, \xi_0) \in WF(f)$ is exclusively contained in the singularity $(\theta_0, t_0) \in WF(\mathcal{R}f)$, where $l_{(\theta_0,t_0)}$ is the line passing through the point x_0 and perpendicular to the vector $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$. This principle is reflected by our formula. Let $(b_0, s_0) \in WF_0(f)$ and consider formula (3.47) for the shearlet transform assuming $\text{supp}\phi_2 \subseteq [-1,1]$. It reads

$$\begin{aligned} \mathcal{S}^{\gamma}_{\psi} f(b,s,a) &= |a|^{\frac{\gamma-2}{2}} \int_{s-|a|^{1-\gamma}}^{s+|a|^{1-\gamma}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(n(v)\cdot b, a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\ \mathrm{d}v \\ &= |a|^{\frac{\gamma-2}{2}} \int_{s-|a|^{1-\gamma}}^{s+|a|^{1-\gamma}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(v,\cdot))(b_1 + s b_2 + \epsilon(a) b_2, a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\ \mathrm{d}v, \end{aligned}$$

with $b = (b_1, b_2)$ and $\epsilon(a) \to 0$ as $a \to 0$. Observe that $\mathcal{S}^{\gamma}_{\psi} f(b_0, s_0, a)$ is the only shearlet coefficient which contains $\mathcal{W}_{\chi_1}(\mathcal{R}^{\text{aff}} f(s_0,\cdot))(n(s_0)\cdot b_0, a)$ as $a \to 0$. This wavelet coefficient is exactly the one containing the information on the singularity (b_0, s_0) since, by Theorem 4.12, $(s_0, n(s_0) \cdot b_0)$ is the only singularity created in $WF(\mathcal{R}f)$ by (b_0, s_0). In fact, we already know that if $(b_0, s_0) \in WF_0(f)$, then $\mathcal{S}_\psi f(b, s, a)$ decays rapidly when a goes to zero unless when $b = b_0$ and $s = s_0$. By the previous observations, also the role of the convolution with the scale-dependent filter

$$\Phi_a(v) = \overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}$$

is made clear.

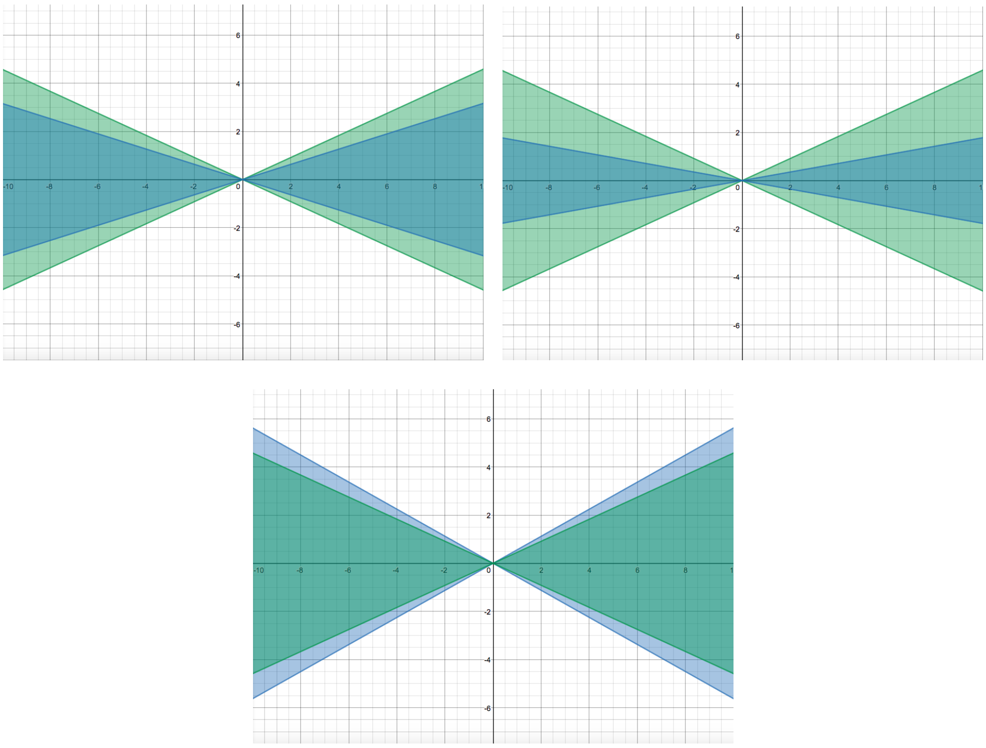

Figure 4.8: The figures depicte the cones of directions $-|a|^{1-\gamma} \leq v \leq |a|^{1-\gamma}$ (green ones) and $-\sqrt{2a+a^2} \leq v \leq \sqrt{2a+a^2}$ (blue ones) for the values $a = 0.1$ and $\gamma = 1/2$, $\gamma = 1/4$ and $\gamma = 3/4$, respectively starting from the bottom on the left . As expected, when $\gamma = 1/2$ and $0 < \gamma < 1/2$ the cone created by the bump function ϕ_2, whose amplitude is controlled by the parameter γ, falls within the cone of influence associated with the singularity $t = 1$. While for $1/2 < \gamma < 1$ we have the viceversa.

The generalization of this kind of arguments to arbitrary distributions is left to future investigation. The idea would be to exploit results in [49] and the geometric construction that relates $WF(\mathcal{R}f)$ and $WF(f)$ based on the Legendre transform given in [61] and also in the online lectures `https://www.icts.res.in/sites/default/files/Jan_Boman_Lecture_Notes_0.pdf`.

Chapter 5

The Shearlet Transform of Distributions

This last chapter is based on an ongoing project with Stevan Pilipović and Nenad Teofanov [10]. Our main results are continuity theorems for the shearlet transform and its transpose, called the shearlet synthesis operator, on various test function spaces. Then, we use these continuity results to develop a distributional framework for the shearlet transform via a duality approach. Precisely, we show that the shearlet transform can be extended as a continuous map from $\mathcal{S}_0'(\mathbb{R}^2)$ into $\mathcal{S}'(\mathbb{S})$, where $\mathcal{S}_0'(\mathbb{R}^2)$ is the Lizorkin distribution space and $\mathcal{S}'(\mathbb{S})$ is the space of distributions of slow growth on $\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}_+$. This work arises from the lack of a complete distributional framework for the shearlet transform in the literature and from the link between the shearlet transform with the Radon and the wavelet transforms, whose distribution theory is a deeply investigated and well known subject in applied mathematics. We refer respectively to [42] and to [38, 48] for the extension of the wavelet transform and the Radon transform to various spaces of distributions. We emphasize the importance of the Lizorkin space, which plays a crucial role in the extension of these two classical transforms and which turns out to be a natural domain for the shearlet transform too. Furthermore, it has been used in [48] to develop a distributional framework for the ridgelet transform based on the intimate connection between the Radon, the ridgelet and the wavelet transforms. We conclude our analysis showing that the shearlet transform of Lizorkin distributions extends the one considered in [51, 34] on the space of tempered distributions (see Theorem 5.10) and allows to write the action of any Lizorkin distribution on any test function as an absolutely convergent integral over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ (see Corollary 5.11).

5.1 The spaces

In this section we recall the spaces that occur in this chapter. We provide all distribution spaces with the strong dual topologies. Besides the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, the already mentioned Lizorkin test function space $\mathcal{S}_0(\mathbb{R}^d)$ plays a crucial role in our analysis. We recall that it consists of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ having all vanishing moments, i.e.

$$\int_{\mathbb{R}^d} x^m \varphi(x)dx = 0,$$

for all $m \in \mathbb{N}^d$. The Lizorkin test function space $\mathcal{S}_0(\mathbb{R}^d)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^d)$ equipped with the relative topology inhered from $\mathcal{S}(\mathbb{R}^d)$. Its dual space $\mathcal{S}_0'(\mathbb{R}^d)$, known as the space of Lizorkin distributions, is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^d)$ by the space of polynomials. Finally, the Fourier Lizorkin space $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ consists of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ that vanish in zero together with all their partial derivatives and it is equipped with the relative topology inhered from $\mathcal{S}(\mathbb{R}^d)$. For the seminorms on $\mathcal{S}(\mathbb{R}^d)$, we make the choice

$$\rho_\nu(\varphi) = \sup_{x\in\mathbb{R}^d, |m|\leq\nu} \langle x\rangle^\nu |\varphi^{(m)}(x)|,$$

for every $\nu \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We observe that, since $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ are closed subspaces of the nuclear space $\mathcal{S}(\mathbb{R}^d)$, they are nuclear as well and

$$\mathcal{S}_0(\mathbb{R}^{d_1})\hat{\otimes}_\varepsilon\mathcal{S}_0(\mathbb{R}^{d_2}) = \mathcal{S}_0(\mathbb{R}^{d_1})\hat{\otimes}_\pi\mathcal{S}_0(\mathbb{R}^{d_2}),$$
$$\hat{\mathcal{S}}_0(\mathbb{R}^{d_1})\hat{\otimes}_\varepsilon\hat{\mathcal{S}}_0(\mathbb{R}^{d_2}) = \hat{\mathcal{S}}_0(\mathbb{R}^{d_1})\hat{\otimes}_\pi\hat{\mathcal{S}}_0(\mathbb{R}^{d_2}),$$

where $d_1, d_2 \in \mathbb{Z}_+$, and $X\hat{\otimes}Y$ denotes the topological tensor product space obtained as the completion of $X \otimes Y$ in the inductive tensor product topology ε or the projective tensor product topology π, see [68] for details. Moreover, we have the following result.

Lemma 5.1 ([10]). *The spaces $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ are closed under translation, dilation, differentiation and multiplication by a polynomial. Moreover, the Fourier transform is an isomorphism between $\mathcal{S}_0(\mathbb{R}^d)$ and $\hat{\mathcal{S}}_0(\mathbb{R}^d)$ and we have the following canonical isomorphisms:*

$$\mathcal{S}_0(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}_0(\mathbb{R}^{d_1})\hat{\otimes}\mathcal{S}_0(\mathbb{R}^{d_2}),$$
$$\hat{\mathcal{S}}_0(\mathbb{R}^{d_1+d_2}) \cong \hat{\mathcal{S}}_0(\mathbb{R}^{d_1})\hat{\otimes}\hat{\mathcal{S}}_0(\mathbb{R}^{d_2}),$$

where $d_1, d_2 \in \mathbb{Z}_+$, and $\hat{\otimes}$ denotes the completion with respect to the ε-topology or the π-topology.

Proof. The proof is based on classical arguments and we omit it (cf. [68, Theorem 51.6] for the canonical isomorphism). □

The next lemma follows directly by Lemma 1.50.

Lemma 5.2 ([10]). *Let $f \in \mathcal{S}_0(\mathbb{R}^d)$. Then, for any given $m \in \mathbb{N}^d$ there exists $g \in \mathcal{S}_0(\mathbb{R}^d)$ such that*

$$\mathcal{F}f(\xi) = \xi^m \mathcal{F}g(\xi), \quad \xi \in \mathbb{R}^d,$$

and vice versa.

Remark 5.3. We consider $f \in \mathcal{S}_0(\mathbb{R})$ and $m \in \mathbb{N}$. Then, by Lemma 1.50, there exists $g \in \mathcal{S}_0(\mathbb{R})$ such that

$$\mathcal{F}f(\xi) = \xi^m \mathcal{F}g(\xi), \quad \xi \in \mathbb{R},$$

and, for every $k \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\langle x\rangle^k |g(x)| \lesssim \rho_{2k+4}(f).$$

Furthermore, $g'(x) = f(x)$, so that $\langle x\rangle^k |g^{(l)}(x)| = \langle x\rangle^k |f^{(l-1)}(x)| \lesssim \rho_k(f)$ for every $k \in \mathbb{N}$ and $1 \leq l \leq k$. Therefore, for every given $\nu_1 \in \mathbb{N}$, there exist $\nu_2 \in \mathbb{N}$ and a constant $C > 0$ such that

$$\rho_{\nu_1}(g) \leq C\rho_{\nu_2}(f).$$

It is not difficult to see that the analogous statement holds true for $d > 1$.

We denote $\mathbb{S} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ and we introduce the space $\mathcal{S}(\mathbb{S})$ of the functions $\Phi \in C^\infty(\mathbb{S})$ such that the seminorms

$$p^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(\Phi) = \sup_{((b_1,b_2),s,a)\in\mathbb{S}} \langle b_1\rangle^{k_1} \langle b_2\rangle^{k_2} \langle s\rangle^{l} \left(a^m + \frac{1}{a^m}\right) \left|\partial^\gamma_a \partial^\beta_s \partial^{\alpha_2}_{b_2} \partial^{\alpha_1}_{b_1} \Phi(((b_1,b_2),s,a)\right| \tag{5.1}$$

are finite for all $k_1, k_2, l, m, \alpha_1, \alpha_2, \beta, \gamma \in \mathbb{N}$. The topology of this space is defined by means of the seminorms (5.1). Its dual $\mathcal{S}'(\mathbb{S})$ will play a crucial role in our definition of the shearlet transform of Lizorkin distributions since it contains the range of this transform. We fix $a^{-3}\mathrm{d}b\mathrm{d}s\mathrm{d}a$ as the standard measure on $\mathbb{S}$. The space of functions $\mathcal{S}(\mathbb{S})$ naturally embeds into $\mathcal{S}'(\mathbb{S})$ and we identify $F \in \mathcal{S}(\mathbb{S})$ with an element of $\mathcal{S}'(\mathbb{S})$ by means of the equality

$$\langle F, \Phi\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_{\mathbb{R}^2} F(b,s,a)\Phi(b,s,a)\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{a^3}, \tag{5.2}$$

for every $\Phi \in \mathcal{S}(\mathbb{S})$.

5.2 The continuity of the shearlet transform on test function spaces

We recall that if χ_1 is an admissible wavelet with all vanishing moments and $\phi_2 \in L^2(\mathbb{R})$, then the function $\psi_1 \in L^2(\mathbb{R})$ defined by

$$\mathcal{F}\psi_1(\tau) = |\tau|^{-1}\mathcal{F}\chi_1(\tau) \tag{5.3}$$

satisfies the conditions (3.38) and (3.41) and ψ defined by

$$\mathcal{F}\psi(\xi_1,\xi_2) = \mathcal{F}\psi_1(\xi_1)\mathcal{F}\psi_2\left(\frac{\xi_2}{\xi_1}\right), \quad (\xi_1,\xi_2)\in\mathbb{R}^2,\, \xi_1\neq 0, \tag{5.4}$$

where $\mathcal{F}\psi_2 = \phi_2$, satisfies the admissible condition (3.52). Then, by Corollary 3.16, for every $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ the shearlet transform of f satisfies

$$\mathcal{S}^\gamma_\psi f(b,s,a) = |a|^{\frac{\gamma-2}{2}} \int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(n(v)\cdot b, a)\overline{\phi_2\left(\frac{v-s}{|a|^{1-\gamma}}\right)}\,\mathrm{d}v,$$

for every $(b,s,a) \in \mathbb{R}^2 \rtimes \mathbb{R} \rtimes \mathbb{R}^\times$. From now on, everytime we consider an admissible vector ψ, we assume that it is of the form (5.4) with $\chi_1 \in \mathcal{S}_0(\mathbb{R})$ and $\phi_2 \in \mathcal{S}(\mathbb{R})$. We observe that, under these assumptions, $\mathcal{F}\psi$ extends to a function belonging to the Fourier Lizorkin space $\hat{\mathcal{S}}_0(\mathbb{R}^2)$ and, with slight abuse of notation, ψ denotes both the admissible vector defined by (5.4) and its Schwartz extension over $\mathbb{R}^2$. Furthermore, for simplicity, we fix $\gamma = 1/2$ and we restrict ourselves to the connected version of the shearlet group, which corresponds to restricting the scale parameter a over $\mathbb{R}_+$. By direct computation, applying the Plancherel theorem and the Fourier slice theorem, we have that

$$\begin{aligned}
\mathcal{S}_\psi f((b_1,b_2),s,a) &= a^{-\frac{3}{4}}\int_{\mathbb{R}} \mathcal{W}_{\chi_1}(\mathcal{R}^{\mathrm{aff}} f(v,\cdot))(b_1+vb_2,a)\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}v \\
&= a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}\mathcal{R}^{\mathrm{aff}} f(v,\cdot)(\tau)\overline{\mathcal{F}\chi_1(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\mathrm{d}\tau\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}v \\
&= a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v, \quad (5.5)
\end{aligned}$$

for every $((b_1,b_2),s,a)\in\mathbb{S}$. We are now ready to state our first main result. Let ψ be an admissible vector of the form as discussed above.

Theorem 5.4 ([10])**.** *The shearlet transform $\mathcal{S}_\psi$ is a continuous operator from $\mathcal{S}_0(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{S})$.*

Proof. We have to show that for every $f\in\mathcal{S}_0(\mathbb{R}^2)$, given $k_1,k_2,l,m,\alpha_1,\alpha_2,\beta,\gamma\in\mathbb{N}$, there exist $\nu\in\mathbb{N}$ and a constant $C>0$ such that

$$\rho^{\alpha_1,\alpha_2,\beta,\gamma}_{k_1,k_2,l,m}(\mathcal{S}_\psi f)\le C\rho_\nu(f).$$

Without loss of generality, we may assume that k_1, k_2 and l are even and $m\ge 1$. In the following C is a generic positive constant which may vary from expression to expression. We divide the proof into five steps. In the first three steps, we show that it is enough to prove that for every $m\in\mathbb{N}$

$$\rho^{0,0,0,0}_{0,0,0,m}(\mathcal{S}_\psi f)\le C\rho_\nu(f),$$

for some $\nu\in\mathbb{N}$ and some constant $C>0$.

1. We start showing that we can assume $\alpha_1=\alpha_2=\beta=\gamma=0$. By formula (5.5), we have that

$$\begin{aligned}
&|\partial_s^\beta\partial_{b_2}^{\alpha_2}\partial_{b_1}^{\alpha_1}\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= Ca^{-\frac{1+2\beta}{4}}|\int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{\alpha_1+\alpha_2}v^{\alpha_2}\mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2^{(\beta)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\eta\mathrm{d}v|, \quad (5.6)
\end{aligned}$$

where C is a positive constant dependent only on α_1 and α_2. Then, by Liebniz formula and Faà di Bruno's formula, we compute

$$\begin{aligned}
&|\partial_a^\gamma\partial_s^\beta\partial_{b_2}^{\alpha_2}\partial_{b_1}^{\alpha_1}\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= C|\int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{\alpha_1+\alpha_2}v^{\alpha_2}\mathcal{F}f(\tau,\tau v)e^{2\pi i\tau(b_1+vb_2)}\partial_a^\gamma\left(a^{-\frac{1+2\beta}{4}}\overline{\mathcal{F}\chi_1(a\tau)\phi_2^{(\beta)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\right)\mathrm{d}\tau\mathrm{d}v| \\
&= C|\sum_{j+l+k=\gamma} c_{j,l,k}(a^{-\frac{1+2\beta}{4}})^{(j)}\int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{\alpha_1+\alpha_2}v^{\alpha_2}\mathcal{F}f(\tau,\tau v)e^{2\pi i\tau(b_1+vb_2)}\partial_a^l\overline{\mathcal{F}\chi_1(a\tau)}\times \\
&\times\partial_a^k\overline{\phi_2^{(\beta)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v| \\
&= C|\sum_{j+l+k=\gamma} c_{j,l,k,\beta}\, a^{-\frac{1+2\beta}{4}-j}\int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{\alpha_1+\alpha_2+l}v^{\alpha_2}\mathcal{F}f(\tau,\tau v)e^{2\pi i\tau(b_1+vb_2)}\overline{(\mathcal{F}\chi_1)^{(l)}(a\tau)}\times \\
&\times\sum C_{m_1,\dots,m_k}\,\overline{\phi_2^{(\beta+m_1+\dots+m_k)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}(v-s)^{m_1+\ldots+m_k}\prod_{p=1}^{k}a^{(-\frac{1}{2}-p)m_p}\,\mathrm{d}\tau\mathrm{d}v|,
\end{aligned}$$

where the second sum is over all k-tuples of nonnegative integers $(m_1, ..., m_k)$ satisfying the constraint $m_1 + 2m_2 + \ldots + km_k = k$. We continue the above chain of equalities as follows

$$
\begin{aligned}
&|\partial_a^\gamma \partial_s^\beta \partial_{b_2}^{\alpha_2} \partial_{b_1}^{\alpha_1} \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= C| \sum_{j+l+k=\gamma} \sum c_{j,l,k,\beta}\ C_{m_1,\ldots,m_k}\ a^{-\frac{1+2\beta}{4}-j-(m_1+\cdots+km_k)} \int_{\mathbb{R}} \int_{\mathbb{R}} \tau^{\alpha_1+\alpha_2+l} v^{\alpha_2} \times \\
&\times \mathcal{F}f(\tau,\tau v) e^{2\pi i \tau(b_1+vb_2)} \overline{(\mathcal{F}\chi_1)^{(l)}(a\tau) \phi_2^{(\beta+m_1+\cdots+m_k)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)\left(\frac{v-s}{a^{\frac{1}{2}}}\right)^{m_1+\ldots+m_k}} \mathrm{d}\tau \mathrm{d}v| \\
&= C| \sum_{j+l+k=\gamma} \sum c_{j,l,k,\beta}\ C_{m_1,\ldots,m_k}\ a^{-\frac{1+2\beta}{4}-j-(m_1+\cdots+km_k)} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}\partial_1^{\alpha_1+l}\partial_2^{\alpha_2} f(\tau,\tau v) \times \\
&\times e^{2\pi i \tau(b_1+vb_2)} \overline{(\mathcal{F}\chi_1)^{(l)}(a\tau) \phi_2^{(\beta+m_1+\cdots+m_k)}\left(\frac{v-s}{a^{\frac{1}{2}}}\right)\left(\frac{v-s}{a^{\frac{1}{2}}}\right)^{m_1+\ldots+m_k}} \mathrm{d}\tau \mathrm{d}v| \\
&\lesssim \sum_{j+l+k=\gamma} \sum \rho^{0,0,0,0}_{0,0,0,\frac{\beta}{2}+j+(m_1+\cdots+km_k)} (\mathcal{S}_\Psi \partial_1^{\alpha_1+l}\partial_2^{\alpha_2} f),
\end{aligned}
$$

with the implied constant dependent only on $\alpha_1, \alpha_2, \beta, \gamma$ and where Ψ denotes the admissible vector of the form (5.4) defined starting from the functions $t \mapsto t^l \chi_1(t)$ and $t \mapsto \phi_2(t) = t^{m_1+\ldots+m_k} \phi_2^{(\beta+m_1+\cdots+m_k)}(t)$, which are still in $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, respectively. Therefore, we can assume $\alpha_1 = \alpha_2 = \beta = \gamma = 0$ because differentiation is a continuous operator on $\mathcal{S}_0(\mathbb{R}^2)$.

2. The next step is to show that we can assume $k_1 = k_2 = 0$. By formula (5.5), for every $N \in \mathbb{N}$, we have that

$$
\begin{aligned}
&|\langle b_1 \rangle^{k_1} \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= \langle b_1 \rangle^{k_1} a^{-\frac{1}{4}} | \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}f(\tau,\tau v) \overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i \tau v b_2} \frac{(1-\frac{\partial^2}{\partial^2 \tau})^N e^{2\pi i \tau b_1}}{\langle 2\pi b_1 \rangle^{2N}} \overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)} \mathrm{d}\tau \mathrm{d}v| \\
&= \frac{\langle b_1 \rangle^{k_1}}{\langle 2\pi b_1 \rangle^{2N}} a^{-\frac{1}{4}} | \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i \tau b_1} (1-\frac{\partial^2}{\partial^2 \tau})^N (\mathcal{F}f(\tau,\tau v) \overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i \tau v b_2}) \overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)} \mathrm{d}\tau \mathrm{d}v| \\
&= \frac{\langle b_1 \rangle^{k_1}}{\langle 2\pi b_1 \rangle^{2N}} | \sum_{j,|l|,k \le 2N} a^{k-\frac{1}{4}} \int_{\mathbb{R}} \int_{\mathbb{R}} P_{j,l,k}(v) b_2^j \partial_1^{l_1} \partial_2^{l_2} \mathcal{F}f(\tau,\tau v) \overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)} e^{2\pi i \tau(b_1+vb_2)} \times \\
&\times \overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)} \mathrm{d}\tau \mathrm{d}v| \\
&\le \frac{\langle b_1 \rangle^{k_1} |b_2|^{2N} a^{2N}}{\langle 2\pi b_1 \rangle^{2N}} | \sum_{j,|l|,k \le 2N} a^{-\frac{1}{4}} \int_{\mathbb{R}} \int_{\mathbb{R}} P_{j,l,k}(v) \partial_1^{l_1} \partial_2^{l_2} \mathcal{F}f(\tau,\tau v) \overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)} e^{2\pi i \tau(b_1+vb_2)} \times \\
&\times \overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)} \mathrm{d}\tau \mathrm{d}v|,
\end{aligned}
$$

for some polynomials $P_{j,l,k}$ with $l = (l_1, l_2) \in \mathbb{N}^2$, where in the last inequality we have assumed $a > 1$ and $|b_2| > 1$. Then, we continue the above computation as follows

$$
\begin{aligned}
&|\langle b_1\rangle^{k_1}\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\leq \frac{\langle b_1\rangle^{k_1}|b_2|^{2N}a^{2N}}{\langle 2\pi b_1\rangle^{2N}}\Big|\sum_{j,|l|,k\leq 2N} a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}(c_0+c_1v+\ldots+c_{p_{j,l,k}}v^{p_{j,l,k}})\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\tau,\tau v)\times \\
&\times\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big| \\
&= \frac{\langle b_1\rangle^{k_1}|b_2|^{2N}a^{2N}}{\langle 2\pi b_1\rangle^{2N}}\Big|\sum_{j,|l|,k\leq 2N}\sum_{m=0}^{p_{j,l,k}} c_m a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}} v^m\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}\times \\
&\times e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big|.
\end{aligned}
$$

Since multiplication by polynomials is a continuous operator on $\mathcal{S}_0(\mathbb{R}^2)$, by Lemma 5.2 for every $l_1, l_2, m \in \mathbb{N}$ there exists $g \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\xi_1,\xi_2) = \xi_1^m\mathcal{F}g(\xi_1,\xi_2)$. Then, we have that

$$
\begin{aligned}
&|\langle b_1\rangle^{k_1}\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\leq \frac{\langle b_1\rangle^{k_1}|b_2|^{2N}a^{2N}}{\langle 2\pi b_1\rangle^{2N}}\Big|\sum_{j,|l|,k\leq 2N}\sum_{m=0}^{p_{j,l,k}} c_m a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}\tau^m v^m\mathcal{F}g(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}\times \\
&\times e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big| \\
&\lesssim \frac{\langle b_1\rangle^{k_1}|b_2|^{2N}}{\langle 2\pi b_1\rangle^{2N}}\sum_{j,|l|,k\leq 2N}\sum_{m=0}^{p_{j,l,k}} c_m\rho^{0,0,0,0}_{0,0,0,2N}(\mathcal{S}_\Psi\partial_2^m g),
\end{aligned}
$$

where Ψ denotes the admissible vector of the form (5.4) defined starting from the functions $t \mapsto t^k\chi_1(t)$, which is still in $\mathcal{S}_0(\mathbb{R})$, and $\phi_2 \in \mathcal{S}(\mathbb{R})$. Analogously, for every $M \in \mathbb{N}$ we have that

$$
\begin{aligned}
&|\langle b_2\rangle^{k_2}\mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&= \langle b_2\rangle^{k_2}a^{-\frac{1}{4}}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)}e^{2\pi i\tau b_1}\frac{(1-\frac{\partial^2}{\partial^2\tau})^M e^{2\pi i\tau vb_2}}{\langle 2\pi vb_2\rangle^{2M}}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big| \\
&= \langle b_2\rangle^{k_2}a^{-\frac{1}{4}}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{e^{2\pi i\tau vb_2}}{\langle 2\pi vb_2\rangle^{2M}}(1-\frac{\partial^2}{\partial^2\tau})^M(\mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)}e^{2\pi i\tau b_1})\times \\
&\times\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big| \\
&= \langle b_2\rangle^{k_2}\Big|\sum_{j,|l|,k\leq 2M} a^{k-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{P_{j,l,k}(v)b_1^j}{\langle 2\pi vb_2\rangle^{2M}}\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\times \\
&\times\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big| \\
&\leq \frac{\langle b_2\rangle^{k_2}|b_1|^{2M}a^{2M}}{\langle 2\pi b_2\rangle^{2M}}\Big|\sum_{j,|l|,k\leq 2M} a^{-\frac{1}{4}}\int_{\mathbb{R}}\int_{\mathbb{R}}(\frac{v^2+1}{v^2})^{2M}P_{j,l,k}(v)\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\tau,\tau v)\times \\
&\times\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac{1}{2}}}\right)}\mathrm{d}\tau\mathrm{d}v\Big|,
\end{aligned}
$$

for some polynomials $P_{j,l,k}$ with $l=(l_1,l_2)\in\mathbb{N}^2$, where in the last inequality we have assumed $a>1$ and $|b_1|>1$. Then, we set $Q_{j,l,k}=P_{j,l,k}(v^2+1)^{2M}$ and we continue the above chain of inequalities as follows

$$\begin{aligned}
&|\langle b_2\rangle^{k_2}\mathcal{S}_\psi f((b_1,b_2),s,a)|\\
&\lesssim \frac{\langle b_2\rangle^{k_2}|b_1|^{2M}a^{2M}}{\langle 2\pi b_2\rangle^{2M}}|\sum_{j,|l|,k\le 2M} a^{-\frac14}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{Q_{j,l,k}(v)}{v^{4M}}\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}\times\\
&\times e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac12}}\right)}\mathrm{d}\tau\mathrm{d}v|.
\end{aligned}$$

Since multiplication by polynomials is a continuous operator on $\mathcal{S}_0(\mathbb{R}^2)$, then by Lemma 5.2 for every $l_1,l_2,M\in\mathbb{N}$ there exists $h\in\mathcal{S}_0(\mathbb{R}^2)$ such that $\partial_1^{l_1}\partial_2^{l_2}\mathcal{F}f(\xi_1,\xi_2)=\xi_2^{4M}\mathcal{F}h(\xi_1,\xi_2)$. Then, we have that

$$\begin{aligned}
&|\langle b_2\rangle^{k_2}\mathcal{S}_\psi f((b_1,b_2),s,a)|\\
&\lesssim \frac{\langle b_2\rangle^{k_2}|b_1|^{2M}a^{2M}}{\langle 2\pi b_2\rangle^{2M}}|\sum_{j,|l|,k\le 2M} a^{-\frac14}\int_{\mathbb{R}}\int_{\mathbb{R}}Q_{j,l,k}(v)\tau^{4M}\mathcal{F}h(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}\times\\
&\times e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac12}}\right)}\mathrm{d}\tau\mathrm{d}v|\\
&=\frac{\langle b_2\rangle^{k_2}|b_1|^{2M}a^{2M}}{\langle 2\pi b_2\rangle^{2M}}|\sum_{j,|l|,k\le 2M}\sum_{m=0}^{p_{j,l,k}} c_m a^{-\frac14}\int_{\mathbb{R}}\int_{\mathbb{R}}v^m\tau^{4M}\mathcal{F}h(\tau,\tau v)\overline{(\mathcal{F}\chi_1)^{(k)}(a\tau)}\times\\
&\times e^{2\pi i\tau(b_1+vb_2)}\overline{\phi_2\left(\frac{v-s}{a^{\frac12}}\right)}\mathrm{d}\tau\mathrm{d}v|\\
&\lesssim \frac{\langle b_2\rangle^{k_2}|b_1|^{2M}}{\langle 2\pi b_2\rangle^{2M}}\sum_{j,|l|,k\le 2M}\sum_{m=0}^{p_{j,l,k}}\rho^{0,0,0,0}_{0,0,0,2M}(\mathcal{S}_\Psi\partial_1^{4M-m}\partial_2^m h),
\end{aligned}$$

where Ψ denotes the admissible vector of the form (5.4) defined starting from the functions $t\mapsto t^k\chi_1(t)$, which is still in $\mathcal{S}_0(\mathbb{R})$, and $\phi_2\in\mathcal{S}(\mathbb{R})$. Observe that, if $m>4M$, then we can apply again Lemma 5.2 in order to have the power of τ greater than the one of v. Therefore, recalling that we have assumed $|b_1|>1$ and $|b_2|>1$, if we choose $2N=2M=k_1+k_2$, we have that

$$\begin{aligned}
\langle b_1\rangle^{k_1}\langle b_2\rangle^{k_2}|\mathcal{S}_\psi f((b_1,b_2),s,a)|^2 &\lesssim \frac{\langle b_1\rangle^{2k_1+k_2}\langle b_2\rangle^{k_1+2k_2}}{\langle b_1\rangle^{2k_1+2k_2}\langle b_2\rangle^{2k_1+2k_2}}\times\\
&\times[\sum_{j,|l|,k\le 2N}\sum_{m=0}^{p_{j,l,k}}c_m\rho^{0,0,0,0}_{0,0,0,2N}(\mathcal{S}_\Psi\partial_2^m g)]\times\\
&\times[\sum_{j,|l|,k\le 2M}\sum_{m=0}^{p_{j,l,k}}C_m\,\rho^{0,0,0,0}_{0,0,0,2M}(\mathcal{S}_\Psi\partial_1^{4M-m}\partial_2^m h)]\\
&\lesssim[\sum_{j,|l|,k\le 2N}\sum_{m=0}^{p_{j,l,k}}c_m\rho^{0,0,0,0}_{0,0,0,2N}(\mathcal{S}_\Psi\partial_2^m g)]\times\\
&\times[\sum_{j,|l|,k\le 2M}\sum_{m=0}^{p_{j,l,k}}C_m\,\rho^{0,0,0,0}_{0,0,0,2M}(\mathcal{S}_\Psi\partial_1^{4M-m}\partial_2^m h)].
\end{aligned}$$

and we conclude that we can assume $k_1 = k_2 = 0$ by Remark 5.3 and since differentiation and multiplication by polynomials are continuous operators on $\mathcal{S}_0(\mathbb{R}^2)$.

3. Now we show that we can assume $l = 0$. By formula (5.5), we have that

$$|\langle s\rangle^l \mathcal{S}_\psi f((b_1,b_2),s,a)| = |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle s\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v|.$$

We divide the computation in the two cases $0 < a < 1$ and $a > 1$. If $a > 1$, by Peetre's inequality we have that

$$\langle s\rangle^l \leq a^{\frac{l}{2}} 2^l \langle \frac{v}{\sqrt{a}}\rangle^l \langle \frac{v-s}{\sqrt{a}}\rangle^l \leq a^{\frac{l}{2}} 2^l \langle v\rangle^l \langle \frac{v-s}{\sqrt{a}}\rangle^l. \tag{5.7}$$

Then, we compute

$$\begin{aligned}
&|\langle s\rangle^l \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\leq a^{\frac{l}{2}} 2^l |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \langle v\rangle^l \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle \frac{v-s}{\sqrt{a}}\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v| \\
&\lesssim a^{\frac{l}{2}} \sum_{m=0}^{p_l} c_m |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} v^m \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle \frac{v-s}{\sqrt{a}}\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v|
\end{aligned}$$

By Lemma 5.2, for every $m \in \mathbb{N}$, there exists $g \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\mathcal{F}f(\xi_1,\xi_2) = \xi_1^m \mathcal{F}g(\xi_1,\xi_2)$. Then, we have that

$$\begin{aligned}
&|\langle s\rangle^l \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\lesssim a^{\frac{l}{2}} \sum_{m=0}^{p_l} c_m |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \tau^m v^m \mathcal{F}g(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle \frac{v-s}{\sqrt{a}}\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v| \\
&\lesssim \sum_{m=0}^{p_l} c_m \rho^{0,0,0,0}_{0,0,0,\frac{l}{2}}(\mathcal{S}_\Psi(\partial_2^m g)),
\end{aligned}$$

where Ψ denotes the admissible shearlet of the form (5.4) defined by the functions χ_1 and $t \mapsto \phi_2(t) = \langle t\rangle^l \phi_2(t)$, which is clearly in $\mathcal{S}(\mathbb{R})$. We now consider the case $0 < a < 1$. By Peetre's inequality we have that

$$\langle s\rangle^l \leq 2^l \langle \frac{v}{\sqrt{a}}\rangle^l \langle \frac{v-s}{\sqrt{a}}\rangle^l \leq a^{-\frac{l}{2}} 2^l \langle v\rangle^l \langle \frac{v-s}{\sqrt{a}}\rangle^l \tag{5.8}$$

and, following computations analogous to the case $a > 1$, we obtain

$$\begin{aligned}
&|\langle s\rangle^l \mathcal{S}_\psi f((b_1,b_2),s,a)| \\
&\lesssim a^{-\frac{l}{2}} \sum_{m=0}^{p_l} c_m |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} v^m \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle \frac{v-s}{\sqrt{a}}\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v| \\
&\lesssim a^{\frac{l}{2}} \sum_{m=0}^{p_l} c_m |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \tau^m v^m \mathcal{F}g(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)} e^{2\pi i\tau(b_1+vb_2)} \langle \frac{v-s}{\sqrt{a}}\rangle^l \overline{\phi_2(\frac{v-s}{\sqrt{a}})} \mathrm{d}\tau \mathrm{d}v| \\
&\lesssim \sum_{m=0}^{p_l} c_m \rho^{0,0,0,0}_{0,0,0,\frac{l}{2}}(\mathcal{S}_\Psi(\partial_2^m g)),
\end{aligned}$$

where, for every $m \in \mathbb{N}$, $g \in \mathcal{S}_0(\mathbb{R}^2)$ is such that $\mathcal{F}f(\xi_1,\xi_2) = \xi_1^m \mathcal{F}g(\xi_1,\xi_2)$. So, we can assume $l = 0$ by Remark 5.3 and since differentiation is a continuous operator on $\mathcal{S}_0(\mathbb{R}^2)$.

4. Now, we consider multiplication by positive powers of a. We have

$$|a^m \mathcal{S}_\psi f((b_1,b_2),s,a)| = |a^m a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)e^{2\pi i\tau(b_1+vb_2)}\phi_2(\frac{v-s}{\sqrt{a}})}\mathrm{d}\tau\mathrm{d}v|.$$

By Lemma 1.50 there exists $g \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\mathcal{F}f(\xi_1,\xi_2) = \xi_1^m \mathcal{F}g(\xi_1,\xi_2)$. Then, we have that

$$\begin{aligned}
&|a^m \mathcal{S}_\psi f((b_1,b_2),s,a)| = |a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}g(\tau,\tau v)\tau^m a^m \overline{\mathcal{F}\chi_1(a\tau)e^{2\pi i\tau(b_1+vb_2)}\phi_2(\frac{v-s}{\sqrt{a}})}\mathrm{d}\tau\mathrm{d}v| \\
&\lesssim a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{F}g(\tau,\tau v)||\mathcal{F}(\chi_1^{(m)})(a\tau)||\phi_2(\frac{v-s}{\sqrt{a}})|\mathrm{d}\tau\mathrm{d}v \\
&\lesssim \rho_2(g)a^{-\frac{3}{4}} \int_{\mathbb{R}} |\mathcal{F}(\chi_1^{(m)})(a\tau)|a\,\mathrm{d}\tau \int_{\mathbb{R}} |\phi_2(\frac{v-s}{\sqrt{a}})|\frac{\mathrm{d}v}{\sqrt{a}} \lesssim \rho_2(g)\|\phi_2\|_1\|\mathcal{F}(\chi_1^{(m)})\|_1 \\
&\lesssim \rho_2(g),
\end{aligned}$$

where we have assumed $a \geq 1$.

5. Finally, we consider multiplication by negative powers of a. We have

$$|a^{-m} \mathcal{S}_\psi f((b_1,b_2),s,a)| = |a^{-m} a^{-\frac{1}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\chi_1(a\tau)e^{2\pi i\tau(b_1+vb_2)}\phi_2(\frac{v-s}{\sqrt{a}})}\mathrm{d}\tau\mathrm{d}v|.$$

By Lemma 1.50, there exists $\varphi_1 \in \mathcal{S}_0(\mathbb{R})$ such that $\mathcal{F}\chi_1(\tau) = \tau^{m+1}\mathcal{F}\varphi_1(\tau)$. Then, we have that

$$\begin{aligned}
&|a^{-m} \mathcal{S}_\psi f((b_1,b_2),s,a)| = |a^{\frac{3}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \tau^{m+1}\mathcal{F}f(\tau,\tau v)\overline{\mathcal{F}\varphi_1(a\tau)e^{2\pi i\tau(b_1+vb_2)}\phi_2(\frac{v-s}{\sqrt{a}})}\mathrm{d}\tau\mathrm{d}v| \\
&\lesssim a^{\frac{3}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{F}(\partial_1^{m+1}f)(\tau,\tau v)||\mathcal{F}\varphi_1(a\tau)||\phi_2(\frac{v-s}{\sqrt{a}})|\mathrm{d}\tau\mathrm{d}v \\
&\lesssim \rho_{m+2}(\partial_1^{m+1}f)a^{\frac{1}{4}} \int_{\mathbb{R}} |\mathcal{F}\varphi_1(a\tau)|a\,\mathrm{d}\tau \int_{\mathbb{R}} |\phi_2(\frac{v-s}{\sqrt{a}})|\frac{\mathrm{d}v}{\sqrt{a}} \lesssim \rho_{m+2}(\partial_1^{m+1}f)\|\mathcal{F}\varphi_1\|_1\|\phi_2\|_1 \\
&\lesssim \rho_{m+2}(\partial_1^{m+1}f),
\end{aligned}$$

where we have assumed $a \leq 1$. Then, we can conclude by Remark 5.3 and since differentiation is a continuous operator on $\mathcal{S}_0(\mathbb{R}^2)$. □

5.3 The shearlet synthesis operator

The reconstruction formula (3.53) suggests to define a linear operator $\mathcal{S}_\psi^t$ which maps functions over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ to functions over the Euclidean plane $\mathbb{R}^2$. Given $\psi \in \mathcal{S}(\mathbb{R}^2)$, we define the shearlet synthesis operator of Φ as the map $\mathcal{S}_\psi^t\Phi \colon \mathbb{R}^2 \to \mathbb{C}$ given by

$$\mathcal{S}_\psi^t\Phi(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \Phi(b,s,a)\ S_{b,s,a}\psi(x)\ \frac{\mathrm{d}a\mathrm{d}s\mathrm{d}b}{a^3}, \qquad x \in \mathbb{R}^2, \tag{5.9}$$

for any function Φ for which the integral converges. We observe that the integral in (5.9) is absolutely convergent if $\Phi \in \mathcal{S}(\mathbb{S})$. Furthermore, if $f \in L^1(\mathbb{R}^2)$ and $\Phi \in \mathcal{S}(\mathbb{S})$, then by Fubini theorem we have that

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)\mathcal{S}_\psi^t\Phi(x)\mathrm{d}x &= \int_{\mathbb{R}^2} f(x)\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \Phi(b,s,a)\ \overline{S_{b,s,a}\psi(x)}\ \frac{\mathrm{d}a\mathrm{d}s\mathrm{d}b}{a^3}\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \Phi(b,s,a)\int_{\mathbb{R}^2} f(x)\overline{S_{b,s,a}\psi(x)}\mathrm{d}x\ \frac{\mathrm{d}a\mathrm{d}s\mathrm{d}b}{a^3} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \mathcal{S}_\psi f(b,s,a)\Phi(b,s,a)\ \frac{\mathrm{d}a\mathrm{d}s\mathrm{d}b}{a^3}. \end{aligned} \tag{5.10}$$

In Theorem 5.5 we will show that the shearlet synthesis operator $\mathcal{S}_\psi^t$ is a continuous operator from $\mathcal{S}(\mathbb{S})$ into $\mathcal{S}_0(\mathbb{R}^2)$. Then, since $L^1(\mathbb{R}^2)$ naturally embeds into $\mathcal{S}'(\mathbb{R}^2)$ and by identification (5.2), we may write equality (5.10) as

$$\langle f, \mathcal{S}_\psi^t\Phi\rangle = \langle \mathcal{S}_\psi f, \Phi\rangle.$$

This dual relation will motivate our definition of the distributional shearlet transform. Furthermore, by the definition of the shearlet synthesis operator, we can rewrite reconstruction formula (3.53) as

$$f = (\mathcal{S}_\psi^t \circ \mathcal{S}_\psi)f, \tag{5.11}$$

where in general the integral in (5.9) has to be interpreted as a weak integral. We observe that if $\psi \in \mathcal{S}(\mathbb{R}^2)$ and $f \in L^1(\mathbb{R}^2)$ with $\mathcal{F}f \in L^1(\mathbb{R}^2)$, then formula (3.53) holds pointwisely. The proof mimics [48, Proposition 3.2].

Theorem 5.5 ([10])**.** *The shearlet synthesis operator $\mathcal{S}_\psi^t$ is continuous from $\mathcal{S}(\mathbb{S})$ into $\mathcal{S}_0(\mathbb{R}^2)$.*

Proof. We start proving the continuity. We need to show that for every $\Phi \in \mathcal{S}(\mathbb{S})$, given $\nu \in \mathbb{N}$, there exist $k_1, k_2, l, m, \alpha_1, \alpha_2, \beta, \gamma \in \mathbb{N}$ and a constant $C > 0$ such that

$$\rho_\nu(S_\psi^t\Phi) \leq C\rho_{k_1,k_2,l,m}^{\alpha_1,\alpha_2,\beta,\gamma}(\Phi).$$

We will use the fact that the families $\hat{\rho}_\nu(\chi) = \rho_\nu(\mathcal{F}\chi)$ and $\hat{\rho}_{k_1,k_2,l,m}^{\alpha_1,\alpha_2,\beta,\gamma}(\Phi) = \rho_{k_1,k_2,l,m}^{\alpha_1,\alpha_2,\beta,\gamma}(\mathcal{F}\Phi)$, where $\mathcal{F}\Phi$ denotes the Fourier transform of Φ with respect to the variable b, are bases of seminorms for the topologies of $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}(\mathbb{S})$, respectively. Furthermore, by Plancherel theorem, Fubini theorem and by equations (5.4) and (5.3) for the expression of the admissible vector, we obtain the following formula for the shearlet synthesis operator

$$\begin{aligned}\mathcal{S}_\psi^t\Phi(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{3/4}\int_{\mathbb{R}^2} \mathcal{F}\Phi(\xi,s,a)\ e^{2\pi i x\cdot\xi}\overline{\mathcal{F}\psi(-a\xi_1, a^{1/2}(-\xi_2+s\xi_1))}\ \mathrm{d}\xi_1\mathrm{d}\xi_2\frac{\mathrm{d}a\mathrm{d}s}{a^3} \\ &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{3/4}\mathcal{F}\Phi(\xi,s,a)\ \overline{\mathcal{F}\psi(-a\xi_1, a^{1/2}(-\xi_2+s\xi_1))}\ \frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{3/4}\mathcal{F}\Phi(\xi,s,a)\ \overline{\mathcal{F}\psi_1(-a\xi_1)\mathcal{F}\psi_2(a^{-1/2}(\xi_2/\xi_1 - s))}\ \frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{3/4}\mathcal{F}\Phi(\xi,s,a)\ \overline{\mathcal{F}\psi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1 - s))}\ \frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/4}\mathcal{F}\Phi(\xi,s,a)\ |\xi_1|^{-1}\overline{\mathcal{F}\chi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1 - s))}\ \frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2. \end{aligned} \tag{5.12}$$

We fix $\alpha \in \mathbb{N}$, $\alpha \geq 1$. By formula (5.12), we have that

$$
\begin{aligned}
&|\partial_{x_1}^{\alpha}(S_\psi^t \Phi)(x_1,x_2)| \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{1}{4}}|\xi_1|^{\alpha-1}|\mathcal{F}\Phi(\xi,s,a)|\ |\mathcal{F}\chi_1(-a\xi_1)||\phi_2(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{[0,\epsilon]} \frac{(1+|\xi|^2)^{N/2}}{(1+|\xi|^2)^{N/2}}(1+s^2)a^{-\frac{13}{4}}|\xi_1|^{\alpha-1}|\mathcal{F}\Phi(\xi,s,a)|\ |\mathcal{F}\chi_1(-a\xi_1)| \\
&|\phi_2(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&+\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{[\epsilon,+\infty)} \frac{(1+|\xi|^2)^{N/2}}{(1+|\xi|^2)^{N/2}}(1+s^2)a^{-\frac{13}{4}}|\xi_1|^{\alpha-1}|\mathcal{F}\Phi(\xi,s,a)|\ |\mathcal{F}\chi_1(-a\xi_1)| \\
&|\phi_2(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&\lesssim \rho^{0,0,0,0}_{N+\alpha-1,N,2,\frac{13}{4}}(\mathcal{F}\Phi)+\rho^{0,0,0,0}_{N+\alpha-1,N,2,0}(\mathcal{F}\Phi),
\end{aligned}
$$

where $N \in \mathbb{N}$, $N > 2$ and $\epsilon > 0$. We can treat $|\partial_{x_2}^{\beta}(S_\psi^t\Phi)(x_1,x_2)|$, $\beta \in \mathbb{N}$, $\beta \geq 1$, with the same approach. Now, we consider multiplication by powers of x_1. We fix $k \in \mathbb{N}$, $k \geq 1$. By formula (5.12), we have that

$$
\begin{aligned}
&|x_1^k(S_\psi^t\Phi)(x_1,x_2)| \\
&=|\int_{\mathbb{R}^2} x_1^k e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/4}\mathcal{F}\Phi(\xi,s,a)\ |\xi_1|^{-1}\times \\
&\times\overline{\mathcal{F}\chi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1-s))}\frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&=|\int_{\mathbb{R}^2}(2\pi i)^{-k}e^{2\pi i x\cdot\xi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/4}\partial_{\xi_1}^k[\mathcal{F}\Phi(\xi,s,a)\ |\xi_1|^{-1}\times \\
&\times\overline{\mathcal{F}\chi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1-s))}]\frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2| \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/4}|\partial_{\xi_1}^k[\mathcal{F}\Phi(\xi,s,a)\ |\xi_1|^{-1}\overline{\mathcal{F}\chi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1-s))}]|\frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2,
\end{aligned}
$$

which is minor-equal of a finite sum of addends of the form

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{1}{4}+k_3-\frac{k_4}{2}}|\partial_{\xi_1}^{k_1}\mathcal{F}\Phi(\xi,s,a)||\xi_1|^{-k_2-k_4}|(\mathcal{F}\chi_1)^{(k_3)}(-a\xi_1)|\times \\
&\times|\xi_2|^{k_4}|\phi_2^{(k_4)}(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{a^3}\mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{13}{4}+k_3-\frac{k_4}{2}}(1+|\xi|^2)^{N/2}(1+s^2)|\partial_{\xi_1}^{k_1}\mathcal{F}\Phi(\xi,s,a)||\xi_1|^{-k_2-k_4}\times \\
&\times|(\mathcal{F}\chi_1)^{(k_3)}(-a\xi_1)||\xi_2|^{k_4}|\phi_2^{(k_4)}(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)(1+|\xi|^2)^{N/2}}\mathrm{d}\xi_1\mathrm{d}\xi_2,
\end{aligned}
$$

where k_1, k_2, k_3, $k_4 \in \mathbb{N}$ are less than k. Since $\mathcal{S}_0(\mathbb{R})$ is closed under multiplications by a polynomial, by Lemma 1.50 for any given $k_3, m \in \mathbb{N}$ there exists $g \in \mathcal{S}_0(\mathbb{R})$ such that

$$(\mathcal{F}\chi_1)^{(k_3)}(-a\xi_1) = -a^m\xi_1^m\mathcal{F}g(-a\xi_1), \quad \xi_1 \in \mathbb{R},$$

and we can continue the chain of inequalities with terms of the form

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{13}{4}+k_3-\frac{k_4}{2}+m}(1+|\xi|^2)^{N/2}(1+s^2)|\partial_{\xi_1}^{k_1}\mathcal{F}\Phi(\xi,s,a)||\xi_1|^{-k_2-k_4+m}|\mathcal{F}g(-a\xi_1)|$$
$$|\xi_2|^{k_4}|\phi_2^{(k_4)}(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)(1+|\xi|^2)^{N/2}}\mathrm{d}\xi_1\mathrm{d}\xi_2.$$

Finally, choosing $N\in\mathbb{N}$, $N>2$, $m\geq k_2+k_4$ and dividing the integral over $\mathbb{R}_+$ as $[0,\epsilon]\cup(\epsilon,+\infty)$, with $\epsilon>0$, we obtain that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{13}{4}+k_3-\frac{k_4}{2}+m}(1+|\xi|^2)^{N/2}(1+s^2)|\partial_{\xi_1}^{k_1}\mathcal{F}\Phi(\xi,s,a)||\xi_1|^{-k_2-k_4+m}|\mathcal{F}g(-a\xi_1)|$$
$$|\xi_2|^{k_4}|\phi_2^{(k_4)}(a^{-1/2}(\xi_2/\xi_1-s))|\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)(1+|\xi|^2)^{N/2}}\mathrm{d}\xi_1\mathrm{d}\xi_2$$
$$\lesssim[\rho^{k_1,0,0,0}_{N+m-k_2-k_4,N+k_4,2,|\frac{13}{4}-k_3+\frac{k_4}{2}-m|}(\mathcal{F}\Phi)+\rho^{k_1,0,0,0}_{N+m-k_2-k_4,N+k_4,2,|-k_3+\frac{k_4}{2}-m|}(\mathcal{F}\Phi)].$$

We can treat $|{x_2}^k(S^t_\psi\Phi)(x_1,x_2)|$, $k\in\mathbb{N}$, $k\geq 1$, with the same approach and we conclude that the shearlet synthesis operator $\mathcal{S}^t_\psi:\mathcal{S}(\mathbb{S})\to\mathcal{S}(\mathbb{R}^2)$ is continuous. Finally, it remains to prove that $S^t_\psi\Phi\in\mathcal{S}_0(\mathbb{R}^2)$, which is equivalent to prove that

$$\lim_{\xi\to 0}\frac{\mathcal{F}S^t_\psi\Phi(\xi)}{|\xi|^k}=0, \tag{5.13}$$

for every $k\in\mathbb{N}$, see [42, Lemma 6.0.4]. We start observing that by formula (5.12) and by the Fourier inversion formula

$$\mathcal{F}\mathcal{S}^t_\psi\Phi(\xi_1,\xi_2)=\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/4}\mathcal{F}\Phi(\xi,s,a)\ |\xi_1|^{-1}\overline{\mathcal{F}\chi_1(-a\xi_1)\phi_2(a^{-1/2}(\xi_2/\xi_1-s))}\ \frac{\mathrm{d}a\mathrm{d}s}{a^3}, \tag{5.14}$$

where we recall that $\mathcal{F}\Phi$ denotes the Fourier transform of Φ with respect to the variable b. We prove that for every $k\in\mathbb{N}$ there exists $N_k>0$ and a constant $C>0$ such that

$$\frac{|\mathcal{F}S^t_\psi\Phi(r\cos\theta,r\sin\theta)|}{r^k}\leq Cr^{N_k},$$

for every $\theta\in[0,2\pi)$. By equation (5.14), we have that

$$|\mathcal{F}S^t_\psi\Phi(r\cos\theta,r\sin\theta)|$$
$$\lesssim\int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-1/2}|\mathcal{F}F((r\cos\theta,r\sin\theta),s,a)|r^{-1}|\cos\theta|^{-1}|\mathcal{F}\chi_1(-ar\cos\theta)|\times$$
$$\times|\phi_2(a^{-1/2}(\tan\theta-s))|\frac{\mathrm{d}a\mathrm{d}s}{a^3},$$

with the implied constant independent of $\theta\in[0,2\pi)$. By Lemma 1.50, for every $m\in\mathbb{N}$ there exists $g\in\mathcal{S}_0(\mathbb{R})$ such that

$$\mathcal{F}\chi_1(-a\tau)=-a^m\tau^m\mathcal{F}g(-a\tau),\quad\tau\in\mathbb{R},$$

and we can continue the above inequality as follows

$$\begin{aligned}
&|\mathcal{F}S^t_\psi\Phi(r\cos\theta, r\sin\theta)| \\
&\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}_+} a^{-\frac{13}{4}+m}|\mathcal{F}F((r\cos\theta, r\sin\theta), s, a)|r^{-1+m}|\cos\theta|^{-1+m}|\mathcal{F}g(-ar\cos\theta)|\times \\
&\times|\phi_2(a^{-1/2}(\tan\theta - s))|(1+s^2)\frac{\mathrm{d}a\mathrm{d}s}{(1+s^2)}.
\end{aligned}$$

Then, choosing $m > 1$ and dividing the integral over $\mathbb{R}_+$ as $[0,\epsilon]\cup(\epsilon,+\infty)$, for some $\epsilon > 0$, we obtain that

$$|\mathcal{F}S^t_\psi\Phi(r\cos\theta, r\sin\theta)| \lesssim r^{-1+m}[\rho^{0,0,0,0}_{0,0,2,\frac{13}{4}}(\mathcal{F}\Phi) + \rho^{0,0,0,0}_{0,0,2,m}(\mathcal{F}\Phi)] \lesssim r^{-1+m},$$

where the implied constant is independent of $\theta \in [0, 2\pi)$. Therefore, if we choose $m > k + 1$, we have that

$$\frac{|\mathcal{F}S^t_\psi\Phi(r\cos\theta, r\sin\theta)|}{r^k} \lesssim \frac{r^{-1+m}}{r^k},$$

which implies that (5.13) holds true for every $k \in \mathbb{N}$ and we conclude that $S^t_\psi\Phi$ belongs to $\mathcal{S}_0(\mathbb{R}^2)$. □

5.4 The shearlet transform on $\mathcal{S}_0'(\mathbb{R}^2)$

We are ready to define the shearlet transform of Lizorkin distributions.

Definition 5.6 ([10])**.** We define the shearlet transform of $f \in \mathcal{S}_0'(\mathbb{R}^2)$ with respect to ψ as the element $\mathcal{S}_\psi f \in \mathcal{S}'(\mathbb{S})$ whose action on test functions is given by

$$\langle \mathcal{S}_\psi f, \Phi\rangle = \langle f, \mathcal{S}^t_{\overline{\psi}}\Phi\rangle, \quad \Phi \in \mathcal{S}(\mathbb{S}).$$

The consistence of Definition 5.6 is guaranteed by Theorem 5.5.

Definition 5.7 ([10])**.** We define the shearlet synthesis operator of $\Phi \in \mathcal{S}'(\mathbb{S})$ with respect to ψ as the element $\mathcal{S}^t_\psi\Phi \in \mathcal{S}_0'(\mathbb{R}^2)$ whose action on test functions is given by

$$\langle \mathcal{S}^t_\psi\Phi, f\rangle = \langle \Phi, \mathcal{S}_{\overline{\psi}}f\rangle, \quad f \in \mathcal{S}_0(\mathbb{R}^2).$$

The consistence of Definition 5.7 is guaranteed by Theorem 5.4. Furthermore, we immediately obtain the following result.

Proposition 5.8 ([10])**.** *The shearlet transform $\mathcal{S}_\psi\colon \mathcal{S}_0'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{S})$ and the shearlet synthesis operator $\mathcal{S}^t_\psi : \mathcal{S}'(\mathbb{S}) \to \mathcal{S}_0'(\mathbb{R}^2)$ are continuous linear operators.*

Proof. The proof follows straightforwardly by Theorem 5.4 and 5.5. □

Finally, we can generalize reconstruction formula (3.53) to Lizorkin distributions.

Proposition 5.9 ([10])**.** *We have that*

$$\mathrm{id}_{\mathcal{S}_0'(\mathbb{R}^2)} = \mathcal{S}^t_\psi \circ \mathcal{S}_\psi.$$

Proof. By Definitions 5.6, 5.7 and by equation (5.11), we have that for every $f \in \mathcal{S}_0'(\mathbb{R}^2)$

$$\langle (\mathcal{S}^t_\psi \circ \mathcal{S}_\psi) f, \phi \rangle = \langle \mathcal{S}_\psi f, \mathcal{S}_{\overline{\psi}} \phi \rangle = \langle f, (\mathcal{S}^t_\psi \circ \mathcal{S}_{\overline{\psi}}) \phi \rangle = \langle f, \phi \rangle,$$

for every $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ and we conclude. □

The next theorem shows that Definition 5.6 is consistent with the definition of shearlet transform for test functions, see § 3.1.2. Furthermore, it shows that our definition generalizes the one considered in [51, 34] on $\mathcal{S}'(\mathbb{R}^2) \subseteq \mathcal{S}_0'(\mathbb{R}^2)$. Here the authors defines the shearlet transform of a tempered distribution f with respect to an admissible vector $\psi \in \mathcal{S}(\mathbb{R}^2)$ as the function on $\mathbb{S}$ given by

$$\mathcal{S}_\psi f(b, s, a) = \langle f, S_{b,s,a}\psi \rangle,$$

which is well-defined since $S_{b,s,a}\psi$ still belongs to $\mathcal{S}(\mathbb{R}^2)$.

Theorem 5.10 ([10])**.** *Let $f \in \mathcal{S}_0'(\mathbb{R}^2)$. The shearlet transform of f is given by the function*

$$\mathcal{S}_\psi f(b, s, a) = \langle f, S_{b,s,a}\psi \rangle,$$

that is, for any $\Phi \in \mathcal{S}(\mathbb{S})$,

$$\langle \mathcal{S}_\psi f, \Phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle f, S_{b,s,a}\psi \rangle \Phi(b, s, a) \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3}.$$

Furthermore, $\mathcal{S}_\psi f(b, s, a)$ is a function of at most polynomial growth on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$.

Proof. Consider $f \in \mathcal{S}_0'(\mathbb{R}^2)$. Since the space of Lizorkin distributions $\mathcal{S}_0'(\mathbb{R}^2)$ is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^2)$ by the space of polynomials, by Schwartz' structural theorem [65, Teorema VI], we can write $f = g^{(\alpha)} + p$, where g is a continuous slowly growing function, $\alpha \in \mathbb{N}^2$ and p is a polynomial. Then, for any $\Phi \in \mathcal{S}(\mathbb{S})$

$$\begin{aligned}
\langle g^{(\alpha)}, \mathcal{S}^t_\psi \Phi \rangle &= (-1)^{|\alpha|} \langle g, \mathcal{S}^t_\psi \Phi^{(\alpha)} \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^2} g(x) \mathcal{S}^t_\psi \Phi^{(\alpha)}(x) \mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^2} g(x) \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ (S_{b,s,a}\psi)^{(\alpha)}(x) \ \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3} \mathrm{d}x \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ (-1)^{|\alpha|} \int_{\mathbb{R}^2} g(x) (S_{b,s,a}\psi)^{(\alpha)}(x) \mathrm{d}x \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3} \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ (-1)^{|\alpha|} \langle g, (S_{b,s,a}\psi)^{(\alpha)} \rangle \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3} \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ \langle g^{(\alpha)}, S_{b,s,a}\psi \rangle \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3}.
\end{aligned}$$

Analogously,

$$\langle p, \mathcal{S}^t_\psi \Phi \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ \langle p, S_{b,s,a}\psi \rangle \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3}.$$

Therefore,

$$\begin{aligned}
\langle f, \mathcal{S}^t_\psi \Phi \rangle = \langle g^{(\alpha)} + p, \mathcal{S}^t_\psi \Phi \rangle &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ \langle g^{(\alpha)} + p, S_{b,s,a}\psi \rangle \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3} \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Phi(b, s, a) \ \langle f, S_{b,s,a}\psi \rangle \frac{\mathrm{d}b \mathrm{d}s \mathrm{d}a}{a^3}
\end{aligned}$$

and we conclude. It remains to prove that $\mathcal{S}_\psi f(b,s,a)$ is a function of slow growth on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$. We divide the proof in three steps. We start considering $\mathcal{S}_\psi f(b,0,1)$ for every $b \in \mathbb{R}^2$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}|\mathcal{S}_\psi f(b,0,1)| = |\langle f, S_{b,0,1}\psi\rangle| &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |(S_{b,0,1}\psi)^{(m)}(x)| \\ &= \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |\psi^{(m)}(x-b)| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \langle x-b\rangle^{-\nu} \lesssim \langle b\rangle^\nu,\end{aligned}$$

where we have used that $\psi \in \mathcal{S}(\mathbb{R}^2)$ and Peetre's inequality. We now consider $\mathcal{S}_\psi f(0,s,1)$ for every $s \in \mathbb{R}$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}&|\mathcal{S}_\psi f(0,s,1)| = |\langle f, S_{0,s,1}\psi\rangle| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |(S_{0,s,1}\psi)^{(m)}(x)| \\ &= \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \Big|\sum_{i=0}^{p_m} c_i s^i S_{0,s,1}\psi^{(m)}(x)\Big| \le \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i |\psi^{(m)}(S_s^{-1}x)| \\ &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i (1+|S_s^{-1}x|^2)^{-\frac{\nu}{2}}.\end{aligned}$$

We recall that, for any $x \in \mathbb{R}^2$ and $M \in GL(2,\mathbb{R})$, we have that

$$|M^{-1}x| \ge \|M\|^{-1}|x|,$$

where $\|M\|$ denotes the spectral norm of the matrix M. Thus, since $\|S_s\| = (1+\frac{s^2}{2} + (s^2+\frac{s^2}{2})^{1/2})^{1/2}$ for all $s \in \mathbb{R}$, we have that

$$\begin{aligned}&|\mathcal{S}_\psi f(0,s,1)| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i (1+(\|S_s\|^{-1}|x|)^2)^{-\frac{\nu}{2}} \\ &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i \langle \|S_s\|^{-1}x\rangle^{-\nu} \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu \sum_{i=0}^{p_m} |c_i||s|^i \|S_s\|^\nu \langle x\rangle^{-\nu} \\ &\lesssim \sup_{|m|\le\nu} \sum_{i=0}^{p_m} |c_i||s|^i \|S_s\|^\nu,\end{aligned}$$

which proves that the function $\mathcal{S}_\psi f(0,s,1)$ is of at most polynomial growth in s. Finally, we consider $\mathcal{S}_\psi f(0,0,a)$ for every $a \in \mathbb{R}_+$. Since $f \in \mathcal{S}_0'(\mathbb{R}^2)$, there exists $\nu \in \mathbb{N}$ such that

$$\begin{aligned}&|\mathcal{S}_\psi f(0,0,a)| = |\langle f, S_{0,0,a}\psi\rangle| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu |(S_{0,0,a}\psi)^{(m)}(x)| \\ &= \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}} |S_{0,0,a}\psi^{(m)}(x)| \le \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}} |\psi^{(m)}(A_a^{-1}x)| \\ &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}} (1+|A_a^{-1}x|^2)^{-\frac{\nu}{2}} \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}} \langle \|A_a\|^{-1}x\rangle^{-\nu},\end{aligned}$$

where $(m_1,m_2) \in \mathbb{N}^2$ and $m_1+m_2 = m$. Since $\|A_a\| = |a|^{\frac{1}{2}}$ for $|a| < 1$, then

$$|\mathcal{S}_\psi f(0,0,a)| \lesssim \sup_{x\in\mathbb{R}^2,\,|m|\le\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}} \langle x\rangle^{-\nu} \lesssim a^{-p},$$

for some $p \in \mathbb{N}$. If $|a| \geq 1$, $\|A_a\| = |a|$ and we have that

$$\begin{aligned}|\mathcal{S}_\psi f(0,0,a)| &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\leq\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}}(1+(\|A_a\|^{-1}|x|)^2)^{-\frac{\nu}{2}}\\ &\lesssim \sup_{x\in\mathbb{R}^2,\,|m|\leq\nu} \langle x\rangle^\nu a^{-m_1-\frac{m_2}{2}-\frac{3}{4}+\nu}\langle x\rangle^{-\nu} \lesssim \sup_{|m|\leq\nu} a^{-m_1-\frac{m_2}{2}-\frac{3}{4}+\nu} \lesssim a^\nu,\end{aligned}$$

which proves that the function $\mathcal{S}_\psi f(0,0,a)$ is of at most polynomial growth in a on $\mathbb{R}_+$. Therefore, by

$$\mathcal{S}_\psi f(b,s,a) = \langle f, S_{b,0,1}S_{0,s,1}S_{0,0,a}\psi\rangle$$

the thesis follows straightforwardly. □

We conclude this chapter showing that, through the shearlet transform, the action of every Lizorkin distribution on any test function can be written as an absolutely convergent integral over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$.

Corollary 5.11 ([10])**.** *Let $f \in \mathcal{S}_0'(\mathbb{R}^2)$. Then, for any $\phi \in \mathcal{S}_0(\mathbb{R}^2)$*

$$\langle f,\phi\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{S}_\psi f(b,s,a)\ \mathcal{S}_{\overline{\psi}}\phi(b,s,a)\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{a^3}.$$

Proof. If $f \in \mathcal{S}_0'(\mathbb{R}^2)$, by Proposition 5.9 and Theorem 5.10 we have that

$$\langle f,\phi\rangle = \langle(\mathcal{S}_\psi^t\circ\mathcal{S}_\psi)f,\phi\rangle = \langle\mathcal{S}_\psi f,\mathcal{S}_{\overline{\psi}}\phi\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \mathcal{S}_\psi f(b,s,a)\ \mathcal{S}_{\overline{\psi}}\phi(b,s,a)\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{a^3},$$

for every $\phi \in \mathcal{S}_0(\mathbb{R}^2)$. The absolutely convergence of the integral follows by the fact that $\mathcal{S}_{\overline{\psi}}\phi \in \mathcal{S}(\mathbb{S})$ and, by Theorem 5.10, $\mathcal{S}_{\overline{\psi}}\phi$ is a function of at most polynomial growth over $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$. This concludes the proof. □

We are strongly convinced that it is sufficient to take $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ in order to obtain continuity results for the shearlet transform like Theorems 5.4 and 5.5. This would allow to generalize Definitions 5.6 and 5.7 and consequently all the results in Section 5.4 enlarging the choice of the admissible vectors ψ. The development of this kind of arguments is under investigation.

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