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Some aspects of Analysis on symmetric spaces and homogeneous trees: from Radon transform to Bergman spaces

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Contents

Introduction

The work of my past three years is focused on the Analysis on two classes of non Euclidean spaces: the symmetric spaces of the noncompact type and the homogeneous trees. The two families of spaces are more linked than their structure might suggest.

In continuity with my master thesis, my work starts from the Radon transform. Given a space X and a family of subsets Ξ, the Radon transform of a function f on X is a function on Ξ defined by the integral of f restricted to every subset in Ξ. In [1] some of my collaborators have solved the unitarization problem for a class of Radon transforms. That is, they prove the existence of a pseudo-differential operator that, precomposed with the Radon transform, extends to a unitary operator $\mathcal{Q}$ on $L^2(X)$. The problem has been inspired by Helgason who addressed and solved it in the case of the polar Radon transform. The spaces X and Ξ analyzed in [1] are transitive G-spaces of a lcsc group G such that (X, Ξ) is a dual pair in the sense of Helgason. The Radon transform for dual pairs is classically defined by Helgason [39]. Under some technical assumptions they prove the existence of the unitarization of the Radon transform. Furthermore they show that $\mathcal{Q}$ intertwines the two quasi regular representations π and $\hat{\pi}$ of G on $L^2(X)$ and $L^2(\Xi)$, respectively. The latter result, under the additional hypothesis of square integrability of π, provides a new inversion formula for the Radon transform.

The hyperbolic disk, and, more in general, the family of symmetric spaces of the noncompact type, together with the family of its horocycles form a classical example of dual pair in the sense of Helgason. Since this setup does not satisfy the technical assumptions of [1], it is natural to investigate the unitarization problem in this context. Such problem is not directly addressed by Helgason who however shows that the composition of the Radon transform with an operator extends to an injective operator [37]. We cannot find a surjectivity result in his wide literature. Another natural context in which it is worthwhile to study the unitarization problem for the Radon transform is the homogeneous tree. Homogeneous trees have some common properties with the hyperbolic disk and also here it is possible to define the horocyclic Radon transform. The question was instead totally open on homogeneous trees.

The two settings are, in a certain way, similar and the problem is the same. Our contribution is the completion of the unitarization result for the symmetric spaces by using techniques designed to fit well in the case of the homogeneous trees. Indeed some intermediate results coincide in the statements and the spaces involved are clearly in relation (see Figure 1), though the techniques used in the proofs are fairly different. Just to give an idea, on symmetric spaces we can count on the Iwasawa decomposition of the associated semisimple Lie group, while in the other case, since there is no decomposition

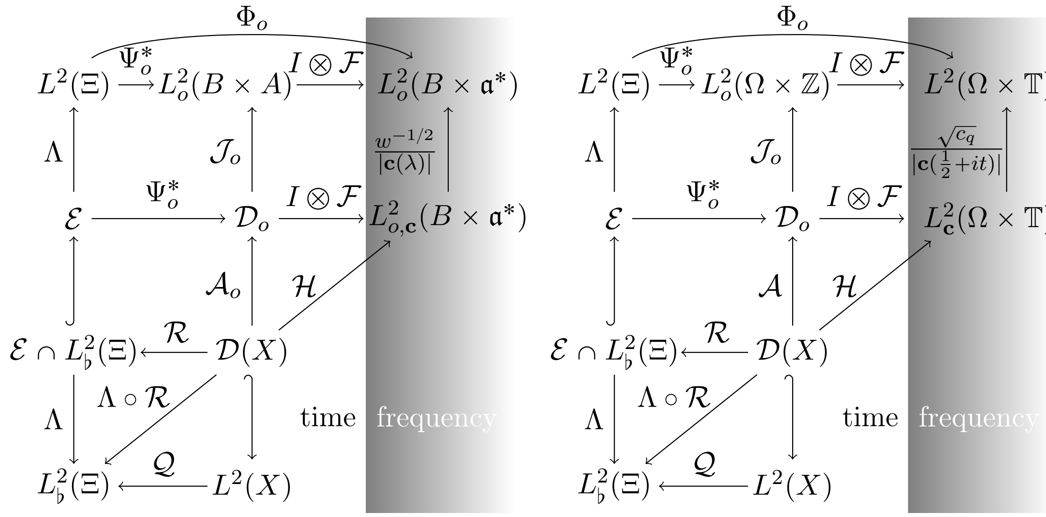

Figure 1: The two diagrams represent the main relations between the spaces and the operators involved in the unitarization of the horocyclic Radon transform on symmetric spaces and homogeneous trees, respectively.

for the group of isometries of the homogeneous tree (still denoted G), we use the easier geometry of the tree.

In both settings, the range of the unitarization of the Radon transform results to be a subspace $L^2_\flat(\Xi) \subseteq L^2(\Xi)$ which keeps track of the symmetries that the Radon transform inherits from the geometry of the spaces. Furthermore we prove that the unitarization $\mathcal{Q}$ intertwines the two quasi regular representations π and $\hat{\pi}$ of G on $L^2(X)$ and $L^2_\flat(\Xi)$, respectively. Unlike the cases studied in [1], in these setups the representation π is not square integrable and then we are not able to provide new inversion formulae for the Radon transform.

I essentially completed the solution of the unitarization problems of the two Radon transforms by using a mix of techniques of [6] and classical results on symmetric spaces and on homogeneous trees. Hence I discussed the results with F. Bartolucci and F. De Mari, which helped me to put the results in the correct perspective. With their experience and expertise, we wrote the paper in a more complete and satisfying way.

This problem motivates the second part of my work. We decided to study the class of square integrable representations on symmetric spaces and homogeneous trees. In the first case, it is well known that square integrable representations have a realization on the holomorphic Bergman spaces. We observed that it is not possible to obtain a unitarization result for the Radon transform on the hyperbolic disk which intertwines a square integrable representation on the Bergman space with a representation on the functions defined on Ξ.

We moved our attention to the case of the homogeneous tree X. Ol'shanskii classifies the square integrable representations of the group G of isometries on X in [45] without exhibiting any realization on spaces of functions on X. In view of the analogy between symmetric spaces and homogeneous trees, it is natural to ask whether a square integrable representation of G can be realized on a space related to the holomorphic

Bergman space. The notion of holomorphic function on X is not clearly stated in literature, but the notion of harmonic function there is well known. In [17], the harmonic Bergman spaces are introduced for functions defined on homogeneous trees. For every finite measure σ on X which is radial w.r.t. a fixed origin $o \in X$ and decreasing w.r.t. the distance from o, and for every $1 \leqslant p \leqslant \infty$ the (harmonic) Bergman space is defined as the space of the harmonic functions in $L^p(\sigma)$. We show that, as in the case of the holomorphic Bergman spaces on the disk, when $p = 2$ they are reproducing kernel Hilbert spaces. We provide a formula for the kernel and we study the boundedness properties of the extension of the projector to $L^p(\sigma)$. For simplicity we focus only on the family of measures which decrease radially exponentially. We show that for every measure of that family the extension of the projector to $L^p(\sigma)$ is bounded if and only if $p > 1$.

The problem has been posed to me by F. De Mari and M. Vallarino. I provided the formula of the kernel and essentially solved the problem of the boundedness of the projector. I drew inspiration from the approach of [25], [49], and [54], but the techniques used in the proofs are fairly different, since the holomorphic kernel on the disk has an easy Taylor decomposition that we do not have. With the help of the expertise of F. De Mari and M. Vallarino, I was able to rewrite in a clearer formulation the results obtained by me in a preliminary form.

The Radon transform

The Radon transform is introduced by J. Radon in the context of inverse problems. In particular he addresses the problem of reconstructing a function starting from its integrals on a family of subsets, typically lines or hyperplanes when the signal is defined on $\mathbb{R}^d$. Radon proves a first reconstruction formula in 1917 in the case of integration on hyperplanes of $\mathbb{R}^3$.

A generalization of the problem then is to reconstruct a function defined on a manifold through the values of its integrals on a family of submanifolds. Helgason, motivated by the example of the polar Radon transform, is the first to link the inversion problem with the theory of homogeneous spaces, by introducing the Radon transform in the context of dual pairs [39].

Given a function f defined on $\mathbb{R}^2$, the polar Radon transform of f on a line of the plane is defined as the integral of f restricted to the line. The polar structure comes from the parametrization of the family of lines by polar coordinates via $[0, 2\pi) \times \mathbb{R}$. Namely,

$$\mathcal{R}^{\mathrm{pol}} f(\theta, t) := \int_{\mathbb{R}} f(t\cos\theta - y\sin\theta, t\sin\theta + y\cos\theta)\mathrm{d}y, \qquad (\theta, t) \in [0, 2\pi) \times \mathbb{R}.$$

This means that $\mathcal{R}^{\mathrm{pol}}$ associates to a function on $\mathbb{R}^2$ a function on $[0, 2\pi) \times \mathbb{R}$. The idea of Helgason is to reduce this problem to the relation between the space $X = \mathbb{R}^2$ where the function is defined and the space of parameters $\Xi = [0, 2\pi) \times \mathbb{R}$. In this case it is possible to see X and Ξ as homogeneous spaces of the group of rigid motions of the plane: $G = \mathbb{R}^2 \times K$, where $K = \{R_\phi \colon \phi \in [0, 2\pi)\}$ with

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \qquad \phi \in [0, 2\pi).$$

The group law of G is

$$(b,\phi)(b',\phi') = (b + R_\phi b', \phi + \phi' \bmod 2\pi)$$

and gives to G structure of semidirect product of $\mathbb{R}^2$ and K, namely $G = \mathbb{R}^2 \rtimes K$. It is easy to see that G acts on $X = \mathbb{R}^2$ via the canonical action $(b,\phi)[x] = b + R_\phi x$. Furthermore the isotropy of G at $o \in \mathbb{R}^2$ is K and then $G/K \simeq X$. Observe that the action of (b,ϕ) on $\mathbb{R}^2$ is affine and maps lines into lines. Hence G acts on Ξ, too, via

$$(b,\phi).(\theta,t) = (\theta + \phi \bmod 2\pi, t + n(\theta)\cdot R_\phi^{-1} b),$$

where ${}^t n(\theta) = (\cos\theta, \sin\theta)$. The isotropy of G at the y-axis $(0,0) \in [0,2\pi) \times \mathbb{R}$ is

$$H = \{((0,b_2),\phi) \colon \phi \in \{0,\pi\}, b_2 \in \mathbb{R}\}$$

and then $\Xi \simeq G/H$. The fact that a point $x \in \mathbb{R}^2$ belongs to a line $\xi \in \Xi$ to the fact that the cosets $x = g_1 K$ and $\xi = g_2 H$ intersect. Helgason extends this notion of intersection to any pair of homogeneous spaces of the same group G.

We generalize this context. Let G be a lcsc group. We consider two G-transitive spaces X and Ξ. We fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote by K and H their stabilizers at G, respectively, so that $X = G/K$ and $\Xi = G/H$. Two elements $x = g_1 K \in X$ and $\xi = g_2 H \in \Xi$ are said to be incident if as cosets in G they intersect. Then it is possible to define

$$\check{x} = \{\xi \in \Xi \colon x \text{ and } \xi \text{ intersect}\} \subseteq \Xi;$$
$$\widehat{\xi} = \{x \in X \colon x \text{ and } \xi \text{ intersect}\} \subseteq X,$$

which are closed subsets by Lemma 1.1 in [39]. Looking back at the example of the polar Radon transform, it is easy to see that $\widehat{\xi}$ consists of the set of points lying on the line ξ, basically the realization in X of the line parametrized by $\xi \in \Xi$. On the other hand, $\check{x}$ is the "sheaf" of (the parameters of) lines in Ξ passing through x. It is immediate to see that if $x = g_1[x_0]$ and $\xi = g_2.\xi_0$ then

$$\check{x} = g_1.\check{x}_0, \qquad \widehat{\xi} = g_2[\widehat{\xi}_0]$$

Suppose that $\widehat{\xi}_0$ carries an H-invariant measure $\mathrm{d}m_0$. We push-forward the measure m_0 to every $\widehat{\xi} = g\widehat{\xi}_0$ by the map $\widehat{\xi}_0 \ni x \mapsto g[x] \in \widehat{\xi}$. Hence we can define the Radon transform by

$$\mathcal{R}f(\xi) = \int_{\widehat{\xi}} f(x)\mathrm{d}m_\xi(x) := \int_{\widehat{\xi}_0} f(g[x])\mathrm{d}m_0(x).$$

An important result on Radon transforms is the unitarization theorem. It replaces the unitary extension of the Radon transform which, in general, is not possible to achieve. Helgason states in [39] the existence of a pseudo-differential operator Λ such that $\Lambda\mathcal{R}^{\mathrm{pol}}$ extends to a unitary operator $\mathcal{Q} \colon L^2(X) \to L^2(\Xi)$.

In [6], F. Bartolucci, F. De Mari, E. De Vito and F. Odone obtain both an unitarization and an intertwining result for a different Radon transform, the affine Radon transform. The techniques used in [6] mimic the approach followed by Helgason to unitarize the polar Radon transform. In [1], the same authors together with G. S. Alberti present a new version of the unitarization theorem for a large class of Radon transforms. They consider the case in which the spaces X and Ξ forms a dual pair of lcsc transitive G-spaces of a lcsc group G. They follow a different approach, based on representation theory. Indeed, they prove that, under some technical assumptions, the unitary extension $\mathcal{Q}$ exists and it intertwines the quasi regular representations π and $\hat{\pi}$ of G on $L^2(X)$ and $L^2(\Xi)$, respectively. The main result is based on a generalization of Schur's lemma due to Duflo and Moore [21] and requires the irriducibility of π. Furthermore they show that, if π is square integrable, then an inversion formula for the Radon transform follows from the unitarization result.

In the first part of my work, presented in Chapter 2 and 3, we investigated which results of [1] extends to new cases. The first setup we decided to investigate has been the hyperbolic disk and, more in general, the noncompact symmetric spaces. The motivation is fairly easy: the space of horocycles defined there, together with the symmetric space, forms a prototypical example of dual pair in the sense of Helgason. Another family of spaces on which we focused are the homogeneous trees, due to their deeply studied relation with symmetric spaces of rank one [16]. Among other correspondences, horocycles are defined on homogeneous trees as well and they form a dual pair with the tree in the sense of Helgason.

The techniques used in [1] cannot be transferred directly to the cases of symmetric spaces and homogeneous trees primarily because the quasi regular representations are not irreducible, much less square integrable. Hence, we adopted in both setups a combination of the classical results of the theory of symmetric spaces and homogeneous trees (presented in [37] and [22], respectively) and the techniques that have been exploited in [6].

Helgason essentially solves the unitarization problem in the case of symmetric spaces. Indeed he proves that there exists an operator Λ which, precomposed with the horocyclic Radon transform, extends to a injective operator on $L^2(X)$. We are not able to find a unitarization result in the wide literature of Helgason's. In our work then we characterize the closed subspace of $L^2(\Xi)$ that, by keeping track of all the symmetries of the Radon transform, is the range of the extension.

The horocyclic Radon transform on homogeneous trees was first introduced by P. Cartier [14] and studied by A. Figà-Talamanca and M.A. Picardello [23], W. Betori, J. Faraut and M. Pagliacci [12], M. Cowling, S. Meda and A.G. Setti [18], J. Cohen, F. Colonna and E. Tarabusi [15], and A. Veca [52], to name a few.

Although the two setups have many common properties and the problem we solve is essentially the same, some steps are fairly different. As we said before, the approach is based on [6]. The crucial point is the relation between the Helgason-Fourier and the horocyclic Radon transforms (that is, the Fourier slice theorem) and the correspondence between the range of the former with the closed subspace $L^2_\flat(\Xi) \subseteq L^2(\Xi)$ which is the range of the unitarization. Horocycles are defined in both setups. We are interested in their parametrization. We recall that the semisimple Lie group G associated to the symmetric space has an Iwasawa decomposition $G = KAN$. Observe that the role played by A on symmetric spaces is, in a certain sense, played by $\mathbb{Z}$ on homogeneous trees. Indeed, on symmetric spaces, horocycles are parametrized by a boundary point and an element of A, while on homogeneous trees by a boundary point and an integer.

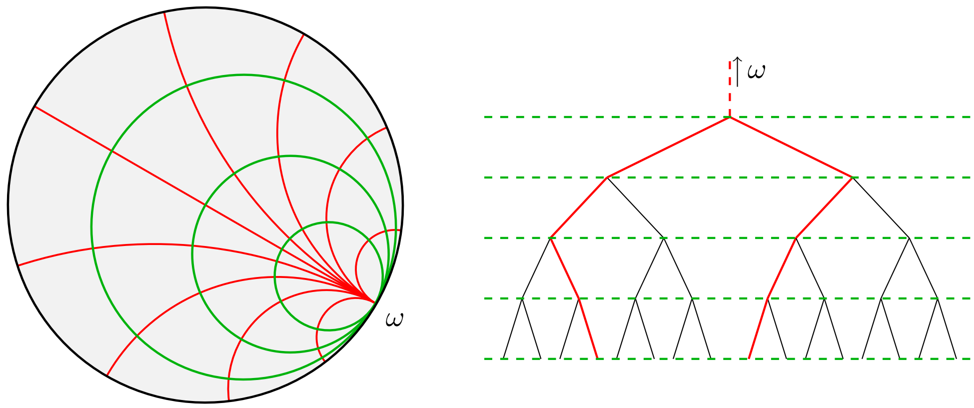

Figure 2: On the left the hyperbolic disk, that is the open unit disk in $\mathbb{C}$. Geodesics are diameters and portions of Euclidean circles orthogonal to the boundary. On the right a part of a 2-homogeneous tree. Here geodesics are (double) infinite chains. In both sides we highlight in red some geodesics lying in the bundle of parallel geodesics ending at the boundary point ω and in green some of the horocylces (on the tree, only portions of them) which are orthogonal to such bundle.

We show that in both cases, the operator Λ, involved in the unitarization, can be expressed as the conjugation of a Fourier multiplier $\mathcal{J}$ on A or on $\mathbb{Z}$ by a diffeomorphism Φ_o^*. The mapping Φ_o^* "transforms" a function on Ξ in a function on the parameters. Furthermore we prove that $\Lambda\mathcal{R}$ extends to $\mathcal{Q}\colon L^2(X) \to L^2_\flat(\Xi)$ in such a way that

$$\hat{\pi}(g)\mathcal{Q} = \mathcal{Q}\pi(g), \qquad g \in G, \tag{1}$$

where $\hat{\pi}$ and π are the quasi regular representations of G on $L^2(X)$ and $L^2_\flat(\Xi)$, respectively.

The Bergman spaces on homogeneous trees

In view of [1], where the authors use the square integrability of π and (1) to invert the Radon transform, it is natural to ask whether it is possible to replace π with a square integrable representation of the group in (1). Let us consider the hyperbolic disk. In this setting, it is well known that each square integrable representation of $\mathrm{SU}(1,1)$ has a realization on a Bergman space. For every $\alpha > 1$, we consider the measure $\mathrm{d}\nu_\alpha(x+iy) = (1-x^2-y^2)^{\alpha-2}\mathrm{d}x\mathrm{d}y$ on the disk. We call Bergman space the closed subspace $\mathcal{A}^p_\alpha(\mathbb{D})$ of $L^p(\mathbb{D},\nu_\alpha)$ consisting of holomorphic functions. The square integrable representations of $\mathrm{SU}(1,1)$ are of the following form. Let $m \in \mathbb{N}$, $m > 1$. The representation π_m of $\mathrm{SU}(1,1)$ on $\mathcal{A}^2_\alpha(\mathbb{D})$ is defined by

$$\pi_m(g)f(x) := (\overline{bx+a})^{-m} f(g^{-1}[x]), \qquad g^{-1} = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathrm{SU}(1,1),\ f \in \mathcal{A}^2_m(\mathbb{D}).$$

To the best of our knowledge, it is not possible to obtain (1) by replacing π with π_m.

It is well known that $\mathcal{A}^2_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space for every $\alpha > 1$ with kernel

$$\mathcal{K}_\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^\alpha}, \qquad z,\, w \in \mathbb{D}. \tag{2}$$

Since $\mathcal{A}^p_\alpha(\mathbb{D})$ is a closed subspace of $L^p(\mathbb{D},\nu_\alpha)$, there exists an orthogonal projection $P_\alpha \colon L^2(\mathbb{D},\nu_\alpha) \to \mathcal{A}^2_\alpha(\mathbb{D})$ and it is immediate to see that

$$P_\alpha f(z) = \int_{\mathbb{D}} f(w)\mathcal{K}_\alpha(z,w)\mathrm{d}\nu_\alpha(w), \qquad f \in \mathcal{A}^2_\alpha(\mathbb{D}).$$

The boundedness properties of the extension of P_α to $L^p(\mathbb{D},\nu_\alpha)$ are known. Indeed, Theorem 4.24 in [54] states that, for every $\alpha > 1$, the extension of P_α is bounded on $L^p(\mathbb{D},\nu_\alpha)$ if and only if $p > 1$. The two fundamental works on this topic are [25] and [49] which, among other results, solve the problem for the extension to the non-weighted Bergman spaces, that is when $\alpha = 2$. Zhu in [54] rewrites this result extending to the case of any α.

It is natural to ask whether this setup is replied on homogeneous trees. The square integrable representations have been classified by G. I. Ol'shanskii in [45], but no explicit realization on Hilbert spaces of functions is given. Further, to the best of our knowledge, a definition of holomorphic function on homogeneous trees is not available. Hence it is not obvious how to define the analogous of holomorphic Bergman spaces.

By borrowing from the mean value property of harmonic functions, on homogeneous trees we say that a function is harmonic if its value at a vertex is equal to the average of the values in the neighbors. In [17], authors introduce Bergman spaces by replacing the requirement for the functions to be harmonic rather than holomorphic. Although it appears to be hard to define a unitary representation on these spaces, we decided to investigate them. In particular we want to determine which properties they have in common with their "continuous" and holomorphic counterpart defined above.

Consider a finite measure σ on the q-homogeneous tree X whose Radon-Nikodym derivative w.r.t. the counting measure is a function, still denoted by σ, which is radial (w.r.t. a fixed origin o) and decreasing. We call it reference measure. For every $1 \leqslant p < \infty$, the Bergman space $\mathcal{A}^p(X,\sigma)$ is the (closed) subspace of $L^p(X,\sigma)$ consisting of harmonic functions. They are Banach spaces and, for $p = 2$, Hilbert spaces. In the work presented in Chapter 4 we show that $\mathcal{A}^2(X,\sigma)$ is a reproducing kernel Hilbert space for every reference measure σ. The discrete structure of the tree, together with the fact that we are considering harmonic functions and not holomorphic functions, makes the formula for the kernel more troublesome than (2). Theorem 4.9 shows that there exists a function Γ (independent of σ), a constant $B_\sigma > 0$, and a positive sequence $\{b_n\}_{n\in\mathbb{N}}$ such that

$$K_\sigma(z,x) = \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2}\sum_{v\in X}\frac{1}{b_{|v|}}\Gamma(v,z,x)(1-q^{|v|-|z|})(1-q^{|v|-|x|}),$$

where $|x| = d(o,x)$, with the canonical distance on X.

In what follows, we use this formula to obtain boundedness results for the Bergman projectors. As in the continuous case, there exists an orthogonal projector $P_\sigma \colon L^2(X,\sigma) \to$

$\mathcal{A}^2(X,\sigma)$ and it can be expressed via the kernel by

$$P_\sigma f(z) = \sum_{x\in X} K_\sigma(z,x) f(x)\sigma(x), \qquad f \in L^2(X,\sigma).$$

We focus on a family of reference measures which appears to be the natural counterpart of the measures considered on the disk. For every $\alpha > 1$, we denote by

$$\mu_\alpha(x) = q^{-\alpha|x|}, \qquad x \in X.$$

The boundedness result for $P_{\mu_\alpha} = P_\alpha$ is obtained as byproduct of a boundedness result for the more general operators:

$$S_{a,b,c} f(z) = q^{-a|z|} \sum_{x\in X} |K_c(z,x)| f(x) q^{-b|x|};$$
$$T_{a,b,c} f(z) = q^{-a|z|} \sum_{x\in X} K_c(z,x) f(x) q^{-b|x|},$$

for $a, b \in \mathbb{R}$ and $c > 1$. Our main results are Theorems 4.10 and 4.11, where we show under which condition on a, b, c, α, p the operators $S_{a,b,c}$ and $T_{a,b,c}$ are bounded on $L^p(X,\mu_\alpha)$. As a consequence of the fact that $P_\alpha = T_{0,\alpha,\alpha}$ we have Theorem 4.15 in which we show that P_α is bounded on $L^p(X,\mu_\alpha)$ if and only if $p > 1$.

A general question that arises form the previous results is: what do we know for the other reference measures? The problem appears to be nontrivial. For the case $p = 1$, we are able to find a counterexample to the boundedness for a large family of reference measures, called optimal, introduced in [17]. On the other hand, we are aware that for a subfamily of optimal measures it is possible to prove that P_σ is bounded on $L^p(X,\sigma)$ for every $p > 1$. The characterization of the class of measures σ for which the boundedness of P_σ on $L^p(X,\sigma)$ is equivalent to $p > 1$ is part of the work I am carrying on with F. De Mari and M. Vallarino.

The work of this thesis is contained in the following list of papers:

1. Francesca Bartolucci and Matteo Monti. Unitarization and inversion formula for the Radon transform for hyperbolic motions. In *2019 13th International conference on Sampling Theory and Applications (SampTA)*, pages 1–5, 2019.

2. Francesca Bartolucci, Filippo De Mari, and Matteo Monti. *Unitarization of the Horocyclic Radon Transform on Symmetric Spaces*, pages 1–54. Springer International Publishing, Cham, 2021.

3. Francesca Bartolucci, Filippo De Mari, and Matteo Monti. Unitarization of the horocyclic Radon transform on homogeneous trees. *Journal of Fourier Analysis and Applications*, 27(5):84, 2021.

4. Filippo De Mari, Matteo Monti, and Maria Vallarino. Boundedness of harmonic Bergman projectors on homogeneous trees, in preparation.

Chapter 1

Preliminaries

This chapter collects several notions from different branches which have the common property of being useful in the following chapters.

The chapter is organized as follows. In Section 1.1 we first recall the basic definitions and results used in Analysis on groups, as the Haar measure, representation theory and the Fourier transform on Abelian groups, and then we present homogeneous spaces. Section 1.2 contains a list of important classes of Lie algebras and the corresponding list of Lie groups. In particular the notion of semisimple Lie group is presented, which will play a crucial role in Chapter 2, together with the Iwasawa decomposition. Finally, in Section 1.3 we present the classical definition of Radon transform and a short version of [1] in which an extension of the classical Radon transform is used to obtain a result which has been the inspiration for Chapter 2 and 3.

1.1 Analysis on Lie groups and homogeneous spaces

The purpose of this section is to recall the basic facts of Analysis on groups and to establish the notation used throughout. In Section 1.1.1 we present measure theory on locally compact groups, which is then used in the particular case of Lie groups. The general reference is [24]. Section 1.1.2 is devoted to a brief summary of group representation theory, with focus on square integrable representations. We refer the reader to [20] and [24] where this material is developed at some length. Then in Section 1.1.3 we resume the theory which leads to define the Fourier transform on Abelian groups. In addition to $\mathbb{R}$, it will be used on $\mathbb{Z}$, $\mathbb{T}$, and on the Abelian component of semisimple Lie groups. In the last section we present homogeneous spaces, on which the analysis is carried out in what follows. General reference for the last two sections is again [24].

1.1.1 Haar measures and modular functions

A Borel measure μ on the topological space X, that is, a meausure on the σ-algebra $\mathcal{B}(X)$ of the Borel sets of X, is called a *Radon measure* if:

(i) μ is finite on compact sets;

(ii) μ is outer regular on the Borel sets, that is for every Borel set E

$$\mu(E) = \inf\{\mu(U) : U \supset E,\ U \text{ open}\};$$

(iii) μ is inner regular on the open sets, that is for every open set U

$$\mu(U) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

Definition 1.1. A *left Haar measure* on the topological group G is a non zero Radon measure μ such that $\mu(xE) = \mu(E)$ for every Borel set $E \subset G$ and every $x \in G$. Similarly for *right Haar measures.*

Of course, the prototype of Haar measure is the Lebesgue measure on the additive group $\mathbb{R}^d$, which is invariant under left (and right) translations. Compactly supported continuous functions on a topological space Y are denoted $C_c(Y)$. An equivalent definition for the left Haar measure μ is to require that for every $f \in C_c(G)$ and $h \in G$,

$$\int_G f(hg)\mathrm{d}\mu(g) = \int_G f(g)\mathrm{d}\mu(g). \tag{1.1}$$

A fundamental result on Haar measures is the following theorem due to A. Weil.

Theorem 1.2 (Theorem 2.10, [24]). *Every locally compact group G has a left Haar measure λ, which is essentially unique in the sense that if μ is any other left Haar measure, then there exists a positive constant C such that $\mu = C\lambda$.*

If we fix a left Haar measure μ on G, then for any $g \in G$ the measure μ_g defined by

$$\mu_g(E) = \mu(Eg)$$

is still a left Haar measure. Therefore by Theorem 1.2 there exists a positive real number, denoted $\Delta(g)$ such that

$$\mu_g = \Delta(g)\mu.$$

The function $\Delta : G \to \mathbb{R}_+$ is called the *modular function* of G.

Proposition 1.3 (Proposition 2.24, [24]). *Let G be a locally compact group with left Haar measure μ. The modular function $\Delta : G \to \mathbb{R}_+$ is a continuous homomorphism into the multiplicative group $\mathbb{R}_+$. Furthermore, for every $f \in L^1(G, \mu)$ we have*

$$\int_G f(gh)\mathrm{d}\mu(g) = \Delta(h)^{-1} \int_G f(g)\mathrm{d}\mu(g).$$

A group for which every left Haar measure is also a right Haar measure, hence for which $\Delta \equiv 1$, is called *unimodular.* Large classes of groups are unimodular, such as the Abelian, compact, nilpotent, semisimple and reductive groups. Many solvable groups, however, are not. Prototypical examples of non unimodular groups are the Iwasawa NA groups, such as the affine "$ax + b$" group that we illustrate in Example 1.1. A practical recipe for the computation of modular functions is given by the following proposition.

Proposition 1.4 (Proposition 2.30, [24]). *If G is a connected Lie group and* Ad *denotes the adjoint action of G on its Lie algebra, then $\Delta(g) = \det(\mathrm{Ad}(g^{-1}))$.*

We present the notion of semidirect product of Lie groups. Suppose that G and H are two Lie groups and that we are given a group homomorphism

$$\tau\colon H \to \mathrm{Aut}(G), \qquad h \mapsto \tau_h$$

such that the map $(g, h) \mapsto \tau_h(g)$ is a smooth map of $G \times H$ into H. Hence, for every $h \in H$ the map τ_h is an invertible Lie group homomorphism of G onto itself, and $\tau_{hk} = \tau_h \circ \tau_k$ for every $h, k \in H$. It is then possible to define the *semidirect product* of G and H. It is the group denoted $G \rtimes H$ whose elements are those of $G \times H$ and where the product is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1\tau_{h_1}(g_2), h_1h_2).$$

It is immediate to check that this is a group law, and indeed smooth, so that $G \rtimes H$ is a Lie group. The neutral element is (e_G, e_H) and inverses are given by

$$(g, h)^{-1} = (\tau_{h^{-1}}(g^{-1}), h^{-1}). \tag{1.2}$$

If we identify G and H with the subsets of $G \rtimes H$ given by $\{(g, e_H) : g \in G\}$ and $\{(e_G, h) : h \in H\}$, respectively, then both G and H are closed subgroups and G is a normal subgroup in $G \rtimes H$.

Example 1.1. The most obvious examples of semidirect product are the two versions of the "$ax + b$" group. We start from the connected version. Namely, it consists in the set $G = \mathbb{R}_+ \times \mathbb{R}$; the multiplication is obtained by thinking of the pair $(a, b) \in G$ as the affine transformation on $\mathbb{R}$ given by $x \mapsto ax + b$, that is

$$x \mapsto ax + b \mapsto a'(ax + b) + b' = (a'a)x + (a'b + b').$$

Therefore, the multiplication law is

$$(a', b')(a, b) = (a'a, a'b + b'). \tag{1.3}$$

The neutral element of the group is clearly $e = (1, 0)$ and the inverse of an element has the form

$$(a, b)^{-1} = (a^{-1}, -a^{-1}b). \tag{1.4}$$

Evidently, both components in (1.3) are smooth in the global coordinates on G, which is then a (connected) Lie group. By using (1.1), it is easy to show that $a^{-2}\mathrm{d}a\mathrm{d}b$ is a left Haar measure for G, indeed if $f \in C_c(G)$ and $(a_1, b_1) \in G$, then

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} f((a_1, b_1)(a, b))\frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \int_{\mathbb{R}_+\times\mathbb{R}} f(a_1a, a_1b + b_1)\frac{\mathrm{d}a\mathrm{d}b}{a^2}\\ &= \int_{\mathbb{R}_+\times\mathbb{R}} f(\alpha, \beta)a_1^{-2}\frac{\mathrm{d}\alpha\mathrm{d}\beta}{a_1^{-2}\alpha^2}\\ &= \int_{\mathbb{R}_+\times\mathbb{R}} f(\alpha, \beta)\frac{\mathrm{d}a\mathrm{d}b}{\alpha^2}.\end{aligned}$$

It is possible to realize the "$ax + b$" group as a matrix group by considering matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \leftrightarrow (a, b), \qquad a > 0,\ b \in \mathbb{R}.$$

Hence we have an isomorphism of "$ax + b$" with a closed Lie subgroup of $\mathrm{GL}(d, \mathbb{R})$. The Lie algebra of such subgroup consist of matrices

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad A, B \in \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathrm{Ad}(a, b) \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A & -bA + aB \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}. \end{aligned}$$

By Proposition 1.4, we have that $\Delta_G(a, b) = \det(Ad((a, b)^{-1})) = a^{-1}$.

A different version of the affine group is usually presented as full or non-connected. Let $G_{\mathrm{full}} = \mathbb{R}^* \times \mathbb{R}$. We endow G_{full} with the same multiplicative law in (1.3). The group is clearly non-connected since $G_{\mathrm{full}} = (\mathbb{R}_+ \times \mathbb{R}) \sqcup (\mathbb{R}_- \times \mathbb{R})$, $a^{-2}\mathrm{d}a\mathrm{d}b$ is still a Haar measure and the modular function is $\Delta_{G_{\mathrm{full}}}(a, b) = |a|^{-1}$.

Other examples of semidirect products are the Iwasawa AN groups (see Section 1.2.2.3) which, in the present notation, should be written $N \rtimes A$.

From now on, the choice of a left Haar measure μ is considered as implicitly made, and hence we write

$$\mathrm{d}g := \mathrm{d}\mu(g).$$

1.1.2 Group representation theory

Below we present the fundamental notions and results concerning the theory of group representations which are used in what follows. We use Chapter 2 of [20] and [24] as general references.

1.1.2.1 Irreducible representations and Schur's Lemma

Let us start by introducing some very basic notation. If $\mathcal{H}$ is an Hilbert space, we denote its scalar product and the associated norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces. We write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the space of bounded linear operators of $\mathcal{H}_1$ into $\mathcal{H}_2$. In the case in which $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ we use $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H}, \mathcal{H})$. We recall that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an isometry if it satisfies $\|Tu\|_{\mathcal{H}_2} = \|u\|_{\mathcal{H}_1}$ for every $u \in \mathcal{H}_1$. Thus, every isometry is injective (since $\ker T = 0$), but not necessarily surjective. A surjective isometry is called a *unitary* operator. Observe that, since

$\|Tu\|_{\mathcal{H}_2} = \langle Tu, Tu\rangle_{\mathcal{H}_2} = \langle u, T^*Tu\rangle_{\mathcal{H}_1}$ for every $u \in \mathcal{H}_1$, the polarization identity implies that T is an isometry if and only if $T^*T = \mathrm{id}_{\mathcal{H}_1}$. This means that if T is a unitary operator, then its inverse coincides with its adjoint. Finally, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, the set

$$\mathcal{U}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is unitary}\}$$

is a group with respect to the composition.

We have now all the necessary elements to present the representation theory. Let G be a locally compact second countable Hausdorff topological group and let $\mathcal{H}$ be a separable Hilbert space.

Definition 1.5. A *unitary representation* of G on the Hilbert space $\mathcal{H}$ is a group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$ continuous in the strong operator topology. The Hilbert space $\mathcal{H}$ is called the *representation space* of π and its dimension is called the *dimension* or the *degree of* π.

Observe that form the previous definition it immediately follows that if π is a unitary representation of G on $\mathcal{H}$, then for every $g, h \in G$:

(i) $\pi(gh) = \pi(g)\pi(h)$ and $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$;

(ii) $g \mapsto \pi(g)u$ is continuous for every $u \in \mathcal{H}$.

From the fact that $\pi(g) \in \mathcal{U}(\mathcal{H})$ and (i), we have that $\|\pi(g)u - \pi(h)u\|_{\mathcal{H}} = \|\pi(h^{-1}g)u - u\|_{\mathcal{H}}$. Therefore, it is sufficient to check the continuity of the maps $g \mapsto \pi(g)u$ at the identity $e \in G$. Furthermore, $\pi(g) \in \mathcal{U}(\mathcal{H})$ implies

$$\|\pi(g)u - u\|^2_{\mathcal{H}} = 2\|u\|^2_{\mathcal{H}} - 2\mathrm{Re}(\langle \pi(g)u, u\rangle_{\mathcal{H}}).$$

Hence if $g \mapsto \langle \pi(g)u, u\rangle_{\mathcal{H}}$ are continuous for every $u \in \mathcal{H}$ then the strong continuity of π is guaranteed. This is because the weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{H})$.

Let π be a unitary representation of G on the Hilbert space $\mathcal{H}$.

Definition 1.6. A subspace $\mathcal{M} \subseteq \mathcal{H}$ is said to be π*-invariant* if $\pi(g)u \in \mathcal{M}$ for every $u \in \mathcal{M}$ and $g \in G$, namely if $\pi(g)(\mathcal{M}) \subseteq \mathcal{M}$ for every $g \in G$.

Definition 1.7. We say that the representation π is irreducible if the only closed π-invariant subspaces in $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.

We present a classical test for the irriducibility of a representation which is useful in the following.

Lemma 1.8 (Proposition 2.47 in [20])**.** *Let π be a unitary representation of the group G on the Hilbert space $\mathcal{H}$. The following two conditions are equivalent:*

(i) the representation π is irreducible;

(ii) the coefficient $g \mapsto \langle u, \pi(g)v\rangle$ is not the zero function whenever $u, v \in \mathcal{H}\backslash\{0\}$.

Example 1.1 (continued)**.** Let G be the affine group "$ax+b$" considered in the first part of Example 1.1. Let us consider the Hilbert space $L^2(\mathbb{R})$; define for every $f \in L^2(\mathbb{R})$ and $(a,b) \in G$

$$\pi(a,b)f(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right), \qquad x \in \mathbb{R}.$$

We have that $\pi(a,b) \in \mathcal{U}(L^2(\mathbb{R}))$ since

$$\|\pi(a,b)f\|_2^2 = \int_{\mathbb{R}} \frac{1}{a} \left| f\left(\frac{x-b}{a}\right)\right|^2 \mathrm{d}x = \int_{\mathbb{R}} |f(y)|^2 \,\mathrm{d}y = \|f\|_2^2.$$

Furthermore, $\pi\colon G \to \mathcal{U}(L^2(\mathbb{R}))$ is a group homomorphism by the fact that $x \mapsto \frac{x-b}{a}$ is the affine transformation associated to $(a,b)^{-1}$, as observed in (1.4), and that the factor $\sqrt{a}$ is multiplicative.

We show that π is not irreducible. We need the Fourier transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} \mathrm{d}x, \qquad \xi \in \mathbb{R}.$$

In the next section we introduce the Fourier transform on Abelian groups which generalizes the notion above. Furthermore, Theorem 1.19 and Theorem 1.20 have to be intended as a general version of the classical Fourier inverse theorem and the Plancherel theorem, respectively. Let $\mathscr{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary extension of $\mathcal{F}$. A straightforward computation shows that for every $f \in L^2(\mathbb{R})$

$$\mathscr{F}(\pi(a,b)f)(\xi) = \sqrt{a} e^{-2\pi i b\xi} \mathscr{F}f(a\xi).$$

Take two non zero $f, g \in L^2(\mathbb{R})$. Then, by Plancherel theorem

$$\begin{aligned}
\int_{\mathbb{R}_+ \times \mathbb{R}} |\langle \pi(a,b)f, g\rangle|^2 \frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \int_{\mathbb{R}_+ \times \mathbb{R}} |\langle \mathscr{F}(\pi(a,b)f), \mathscr{F}g\rangle|^2 \frac{\mathrm{d}a\mathrm{d}b}{a^2} \\
&= \int_{\mathbb{R}_+ \times \mathbb{R}} a \left| \int_{\widehat{\mathbb{R}}} e^{-2\pi i b\xi} \mathscr{F}f(a\xi)\overline{\mathscr{F}g(\xi)} \right|^2 \frac{\mathrm{d}a\mathrm{d}b}{a^2} \\
&= \int_{\mathbb{R}_+ \times \mathbb{R}} \left|(\mathscr{F}^{-1}\omega_a)(-b)\right|^2 \mathrm{d}b \frac{\mathrm{d}a}{a},
\end{aligned}$$

where $\omega_a(\xi) = \mathscr{F}f(a\xi)\overline{\mathscr{F}g(\xi)}$. Then we can apply again Plancherel theorem obtaining

$$\begin{aligned}
\int_{\mathbb{R}_+ \times \mathbb{R}} |\langle \pi(a,b)f, g\rangle|^2 \frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} |\omega_a(\xi)|^2 \,\mathrm{d}\xi \frac{\mathrm{d}a}{a} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} |\mathscr{F}f(a\xi)|^2 \frac{\mathrm{d}a}{a} \right) |\mathscr{F}g(\xi)|^2 \,\mathrm{d}\xi. \qquad (1.5)
\end{aligned}$$

Define now the following (Hardy) spaces

$$\mathscr{H}_+(\mathbb{R}) = \{f \in L^2(\mathbb{R})\colon \mathscr{F}f(\xi) = 0 \text{ if } \xi < 0\},$$
$$\mathscr{H}_-(\mathbb{R}) = \{f \in L^2(\mathbb{R})\colon \mathscr{F}f(\xi) = 0 \text{ if } \xi > 0\}.$$

Observe that if we choose $f \in \mathscr{H}_+(\mathbb{R})$ and $g \in \mathscr{H}_-(\mathbb{R})$, then, since $a > 0$, it follows from (1.5) that $\|\langle \pi(\cdot)f, g\rangle\|_{L^2(G)} = 0$, that is $\langle \pi(\cdot)f, g\rangle = 0$ for a.e. $g \in G$. From Lemma 1.8 we have that π is not irreducible. Furthermore, it is possible to show that the Hardy spaces are (the only) π-invariant closed subspaces of $L^2(\mathbb{R})$ and that the restrictions of π to $\mathscr{H}_+(\mathbb{R})$ and to $\mathscr{H}_-(\mathbb{R})$ are irreducible.

Let us consider the non-connected version of the affine group. We define the unitary representation $\pi_{\text{full}} \colon G_{\text{full}} \to \mathcal{U}(L^2(\mathbb{R}))$ by

$$\pi_{\text{full}}(a,b)f(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x-b}{a}\right), \qquad f \in L^2(\mathbb{R}),\ (a,b) \in G_{\text{full}}.$$

By the same approach used in the connected case, we have that

$$\begin{aligned}\int_{\mathbb{R}^*\times\mathbb{R}} |\langle \pi(a,b)f, g\rangle|^2 \frac{\mathrm{d}a\mathrm{d}b}{a^2} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} |\mathscr{F}f(a\xi)|^2 \frac{\mathrm{d}a}{|a|} \right) |\mathscr{F}g(\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^*} |\mathscr{F}f(a)|^2 \frac{\mathrm{d}a}{|a|} \int_{\mathbb{R}} |\mathscr{F}g(\xi)|^2 \,\mathrm{d}\xi, \qquad (1.6)\end{aligned}$$

because the measure $|a|^{-1}\mathrm{d}a$ is invariant under the transformation $a \mapsto a/\xi$ for any $\xi \neq 0$. Hence if f and g are not the zero function, then (1.6) cannot be zero. This proves that π_{full} is irreducible.

The following is a classical result for the case of Abelian groups which is important for the theory presented in the next section.

Proposition 1.9 (Corollary 3.6 in [24])**.** *If the group G is Abelian, then every irreducible representation π of G is one dimensional, that is $\mathcal{H}_\pi \simeq \mathbb{C}$.*

Let π_1 and π_2 be two unitary representations of G on the separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Definition 1.10. We say that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an *intertwining operator* for π_1 and π_2 if it satisfies

$$T \circ \pi_1(g) = \pi_2(g) \circ T, \qquad g \in G.$$

We denote by $C(\pi_1, \pi_2)$ the space of intertwining operators of π_1 and π_2. If $\pi_1 = \pi_2$ we write $C(\pi, \pi) = C(\pi)$.

Definition 1.11. The representations π_1 and π_2 are called *unitarily equivalent* if there exists a unitary operator $C(\pi_1, \pi_2) \neq \varnothing$ and we write $\pi_1 \sim \pi_2$.

The next result is one of the fundamental theorems in representation theory. It gives a necessary and sufficient conditions for a representation to be irreducible and it describes the set $C(\pi_1, \pi_2)$ if both representations are irreducible.

Theorem 1.12 (Schur's Lemma, Lemma 3.5 in [24])**.** *Let π_1 and π_2 be two unitary representations of G on the separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively and let $T \in C(\pi_1, \pi_2)$.*

(i) Suppose that $\pi_1 = \pi_2 = \pi$. Then, π is irreducible if and only if $C(\pi)$ contains only scalar multiples of the identity.

(ii) Suppose that π_1 is irreducible. Then $T = \lambda S$ for some $\lambda \geqslant 0$ and some isometry S. If in addition π_2 is irreducible, then

$$\dim C(\pi_1, \pi_2) = \begin{cases} 1, & \text{if } \pi_1 \sim \pi_2, \\ 0, & \text{if } \pi_1 \nsim \pi_2. \end{cases}$$

1.1.2.2 Square integrable representations

Square integrable representations are a class of representations very useful in applications. Indeed it is possible to reconstruct a vector $u \in \mathcal{H}$ of the Hilbert space associated to the square integrable representation π by knowing all its projections $\langle u, \pi(g)\psi\rangle_{\mathcal{H}}$, $g \in G$, where $\psi \in \mathcal{H}$ is a fixed vector.

Let G be a locally compact second countable group and let π be a unitary representation of G on the separable Hilbert space $\mathcal{H}$. Given a vector $\psi \in \mathcal{H}$, we define the *voice transform* of π with respect to ψ as the operator $\mathcal{V}_\psi \colon \mathcal{H} \to L^\infty(G) \cap C(G)$ defined for any $u \in \mathcal{H}$ by

$$\mathcal{V}_\psi u(g) := \langle u, \pi(g)\psi\rangle_{\mathcal{H}}, \qquad g \in G.$$

The continuity of $\mathcal{V}_\psi u$ as a function on G is guaranteed by the continuity of π and the remark after Definition 1.5, its boundedness by the Cauchy-Schwarz inequality on $\mathcal{H}$.

Definition 1.13. A vector $\psi \in \mathcal{H}$ is called *admissible* for the representation π if the voice transform associated to ψ is an isometry from $\mathcal{H}$ into $L^2(G)$, where G is endowed with the Haar measure.

In the literature a vector is sometimes said to be admissible even when the voice transform is a scalar multiple of an isometry. The two definitions are different but clearly related by the multiplication by a constant.

Definition 1.14. If π is irreducible and admits an admissible vector, then we say that π is *square integrable.*

In order to state the fundamental result on square integrable representations, we need to recall the definition of weak-integral.

Definition 1.15. If $\Psi \colon G \to \mathcal{H}$ is a map such that

$$u \mapsto \int_G \langle u, \Psi(g)\rangle_{\mathcal{H}} \mathrm{d}g$$

is a continuous linear functional on $\mathcal{H}$, then we define as *weak integral* of Ψ, and we denote it by

$$\int_G \Psi(g)\mathrm{d}g \in \mathcal{H},$$

the unique element in $\mathcal{H}$ satisfying for every $u \in \mathcal{H}$

$$\langle u, \int_G \Psi(g)\mathrm{d}g\rangle_{\mathcal{H}} = \int_G \langle u, \Psi(g)\rangle_{\mathcal{H}} \mathrm{d}g.$$

The existence and uniqueness of such element is guaranteed by the Riesz representation theorem, see Theorem II.4 in [47]. We are now in a position to state the fundamental result on square integrable representations; thanks to which it is possible to reconstruct an element in the representation space by its coefficients under an admissible vector.

Theorem 1.16 (Theorem 2.25 in [26]). *Suppose that π has an admissible vector $\psi \in \mathcal{H}$. Then, for any $u \in \mathcal{H}$ we have the reproducing formula*

$$u = \int_G \mathcal{V}_\psi u(g)\pi(g)\psi \mathrm{d}g, \tag{1.7}$$

where the right-hand side is interpreted as weak-integral, and

$$\|u\|_{\mathcal{H}}^2 = \int_G |\mathcal{V}_\psi u(g)|^2 \mathrm{d}g.$$

Example 1.1 (continued). We present an important example of square integrable representation: the representation π_{full} introduced in Section 1.1.2. We say that $\psi \in L^2(\mathbb{R})$ satisfies the Calderón equation if

$$\int_{\mathbb{R}} |\mathscr{F}\psi(a)|^2 \frac{\mathrm{d}a}{|a|} = 1. \tag{1.8}$$

From (1.6) it follows that the voice transform $\mathcal{W}_\psi f(a,b) = \langle f, \pi(a,b)\psi\rangle_{L^2(\mathbb{R})}$ is an isometry from $L^2(\mathbb{R})$ into $L^2(G_{\mathrm{full}})$ if and only if ψ satisfies the Calderón condition. Thus the Calderón equation selects the admissible vectors, which, in this particular example, are called 1D-*wavelets*. The voice transform associated to the affine group is called wavelet and this is also the reason for the different notation. The reproducing formula (1.7) reads

$$f(x) = \int_{\mathbb{R}^* \times \mathbb{R}} \mathcal{W}_\psi f(a,b)\pi_{\mathrm{full}}(a,b)\psi(x)\frac{\mathrm{d}a\mathrm{d}b}{a^2}.$$

We refer to [19], [29], [44] for further reading on this topic.

1.1.3 The Fourier transform on Abelian groups

In this section we focus on the analysis on Abelian groups, with particular attention to the definition of Fourier transform. The general reference is [24].

Let G be a locally compact Abelian group. Proposition 1.9 states that every irreducible representation of G is one-dimensional. This means that the Hilbert space associated to an irreducible representation π is $\mathcal{H}_\pi = \mathbb{C}$ and $\pi(g)u = \xi(g)u$ for every $u \in \mathbb{C}$, where ξ is a continuous homomorphism of G into $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ endowed with the complex product. The homomorphism $\xi : G \to \mathbb{T}$ is called *character* of G and the set of characters is denoted by $\widehat{G}$. For reasons of duality which will be clearer in the following, the notation

$$\langle g, \xi\rangle = \xi(g), \qquad g \in G,\, \xi \in \widehat{G},$$

is preferred.

It is straightforward to see that $\widehat{G}$ is an Abelian group under pointwise multiplication. Hence, $\widehat{G}$ is usually called the *dual group* of G. Furthermore it is possible to see that in general if G is discrete, than $\widehat{G}$ is compact and vice versa. The next result collects some classical examples.

Proposition 1.17 (Theorem 4.6 and Corollary 4.8 in [24]). *The following groups are isomorphic:*

(i) $\widehat{\mathbb{R}^d} \simeq \mathbb{R}^d$ *through the paring* $\langle x, \xi \rangle = e^{2\pi i \langle \xi, x \rangle}$*;*

(ii) $\widehat{\mathbb{T}} \simeq \mathbb{Z}$ *through the paring* $\langle \alpha, n \rangle = \alpha^n$*;*

(iii) $\widehat{\mathbb{Z}} \simeq \mathbb{T}$ *through the paring* $\langle n, \alpha \rangle = \alpha^n$*.*

Definition 1.18. The *Fourier transform* on $L^1(G)$ is the map $\mathcal{F}\colon L^1(G) \to C(\widehat{G})$ defined by

$$\mathcal{F}f(\xi) := \int_G \overline{\langle g, \xi \rangle} f(g) \mathrm{d}g.$$

Example 1.2. We show how classical groups give rise to classical transforms.

(i) If $G = \mathbb{R}^d$, then the Fourier transform obtained is the classical Euclidean Fourier transform defined on $f \in L^1(\mathbb{R}^d)$ as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2i\pi\langle x, \xi \rangle} \mathrm{d}x, \qquad \xi \in \mathbb{R}^d;$$

(ii) If $G = \mathbb{T}$, endowed with the normalized measure such that $|\mathbb{T}| = 1$, then the Fourier transform obtained is the classical Fourier series defined on a T-periodic function, $T > 0$. Namely, if we consider a function f on $\mathbb{T} = \{e^{2i\pi t/T} : t \in [0, T]\}$ (endowed with the normalized Haar measure $\mathrm{d}t/T$) as a T-periodic function, then its Fourier series is determined by

$$\hat{f}(n) = \int_0^T f(t) e^{-i\frac{2\pi}{T}nt} \frac{\mathrm{d}\theta}{T}, \qquad n \in \mathbb{Z}; \tag{1.9}$$

(iii) If $G = \mathbb{Z}$, endowed with the counting measure, then for every $T > 0$ the Fourier transform can be expressed as the T-periodic function defined on $L^1(\mathbb{Z})$ by

$$\mathcal{F}f(t) = \sum_{n \in \mathbb{Z}} f(n) e^{i\frac{2\pi}{T}nt}, \qquad t \in [0, T]. \tag{1.10}$$

Observe that, differently from the canonical choice made in the previous example, we choose to parametrize $\mathbb{T}$ with $[0, T]$ through $t \mapsto e^{-2i\pi t/T}$. In fact, there is no substantial difference from the canonical parametrization obtained by conjugating the previous one. The reason for this choice will be clearer after Theorem 1.19.

The following two results consist in generalizations of fundamental facts in the theory of classical Euclidean Fourier transform.

Theorem 1.19 (Theorem 4.22 in [24]). *Let $f \in L^1(G)$ such that $\mathcal{F}f \in L^1(\widehat{G})$. The following inversion formula holds:*

$$f(g) = \int_{\widehat{G}} \mathcal{F}f(\xi)\langle g, \xi\rangle \mathrm{d}g, \qquad g \in G.$$

As a consequence, we have that the choice made in the parametrization of $\mathbb{T}$ in (1.10) allows us to consider the Fourier transform on $\mathbb{Z}$ as a kind of inverse of the Fourier series in (1.9). Namely if $f \in L^1(\mathbb{T})$ and $\hat{f} \in L^1(\mathbb{Z})$ then

$$\mathcal{F}\hat{f}(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{i\frac{2\pi}{T}nt} = f(t).$$

By duality, it is possible to consider the Fourier series in (1.9) as the inverse of the Fourier transform in (1.10), as well.

Theorem 1.20 (Theorem 4.26 in [24]). *The Fourier transform on $L^1(G) \cap L^2(G)$ extends to a unitary isomorphism $\mathscr{F}\colon L^2(G) \to L^2(\widehat{G})$, that is*

$$\|\mathscr{F}f\|_{L^2(\hat{G})} = \|f\|_{L^2(G)}, \qquad f \in L^2(G). \tag{1.11}$$

1.1.4 Homogeneous spaces

Homogeneous spaces represent the setting on which we operate from now on. Here we collect some basic facts about them. From now on, every group G considered is assumed to be topological.

Definition 1.21. A group G *acts* on a set X if there exists a map, called *action*, $G \times X \ni (g, x) \mapsto g[x] \in X$ such that:

(i) the neutral element $e \in G$ satisfies $e[x] = x$ for every $x \in X$;

(ii) $h[g[x]] = hg[x]$ for every $h, g \in G$ and $x \in X$.

Fix $x \in X$, two important definitions associated to an action follows:

(i) the *orbit* of $x \in X$ is the set $\mathcal{O}_x = \{g[x] \colon g \in G\} \subseteq X$;

(ii) the *isotropy* of $x \in X$ is the set $H_x = \{g \in G \colon g[x] = x\} \subseteq G$.

When a group G acts on a set X, we say that X is a G-*space*. We say that X is a *transitive* G-space, or equivalently that the action is transitive, if for every $x, y \in X$ there exists $g \in G$ such that $g[x] = y$.

The quotient spaces G/H represent a classical example of transitive G-spaces, since G acts by left multiplication on G/H. If X is a transitive G-space, $x_0 \in X$ is fixed and $H = H_{x_0} \subseteq G$ is the isotropy of G at x_0. Define the map $\Phi\colon G/H \to X$ by $\Phi(gH) = g[x_0]$; it is easy to see that Φ is a bijection. In particular, if G is a locally compact and second countable group (from now on abbreviated by lcsc), then G/H is a lcsc topological space. If in addition, X is a lcsc transitive G-space and the action of G on X is continuous with respect to the product topology of $G \times X$, then Φ is an homeomorphism by Proposition 2.46 in [24], so that X is homeomorphic to the quotient space G/H. In such a case, we say that X is a *homogenous space*. Let us observe that if we choose a different reference point $x_0' = g_0 x_0$ for some $g_0 \in G$ (which exists because we are assuming the action to be transitive), it is sufficient to replace H with $H' = g_0 H g_0^{-1}$. The map $g \mapsto g_0 g g_0^{-1}$ induces a G-equivariant homeomorphism between G/H and G/H'. Since the topology on G/H is the quotient topology then the identification map is actually a homeomorphism.

In the following, the Radon transform is studied in different settings. The basic spaces X and Ξ which are involved in the theory of Radon transform are homogeneous spaces of the same group G. From the point of view of Analysis, the natural question arises whether the homogeneous space G/H admits a G-invariant Radon measure or not. The answer to this question is contained in Theorem 1.22 below, which relates integration on G to an iterated integral, first on H and then on G/H. These formulae are achieved by means of the natural projection operator $P : C_c(G) \to C_c(G/H)$, also known as Weil's mean opearator, defined by

$$Pf(gH) = \int_H f(gh)\mathrm{d}h,$$

which is well defined by the left invariance of $\mathrm{d}h$, the Haar measure on H. Furthermore, it is possible to see that P is continuous and surjective.

We are now in a position to state the classical result also known as the Weil's decomposition theorem. Here Δ_G and Δ_H are the modular functions of G and H, respectively.

Theorem 1.22 (Theorem 2.51, [24])**.** *Let G be a locally compact group and H a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor, and if the factor is suitably chosen then*

$$\int_G f(g)\mathrm{d}g = \int_{G/H} Pf(gH)\mathrm{d}\mu(gH) = \int_{G/H}\int_H f(gh)\mathrm{d}h\mathrm{d}\mu(gH)\ ,$$

for every $f \in C_c(G)$.

Hence, there always exists a G-invariant Radon measure on G/H whenever H is compact, since $\Delta_G|_H = \Delta_H \equiv 1$. Indeed, the image of H under both modular functions is a compact subgroup of the multiplicative group of positive reals, namely $\{1\}$.

Although many homogeneous spaces do not admit invariant measures (for example $\mathbb{R}$ as a homogeneous space of the "$ax + b$" group), all of them admit strongly quasi invariant measures. If μ is a measure on $X = G/H$ and we write $\mu^g(E) = \mu(gE)$ for $E \in \mathcal{B}(X)$, we say that μ is a *quasi invariant measure* if all the μ^g are equivalent, that is, mutually absolutely continuous. We say that μ is *strongly quasi invariant* if there exists a continuous function $\lambda : G \times X \to (0, +\infty)$ such that

$$\mathrm{d}\mu^g(x) = \lambda(g, x)\mathrm{d}\mu(x), \qquad x \in X, g \in G.$$

In other words, the requirement is that the Radon-Nikodym derivative $(\mathrm{d}\mu^g/\mathrm{d}\mu)(x)$ is jointly continuous in g and x. As mentioned, all homogeneous spaces admit strongly quasi invariant measures (see Proposition 2.56 and Theorem 2.58 in [24]).

Finally, we recall the following important notion.

Definition 1.23. Let G be a lcsc group and let $X \simeq G/H$ be a homogeneous space, where H is a closed subgroup of G. Fix $x_0 \in X$. A *Borel section* is a measurable map $s\colon X \to G$ satisfying $s(x)[x_0] = x$ for every $x \in X$ and $s(x_0) = e$.

Theorem 5.11 in [51] states that if G is second countable then there exists a Borel section.

1.2 The structure of semisimple Lie algebras and Lie groups

In this section we present some classes of Lie algebras and Lie groups, with focus on semisimple Lie groups. In the last part we present the example of the (semisimple) Lie group $\mathrm{SL}(n, \mathbb{R})$ which plays an important role in Chapter 2. Classical references with a wider scope are [35], [36], [43] and [50].

1.2.1 Classes of Lie algebras and Lie groups

1.2.1.1 Classes of Lie algebras

We fix a Lie algebra $\mathfrak{g}$ and define recursively

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}', \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

Each $\mathfrak{g}^j$ is an ideal in $\mathfrak{g}$. The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

is called the *derived series* or the *commutator series* of $\mathfrak{g}$. A Lie algebra is *solvable* if $\mathfrak{g}^j = 0$ for some positive integer j.

Now we consider another recursive definition in $\mathfrak{g}$:

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}', \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

Again, each $\mathfrak{g}_j$ is an ideal in $\mathfrak{g}$. The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$$

is called the *lower central series* of $\mathfrak{g}$. A Lie algebra is *nilpotent* if $\mathfrak{g}_j = 0$ for some positive integer j.

Evidently, an Abelian Lie algebra is nilpotent and a nilpotent Lie algebra is solvable. Both inclusions of these classes are proper.

Now we present the class of Lie algebras we are most interested in: semisimple ones. Roughly speaking, a *simple* Lie algebra is a Lie algebra that is as far from being Abelian as possible. This is formalized by saying that a Lie algebra is simple if it is not Abelian and contains no proper Abelian ideals. A *semisimple* Lie algebra $\mathfrak{g}$ is then the Lie algebra direct sum of (all its) simple ideals (Corollary 6.3, Ch.II in [35]). This turns out to be equivalent to the nondegeneracy of the Killing form B of $\mathfrak{g}$, thanks to a theorem of Elie Cartan. The latter property is perhaps more efficient when it comes to calculations, and we shall adopt it as definition.

Definition 1.24. A real Lie algebra is called *semisimple* if its Killing form is nondegenerate, i.e.

$$\mathfrak{g}^{\perp} := \{X \in \mathfrak{g} : B(X,Y) = 0 \text{ for all } Y \in \mathfrak{g}\} = \{0\}$$

It is *simple* if it is semisimple and has no nontrivial ideals.

A Lie algebra $\mathfrak{g}$ is said *reductive* if for every ideal $\mathfrak{a}$ of $\mathfrak{g}$ there is an ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

A real Lie algebra $\mathfrak{g}$ of square matrices with entries in either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ which is closed under $X \mapsto X^* = {}^t\overline{X}$ is reductive (Proposition 1.56 in [43]). Evidently, a semisemple Lie algebra is reductive. The converse is of course not true. For example, $\mathfrak{gl}(n,\mathbb{R})$ is reductive but not semisimple.

Take a Lie subalgebra $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ and let K^* denote the connected Lie subgroup of $\mathrm{Int}(\mathfrak{g})$ (which is the connected Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ having $\mathrm{ad}(\mathfrak{g})$ as Lie algebra) whose Lie algebra is $\mathrm{ad}(\mathfrak{k}) \subset \mathrm{ad}(\mathfrak{g})$. We say that $\mathfrak{k}$ is *compactly contained* in $\mathfrak{g}$ if K^* is compact, and that $\mathfrak{g}$ is *compact* if it is compactly contained in itself, that is, if $\mathrm{Int}(\mathfrak{g})$ is compact. Notice that, as a consequence, the Lie algebra of a compact Lie group G is compact because $\mathrm{Int}(\mathfrak{g})$ is the continuous image of G under Ad, hence compact.

1.2.1.2 Classes of Lie Groups

For all the classes of Lie algebras that we have introduced in the previous section, the corresponding classes of Lie groups are defined by requiring the Lie algebras to satisfy the corresponding property.

Definition 1.25. A Lie group is called either *solvable, nilpotent, semisimple* or *reductive* if such is its Lie algebra.

The structure of solvable Lie groups is investigated in the recent monography [5], while that of nilpotent Lie groups is the subject of many books, among which we mention the classical [28]. We content ourselves with some very basic facts described in the next result.

Proposition 1.26 (Corollary 1.103 and Theorem 1.104 in [43])**.** *(i) If $\mathfrak{g}$ is solvable, then there exists a simply connected Lie group G diffeomorphic to $\mathbb{R}^n$ whose Lie algebra is $\mathfrak{g}$.*

(ii) If N is a nilpotent, simply connected Lie group with Lie algebra $\mathfrak{n}$, then $\exp\colon \mathfrak{n} \to N$ *is a diffeomorphism.*

1.2.2 Decompositions

Classically, three decompositions play a crucial role in the theory of symmetric spaces: the Cartan, Iwasawa and Bruhat decompositions. The latter, however, will not be relevant for our purposes and we therefore shall not recall it.

1.2.2.1 Cartan decomposition

It is fair to say that large parts of the structure theory of semisimple Lie algebras and Lie groups rests on the notion of *Cartan involution*. For matrix Lie algebras, we have already seen a very effective way of testing reductivity, hence semisimplicity, based on the map $X \to {}^t\overline{X} = X^*$. Better properties are enjoyed by the variant

$$\theta(X) = -{}^t\overline{X}, \tag{1.12}$$

which is an *involution*, that is, an automorphism of the Lie algebra whose square is the identity. Let B denote the Killing form of a matrix Lie algebra. The involution θ has the property that $B_\theta(X,Y) = -B(X,\theta Y)$ is symmetric and positive definite. This turns out to be a particular case of the following definition.

Definition 1.27. An involution θ of a real semisimple Lie algebra $\mathfrak{g}$ such that the symmetric bilinear form

$$B_\theta(X,Y) = -B(X,\theta Y)$$

is positive definite is called a *Cartan involution.*

From Theorem 6.16 in [43], we know that every real semisimple Lie algebra $\mathfrak{g}$ has a Cartan involution and any two such are conjugate via $\mathrm{Int}(\mathfrak{g})$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let θ be a Cartan involution. The involution gives rise to the vector space direct sum

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \tag{1.13}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of $\mathfrak{g}$ relative to θ, respectively. Thus

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{1.14}$$

From (1.13) and (1.14) it follows that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to both B and B_θ. Furthermore, since B_θ is positive definite, it follows that

$$B|_{\mathfrak{k}\times\mathfrak{k}} < 0, \qquad B|_{\mathfrak{p}\times\mathfrak{p}} > 0. \tag{1.15}$$

Definition 1.28. A decomposition $\mathfrak{g}$ in a vector space direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ which satisfies (1.14) and (1.15) is called a *Cartan decomposition* of $\mathfrak{g}$.

The most important consequence of the Cartan decomposition at the Lie algebra level is the following version at the Lie group level, a general *polar decomposition.* If K is the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$, then there exists an automorphism Θ of G such that $d\Theta = \theta$ and $\Theta^2 = \mathrm{id}$. Furthermore, K is the subgroup of G consisting of the fixed points of Θ and the map $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism. See Theorem 6.31 in [43].

1.2.2.2 The Iwasawa Decomposition of a Semisimple Lie Algebra

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan involution θ and associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Fix a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Thus, $\mathfrak{a}$ is a vector subspace of $\mathfrak{p}$ such that $[\mathfrak{a}, \mathfrak{a}] = \{0\}$ and is maximal with this property. It is easy to check that

$$(\mathrm{ad}X)^* = -\mathrm{ad}(\theta X)$$

where the adjoint $(\cdot)^*$ is taken w.r.t the inner product B_θ, so that the elements of $\mathrm{ad}(\mathfrak{p})$ are self-adjoint. This entails that the set $\{\mathrm{ad}H : H \in \mathfrak{a}\}$ is a commuting family of self-adjoint linear maps. Therefore, $\mathfrak{g}$ is the B_θ-orthogonal direct sum of their joint eigenspaces, all the eigenvalues of which are real and depend linearly on H. For any fixed $\alpha \in \mathfrak{a}^*$, we write

$$\mathfrak{g}_\alpha = \{H \in \mathfrak{g} : (\mathrm{ad}H)X = \alpha(H)X, \text{ for all } H \in \mathfrak{a}\}$$

and we say that $\alpha \neq 0$ is a *restricted root*, or simply a *root* of the pair $(\mathfrak{g}, \mathfrak{a})$, whenever $\mathfrak{g}_\alpha \neq \{0\}$. The set of restricted roots is Σ and the spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Sigma$ are called *(restricted) root spaces.*

An element $H \in \mathfrak{a}$ is called *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, otherwise it is *singular*. The set $\mathfrak{a}'$ of regular elements is the complement in $\mathfrak{a}$ of finitely many hyperplanes and its connected components are called the *Weyl chambers*. We fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and we declare a root α to be *positive* if it has positive values on $\mathfrak{a}^+$. A root is *simple* if it cannot be written as sum of positive roots. The set Δ of simple roots turns out to be a basis of $\mathfrak{a}^*$. Thus, there are exactly $\ell = \dim \mathfrak{a}$ simple roots. This number is of utmost importance and is called the *real rank* of $\mathfrak{g}$. We order the elments in $\mathfrak{a}^*$, hence the roots in Σ, *lexicographically* with respect to an ordering $\delta_1, \ldots, \delta_\ell$ of the simple roots. This means that $\lambda = \sum a_j \delta_j$ is positive (written $\lambda > 0$) if the first non-zero coefficient a_k is positive. Together with $\mathfrak{g}$, θ and $\mathfrak{a}$ we assume that an ordering "$>$" has been fixed on $\mathfrak{a}^*$ by choosing a labeling of the simple roots relative to a fixed Weyl chamber $\mathfrak{a}^+$. We consequently denote by Σ^+ and Σ^- the positive and negative roots, respectively. Clearly, $\Sigma = \Sigma^+ \cup \Sigma^-$, a disjoint union.

Theorem 1.29 (Proposition 6.43 in [43]). *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the semisimple Lie algebra $\mathfrak{g}$, and fix a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and an ordering on $\mathfrak{a}^*$. The vector space direct sum*

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \tag{1.16}$$

is a nilpotent Lie algebra and $\mathfrak{g}$ decomposes as the vector space direct sum

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}. \tag{1.17}$$

Furthermore, $\mathfrak{a} + \mathfrak{n}$ is solvable and $[\mathfrak{a} + \mathfrak{n}, \mathfrak{a} + \mathfrak{n}] = \mathfrak{n}$.

The vector space direct sum (1.17) is called the *Iwasawa decomposition* of $\mathfrak{g}$ relative to the choice of $(\theta, \mathfrak{a}, \mathfrak{a}^+)$. The fact that $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$ is clear from the very definition of root space, because $[\mathfrak{a}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$. This exhibits $\mathfrak{a} + \mathfrak{n}$ as a semidirect product.

1.2.2.3 The Iwasawa decomposition of a semisimple Lie group

We are finally in a position to state the Iwasawa decomposition at the group level.

Theorem 1.30 (Theorem 6.46 in [43]). *Let G be a connceted semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of its Lie algebra. Let K, A and N be the connected subgroups of G whose Lie algebras are $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism. The groups A and N are simply connected and AN is solvable.*

Observe that AN is in fact a semidirect product. Indeed, A acts on N by conjugation, as is most rapidly seen by observing that $\mathrm{Ad}a(X) \in \mathfrak{g}_\alpha$ if $X \in \mathfrak{g}_\alpha$ for any root $\alpha \in \Sigma$ and for all $a \in A$. Indeed, for any $H \in \mathfrak{a}$, since $\mathfrak{a}$ is Abelian, one has

$$[H, \mathrm{Ad}a(X)] = \mathrm{Ad}a\left([\mathrm{Ad}a^{-1}(H), X]\right) = \mathrm{Ad}a\left([H, X]\right) = \alpha(H)\mathrm{Ad}a(X).$$

Therefore $\mathrm{Ad}a$ preserves root spaces and in particular it preserves $\mathfrak{n}$. Thus A acts on $\mathfrak{n}$ via the adjoint action and, passing to exponentials, it acts on N by conjugation. This is tantamount to saying that A normalizes N inside G. Hence $NA = AN$ is the semidirect product $N \rtimes A$.

The interpretation of the above theorem is that any element in G can be written uniquely as a product $g = kan$ with $k \in K$, $a \in A$ and $n \in N$. These groups are called *Iwasawa subgroups* of G, as any of their conjugates. The decomposition is normally expressed in the short form $G = KAN$. Actually, the result entails three similar decompositions, that is $G = KNA$, $G = ANK$ and $G = NAK$, where it is to be understood that each element may be written in a unique way as a product of factors in the three Iwasawa subgroups in any of the indicated orders. The groups K, A and N are always the same but the factors of each element are not. This means that if one changes the decomposition, that is, the order, then the single factors of the same element may (and do) change, see formulæ (1.23) in Section 1.2.3. The letters K, A and N stand for compact, Abelian and nilpotent, respectively.

In the theory of symmetric spaces, two decompositions are more used than the others, and for these an *ad hoc* notation is introduced. It is common to write

$$g = k \exp H(g) n \tag{1.18}$$

and

$$g = n \exp A(g) k, \tag{1.19}$$

whereby the N and K components appearing in (1.18) and (1.19) are different. Evidently, $H(g), A(g) \in \mathfrak{a}$ but in general $H(g) \neq A(g)$ (unless $g \in A$).

Let M and M' denote the centralizer and normalizer of $\mathfrak{a}$ in K, respectively. This means that

$$M = \left\{m \in K : \mathrm{Ad}m(H) = H \text{ for all } H \in \mathfrak{a}\right\},$$
$$M' = \left\{w \in K : \mathrm{Ad}w(H) \in \mathfrak{a} \text{ for all } H \in \mathfrak{a}\right\}.$$

Passing to exponentials, it follows that if $m \in M$, then $mam^{-1} = a$ for all $a \in A$ and if $w \in M'$, then $waw^{-1} \in A$ for all $a \in A$.

Definition 1.31. The quotient group $W = M'/M$ is called the *Weyl group* of (G, K).

The compact Lie groups M and M' have the same Lie algebra, namely $\mathfrak{m}$, so that W is in fact a finite group. We observe *en passant* that the Weyl group W acts on Σ by

$$(w \cdot \alpha)(H) = \alpha(\mathrm{Ad}w^{-1}H), \qquad H \in \mathfrak{a}, \tag{1.20}$$

because for any root vector $X \in \mathfrak{g}_\alpha$ and any $w \in M'$

$$[H, \mathrm{Ad}w(X)] = \mathrm{Ad}w\left([\mathrm{Ad}w^{-1}(H), X]\right) = (w \cdot \alpha)(H)\mathrm{Ad}w(X).$$

Hence $\mathrm{Ad}w(X) \in \mathfrak{g}_{w\cdot\mathfrak{a}}$ so that the corresponding root space is not zero. Notice that the right hand side of (1.20) does not change if w is replaced with wm with $m \in M$, so it is really a definition on W and not just on M'.

Assume now that the data $\mathfrak{g}, \theta, \mathfrak{a}, \mathfrak{a}^+$ have been fixed. It is important to recall that the exponential mapping $\exp\colon \mathfrak{a} \to A$ is a diffeomorphism. This justifies the notation

$$\log a, \qquad a \in A,$$

to mean the only element in $\mathfrak{a}$ such that $\exp(\log a) = a$.

For any $\alpha \in \Sigma$, the vector space dimension of $\mathfrak{g}_\alpha$ is called the *multiplicity* of α and is usually denoted m_α. Multiplicities may well vary, depending on the group. The following element of $\mathfrak{a}^*$ plays a crucial role in the theory:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha. \tag{1.21}$$

Though it might appear somewhat exotic, this linear functional on $\mathfrak{a}$ naturally appears in relation with the semidirect product structure of the Iwasawa group AN, which arises from the fact that A acts on $\mathfrak{n}$ via the adjoint action and hence by conjugation on N, this fact is presented in Section 2.2.

1.2.3 An example of semisimple Lie group: $\mathrm{SL}(d, \mathbb{R})$

In this last part we show a canonical example of semisimple Lie group: $\mathrm{SL}(d, \mathbb{R})$. In Chapter 2 we focus on the case $d = 2$ together with the group $\mathrm{SU}(1, 1)$, highlighting their relations with the symmetric spaces of upper half plane and hyperbolic disk, respectively. We start by analyzing its Lie algebra.

Let $d \geqslant 2$. We denote by $\mathfrak{gl}(d, \mathbb{R})$ the Lie algebra $M_d(\mathbb{R})$ and we put

$$\mathfrak{sl}(d, \mathbb{R}) = \{X \in \mathfrak{gl}(d, \mathbb{R})\colon \mathrm{tr}X = 0\}.$$

The Lie algebra $\mathfrak{sl}(d, \mathbb{R})$ represents a classical example of semisimple Lie algebra, and then $\mathrm{SL}(d, \mathbb{R})$ of semisimple Lie group.

The Cartan decomposition associated to the standard involution (1.12) reads

$$\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{so}(d, \mathbb{R}) + \mathrm{Sym}_0(d),$$

where

$$\mathfrak{so}(d, \mathbb{R}) = \{X \in \mathfrak{gl}(d, \mathbb{R})\colon X + X^* = 0\}$$

is a compact Lie algebra and $\mathfrak{p} = \mathrm{Sym}_0(d)$ is the space of $d \times d$ symmetric and traceless real matrices. The Cartan involution Θ for $\mathrm{SL}(d, \mathbb{R})$ is then

$$\Theta g = {}^t g^{-1}$$

as for all matrix groups with real entries. Hence $K = \mathrm{SO}(d)$. The diffeomorphism $(k, X) \mapsto k \exp X$ of $\mathrm{SO}(d) \times \mathrm{Sym}_0(d) \to \mathrm{SL}(d, \mathbb{R})$ is just the classical polar decomposition.

Let $\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{so}(d, \mathbb{R}) + \mathrm{Sym}_0(d)$ be the Cartan decomposition of $\mathfrak{sl}(d, \mathbb{R})$ described above, with Cartan involution $\theta X = -{}^t X$. The natural maximal Abelian subspace of $\mathrm{Sym}_0(d)$ is the $(d-1)$-dimensional vector space consisting of the diagonal matrices $\mathrm{diag}(a_1, \ldots, a_d)$ with $a_1 + \cdots + a_d = 0$. Thus, the real rank of $\mathfrak{sl}(d, \mathbb{R})$ is $d - 1$. Let E_{ij} denote the matrix whose only non-zero entry is 1 at position ij. Then, for $H = \mathrm{diag}(a_1, \ldots, a_d)$ and $i \neq j$

$$[H, E_{ij}] = (a_i - a_j) E_{ij}$$

and in fact E_{ij} spans a root space provided that $i \neq j$. It is customary to introduce the linear functionals $e_k(\cdot)$ on $\mathfrak{a}$, with $1 \leqslant k \leqslant d$, via $e_k(\mathrm{diag}(a_1, \ldots, a_d)) = a_k$. Thus, for $i \neq j$ the (restricted) root $\alpha_{ij} = e_i - e_j$ acts on $H = \mathrm{diag}(a_1, \ldots, a_d)$ by

$$\alpha_{ij}(H) = a_i - a_j,$$

and we write in simplified form $\mathfrak{g}_{ij}$ in place of $\mathfrak{g}_{\alpha_{ij}}$ for the root space

$$\mathfrak{g}_{ij} = \mathrm{span}\{E_{ij}\}, \qquad i \neq j.$$

For $i < j$ the matrix E_{ij} is upper triangular, and for $i > j$ it is lower triangular. A natural choice of Weyl chamber is

$$\mathfrak{a}^+ = \Big\{ \mathrm{diag}(a_1, \ldots, a_d) : a_1 > a_2 > \cdots > a_d \Big\}.$$

It is immediate to check that for $j = 1, \ldots, d-1$ the roots $\delta_j = e_j - e_{j+1}$ are the simple ones and that the set of positive roots is

$$\Sigma^+ = \{\alpha_{ij} : i < j\}. \tag{1.22}$$

It follows that the nilpotent Iwasawa Lie algebra $\mathfrak{n}$ defined in (1.16) is just the Lie algebra of strictly upper triangular matrices. Notice that $\mathfrak{g}_0 = \mathfrak{a}$, that is, $\mathfrak{m} = \{0\}$ and that $\dim \mathfrak{g}_\alpha = 1$ for every restricted root $\alpha \in \Sigma$.

At the group level, A is the group of diagonal matrices with positive entries and determinant 1, namely

$$\mathrm{diag}(\mathrm{e}^{a_1}, \ldots, \mathrm{e}^{a_d}), \qquad a_1 + \cdots + a_d = 0,$$

and N is the group of unipotent upper triangular matrices, namely those of the form

$$\begin{bmatrix} 1 & a_{12} & \cdots & & \cdots & a_{1,d} \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & 1 & a_{d-1,d} \\ 0 & \cdots & \cdots & & 0 & 1 \end{bmatrix}.$$

In the special case of $\mathrm{SL}(2,\mathbb{R})$, the following explicit formulæ may be checked directly for the Iwasawa KAN- and NAK-decompositions respectively:

$$\begin{aligned}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{bd+ac}{c^2+d^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \sqrt{c^2+d^2} \end{bmatrix} \begin{bmatrix} \frac{d}{\sqrt{c^2+d^2}} & -\frac{c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{bmatrix}.\end{aligned} \tag{1.23}$$

Thus, if g is parametrized as above, then the functions in (1.18) and (1.19) take the form

$$H(g) = \frac{1}{2}\log(a^2+c^2), \qquad A(g) = \frac{1}{2}\log(c^2+d^2).$$

It is very easy to see that any orthogonal matrix $w \in \mathrm{SO}(d)$ with the property that each row and each column has exactly one non-zero entry conjugates any diagonal matrix into another such. All such matrices are therefore in M' provided that they are orthogonal, and this forces each non-zero entry to be ± 1. Further, the only matrices with determinant 1 that commute with all diagonal matrices with determinant 1 are precisely the diagonal matrices and these are in $\mathrm{SO}(d)$ if and only if they are of the form $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_d)$ with $\varepsilon_j = \pm 1$ and $\prod_j \varepsilon_j = 1$. This leads to the identification of $W = M'/M$ with the set of permutation matrices, that is, with the symmetric group Σ_d.

As we have seen in (1.22), Σ^+ consists of all the linear operators α_{ij} having $j > i$. Since every $\mathfrak{g}_{\alpha_{ij}}$ is one dimensional, we have that $m_\alpha = 1$ for every $\alpha \in \Sigma^+$. Hence for $H = \mathrm{diag}(a_1, \ldots, a_d)$, we have

$$\rho(H) = \frac{1}{2}\sum_{1\leqslant i<j\leqslant d} \alpha_{ij}(H) = \frac{1}{2}\sum_{k=1}^{d}(d+1-2k)a_k = -\sum_{k=1}^{d} k a_k,$$

where we used $a_1 + \cdots + a_d = 0$.

Another interesting example is that of $\mathrm{SU}(1,1)$, that is the group

$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{R},\ |a|^2 - |b|^2 = 1 \right\},$$

whose Lie algebra $\mathfrak{su}(1,1)$ is

$$\mathfrak{su}(1,1) = \left\{ \begin{bmatrix} i\theta & t+is \\ t-is & -i\theta \end{bmatrix} : \theta, t, s \in \mathbb{R} \right\}.$$

The Cartan involution is the usual one and yields

$$\mathfrak{k} = \left\{ \begin{bmatrix} i\theta & 0 \\ 0 & -i\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}, \qquad \mathfrak{p} = \left\{ \begin{bmatrix} 0 & t+is \\ t-is & 0 \end{bmatrix} : \theta, t, s \in \mathbb{R} \right\}.$$

There is an elementary, yet fundamental, isomorphism $\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$, hence $\mathfrak{su}(1,1) \simeq \mathfrak{sl}(2,\mathbb{R})$, which we now describe. Consider the matrix

$$\Lambda = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix},$$

which is in $\mathrm{GL}(2,\mathbb{C})$ but *not* in $\mathrm{SU}(1,1)$, of course. The maps

$$\begin{aligned} &\Psi\colon \mathrm{SU}(1,1)\to \mathrm{SL}(2,\mathbb{R}), && \Psi(g)=\Lambda g\Lambda^{-1}, \\ &\psi\colon \mathfrak{su}(1,1)\to \mathfrak{sl}(2,\mathbb{R}), && \psi(X)=\Lambda X\Lambda^{-1} \end{aligned} \tag{1.24}$$

are a Lie group and Lie algebra isomorphisms, respectively. It is easy to check that

$$\psi\left(\begin{bmatrix} i\theta & 0 \\ 0 & -i\theta \end{bmatrix}\right)=\begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}, \qquad \psi\left(\begin{bmatrix} 0 & t+is \\ t-is & 0 \end{bmatrix}\right)=\begin{bmatrix} s & t \\ t & -s \end{bmatrix}.$$

This proves that under ψ the Cartan decomposition of $\mathfrak{su}(1,1)$ is mapped onto the Cartan decomposition of $\mathfrak{sl}(2,\mathbb{R})$. Furthermore, Iwasawa decomposition of $\mathrm{SU}(1,1)$ can be found from $\mathrm{SL}(2,\mathbb{R})$'s by inverting Ψ.

1.3 The Radon transform

In this section we present the result of which we provide different versions in Chapters 2 and 3: Theorem 1.35. Such result is classically called unitarization theorem for the Radon transform and has been proved first by Helgason in the setup of the polar Radon transform.

The first part of this section is devoted to the introduction of the polar Radon transform, while in the second part a generalization of the Radon transform due to Helgason [39] is followed by a result due to G. Alberti, F. Bartolucci, F. De Mari, E. De Vito [1] which represents a first extension of the unitarization theorem. Other general references are [6] and [2].

1.3.1 The classical Radon transform

Given a function on a space X and a family of subsets of X, the Radon transform is a function which associates to every subset of the family the integral of the function restricted to the subset. The Radon transform has been originally introduced by J. Radon [46] for integrals on lines and planes in the case of functions defined on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. It was later generalized on subspaces of every dimension in $\mathbb{R}^d$. In the following we are interested in the case of integration on hyperplanes in $\mathbb{R}^d$ and then for simplicity we focus our attention on the case of lines in $\mathbb{R}^2$. Since the Radon transform is a function defined on the family of hyperplanes, it is necessary to fix a parametrization for hyperplanes; there are several choices which give rise to different definitions of Radon transforms. The most general choice for the parametrization is to associate to every $(n,t)\in(\mathbb{R}^d\backslash\{0\})\times\mathbb{R}$ the hyperplane

$$[n;t]=\{x\in\mathbb{R}^d\colon x\cdot n=t\}.$$

Under this parametrization, the Radon transform of $f\in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{R}f(n,t):=\frac{1}{|n|}\int_{\{x\cdot n=t\}} f(x)\mathrm{d}m(x) \qquad \text{a.e. } (n,t)\in(\mathbb{R}^d\backslash\{0\})\times\mathbb{R},$$

where m is the Euclidean measure on the hyperplane $[n;t]$. Observe that $\mathcal{R}f$ is well defined for every $f \in L^1(\mathbb{R}^d)$. In fact, if $n \in \mathbb{R}^d\backslash\{0\}$ then $\mathcal{R}f(n,t) < +\infty$ for a.e. $t \in \mathbb{R}$, since by Fubini's theorem

$$\int_{\mathbb{R}^d} |f(x)|\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\{x\cdot n=t\}} |f(x)|\mathrm{d}m(x) \right) \mathrm{d}t < +\infty.$$

We state a crucial result for the Radon transform which highlights its relation with the Euclidean Fourier transform.

Proposition 1.32 (Fourier slice theorem, [39]). *For any $f \in L^1(\mathbb{R}^d)$ we have*

$$(I \otimes \mathcal{F})(\mathcal{R}f)(n,\tau) = \mathcal{F}f(\tau n), \qquad n \in \mathbb{R}^d\backslash\{0\},\ \tau \in \mathbb{R}.$$

Here the Fourier transform on the right hand side is on $L^1(\mathbb{R}^d)$, whereas the one on the left hand side is one-dimensional and acts on the variabile t.

The parametrization we have chosen is clearly not injective, indeed $[n;t] = [\lambda n; \lambda t]$, for every $\lambda \in \mathbb{R}\backslash\{0\}$. From the non-injectivity of the parametrization it follows that

$$\mathcal{R}f(\lambda n, \lambda t) = |\lambda|^{-1}\mathcal{R}f(n,t).$$

Clearly $\mathcal{R}f$ is completely defined by choosing a unique representative for each hyperplane.

We focus our attention on a different version of the Radon transform obtained by choosing a suitable parametrization of the family of hyperplanes of $\mathbb{R}^d$. Observe that

$$\{\text{hyperplanes of } \mathbb{R}^d\} \simeq \mathbb{P}^{d-1} \times \mathbb{R}.$$

The canonical choice is given by parametrizing $\mathbb{P}^{d-1}$ by the two-fold covering of the unitary sphere $S^{d-1} \subseteq \mathbb{R}^d$. Define $\Theta^{d-1} = [0,\pi]^{d-2} \times [0,2\pi)$. For all $\theta \in \Theta^{d-1}$ we write inductively

$${}^t\theta = (\theta_1, {}^t\hat{\theta}), \qquad \theta_1 \in [0,\pi],\ \hat{\theta} \in \Theta^{d-2},$$

and then we put

$$n(\theta) = (\cos\theta_1, \sin\theta_1 {}^t n(\hat{\theta})),$$

where $n(\hat{\theta}) \in S^{d-2}$ corresponds to the previous inductive step. We shall use the parametrization induced by $n\colon \Theta^{d-1} \to S^{d-1}$. The Radon transform that we obtain by adopting this parametrization is the most common realization and is called polar Radon transform.

Definition 1.33. The *polar Radon transform* of $f \in L^1(\mathbb{R}^d)$ is $\mathcal{R}^{\mathrm{pol}}f\colon \Theta^{d-1} \times \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{R}^{\mathrm{pol}}f(\theta,t) := \mathcal{R}f(n(\theta),t) = \int_{\{n(\theta)\cdot x=t\}} f(x)\mathrm{d}m(x), \qquad (\theta,t) \in \Theta^{d-1} \times \mathbb{R}.$$

For our purposes, it is sufficient to focus our attention on the case $d = 2$ in which the polar Radon transform of $f \in L^1(\mathbb{R}^2)$ can be written as

$$\mathcal{R}^{\mathrm{pol}}f(\theta,t) = \int_{\mathbb{R}} f(t\cos\theta - y\sin\theta, t\sin\theta + y\cos\theta)\mathrm{d}y, \tag{1.25}$$

where, as above, the equality holds for a.e. $(\theta, t) \in S^1 \times \mathbb{R}$. As observed in the d-dimensional case, S^1 is a two-fold covering of $\mathbb{P}^2$ and thus the parametrization is still non-injective, indeed

$$\mathcal{R}^{\mathrm{pol}} f(\theta, t) = \mathcal{R}^{\mathrm{pol}} f(\theta + \pi, -t). \tag{1.26}$$

This new realization of the Radon transform yields a new version of Proposition 1.32.

Proposition 1.34 (Fourier slice theorem 2, Proposition 6 in [6])**.** *Define $\psi\colon [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^2$ by $\psi(\theta, \tau) = \tau n(\theta)$. For every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ there exists a negligible set $E \subseteq [0, 2\pi)$ such that for all $\theta \notin E$ the function $\mathcal{R}^{\mathrm{pol}} f(\theta, \cdot)$ is in $L^2(\mathbb{R})$ and satisfies*

$$\mathcal{R}^{\mathrm{pol}} f(\theta, \cdot) = \mathcal{F}^{-1}[\mathcal{F} f \circ \psi(\theta, \cdot)].$$

It is not possible to extend directly the Radon transform to an isometry on $L^2(\mathbb{R}^2)$. A classical result in the theory of Radon transform shows that there exists a pseudo-differential operator whose precomposition with $\mathcal{R}^{\mathrm{pol}}$ extends to a unitary operator onto the closed subspace of functions

$$L^2_e([0, 2\pi) \times \mathbb{R}) = \{F \in L^2([0, 2\pi) \times \mathbb{R})\colon F(\theta, t) = F(\theta + \pi, -t)\}$$

of $L^2([0, 2\pi) \times \mathbb{R})$ endowed with the measure $\mathrm{d}\theta \mathrm{d}t/2$. Consider the subspace

$$\mathcal{D} = \{F \in L^2([0, 2\pi) \times \mathbb{R})\colon \int_{[0,2\pi)\times\mathbb{R}} |\tau| |(I \otimes \mathcal{F}) F(\theta, \tau)|^2 \mathrm{d}\theta \mathrm{d}\tau < +\infty\}$$

of $L^2([0, 2\pi) \times \mathbb{R})$ and define the operator $\mathcal{I}^{\mathrm{pol}}\colon \mathcal{D} \to L^2([0, 2\pi) \times \mathbb{R})$ by

$$(I \otimes \mathcal{F})(\mathcal{I}^{\mathrm{pol}} F)(\theta, t) = |\tau|^{\frac{1}{2}} (I \otimes \mathcal{F}) F(\theta, t),$$

that is a Fourier multiplier with respect to the second coordinate. Since $\tau \mapsto |\tau|^{\frac{1}{2}}$ is a strictly positive (almost everywhere) Borel function on $\mathbb{R}$, the spectral theorem for unbounded operators, see Theorem VIII.6 in [47], shows that $\mathcal{D}$ is dense and that $\mathcal{I}^{\mathrm{pol}}$ is a positive self-adjoint injective operator. It is possible to see that the operator $\mathcal{I}^{\mathrm{pol}}$ acts on the second coordinate as the inverse of the Riesz potential with exponent $1/2$ on $L^2(\mathbb{R})$. We show that if $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, then $\mathcal{R}^{\mathrm{pol}} f \in L^2([0, 2\pi) \times \mathbb{R})$, indeed

$$\begin{aligned}
\frac{1}{2} \int_{[0,2\pi)\times\mathbb{R}} |\mathcal{R}^{\mathrm{pol}} f(\theta, t)|^2 \mathrm{d}\theta \mathrm{d}t &= \frac{1}{2} \int_0^{2\pi} \int_{\mathbb{R}} |(I \otimes \mathcal{F}) \mathcal{R}^{\mathrm{pol}} f(\theta, \tau)|^2 \mathrm{d}\tau \mathrm{d}\theta \\
&= \frac{1}{2} \int_{[0,2\pi)\times\mathbb{R}} |\mathcal{F} f(\tau n(\theta))|^2 \mathrm{d}\theta \mathrm{d}\tau \\
&\leqslant \frac{1}{2} \int_0^{2\pi} \int_{|\tau| \leqslant 1} |\mathcal{F} f(\tau n(\theta))|^2 \mathrm{d}\tau \mathrm{d}\theta + \frac{1}{2} \int_0^{2\pi} \int_{|\tau| > 1} |\tau| |\mathcal{F} f(\tau n(\theta))|^2 \mathrm{d}\tau \mathrm{d}\theta \\
&\leqslant \pi \|f\|_1^2 + \frac{1}{2} \int_{\mathbb{R}^2} |\mathcal{F} f(\xi)|^2 \mathrm{d}\xi = \pi^2 \|f\|_1^2 + \|f\|_2^2 < +\infty,
\end{aligned}$$

where we used Plancherel theorem (1.11) and Proposition 1.34. Furthermore it is easy to see that $\mathcal{R}^{\mathrm{pol}} f \in \mathcal{D}$ for every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ since

$$\begin{aligned}
\int_{[0,2\pi)\times\mathbb{R}} |\tau| |(I \otimes \mathcal{F}) \mathcal{R}^{\mathrm{pol}} f(\theta, \tau)|^2 \mathrm{d}\theta \mathrm{d}\tau &= \int_{[0,2\pi)\times\mathbb{R}} |\tau| |\mathcal{F} f(\tau n(\theta))|^2 \mathrm{d}\theta \mathrm{d}\tau \\
&= 2\|f\|_2^2 < +\infty.
\end{aligned}$$

Hence we can consider the composite operator

$$\mathcal{I}^{\mathrm{pol}}\mathcal{R}^{\mathrm{pol}}: L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \to L^2([0, 2\pi) \times \mathbb{R}).$$

We are now in a position to state one of the fundamental results in Radon transform theory, which will be of inspiration for the results in the next section and then in the following chapters.

Theorem 1.35 (Unitarization of the polar Radon transform, Theorem 4.1 [39]). *The composite operator $\mathcal{I}^{\mathrm{pol}}\mathcal{R}^{\mathrm{pol}}$ from $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ into $L^2([0, 2\pi) \times \mathbb{R})$ extends to a unique unitary operator*

$$\mathcal{Q}: L^2(\mathbb{R}^2) \to L^2_e([0, 2\pi) \times \mathbb{R}).$$

Since the evenness condition (1.26) is a closed condition, the image of the operator $\mathcal{Q}$ can not be the whole L^2-space but just the subspace $L^2_e([0, 2\pi) \times \mathbb{R})$.

1.3.2 Unitarization of the Radon transform between dual pairs

This section is devoted to the generalization of the notion of Radon transform due to Helgason [39] and to the extension of Theorem 1.35 to this new class of transforms. This different unitarization of Radon transform is presented in [1] in which, under some hypothesis on the spaces involved, the authors derive new inversion formulae for the Radon transform from it. The notion of Radon transform that we present has been introduced by Helgason in [39] for a large class of pairs of homogeneous spaces of the same locally compact group. Below we consider the adaptation of this notion to the cases presented in [1].

Helgason considers two transitive G-spaces of a lcsc group G, X and Ξ, which represent the space on which our functions are defined and the parameters of the family of submanifolds on which we want to integrate, respectively. We denote the two actions of $g \in G$ on X and Ξ by:

$$x \mapsto g[x], \quad \xi \mapsto g.\xi, \qquad x \in X,\ \xi \in \Xi.$$

In view of the desired intertwining result, we need to introduce the two quasi regular representations of G which are involved. We suppose that X and Ξ carry relatively G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$ with characters α and β, respectively. The group G acts unitarily on $L^2(X, \mathrm{d}x)$ via the quasi regular representation defined by

$$\pi(g)f(x) = \alpha(g)^{-\frac{1}{2}} f(g^{-1}[x]), \qquad g \in G,\ f \in L^2(X, \mathrm{d}x),$$

and on $L^2(\Xi, \mathrm{d}\xi)$ via the representation $\hat{\pi}$ defined by

$$\hat{\pi}(g)F(\xi) = \beta(g)^{-\frac{1}{2}} F(g^{-1}.\xi), \qquad g \in G,\ F \in L^2(\Xi, \mathrm{d}\xi).$$

Now we define the Radon transform. Fix two origins $x_0 \in X$ and $\xi_0 \in \Xi$. We denote by$K$ and H the isotropy subgroups of G at x_0 and ξ_0, respectively, so that

$$X \simeq G/K, \qquad \Xi \simeq G/H.$$

We put

$$\check{x}_0 = K.\xi_0 \subseteq \Xi, \qquad \hat{\xi}_0 = H[x_0] \subseteq X.$$

In order to define the Radon transform we assume that there exists an H-invariant measure m_0 on $\hat{\xi}_0$ and a Borel section $\sigma\colon \Xi \to G$. In such way, we can "transport" the definition of $\hat{\xi}_0$ and $\check{x}_0$ to every $\xi \in \Xi$ and $x \in X$, respectively, by defining

$$\hat{\xi} = \sigma(\xi)[\hat{\xi}_0] \subseteq X, \qquad \check{x} = s(x).\check{x}_0 \subseteq \Xi,$$

which are closed subsets by Lemma 1.1 in [39].

We assume the *transversality condition*, that is we ask for the maps $\xi \mapsto \hat{\xi}$ and $x \mapsto \check{x}$ to be injective, so that (X, Ξ) is a dual pair in the sense of Helgason. In addition to the cases considered in the following, we refer to [39] for numerous examples of dual pairs (X, Ξ). Example 1.3 shows that the polar Radon transform (together with its dual transform) can be obtained in this weakened framework starting from the similitude group of the plane. The last part of this chapter is devoted to a different example in which another dual pair is presented.

We are now ready to define the Radon transform for a dual pair according to Helgason [39]. The idea is to use the Borel section σ to push-forward the measure m_0 from $\hat{\xi}_0$ to the manifolds $\hat{\xi}$.

Definition 1.36. The *Radon transform* of f is the map $\mathcal{R}f\colon \Xi \to \mathbb{C}$ defined by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x) := \int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x),$$

whenever the integral converges.

It is possible to define the dual Radon transform, or back-projection, of a function defined on Ξ by considering the integration of the restriction to $\check{x}$, for every $x \in X$. In the following dual Radon transform does not enter into play, so we refer to [1] and [39] for a more in-depth analysis on it.

The main result of [1] holds under some assumptions. Before presenting it, we show as the polar Radon transform satisfies the hypothesis we assumed throughout. First of all, we recall the main assumptions that we have made until now on the transitive G-spaces X and Ξ:

(A1) the spaces X and Ξ carry relatively G-invariant measures $\mathrm{d}X$ and $\mathrm{d}\xi$, respectively;

(A2) the H-transitive space $\hat{\xi}_0 = H[x_0] \subseteq X$ carries a relatively H-invariant measure m_0, with character γ;

(A3) the pair (X, Ξ) is a dual pair in the sense of Helgason, that is the transversality condition holds.

We present now an example in which assumptions (A1)-(A3) are satisfied and the Radon transform from Definition 1.36 is the polar Radon transform.

Example 1.3. We consider the (connected component of the identity of the) similitude group $SIM(2)$ of the plane. That is, $SIM(2) = \mathbb{R}^2 \rtimes K$ with $K = \{R_\phi a \in \mathrm{GL}(2, \mathbb{R})\colon \phi \in [0, 2\pi), a \in \mathbb{R}^+\}$ where

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

By the identification $K \simeq [0, 2\pi) \times \mathbb{R}^+$, we write (b, ϕ, a) for the elements in $SIM(2)$ and the group law becomes

$$(b, \phi, a)(b', \phi', a') = (b + R_\phi a b', \phi + \phi' \mod 2\pi, aa').$$

Let $\mathrm{d}b$, $\mathrm{d}\phi$ and $\mathrm{d}a$ be the left Haar measure on $\mathbb{R}^2$, $[0, 2\pi)$ and $\mathbb{R}^+$, respectively. It is easy to see that the left Haar measure on $SIM(2)$ is

$$\mathrm{d}\mu(b, \phi, a) = a^{-3}\mathrm{d}b\mathrm{d}\phi\mathrm{d}a.$$

We put $X = \mathbb{R}^2$ and $x_0 = o \in \mathbb{R}^2$. The group acts transitively on $\mathbb{R}^2$ by

$$(b, \phi, a)[x] = R_\phi a x + b$$

and $\mathbb{R}^2 = X \simeq G/K$. Furthermore, since for every Borelian set E of $\mathbb{R}^2$ we have

$$|(b, \phi, a)[E]| = |b + R_\phi a E| = |\det R_\phi a||E| = a^2|E|,$$

a relatively G-invariant measure on X is the Lebesgue measure with character $\alpha(b, \phi, a) = a^{-2}$.

We need to choose the space Ξ and the corresponding subgroup $H \subseteq SIM(2)$. We choose $\Xi = [0, \pi) \times \mathbb{R}$ on which $SIM(2)$ acts transitively through the affine action 1.3 on the lines of the plane parametrized by Ξ via (1.25), that is:

$$(b, \phi, a).(\theta, t) = (\theta + \phi \mod 2\pi, a(t + w(\theta) \cdot a^{-1} R_\phi^{-1} b)),$$

where $w(\theta) = {}^t(\cos\theta, \sin\theta)$, or equivalently by (1.2)

$$\begin{aligned}(b, \phi, a)^{-1}.(\theta, t) &= (-a^{-1}R_\phi^{-1}b, -\phi \mod \pi, a^{-1}).(\theta, t) \\ &= (\theta - \phi \mod \pi, \frac{t + w(\theta) \cdot b}{a}).\end{aligned}$$

The isotropy at the point $\xi_0 = (0, 0) \in \Xi$ is

$$H = \{((0, b_2), \phi, a) \in SIM(2)\colon b_2 \in \mathbb{R}, \phi \in \{0, \pi\}, a \in \mathbb{R}^+\}.$$

Thus, $[0, \pi) \times \mathbb{R} \simeq SIM(2)/H$ and

$$\int_\Xi F((b, \phi, a)^{-1}.(\theta, t))\mathrm{d}\theta\mathrm{d}t = a \int_\Xi F(\theta, t)\mathrm{d}\theta\mathrm{d}t, \qquad F \in L^1(\Xi, \mathrm{d}\theta\mathrm{d}t).$$

The relatively $SIM(2)$-invariant measure on Ξ is then the Lebesgue measure $\mathrm{d}\theta\mathrm{d}t$ with $\beta(b, \phi, a) = a$. This means that (A1) is satisfied.

It is easy to verify by direct computation that

$$\hat{\xi}_0 = H[x_0] = \{(0, b_2) \in \mathbb{R}^2 \colon b_2 \in \mathbb{R}\} \simeq \mathbb{R};$$
$$\check{x}_0 = K.\xi_0 = \{(\theta, 0) \in \Xi \colon \theta \in [0, \pi)\} \simeq [0, \pi),$$

and that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is relatively H-invariant with character $\gamma((0, b_2), \phi, a)) = a$, so that (A2) is satisfied.

We define and $\sigma \colon \Xi \to SIM(2)$ by

$$\sigma(\theta, t) = (tw(\theta), \theta, 1),$$

it is immediate to see that σ is a Borel section. Then we can define

$$\widehat{(\theta, t)} = \sigma(\theta, t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 \colon x \cdot w(\theta) = t\},$$

which is the set of all points laying on the line of equation $x \cdot w(\theta) = t$, namely the line that is uniquely parametrized by $(\theta, t) \in \Xi$ under the parametrization used in (1.25). In a similar way we find

$$\check{x} = s(x).\check{x}_0 = \{(\theta, t) \in \Xi \colon t - w(\theta) \cdot x = 0\},$$

which parametrizes the set of all lines of the plane passing through the point x.

The transversality condition is satisfied since $x \mapsto \check{x}$ maps a point in the set of all lines passing through that point and $(\theta, t) \mapsto \widehat{(\theta, t)}$ maps a line to the set of points lying on that line, so that they are both clearly injective. Hence $(\mathbb{R}^2, [0, \pi) \times \mathbb{R})$ is a dual pair in the sense of Helgason, as asked in (A3).

Finally we observe that the Radon transform defined as in Definition 1.36 coincides with the polar Radon transform. Let $f \colon \mathbb{R}^2 \to \mathbb{C}$. The Radon transform of f is the map $\mathcal{R}f \colon \Xi \to \mathbb{C}$ defined by

$$\begin{aligned}\mathcal{R}f(\theta, t) :&= \int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x) \\ &= \int_{\mathbb{R}} f(t\cos\theta - y\sin\theta, t\sin\theta + y\cos\theta)\mathrm{d}y = \mathcal{R}^{\mathrm{pol}}f(\theta, t).\end{aligned}$$

Hence we have shown that the polar Radon transform can be defined by following the general theory of the Radon transform for dual pairs and belongs to the family of transforms that are considered in [1]. Observe that, unlike the classical polar Radon transform presented in Section 1.3.1, the angle θ belongs to the smaller interval $[0, \pi)$. This may be assumed in order to make the map $(\theta, t) \mapsto \widehat{(\theta, t)}$ injective. It is clear that Theorem 1.35 is still valid without the evenness condition (1.26) and removing the constant $1/2$ from the density of the measure. Hence the operator $\mathcal{Q}$ maps $L^2(\mathbb{R}^2)$ onto $L^2([0, \pi) \times \mathbb{R}) = L^2(\Xi)$.

From now on, we suppose that assumptions (A1)-(A3) are satisfied. We write here all the other hypothesis we need for the main results:

(A4) there exists a Borel section $\sigma \colon \Xi \to G$ such that

$$(g, \xi) \to \gamma(\sigma(\xi)^{-1} g \sigma(g^{-1}.\xi))$$

extends to a positive character of G independent of ξ;

(A5) the quasi regular representation π of G acting on $L^2(X, \mathrm{d}x)$ is irreducible and square integrable;

(A6) the quasi regular representation $\hat{\pi}$ of G acting on $L^2(\Xi, \mathrm{d}\xi)$ is irreducible;

(A7) there exists a non-trivial π-invariant subspace $\mathcal{A} \subseteq L^2(X, \mathrm{d}x)$ such that

$$f(\sigma(\xi)[\cdot]) \in L^1(\hat{\xi}_0, m_0) \quad \text{for almost all } \xi \in \Xi; \tag{1.27}$$

$$\mathcal{R}f := \int_{\hat{\xi}_0} f(\sigma(\cdot)[x])\mathrm{d}m_0(x) \in L^2(\Xi, \mathrm{d}\xi), \quad \text{for all } f \in \mathcal{A}, \tag{1.28}$$

and the map $f \mapsto \mathcal{R}f$ is a closable operator from $\mathcal{A}$ to $L^2(\Xi, \mathrm{d}\xi)$.

The next results hold under the hypotheses (A1)-(A7).

Lemma 1.37 (Lemma 3.1 and Lemma 3.5 in [1])**.** *We define the character* χ *by*

$$\chi(g) = \beta(g)^{-\frac{1}{2}}\alpha(g)^{\frac{1}{2}}\gamma(g\sigma(g^{-1}.\xi_0))^{-1}, \qquad g \in G. \tag{1.29}$$

The following statements hold true:

(i) the restriction of $\mathcal{R}$ *to* $\mathcal{A}$ *is a densely defined operator from* $\mathcal{A}$ *into* $L^2(\Xi, \mathrm{d}\xi)$ *satisfying*

$$\mathcal{R}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\mathcal{R}, \qquad g \in G;$$

(ii) the closure $\overline{\mathcal{R}}$ *of* $\mathcal{R}$ *is a densely defined operator satisfying*

$$\overline{\mathcal{R}}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\overline{\mathcal{R}}, \qquad g \in G.$$

By observing that the previous lemma shows that $\mathcal{R}$ is a semi-invariant operator[1] with weight given by (1.29), we are in a position to state and prove the main result in [1]. We stress that its proof does not use the transversality condition on (X, Ξ), that is (A3), and the square integrability of π; the irreducibility of $\hat{\pi}$ is only needed in the last claim of the theorem.

Theorem 1.38 (Theorem 3.9, [1])**.** *There exists a unique positive self-adjoint operator*

$$\mathcal{I}\colon \mathrm{dom}(\mathcal{I}) \supseteq \mathrm{Im}\overline{\mathcal{R}} \to L^2(\Xi, \mathrm{d}\xi),$$

semi-invariant with weight $\zeta = \chi^{-1}$ *with the property that the composite operator* $\mathcal{I}\overline{\mathcal{R}}$ *extends to an isometry* $\mathcal{Q}\colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ *intertwining* π *and* $\hat{\pi}$*, namely*

$$\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, \qquad g \in G.$$

Furthermore, if $\hat{\pi}$ *is irreducible, then* $\mathcal{Q}$ *is a unitary operator and* π *and* $\hat{\pi}$ *are equivalent representations.*

[1] According to the classical work presented by Duflo and Moore in [21], a densely defined closed operator T from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$ is called semi-invariant with weight ζ if it satisfies

$$\pi_2(g)T\pi_1(g)^{-1} = \zeta(g)T, \qquad g \in G,$$

where ζ is a character of G and π_1 and π_2 are unitary representations of G acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

The above result represents a generalizations of Theorem 1.35. In the next chapters we focus on different versions, in the context of spaces in which assumptions (A1)-(A7) are not satisfied. In particular in Chapters 2-3 we present new versions of Theorem 1.38 in settings in which the representation π is not irreducible.

In [1], the authors provide an inversion formula for the Radon transform as direct consequence of Theorem 1.38 under the assumption that π is square integrable.

Theorem 1.39 (Theorem 6, [1])**.** *Let $\psi \in L^2(X, \mathrm{d}x)$ be an admissible vector for π such that $\mathcal{Q}\psi \in \mathrm{dom}(\mathcal{I})$ and set $\Psi = \mathcal{I}\mathcal{Q}\psi$. Then, for any $f \in \mathrm{dom}(\overline{\mathcal{R}})$,*

$$f = \int_G \chi(g)\langle \overline{\mathcal{R}} f, \hat{\pi}(g)\Psi\rangle \pi(g)\psi \mathrm{d}\mu(g), \tag{1.30}$$

where the integral is weakly convergent, and

$$\|f\|^2 = \int_G \chi(g)^2 |\langle \overline{\mathcal{R}} f, \hat{\pi}(g)\Psi\rangle|^2 \mathrm{d}\mu(g).$$

If, in addition, $\psi \in \mathrm{dom}(\overline{\mathcal{R}})$, then $\psi = \mathcal{I}^2\overline{\mathcal{R}}\psi$.

Example 1.3 (continued)**.** We continue Example 1.3. We have already proved that the setting of the polar Radon transform satisfies (A1)-(A3). We next show that (A4)-(A7) are satisfied.

By a simple calculation, we can see that

$$\gamma(\sigma(\theta,t)(b,\phi,a)\sigma((b,\phi,a)^{-1}.(\theta,t))) = a, \qquad (\theta,t) \in \Xi,\ (b,\phi,a) \in SIM(2),$$

and this means that γ can be extended to a positive character of G which does not depend on (θ, t), as required in (A4).

The group $SIM(2)$ acts on $L^2(\mathbb{R}^2)$ by means of the representation π defined by

$$\pi(b,\phi,a)f(x) = a^{-\frac{1}{2}} f(a^{-1}R_\phi^{-1}(x-b)). \tag{1.31}$$

The representation π is unitary and irreducible, as it can be seen by passing through the equivalent definition in frequency domain

$$\mathcal{F}[\pi(b,\phi,a)f](\omega) = ae^{-2\pi i b\cdot\omega}\mathcal{F}f(aR_\phi^{-1}\omega), \qquad \omega \in \mathbb{R}^2, \tag{1.32}$$

together with Lemma 1.8 and Plancherel formula (1.11), as we do for the (full) "$ax+b$"-representation in Section 1.1.2. It is also known [3] that π is square integrable and thus satisfies (A5).

Furthermore, $SIM(2)$ acts on $L^2([0,\pi) \times \mathbb{R})$ via the unitary quasi regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b,\phi,a)F(\theta,t) = a^{-\frac{1}{2}} F\left(\theta - \phi \bmod \pi, \frac{t - w(\theta)\cdot b}{a}\right), \tag{1.33}$$

which is irreducible, as we want in (A6).

It remains to prove that (A7) is satisfied, that is the existence of a non-trivial π-invariant subspace $\mathcal{A} \subseteq L^2(\mathbb{R}^2)$ such that (1.27) and (1.28) holds and $f \mapsto \mathcal{R}f$ is a closable operator from $\mathcal{A}$ to $L^2(\Xi, \mathrm{d}\xi)$. Observe that for $f \in L^1(\mathbb{R}^2)$ by Theorem 1.34 we have

$$\begin{aligned}\int_{[0,\pi)\times\mathbb{R}} |\mathcal{R}^{\mathrm{pol}} f(\theta,t)|^2 \mathrm{d}\theta\mathrm{d}t &= \int_{[0,\pi)\times\mathbb{R}} |(I\otimes\mathcal{F})(\mathcal{R}^{\mathrm{pol}} f)(\theta,\tau)|^2 \mathrm{d}\theta\mathrm{d}\tau \\ &= \int_{[0,\pi)\times\mathbb{R}} |\mathcal{F}f(\tau w(\theta))|^2 \mathrm{d}\theta\mathrm{d}\tau \\ &= \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi_1,\xi_2)|^2}{\sqrt{\xi_1^2+\xi_2^2}} \mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned}$$

We are thus led to consider

$$\mathcal{A}^{\mathrm{pol}} = \{f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \colon \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi_1,\xi_2)|^2}{\sqrt{\xi_1^2+\xi_2^2}} \mathrm{d}\xi_1\mathrm{d}\xi_2 < +\infty\},$$

which is π-invariant by (1.32) and by definition $\mathcal{R}^{\mathrm{pol}} f \in L^2([0,\pi)\times\mathbb{R})$ for every $f \in \mathcal{A}^{\mathrm{pol}}$. Furthermore, we refer to Example 1 in [1] to show that $\mathcal{R}^{\mathrm{pol}}$ restricted to $\mathcal{A}^{\mathrm{pol}}$ is closable.

Hence, we have that the pair $(\mathbb{R}^2, [0,\pi)\times\mathbb{R})$ satisfies assumpitons (A1)-(A7). By applying Lemma 1.37 to $\mathcal{R}^{\mathrm{pol}}$ we have that $\overline{\mathcal{R}^{\mathrm{pol}}}$ is a semi-invariant operator from $\mathcal{A}^{\mathrm{pol}}$ to $L^2([0,\pi)\times\mathbb{R})$ with weight $\chi(b,\phi,a) = a^{-1/2}$. From Theorem 1.38 there exists a positive self-adjoint operator $\mathcal{I}\colon \mathrm{dom}(\mathcal{I}) \supseteq \mathrm{Im}(\overline{\mathcal{R}^{\mathrm{pol}}}) \to L^2([0,\pi)\times\mathbb{R})$, semi-invariant with weight $\chi(g)^{-1} = a^{1/2}$, such that $\mathcal{I}\overline{\mathcal{R}^{\mathrm{pol}}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2) \to L^2([0,\pi)\times\mathbb{R})$ intertwining the representations π and $\hat{\pi}$ defined in (1.31) and (1.33), respectively. Hence

$$\begin{aligned}&\mathcal{I}\mathcal{R}^{\mathrm{pol}} f = \mathcal{Q}f, && f \in \mathcal{A}^{\mathrm{pol}},\\ &\mathcal{Q}^*\mathcal{Q}f = f, && f \in L^2(\mathbb{R}^2),\\ &\mathcal{Q}\mathcal{Q}^*F = F, && F \in L^2([0,\pi)\times\mathbb{R}),\\ &\pi(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, && g \in SIM(2).\end{aligned}$$

It follows from Theorem 1.35 that the operator $\mathcal{I}$ we are looking for is strictly related to $\mathcal{I}^{\mathrm{pol}}$. In fact, the only variation is that in the domain $\mathcal{D}$ we substitute the evenness condition (1.26) for functions defined on $[0,2\pi)\times\mathbb{R}$ by considering only functions defined on $[0,\pi)\times\mathbb{R}$. As a consequence of such choice, we have that the extension $\mathcal{Q}$ is such that $\mathrm{Im}(\mathcal{Q}) = L^2([0,\pi)\times\mathbb{R})$.

It is known that π is square integrable and the corresponding voice transform gives rise to $2D$-directional wavelets [4]. An admissible vector is a function $\psi \in L^2(\mathbb{R}^2)$ satisfying the following admissibility condition

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\xi_1,\xi_2)|^2}{\xi_1^2+\xi_2^2} \mathrm{d}\xi_1\mathrm{d}\xi_2 = 1.$$

Fix an admissible vector $\psi \in L^2(\mathbb{R}^2)$ and put $\Psi = \mathcal{I}\mathcal{Q}\psi$. Given $f \in \mathcal{A}^{\mathrm{pol}}$, define $\mathcal{G}(b,\phi,a) = a^{\frac{1}{2}}\langle \mathcal{R}^{\mathrm{pol}} f, \hat{\pi}(b,\phi,a)\Phi\rangle$, then (1.30) reads

$$f(x) = \int_{\mathbb{R}^2\rtimes([0,2\pi)\times\mathbb{R}^+)} \mathcal{G}(b,\phi,a)\psi\left(R_\phi^{-1}\frac{x-b}{a}\right) \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5}.$$

1.3.2.1 Radon Transform for Hyperbolic Motions

In the following we present another example. In fact we show that the general previous results may be applied to the Radon transform associated to the group of hyperbolic motions of the plane. The results we present are collected in [9].

We consider the semidirect product $G = \mathbb{R}^2 \rtimes K$, with

$$K = \{aA_s\Omega_\epsilon \in \mathrm{GL}(2,\mathbb{R}) : a \in \mathbb{R}^*,\ s \in \mathbb{R},\ \epsilon \in \{-1,1\}\}$$

where

$$A_s = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}, \qquad \Omega_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and Ω_1 is the identity matrix. We denote by C_2 the multiplicative group $\{-1,1\}$. Under the identification $K \simeq \mathbb{R} \times \mathbb{R}^* \times C_2$, we write (b,s,a,ϵ) for elements in G, so that the group law becomes

$$(b,s,a,\epsilon)(b',s',a',\epsilon') = (b + aA_s\Omega_\epsilon b', s+s', aa', \epsilon\,\epsilon').$$

A left Haar measure of G is $\mathrm{d}\mu(b,s,a,\epsilon) = |a|^{-3}\mathrm{d}b\mathrm{d}s\mathrm{d}a\mathrm{d}\epsilon$, where $\mathrm{d}b$, $\mathrm{d}s$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^*$, respectively and $\mathrm{d}\epsilon$ is the counting measure on C_2.

The group G acts transitively on $X = \mathbb{R}^2$ by the canonical action

$$(b,s,a,\epsilon)[x] = b + aA_s\Omega_\epsilon x, \quad (b,s,a,\epsilon) \in G,\ x \in X.$$

The isotropy at the origin $x_0 = 0$ is the closed subgroup $\{(0,k) : k \in K\} \simeq K$, so that $X \simeq G/K$ and the Lebesgue measure $\mathrm{d}x$ on X is relatively G-invariant with positive character $\alpha(b,s,a,\epsilon) = |a|^2$. It is possible to parametrize lines in the plane, except those with slope -1 or 1, by the space of parameters $\Xi = C_2 \times \mathbb{R} \times \mathbb{R}$. In fact, considering a triple $(\eta,u,t) \in \Xi$, the vector u parametrizes the slope, whereas the choice $\eta = 1$ ($\eta = -1$) corresponds to slope > 1 (< 1) and fixes the $x-$axis (y-axis) as reference line; finally, t parametrizes the intersection of the line with the reference axis. We refer to Figure 1.1 for a graphic realization of the parametrization. The group G is a subgroup of affine transformations of the plane and thus maps lines into lines. Its action on this set of lines is given by

$$(b,s,a,\epsilon)^{-1}.(\eta,u,t) = \left(\epsilon\,\eta, u+s, \frac{t - \Omega_\eta w(u)\cdot b}{a}\right),$$

where $w(u) = {}^t(\cosh u, \sinh u)$, and is easily seen to be transitive. The isotropy at $\xi_0 = (1,0,0)$ is

$$H = \{((0,b_2),0,a,1) : b_2 \in \mathbb{R}, a \in \mathbb{R}^*\}.$$

Thus, $\Xi \simeq G/H$. An immediate calculation gives that the measure $\mathrm{d}\xi = \mathrm{d}\eta\mathrm{d}u\mathrm{d}t$, where $\mathrm{d}u$ and $\mathrm{d}t$ are the Lebesgue measures on $\mathbb{R}$ and $\mathrm{d}\eta$ is the counting measure on C_2, is a G-relatively invariant measure on Ξ with positive character $\beta(b,s,a,\epsilon) = |a|$.

Consider now the Borel section $\sigma\colon \Xi \to G$ defined by

$$\sigma(\eta,u,t) = (t\,\Omega_\eta w(-u), -u, 1, \eta).$$

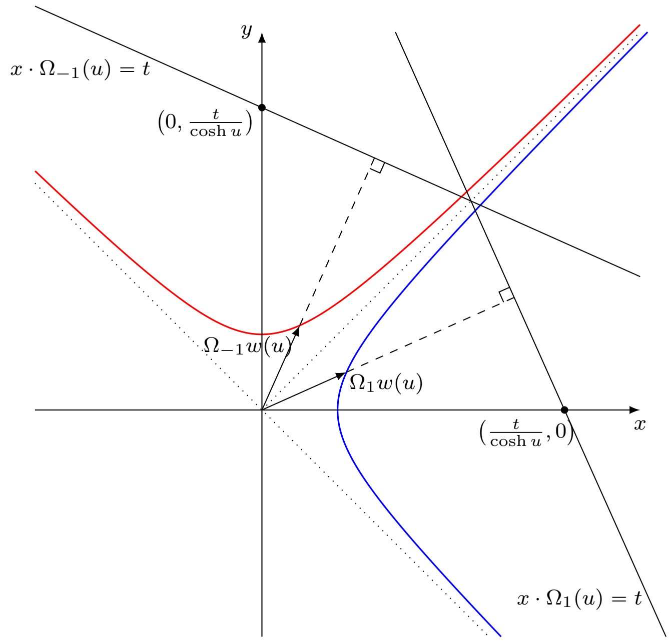

Figure 1.1: The parametrization of all the lines in $\mathbb{R}^2$ having slope different from 1 or -1 under Ξ. In particular, the two lines which are represented are parameteized by $(1, u, t)$ and $(-1, u, t)$ for $u, t > 0$.

By direct computation

$$\hat{\xi}_0 = H[x_0] = \{(0, b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R}.$$

It is immediate to see that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is relatively H-invariant with character $\gamma((0, b_2), 0, a, 1) = |a|$ and that $\gamma(\sigma(\eta, u, t)) = 1$ for all $(\eta, u, t) \in \Xi$, so that $(g, \xi) \mapsto \gamma\big(\sigma(\xi)^{-1} g \sigma(g^{-1}.\,\xi)\big)$ extends to a positive character of G independent of ξ, as required in (A4). Further, we have that

$$\widehat{(\eta, u, t)} = \sigma(\eta, u, t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x \cdot \Omega_\eta w(u) = t\},$$

which is the set of all points laying on the line of equation $x \cdot \Omega_\eta w(u) = t$. Therefore, the submanifolds over which we integrate functions are lines in $\mathbb{R}^2$, except those with slope -1 or 1, and are parametrized by Ξ through the injective map $(\eta, u, t) \mapsto \widehat{(\eta, u, t)}$.

The group G acts on $L^2(X)$ by means of the unitary representation π defined by

$$\pi(b, s, a, \epsilon) f(x) = |a|^{-1} f(a^{-1} \Omega_\epsilon^{-1} A_s^{-1}(x - b)).$$

The dual action $\mathbb{R}^2 \times K \ni (\eta, k) \mapsto {}^t k \eta$ has a single open orbit $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : |x| \neq |y|\}$ for ${}^t(1, 0) \in \mathbb{R}^2$ of full measure and the stabilizer $K_{(1,0)} = \{(0, 1, 1)\}$ is compact. Then, by a result due to Führ in [26], the representation π is square integrable. Furthermore, G acts on $L^2(\Xi, \mathrm{d}\xi)$ by means of the quasi regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b, s, a, \epsilon) F(\eta, u, t) = |a|^{-\frac{1}{2}} F\left(\epsilon\, \eta, u + s, \frac{t - \Omega_\eta w(u) \cdot b}{a}\right), \quad F \in L^2(\Xi, \mathrm{d}\xi).$$

By Mackey imprimitivity theorem [24], one can show that also $\hat{\pi}$ is irreducible. The proof, although not trivial, is based on classical arguments and we omit it. Hence assumptions (A1)-(A6) are satisfied.

By Definition 1.36 we compute the Radon transform between the homogeneous spaces X and Ξ and we obtain

$$\mathcal{R}^{\mathrm{hyp}} f(\eta,u,t) = \int_{\mathbb{R}} f(\Omega_\eta {A_{-u}}^t(t,y))\mathrm{d}y, \tag{1.34}$$

which maps any $(\eta,u,t)\in\Xi$ in the integral of f over the line parametrized by (η,u,t) through the map $(\eta,u,t)\mapsto \widehat{(\eta,u,t)}$, i.e. the line of equation $x\cdot\Omega_\eta w(u)=t$. Observe that, by Fubini's theorem, the integral (1.34) converges for any $f\in L^1(\mathbb{R}^2)$. Then, we define

$$\mathcal{A}^{\mathrm{hyp}} = \{f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi_1,\xi_2)|^2}{\sqrt{|\xi_1^2-\xi_2^2|}}\mathrm{d}\xi_1\mathrm{d}\xi_2 < +\infty\},$$

which is π-invariant and is such that $\mathcal{R}^{\mathrm{hyp}}f\in L^2(\Xi,\mathrm{d}\xi)$ for all $f\in\mathcal{A}^{\mathrm{hyp}}$. Furthermore, it is possible to show that $\mathcal{R}^{\mathrm{hyp}}$, regarded as operator from $\mathcal{A}^{\mathrm{hyp}}$ to $L^2(\Xi,\mathrm{d}\xi)$, is closable. In order to determine the subspace $\mathcal{A}^{\mathrm{hyp}}$ and to prove that $\mathcal{R}^{\mathrm{hyp}}\colon\mathcal{A}^{\mathrm{hyp}}\to L^2(\Xi,\mathrm{d}\xi)$ is closable, we adapt Theorem 1.32 to our context, precisely

$$(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{hyp}}f(\eta,u,\tau) = \mathcal{F}f(\tau\,\Omega_\eta w(u)),$$

for every $f\in L^1(\mathbb{R}^2)$ and $(\eta,u,\tau)\in\Xi$, where I is the identity operator on $L^2(C_2\times\mathbb{R},\mathrm{d}\eta\mathrm{d}u)$.

It is worth observing that when we fix $\eta=1$ ($\eta=-1$) in (1.34) we are restricting the integration of f over all lines with slope >1 (<1). Then, for $\eta=1$ and $\eta=-1$ we have the limited angle horizontal and vertical Radon transforms, respectively. We will see in the following how these two different contributions enter in the inversion formula (1.30) when we reconstruct an unknown function from its Radon transform.

Applying Lemma 1.37, $\mathcal{R}^{\mathrm{hyp}}\colon\mathcal{A}^{\mathrm{hyp}}\to L^2(\Xi,\mathrm{d}\xi)$ is a densely defined operator which intertwines the representations π and $\hat{\pi}$ up to the positive character $\chi(b,s,a,\epsilon) = |a|^{-1/2}$, namely

$$\hat{\pi}(b,s,a,\epsilon)\mathcal{R}^{\mathrm{hyp}}\pi(b,s,a,\epsilon)^{-1} = |a|^{-1/2}\mathcal{R}^{\mathrm{hyp}},$$

for all $(b,s,a,\epsilon)\in G$.

The composition of $\mathcal{R}^{\mathrm{hyp}}$ with a positive selfadjoint operator $\mathcal{I}^{\mathrm{hyp}}$ satisfying

$$\hat{\pi}(b,s,a,\epsilon)\mathcal{I}^{\mathrm{hyp}}\hat{\pi}(b,s,a,\epsilon)^{-1} = |a|^{1/2}\mathcal{I}^{\mathrm{hyp}}$$

can be extended to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2)\to L^2(\Xi,\mathrm{d}\xi)$ intertwining the irreducible representations π and $\hat{\pi}$.

We can provide an explicit formula for $\mathcal{I}^{\mathrm{hyp}}$. We consider the subspace $\mathcal{D}$ of $L^2(\Xi,\mathrm{d}\xi)$ of the functions F such that

$$\int_{\mathbb{R}\times\mathbb{R}} |\tau||(I\otimes\mathcal{F})F(\eta,u,\tau)|^2\,\mathrm{d}u\mathrm{d}\tau < +\infty, \qquad \eta=-1,1,$$

and we define the operator $\mathcal{J}$ on $\mathcal{D}$ by

$$(I \otimes \mathcal{F})\mathcal{J}F(\eta, u, \tau) = |\tau|^{\frac{1}{2}}(I \otimes \mathcal{F})F(\eta, u, \tau),$$

a Fourier multiplier with respect to the last variable. A direct calculation shows that $\mathcal{J}$ is a densely defined positive selfadjoint operator with the property

$$\hat{\pi}(b, s, a, \epsilon)\mathcal{J}\hat{\pi}(b, s, a, \epsilon)^{-1} = |a|^{1/2}\mathcal{J}.$$

By [21, Theorem 1], there exists $c > 0$ such that $\mathcal{I}^{\mathrm{hyp}} = c\mathcal{J}$ and we show that $c = 1$.

It is possible to prove that the admissible vectors for π are the functions $\psi \in L^2(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\xi_1, \xi_2)|^2}{|\xi_1^2 - \xi_2^2|} \mathrm{d}\xi_1 \mathrm{d}\xi_2 = 1. \tag{1.35}$$

The voice transform is then $(\mathcal{V}_\psi f)(g) = \langle f \pi(g)\psi \rangle$, and is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(G, \mathrm{d}\mu)$ provided that ψ satisfies the admissible condition (1.35). If $\mathcal{Q}\psi \in \operatorname{dom} \mathcal{I}^{\mathrm{hyp}}$, by equation (1.30), we have that

$$\begin{aligned}(\mathcal{V}_\psi f)(b, s, a, \epsilon) = &\int_{\mathbb{R}\times\mathbb{R}} \mathcal{R}^{\mathrm{hyp}} f(1, u, t) \overline{\Psi(\epsilon, u + s, \frac{t - w(u)\cdot b}{a})} \frac{\mathrm{d}u\mathrm{d}t}{|a|} \\ &+ \int_{\mathbb{R}\times\mathbb{R}} \mathcal{R}^{\mathrm{hyp}} f(-1, u, t) \overline{\Psi(-\epsilon, u + s, \frac{t - \Omega_{-1}w(u)\cdot b}{a})} \frac{\mathrm{d}u\mathrm{d}t}{|a|},\end{aligned} \tag{1.36}$$

for any $f \in \mathcal{A}$, where $\Psi = \mathcal{I}^{\mathrm{hyp}}\mathcal{Q}\psi$. Note that the coefficients depend on f only through its Radon transform and do not involve the operator $\mathcal{I}^{\mathrm{hyp}}$ as applied to the function. Hence, the inversion formula for the voice transform in Theorem 1.39 allows to reconstruct an unknown function $f \in \mathcal{A}^{\mathrm{hyp}}$ from its Radon transform by computing the coefficients $(\mathcal{V}_\psi f)(b, s, a, \epsilon)$ by means of (1.36). It is worth observing that the different contributions in (1.36) with $\eta = 1$ and $\eta = -1$ reconstruct the frequency projections of f onto the horizontal cone $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2/\xi_1| < 1\}$ and onto the vertical cone $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1/\xi_2| < 1\}$, respectively. Moreover, we choose $\Psi(\eta, u, t) = \Psi_{2,\eta}(u)\Psi_1(t)$ such that Ψ_1, $\Psi_{2,1}$, and $\Psi_{2,-1}$ are 1D-wavelets, that is admissible vectors for the square integrable representation of the affine group. We recall that the previous condition is equivalent to satisfy the Calederón condition (1.8). Then we obtain a formula for the voice transform which involves only integral transforms applied to the Radon transform of the function, precisely a 1D-wavelet transform introduced in Example 1.1, followed by a convolution and it reads

$$(\mathcal{V}_\psi f)(b, s, a, \epsilon) = \frac{1}{\sqrt{|a|}} \sum_{\eta=-1,1} \mathcal{W}_{\Psi_{2,\eta\epsilon}} \left[u \mapsto \mathcal{W}_{\Psi_1}(\mathcal{R}^{\mathrm{hyp}} f(\eta, u, \bullet))(a, w(u)\cdot b) \right] (1, s).$$

By substituting the value of the voice transform in (1.30), we obtain an the desired inversion formula.

Chapter 2

Radon transform on symmetric spaces

In this chapter we present the first extension to a different setting of Theorem 1.38: the hyperbolic disk and, more in general, every (noncompact) symmetric space. The idea to study these cases comes from the fact that a symmetric space, together with the family of horocycles, represents a canonical example of dual pair in the sense of Helgason. Roughly speaking, a symmetric space of the noncompact type is a homogeneous spaces G/K where G is a connected semisimple Lie group G with finite center, and K is a maximal compact subgroup of G. Horocycles Ξ are a family of subsets of X which play the role of hyperplanes in the Euclidean space. It is possible to see that $\Xi \simeq G/H$ is a homogeneous space for some $H \subseteq G$.

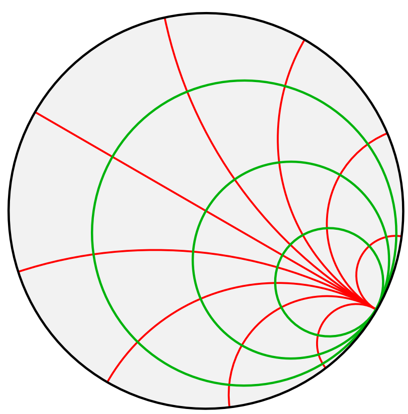

Figure 2.1: The hyperbolic disk is the complex open unit disk endowed with a Riemannian structure for which geodesics are diameters and portions of Euclidean circles orthogonal to the boundary. The red curves in the figure are part of the same bundle of parallel geodesics. The green circles are in the family of horocycles orthogonal to the given bundle.

The horocyclic Radon transform on symmetric spaces is considered by Helgason in several papers [33], [34], [36], [37]. His setting, however, does not fit all the assumptions

(A1)-(A7) of the previous chapter. Indeed the quasi regular representation π of G on $L^2(X)$ is not irreducible. For this reason, it is not possible to repeat the proof of Theorem 1.38 that makes use of the generalization of Schur's lemma due to Duflo and Moore, which requires the irridducibility of π, see [1] and [21]. We adopt a combination of the approach followed by Helgason in the context of symmetric spaces [37] and the techniques that have been developed in [6], in which the unitarization of the affine and polar Radon tranforms is proved without making use of the irridducibility of π.

We are well aware that the unitarization problem was already considered and essentially solved by Helgason in [37]. Precisely, he constructs a pseudo-differential operator Λ and he proves that the pre-composition with the horocyclic Radon transform yields an isometric operator, see Theorem 3.9 in Chap. II in [37]. Here, we prove that the composition $\Lambda\mathcal{R}$ can actually be extended to a unitary operator $\mathcal{Q}\colon L^2(X, \mathrm{d}x) \to L^2_\flat(\Xi, \mathrm{d}\xi)$, where $\mathrm{d}x$ and $\mathrm{d}\xi$ are the G-invariant measures and where $L^2_\flat(\Xi, \mathrm{d}\xi)$ is a closed subspace of $L^2(\Xi, \mathrm{d}\xi)$ which accounts for the Weyl symmetries. Furthermore, we show that $\mathcal{Q}$ intertwines the quasi regular representations π and $\hat{\pi}$.

Classically, a horocycle is parametrized by an element of the boundary K/M and an element of the Abelian subgroup A. Such parametrization is actually made w.r.t. the origin of the symmetric space. Part of our contribution is to introduce a different parametrization w.r.t. each reference point $x \in X$, namely

$$\Psi_x\colon K/M \times A \to \Xi\colon (kM, a) \mapsto \kappa_x(k)aN[x],$$

see (2.10) for details. The first consequences are a different expression of the range of the Helgason-Fourier transform (Theorem 2.17), the definition of $L^2_\flat(\Xi)$ and the inclusion in it of the range of the horocyclic Radon transform (Corollary 2.25). We introduce a Fourier multiplier $\mathcal{J}_o$ on the functions defined on $K/M \times A$. The operator Λ is obtained by "transporting" $\mathcal{J}_o$ on the functions defined on Ξ via Ψ_o. Namely, $\Lambda = \Psi_o^{*-1}\mathcal{J}_o\Psi_o^*$, where Ψ_o^* is the pull-back of Ψ_o. The key which permits us to prove the surjectivity of $\mathcal{Q}$ is Proposition 2.24 which gives us a relation between the symmetries satisfied by the Radon transform and by the Helgason-Fourier transform of a function. In this way, we can use an adaptation of Helgason-Fourier unitary extension theorem (Theorem 2.17) to prove the surjectivity of $\mathcal{Q}$. The symmetries that play a crucial role in the description of the two ranges are expressed w.r.t. every reference point. For this reason, we need to stress the dependence of every notion from the reference point and to keep track of it. Although Figure 2.5 contains a complete map of the operators that come into play, Figures 2.3 and 2.4 may be more understandable at first glance.

This chapter solves the problem in the context of the horocyclic Radon transform on symmetric spaces. Two naturally related problems are the horocyclic Radon transform on homogeneous trees, addressed and solved in Chapter 3, and the geodesic Radon transform. The latter is commonly called X-ray transform and has been introduced and inverted by Helgason on the hyperbolic space $\mathbb{H}^d$, see Theorem 3.12 in Chap.I in [37], and on symmetric spaces of the noncompact type by Rouvière [48]. Although it is not in general true that a horocycle has codimension one in the symmetric space, the horocyclic Radon transform can be seen as the analog of the Euclidean Radon transform on hyperplanes in $\mathbb{R}^d$, whereas the X-ray transform is the analog of the Radon on lines in $\mathbb{R}^d$.

The chapter is divided as follows. In the first section, we recall basic notions on symmetric spaces and we focus on these of the noncompact type. For these, the notion of boundary and horocycle is presented. In Section 2.2, we endow the manifolds we are interested in with measures and we introduce the Helgason-Fourier transform. The Radon transform is presented in the last section together with its unitarization.

2.1 Symmetric spaces

In this section we introduce the notion of symmetric space and, focusing on a specific class of symmetric spaces, we analyze the boundary and the family of horocycles, which can be though of as the analog of hyperplanes. A general reference for the whole section is [8] of which this represents a synthesis.

The first part of the section makes use of terminology of Sections 1.1 and 1.2. After a formal definition, we analyze the classification of symmetric spaces which divides them in three categories: of the Euclidean, compact and noncompact type. We are interested in the last type. The classical setup for a symmetric space is a G-transitive space where K is the isotropy subgroup of G at a fixed reference point $o \in X$. In this sense, fixing a maximal compact subgroup K corresponds to fixing a reference point in X. The natural reference for the material in this section is the celebrated monography [35] by Helgason, of which this is a synthesis. Other sources are for example [41], [53].

A prototypical example of noncompact symmetric space is the hyperbolic disk, it will be a reference for the whole chapter and, when it is possible, we use it to give a concrete idea of our theoretical approach. In Section 2.1.2 we present the notion of boundary of a noncompact symmetric space. In Section 2.1.3 we illustrate a very useful fact: the independence of every notion from the reference point we choose by fixing K. In fact, by considering a different reference point $x \in X$ and then a different maximal compact subgroup K_x we have a new Iwasawa decomposition of G which conserves the factors A and N, namely G factorizes in K_x, A, and N as in Theorem 1.30. This fact leads to express every notion w.r.t. each reference point. In the following this helps keep track of all the symmetries of the Helgason-Fourier and Radon transforms.

Finally, last part of the section is devoted to the family of horocycles. Horocycles are the natural counterpart of Euclidean hyperplanes in symmetric spaces, since they are sets of points orthogonal to a bundle of parallel geodesics. We see that, classically, a horocycle is parametrized by an element of the boundary and an element of A. By changing the reference point, it is possible to find several parametrizations through the same sets of parameters. The freedom to pass from a parametrization to an other plays a crucial role in the following.

In the whole chapter, we often consider different homogeneous spaces of the same group G. For clarity, we shall thus adopt notational variations to distinguish among different actions, such as $g[x]$ or $g.x$ or $g \cdot x$ or $g\langle x\rangle$ and so forth.

2.1.1 Types of symmetric spaces

We start by recalling very basic facts and notions on symmetric spaces. We introduce the definition of (Riemannian globally) symmetric space, followed by a list of examples.

We focus on the classical (noncompact) example of the hyperbolic disk, for which we describe the details. Other examples are presented in a more complete version in [8].

We present a classification of symmetric spaces which is based on properties of (the Lie algebra of) its group of isometries and reflects the curvature of the space. It consists in: Euclidean, compact and noncompact. In the rest of the chapter we focus only on those of the noncompact case.

Definition 2.1. The Riemannian manifold $\mathcal{M}$ is a Riemannian globally symmetric space if each $p \in \mathcal{M}$ is an isolated fixed point of an isometry σ_p of $\mathcal{M}$ that is involutive ($\sigma_p^2 = \mathrm{id}$).

Symmetric spaces are homogeneous spaces. Let $\mathcal{M}$ be a Riemannian globally symmetric space. We denote $I(\mathcal{M})$ the group of isometries of $\mathcal{M}$; $I(\mathcal{M})$ endowed with the compact-open topology[1] is a Lie group. Furthermore, if $G = I_0(\mathcal{M})$ is the connected component of the identity of $I(\mathcal{M})$ and $p_0 \in \mathcal{M}$, then the isotropy subgroup K of G at p_0 is compact and $\mathcal{M} \simeq G/K$.

Example 2.1 (The unit disk)**.** The unit disk is a canonical example which deserves a deeper analysis. It could be useful to keep in mind this example in order to have a concrete counterpart of all the theoretical structure we develop.

By hyperbolic disk, or unit disk, we mean the manifold $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the Riemannian metric given by the inner product

$$\langle u, v\rangle_z = \frac{(u, v)}{(1 - |z|^2)^2}$$

where $u, v \in T_z(\mathcal{M})$ are tangent vectors at $z \in \mathcal{M}$.

The group $G = \mathrm{SU}(1, 1)$ acts on $\mathcal{M}$ by means of the *Möbius action*, namely

$$g[z] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} [z] = \frac{az + b}{cz + d}, \qquad g \in G,\ z \in \mathbb{D}.$$

The Iwasawa subgroups obtained from the Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$ under the Lie group isomorphism introduced in (1.24) are explicitly given by

$$K = \left\{k_\theta = \begin{bmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{bmatrix} : \theta \in [0, 2\pi)\right\}, \qquad A = \left\{a_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} : t \in \mathbb{R}\right\} \tag{2.1}$$

$$N = \left\{n_s = \begin{bmatrix} 1 + is & -is \\ is & 1 - is \end{bmatrix} : s \in \mathbb{R}\right\}.$$

The action of G on $\mathbb{D}$ is transitive since for every $z \in \mathbb{D}$ it is sufficient to consider the matrix

$$g_z = \frac{1}{\sqrt{1 - |z|^2}} \begin{bmatrix} 1 & z \\ \overline{z} & 1 \end{bmatrix} \in \mathrm{SU}(1, 1)$$

which maps o in z via the Möbius action. Of course the isotropy of G at $o \in \mathcal{M}$ is K and $\mathcal{M} \simeq G/K$. The group $\mathrm{SU}(1, 1)$ actually is the connected component of the

[1] The compact-open topology on $I(\mathcal{M})$ is the smallest topology in which all the sets $\{g \in I(\mathcal{M}) : g(C) \subset U\}$ are open, where C varies in the compacta of $\mathcal{M}$ and U in the open sets. Its structure is not important for us.

identity of $I(\mathbb{D})$. There are other isometries which do not arise from $\mathrm{SU}(1,1)$, as for example $g\phi g^{-1}$ where $g \in \mathrm{SU}(1,1)$ and $\phi(z) = \overline{z}$.

As for the isometric involutions, consider first the Möbius action induced by

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

which is the map $z \mapsto -z$. This fixes only o and is thus a global involution of which o is an isolated fixed point. A global involution fixing only the point $z_0 \in \mathbb{D}$ is given by the Möbius action of the $\mathrm{SU}(1,1)$ element $g_{z_0} J g_{z_0}^{-1}$, where g_{z_0} is defined by (**??**). The isometric property of such maps follows from the fact that Möbius actions of elements of G are conformal maps and $z \mapsto \overline{z}$ is an isometry, too.

Example 2.2. Below is a list of classical examples (see [8] for further examples):

- the Euclidean space $\mathbb{R}^n$;
- the Euclidean sphere $S^{n-1} \subseteq \mathbb{R}^n$;
- the upper half plane in $\mathbb{C}$, that is $\mathcal{H}_+ = \{x + iy \colon y > 0\}$. This manifold has a structure similar to the hyperbolic disk. It is in fact another model of the 2-dimensional hyperbolic space. The relation is given by the fact that $\mathcal{H}_+ \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and by the Lie group isomorphism (1.24) between $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(1,1)$.
- the space of positive definite symmetric d-dimensional matrices $\mathrm{SP}(d,\mathbb{R})$, which is a generalization of the upper half plane in higher dimension. Indeed, it is a homogeneous space of $\mathrm{SL}(d,\mathbb{R})$. It is important to observe that for a general $d > 2$ there are no isometries between $\mathbb{H}^d = \mathrm{SO}(d,1)/\mathrm{SO}(d)$ and $\mathrm{SP}(d,\mathbb{R})$, because the former has constant curvature while the latter has not. Thus, this cannot be interpreted as a generalization of the hyperbolic disk.

A crucial point in the theory of symmetric spaces is the following. Let $\mathcal{M}$ be a symmetric space, $\mathfrak{g}$ be the Lie algebra of $I(\mathcal{M})_0$ and $s = \mathrm{d}\sigma_e$, then by Theorem 3.3, Chap. IV [35]:

(i) $\mathfrak{g}$ is a real Lie algebra;

(ii) s is an involutive automorphism of $\mathfrak{g}$;

(iii) the fixed points $\mathfrak{k}$ of s form a Lie algebra compactly contained in $\mathfrak{g}$,

where (iii) holds because K is compact (see Section 1.2.1.1 for the definition of compact containment of a Lie algebra). A pair $(\mathfrak{g}, s)$ satisfying (i), (ii), and (iii) above is called an *orthogonal symmetric Lie algebra*. Denote $\mathfrak{z}$ by the center of $\mathfrak{g}$, if in addition $(\mathfrak{g}, s)$ is such that

(iv) $\mathfrak{k} \cap \mathfrak{z} = \{0\}$,

then $(\mathfrak{g}, s)$ is called *effective*. Fix such a pair and consider the decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{e}$ into the $+1$ and -1 eigenspaces with respect to s. Motivated by the important decomposition result stated below in Theorem 2.2, one introduces the following terminology:

(a) if $\mathfrak{g}$ is compact and semisimple, then $(\mathfrak{g}, s)$ is said to be of the *compact type*;

(b) if $\mathfrak{g}$ is noncompact and semisimple and if $\mathfrak{g} = \mathfrak{u} + \mathfrak{e}$ is a Cartan decomposition, then $(\mathfrak{g}, s)$ is said to be of the *noncompact type*;

(c) if $\mathfrak{e}$ is an Abelian ideal in $\mathfrak{g}$, then $(\mathfrak{g}, s)$ is said to be of the *Euclidean type*.

Theorem 2.2 (Theorem 1.1, Chap. V, [35]). *Suppose that $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra. Then there exist ideals $\mathfrak{g}_0$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ such that*

(i) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_- + \mathfrak{g}_+$, a Lie algebra direct sum;

(ii) $\mathfrak{g}_0$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are invariant under s and orthogonal with respect to the Killing form;

(iii) the pairs $(\mathfrak{g}_0, s_0)$, $(\mathfrak{g}_+, s_+)$ and $(\mathfrak{g}_-, s_-)$ are effective orthogonal symmetric Lie algebras of the Euclidean, compact and noncompact type, respectively.

The involutions s_0, s_+ and s_- are those that arise by restricting s to the corresponding ideals. The above result is of course of central importance because it allows to study separately the various cases. Clearly, the decomposition yields a corresponding decomposition of a symmetric space based on topological properties of (the connected component of) its isometries Lie group and on algebraic properties of the respective Lie algebra. The previous decomposition induces the notions of symmetric space of Euclidean, compact and noncompact types. The Euclidean space, the sphere and the unit disk are the prototypical examples of such spaces. There is a remarkable duality between compact and noncompact types in which we are not interested. We content ourselves with mentioning that the compact types have positive sectional curvature and the noncompact ones have negative sectional curvature.

Since we are only interested in noncompact globlally symmetric spaces, we focus on the corresponding structural assumptions. To this end, we need yet another piece of terminology and we also slightly change the current notation to tune into the noncompact case. Any pair (G, K) where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and where K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$ is said to be associated to the (effective) orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$, and will be called of the noncompact type if such is $(\mathfrak{g}, \theta)$. Thus, from now on we fix an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ of the noncompact type, so that the eigenspace decomposition relative to θ, namely $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, is a Cartan decomposition. The next result is a cornerstone in the theory.

Theorem 2.3 (Theorem 1.1, Chap. VI, [35]). *With the notation above, suppose that (G, K) is any pair associated with the effective orthogonal symmetric Lie algebra of the noncompact type $(\mathfrak{g}, \theta)$. Then:*

(i) K is connected, closed and contains the center Z of G. Moreover, K is compact if and only if Z is finite. In this case, K is a maximal compact subgroup of G;

(ii) there exists an involutive analytic automorphism Θ of G whose fixed point set is K and whose differential at the identity $e \in G$ is θ; the pair (G,K) is a Riemannian symmetric pair[2] *and thus G/K is a globally symmetric space;*

(iii) the mapping $\varphi\colon (X,k) \mapsto (\exp X)k$ is a diffeomorphism of $\mathfrak{p} \times K$ onto G and the mapping Exp *is a diffeomorphism*[3] *of $\mathfrak{p}$ onto the globally symmetric space G/K.*

Assumption. From now on, let G be a connected semisimple Lie group with finite center and $X = G/K$ the associated symmetric space of the noncompact type, where K is a maximal compact subgroup of G. We also fix an Iwasawa decomposition $G = KAN$ and we denote by M the centralizer of A in K.

Such Assumption is satisfied by the hyperbolic disk $\mathbb{D}$ we have seen in Example 2.1 because $\mathrm{SU}(1,1)$ is semisimple and noncompact.

2.1.2 Boundary of a symmetric space

We recall again that our basic example of noncompact symmetric space is the unit disk $\mathbb{D}$, which has a rather obvious (topological) boundary, namely the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The notion of boundary of a symmetric space is highly nontrivial. For a deep study on the matter, we refer to the classical paper of Furstenberg [27] in which a detailed motivation of Definition 2.4 below may be found. For our purposes, some heuristics and some basic observations will suffice.

Notice first that $\mathbb{D}$ and S^1 are orbits of the Möbius action of $G = \mathrm{SU}(1,1)$ on $\mathbb{C}$. We already know that $\mathbb{D}$ is an orbit. Further, AN fixes 1 (easy to check) and K moves it along the unit circle, via

$$k_{\theta/2} \cdot 1 = \begin{bmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{bmatrix} \cdot 1 = e^{i\theta}, \tag{2.2}$$

so that the G-orbit of 1 is S^1. The formula (2.2) shows also that the elements $k_{\theta/2}$ when θ is any multiple of 2π fix 1. These are $\pm I$, namely the elements of M, the centralizer of A in K. Therefore, the stabilizer of 1 is the group $P = MAN$ and $S^1 \simeq G/P$. By means of the Iwasawa decomposition we may write

$$S^1 \simeq KAN/MAN$$

and the natural question arises whether this is the same as K/M or not. In the case at hand this is quite clearly so because K acts transitively with isotropy M. This actually holds more generally in the sense that

$$G/P = KAN/MAN \simeq K/M.$$

[2]The pair (G,H), with H closed subgroup of the connected Lie group G, is called a symmetric pair if there exists an involutive analytic automorphism σ of G such that $(\mathrm{Fix}(\sigma))_0 \subseteq H \subseteq \mathrm{Fix}(\sigma)$, where $\mathrm{Fix}(\sigma)$ is the set of elements fixed by σ. If in addition the group $\mathrm{Ad}_G(H)$ is compact, then (G,H) is called a Riemannian symmetric pair, see [35]. Proposition 3.4, Chap. IV in [35] states that then G/K is a globally symmetric space.

[3]The exponential mapping Exp, quoted for completeness, is just the Riemannian exponential mapping (see for instance [35]) and will play no explicit role in what follows.

Indeed, K acts on the coset space G/P in the natural fashion $k \cdot gP = (kg)P$ and by the Iwasawa decomposition $k \in P = MAN$ if and only if $k \in M$. Hence the isotropy at the coset $\{P\}$ is M. Further, again by the Iwasawa decomposition, the action is transitive, and we conclude that $G/P \simeq K/M$. The reverse point of view (that of G acting on K/M with isotropy P) will be illustrated below in (2.7), where the explicit action of G on K/M is given.

Definition 2.4. The *boundary* of X is the coset space $B := K/M$.

We remark here *en passant* that M, which will play an important role below, normalizes N, that is

$$mNm^{-1} = N, \qquad m \in M. \tag{2.3}$$

To see this, look at the Lie algebra level. If α is a positive root and $X \in \mathfrak{g}_\alpha$, then for every $H \in \mathfrak{a}$ it is

$$[H, \mathrm{Ad}mX] = \mathrm{Ad}m[\mathrm{Ad}m^{-1}H, X] = \mathrm{Ad}m[H, X] = \alpha(H)\mathrm{Ad}mX,$$

so that $\mathrm{Ad}m(\mathfrak{g}_\alpha) \subset \mathfrak{g}_\alpha$.

An other normalization property that involves N is that for any $\alpha \in A$ and any $\nu \in N$ it holds

$$\alpha\nu aN = aN\alpha\nu. \tag{2.4}$$

This, in turn, follows from choosing $\nu' \in N$ such that $\nu'\alpha = \alpha\nu$, which gives

$$\alpha\nu aN = \alpha aa^{-1}\nu aN = \alpha aN\alpha^{-1}\alpha = a\alpha N\alpha^{-1}\alpha = aN\alpha = aN\nu'\alpha = aN\alpha\nu.$$

2.1.3 Changing the reference point

In what follows, it will be useful to change the reference point of both the symmetric space X and its boundary. Although conceptually very well known and somehow trivial, the actual explicit determination of what happens when doing so is not to be found in the literature, to the best of our knowledge. In order to see how the various decompositions are affected by changing the origin of our spaces, it is convenient to introduce Borel sections and occasionaly adopt a slightly different notation for the (various) G-actions.

The action of G on $X = G/K$ will be written $g[x]$, namely

$$g[x] = g[hK] = ghK.$$

For any fixed $x_0 \in X = G/K$, we denote by $s_{x_0} \colon X \to G$ the Borel section relative to x_0. Such Borel section always exists since G is second countable, see Section 1.1.4.

We next show how, in the present context, a Borel section associated to $o = eK \in G/K$ can be determined quite explicitly. Since K is the isotropy subgroup of G at o, the map $\beta \colon gK \mapsto g[o]$ is a diffeomorphism of G/K onto X. Furthermore, by the Iwasawa decomposition of G (Theorem 1.30), each element of $g \in G$ can be written as the product $g = nak$ for exactly one triple $(n, a, k) \in N \times A \times K$, and the correspondence $(n, a, k) \leftrightarrow nak$ is a diffeomorphism with G. Hence each class in G/K has a representative of the form naK with unique $a \in A$ and $n \in N$, so that the mapping $\psi \colon G/K \to NA$ given by $naK \mapsto na$ is a diffeomorphism. It follows that the measurable, actually smooth, map

$$\psi \circ \beta^{-1} \colon X \longrightarrow NA$$

is a Borel section. Indeed, $\psi \circ \beta^{-1}(o) = \psi(K) = e$ and, by construction, for every $x \in X$, it holds $\psi \circ \beta^{-1}(x)[o] = x$. From now on, we will denote by s_o the Borel section $\psi \circ \beta^{-1}$ with image $NA \subseteq G$.

Fix now $x \in X$ and let K_x be the isotropy of G at $x \in X$. Evidently,

$$K_x = s_o(x) K s_o(x)^{-1}.$$

It is then possible to write an Iwasawa decomposition w.r.t. the subgroup K_x. In fact,

$$G = s_o(x) G s_o(x)^{-1} = s_o(x) KAN s_o(x)^{-1} = s_o(x) K s_o(x)^{-1} AN = K_x AN,$$

because, as observed earlier, $s_o(x) \in AN$. By using the same approach, one obtains the various versions of the Iwasawa decomposition where the factors appear in a different order. It is worth observing that the subgroups A and N are independent of the maximal compact subgroup K_x, but the individual factors appearing in the decomposition of a fixed element $g \in G$ are not. Given $g \in G$, we extend the notation introduced in (1.18) and (1.19) by denoting with $H_x(g)$, $A_x(g)$ the elements of $\mathfrak{a}$ uniquely determined by

$$g \in K_x \exp H_x(g) N, \qquad g \in N \exp A_x(g) K_x, \tag{2.5}$$

and furthermore we denote by $\kappa_x(g)$ the unique element in K_x such that $g \in \kappa_x(g) AN$. Clearly,

$$A_x(g^{-1}) = -H_x(g). \tag{2.6}$$

Once the point $x \in X$ has been fixed, a Borel section $s_x \colon X \to G$ can also be fixed, so that for every $y \in X$, $s_x(y)[x] = y$ and $s_x(x) = e$. As before, it may be arranged that $s_x(y) \in NA = AN$. Also, we denote by M_x the centralizer of A in K_x, so that $M_x = s_o(x) M s_o(x)^{-1}$. The following technical observation will be useful below.

Lemma 2.5. *For any x, $y \in X$ we have*

$$\kappa_y \circ \kappa_x \big|_{K_y} = id_{K_y}.$$

In particular, if $k_x = \kappa_x(k_y)$ for some $k_y \in K_y$, then $k_y = \kappa_y(k_x)$.

Proof. Let $k_y \in K_y$. Then according to the Iwasawa decomposition $K_x AN$ it is $k_y = \kappa_x(k_y) an$, that is $\kappa_x(k_y) = k_y (an)^{-1} \in K_y AN$. So that $\kappa_y(\kappa_x(k_y))$ is precisely k_y, as desired. □

The action of G on the boundary $B = K/M$ is induced by the decomposition $G/P = KAN/MAN$ in the sense that if $g \in G$ and $kM \in B$ then

$$g \langle kM \rangle := \kappa_o(gk) M. \tag{2.7}$$

Consider now the action of K_x. From the definition (2.7) and by Lemma 2.5, with $y = o$, for any $k \in K$ it is

$$\kappa_x(k)\langle M\rangle = \kappa_o(\kappa_x(k))M = kM.$$

Thus the action of K_x on the boundary is transitive. Next, observe that an element $k_x = s_o(x)ks_o(x)^{-1}$ stabilizes $M \in K/M$ if and only if $\kappa_o(s_o(x)ks_o(x)^{-1}) \in M$, which means $s_o(x)k \in MAN$. This, together with the fact that M normalizes AN, implies that $k \in M$, hence $k_x \in M_x$. Therefore the isotropy group of K_x at M is M_x. This shows that the map induced by κ_o on K_x/M_x, which we denote $\kappa_{x,o}$, namely

$$\kappa_{x,o} : K_x/M_x \to K/M, \qquad k_xM_x \mapsto \kappa_{x,o}(k_xM_x) := \kappa_o(k_x)M, \tag{2.8}$$

is a diffeomorphism. Furthermore, kM and $\kappa_x(k)M_x$ determine the same boundary point, because by (2.8) $\kappa_o(\kappa_x(k))M = kM$. By Lemma 2.5 the inverse of $\kappa_{x,o}$ is the map

$$\kappa_{o,x} : K/M \to K_x/M_x, \qquad kM \mapsto \kappa_{o,x}(kM) := \kappa_x(k)M_x.$$

2.1.4 Horocycles

A hyperplane in $\mathbb{R}^n$ is orthogonal to a family of parallel lines. What is a reasonable analog of this in, say, Riemannian geometry? Since geodesics are very natural generalizations of lines, a possible answer is given by a manifold that is orthogonal to families of parallel geodesics. In the context of symmetric spaces, such manifolds are called *horocycles*, sometimes also *horospheres*.

Let us see what this idea leads to in the context of the unit disk, our basic example of noncompact symmetric space. The origin in $\mathbb{D}$ will be denoted o. If $\gamma\colon [a,b] \to \mathbb{D}$ is a smooth curve with $\gamma(a) = o$ and $\gamma(b) = x \in (-1,1)$ is a point on the real axis, then the simple inequality

$$\frac{\dot{x}(t)^2}{(1-x(t)^2)^2} \leqslant \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{(1-x(t)^2-y(t)^2)^2}$$

shows that straight real lines through the origin are geodesics. We observe *en passant* that since $\gamma_0(t) = (tx, 0)$ with $t \in [0,1]$ is such a straight line, then[4]

$$\begin{aligned} d(o,x) = L(\gamma_0) &= \int_0^1 \sqrt{\mathcal{G}_{\gamma_0(t)}(\dot{\gamma}_0(t), \dot{\gamma}_0(t))}\mathrm{d}t = \int_0^1 \sqrt{\frac{4(|x|,|x|)}{(1-|\gamma_0(t)|^2)^2}}\mathrm{d}t \\ &= \int_0^1 \frac{2|x|}{1-t^2|x|}\,\mathrm{d}t = \log\frac{1+|x|}{1-|x|}. \end{aligned}$$

[4]The definition of the length of a geodesic γ is defined by

$$L(\gamma) = \int_0^1 \sqrt{\mathcal{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\mathrm{d}t,$$

where $\mathcal{G}$ is the metric tensor. In the case of the disk, the metric tensor has the form

$$\mathcal{G}_z = \frac{4}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}.$$

As we know, $G = \mathrm{SU}(1,1)$ acts by isometries via the Möbius action on $\mathbb{D}$. Such maps are conformal and map circles and lines into circles and lines. Hence the geodesics in $\mathbb{D}$ are circular arcs perpendicular to the boundary $|z| = 1$. All circular arcs perpendicular to the same point at the boundary may be seen as parallel lines and thus a natural notion of horocycle in this context is that of Euclidean circle tangent to the boundary (except the point on S^1) because such a circle is of course perpendicular to all the above parallel geodesics.

The circle through the origin and tangent to the boundary at $1 \in \mathbb{C}$ is therefore the prototype of horocycle. Observe that for $n_s \in N$ as in (2.1)

$$n_s[o] = \begin{bmatrix} 1+is & -is \\ is & 1-is \end{bmatrix}[o] = \frac{-is}{1-is} = \frac{s}{s+i} = \frac{s^2}{s^2+1} - i\frac{s}{s^2+1}$$

and an easy calculation shows that these are precisely the points on the circle of radius 1/4 centered at $1/2 \in \mathbb{C}$ that are contained in $\mathbb{D}$. Furthermore, as $s \to \pm\infty$ one gets the boundary point $b_0 = 1 \in \mathbb{C}$. We have obtained the basic horocycle, which will be denoted ξ_o, as the N-orbit $N[o]$.

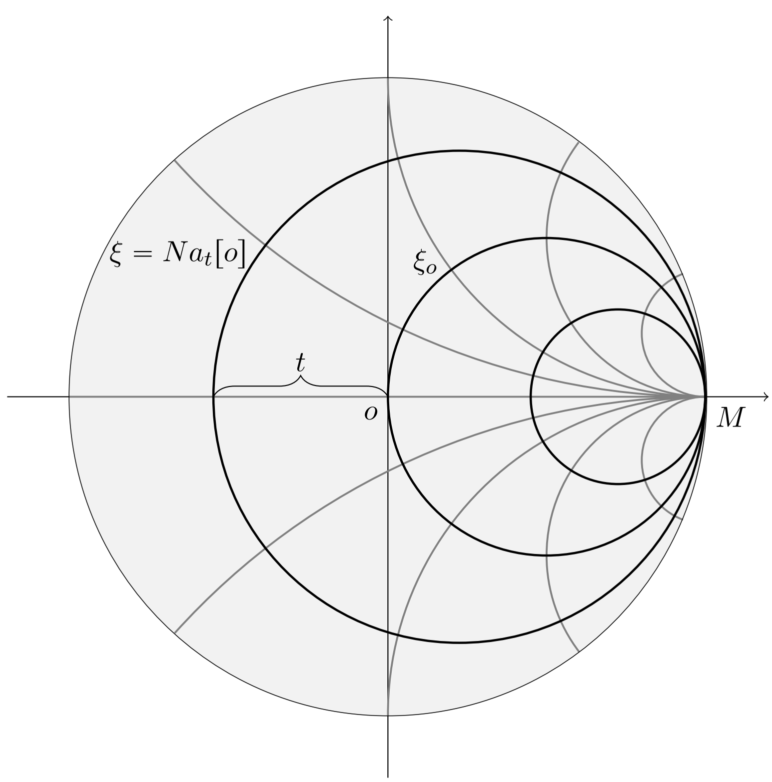

Figure 2.2: The basic horocycle ξ_o in the unit disk and the horocycle ξ tangent to the boundary at 1 and with distance $-t$ from the origin o. In gray, the sheaf of parallel geodesics perpendicular to ξ_o and ξ.

Other horocycles tangent to b_0 are the orbits $Na_t[o] = a_tN[o]$ where of course

$$a_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

is any member of A (recall that A normalizes N). This is because

$$a_t[o] = \tanh t \in (-1, 1)$$

parametrizes any other point on the geodesic line $(-1, 1) \subset \mathbb{C}$ and an easy calculation shows that its N-orbit is just the circle through that point and tangent to b_0 (see Fig. 2.2). It is clear that by acting with the rotation group one gets all other horocycles, that is, all the Euclidean circles in $\mathbb{D}$ tangent to the boundary. Thus, any other horocycle ξ can be written in the form $ka \cdot \xi_0$ with $k \in K$ and $a \in A$. But this means

$$\xi = (ka)N(ka)^{-1}(ka[o]),$$

which exhibits ξ as an orbit of a group conjugate to N, namely $(ka)N(ka)^{-1}$. This motivates the Definition 2.6 below.

Definition 2.6 (§1, Chap.II in [37])**.** A *horocycle* in X is any orbit of any subgroup of G conjugate to N, that is an orbit $N^g[x]$ where $x \in X$, $g \in G$ and $N^g = gNg^{-1}$. We shall denote by Ξ the set of all horocycles in X.

By Theorem 1.1 in Chap.II in [37], horocycles are closed submanifolds of X. We need a more manageable parametrization of horocycles. Observe that horocycles form a homogeneous space of the group G. Indeed, the G-action on X maps horocycles to horocycles and in fact the group G acts transitively on Ξ by

$$(g, N^h[x]) \mapsto g.(N^h[x]) := gN^h[x] = N^{gh}[g[x]] \in \Xi.$$

We fix $x \in X$ and we consider the horocycle $\xi = N[x]$. By Theorem 1.1 in Chap.II in [37], the isotropy at ξ is M_xN and therefore

$$\Xi \simeq G/M_xN$$

under the diffeomorphism $gM_xN \mapsto gN[x]$. Furthermore, by Proposition 1.4 in Chap.II in [37], $(K_x/M_x) \times A$ is diffeomorphic to G/M_xN under the mapping

$$(k_xM_x, a) \mapsto k_xaM_xN.$$

Therefore, for each horocycle $\xi \in \Xi$ there exist unique $k_xM_x \in K_x/M_x$ and $a \in A$ such that

$$\xi = k_xaN[x]. \tag{2.9}$$

Observe that (2.9) gives us a different parametrization of Ξ for every point $x \in X$. This is a crucial point for us because it lets us easily change the reference point as we have seen above. We need an extra step, since until now every parametrization is made by different parameters: in fact we are still using a parametrization of the boundary,

K_x/M_x, which depends on the reference point x. Then, since K/M is diffeomorphic to K_x/M_x under the mapping $\kappa_{o,x}(kM) = \kappa_x(k)M_x$, we define the diffeomorphism

$$\Psi_x \colon K/M \times A \longrightarrow \Xi, \quad (kM, a) \mapsto \kappa_x(k)aN[x]. \tag{2.10}$$

The mappings Ψ_x play a crucial role in our work, actually they are our way to express every horocycle by choosing a point on the boundary and an element in A w.r.t. a given reference vertex $x \in X$. There is a geometric meaning for this parametrization. Looking at Figure 2.2, it is straightforward to note that a boundary element b fixes a family of horocycles; namely, all those that are tangent to the boundary at b. This family is what in differential geometry is called a foliation of the manifold X. The parameter $a \in A$ allows us to move in the foliation and to select a horocycle. The meaning of a is a slightly more complicated than that of b. In the case of the disk, it can be thought of as a "signed" distance from the origin, that is $t \in \mathbb{R}$ such that $a = a_t$ is null when the horocycle passes through the reference point x (in the figure $x = o$), positive when x is "external" to the circle, and negative when x is internal. Clearly, this intuition cannot be extended to higher rank; that is, to the case in which $A \simeq \mathbb{R}^d$ with $d > 1$; in such cases the best interpretation is to consider the foliation determined by a boundary point as a space isomorphic to $\mathbb{R}^d$.

From the previous considerations, it is clear that the element in A must depend on the choice of the reference vertex, while the boundary element should not. And, in a certain sense, this holds true. Indeed the boundary point $kM \in K/M$ which identifies the horocycle $\xi = \kappa_x(k)aN[x]$ through (2.10) is independent of the choice of the reference point $x \in X$. Namely, for every $x, y \in X$

$$\Psi_x(kM, a) = \Psi_y(kM, a')$$

for some $a' \in A$. Indeed, if $\xi = k_x aN[x]$ and if we pick $y \in X$, hence $k_y M_y \in K_y/M_y$ and $a' \in A$ such that $\xi = k_y a' N[y]$, then $k_y M_y = \kappa_y(k_x)M_y$ and this identifies the boundary point $\kappa_o(k_x)M$. Indeed, by the K_yAN- and KAN-Iwasawa decompositions of k_x, we have that

$$\kappa_y(k_x) \in k_x AN = \kappa_o(k_x)AN,$$

so that

$$\kappa_{y,o}(\kappa_y(k_x)M_y) = \kappa_o(\kappa_y(k_x))M = \kappa_o(k_x)M.$$

We shall say that $\Psi_x(kM, a)$ represents the horocycle with *normal* kM and *composite distance* $\log a$ from x (see below, Definition 2.8). We stress that the normal of a horocycle is independent of the choice of $x \in X$. The composite distance, however, is different for different reference points.

This parametrization generalizes the geometric picture in $\mathbb{D}$, where a horocycle $\xi = ka_t N[o]$ is identified by the boundary point $kM \in K/M$ to which it is tangent and the "signed distance" t from the reference point, see Fig. 2.2.

Now we present a result which relates the definition of horocycle with the parametrization we prefer to use.

Proposition 2.7. *Fix a reference point $x \in X$. The horocycle through $y \in X$ with normal kM is $N^{\kappa_x(k)}[y]$.*

Proof. An equivalent statement is that, writing $k = \kappa_o(k_x)$ with $k_x \in K_x$, the horocycle through y with normal $\kappa_o(k_x)M$ is $k_x N k_x^{-1}[y]$ because $k_x = \kappa_x(k)$ by Lemma 2.5. Since $k = \kappa_o(k_x)$, then kM and $k_x M_x$ identify the same boundary point and a horocycle with normal kM has the form $\xi = k_x a M_x N$ as in (2.9). If this represents a horocycle through y, then there exists $g \in G$ such that

$$\xi = gNg^{-1}[y] = \kappa_x(g)N\kappa_x(g)^{-1}[y].$$

Now observe that there exist $\alpha \in A$ and $\nu \in N$ such that $\kappa_x(g)^{-1}[y] = \nu\alpha[x]$, then $\xi = \kappa_x(g)\alpha N[x]$. Thus, since $\xi = k_x a N[x]$, we have that

$$\kappa_x(g)\alpha N[x] = k_x a N[x],$$

which by (2.9) implies $\kappa_x(g)M_x = k_x M_x$. Hence $\kappa_x(g) = k_x m_x$ for some $m_x \in M_x$. However, (2.3) implies at once that $m_x N m_x^{-1} = N$, and hence $N^{\kappa_x(g)} = N^{k_x}$. □

The previous result allows to outline a horocycle starting from a boundary point and an element of the symmetric space. The next determines a notation we use in the following. We prefer to use b for a boundary element whenever it is not important the boundary parametrization we adopt.

Definition 2.8. Fix a reference point $x \in X$ and choose $y \in X$ and $b \in K/M$, so that by Proposition 2.7 the horocycle $\xi = \xi(y, b)$ passing through y with normal $b = kM$ is uniquely determined, and hence there exists a unique $a \in A$ such that

$$\xi(y, kM) = \kappa_x(k)aN[x].$$

We denote by $A_x(y, b) \in \mathfrak{a}$ the *composite distance* of the horocycle $\xi(y, b)$ from $x \in X$, namely

$$A_x(y, b) = \log a,$$

The function A_x introduced above is central in our work; it is a generalization of the function $A \colon X \times K/M \to A$ considered by Helgason in §3, Chap. II [37]. Actually, the two functions coincide when $x = o$ and A_x is a generalization which keeps track of the reference vertex. The reader is warned not to confuse the composite distance $A_x(y, b)$, which depends on $(y, b) \in X \times B$, with the Abelian component $A_x(g)$ of g in the NAK_x-Iwasawa decomposition, which is a function on G (see (2.5)). A relation between the two does exist, as pointed out in the next lemma, where we collect several properties of the composite distance which will play a crucial role in our work.

Lemma 2.9. *Fix a reference point $x \in X$. Then:*

(i) for any $k_x \in K_x$ and $g \in G$ we have

$$A_x(g[x]\,, \kappa_o(k_x)M) = A_x(k_x^{-1}g), \tag{2.11}$$

where the right-hand side is defined by (2.5)*;*

(ii) for any $y \in X$, $kM \in K/M$ and $g \in G$ we have

$$A_x(y, kM) = A_{g[x]}(g[y]\,, g\langle kM\rangle); \tag{2.12}$$

(iii) for any $y, z \in X$ and $kM \in K/M$ we have

$$A_x(y, kM) = A_x(z, kM) + A_z(y, kM). \tag{2.13}$$

Proof. We start by proving (i). Let $k_x \in K_x$ and $g \in G$. By Proposition 2.7 and (ii) of Lemma 2.5, the horocycle passing through $g[x]$ with normal $\kappa_o(k_x)M$ is $k_x N k_x^{-1} g[x]$. By Definition 2.8, we have that

$$k_x N k_x^{-1} g[x] = k_x \exp(A_x(g[x], \kappa_o(k_x)M))N[x],$$

and so $k_x^{-1} g \in N \exp(A_x(g[x], \kappa_o(k_x)M))K_x$. This proves (i).

Now we want to show (2.12). For simplicity, we first prove the statement in the case $x = o$. Let $y \in X$, $kM \in K/M$ and $g \in G$. By Proposition 2.7, and the fact that A normalizes N, the horocycle passing through $g[y]$ with normal $g\langle kM\rangle = \kappa_o(gk)M$ (see (2.7)) is

$$N^{\kappa_o(gk)} g[y] = \kappa_o(gk) N \kappa_o(gk)^{-1} g[y] = gkN(gk)^{-1} g[y].$$

By the diffeomorphism given in (2.9), there exist $h \in K_{g[o]}$ and $a \in A$ such that

$$gkNk^{-1}[y] = haNg[o], \tag{2.14}$$

and thus, by definition

$$a = \exp(A_{g[o]}(g[y]\,, g\langle kM\rangle)).$$

We need to show that $a = \exp(A_o(y, kM))$. Since $K_{g[o]} = gKg^{-1}$, we have $h = gk_1g^{-1}$ for some $k_1 \in K$ and we claim that

$$k_1 \kappa_o(g^{-1})M = kM. \tag{2.15}$$

By (2.14) we have that

$$k_1 g^{-1} aN s_o(g[o])[o] = k_1 g^{-1} aNg[o] = kNk^{-1}[y] = kNs_o(k^{-1}[y])[o].$$

Since s_o takes values in AN and writing the NAK decomposition of g^{-1}, there exist $a', a'' \in A$ such that

$$k_1 \kappa_o(g^{-1}) a' N[o] = ka'' N[o].$$

Hence, by (2.9) we have that $k_1 \kappa_o(g^{-1})M = kM$, that is the claim (2.15). Therefore, for some $m \in M$ the right-hand side of (2.14) is

$$\begin{aligned} haNg[o] &= gkm\kappa_o(g^{-1})^{-1} g^{-1} aNg[o] \\ &= gkmaN\left(\kappa_o(g^{-1})^{-1} g^{-1}\right) g[o] \\ &= gkmaN\kappa_o(g^{-1})^{-1}[o] \\ &= gkmaN[o] = gkaN[o] \end{aligned}$$

where in the second line we have used that $\kappa_o(g^{-1})^{-1} g^{-1} \in AN$ and then (2.4). Summarizing, we have shown that

$$gkNk^{-1} s_o(y)[o] = gkaN[o].$$

By taking $e \in N$ on the left, there must be $n \in N$ such that $s_o(y)[o] = kan[o]$, so that $(kan)^{-1}s_o(y) \in K$, whence $k^{-1}s_o(y) \in Kan$. This shows that

$$a = \exp(A_o(k^{-1}s_o(y))) = \exp(A_o(y, kM)),$$

where the second equality follows by item (i). This concludes (ii) in the case $x = o$. The general case follows by the latter. Indeed, by applying it with $s_o(x)$ and $gs_o(x)$, respectively in the first and the second equality, we obtain

$$A_x(y, kM) = A_o(s_o(x)^{-1}[y], s_o(x)^{-1}\langle kM\rangle) = A_{g[x]}(g[y], g\langle kM\rangle) .$$

This proves (ii).

It remains to prove (iii). For simplicity we start by proving it for $x = o$, the general case follows. Let $y, z \in X$ and $kM \in K/M$. By the definition of s_z, we have that $s_z(o)^{-1} = s_o(z)$ and $K = s_z(o)K_z s_z(o)^{-1}$. Observe that, by the K_zAN Iwasawa decomposition of k

$$s_z(o)k \in s_z(o)\kappa_z(k)AN = s_z(o)\kappa_z(k)s_z(o)^{-1}AN,$$

and then

$$\kappa_o(s_z(o)k) = s_z(o)\kappa_z(k)s_z(o)^{-1}.$$

Furthermore, $s_y(o)k \in K \exp(H_o(s_y(o)k))N$, so that

$$s_z(o)kk^{-1}s_y(o)^{-1} \in s_z(o)\kappa_z(k)s_z(o)^{-1}N \exp(H_o(s_z(o)k) - H_o(s_y(o)k))K. \qquad (2.16)$$

Now, observe that by (2.6) and (i) it is possible to rewrite

$$\begin{aligned} H_o(s_z(o)k) - H_o(s_y(o)k) &= A_o(k^{-1}s_y(o)^{-1}) - A_o(k^{-1}s_z(o)^{-1}) \\ &= A_o(s_y(o)^{-1}[o], kM) - A_o(s_z(o)^{-1}[o], kM) \\ &= A_o(y, kM) - A_o(z, kM). \end{aligned}$$

Hence, (2.16) becomes

$$s_z(o)s_y(o)^{-1} \in s_z(o)\kappa_z(k)s_z(o)^{-1}N \exp(A_o(y, kM) - A_o(z, kM))K ,$$

and by conjugating by $s_z(o)^{-1} \in AN$

$$\begin{aligned} s_y(o)^{-1}s_z(o) &\in \kappa_z(k)s_z(o)^{-1}N \exp(A_o(y, kM) - A_o(z, kM))Ks_z(o) \\ &= \kappa_z(k)N \exp(A_o(y, kM) - A_o(z, kM))s_z(o)^{-1}Ks_z(o) \\ &= \kappa_z(k)N \exp(A_o(y, kM) - A_o(z, kM))K_z, \end{aligned}$$

where in the first equality we use (2.4). Finally, we observe that $s_y(o)^{-1}s_z(o) = s_o(y)s_z(o) = s_z(y)$ and then

$$\kappa_z(k)^{-1}s_z(y) \in N \exp(A_o(y, kM) - A_o(z, kM))K_z.$$

Therefore, by item (i) of Lemma 2.5 and item (i) above

$$A_o(y, kM) - A_o(z, kM) = A_z(\kappa_z(k)^{-1}s_z(y)) = A_z(y, kM) .$$

This proves the case $x = o$. The general case trivially follows:

$$\begin{aligned} A_x(z, kM) + A_z(y, kM) &= A_o(z, kM) - A_o(x, kM) + A_o(y, kM) - A_o(z, kM) \\ &= A_x(y, kM). \end{aligned}$$

Hence this conclude the proof of the lemma. □

We recall that, under the analytic point of view, mappings Ψ_x play a very important role for us. We show how the previous lemma reflects on them. In particular we show how the action of $g \in G$ on a horocycle reflects on its parametrization and how two parametrizations from different reference points are related.

Corollary 2.10. *Let $x \in X$. Then:*

(i) for every $g \in G$, the action on horocycles reads

$$g.\Psi_x(kM, a) = \Psi_{g[x]}(g\langle kM\rangle, a), \qquad (kM, a) \in K/M \times A; \tag{2.17}$$

(ii) if $y \in X$ is an other reference point, then the following "change of variables" holds true for every $(kM, a) \in K/M \times A$

$$(\Psi_y^{-1} \circ \Psi_x)(kM, a) = (kM, a \exp(A_y(x, kM))). \tag{2.18}$$

Proof. Let $x \in X$. By Definition 2.8, for every $(kM, a) \in K/M \times A$ and $z \in X$

$$z \in \Psi_x(kM, a) \quad \Longleftrightarrow \quad A_x(z, kM) = \log a. \tag{2.19}$$

Then, by (2.19) together with (2.12) it follows that

$$\begin{aligned} z \in g.\Psi_x(kM, a) \quad &\Longleftrightarrow \quad g^{-1}[z] \in \Psi_x(kM, a) \\ &\Longleftrightarrow \quad \log a = A_x(g^{-1}[z], kM) \\ &\Longleftrightarrow \quad \log a = A_{g[x]}(z, g\langle kM\rangle) \\ &\Longleftrightarrow \quad z \in \Psi_{g[x]}(g\langle kM\rangle, a). \end{aligned}$$

Hence (2.17) follows.

It remains to prove (ii). If $y \in X$, then by (2.19) and (2.13) we have that

$$\begin{aligned} z \in \Psi_x(kM, a) \quad &\Longleftrightarrow \quad \log a = A_x(z, kM) \\ &\Longleftrightarrow \quad \log a = A_x(y, kM) + A_y(z, kM) \\ &\Longleftrightarrow \quad \log(a \exp(-A_x(y, kM))) = A_y(z, kM) \\ &\Longleftrightarrow \quad z \in \Psi_y(kM, a \exp(A_y(x, kM))), \end{aligned}$$

where in the last equivalence we use the equality $A_y(x, kM) = -A_x(y, kM)$, which follows immediately by (2.13). Hence, we have (2.18). □

2.2 Analysis on symmetric spaces

We collect in this section the analytic ingredients that come into play. Apart from the basic measures and function spaces, we introduce the Helgason-Fourier transform and recall the results that we use throughout. The main references are, beyond [8], [36], [37].

In particular, in the first part of the section we introduce all the measures and function spaces we need in the following. We present the classical fact of how, through Theorem 1.30, the Haar measure of a semisimple Lie group G decomposes w.r.t. to the Haar measures of its Iwasawa components. The G-invariant measure on the symmetric space $X \simeq G/K$ is immediate. Our main contribution here is the stressing of the fact that the (compact) boundary is endowed with different K-invariant (probability) measures, each relating to a different reference point. Furthermore, we clarify in formulae their G-invariance and the relation between them, namely their Radon-Nikodym derivative. These facts are crucial for us, since they allow to express symmetry properties in the definition of the functions spaces involved with the range of Helgason-Fourier and Radon transforms. Finally, in Section 2.2.1.4 we present the G-invariant measure on Ξ. In this case the matter of the reference point is different: indeed, it is still true that from every reference point, that is, from every possible parametrization of Ξ under $K/M \times A$, a different G-invariant measure can be defined, but in this case all these measures coincide as measures on Ξ. Furthermore, the $(L^2(\Xi), L^2_x(K/M \times A))$-pullback $\Psi_x^* F$ of F by Ψ_x is introduced: its main role in the following is to "transport" a function on Ξ in a more manageable function on $K/M \times A$ under the parametrization we need, keeping track of the density of the measure on Ξ.

Section 2.2.2 is a concise recall of the Fourier transform $\mathcal{H}$ introduced by Helgason on symmetric spaces, which we call Helgason-Fourier transform in order to distinguish it from the Euclidean Fourier transform. Furthermore, we state a different formulation of Theorem 1.5 in [37], that is the classical unitary extension of $\mathcal{H}$ on $L^2(X)$. The new formulation makes use of a symmetry property we denote $\sharp$ which is used to define the function space $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$, that is the range of the unitary extension of $\mathcal{H}$.

2.2.1 Measures

This section is devoted to the measures that will be involved in what follows, which are the measures on the spaces X, B and Ξ. These are used to define the function spaces that we are interested in, among which the L^2-spaces that carry the regular representations.

2.2.1.1 Measures on semisimple Lie groups of the noncompact type

Let G a semisimple Lie group. By Theorem 1.2, there exists a (left) Haar measure on G, unique up to multiplication by a positive constant. We recall that by Theorem 1.30 there exist subgroups K, A, and N of G such that $G = KAN = NAK$. Since each subgroup carries a Haar measure, the natural question arises whether it is possible to write the Haar measure of G using the Haar measures of the three subgroups involved, which are all, individually, unimodular.

Since K is compact, we normalize its Haar measure in such a way that the total measure is 1. The Haar measure on A is obtained by starting from the (positive) measure that any Riemannian manifold inherits from its metric, see e.g. Chap. I in [36]. The invariant metric is obtained by taking the restriction to $\mathfrak{a}\times\mathfrak{a}$ of the Killing form, which is positive definite on $\mathfrak{p}\times\mathfrak{p}\supset\mathfrak{a}\times\mathfrak{a}$, whereby $\mathfrak{a}$ is identified with the tangent space to A at the identity. The standard normalization is to multiply the Riemannian measure by $(2\pi)^{-\ell/2}$, where $\ell=\dim A$. As for N, we normalize its Haar measure $\mathrm{d}n$ so that

$$\int_{\overline{N}} e^{-2\rho(H(\overline{n}))}\mathrm{d}\overline{n}=1,$$

where $\overline{N}=\Theta(N)$ and $\mathrm{d}\overline{n}$ is the pushforward of $\mathrm{d}n$ under Θ (the involution Θ is the one whose existence is guaranteed by Theorem 2.3 (ii)). The convergence of the above integral is no trivial matter, and is discussed in detail in [36].

Proposition 2.11 (Proposition 5.1, Chap. I, [36])**.** *Let* $\mathrm{d}k$, $\mathrm{d}a$ *and* $\mathrm{d}n$ *be left-invariant Haar measures on* K, A *and* N, *respectively. Then the left Haar measure* $\mathrm{d}g$ *on* G *can be normalized so that*

$$\begin{aligned}\int_G f(g)\mathrm{d}g &= \int_{K\times A\times N} f(kan)e^{2\rho\log a}\mathrm{d}k\mathrm{d}a\mathrm{d}n\\ &= \int_{N\times A\times K} f(nak)e^{-2\rho(\log a)}\mathrm{d}n\mathrm{d}a\mathrm{d}k\\ &= \int_{A\times N\times K} f(ank)\mathrm{d}a\mathrm{d}n\mathrm{d}k\end{aligned}$$

for every $f\in C_c(G)$.

The case of the group AN deserves a separate comment. We recall that AN is in fact a semidirect product since A acts on N by conjugation from the observation after Theorem 1.30. Furthermore, for any $H\in\mathfrak{a}$ and any root vector $X_\alpha\in\mathfrak{g}_\alpha$ it holds

$$\mathrm{Ad}(\exp H)(X_\alpha)=\mathrm{e}^{\mathrm{ad}H}(X_\alpha)=\sum_0^\infty\frac{(\mathrm{ad}H)^k}{k!}X_\alpha=\mathrm{e}^{\alpha(H)}X_\alpha.$$

It follows that upon choosing a basis of m_α root vectors for each positive root α it is

$$\det\mathrm{Ad}(\exp H)|_{\mathfrak{n}}=\prod_{\alpha>0}\mathrm{e}^{m_\alpha\alpha(H)}$$

or, using (1.21),

$$\det\mathrm{Ad}a|_{\mathfrak{n}}=\mathrm{e}^{2\rho(\log a)}.$$

Proposition 1.4 now entails that the modular function of the AN Iwasawa group is

$$\Delta(na)=\mathrm{e}^{-2\rho(\log a)}. \tag{2.20}$$

Indeed, in the computation of $\det\mathrm{Ad}(na)$ on $\mathfrak{n}+\mathfrak{a}$, all is relevant is the action of $\mathrm{Ad}a$ on $\mathfrak{n}$ because the action of $\mathrm{Ad}a$ is unimodular on $\mathfrak{a}$ since A is Abelian, the action of $\mathrm{Ad}n$ is unimodular on $\mathfrak{n}$ because N is nilpotent and that of $\mathrm{Ad}n$ on $\mathfrak{a}$ is again unimodular because its projection on $\mathfrak{a}$ is the identity (see also Corollary 5.2 in Chap. I in [36]).

2.2.1.2 Measures on X

A classical notation in analysis on manifolds is to put $\mathcal{D}(X)$ instead of $C_c(X)$. It is not just about symbols, the canonical use of $\mathcal{D}(X)$ is due to the topology with which it is endowed (see Chap.II in [42]). In what follows we do not make use of the topology but we have decided to use $\mathcal{D}(X)$ anyway as a matter of consistency with the results that are cited.

Now, our purpose is to determine an explicit G-invariant measure on the symmetric space $X = G/K$, whose existence is guaranteed by the fact that K is compact (see the comment after Theorem 1.22). Recall that, by Proposition 2.11, if $g = nak$, then the Haar measure of G can be normalized so that

$$\mathrm{d}g = e^{-2\rho(\log a)}\mathrm{d}n\mathrm{d}a\mathrm{d}k,$$

where $\mathrm{d}k$, $\mathrm{d}a$, and $\mathrm{d}n$ are the Haar measures on K, A and N that have been fixed in the previous paragraph.

Observe that by the fact that $X \simeq G/K$, functions on X are in a bijective correspondence with right-K-invariant functions on G. Indeed, if f is a function on X, then $F(g) = f(g[o])$ is a right-K-invariant function on G, and vice versa. Then the G-invariant measure $\mathrm{d}x$ on X is the pushforward of $\mathrm{d}g$ under the canonical projection $G \to G/K$. Thus, for any $f \in \mathcal{D}(X)$

$$\int_X f(x)\mathrm{d}x = \int_G f(g[o])\mathrm{d}g = \int_{NA} f(na[o])e^{-2\rho(\log a)}\mathrm{d}n\mathrm{d}a.$$

We henceforth denote by $L^2(X)$ the Lebsegue space of square integrable (equivalence classes of) functions with respect to this measure. The *quasi regular representation* π of G on $L^2(X)$ is then defined in the usual way, namely

$$\pi(g)f(x) := f(g^{-1}[x]), \qquad f \in L^2(X),\, g \in G.$$

It is a unitary non-irreducible representation. Actually, it is possible to construct a family of Hilbert spaces in which $L^2(X)$ can be decomposed as a direct integral, whereby the restriction of π to each of them is irreducible. These are the spherical principal series representations, discussed in Chap. VI in [37]. It is also well known that π is not square integrable.

2.2.1.3 Measures on the boundary

We shall now define positive measures on the boundary B by using its various possible parametrizations. Since K and M are compact subgroups of G, there exists a K-invariant probability measure μ^o on $B = K/M$, see the comment below Theorem 1.22. The choice of this measure is such that Weil's decomposition holds, assuming that we normalize the Haar measure of M in such a way that the total measure is 1.

We stress the fact that the measure above is just one of the possible choice. Indeed for every other reference point $x \in X$ the analogous objects K_x, M_x and μ^x can be introduced. The relation between μ^o and μ^x can be determined explicitly. We consider the diffeomorphism $T_x\colon K \to K_x$ defined by $k \mapsto s_o(x)ks_o(x)^{-1}$. Its restriction to

M is a diffeomorphism between M and M_x. Hence, T_x induces the diffeomorphism $\tilde{T}_x\colon K/M \to K_x/M_x$ defined by

$$\tilde{T}_x(kM) = T_x(k)M_x = s_o(x)ks_o(x)^{-1}M_x = s_o(x)kMs_o(x)^{-1} .$$

Let $(\tilde{T}_x)_*(\mu^o)$ be the pushfoward of the measure μ^o under $\tilde{T}_x$. Clearly, $(\tilde{T}_x)_*(\mu^o)$ is a K_x-invariant probability measure on K_x/M_x and therefore $\mu^x = (\tilde{T}_x)_*(\mu^o)$. As we saw in (2.8), K_x/M_x is diffeomorphic to the boundary K/M through the map induced by κ_o. Therefore, we can consider the following K_x-invariant probability measure on the boundary $B = K/M$

$$\nu^x := (\kappa_o)_*(\mu^x).$$

It is worth observing that $\nu^o = \mu^o$ and the following relation follows

$$\nu^x = (\kappa_o \circ \tilde{T}_x)_*(\nu^o) .$$

Lemma 2.12. *The measure ν^o is G-quasi invariant. Let $F \in C(K/M)$ and $g \in G$,*

$$\int_{K/M} F(g^{-1}\langle kM\rangle)\mathrm{d}\nu^o(kM) = \int_{K/M} F(kM)e^{-2\rho(H_o(gk))}\mathrm{d}\nu^o(kM). \tag{2.21}$$

Proof. By Lemma 5.19 in Chap.I in [37], for every $H \in C(K)$ and $g \in G$,

$$\int_K H(\kappa_o(g^{-1}k))\mathrm{d}k = \int_K H(k)e^{-2\rho(H_o(gk))}\mathrm{d}k. \tag{2.22}$$

A function $F \in C(K/M)$ will now be regarded as an M-right invariant continuous function on K. By our choice of ν^o, Theorem 1.22 holds and hence

$$\begin{aligned}
\int_K F(k)\mathrm{d}k &= \int_{K/M}\int_M F(kM\,m)\mathrm{d}m\mathrm{d}\nu^o(kM) \\
&= \int_{K/M} F(kM)\int_M \mathrm{d}m\mathrm{d}\nu^o(kM) \\
&= \int_{K/M} F(kM)\mathrm{d}\nu^o(kM),
\end{aligned}$$

where we have used the normalization of the Haar measure of M. The function $k \mapsto F(g^{-1}\langle k\rangle) = F(\kappa_o(g^{-1}k))$ is M-invariant by $\kappa_o(g^{-1}km) = \kappa_o(g^{-1}k)m$. Since $m \in M$ commutes with A and N,

$$gkm \in \kappa_o(gk)m\exp(H_o(gk))N$$

and so $k \mapsto H_o(gk)$ is M-invariant. It follows that $k \mapsto F(k)e^{-2\rho(H_o(gk))}$ is also M-invariant. The assertion follows by applying (2.22) to F in place of H and then rewriting the integrals over K of the M-invariant functions as integrals over K/M w.r.t. ν^o as before. ∎

Now we investigate the relation between the different boundary measures introduced above. If $F \in C(K/M)$ and $x \in X$, then

$$\begin{aligned}\int_{K/M} F(kM)\mathrm{d}\nu^x(kM) &= \int_{K/M} F(\kappa_o(\tilde{T}_x(kM)))\mathrm{d}\nu^o(kM)\\ &= \int_{K/M} F(\kappa_o(s_o(x)k)M)\mathrm{d}\nu^o(kM)\\ &= \int_{K/M} F(kM)e^{-2\rho(H_o(s_o(x)^{-1}k))}\mathrm{d}\nu^o(kM)\\ &= \int_{K/M} F(kM)e^{2\rho(A_o(x,kM))}\mathrm{d}\nu^o(kM)\end{aligned}$$

by Lemma 2.12 and then applying item (i) of Lemma 2.9 together with (2.6) , since

$$-H_o(s_o(x)^{-1}k) = A_o(k^{-1}s_o(x)) = A_o(s_o(x)[o], kM) = A_o(x, kM).$$

By expressing the integral of a function on K/M with respect to either ν^x or ν^y as above and then using (2.13) in the form

$$A_o(x, kM) = A_o(y, kM) + A_y(x, kM),$$

the Radon-Nikodym derivative between the measures ν^x and ν^y is

$$\frac{\mathrm{d}\nu^x}{\mathrm{d}\nu^y}(kM) = e^{2\rho(A_y(x,kM))}. \tag{2.23}$$

Let $x \in X$, $g \in G$ and $F \in C(K/M)$. Using first (2.23) with $y = o$ and then (2.21)

$$\begin{aligned}\int_{K/M} F(g^{-1}\langle kM\rangle)\mathrm{d}\nu^x(kM) &= \int_{K/M} F(g^{-1}\langle kM\rangle)e^{2\rho(A_o(x,kM))}\mathrm{d}\nu^o(kM)\\ &= \int_{K/M} F(kM)e^{2\rho(A_o(x,g\langle kM\rangle))}e^{-2\rho(H_o(gk))}\mathrm{d}\nu^o(kM).\end{aligned}$$

Now observe that, by (2.11) and (2.12),

$$\begin{aligned}A_o(x, g\langle kM\rangle) - H_o(gk) &= A_{g^{-1}[o]}(g^{-1}[x], kM) + A_o(k^{-1}g^{-1})\\ &= A_{g^{-1}[o]}(g^{-1}[x], kM) + A_o(g^{-1}[o], kM)\\ &= A_o(g^{-1}[x], kM),\end{aligned}$$

the latter equality being just (2.13) from Lemma 2.9. Hence, we obtain a sort of dual relation between the G-action on the boundary and that on the reference points of the boundary measures, namely

$$\int_{K/M} F(g^{-1}\langle kM\rangle)\mathrm{d}\nu^x(kM) = \int_{K/M} F(kM)\mathrm{d}\nu^{g^{-1}[x]}(kM). \tag{2.24}$$

2.2.1.4 Measures on Ξ

Finally, in order to develop the theory in which we are interested, we need to introduce a G-invariant measure on Ξ. We denote by σ the measure on A with density $e^{2\rho(\log a)}$ with respect to the Haar measure $\mathrm{d}a$. For every $x \in X$, we can endow Ξ with the measure $\mathrm{d}\xi$ obtained as the pushforward of the measure $\nu^x \otimes \sigma$ on $K/M \times A$ by means of the map Ψ_x, i.e.

$$\mathrm{d}\xi = \Psi_{x*}(\nu^x \otimes \sigma).$$

It turns out that $\mathrm{d}\xi$ is independent of the choice of $x \in X$. We denote by $L^1(\Xi)$ and $L^2(\Xi)$ the spaces of absolutely integrable functions and square integrable functions with respect to the measure $\mathrm{d}\xi$, respectively. By definition, for every $F \in L^1(\Xi)$

$$\begin{aligned}\int_\Xi F(\xi)\mathrm{d}\xi &= \int_{K/M\times A} (F \circ \Psi_x)(kM, a)\mathrm{d}(\nu^x \otimes \sigma)(kM, a)\\ &= \int_{K/M\times A} (F \circ \Psi_x)(kM, a)e^{2\rho(\log a)}\mathrm{d}\nu^x(kM)\mathrm{d}a\ .\end{aligned}$$

It is easy to verify that $\mathrm{d}\xi$ is G-invariant. We point out that Helgason introduced this measure w.r.t. $o \in X$, see Lemma 3.1 in Chap. II in [37]. Since in our treatment it is important to change the reference point the expression above suits our needs.

The group G acts on $L^2(\Xi)$ via the quasi regular representation $\hat{\pi}\colon G \to \mathcal{U}(L^2(\Xi))$ defined by

$$\hat{\pi}(g)F(\xi) := F(g^{-1}.\xi), \qquad F \in L^2(\Xi),\ g \in G.$$

Equivalently, given $x \in X$, by (2.17)

$$(\hat{\pi}(g)F) \circ \Psi_x(kM, a) = F \circ \Psi_{g^{-1}[x]}(g^{-1}\langle kM\rangle, a), \tag{2.25}$$

for every $(kM, a) \in K/M \times A$ and $g \in G$. We see in the following that $\hat{\pi}$ is not irreducible.

We need to introduce some more notation. We denote by $\Delta^{-\frac{1}{2}}$ the map on $K/M \times A$ defined by

$$\Delta^{-\frac{1}{2}}(kM, a) = \mathrm{e}^{\rho(\log a)}.$$

The reason for such notation resides in the fact that this function has the same expression of the inverse of the square root of the modular function of the AN Iwasawa group, see (2.20).

Finally, for every $x \in X$, we introduce the space $L^2_x(K/M \times A)$ of square integrable functions on $K/M \times A$ w.r.t. the measure $\nu^x \otimes \mathrm{d}a$. For every $F \in L^2(\Xi)$, we denote by $\Psi_x^* F$ the $(L^2(\Xi), L^2_x(K/M \times A))$-pull-back of F by Ψ_x, that is, we introduce the unitary operator $\Psi_x^*\colon L^2(\Xi) \to L^2_x(K/M \times A)$ given by

$$\Psi_x^* F(kM, a) = (\Delta^{-\frac{1}{2}} \cdot (F \circ \Psi_x))(kM, a)$$

for almost every $(kM, a) \in K/M \times A$. In order to see that Ψ_x^* is unitary, observe that for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}&\int_{K/M\times A} |\Psi_x^* F(kM,a)|^2 \mathrm{d}\nu^x(kM)\mathrm{d}a \\ &= \int_{K/M\times A} |(\Delta^{-\frac{1}{2}} \cdot (F\circ\Psi_x))(kM,a)|^2 \mathrm{d}\nu^x(kM)\mathrm{d}a \\ &= \int_{K/M\times A} |(F\circ\Psi_x)(kM,a)|^2 e^{2\rho(\log a)} \mathrm{d}\nu^x(kM)\mathrm{d}a \\ &= \int_\Xi |F(\xi)|^2 \mathrm{d}\xi = \|F\|^2_{L^2(\Xi)},\end{aligned}$$

so that Ψ_x^* is an isometry from $L^2(\Xi)$ into $L^2_x(K/M\times A)$. Surjectivity is also clear.

2.2.2 The Helgason-Fourier transform

The Helgason-Fourier transform was defined by Helgason in analogy with the Fourier transform on Euclidean spaces in polar coordinates. We briefly recall its definition and its main features.

Definition 2.13 (§1, Chap. III, [37])**.** The *Helgason-Fourier transform* of $f \in \mathcal{D}(X)$ is the function $\mathcal{H}f : K/M \times \mathfrak{a}^* \longrightarrow \mathbb{C}$ defined by

$$\mathcal{H}f(kM,\lambda) = \int_X f(x) e^{(-i\lambda+\rho)(A_o(x,kM))} \mathrm{d}x.$$

As the Euclidean Fourier transform, the Helgason-Fourier transform extends to a unitary operator on $L^2(X)$. The Plancherel measure involves the *Harish-Chandra* **c** *function*, a cornerstone in the analysis on symmetric spaces [31], [32]. It is a meromorphic function $\mathbf{c}\colon \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ defined on the complexified dual space $\mathfrak{a}^*_{\mathbb{C}}$ for which various formulae are available (see e.g. [38]). It may thus be restricted to the real space $\mathfrak{a}^*$. As an example, in the case of the unit disk, if $\mathrm{Re}(i\lambda) > 0$, then

$$\mathbf{c}(\lambda) = \pi^{-1/2} \frac{\Gamma(\frac{1}{2} i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+1))},$$

so that

$$|\mathbf{c}(\lambda)|^{-2} = \frac{\pi\lambda}{2} \tanh\left(\frac{\pi\lambda}{2}\right).$$

We denote by $L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)$ the space of the functions on $K/M\times\mathfrak{a}^*$ that are square integrable w.r.t. the measure $w^{-1}\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\nu^o\mathrm{d}\lambda$, where w stands for the cardinality of the Weyl group W.

Proposition 2.14 (§1, Chap. III, [37])**.** *For every* $f_1, f_2 \in \mathcal{D}(X)$

$$\int_x f_1(x)\overline{f_2(x)}\mathrm{d}x = \int_{\mathfrak{a}^*\times K/M} \mathcal{H}f_1(kM,\lambda)\overline{\mathcal{H}f_2(kM,\lambda)}\mathrm{d}\nu^o(kM)\frac{\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2}. \tag{2.26}$$

The rest of the paragraph is devoted to state the Plancherel theorem for the Helgason-Fourier transform.

Property $\sharp$. We say that a function $F \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$ satisfies Property $\sharp$ if for every $x \in X$ the function

$$\mathfrak{a}^* \ni \lambda \longmapsto \int_{K/M} e^{(\rho+i\lambda)(A_o(x,kM))} F(kM,\lambda)\mathrm{d}\nu^o(kM) \tag{2.27}$$

is W-invariant almost everywhere (see the comments after (1.20) for the W-action on $\mathfrak{a}^*$).

We denote by $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$ the space of functions F in $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$ satisfying Property $\sharp$. We observe that the integral in (2.27) is absolutely convergent for almost every $\lambda \in \mathfrak{a}^*$. By Fubini theorem, for every $F \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$ we have that

$$\|F\|^2_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)} = \int_{\mathfrak{a}^*}\int_{K/M} |F(kM,\lambda)|^2 \mathrm{d}\nu^o(kM)\frac{\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2} < +\infty.$$

Thus, the function $F(\cdot,\lambda)$ is in $L^2(K/M,\nu^o) \subseteq L^1(K/M,\nu^o)$ for almost every $\lambda \in \mathfrak{a}^*$ and, since $\rho(A_o(x,\cdot))$ is bounded on K/M, the integrability properties of $F(\cdot,\lambda)$ continue to hold for the function $e^{(\rho+i\lambda)(A_o(x,\cdot))}F(\cdot,\lambda)$.

Remark 2.15. We observe *en passant* that the Helgason-Fourier definition depends on the choice of $o \in X$. Actually, it is possible to define it w.r.t. to every reference point $x \in X$ on $f \in \mathcal{D}(X)$ by

$$\mathcal{H}_x f(kM,\lambda) = \int_X f(y)e^{(-i\lambda+\rho)(A_x(y,kM))}\mathrm{d}y, \qquad (kM,\lambda) \in K/M \times \mathfrak{a}^*.$$

An easy application of Lemma 2.9 (iii) clearly reveals a relation between $\mathcal{H}$ and $\mathcal{H}_x$, namely

$$\mathcal{H}_x f(kM,\lambda) = e^{(-\rho+i\lambda)A_o(x,kM)}\mathcal{H}f(kM,\lambda).$$

Hence, by using the Radon-Nikodym derivative (2.23) of ν^x w.r.t. ν^o, Property $\sharp$ for $\mathcal{H}$ consists in the a.e. W-invariance of

$$\mathfrak{a}^* \ni \lambda \longmapsto \int_{K/M} \mathcal{H}_x(kM,\lambda)\mathrm{d}\nu^x(kM),$$

for every $x \in X$. This remark is not used in the following but however it represents an explanation of the symmetries required in Property $\sharp$.

Every function $F \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$ is uniquely determined by its restriction on $K/M \times \mathfrak{a}^*_+$. Here $\mathfrak{a}^*_+$ denotes the *positive Weyl chamber*

$$\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* : A_\lambda \in \mathfrak{a}^+\},$$

where A_λ represents λ via the Killing form, in the sense that $\lambda(H) = B(A_\lambda, H)$. If we suppose that F, $G \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$ are such that $F_1|_{K/M\times\mathfrak{a}^*_+} = F_2|_{K/M\times\mathfrak{a}^*_+}$, then

$$\begin{aligned}&\int_{K/M} e^{(\rho+is\lambda)(A_o(x,kM))}(F_1 - F_2)(kM,s\lambda)\mathrm{d}\nu^o(kM)\\ &= \int_{K/M} e^{(\rho+i\lambda)(A_o(x,kM))}(F_1 - F_2)(kM,\lambda)\mathrm{d}\nu^o(kM) = 0\end{aligned}$$

for a. e. $\lambda \in \mathfrak{a}^*_+$ and for every $s \in W$. Therefore, by Lemma 5.3 in Chap. II in [37], we can conclude that $F_1 - F_2 = 0$ in $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$.

By the Paley-Wiener theorem for the Helgason Fourier transform (Theorem 5.1 in Chap. III in [37]), $\mathcal{H}f \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$ for every $f \in \mathcal{D}(X)$, so that $\mathcal{H}f$ is uniquely determined by its restriction on $K/M \times \mathfrak{a}^*_+$. We denote by $L^2_{o,\mathbf{c}}\left(K/M \times \mathfrak{a}^*_+\right)$ the space of the functions on $K/M \times \mathfrak{a}^*_+$ that are square integrable w.r.t. the measure $|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\nu^o \mathrm{d}\lambda$ and the Plancherel theorem for the Helgason-Fourier transform reads:

Theorem 2.16 (Theorem 1.5, Chap. III, [37])**.** *The restricted Helgason-Fourier transform $f \mapsto \mathcal{H}f|_{K/M\times\mathfrak{a}^*_+}$ extends to a unitary operator $\mathscr{H}$ from $L^2(X)$ onto $L^2_{o,\mathbf{c}}\left(K/M \times \mathfrak{a}^*_+\right)$.*

For our purpose we sometimes use a different terminology from Helgason. In particular, we need a different version of Theorem 2.16, that better suits our needs.

By the Plancherel formula (2.26), $\mathcal{H}$ is an isometry from $\mathcal{D}(X)$ into $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$. Furthermore, we show that, by Theorem 2.16, $\mathcal{H}(\mathcal{D}(X))$ embeds densely in $L^2_{o,\mathbf{c}}(K/M\times \mathfrak{a}^*)^\sharp$. Let $F \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$ be such that $\langle F, \mathcal{H}f\rangle_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)} = 0$ for every $f \in \mathcal{D}(X)$. Then, by Fubini theorem we have that

$$\begin{aligned}
0 &= \frac{1}{w}\int_{\mathfrak{a}^*}\int_{K/M} F(kM,\lambda)\overline{\int_X f(x)e^{(-i\lambda+\rho)(A_o(x,kM))}\mathrm{d}x}\,\mathrm{d}\nu^o(kM)\frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}\\
&= \frac{1}{w}\int_{\mathfrak{a}^*}\int_X\int_{K/M} F(kM,\lambda)e^{(i\lambda+\rho)(A_o(x,kM))}\mathrm{d}\nu^o(kM)\overline{f(x)}\mathrm{d}x\frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}\\
&= \int_{\mathfrak{a}^*_+}\int_X\int_{K/M} F(kM,\lambda)e^{(i\lambda+\rho)(A_o(x,kM))}\mathrm{d}\nu^o(kM)\overline{f(x)}\mathrm{d}x\frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2}\\
&= \int_{\mathfrak{a}^*_+}\int_{K/M} F(kM,\lambda)\overline{\mathcal{H}f(kM,\lambda)}\mathrm{d}\nu^o(kM)\frac{\mathrm{d}\lambda}{|\mathbf{c}(\lambda)|^2},
\end{aligned} \tag{2.28}$$

where we use that F satisfies Property $\sharp$ and $|\mathbf{c}|^2$ is W-invariant. Hence, (2.28) yields

$$\langle F|_{K/M\times\mathfrak{a}^*_+}, \mathcal{H}f|_{K/M\times\mathfrak{a}^*_+}\rangle_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*_+)} = \langle F|_{K/M\times\mathfrak{a}^*_+}, \mathscr{H}f\rangle_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*_+)} = 0,$$

for every $f \in \mathcal{D}(X)$, and Theorem 2.16 implies that $F \equiv 0$ a.e. on $K/M\times\mathfrak{a}^*_+$. Therefore, $F = 0$ in $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)$ and $\mathcal{H}(\mathcal{D}(X))$ embeds densely in $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$.

This leads us to state the following formulation of Theorem 2.16.

Theorem 2.17. *The Helgason-Fourier transform $\mathcal{H}$ extends to a unitary operator $\mathscr{H}$ from $L^2(X)$ onto $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$.*

In what follows, we always consider $\mathscr{H}$ taking values in $L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)^\sharp$.

2.3 Unitarization of Radon transform

In this section we introduce the horocyclic Radon transform, we study its range, and we investigate its intertwining properties with the quasi regular representations π and $\hat{\pi}$ of G. In the last part we present our main result: the so called unitarization theorem for the horocyclic Radon transform.

The horocyclic Radon transform of a signal is defined as a function on Ξ. The machinery we developed in the previous sections leads us to consider it as a function on $K/M \times A$ in many ways, that is, w.r.t. every reference point. Section 2.3.1 contains two important results: a classical one that is the Fourier slice theorem (§5, Chap. III in [37]) and Proposition 2.24. The latter is the result on which Section 2.3.2 is based, indeed it links Property $\sharp$ of the previous section with Property $\flat$ introduced below, providing a characterization of $L^2_\flat(\Xi)$, that is the range of the unitarization of the horocyclic Radon transform.

Section 2.3.2 contains our main result, that is Theorem 2.28. In order to define the operator Λ involved in the unitarization of $\mathcal{R}$, we need some technicalities. The operator Λ is the same introduced by Helgason in [37] on a dense domain of $L^2(\Xi)$. In addition, we show that Λ maintains Property $\flat$. Roughly speaking, Λ is the conjugation of a densely defined Fourier multiplier $\mathcal{J}_o$ by $\Delta^{-1/2}$. The last sentence must be interpreted: indeed the conjugation cannot be expressed directly but we need to use the pull-back Ψ_o^* (and its inverse), see Figure 2.4. We stress that if we replace o with $x \in X$, the so obtained operator Λ is the same.

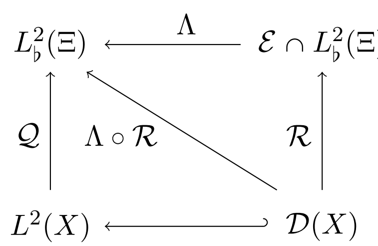

Figure 2.3: The operator Λ is defined on the dense subset $\mathcal{E} \subseteq L^2(\Xi)$ and maintains Property $\flat$. Its precomposition with $\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$ on $L^2(X)$.

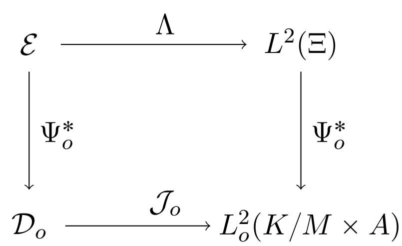

Figure 2.4: The Fourier multiplier $\mathcal{J}_o$ is defined on the dense subset $\mathcal{D}_o \subseteq L^2_o(K/M \times A)$ and is "transported" on functions defined on Ξ by the conjugation with Ψ_o^*.

The presence of the factor $\Delta^{-1/2}$ represents a difference from the setup of the previous chapter, indeed in the classical unitarization examples presented in [1], the operator Λ is always a pure Fourier multiplier.

Finally, we show that the unitary extension $\mathcal{Q}$ of $\Lambda\mathcal{R}$ intertwines the two quasi regular representations of G on $L^2(X)$ and $L^2(\Xi)$, respectively.

We refer the reader to compare the two diagrams in Figures 2.3 and 2.4 with the final on in Figure 2.5, which is completed with results proved below.

2.3.1 The horocyclic Radon transform

With horocyclic Radon transform we mean the integration of a signal on each horocycle of Ξ. Because horocycles admit (several) explicit parametrizations, we define the horocyclic Radon transform appealing directly to the basic parametrization, as clarified in the definition that follows.

Definition 2.18. The *horocyclic Radon transform* $\mathcal{R}f$ of a function $f \in \mathcal{D}(X)$ is the map $\mathcal{R}f : \Xi \to \mathbb{C}$ defined by

$$(\mathcal{R}f \circ \Psi_o)(kM, a) = \int_N f(kan[o])\mathrm{d}n,$$

for every $(kM, a) \in K/M \times A$.

If we change parametrization, and use equality (2.18), for any $x \in X$ we obtain the equivalent definition

$$\begin{aligned}(\mathcal{R}f \circ \Psi_x)(kM, a) &= (\mathcal{R}f \circ \Psi_o)(kM, a\exp(A_o(x, kM))) \\ &= \int_N f(ka\exp(A_o(x, kM))n[o])\mathrm{d}n. \end{aligned} \tag{2.29}$$

Definition 2.19. Let $f \in \mathcal{D}(X)$. We denote by $\mathcal{A}f$ the map $\mathcal{A}f : K/M \times A \to \mathbb{C}$ defined by

$$\mathcal{A}f(kM, a) := \Psi_o^*(\mathcal{R}f)(kM, a) = (\Delta^{-\frac{1}{2}} \cdot (\mathcal{R}f \circ \Psi_o))(kM, a).$$

It is worth observing that if the function f is K-left-invariant, then $\mathcal{A}f$ coincides with the *Abel transform* of f introduced by Helgason in Chap. IV in [37].

We need to introduce the Fourier transform on the Abelian group A. It is possible to see that, in the terminology of Section 1.1.3, $\widehat{A} \simeq \mathfrak{a}^*$ and $\langle a, \lambda \rangle = e^{i\lambda(\log a)}$ for every $a \in A$ and $\lambda \in \mathfrak{a}^*$. Hence Definition 1.18 reads as follows.

Definition 2.20. Let $s \in L^1(A)$. The *Fourier transform* $\mathcal{F}s$ of s is defined on $\mathfrak{a}^*$ by

$$\mathcal{F}s(\lambda) = \int_A s(a)e^{-i\lambda(\log a)}\mathrm{d}a\ .$$

We denote by R the *regular representation* of A on $L^2(A)$, which is defined for every $s \in L^2(A)$ and for every $\alpha \in A$ by

$$R_\alpha s(a) = s(\alpha^{-1}a), \quad a \in A.$$

Furthermore, we denote by M the representation of A on $L^2(\mathfrak{a}^*)$ defined for every $r \in L^2(\mathfrak{a}^*)$ and for every $\alpha \in A$ by

$$M_\alpha r(\lambda) = e^{-i\lambda(\log\alpha)}r(\lambda), \quad \lambda \in \mathfrak{a}^*.$$

Proposition 2.21 (§7.2, Chap. 5, [40])**.** *The Fourier transform $\mathcal{F} : L^2(A) \to L^2(\mathfrak{a}^*)$ intertwines the regular representation R with the representation M, i.e.*

$$\mathcal{F}R_\alpha = M_\alpha\mathcal{F},$$

for every $\alpha \in A$.

We are now ready to recall the result which relates the Helgason-Fourier transform with the horocyclic Radon transform. We refer to Proposition 2.22 as the Fourier Slice Theorem for the horocyclic Radon transform in analogy with Theorem 1.34 for the polar Radon transform.

Proposition 2.22 (§5, Chap. III, [37]). *For every $f \in \mathcal{D}(X)$ and $kM \in K/M$, the function $a \mapsto \mathcal{A}f(kM,a)$ is in $L^1(A)$ and*

$$(I \otimes \mathcal{F})\mathcal{A}f(kM,\lambda) = \mathcal{H}f(kM,\lambda), \tag{2.30}$$

for almost every $\lambda \in \mathfrak{a}^*$.

Let $f \in \mathcal{D}(X)$. By the Paley-Wiener theorem for the Helgason Fourier transform (Theorem 5.1 in Chap. III in [37]), $\mathcal{H}f$ is rapidly decreasing in the variable $\lambda \in \mathfrak{a}^*$ uniformly over K/M, that is for every $n \in \mathbb{N}$

$$|||\mathcal{H}f|||_n := \sup_{kM \in K/M,\, \lambda\in\mathfrak{a}^*} (1+|\lambda|)^n |\mathcal{H}f(kM,\lambda)| < +\infty.$$

By Theorem 1.20 and Proposition 2.22, we have that

$$\begin{aligned}
\int_\Xi |\mathcal{R}f(\xi)|^2 \mathrm{d}\xi &= \int_{K/M\times A} |\Psi_o^*(\mathcal{R}f)(kM,a)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}a \\
&= \int_{K/M\times \mathfrak{a}^*} |(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{R}f))(kM,\lambda)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M\times \mathfrak{a}^*} |\mathcal{H}f(kM,\lambda)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M\times \mathfrak{a}^*} \frac{(1+|\lambda|)^{2n}|\mathcal{H}f(kM,\lambda)|^2}{(1+|\lambda|)^{2n}} \mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&\leqslant |||\mathcal{H}f|||_n^2 \int_{\mathfrak{a}^*} \frac{1}{(1+|\lambda|)^{2n}} \mathrm{d}\lambda < +\infty,
\end{aligned}$$

for every $n > \dim A/2$. Therefore, $\mathcal{R}f \in L^2(\Xi)$ for every $f \in \mathcal{D}(X)$.

The horocyclic Radon transform intertwines the regular representations π and $\hat{\pi}$ of G.

Proposition 2.23. *For every $g \in G$ and $f \in \mathcal{D}(X)$*

$$\mathcal{R}(\pi(g)f) = \hat{\pi}(g)(\mathcal{R}f)\ .$$

Proof. Let $g \in G$ and $f \in \mathcal{D}(X)$. It is sufficient to show that $\mathcal{R}(\pi(g)f)\circ\Psi_o = \hat{\pi}(g)(\mathcal{R}f)\circ\Psi_o$ on $K/M \times A$. Let $(kM,a) \in K/M \times A$. Then

$$\begin{aligned}
\mathcal{R}(\pi(g)f)\circ\Psi_o(kM,a) &= \int_N \pi(g)f(kan[o])\mathrm{d}n \\
&= \int_N f(g^{-1}kan[o])\mathrm{d}n \\
&= \int_N f(\kappa_o(g^{-1}k)\exp(H_o(g^{-1}k))an[o])\mathrm{d}n,
\end{aligned}$$

where we used the decomposition $g^{-1}k \in \kappa_o(g^{-1}k)\exp(H_o(g^{-1}k))N$ and the fact that A normalizes N. Now, by (2.6), (2.11) and (2.13), we have

$$H_o(g^{-1}k) = -A_o(k^{-1}g) = -A_o(g[o],kM) = A_{g[o]}(o,kM)\ .$$

Finally, by $g^{-1}(kM) = \kappa_o(g^{-1}k)M$ and (2.29) we have that

$$\begin{aligned}\mathcal{R}(\pi(g)f) \circ \Psi_o(kM, a) &= \int_N f(\kappa_o(g^{-1}k)\exp(A_{g[o]}(o, kM))an[o])\mathrm{d}n \\ &= \int_N f(\kappa_o(g^{-1}k)\exp(A_o(g^{-1}[o], g^{-1}\langle kM\rangle))an[o])\mathrm{d}n \\ &= \mathcal{R}f \circ \Psi_{g^{-1}[o]}(g^{-1}\langle kM\rangle, a) \\ &= (\hat{\pi}(g)\mathcal{R}f) \circ \Psi_o(kM, a),\end{aligned}$$

where we used the action of G on Ξ given in (2.25). □

We now introduce a closed subspace of $L^2(\Xi)$ which will be crucial because it is the range of the unitarization of the horocyclic Radon transform. To do it, we introduce a property which will play a role similar to Property ♯. Roughly speaking, we can think of Property ♭ above as the time analog of the Property ♯ in frequency, under $(I\otimes\mathcal{F})\Psi_x$, for every $x \in X$, as it is better clarified in Proposition 2.24 and in Figure 2.5.

By definition, for every $x \in X$ and every $F \in L^2(\Xi)$

$$\|F\|^2_{L^2(\Xi)} = \int_{K/M}\int_A |\Psi_x^* F(kM, a)|^2 \mathrm{d}a\mathrm{d}\nu^x(kM) < +\infty.$$

So that, the function $\Psi_x^*F(kM, \cdot)$ is in $L^2(A)$ for almost every $kM \in K/M$. Then, by Plancherel formula 1.11 and Fubini theorem

$$\begin{aligned}\|F\|^2_{L^2(\Xi)} &= \int_{K/M\times A} |\Psi_x^* F(kM, a)|^2 \mathrm{d}\nu^x(kM)\mathrm{d}a \\ &= \int_{K/M\times\mathfrak{a}^*} |(I\otimes\mathcal{F})\Psi_x^* F(kM, \lambda)|^2 \mathrm{d}\nu^x(kM)\mathrm{d}\lambda \\ &= \int_{\mathfrak{a}^*}\int_{K/M} |(I\otimes\mathcal{F})\Psi_x^* F(kM, \lambda)|^2 \mathrm{d}\nu^x(kM)\mathrm{d}\lambda < +\infty.\end{aligned}$$

So that, for almost every $\lambda \in \mathfrak{a}^*$ the function $(I\otimes\mathcal{F})\Psi_x^*F(\cdot, \lambda)$ is in $L^2(K/M, \nu^x) \subseteq L^1(K/M, \nu^x)$ and

$$\left|\int_{K/M} (I\otimes\mathcal{F})\Psi_x^* F(kM, \lambda)\mathrm{d}\nu^x(kM)\right| \leqslant \int_{K/M} |(I\otimes\mathcal{F})\Psi_x^* F(kM, \lambda)|\mathrm{d}\nu^x(kM),$$

which is finite.

Property ♭. We say that a function $F \in L^2(\Xi)$ satisfies Property ♭ if for every $x \in X$ the function

$$\mathfrak{a}^* \ni \lambda \longmapsto \int_{K/M} (I\otimes\mathcal{F})\Psi_x^* F(kM, \lambda)\mathrm{d}\nu^x(kM)$$

is W-invariant almost everywhere.

We denote by $L^2_\flat(\Xi)$ the space of functions $F \in L^2(\Xi)$ satisfying Property ♭. Notice that by the considerations above, the integral appearing in Property ♭ is finite for almost every $\lambda \in \mathfrak{a}^*$. Our main results in Section 2.3.2 are based on the characterization of $L^2_\flat(\Xi)$ given in Proposition 2.24 below. We denote by $L^2_o(K/M \times \mathfrak{a}^*)$ the space of square integrable functions on $K/M \times \mathfrak{a}^*$ w.r.t. the measure $\nu^o \otimes \mathrm{d}\lambda$.

Proposition 2.24. *The operator Φ_o defined on $F \in L^2(\Xi)$ by*

$$\Phi_o F(kM,\lambda) = (I \otimes \mathcal{F})\Psi_o^* F(kM,\lambda), \qquad a.e.\,(kM,\lambda) \in K/M \times \mathfrak{a}^*,$$

is an isometry from $L^2(\Xi)$ into $L^2_o(K/M \times \mathfrak{a}^*)$. Furthermore, a function F belongs to $L^2_\flat(\Xi)$ if and only if $\Phi_o F$ satisfies Property $\sharp$.

Proof. By Parseval identity, for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}
&\int_{K/M\times\mathfrak{a}^*} |\Phi_o F(kM,\lambda)|^2 \,\mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M}\int_{\mathfrak{a}^*} |(I\otimes\mathcal{F})\Psi_o^* F(kM,\lambda)|^2 \,\mathrm{d}\lambda\mathrm{d}\nu^o(kM) \\
&= \int_{K/M\times A} |\Psi_o^* F(kM,a)|^2 \,\mathrm{d}\nu^o(kM)\mathrm{d}a = \|F\|^2_{L^2(\Xi)},
\end{aligned}$$

so that Φ_o is an isometry from $L^2(\Xi)$ into $L^2_o(K/M \times \mathfrak{a}^*)$. Now, let $F \in L^2(\Xi)$. By equation (2.18) and by the definition of the regular representation R of A, for almost every $kM \in K/M$ and $\lambda \in \mathfrak{a}^*$ we have that

$$\begin{aligned}
\Phi_o F(kM,\lambda) &= (I\otimes\mathcal{F})\Psi_o^* F(kM,\lambda) = (I\otimes\mathcal{F})(\Delta^{-\frac12}\cdot(F\circ\Psi_o))(kM,\lambda) \\
&= e^{\rho(A_o(x,kM))}(I\otimes\mathcal{F})(I\otimes R_{\exp(A_x(o,kM))^{-1}})(\Delta^{-\frac12}\cdot(F\circ\Psi_x))(kM,\lambda).
\end{aligned}$$

Therefore, by Proposition 2.21 we obtain

$$\begin{aligned}
\Phi_o F(kM,\lambda) &= e^{\rho(A_o(x,kM))}(I\otimes M_{\exp(A_x(o,kM))^{-1}})(I\otimes\mathcal{F})(\Delta^{-\frac12}\cdot(F\circ\Psi_x))(kM,\lambda) \\
&= e^{(\rho-i\lambda)(A_o(x,kM))}(I\otimes\mathcal{F})(\Delta^{-\frac12}\cdot(F\circ\Psi_x))(kM,\lambda) \\
&= e^{(\rho-i\lambda)(A_o(x,kM))}(I\otimes\mathcal{F})\Psi_x^* F(kM,\lambda). \qquad (2.31)
\end{aligned}$$

Now, for every $x \in X$ and for almost every $\lambda \in \mathfrak{a}^*$, (2.31) yields

$$\begin{aligned}
&\int_{K/M} e^{(\rho+i\lambda)(A_o(x,kM))}\Phi_o F(kM,\lambda)\mathrm{d}\nu^o(kM) \\
&= \int_{K/M} e^{(\rho+i\lambda)(A_o(x,kM))}e^{(\rho-i\lambda)(A_o(x,kM))}(I\otimes\mathcal{F})\Psi_x^* F(kM,\lambda)\mathrm{d}\nu^o(kM) \\
&= \int_{K/M} (I\otimes\mathcal{F})\Psi_x^* F(kM,\lambda)e^{2\rho(A_o(x,kM))}\mathrm{d}\nu^o(kM) \\
&= \int_{K/M} (I\otimes\mathcal{F})\Psi_x^* F(kM,\lambda)\mathrm{d}\nu^x(kM). \qquad (2.32)
\end{aligned}$$

Equality (2.32) allows us to conclude that F satisfies Property $\flat$ if and only if $\Phi_o F$ satisfies Property $\sharp$ and this concludes our proof.

□

Corollary 2.25. *For every $f \in \mathcal{D}(X)$,*

$$\Phi_o(\mathcal{R}f) = \mathcal{H}f$$

in $L^2_o(K/M \times \mathfrak{a}^*)$ and $\mathcal{R}f \in L^2_\flat(\Xi)$.

Proof. The proof follows immediately by Proposition 2.22 and the fact that the Helgason-Fourier transform satisfies Property ♯. □

Some comments are in order. Proposition 2.24 with Corollary 2.25 shows the link between the range of the Radon transform with the range of the Helgason-Fourier transform, which will play a crucial role in our main result. The range $\mathcal{R}(\mathcal{D}(X))$ has already been completely characterized in Chap. IV in [37]. As it will be made clear in the following, Property ♭ better suits our needs.

2.3.2 Unitarization and intertwining

In order to obtain the unitarization for the horocyclic Radon transform that we are after, we need some technicalities.

We put

$$\mathcal{D}_o = \{\varphi \in L^2_o(K/M \times A) : (I \otimes \mathcal{F})\varphi \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)\}$$

and we define the operator $\mathcal{J}_o \colon \mathcal{D}_o \subseteq L^2_o(K/M \times A) \to L^2_o(K/M \times A)$ as the Fourier multiplier

$$(I \otimes \mathcal{F})(\mathcal{J}_o\varphi)(kM, \lambda) = \frac{1}{\sqrt{w}\,|\mathbf{c}(\lambda)|}(I \otimes \mathcal{F})\varphi(kM, \lambda), \quad \text{a.e. } (kM, \lambda) \in K/M \times \mathfrak{a}^*.$$

We define the set of functions

$$\mathcal{E} = \{F \in L^2(\Xi) : \Phi_o F \in L^2_{o,\mathbf{c}}(K/M \times \mathfrak{a}^*)\}$$

and we consider the operator $\Lambda \colon \mathcal{E} \subseteq L^2(\Xi) \to L^2(\Xi)$ given by

$$\Lambda F = \Psi_o^{*-1} \mathcal{J}_o \Psi_o^* F.$$

As a direct consequence of the definition of Λ and $\mathcal{J}_o$, for every $F \in \mathcal{E}$ and for almost every $(kM, \lambda) \in K/M \times \mathfrak{a}^*$ we have (see the rightmost block in Fig. 2.5)

$$\begin{aligned}\Phi_o(\Lambda F)(kM, \lambda) &= (I \otimes \mathcal{F})(\mathcal{J}_o \Psi_o^* F)(kM, \lambda) \\ &= \frac{1}{\sqrt{w}\,|\mathbf{c}(\lambda)|}(I \otimes \mathcal{F})(\Psi_o^* F)(kM, \lambda) \\ &= \frac{1}{\sqrt{w}\,|\mathbf{c}(\lambda)|}\Phi_o F(kM, \lambda). \end{aligned} \tag{2.33}$$

The operator Λ intertwines the regular representation $\hat{\pi}$ as shown by the next proposition.

Proposition 2.26. *The subspace $\mathcal{E}$ is $\hat{\pi}$-invariant and for all $F \in \mathcal{E}$ and $g \in G$*

$$\hat{\pi}(g)\Lambda F = \Lambda\hat{\pi}(g)F. \tag{2.34}$$

Proof. We consider $F \in \mathcal{E}$, $g \in G$ and we prove that $\hat{\pi}(g)F \in \mathcal{E}$. By (2.25)

$$\hat{\pi}(g)F \circ \Psi_o(kM, a) = F \circ \Psi_{g^{-1}[o]}(g^{-1}\langle kM \rangle, a)$$

for almost every $(kM, a) \in K/M \times A$. Therefore, we have

$$\Psi_o^*(\hat{\pi}(g)F)(kM, a) = \Psi_{g^{-1}[o]}^* F(g^{-1}\langle kM\rangle, a)$$

and consequently by equation (2.31)

$$\begin{aligned}\Phi_o(\hat{\pi}(g)F)(kM,\lambda) &= (I \otimes \mathcal{F}_A)(\Psi_{g^{-1}[o]}^* F)(g^{-1}\langle kM\rangle, \lambda)\\ &= e^{(\rho - i\lambda)(A_{g^{-1}[o]}(o, g^{-1}\langle kM\rangle))}\Phi_o(F)(g^{-1}\langle kM\rangle, \lambda) \qquad (2.35)\end{aligned}$$

for almost every $(kM, \lambda) \in K/M \times \mathfrak{a}^*$. By equations (2.35), (2.24) and (2.23)

$$\begin{aligned}&\int_{K/M\times\mathfrak{a}^*} |\Phi_o(\hat{\pi}(g)F)(kM,\lambda)|^2 \frac{\mathrm{d}\nu^o(kM)\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2}\\ &= \int_{\mathfrak{a}^*}\int_{K/M} |\Phi_o(F)(g^{-1}\langle kM\rangle,\lambda)|^2 e^{2\rho(A_{g^{-1}[o]}(o,g^{-1}\langle kM\rangle))}\frac{\mathrm{d}\nu^o(kM)\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2}\\ &= \int_{K/M\times\mathfrak{a}^*} |\Phi_o F(kM,\lambda)|^2 e^{2\rho(A_{g^{-1}[o]}(o,kM))}\frac{\mathrm{d}\nu^{g^{-1}[o]}(kM)\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2}\\ &= \int_{K/M\times\mathfrak{a}^*} |\Phi_o F(kM,\lambda)|^2 \frac{\mathrm{d}\nu^o(kM)\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2} < +\infty\end{aligned}$$

and we conclude that $\hat{\pi}(g)F \in \mathcal{E}$. We next prove the intertwining property (2.34). We have already observed that, by Proposition 2.24, it is enough to prove that

$$\Phi_o(\hat{\pi}(g)\Lambda F) = \Phi_o(\Lambda\hat{\pi}(g)F)$$

for every $g \in G$ and $F \in \mathcal{E}$. By equations (2.35) and (2.33), for almost every $(kM, \lambda) \in K/M \times \mathfrak{a}^*$, we have the chain of equalities

$$\begin{aligned}\Phi_o(\hat{\pi}(g)\Lambda F)(kM,\lambda) &= e^{(\rho-i\lambda)(A_{g^{-1}[o]}(o,g^{-1}\langle kM\rangle))}\Phi_o(\Lambda F)(g^{-1}\langle kM\rangle,\lambda)\\ &= \frac{1}{\sqrt{w}\,|\mathbf{c}(\lambda)|} e^{(\rho-i\lambda)(A_{g^{-1}[o]}(o,g^{-1}\langle kM\rangle))}\Phi_o(F)(g^{-1}\langle kM\rangle,\lambda)\\ &= \frac{1}{\sqrt{w}\,|\mathbf{c}(\lambda)|}\Phi_o(\hat{\pi}(g)F)(kM,\lambda) = \Phi_o(\Lambda\hat{\pi}(g)F)(kM,\lambda),\end{aligned}$$

which proves the intertwining relation. □

The next result follows directly by Proposition 2.24 and equation (2.33).

Corollary 2.27. *For every $F \in \mathcal{E}$, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $F \in L^2_\flat(\Xi)$.*

Proof. By Proposition 2.24, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $\Phi_o(\Lambda F)$ satisfies Property $\sharp$. By (2.33) and since $\lambda \mapsto |\mathbf{c}(\lambda)|$ is W-invariant, $\Phi_o(\Lambda F)$ satisfies Property $\sharp$ if and only if $\Phi_o(F)$ satisfies Property $\sharp$, which is equivalent to $F \in L^2_\flat(\Xi)$. This concludes the proof. □

We are now in a position to prove our main result.

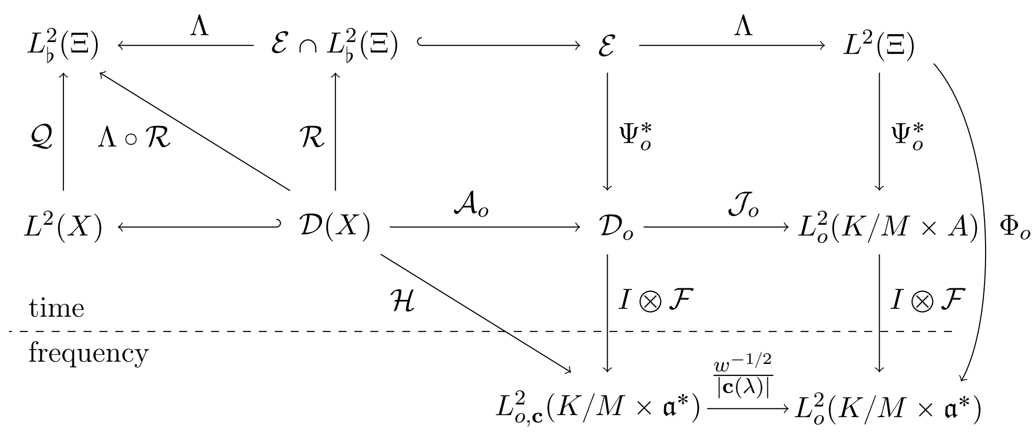

Figure 2.5: Spaces and operators that come into play in our construction.

Theorem 2.28. *The composite operator* $\Lambda\mathcal{R}$ *extends to a unitary operator*

$$\mathcal{Q}\colon L^2(X) \longrightarrow L^2_\flat(\Xi)$$

which intertwines the representations π *and* $\hat{\pi}$, *i.e.*

$$\hat{\pi}(g)\mathcal{Q} = \mathcal{Q}\pi(g), \qquad g \in G. \tag{2.36}$$

Theorem 2.28 implies that π and the restriction $\hat{\pi}|_{L^2_\flat(\Xi)}$ of $\hat{\pi}$ to $L^2_\flat(\Xi)$ are unitarily equivalent representations. Moreover, $\hat{\pi}|_{L^2_\flat(\Xi)}$ (and then $\hat{\pi}$) is not irreducible, too.

Proof. We first show that $\Lambda\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$ from $L^2(X)$ onto $L^2(\Xi)$. It might be useful to keep in mind see the leftmost block in Fig. 2.5. Let $f \in \mathcal{D}(X)$, by the Fourier Slice Theorem (2.30), the Plancherel formula(1.11) and the definition of $\mathcal{J}_o$ and Λ, we have that

$$\begin{aligned}
\|f\|^2_{L^2(X)} &= \|\mathcal{H}f\|^2_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)^\sharp} \\
&= \|(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{R}f))\|^2_{L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)^\sharp} \\
&= \int_{K/M\times\mathfrak{a}^*} |(I\otimes\mathcal{F})(\mathcal{J}_o\Psi_o^*(\mathcal{R}f))(kM,\lambda)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M\times\mathfrak{a}^*} |(I\otimes\mathcal{F})(\Psi_o^*(\Lambda\mathcal{R}f))(kM,\lambda)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M\times A} |\Psi_o^*(\Lambda\mathcal{R}f)(kM,a)|^2 \mathrm{d}\nu^o(kM)\mathrm{d}a \\
&= \|\Lambda\mathcal{R}f\|^2_{L^2(\Xi)}.
\end{aligned}$$

Hence, $\Lambda\mathcal{R}$ is an isometric operator from $\mathcal{D}(X)$ into $L^2(\Xi)$. Since $\mathcal{D}(X)$ is dense in $L^2(X)$, $\Lambda\mathcal{R}$ extends to a unique isometry from $L^2(X)$ onto the closure of $\mathrm{Ran}(\Lambda\mathcal{R})$ in $L^2(\Xi)$. We must show that $\Lambda\mathcal{R}$ has dense image in $L^2_\flat(\Xi)$. The inclusion $\mathrm{Ran}(\Lambda\mathcal{R}) \subseteq$

$L^2_\flat(\Xi)$ follows immediately from Corollary 2.25 and Corollary 2.27. Let $F \in L^2_\flat(\Xi)$ be such that $\langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} = 0$ for every $\mathcal{D}(X)$. By the Plancherel formula (1.11) and the Fourier Slice Theorem (2.30) we have that

$$\begin{aligned}
0 &= \langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} \\
&= \int_{K/M\times A} (F\circ\Psi_o)(kM,a)\overline{(\Lambda\mathcal{R}f\circ\Psi_o)(kM,a)}e^{2\rho(\log a)}\mathrm{d}\nu^o(kM)\mathrm{d}a \\
&= \int_{K/M\times A} (\Psi_o^*F)(kM,a)\overline{(\mathcal{J}_o\Psi_o^*(\mathcal{R}f))(kM,a)}\mathrm{d}\nu^o(kM)\mathrm{d}a \\
&= \int_{K/M\times\mathfrak{a}^*} \Phi_o(F)(kM,\lambda)\overline{(I\otimes\mathcal{F})(\mathcal{J}_o\Psi_o^*(\mathcal{R}f))(kM,\lambda)}\mathrm{d}\nu^o(kM)\mathrm{d}\lambda \\
&= \int_{K/M\times\mathfrak{a}^*} \Phi_o(F)(kM,\lambda)\overline{(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{R}f))(kM,\lambda)}\frac{\mathrm{d}\nu^o(kM)\mathrm{d}\lambda}{\sqrt{w}|\mathbf{c}(\lambda)|} \\
&= \int_{K/M\times\mathfrak{a}^*} \sqrt{w}|\mathbf{c}(\lambda)|\Phi_o(F)(kM,\lambda)\overline{\mathcal{H}f(kM,\lambda)}\frac{\mathrm{d}\nu^o(kM)\mathrm{d}\lambda}{w|\mathbf{c}(\lambda)|^2}.
\end{aligned}$$

For simplicity, we denote by ΘF the function on $K/M\times\mathfrak{a}^*$ defined as

$$\Theta F(kM,\lambda) = \sqrt{w}|\mathbf{c}(\lambda)|\Phi_o(F)(kM,\lambda), \qquad \text{a.e. } (kM,\lambda)\in K/M\times\mathfrak{a}^*.$$

Hence we have proved that $\langle\Theta F, \mathcal{H}f\rangle = 0$ for every $f\in\mathcal{D}(X)$. The next two facts follow immediately by Proposition 2.24. Since Φ_o is an isometry from $L^2(\Xi)$ into $L^2_o(K/M\times\mathfrak{a}^*)$, the function ΘF belongs to $L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)$. Further, since $F\in L^2_\flat(\Xi)$ and since $\lambda\mapsto|\mathbf{c}(\lambda)|$ is W-invariant, then $\Theta F\in L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)^\sharp$. By Theorem 2.16, $\mathcal{H}(\mathcal{D}(X))$ is dense in $L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)^\sharp$. Hence, $\Theta F = 0$ in $L^2_{o,\mathbf{c}}(K/M\times\mathfrak{a}^*)^\sharp$ and then $\Phi_o(F) = 0$ in $L^2_o(K/M\times\mathfrak{a}^*)$. Since Φ_o is an isometry from $L^2(\Xi)$ into $L^2_o(K/M\times\mathfrak{a}^*)$, then $F = 0$ in $L^2(\Xi)$. Therefore, $\overline{\mathrm{Ran}(\Lambda\mathcal{R})} = L^2_\flat(\Xi)$ and $\Lambda\mathcal{R}$ extends uniquely to a surjective isometry

$$\mathcal{Q}\colon L^2(X)\longrightarrow L^2_\flat(\Xi).$$

Observe that $\mathcal{Q}f = \Lambda\mathcal{R}f$ for every $f\in\mathcal{D}(X)$. The intertwining property (2.36) follows immediately from Proposition 2.23 and Proposition 2.26. □

Chapter 3

Radon transform on homogeneous trees

This chapter contains the results published in [7]. The purpose of the work is to write a unitarization result, that is a version of Theorem 1.35 and 1.38, in the setup of the horocyclic Radon transform on homogeneous trees.

The idea of considering homogeneous trees is motivated by the fact that they are a family of discrete spaces having a lot of similarities with the symmetric spaces of the noncompact type of rank 1 analyzed in the previous chapter. Among others, the definitions of boundary and horocycle are developed on the homogeneous trees. Consequently, the Radon transform is defined on horocycles. After the results of Chapter 2, a natural question is if the similarity of the two setting leads us to an analog result for the horocyclic Radon transform on homogeneous trees. Indeed, this setting, as the one in the previous chapter, does not satisfy the assumptions of [1].

The horocyclic Radon transform $\mathcal{R}$ on homogeneous trees was introduced by P. Cartier [14] and studied by A. Figà-Talamanca and M.A. Picardello [23], W. Betori, J. Faraut and M. Pagliacci [12], M. Cowling, S. Meda and A.G. Setti [18], J. Cohen, F. Colonna and E. Tarabusi [15], and A. Veca [52], to name a few. Some of the typical issues considered are inversion formulæ and range problems.

We consider the group G of the isometries of the homogeneous tree X and we show that the counting measure on X is G-invariant and horocycles Ξ are endowed with a G-invariant measure, too. The main result of our work is to provide a formula for the pseudo-differential operator Λ such that $\Lambda\mathcal{R}$ extends to a unitary operator $\mathcal{Q}\colon L^2(X)\colon L^2_\flat(\Xi)$, where even in this case $L^2_\flat(\Xi)$ keeps track of symmetries naturally satisfied by the structure of the Radon transform. Furthermore we show that $\mathcal{Q}$ intertwines the two quasi regular representation π and $\hat{\pi}$ of G on $L^2(X)$ and $L^2_\flat(\Xi)$, respectively.

Our approach is similar to the previous chapter, but the techniques and the ideas behind the results are sometimes very different. Also in this setup, horocycles can be parametrized by an element of the boundary Ω and an integer number (instead of an element of the Abelian subgroup A) w.r.t. every reference vertex $x \in X$. We keep track of these parametrizations with the family of functions

$$\Psi_v\colon \Omega \times \mathbb{Z} \to \Xi$$

defined in (3.3). Again, the key of our work lies in the freedom of changing the reference point, which permits us to relate the range of the Helgason-Fourier transform and $L^2_\flat(\Xi)$, the image of the unitarization $\mathcal{Q}$. This relation is highlighted in Proposition 3.9 which shows how these spaces are related by the Fourier transform of Ψ_o^*, the pull-back of Ψ_o. Given a Fourier multiplier $\mathcal{J}_o$ on functions defined on $\Omega \times \mathbb{Z}$, the main operator is $\Lambda = \Psi_o^{*-1}\mathcal{J}_o\Psi_o^*$. Also on the homogeneous trees, the definition of Λ can be seen to be independent of the choice of the reference vertex $o \in X$.

The chapter is organized in three sections. In Section 3.1, we present the main notions and the relevant results in the theory of homogeneous trees, presenting notions as boundary and horocycle. In Section 3.2, we introduce measures involved in our analysis, we present the group of automorphisms on the tree and its quasi regular representations, and then, we give a brief overview of the Helgason-Fourier transform. In Section 3.3, we recall the horocyclic Radon transform on homogeneous trees, we present its link with the Helgason-Fourier transform and we show its intertwining properties with quasi regular representations. Finally, we prove the unitarization theorem for the horocyclic Radon transform.

3.1 Homogeneous trees

In this section we recall the basic definitions and facts on homogeneous trees that will be used throughout, focusing on the space of horocycles. We refer again to [12], [18] and [22].

Section 3.1.1 contains the very basic definitions on homogeneous trees and fixes terminology used throughout. It has to be considered as a concise introduction for Chapter 4, too. Then we present the boundary Ω of a homogeneous tree in Section 3.1.2. It is one of the clearest point of contact with symmetric spaces of rank one, and in particular with the hyperbolic disk. Finally, we introduce the family of horocycles and their different parametrization through $\Omega \times \mathbb{Z}$ w.r.t. every possible reference vertex. We stress the similitude with the hyperbolic disk where horocycles are parametrized by the product of the boundary with the subgroup $A \simeq \mathbb{R}$.

3.1.1 Preliminaries

A *graph* is a pair $(X, \mathfrak{E})$, where X is the discrete set of *vertices* and $\mathfrak{E}$ is the family of *edges*, where an edge is a two-element subset of X. We often think of an edge as a segment joining two vertices. If two vertices are joint by a segment, that is they belong to the same edge, they are called *adjacent*. A *tree* is an undirected[1], connected, loop-free graph. In this chapter and in the following we are interested in homogeneous trees. A *q-homogeneous* tree is a tree in which each vertex has exactly $q + 1$ adjacent vertices. If $q \geqslant 1$, a q-homogeneous tree is infinite. From now on, we suppose $q \geqslant 2$ in order to exclude trivial cases, that is, segments and lines. Furthermore, with a slight abuse of notation, we shall often call the set of vertices a tree, implying the structure of edges which is totally clear on homogeneous trees.

[1] A graph is undirected if edges are unordered pairs.

Given $u, v \in X$ with $u \neq v$, we denote by $[u, v]$ the unique ordered t-uple $(x_0 = u, x_1, \ldots, x_{t-1} = v) \in X^t$, where $\{x_i, x_{i+1}\} \in \mathfrak{E}$ and all the x_i are distinct. We call $[u, v]$ a (finite) *t-chain* and we think of it as a path starting at u and ending at v or, equivalently, as the finite sequence of consecutive 2-chains $[u, x_1], [x_1, x_2], \ldots, [x_{t-2}, v]$. With slight abuse of notation, if $[u, v] = (x_0, \ldots, x_{t-1})$ we write $u, v, x_i \in [u, v]$ and $[u, v] = [u, x_i] \cup [x_i, v]$, $i \in \{1, \ldots, t-2\}$. In particular, if u and v are adjacent, both $[u, v], [v, u] \in X^2$ are oriented, unlike the edge $\{u, v\} \in \mathfrak{E}$ which is not. A homogeneous tree X carries a natural distance $d \colon X \times X \to \mathbb{N}$, where for every $u, v \in X$ the distance $d(u, v)$ is the number of 2-chains in the path $[u, v]$. Thus, paths minimize distance and are geodesics of the tree.

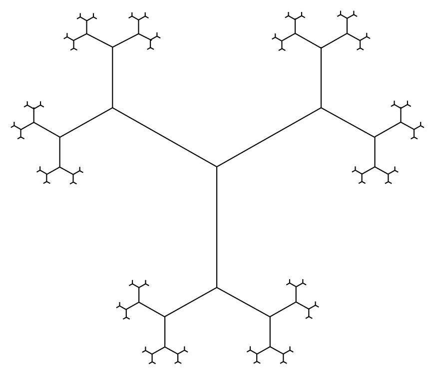

Figure 3.1: A portion of a 2-homogeneous tree

3.1.2 The boundary of a homogeneous tree

An *infinite chain* is an infinite sequence $(x_i)_{i \in \mathbb{N}}$ of vertices of X such that, for every $i \in \mathbb{N}$, $d(x_i, x_{i+1}) = 1$ and $x_i \neq x_{i+2}$. We denote by $c(X)$ the set of infinite chains on X. We say that two chains $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ are equivalent if there exist $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $x_i = y_{i+m}$ for every $i \geqslant N$ and, in such case, we write $(x_i)_{i \in \mathbb{N}} \sim (y_i)_{i \in \mathbb{N}}$. The *boundary* of X is the space Ω of equivalence classes $c(X)/\sim$. Observe that an infinite chain identifies uniquely a point of the boundary, which may be thought of as a point at infinity. In fact, it is well known [16] that a homogeneous tree of even order $q+1$ can be isometrically embedded in the unit disk, the latter endowed with its hyperbolic metric, in such a way that the limit points of infinite chains correspond a.e. to the points of the unit circle, the topological boundary of the unit disk.

We denote by p the canonical projection of $c(X)$ onto Ω. For $v \in X$ and $\omega \in \Omega$ we write $[v, \omega)$ for the unique chain $(x_i)_{i \in \mathbb{N}}$ starting at v, i.e. $x_0 = v$, and "pointing at" the boundary point ω, i.e. $p((x_i)_{i \in \mathbb{N}}) = \omega$. Furthermore, given $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, we denote by (ω_1, ω_2) the unique infinite sequence of (distinct) vertices $(x_i)_{i \in \mathbb{Z}}$ such that $(x_{-i})_{i \in \mathbb{N}} \in \omega_1$ and $(x_i)_{i \in \mathbb{N}} \in \omega_2$, and we call it a doubly infinite chain. The boundary Ω is endowed with the topology (independent of the reference point v) generated by the open sets

$$\Omega_v(u) = \{\omega \in \Omega : u \in [v, \omega)\}, \qquad u \in X.$$

With this topology, Ω is a compact topological space. For later use, we remark that in every class $\omega \in \Omega$, there is a unique infinite chain $[v, \omega)$ starting at v, and we denote by

$$\Gamma_v = \{[v, \omega) : \omega \in \Omega\}.$$

the set of all infinite chains starting at v. Clearly Γ_v and Ω may be identified.

3.1.3 Horocycles

A 2-chain $[v, u]$ is said to be positively oriented with respect to $\omega \in \Omega$ if $u \in [v, \omega)$, otherwise we say that $[v, u]$ is negatively oriented.

For $\omega \in \Omega$ and $v, u \in X$, we denote by $\kappa_\omega(v, u) \in \mathbb{Z}$ the so-called *horocyclic index* of v and u w.r.t. ω, namely the number of positively oriented (w.r.t. ω) 2-chains in $[v, u]$ minus the number of negatively oriented (w.r.t. ω) 2-chains in $[v, u]$. Horocyclic index $\kappa_\omega(v, \cdot)$ on the tree is the analog of the function $A_o(\cdot, kM)$ on symmetric spaces introduced in Definition 2.8, indeed,as we will see below, in both the cases horocycles can be outlined as counterimages of an integer or an element of the abelian subgroup A through $\kappa_\omega(v, \cdot)$ and $A_o(\cdot, kM)$, respectively. An easy idea of the geometric meaning of the horocyclic index is given by Figure 2.2 in which the tree is subdivided in layers from the value of the horocyclic index w.r.t. a point at infinity which lies above.

Clearly, $|\kappa_\omega(v, u)| \leqslant d(v, u)$. It is easy to verify that, for every $v, u, x \in X$ and for every $\omega \in \Omega$,

$$\kappa_\omega(v, x) = \kappa_\omega(v, u) + \kappa_\omega(u, x). \tag{3.1}$$

Furthermore, we have the following result, which relates the definition of horocyclic index that we adopt with the one presented in [18], which will be useful in the following.

Proposition 3.1. *Let $v \in X$ and $\omega \in \Omega$. If $[v, \omega) = (x_i)_{i \in \mathbb{N}}$, then for every $x \in X$*

$$\kappa_\omega(v, x) = \lim_{i \to \infty} (i - d(x, x_i)).$$

Proof. We fix $x \in X$ and we observe that

$$\lim_{i \to \infty} (i - d(x, x_i)) = \lim_{i \to \infty} (d(v, x_i) - d(x, x_i)).$$

Since $[v, \omega) \sim [x, \omega)$, then there exists $N \in \mathbb{N}$ such that $x_N \in [x, \omega)$ and $x_{N-1} \notin [x, \omega)$, with the understanding that if $v \in [x, \omega)$ then $N = 0$. Thus, for all $i \geqslant N$

$$d(v, x_i) - d(x, x_i) = d(v, x_N) - d(x, x_N)$$

and then

$$\lim_{i \to \infty} (d(v, x_i) - d(x, x_i)) = d(v, x_N) - d(x, x_N).$$

Furthermore, $[v, x] = [v, x_N] \cup [x_N, x]$, where $[v, x_N]$ is the union of positively oriented 2-chains and $[x_N, x]$ is the union of negatively oriented 2-chains. Hence,

$$\kappa_\omega(v, x) = d(v, x_N) - d(x, x_N) = \lim_{i \to \infty} (d(v, x_i) - d(x, x_i)) = \lim_{i \to \infty} (i - d(x, x_i))$$

and this concludes the proof. $\square$

We are now in a position to introduce the horocycles.

Definition 3.2. For $\omega \in \Omega$, $v \in X$ and $n \in \mathbb{Z}$, the *horocycle* tangent to ω of index n with respect to the vertex v is the subset of X defined as

$$h^v_{\omega,n} = \{x \in X : \kappa_\omega(v,x) = n\}.$$

We denote by Ξ the set of horocycles.

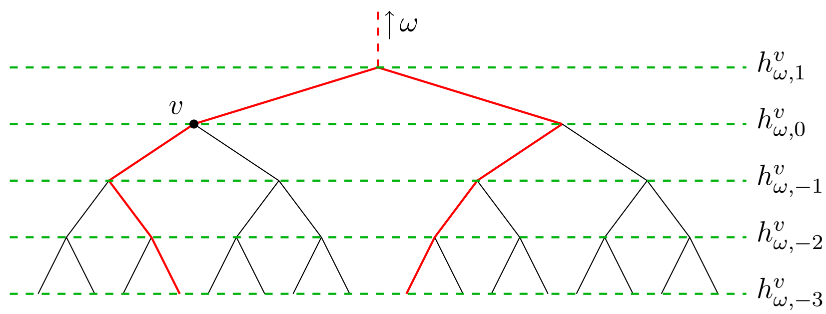

Figure 3.2: A part of a 2-homogeneous tree. It is possible to see a relation with Figure 2.1. The bundle of (red) parallel geodesics here is the family of all the geodesics "ending" in $\omega \in \Omega$; we represent only two of them. The tree is covered by the family of horocycles "tangent" to ω which here are represented as unions of vertices lying on horizontal layers (dashed green lines). Actually, every layer is just a portion of a (infinite) horocycle.

It follows immediately from (3.1) that for every $v, u \in X$, $n \in \mathbb{Z}$ and $\omega \in \Omega$

$$h^v_{\omega,n} = h^u_{\omega,n+\kappa_\omega(u,v)}. \tag{3.2}$$

Hence the mapping $(v,\omega,n) \mapsto h^v_{\omega,n}$ is not injective and so Ξ is not well parametrized by $X \times \Omega \times \mathbb{Z}$. However, for fixed $v \in X$, the map $(\omega,n) \mapsto h^v_{\omega,n}$ is actually bijective, so that Ξ may be identified with $\Omega \times \mathbb{Z}$. Formally, for every $v \in X$, there is a bijection

$$\Psi_v \colon \Omega \times \mathbb{Z} \to \Xi, \qquad \Psi_v(\omega,n) = h^v_{\omega,n} \tag{3.3}$$

and, for every fixed $\omega \in \Omega$, X can be covered disjointly as

$$X = \bigcup_{n\in\mathbb{Z}} h^v_{\omega,n}.$$

This is clear also through Figure 3.2, where we fix the point at infinity and the integer element outlines the "height" of the horocylce w.r.t. a reference vertex, $v \in X$.

By equality (3.2), for each pair of vertices u, $v \in X$

$$\Psi_u^{-1} \circ \Psi_v(\omega,n) = (\omega, n + \kappa_\omega(u,v)).$$

The topology that Ξ inherits as product of Ω and $\mathbb{Z}$ is proved to be independent of the choice of $v \in X$.

At this point, the analogy with symmetric spaces can be highlighted. Horocycles are parametrized by $\Omega \times \mathbb{Z}$ in different way, namely for every $v \in X$ the function Ψ_v corresponds to a different parametrization. The notation for the parametrization is the same of Chapter 2, the parameters of course are not. Although the correspondence of the boundary of, say, the disk with Ω is clear, the role of the subgroup $A \simeq \mathbb{R}$ of $\mathrm{SU}(1,1)$ is played by $\mathbb{Z}$. This is one of the reason for which homogeneous trees can be seen as the discrete counterpart of hyperbolic disk and, in general, symmetric spaces of rank one.

3.2 Analysis on homogeneous trees

The aim of this section is to describe the analysis on homogeneous trees, from the measures involved to the Fourier transform defined on X. Standard references for these are [12], [18] and [22].

We start by showing that the machinery of dual pairs devised by Helgason (see Section 1.3.1) is applicable to homogeneous trees and horocycles. In particular both the homogeneous tree X and the family of horocycles Ξ are homogeneous spaces of the group G of isometries on X and carry G-invariant measure. For every $x \in X$, we endow the (compact) boundary Ω with a K_x-invariant (probability) measure, where K_x is the isotropy subgroup of G at x. As on the boundary of symmetric spaces, we analyze the relation between them. Even in this case, the possibility of changing reference point is crucial; in particular it helps in the expression of the ranges of Helgason-Fourier and Radon transforms. Section 3.2.3 is devoted to a brief overview of the Helgason-Fourier transform. Finally, in Section 3.2.4 we give the definitions of the quasi regular representations π and $\hat{\pi}$ on $L^2(X)$ and $L^2(\Xi)$, respectively, highlighting the fact that π is not irreducible.

3.2.1 Group actions

From now on, we fix a vertex $o \in X$. As in the previous chapter, this is not a privileged point because every vertex has the same properties in a homogeneous tree. Fixing o will avoid confusion when in the following we would like to stress the changing of the reference point. Furthermore, we denote $|x| = d(o,x)$ for every $x \in X$.

Let G be the group of isometries on X, that is the group of bijections $g\colon X \to X$ which preserve the distance d. The group G is unimodular and locally compact, and acts transitively on X by the action

$$(g,x) \longmapsto g[x] := g(x), \quad g \in G.$$

We denote by K the stability subgroup at o. It turns out that K is a maximal compact subgroup of G and under the canonical bijection $gK \mapsto g[o]$ we have the identification $X \simeq G/K$.

The group G acts on the boundary as well. Indeed, it is easy to see that if $g \in G$ and $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$, then $(g[x_i])_{i\in\mathbb{N}} \sim (g[y_i])_{i\in\mathbb{N}}$, so that the transitive action of G on X induces a transitive action of G on Ω. Indeed, if $(x_i)_{i\in\mathbb{N}} \in c(X)$, then $(g[x_i])_{i\in\mathbb{N}} \in c(X)$ as well, because

$$d(g[x_i], g[x_{i+1}]) = d(x_i, x_{i+1}) = 1$$

and $g[x_i] \neq g[x_{i+2}]$ since g is an isometry. Furthermore, $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ implies that there exist $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $d(g[x_i], g[y_{i+m}]) = d(x_i, y_{i+m}) = 0$ for every $i \geqslant N$ and then $(g[x_i])_{i\in\mathbb{N}} \sim (g[y_i])_{i\in\mathbb{N}}$. Precisely, the group G acts on Ω by the action

$$(g, \omega) \longmapsto g\langle\omega\rangle := p((g[x_i])_{i\in\mathbb{N}}), \qquad \omega = p((x_i)_{i\in\mathbb{N}}).$$

This, in turn, induces a transitive action of K on the set Γ_o of infinite chains starting at o by means of

$$(k, [o, \omega)) \longmapsto [o, k\langle\omega\rangle), \quad \omega \in \Omega.$$

We fix $\omega_0 \in \Omega$ and we denote by K_{ω_0} the stabilizer of $[o, \omega_0)$ in K, so that $\Gamma_o \simeq K/K_{\omega_0}$.

The group G of isometries of X acts transitively also on the space Ξ of horocycles through the action on vertices because the G-action maps horocycles in themselves. Indeed, if $\xi \in \Xi$, $\xi = h^v_{\omega,n}$, with $v \in X$, $\omega \in \Omega$, $n \in \mathbb{N}$ and $[v, \omega) = (x_i)_{i\in\mathbb{N}}$, then for every $g \in G$

$$\begin{aligned} g[\xi] = \{g[x] : x \in X,\ \kappa_\omega(v, x) = n\} &= \{g[x] : x \in X,\ \lim_{i\to\infty}(i - d(x, x_i)) = n\} \\ &= \{x \in X : \lim_{i\to\infty}(i - d(g^{-1}[x], x_i)) = n\} \\ &= \{x \in X : \lim_{i\to\infty}(i - d(x, g[x_i])) = n\} \\ &= \{x \in X : \kappa_{g\langle\omega\rangle}(g[v], x) = n\} \\ &= h^{g[v]}_{g\langle\omega\rangle,n}, \end{aligned} \tag{3.4}$$

by Proposition 3.1. Basically, we proved that

$$\kappa_\omega(v, x) = \kappa_{g\langle\omega\rangle}(g[v], g[x]). \tag{3.5}$$

Therefore, from (3.4), G acts transitively on Ξ by

$$(g, h^v_{\omega,n}) \longmapsto g.h^v_{\omega,n} := h^{g[v]}_{g\langle\omega\rangle,n}.$$

Fix next $\omega_0 \in \Omega$ and consider the horocycle

$$\xi_0 = h^o_{\omega_0,0} = \{x \in X : \kappa_{\omega_0}(o, x) = 0\}.$$

If $[o, \omega_0) = (x_i)_{i\in\mathbb{N}}$, then

$$g.\xi_0 = \{x \in X : \lim_{i\to\infty}(i - d(x, g[x_i])) = 0\}.$$

Hence, the isotropy subgroup at ξ_0 is $H = \bigcup_{j=0}^{\infty} H_j$, where H_j is the subgroup of isometries fixing the sub-path $[x_j, \omega_0) \in c(X)$. Therefore, $\Xi \simeq G/H$. Observe that H is the isotropy subgroup of G at $h^o_{\omega_0,n}$ for every $n \in \mathbb{Z}$. Thus, by (3.2), H is the isotropy subgroup of G at every horocycle tangent to ω_0, namely at $h^v_{\omega_0,n}$ for every $n \in \mathbb{Z}$ and $v \in X$.

Let $\tau \in G$ be a one-step translation along (ω_1, ω_0), with $\omega_1 \in \Omega\backslash\{\omega_0\}$, where ω_0 is as in the definition of H (see [22] for further details on the one-step translations in G). Let $v \in (\omega_0, \omega_1)$ and assume $\tau(v) \in [v, \omega_0)$. Furthermore, denote by A the subgroup of G generated by the powers of τ. It is easy to see that the group A acts on H by conjugation. Indeed, for every $m \in \mathbb{Z}$ and $g \in H$, g stabilizes every $h^v_{\omega_0,n}$ and hence

$$\tau^m g \tau^{-m}.\,\xi_0 = \tau^m g.\,h^{\tau^{-m}[o]}_{\omega_0,0} = \tau^m.\,h^{\tau^{-m}[o]}_{\omega_0,0} = h^{\tau^m\tau^{-m}[o]}_{\omega_0,0} = \xi_0,$$

where we use that $\tau^m \cdot \omega_0 = \tau^{-m} \cdot \omega_0 = \omega_0$. It has been proved in [52] that the resulting semidirect product $H \rtimes A$ has modular function

$$\Delta(h, \tau^m) = q^{-m}.$$

With slight abuse of notation, we write $\Delta^{-\frac{1}{2}}$ for the function $\Delta^{-\frac{1}{2}} : \mathbb{Z} \to \mathbb{R}_+$ defined by

$$\Delta^{-\frac{1}{2}}(n) := q^{\frac{n}{2}},$$

and in what follows, the same notation is used for its trivial extension to $\Omega \times \mathbb{Z}$. It is straightforward to see that $n \mapsto q^{\frac{n}{2}}$ plays the role of $a \mapsto e^{\rho(\log a)}$ in (2.20).

3.2.2 Measures

We endow X with the counting measure $\mathrm{d}x$ which is trivially G-invariant, and we denote by $L^2(X)$ the Hilbert space of square integrable functions with respect to $\mathrm{d}x$.

As far as Ω is concerned, recall that Ω is identified with Γ_o on which K acts transitively. Therefore, Γ_o admits a unique K-invariant probability measure μ^o. We denote by ν^o the measure on Ω obtained as the push-forward of μ^o by means of the canonical projection $p_{|\Gamma_o} : \Gamma_o \to \Omega$. It has been shown in [22] that

$$\nu^o(\Omega_o(u)) = \frac{q}{(q+1)q^{d(o,u)}}, \qquad u \neq o.$$

The measure ν^o is G-quasi invariant and, by definition, the Poisson kernel $p_o(g, \omega)$ is the associated Radon-Nikodym derivative $d\nu^o(g^{-1}\langle\omega\rangle)/d\nu^o(\omega)$, i.e. for every F in $L^1(\Omega, \nu^o)$

$$\int_\Omega F(g^{-1}\langle\omega\rangle)\mathrm{d}\nu^o(\omega) = \int_\Omega F(\omega)p_o(g^{-1}, \omega)\mathrm{d}\nu^o(\omega), \qquad g \in G. \tag{3.6}$$

It is possible to prove [22] that

$$p_o(g, \omega) = q^{\kappa_\omega(o, g[o])}.$$

Since ν^o is K-invariant, we may write $p_o(gK, \omega)$ instead of $p_o(g, \omega)$. For every other choice of the reference vertex $v \in X$ the analog objects K_v (the isotropy subgroup of G at v), Γ_v, μ^v, ν^v, p_v can be introduced. It turns out that the measure ν^o is absolutely continuous with respect to ν^v. Precisely

$$\int_\Omega F(\omega)\mathrm{d}\nu^o(\omega) = \int_\Omega F(\omega)q^{\kappa_\omega(v,o)}\mathrm{d}\nu^v(\omega),$$

for every $F \in L^1(\Omega, \nu^o)$. Therefore, we can endow the boundary Ω with infinitely many measures which are absolutely continuous with respect to each other.

In order to adequately describe the measure on Ξ relative to which we form the Lebesgue spaces $L^1(\Xi)$ and $L^2(\Xi)$, we need the parametrization (3.3). The idea is to define compatible measures on $\Omega \times \mathbb{Z}$ and Ξ in the sense that the natural pull-back of functions induced by the mapping $\Psi_v\colon \Omega \times \mathbb{Z} \to \Xi$ induces a unitary operator Ψ_v^* of the corresponding L^2 spaces. To this end, we consider the measure on $\mathbb{Z}$ with density q^n with respect to the counting measure $\mathrm{d}n$. We fix $v \in X$ and endow Ξ with the measure λ obtained as the push-forward of the measure $\nu^v \otimes q^n \mathrm{d}n$ on $\Omega \times \mathbb{Z}$ by means of the map Ψ_v, i.e.

$$\lambda = \Psi_{v*}(\nu^v \otimes q^n \mathrm{d}n),$$

which is independent of the choice of the vertex v (see [12]). We denote by $L^1(\Xi)$ and $L^2(\Xi)$ the spaces of absolutely integrable functions and square integrable functions with respect to λ, respectively. Thus, by definition of λ, for every $F \in L^1(\Xi)$

$$\int_{\Xi} F(\xi)\mathrm{d}\lambda(\xi) = \int_{\Omega\times\mathbb{Z}} (F \circ \Psi_v)(\omega, n) q^n \mathrm{d}\nu^v(\omega)\mathrm{d}n.$$

It is easy to verify that λ is G-invariant.

For every $v \in X$, let $L^2_v(\Omega \times \mathbb{Z})$ be the space of square integrable functions w.r.t. the measure $\nu^v \otimes \mathrm{d}n$. For every $F \in L^2(\Xi)$, we denote by $\Psi_v^* F$ the $(L^2(\Xi), L^2_v(\Omega \times \mathbb{Z}))$-pull-back of F by Ψ_v, which involves the function $\Delta^{-\frac{1}{2}}$ introduced in the previous subsection, namely

$$\Psi_v^* F(\omega, n) = (\Delta^{-\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, n),$$

for almost every $(\omega, n) \in \Omega \times \mathbb{Z}$. Clearly, Ψ_v^* is a unitary operator from $L^2(\Xi)$ into $L^2_v(\Omega \times \mathbb{Z})$. Indeed, for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}
\int_{\Omega\times\mathbb{Z}} |\Psi_v^* F(\omega, n)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n &= \int_{\Omega\times\mathbb{Z}} \left|(\Delta^{-\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, n)\right|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n \\
&= \int_{\Omega\times\mathbb{Z}} |(F \circ \Psi_v)(\omega, n)|^2 q^n \mathrm{d}\nu^v(\omega)\mathrm{d}n \\
&= \int_{\Xi} |F(\xi)|^2 \mathrm{d}\lambda(\xi) = \|F\|^2_{L^2(\Xi)}
\end{aligned}$$

and then Ψ_v^* is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega \times \mathbb{Z})$. Surjectivity is also clear.

3.2.3 The Helgason-Fourier transform on homogeneous trees

The Helgason-Fourier transform can be defined on homogeneous trees (see [18], [22], [23]) in analogy with the setup of symmetric spaces, see Section 2.2.2 and [37]. We briefly recall its definition and its main features. We put $T = 2\pi/\log(q)$, $\mathbb{T} = \mathbb{R}/T\mathbb{Z} \simeq [0, T)$ and we denote by $\mathrm{d}t$ the normalized Lebesgue measure on $\mathbb{T}$.

Definition 3.3. The *Helgason-Fourier transform* of $f \in C_c(X)$ is the function $\mathcal{H}f : \Omega \times \mathbb{T} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{H}f(\omega, t) = \sum_{x\in X} f(x) q^{(\frac{1}{2}+it)\kappa_\omega(o,x)}, \qquad (\omega, t) \in \Omega \times \mathbb{T}. \tag{3.7}$$

As the Euclidean Fourier transform, the Helgason-Fourier transform extends to a unitary operator on $L^2(X)$ (see [22], [23]). The Plancherel measure involves a version of the Harish-Chandra $\mathbf{c}$-function inspired by the symmetric space construction (see Section 2.2.2 and for a concise exposition [30]), namely the meromorphic function

$$\mathbf{c}(z) = \frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \frac{q^{1-z} - q^{z-1}}{q^{\frac{1}{2}-z} - q^{z-\frac{1}{2}}}, \qquad z \in \mathbb{C} \text{ with } q^{2z-1} \neq 1.$$

We put

$$c_q = \frac{q}{2(q+1)} \tag{3.8}$$

and we denote by $L^2_{\mathbf{c}}(\Omega \times \mathbb{T})^\sharp$ the space of functions F in $L^2_{\mathbf{c}}(\Omega \times \mathbb{T})$, the space of square integrable functions on $\Omega \times \mathbb{T}$ w.r.t. the measure $c_q\, |\mathbf{c}(1/2 + it)|^{-2}\, \mathrm{d}\nu^o \mathrm{d}t$, satisfying the symmetry

$$\int_\Omega p_o(x,\omega)^{\frac{1}{2}-it} F(\omega,t) \mathrm{d}\nu^o(\omega) = \int_\Omega p_o(x,\omega)^{\frac{1}{2}+it} F(\omega,-t) \mathrm{d}\nu^o(\omega), \tag{3.9}$$

for every $x \in X$ and for almost every $t \in \mathbb{T}$.

As in the Euclidean case and in Theorem 2.16, the Helgason-Fourier transform extends to a unitary operator on $L^2(X)$.

Theorem 3.4 ([22])**.** *The Helgason-Fourier transform $\mathcal{H}$ extends to a unitary isomorphism $\mathscr{H}$ from $L^2(X)$ onto $L^2_{\mathbf{c}}(\Omega \times \mathbb{T})^\sharp$.*

3.2.4 Representations

Here we present the two representations of G which we are interesting in: the one on $L^2(X)$ and the one on $L^2(\Xi)$ which we denote by π and $\hat{\pi}$, respectively. Furthermore, we show that π is not irreducible. The representation $\hat{\pi}$ is not irreducible, too, but we do not show it since, as a consequence of our main result, Theorem 3.14, we have that π is unitary equivalent to a subrepresentation of $\hat{\pi}$.

3.2.4.1 Quasi regular representation on $L^2(X)$

Recall that X is endowed with the counting measure $\mathrm{d}x$ which is trivially G-invariant. Thus, the group G acts on $L^2(X)$ by the quasi regular representation $\pi\colon G \longrightarrow \mathcal{U}(L^2(X))$ defined by

$$\pi(g)f(x) := f(g^{-1}[x]), \qquad f \in L^2(X),\ g \in G,$$

where $\mathcal{U}(L^2(X))$ denotes the group of unitary operators of $L^2(X)$.

The representation π is not directly treated in the analysis on G: the regular representation on $L^2(G)$ is preferred. For this reason we have not found a clear statement about the irriducibility of π. We present now a short proof that π is not irreducible.

We use the characterization of irreducibility given by Lemma 1.8, for which the representation π is not irreducible if and only if there exist two functions h_1, $h_2 \in$

$L^2(X)\backslash\{0\}$ such that $\langle h_1, \pi(\,\cdot\,)h_2\rangle_{L^2(X)}$ vanishes identically on G. We start by proving that for $f \in L^2(X)$ and $g \in G$, the action of G on X in frequency reads

$$\mathscr{H}(\pi(g)f)(\omega,t) = q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,g[o])}\mathscr{H}f(g^{-1}\langle\omega\rangle,t), \quad (\omega,t)\in\Omega\times\mathbb{T}. \tag{3.10}$$

By the density of $C_c(X)$ in $L^2(X)$, it is sufficient to prove (3.10) for $f \in C_c(X)$. Indeed,

$$\begin{aligned}\mathcal{H}(\pi(g)f)(\omega,t) &= \sum_{x\in X} f(g^{-1}[x])q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,x)}\\ &= \sum_{x\in X} f(x)q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,g[x])}\\ &= q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,g[o])}\sum_{x\in X} f(x)q^{\left(\frac{1}{2}+it\right)\kappa_\omega(g[o],g[x])}\\ &= q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,g[o])}\sum_{x\in X} f(x)q^{\left(\frac{1}{2}+it\right)\kappa_{g^{-1}\langle\omega\rangle}(o,x)}\\ &= q^{\left(\frac{1}{2}+it\right)\kappa_\omega(o,g[o])}\mathcal{H}f(g^{-1}\langle\omega\rangle,t),\end{aligned}$$

where we used (3.1) and (3.5).

Now we want to find two not vanishing functions of $L^2(X)$ whose corresponding coefficient vanishes identically on G. We introduce the subset

$$A := \Omega\times\left[\frac{T}{4},\frac{3T}{4}\right] \subseteq \Omega\times\mathbb{T} = \Omega\times[0,T).$$

Take $f \in L^2(X)$. We know that $\mathscr{H}f \in L^2_{\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$ and if we multiply $\mathscr{H}f$ by the characteristic function of A or A^c it remains in $L^2_{\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$ since (3.9) is true if the function is restricted to a symmetric subset. We therefore choose

$$h_1 = \mathscr{H}^{-1}(\chi_A\mathscr{H}f), \quad h_2 = \mathscr{H}^{-1}(\chi_{A^c}\mathscr{H}f).$$

Observe that the coefficient associated to h_1 and h_2 is

$$\begin{aligned}\langle h_1,\pi(g)h_2\rangle_{L^2(X)} =&\langle\chi_A\mathscr{H}f,\mathscr{H}(\pi(g)h_2)\rangle_{L^2_{\mathbf{c}}(\Omega\times\mathbb{T})^\sharp}\\ =&\int_{\Omega\times\mathbb{T}}\chi_A(\omega,t)\chi_{A^c}(g^{-1}\langle\omega\rangle,t)\mathscr{H}f(\omega,t)\overline{\mathscr{H}f(g^{-1}\langle\omega\rangle,t)}\\ &\times q^{\left(\frac{1}{2}-it\right)\kappa_\omega(o,g[o])}\frac{c_q\mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} = 0.\end{aligned}$$

Finally by Lemma 1.8, we conclude that π is not irreducible.

3.2.4.2 Quasi regular representation on $L^2(\Xi)$

Similarly, since λ is G-invariant, the group G acts on $L^2(\Xi)$ by the quasi regular unitary representation $\hat{\pi}\colon G \longrightarrow \mathcal{U}(L^2(\Xi))$ defined by

$$\hat{\pi}(g)F(\xi) := F(g^{-1}.\xi), \qquad F\in L^2(\Xi),\ g\in G.$$

3.3 Unitarization of Radon transform

In this last section, we present the horocyclic Radon transform and the main operators which come into play in our main result: the extension of Theorem 1.35, presented in Theorem 3.14.

Below we recall the definition of the horocyclic Radon transform on homogeneous trees and its fundamental properties. As already mentioned, the horocyclic Radon transform is precisely the Radon transform *à la* Helgason relative to the dual pair (X, Ξ). As for symmetric spaces, the case of homogeneous trees is not covered by the general setup considered in [1] since the quasi regular representation π of G on $L^2(X)$ is not irreducible. For this reason, we can not apply the results presented in Section 1.3.2 in order to obtain a unitarization theorem and we therefore adopt an approach which mimics the one used in [46] and [6] in the case of the polar and the affine Radon transforms, respectively.

We recall a version of the Fourier slice theorem for homogeneous trees (Proposition 3.7). Its role is to relate the Helgason-Fourier transform with the horocyclic Radon transform. As in Chapter 2, the key of the proof of the unitarization result can be outlined in Proposition 3.9 and then in the role of Φ_o. Roughly speaking, Proposition 3.9 highlights the relation between the range of the Helgason-Fourier tranform and the range of the Radon transform; more precisely, the range of the unitarization of the Radon transform, $L^2_\flat(\Xi)$. This space is not the full L^2 space of Ξ but it keeps track of the symmetries which are naturally satisfied by the Radon transform.

Section 3.3.2 contains our main result, Theorem 3.14. The operator Λ involved in the unitarization of the Radon transform is the conjugation by Ψ_o of a Fourier multiplier $\mathcal{J}_o$ on $\mathbb{Z}$. The operator $\mathcal{J}_o$, as Ψ_o, depends on the choice of the reference vertex o. Anyway, it is important to stress that Λ is independent of the reference vertex. The intertwining property of $\mathcal{Q}$ follows from the fact that the Radon transform (Proposition 3.8) and operator Λ (Proposition 3.12) are invariant under the action of G.

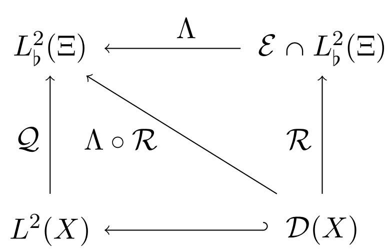

Figure 3.3: The operator Λ is defined on the dense subset $\mathcal{E} \subseteq L^2(\Xi)$ and maintains Property $\flat$. Its precomposition with $\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$ on $L^2(X)$.

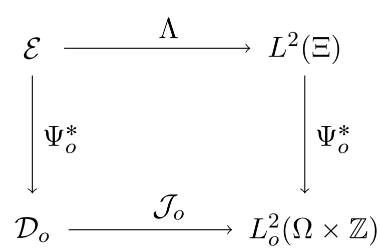

Figure 3.4: The Fourier multiplier $\mathcal{J}_v$ is defined on the dense subset $\mathcal{D}_v \subseteq L^2_o(\Omega \times \mathbb{Z})$ and is "transported" on functions defined on Ξ by the conjugation with Ψ_v^*.

3.3.1 The horocyclic Radon transform

The definition of the horocyclic Radon transform is quite immediate. From the point of view of Helgason, the crucial fact is that the counting measure restricted to the horocycle ξ_0 is H-invariant, where H is the isotropy subgroup of G at ξ_0.

Definition 3.5. The horocyclic Radon transform $\mathcal{R}f$ of a function $f \in C_c(X)$ is the map $\mathcal{R}f : \Xi \to \mathbb{C}$ defined by

$$\mathcal{R}f(\xi) = \sum_{x \in \xi} f(x).$$

We recall that for every $v \in X$ there exists a bijection $\Psi_v \colon \Omega \times \mathbb{Z} \to \Xi$ given by $(\omega, n) \mapsto h^v_{\omega,n}$ and we shall write $\mathcal{R}_v f = \mathcal{R}f \circ \Psi_v$.

Definition 3.6. The Abel transform $\mathcal{A}f$ of a function $f \in C_c(X)$ is the map $\mathcal{A}f : \Omega \times \mathbb{Z} \to \mathbb{C}$ defined by

$$\mathcal{A}f(\omega, n) = \Psi_o^*(\mathcal{R}f)(\omega, n) = (\Delta^{-\frac{1}{2}} \cdot \mathcal{R}_o f)(\omega, n).$$

We need the Fourier transform on $L^2(\mathbb{Z})$. As we have seen in Section 1.1.3, the definition of the Fourier transform on $\mathbb{Z}$ strongly depends on the choice we made on the parametrization of $\mathbb{T}$. In fact, such parametrization determines the period of the Fourier transform. The choice which better suits our needs is

$$[0, T] \ni t \mapsto e^{-i\frac{2\pi}{T}t} = q^{-it} \in \mathbb{T}.$$

We have already discussed the choice to consider the negative power in Section 1.1.3, while the decision to express the parametrization as power of q is determined by the expression of the Helgason-Fourier transform (3.7).

We denote by L^2_T the space of T-periodic functions f on $\mathbb{R}$ such that

$$\|f\|^2_{L^2_T} = \int_0^T |f(t)|^2 \mathrm{d}t < +\infty.$$

Let $s \in L^2(\mathbb{Z})$, the Fourier transform $\mathcal{F}s$ of s is then defined as the T-periodic function having Fourier coefficients $(s(n))_{n\in\mathbb{Z}}$. Precisely,

$$\mathcal{F}s = \sum_{n \in \mathbb{Z}} s(n) q^{in\cdot},$$

where the series converges in L^2_T. The Plancherel formula (1.11), also called Parseval identity in this case, reads

$$\|\mathcal{F}s\|^2_{L^2_T} = \sum_{n \in \mathbb{Z}} |s(n)|^2.$$

Furthermore, if $s \in L^1(\mathbb{Z})$, for almost every $t \in \mathbb{T}$

$$\mathcal{F}s(t) = \sum_{n \in \mathbb{Z}} s(n) q^{int}.$$

We are now ready to state the result which relates the Helgason-Fourier transform with the horocyclic Radon transform. For the reader's convenience, we include the proof.

Proposition 3.7 (Fourier slice theorem, version I, [13, 18])**.** *For every $f \in C_c(X)$ and $\omega \in \Omega$, $\mathcal{A}f(\omega,\cdot) \in L^1(\mathbb{Z})$ and*

$$(I \otimes \mathcal{F})\mathcal{A}f(\omega,t) = \mathcal{H}f(\omega,t), \tag{3.11}$$

for almost every $t \in \mathbb{T}$.

We refer to Proposition 3.7 as the Fourier Slice Theorem for the horocyclic Radon transform in analogy with the polar Radon transform, see Theorem 1.34, and with the symmetric spaces case, see Theorem 2.22.

Let $f \in C_c(X)$ and $v \in X$. By Parseval identity and Proposition 3.7 we have that

$$\begin{aligned}\int_{\Xi} |\mathcal{R}f(\xi)|^2 \mathrm{d}\lambda(\xi) &= \int_{\Omega\times\mathbb{Z}} |\Psi_o^*(\mathcal{R}f)(\omega,n)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}n \\ &= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})\mathcal{A}f(\omega,t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t \\ &= \int_{\Omega\times\mathbb{T}} |\mathcal{H}f(\omega,t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t.\end{aligned}$$

Since f has finite support, then by the definition of the Helgason-Fourier transform, the inequality $|\kappa_\omega(o,x)| \leqslant |x|$ and $\nu^o(\Omega) = 1$ the above leads to

$$\begin{aligned}\int_{\Xi} |\mathcal{R}f(\xi)|^2 \mathrm{d}\lambda(\xi) &= \int_{\Omega\times\mathbb{T}} \Big|\sum_{x\in\mathrm{supp}f} f(x) q^{(\frac{1}{2}+it)\kappa_\omega(o,x)}\Big|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t \\ &\leqslant \int_{\Omega} \left(\sum_{x\in\mathrm{supp}f} |f(x)| q^{\frac{\kappa_\omega(o,x)}{2}}\right)^2 \mathrm{d}\nu^o(\omega) \\ &\leqslant \left(\sum_{x\in\mathrm{supp}f} |f(x)|^2\right) \sum_{x\in\mathrm{supp}f} \int_{\Omega} q^{\kappa_\omega(o,x)} \mathrm{d}\nu^o(\omega) \\ &\leqslant \left(\sum_{x\in\mathrm{supp}f} |f(x)|^2\right) \sum_{x\in\mathrm{supp}f} q^{|x|} < +\infty.\end{aligned}$$

Therefore, $\mathcal{R}f \in L^2(\Xi)$ for every $f \in C_c(X)$. The horocyclic Radon transform intertwines the regular representations of G on $L^2(X)$ and $L^2(\Xi)$. This result is a direct consequence of the fact that X and Ξ carry G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\lambda$.

Proposition 3.8. *For every $g \in G$ and $f \in C_c(X)$*

$$\mathcal{R}(\pi(g)f) = \hat{\pi}(g)(\mathcal{R}f).$$

Proof. For all $g \in G$ and $f \in C_c(X)$

$$\mathcal{R}(\pi(g)f)(\xi) = \sum_{x\in\xi} f(g^{-1}[x]) = \sum_{y\in g^{-1}.\xi} f(y) = \hat{\pi}(g)(\mathcal{R}f)(\xi),$$

for every $\xi \in \Xi$. ∎

We now introduce a closed subspace of $L^2(\Xi)$ which will play a crucial role because it is the range of the unitarization of the horocyclic Radon transform.

For every $F \in L^2(\Xi)$

$$\|F\|^2_{L^2(\Xi)} = \int_\Omega \sum_{n\in\mathbb{Z}} |\Psi_o^* F(\omega, n)|^2 \mathrm{d}\nu^o(\omega) < +\infty.$$

Hence, the function $\Psi_o^* F(\omega, \cdot)$ is in $L^2(\mathbb{Z})$ for almost every $\omega \in \Omega$. Moreover, by Parseval identity and Fubini theorem

$$\begin{aligned}\|F\|^2_{L^2(\Xi)} &= \int_{\Omega\times\mathbb{Z}} |\Psi_o^* F(\omega, n)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}n \\ &= \int_\mathbb{T}\int_\Omega |(I\otimes\mathcal{F})\Psi_o^* F(\omega, t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t < +\infty.\end{aligned}$$

Then, for almost every $t \in \mathbb{T}$ the function $(I\otimes\mathcal{F})\Psi_o^* F(\cdot, t)$ is in $L^2(\Omega, \nu^o)$ and

$$|\int_\Omega (I\otimes\mathcal{F})\Psi_o^* F(\omega, t)\mathrm{d}\nu^o(\omega)| \leqslant \int_\Omega |(I\otimes\mathcal{F})\Psi_o^* F(\omega, t)|\mathrm{d}\nu^o(\omega) < +\infty.$$

We denote by $L^2_\flat(\Xi)$ the space of functions in $L^2(\Xi)$ satisfying the symmetry condition

$$\int_\Omega (I\otimes\mathcal{F})\Psi_o^* F(\omega, t)\mathrm{d}\nu^o(\omega) = \int_\Omega (I\otimes\mathcal{F})\Psi_o^* F(\omega, -t)\mathrm{d}\nu^o(\omega) \tag{3.12}$$

for every $v \in X$ and for almost every $t \in \mathbb{T}$.

Our main results in Section 3.3.2 are based on the following characterization of $L^2_\flat(\Xi)$. We denote by $L^2(\Omega \times \mathbb{T})$ the space of square integrable functions on $\Omega \times \mathbb{T}$ w.r.t. the measure $\nu^o \otimes \mathrm{d}t$.

Proposition 3.9. *Let $v \in X$. We define the operator Φ_v on $F \in L^2(\Xi)$ by*

$$\Phi_v F(\omega, t) = (I\otimes\mathcal{F})\Psi_v^* F(\omega, t) = (I\otimes\mathcal{F})(\Delta^{-\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, t),$$

for a.e. $(\omega, t) \in \Omega \times \mathbb{T}$. Then Φ_o is an isometry from $L^2(\Xi)$ into $L^2(\Omega \times \mathbb{T})$. Furthermore,

$$\Phi_o F(\omega, t) = p_o(v, \omega)^{\frac{1}{2}+it} \Phi_v F(\omega, t), \tag{3.13}$$

for a.e. $(\omega, t) \in \Omega \times \mathbb{T}$. Finally, a function F belongs to $L^2_\flat(\Xi)$ if and only if $\Phi_o F$ satisfies (3.9) *for every $x \in X$ and almost every $t \in \mathbb{T}$.*

The last statement can be generalized. Indeed, it can be proved that $F \in L^2_\flat(\Xi)$ if and only if $\Phi_v F$ satisfies (3.9) for every $x \in X$ and almost every $t \in \mathbb{T}$, for at least one (and hence for every) $v \in X$. We refer to [7].

Proof. By Parseval identity, for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}\int_{\Omega\times\mathbb{T}} |\Phi_o F(\omega, t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t &= \int_\Omega\int_\mathbb{T} |(I\otimes\mathcal{F})\Psi_o^* F(\omega, t)|^2 \mathrm{d}t\mathrm{d}\nu^o(\omega) \\ &= \int_{\Omega\times\mathbb{Z}} |\Psi_o^* F(\omega, n)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}n = \|F\|^2_{L^2(\Xi)},\end{aligned}$$

so that Φ_o is an isometry from $L^2(\Xi)$ into $L^2(\Omega \times \mathbb{T})$. Now, let $v \in X$ and $F \in L^2(\Xi)$. For almost every $\omega \in \Omega$ we have that

$$\begin{aligned}
0 &= \lim_{N\to+\infty} \int_0^T \Big| \sum_{n=-N}^{N} F \circ \Psi_o(\omega, n) q^{\frac{n}{2}} q^{int} - \Phi_o F(\omega, t)\Big|^2 \mathrm{d}t \\
&= \lim_{N\to+\infty} \int_0^T \Big| \sum_{n=-N}^{N} F \circ \Psi_v(\omega, n + \kappa_\omega(v, o)) q^{\frac{n}{2}} q^{int} - \Phi_o F(\omega, t)\Big|^2 \mathrm{d}t \\
&= \lim_{N\to+\infty} \int_0^T \Big| \sum_{m=-N+\kappa_\omega(v,o)}^{N+\kappa_\omega(v,o)} F \circ \Psi_v(\omega, m) q^{(\frac{1}{2}+it)(m-\kappa_\omega(v,o))} - \Phi_o F(\omega, t)\Big|^2 \mathrm{d}t \\
&= \lim_{N\to+\infty} \int_0^T \Big| q^{(\frac{1}{2}+it)\kappa_\omega(o,v)} \sum_{m=-N+\kappa_\omega(v,o)}^{N+\kappa_\omega(v,o)} F \circ \Psi_v(\omega, m) q^{\frac{m}{2}} q^{imt} - \Phi_o F(\omega, t)\Big|^2 \mathrm{d}t
\end{aligned}$$

and, since

$$\Phi_v F(\omega, t) = \lim_{N\to+\infty} \sum_{m=-N+\kappa_\omega(v,o)}^{N+\kappa_\omega(v,o)} F \circ \Psi_v(\omega, m) q^{\frac{m}{2}} q^{imt}$$

in L^2_T, we conclude that relation (3.13) holds true. Finally, let $F \in L^2(\Xi)$. For every $x \in X$ and for almost every $t \in \mathbb{T}$, (3.13) yields

$$\begin{aligned}
\int_\Omega p_o(x, \omega)^{\frac{1}{2}-it} \Phi_o F(\omega, t) \mathrm{d}\nu^o(\omega) &= \int_\Omega p_o(x, \omega)^{\frac{1}{2}-it} p_o(x, \omega)^{\frac{1}{2}+it} \Phi_x F(\omega, t) \mathrm{d}\nu^o(\omega) \\
&= \int_\Omega \Phi_x F(\omega, t) p_o(x, \omega) \mathrm{d}\nu^o(\omega) \\
&= \int_\Omega \Phi_x F(\omega, t) \mathrm{d}\nu^x(\omega).
\end{aligned}$$

Then, for every $x \in X$ and almost every $t \in \mathbb{T}$

$$\int_\Omega p_o(x, \omega)^{\frac{1}{2}-it} \Phi_o F(\omega, t) \mathrm{d}\nu^o(\omega) = \int_\Omega (I \otimes \mathcal{F}) \Psi_x^* F(\omega, t) \mathrm{d}\nu^x(\omega).$$

This equality allows us to conclude that $F \in L^2_\flat(\Xi)$ if and only if $\Phi_o F$ satisfies (3.9) and this concludes our proof. □

Corollary 3.10. *For every $f \in C_c(X)$,*

$$\Phi_o(\mathcal{R} f) = \mathcal{H} f \tag{3.14}$$

in $L^2(\Omega \times \mathbb{T})$ and $\mathcal{R} f \in L^2_\flat(\Xi)$.

Proof. The proof follows immediately by Proposition 3.7 and the fact that the Helgason-Fourier transform satisfies (3.9). □

Some comments are in order. Proposition 3.9 with Corollary 3.10 shows that $\mathcal{R}(C_c(X)) \subseteq L^2_\flat(\Xi)$ and it highlights the link between the range of the Radon transform with the range of the Helgason-Fourier transform, which will play a crucial role in our main result. The range $\mathcal{R}(C_c(X))$ has already been completely characterized in [15]. We recall the result in [15] for completeness and in order to understand the relation with $L^2_\flat(\Xi)$.

Theorem 3.11 (Theorem 1, [15])**.** *The range of the horocyclic Radon transform on the space of functions with finite support on X is the space of continuous compactly supported functions on Ξ satisfying the following two conditions*

(i) for some $v \in X$, hence for every $v \in X$, $\sum_{n\in\mathbb{Z}} F \circ \Psi_v(\omega, n)$ is independent of $\omega \in \Omega$;

(ii) for every $v \in X$ and $n \in \mathbb{Z}$

$$\int_\Omega \Psi_v^* F(\omega, n)\mathrm{d}\nu^v(\omega) = \int_\Omega \Psi_v^* F(\omega, -n)\mathrm{d}\nu^v(\omega). \tag{3.15}$$

It is worth observing that condition (3.12) is the equivalent on the frequency side of equation (3.15) for continuous compactly supported functions on Ξ. As it will be made clear in the next section, condition (3.12) better suits our needs.

3.3.2 Unitarization and intertwining

In order to obtain the unitarization for the horocyclic Radon transform that we are after, we need some technicalities. Figure 3.5 below might help the reader to keep track of all the spaces and operators involved in our construction.

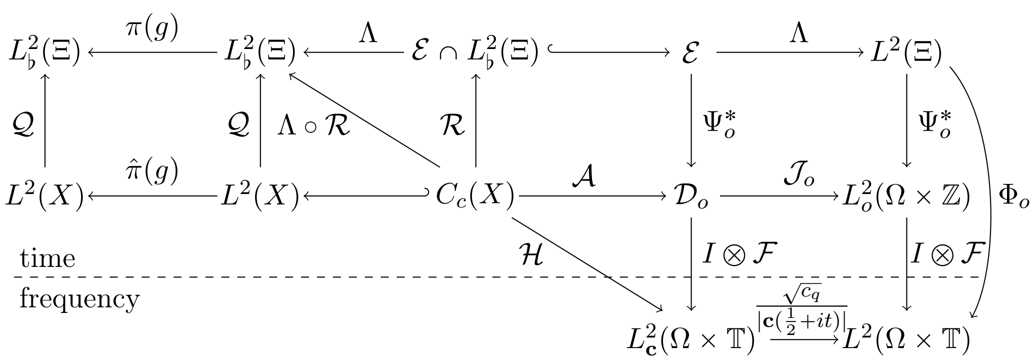

Figure 3.5: Spaces and operators that come into play in our construction.

We set

$$\mathcal{D}_o = \{\varphi \in L^2_o(\Omega \times \mathbb{Z}) : (I \otimes \mathcal{F})\varphi \in L^2_{\mathbf{c}}(\Omega \times \mathbb{T})\}$$

and we define the operator $\mathcal{J}_o \colon \mathcal{D}_o \subseteq L^2_o(\Omega \times \mathbb{Z}) \to L^2_o(\Omega \times \mathbb{Z})$ as the Fourier multiplier

$$(I \otimes \mathcal{F})(\mathcal{J}_o\varphi)(\omega, t) = \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}(I \otimes \mathcal{F})\varphi(\omega, t), \quad \text{a.e.}\,(\omega, t) \in \Omega \times \mathbb{T},$$

where c_q is given by (3.8). We define the set of functions

$$\mathcal{E} = \{F \in L^2(\Xi) : \Phi_o F \in L^2_{\mathbf{c}}(\Omega \times \mathbb{T})\}$$

and we consider the operator $\Lambda\colon \mathcal{E} \subseteq L^2(\Xi) \to L^2(\Xi)$ given by

$$\Lambda F = \Psi_o^{*-1} \mathcal{J}_o \Psi_o^* F.$$

It is possible to observe, see [7], that the construction of Λ, despite the definitions of Ψ_o, $\mathcal{D}_o$ and $\mathcal{J}_o$, is independent of the reference point that we fixed as origin; namely, if $\mathcal{J}_v$ is defined in the same way, for an other $v \in X$, then $\Lambda = \Psi_v^{*-1} \mathcal{J}_v \Psi_v^*$.

As a direct consequence of the definition of Λ and $\mathcal{J}_o$, for every $F \in \mathcal{E}$ and for almost every $(\omega, t) \in \Omega \times \mathbb{T}$ we have that

$$\begin{aligned} \Phi_o(\Lambda F)(\omega,t) &= (I \otimes \mathcal{F})(\mathcal{J}_o \Psi_o^* F)(\omega,t) \\ &= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2}+it)\right|}(I \otimes \mathcal{F})(\Psi_o^* F)(\omega,t) \\ &= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2}+it)\right|}\Phi_o F(\omega,t). \end{aligned} \tag{3.16}$$

The operator Λ intertwines the regular representation $\hat{\pi}$ as shown by the next proposition.

Proposition 3.12. *The subspace $\mathcal{E}$ is $\hat{\pi}$-invariant and for all $F \in \mathcal{E}$ and $g \in G$*

$$\hat{\pi}(g)\Lambda F = \Lambda \hat{\pi}(g) F. \tag{3.17}$$

Proof. We consider $F \in \mathcal{E}$, $g \in G$ and we prove that $\hat{\pi}(g)F \in \mathcal{E}$. We observe that

$$\hat{\pi}(g)F \circ \Psi_o(\omega,n) = F \circ \Psi_{g^{-1}[o]}(g^{-1}\langle\omega\rangle, n)$$

for almost every $(\omega, n) \in \Omega \times \mathbb{Z}$. Therefore, we have

$$\Psi_o^*(\hat{\pi}(g)F)(\omega,n) = \Psi_{g^{-1}[o]}^* F(g^{-1}\langle\omega\rangle, n)$$

and consequently

$$\Phi_o(\hat{\pi}(g)F)(\omega,t) = \Phi_{g^{-1}[o]} F(g^{-1}\langle\omega\rangle, t) \tag{3.18}$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$. By equations (3.6), (3.13) and (3.18)

$$\begin{aligned} &\int_{\Omega\times\mathbb{T}} |\Phi_o(\hat{\pi}(g)F)(\omega,t)|^2 \frac{c_q \mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\ &= \int_{\mathbb{T}}\int_{\Omega} |\Phi_{g^{-1}[o]} F(g^{-1}\langle\omega\rangle,t)|^2 \frac{c_q \mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\ &= \int_{\mathbb{T}}\int_{\Omega} |\Phi_{g^{-1}[o]} F(\omega,t)|^2 p_o(g^{-1}[o],\omega) \frac{c_q \mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\ &= \int_{\Omega\times\mathbb{T}} |\Phi_o F(\omega,t)|^2 \frac{c_q \mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} < +\infty \end{aligned}$$

and we conclude that $\hat{\pi}(g)F \in \mathcal{E}$. We next prove the intertwining property (3.17). We have already observed that, by Proposition 3.9, it is enough to prove that

$$\Phi_o(\hat{\pi}(g)\Lambda F) = \Phi_o(\Lambda \hat{\pi}(g) F)$$

for every $g \in G$ and $F \in \mathcal{E}$. By equations (3.16) and (3.18), for almost every $(\omega, t) \in \Omega \times \mathbb{T}$, we have the chain of equalities

$$\begin{aligned}\Phi_o(\hat{\pi}(g)\Lambda F)(\omega, t) &= \Phi_{g^{-1}[o]}(\Lambda F)(g^{-1}\langle\omega\rangle, t) \\ &= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2}+it)\right|}\Phi_{g^{-1}[o]}F(g^{-1}\langle\omega\rangle, t) \\ &= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2}+it)\right|}\Phi_o(\hat{\pi}(g)F)(\omega, t) = \Phi_o(\Lambda\hat{\pi}(g)F)(\omega, t),\end{aligned}$$

which proves the intertwining relation. □

The next result follows directly by Proposition 3.9 and equation (3.16).

Corollary 3.13. *For every $F \in \mathcal{E}$, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $F \in L^2_\flat(\Xi)$.*

Proof. By Proposition 3.9, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $\Phi_o(\Lambda F)$ satisfies (3.9). By (3.16) and since $t \mapsto |\mathbf{c}(1/2 + it)|$ is even, $\Phi_o(\Lambda F)$ satisfies (3.9) if and only if $\Phi_o F$ satisfies (3.9), which is equivalent to $F \in L^2_\flat(\Xi)$. This concludes the proof. □

We are now in a position to prove our main result.

Theorem 3.14. *The composite operator $\Lambda\mathcal{R}$ extends to a unitary operator*

$$\mathcal{Q}\colon L^2(X) \longrightarrow L^2_\flat(\Xi)$$

which intertwines the representations π and $\hat{\pi}$, i.e.

$$\hat{\pi}(g)\mathcal{Q} = \mathcal{Q}\pi(g), \qquad g \in G. \tag{3.19}$$

Theorem 3.14 implies that $\hat{\pi}$ is not irreducible, too. In particular, it is not irreducible the subrepresentation of $\hat{\pi}$ obtained by restricting it to the (closed) subspace $L^2_\flat(\Xi)$.

Proof. We first show that $\Lambda\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$ from $L^2(X)$ onto $L^2_\flat(\Xi)$. Let $f \in C_c(X)$. By the Fourier Slice Theorem (3.11), the Parseval identity and the definition of Λ, we have that

$$\begin{aligned}\|f\|^2_{L^2(X)} &= \|\mathcal{H}f\|^2_{L^2_{\mathfrak{c}}(\Omega\times\mathbb{T})^\sharp} \\ &= \|(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{R}f))\|^2_{L^2_{\mathfrak{c}}(\Omega\times\mathbb{T})^\sharp} \\ &= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})(\mathcal{J}_o\Psi_o^*(\mathcal{R}f))(\omega,t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t \\ &= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})(\Psi_o^*(\Lambda\mathcal{R}f))(\omega,t)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}t \\ &= \int_{\Omega\times\mathbb{Z}} |\Psi_o^*(\Lambda\mathcal{R}f)(\omega,n)|^2 \mathrm{d}\nu^o(\omega)\mathrm{d}n \\ &= \|\Lambda\mathcal{R}f\|^2_{L^2(\Xi)}.\end{aligned}$$

Hence, $\Lambda\mathcal{R}$ is an isometric operator from $C_c(X)$ into $L^2(\Xi)$. Since $C_c(X)$ is dense in $L^2(X)$, $\Lambda\mathcal{R}$ extends to a unique isometry from $L^2(X)$ onto the closure of $\mathrm{Ran}(\Lambda\mathcal{R})$ in $L^2(\Xi)$. We must show that $\Lambda\mathcal{R}$ has dense image in $L^2_\flat(\Xi)$. The inclusion $\mathrm{Ran}(\Lambda\mathcal{R}) \subseteq L^2_\flat(\Xi)$ follows immediately from Corollary 3.10 and Corollary 3.13. Let $F \in L^2_\flat(\Xi)$ be such that $\langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} = 0$ for every $f \in C_c(X)$. By the Parseval identity and the Fourier Slice Theorem (3.11) we have that

$$\begin{aligned}
0 &= \langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} \\
&= \int_{\Omega\times\mathbb{Z}} (F\circ\Psi_o)(\omega,n)\overline{(\Lambda\mathcal{R}f\circ\Psi_o)(\omega,n)}q^n \mathrm{d}\nu^o(\omega)\mathrm{d}n \\
&= \int_{\Omega\times\mathbb{Z}} (\Psi_o^*F)(\omega,n)\overline{(\mathcal{J}_v\Psi_o^*(\mathcal{R}f))(\omega,n)}\mathrm{d}\nu^o(\omega)\mathrm{d}n \\
&= \int_{\Omega\times\mathbb{T}} \Phi_o(F)(\omega,t)\overline{(I\otimes\mathcal{F})(\mathcal{J}_v\Psi_o^*(\mathcal{R}f))(\omega,t)}\mathrm{d}\nu^o(\omega)\mathrm{d}t \\
&= \int_{\Omega\times\mathbb{T}} \Phi_o(F)(\omega,t)\overline{(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{R}f))(\omega,t)}\frac{\sqrt{c_q}\mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|} \\
&= \int_{\Omega\times\mathbb{T}} \frac{|\mathbf{c}(\frac{1}{2}+it)|}{\sqrt{c_q}}\Phi_o(F)(\omega,t)\overline{\mathcal{H}f(\omega,t)}\frac{c_q\mathrm{d}\nu^o(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2}.
\end{aligned}$$

For simplicity of notation, we denote by ΘF the function on $\Omega\times\mathbb{T}$ defined as

$$\Theta F(\omega,t) = \frac{|\mathbf{c}(\frac{1}{2}+it)|}{\sqrt{c_q}}\Phi_o(F)(\omega,t), \qquad \text{a.e. } (\omega,t)\in\Omega\times\mathbb{T}.$$

Hence we have proved that $\langle\Theta F, \mathcal{H}f\rangle = 0$ for every $f \in C_c(X)$. The following two facts follow immediately from Proposition 3.9. Since Φ_o is an isometry from $L^2(\Xi)$ into $L^2(\Omega\times\mathbb{T})$, then ΘF belongs to $L^2_{\mathbf{c}}(\Omega\times\mathbb{T})$. Furthermore, since $F \in L^2_\flat(\Xi)$ and since $t \mapsto |\mathbf{c}(1/2+it)|$ is even, then $\Theta F \in L^2_{o,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$. By Theorem 3.4, $\mathcal{H}(C_c(X))$ is dense in $L^2_{o,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$. Thus, $\Theta F = 0$ in $L^2_{o,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$ and then $\Phi_o(F) = 0$ in $L^2(\Omega\times\mathbb{T})$. Since Φ_o is an isometry from $L^2(\Xi)$ into $L^2(\Omega\times\mathbb{T})$, then $F = 0$ in $L^2(\Xi)$. Therefore, $\overline{\mathrm{Ran}(\Lambda\mathcal{R})} = L^2_\flat(\Xi)$ and $\Lambda\mathcal{R}$ extends uniquely to a surjective isometry

$$\mathcal{Q}\colon L^2(X) \longrightarrow L^2_\flat(\Xi).$$

Observe that $\mathcal{Q}f = \Lambda\mathcal{R}f$ for every $f \in C_c(X)$. Then, the intertwining property (3.19) follows immediately from Proposition 3.8 and Proposition 3.12. □

As a byproduct, one obtains an extended Fourier Slice Theorem.

Proposition 3.15 (Fourier Slice Theorem, version II). *For every $f \in L^2(X)$*

$$(I\otimes\mathcal{F})(\Psi_o^*(\mathcal{Q}f))(\omega,t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathscr{H}f(\omega,t)$$

for almost every $(\omega,t)\in\Omega\times\mathbb{T}$.

Proof. For every $f \in C_c(X)$, by (3.14) and (3.16) we have that

$$\begin{aligned}(I \otimes \mathcal{F})(\Psi_o^*(\mathcal{Q}f))(\omega,t) &= \Phi_o(\mathcal{Q}f)(\omega,t)\\ &= \Phi_o(\Lambda\mathcal{R}f)(\omega,t)\\ &= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\Phi_o(\mathcal{R}f)(\omega,t)\\ &= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathcal{H}f(\omega,t),\end{aligned}$$

for almost every $(\omega,t) \in \Omega \times \mathbb{T}$. Let $f \in L^2(X)$, since $C_c(X)$ is dense in $L^2(X)$, then there exists a sequence $(f_m)_m \subseteq C_c(X)$ such that $f_m \to f$ in $L^2(X)$. Then, since $\mathcal{Q}$ is a unitary operator from $L^2(X)$ onto $L^2_\flat(\Xi)$ and Φ_o is an isometry from $L^2(\Xi)$ into $L^2(\Omega \times \mathbb{T})$, then $\Phi_o(\mathcal{Q}f_m) \to \Phi_o(\mathcal{Q}f)$ in $L^2(\Omega \times \mathbb{T})$. Since $f_m \in C_c(X)$ for every $m \in \mathbb{N}$,

$$(I \otimes \mathcal{F})(\Psi_o^*(\mathcal{Q}f_m))(\omega,t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathcal{H}f_m(\omega,t),$$

for almost every $(\omega,t) \in \Omega \times \mathbb{T}$. Hence, passing to a subsequence if necessary, for almost every $(\omega,t) \in \Omega \times \mathbb{T}$

$$\lim_{m\to+\infty} \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathcal{H}f_m(\omega,t) = (I \otimes \mathcal{F})(\Psi_o^*(\mathcal{Q}f))(\omega,t).$$

Therefore, passing to a subsequence if necessary, for almost every $(\omega,t) \in \Omega \times \mathbb{T}$

$$(I \otimes \mathcal{F})(\Psi_o^*(\mathcal{Q}f))(\omega,t) = \lim_{m\to+\infty} \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathcal{H}f_m(\omega,t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\mathscr{H}f(\omega,t)$$

and this concludes our proof. □

Chapter 4

Harmonic Bergman projectors on homogeneous trees

This chapter contains an ongoing work with Filippo De Mari and Maria Vallarino, which is in its final stage but still not submitted.

Our first purpose was to answer the question: given a q-homogeneous tree X ($q \geqslant 2$) and its group of isometries G, do we know how to realize a square integrable representation of G? The square integrable representations of G have been classified in [22] and [45] but there are no explicit formulae for them. On the other hand, it is well known that, on the analog setting of the hyperbolic disk, square integrable representations of $\mathrm{SU}(1,1)$ have a realization on Bergman spaces [42]. The definition of Bergman space on the hyperbolic disk is given for holomorphic functions. To the best of our knowledge, the definition of holomorphic function is not clearly stated for functions defined on homogeneous trees.

The notion of harmonic function is stated for functions defined on homogeneous trees by starting from the mean value property. That is, a function on X is said to be harmonic if the mean of its values on the neighbors of a vertex coincides with the value at the vertex, for every vertex. J. Cohen, F. Colonna, M. Picardello, and D. Singman present the harmonic version of Bergman spaces on homogeneous trees in [17]. They consider a family of reference measures which consist of finite measures absolutely continuous w.r.t. the counting measure and whose Radon-Nikodym derivative σ is a radial (w.r.t. a fixed origin $o \in X$) positive decreasing function on X. Thus, they define the Bergman space $\mathcal{A}^p(\sigma)$ as the closed subspace of $L^p(\sigma)$ consisting of harmonic functions. The request for the measure σ to be finite is necessary in order to avoid the case in which the Bergman space consists only of the null function.

In the context of hyperbolic disk, when $p = 2$ Bergman spaces are RKHS, and the holomorphic Bergman kernel is known as well as the properties of the associated projector. Indeed, the extension of the holomorphic Bergman projector to the (weighted) L^p-spaces is bounded if and only if $p > 1$, see [25], [49] and [54]. Furthermore, it is weakly continuous on L^1, see [10] and [11]. In our work, first of all, we show that $\mathcal{A}^2(\sigma)$ is a reproducing kernel Hilbert space for every reference measure σ and we provide an explicit formula for the kernel K_σ in Theorem 4.14. Then, focusing on a class of reference measures, we prove that the extension of the projector

$$P_\sigma f(z) = \sum_{x \in X} K_\sigma(z,x) f(x) \sigma(x)$$

to an operator from $L^p(\sigma)$ to $\mathcal{A}^p(\sigma)$ is bounded if and only if $p > 1$.

The class of measures we focus on coincides with the request of σ to have an exponential decreasing, namely we shall consider the measures defined by

$$\mu_\alpha(x) = q^{-\alpha|x|}, \qquad \alpha > 1,\, x \in X, \tag{4.1}$$

where $|x| \in \mathbb{N}$ denotes the distance of $x \in X$ from the origin o. This family of measures appears to be the natural counterpart of the measures $(1 - |x+iy|^2)^{\alpha-2}\mathrm{d}x\mathrm{d}y$, $\alpha > 1$, considered on the hyperbolic disk in the definition of the weighted (holomorphic) Bergman spaces. Furthermore, the relation between different measures of the type in (4.1) allows us to obtain intermediate results on the operator

$$T_{a,b,c} f(z) = q^{-a|z|} \sum_{x \in X} K_c(z,x) f(x) q^{-b|x|},$$

where K_c is the kernel of $\mathcal{A}^2(\mu_c)$, $c > 1$. The fact that the extension of $P_{\mu_\alpha} = P_\alpha$ to $L^p(\mu_\alpha) = L^p_\alpha$ is bounded if and only if $p > 1$ follows from the boundedness results for $T_{a,b,c}$ on L^p_α.

A first natural question is whether the same holds for a general reference measure σ. In [17], the authors consider the optimal measures, a subset of the reference measures. Roughly speaking, they are measures which decrease fast as the distance from the origin increase. The measures introduced in (4.1) are optimal. We are aware that if σ is an optimal measure, then P_σ is not bounded on $L^1(\sigma)$. Furthermore, we know that for a large class of optimal measures P_σ is bounded on $L^p(\sigma)$ for every $p > 1$. Another natural question we would like to investigate is whether, and for which σ, the operator P_σ is weak type $(1,1)$, that is when it is bounded as operator $P_\sigma \colon L^1(\sigma) \to L^{1,\infty}(\sigma)$, where $L^{1,\infty}(\sigma)$ denotes the Lorentz space.

4.1 Harmonic Bergman spaces on homogeneous trees

In what follows we consider a q-homogeneous tree X with $q \geqslant 2$ and we fix $o \in X$ as origin.

We start by integrating the preliminaries in Section 3.1.1 with some very basic notation on the homogeneous tree which is not used in the previous chapter. If $v \in X$, then we denote by $S(v,n)$ and $B(v,n)$ the *sphere* and then *ball* centered at y with radius $n \in \mathbb{N}$, respectively, i.e.

$$S(v,n) := \{x \in X \colon d(v,x) = n\} \qquad \text{and} \qquad B(v,n) := \{x \in X \colon d(v,x) \leqslant n\}.$$

It is straightforward to check that

$$|S(v,n)| = \begin{cases} 1, & n = 0; \\ (q+1)q^{n-1}, & n \neq 0, \end{cases} \qquad \text{and} \qquad |B(v,n)| = \frac{q^{n+1} + q^n - 2}{q-1}.$$

We denote by $|v| = d(o, v)$ the distance of $v \in X$ from o. If $v \neq o$, then we define the *sector of* v as the subset

$$T_v := \{x \in X \colon [o, v] \subseteq [o, x]\},$$

where $[o, v]$ is the chain defined in Section 3.1.1, and we adopt the convention $T_o = X$. Moreover, we call *predecessor of* v the unique neighbor $p(v) \in X$ of v such that $|p(v)| = |v| - 1$ and *sons* of v the neighbors of v that are different from the predecessor; in other words sons are those neighbors of v having distance from the origin equal to $|v| + 1$. We denote the set of all the sons of v as $s(v) \subseteq X$. Observe that o has no predecessor and $s(o) = S(o, 1)$, while $|s(v)| = q$ for every $v \in X \backslash \{o\}$. Furthermore, in what follows, we consider the predecessor as a function $p \colon X \backslash \{o\} \to X$ so that we can denote by $p^\ell \colon X \backslash B(o, \ell - 1) \to X$ the ℓ-th predecessor.

In Section 4.1.1 we present some introductory fact for harmonic functions on homogeneous trees. In particular we want to extend a function which is locally harmonic, say on a ball, to a harmonic function on X. The discrete structure of the tree allows to build the extension in several ways. We choose to extend a function which is harmonic on the ball $B(o, n)$ to a harmonic function on X which is "radial" on sectors T_y generated by vertices $y \in S(o, n + 1)$. The harmonic extension that we adopt coincides with the function on $B(o, n+1)$ and is constant on the sets $T_y \cap S(o, m)$, with $y \in S(o, n+1)$ and $m > n$. Clearly, we choose to consider functions which are harmonic in a ball centered in o because we will use it in what follows, but the center does not play a crucial role. It is possible to obtain an extension for functions defined on every ball or even on suitable, say connected, subsets of X.

Since there are no harmonic functions that are in L^p w.r.t. the counting measure, we need to endow the tree with finite measures. In Section 4.1.2 we recall the family of measures, called reference measures, considered in [17] in the definition of the harmonic Bergman spaces. For every $1 \leqslant p < +\infty$ we denote by $\mathcal{A}^p(\sigma)$ the Bergman space associated to the measure σ. They are Banach spaces and when $p = 2$ they are Hilbert spaces with the scalar product inherited from $L^2(\sigma)$. In Section 4.1.3 we provide an orthonormal basis for every $\mathcal{A}^2(\sigma)$.

4.1.1 Harmonic functions on homogeneous trees

Definition 4.1. A complex valued function f on X is *harmonic* if its value at a vertex coincides with the average of its values on its neighbors, namely, for every $v \in X$

$$f(v) = \frac{1}{q+1} \sum_{u \sim v} f(u). \tag{4.2}$$

Equivalently, f is harmonic if $Lf = 0$, where L is the combinatorial Laplacian defined by

$$Lf(v) := \frac{1}{q+1} \sum_{u \sim v} f(u) - f(v) \qquad \forall v \in X.$$

We say that a complex valued function f is harmonic on a subset $Y \subset X$ of X if $Lf = 0$ on Y.

The harmonicity property (4.2) implies a certain rigidity for the function. In particular, the value of a harmonic function at a vertex $y \in X$ "propagates" to every layer of the sector T_y, as showed in the following. The following proposition is a modified version of Lemma 4.1 in [17]. In this lemma, the authors show that a function which is harmonic and radial on a sector T_y, $y \in X \backslash \{o\}$, is completely described by its values at y and $p(y)$. We consider a harmonic function on the sector T_y without the radial condition and we formulate the conclusion just for the average on $S(o,n)$, $n \geqslant |y|$, instead of for each vertex of the sector.

Proposition 4.2 (Lemma 4.1 in [17]). *Let $y \in X \backslash \{o\}$. If $f \colon X \to \mathbb{C}$ is harmonic on T_y, then for every $n \in \mathbb{N}$, $n \geqslant |y|$, we have*

$$\sum_{\substack{|x|=n \\ x \in T_y}} f(x) = \left(\sum_{j=0}^{n-|y|} q^j \right) f(y) - \left(\sum_{j=0}^{n-|y|-1} q^j \right) f(p(y)). \tag{4.3}$$

Furthermore, if a function $f \colon X \to \mathbb{C}$ is radial on T_y and satisfies (4.3) *for every $n \geqslant |y|$, then f is harmonic on T_y.*

From Proposition 4.2 we deduce a generalization of formula (4.2).

Corollary 4.3. *Let f be a harmonic function on X. Then the following mean value property holds true: for every $n \in \mathbb{N} \backslash \{0\}$*

$$f(o) = \frac{1}{|S(o,n)|} \sum_{|x|=n} f(x). \tag{4.4}$$

We introduce a technique which permits to extend a function which is harmonic on a ball (for simplicity centered in o) which will be useful in what follows. Let $n \in \mathbb{N}$, $n \geqslant 1$, and g be a function on X which is harmonic on $B(o, n+1)$. It is easy to see that there are infinite ways to extend g to a harmonic function on all X which coincides with g on $B(o,n)$. For our purposes, we choose the next (fairly standard) extension. We define g_n^H on X such that g_n^H is radial restricted on T_y for every $y \in S(o, n+1)$ and harmonic on X.

We suppose that such extension g_n^H exists and we provide an explicit formula for it through Proposition 4.2. Let $x \in X \backslash B(o,n)$. There exists a unique $y \in S(o, n+1)$ such that $x \in T_y$, and $y = p^{|x|-n-1}(x)$. Since the function g_n^H is supposed to be radial and harmonic on T_y, by Proposition 4.2 we have that

$$\begin{aligned}
g_n^H(x) &= \frac{1}{|S(o,|x|)\cap T_y|}\sum_{\substack{|z|=|x|,\\ z\in T_y}} g_H(z)\\
&= q^{|y|-|x|}\left[\left(\sum_{j=0}^{|x|-|y|} q^j\right)g(y)-\left(\sum_{j=0}^{|x|-|y|-1} q^j\right)g(p(y))\right]\\
&= q^{n-1-|x|}\left[\left(\sum_{j=0}^{|x|-n-1} q^j\right)g(p^{|x|-n-1}(x))-\left(\sum_{j=0}^{|x|-n-2} q^j\right)g(p^{|x|-n}(x))\right]\\
&= \left(\sum_{j=0}^{|x|-n-1} q^{-j}\right)g(p^{|x|-n-1}(x))-\left(\sum_{j=1}^{|x|-n-1} q^{-j}\right)g(p^{|x|-n}(x)).
\end{aligned}$$

For simplicity we introduce the notation

$$a_n = \sum_{j=0}^{n} q^{-j} = \frac{q-q^{-n}}{q-1}, \qquad n \in \mathbb{N},$$

and we put $a_{-1} = 0$. Hence the extension is defined by

$$g_n^H(x) = \begin{cases} g(x), & |x| \leqslant n;\\ a_{|x|-n-1}g(p^{|x|-n-1}(x)) - \left(a_{|x|-n-1}-1\right) g(p^{|x|-n}(x)), & |x| > n. \end{cases}$$

The function g_n^H defined above is harmonic on X by Proposition 4.2 and by using the fact that

$$X = B(o,n) \cup \bigcup_{y\in S(o,n+1)} T_y.$$

Observe that we do not lose the fact that g_n^H is harmonic on $B(o,n)$ because, by the fact that $a_0 = 1$ and $a_{-1} = 0$, $g_n^H = g$ on $B(o,n+1)$, and not only on $B(o,n)$. Furthermore, the extension g_n^H is radial on every sector "starting" from $S(o,n+1)$ by construction.

It is worth mentioning that, in Definition 7 in [17], the authors introduce an operator RH_v which "radialize" a function defined on X on the sector T_v and which maintains harmonicity.

4.1.2 Harmonic Bergman spaces $\mathcal{A}^p_\sigma$

Homogeneous trees are classically endowed with the counting measure. The main advantage of such measure is the invariance under the group of isometries of the tree, as we have seen in the previous chapter. Let $p \geqslant 1$. The only harmonic function that is p-summable w.r.t. the counting measure is the null function, as we show in the following statement.

Proposition 4.4. *If f is a harmonic function in $L^p(X)$, then f is the null function.*

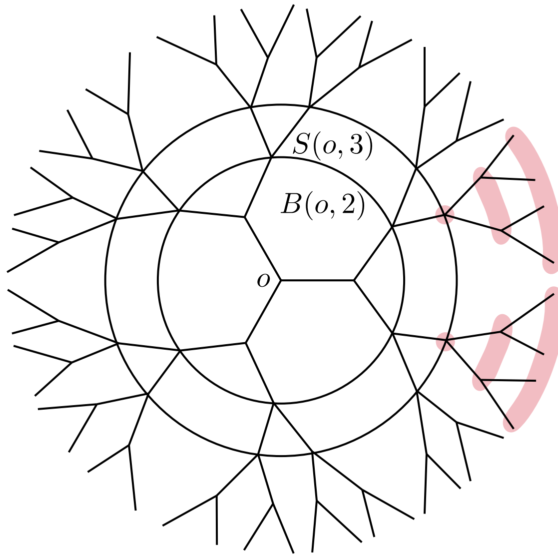

Figure 4.1: The function g is harmonic on $B(o,2)$, that is the set of vertices in green area. The function g_2^H is obtained by "extending" the values of g in $S(o,3)$ (blue area) along their sectors such that g_2^H is harmonic and constant on vertices lying on the same red arc, that is on the "layers" of the sectors.

Proof. Suppose that f is harmonic. We have that

$$\begin{aligned}\sum_{x\in X}|f(x)|^p =&\sum_{n=0}^{+\infty}\sum_{|x|=n}|f(x)|^p\\ =&\frac{1}{(q+1)^p}\sum_{n=0}^{+\infty}\sum_{|x|=n}\left|\sum_{y\sim x}f(y)\right|^p\\ \leqslant&\frac{(q+1)^{p-1}}{(q+1)^p}\sum_{n=0}^{+\infty}\sum_{|x|=n}\sum_{y\sim x}|f(y)|^p\\ =&\frac{1}{q+1}(q+1)\|f\|^p = \|f\|^p_{L^p(X)} < +\infty,\end{aligned}$$

since every vertex is neighbor of exactly $q+1$ other vertices. Hence the unique inequality in the computation above is an equality, so that

$$(q+1)^{p-1}\sum_{y\sim x}|f(y)|^p = \left|\sum_{y\sim x}f(y)\right|^p = (q+1)^p|f(x)|^p,$$

which means that $|f|^p$ is harmonic, too. If f is not the null function, then there exists $v\in X$ such that $f(v)\neq 0$. Hence by Corollary 4.3, we have

$$\sum_{x\in X}|f(x)|^p = \sum_{n=0}^{+\infty}\sum_{d(v,x)=n}|f(x)|^p = |f(v)|^p\sum_{n=0}^{+\infty}|S(v,n)| = +\infty,$$

which is a contradiction. Hence $f=0$. ∎

If we want to work with Hilbert spaces of harmonic functions, the previous proposition leads to consider finite measures on X. Harmonic Bergman spaces have been introduced [17] on a q-homogeneous tree X for the following class of measures.

Definition 4.5. A *reference measure* on X is a finite measure that is absolutely continuous w.r.t. the counting measure and whose Radon-Nikodym derivative σ is a radial positive decreasing function on X. With a slight abuse of notation we denote by σ the reference measure, too. Given a reference measure σ on X for every $p \in [1, \infty)$ the *Bergman space* $\mathcal{A}^p(\sigma)$ is the space of harmonic functions on X such that

$$\|f\|^p_{\mathcal{A}^p(\sigma)} := \sum_{x \in X} |f(x)|^p \sigma(x) < +\infty.$$

If σ is a reference measure on X, and if we denote by σ_n the value of σ on the sphere $S(0, n)$, then

$$\sigma_0 + \frac{q+1}{q} \sum_{n=1}^{+\infty} \sigma_n q^n < +\infty.$$

From now on, for a reference measure σ, we put $B_\sigma := \sigma(X) < +\infty$.

Example 4.1. Let $\alpha > 1$. An interesting example of reference measure is the function

$$\mu_\alpha(x) = q^{-\alpha|x|}, \qquad x \in X.$$

Indeed, σ is radial, positive and decreasing. Furthermore,

$$B_{\mu_\alpha} = 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(1-\alpha)n} = 1 + \frac{q+1}{q} \frac{q^{1-\alpha}}{1 - q^{1-\alpha}} = \frac{1 + q^{-\alpha}}{1 - q^{1-\alpha}} < +\infty.$$

Given a reference measure σ, we introduce a decreasing sequence $(b_m)_{m \in \mathbb{N}}$ which collects some important information on σ, as we see in the following result. For every $n \in \mathbb{N}$, we define

$$b_n = \sum_{m=n+1}^{+\infty} \left[\sigma_m \left(\sum_{k=0}^{m-n-1} q^k \right) \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \right]. \tag{4.5}$$

The sum are finite because σ is a finite measure on X.

The next lemma is a technical result that is very useful in what follows. Roughly speaking, we can say that the harmonic extension g_n^H of a function g harmonic on $B(o, n)$ and null on $S(o, n)$ "localizes" the functions in $\mathcal{A}^2(\sigma)$ on $B(o, n+1)$. Indeed the scalar product of g_n^H with a function $f \in \mathcal{A}^2(\sigma)$ is completely determined by the values that the two functions assume in $B(o, n+1)$. In particular, it involves only values of g_n^H at vertices in which it coincides with g.

Lemma 4.6. *Let $n \in \mathbb{N}$ and g be a function on X which is harmonic on $B(o, n)$ and vanishes on $S(o, n)$. Then there exists a constant $b'_n > 0$ such that for every $f \in \mathcal{A}^2(\sigma)$*

$$\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \langle f|_{B(o,n)}, g|_{B(o,n)} \rangle_{\mathcal{A}^2(\sigma)} + \sum_{|y|=n+1} \big(b_n f(y) - b'_n f(p(y)) \big) \overline{g(y)},$$

where b_n is defined in (4.5).

The constant b'_m has a structure similar to b_m, as can be seen in the proof below, but we are not interested in it.

Proof. Let $f \in \mathcal{A}^2(\sigma)$. We have that

$$\langle f, g_n^H \rangle_{\mathcal{A}^2(\sigma)} = \sum_{x \in B(o,n)} f(x)\overline{g(x)}\sigma(x) + \sum_{m=n+1}^{+\infty} \sigma_m \sum_{|x|=m} f(x)\overline{g_n^H(x)}.$$

Observe that from the definition of g_n^H and Proposition 4.2

$$\begin{aligned}
\sum_{|x|=m} f(x) g_n^H(x) &= \sum_{|y|=n+1} \sum_{\substack{|x|=m \\ x \in T_y}} f(x)\overline{g_n^H(x)} \\
&= \sum_{|y|=n+1} \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \left[\left(\sum_{k=0}^{m-n-1} q^k \right) f(y) - \left(\sum_{k=0}^{m-n-2} q^k \right) f(p(y)) \right] \overline{g(y)} \\
&= \sum_{|y|=n+1} \left[f(y) q^{m-n-1} \left(\sum_{j=0}^{m-n-1} q^{-j} \right)^2 - f(p(y)) \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \left(\sum_{k=0}^{m-n-2} q^k \right) \right] \overline{g(y)}.
\end{aligned}$$

Then by summing for m, we have

$$\begin{aligned}
\sum_{m=n+1}^{+\infty} \sigma_m \sum_{|x|=m} f(x)\overline{g_n^H}(x) &= \sum_{|y|=n+1} \left[f(y)\overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m q^{m-n-1} \left(\sum_{j=0}^{m-n-1} q^{-j} \right)^2 \right] \\
&\quad - \sum_{|y|=n+1} \left[f(p(y))\overline{g(y)} \sum_{m=n+1}^{+\infty} \sigma_m \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \left(\sum_{k=0}^{m-n-2} q^k \right) \right] \\
&= \sum_{|y|=n+1} \big(b_n f(y) - B'_n f(p(y)) \big) \overline{g(y)}.
\end{aligned}$$

□

4.1.3 Orthonormal basis of $\mathcal{A}^2(\sigma)$

We now focus on the case $p = 2$. The goal of this section is the construction of an orthonormal basis for the space $\mathcal{A}^2(\sigma)$.

Let us consider the linear spaces

$$W_v := \Big\{ \varphi \colon s(v) \to \mathbb{C} \colon \sum_{z \in s(v)} \varphi(z) = 0 \Big\} \simeq \begin{cases} \mathbb{C}^q, & v = o, \\ \mathbb{C}^{q-1}, & v \in X \backslash \{o\}. \end{cases}$$

For convenience we introduce the following intervals of integer numbers: for every $v \in X$ we set $I_v = \{1, \ldots, |s(v)|\}$. We fix orthonormal basis $\{e_{v,j}\}_{j \in I_v}$ of W_v w.r.t. to the scalar product

$$\langle \varphi, \psi \rangle_{W_v} = \sum_{y \in s(v)} \varphi(y)\overline{\psi(y)}.$$

Let $v \in X$ and $j \in I_v$. We consider the function defined by $E_{v,j}(x) = e_{v,j}\mathbb{1}_{s(v)}$. It is easy to see that $E_{v,j}$ is harmonic and null on $B(o,|v|)$. We denote the harmonic extension of $E_{v,j}$ by $f_{v,j} = (E_{v,j})^H_{|v|}$, namely

$$f_{v,j}(x) = \begin{cases} 0, & \text{if } x \notin T_v \backslash \{v\}, \\ a_{|x|-|v|-1} e_{v,j}(p^{|x|-|v|-1}(x)), & \text{otherwise.} \end{cases}$$

Hence $f_{v,j}$ is harmonic for every $v \in X$ and $j \in I_v$. Furthermore $f_{v,j}$ is bounded, since $|f_{v,j}| \leqslant (1-q^{-1})^{-1}\|e_{v,j}\|_{W_v,\infty}$, and then $f_{v,j} \in \mathcal{A}^2(\sigma)$ for every reference measure σ. Observe that $f_{v,j}(v) = E_{v,j}(v) = 0$ for every $v \in X$ and $j \in I_v$. More precisely, if $v \neq o$, then

$$\operatorname{supp} f_{v,j} \subseteq T_v \backslash \{v\}. \tag{4.6}$$

Finally, we denote by $f_0 = \mathbb{1}_X$. Of course, $f_0 \in \mathcal{A}^2(\sigma)$, too.

Notice that the family

$$\mathcal{F} = \{f_0\} \cup \{f_{v,j} \colon v \in X,\, j \in I_v\} \subseteq \mathcal{A}^2(\sigma) \tag{4.7}$$

is independent of the choice of the reference measure σ. In the following we prove that $\mathcal{F}$ is an orthogonal system in every $\mathcal{A}^2(\sigma)$. In the proofs we use that $(e_{v,j})_{j \in I_v}$ are orthonormal and that the harmonic extension that we have introduced is radial on sectors.

Proposition 4.7. *The family $\mathcal{F}$ is a complete orthogonal system in $\mathcal{A}^2(\sigma)$ for every reference measure σ.*

Proof. Fix a reference measure σ. The fact that f_0 is orthogonal to every function of the family follows from the harmonicity of $f_{v,j}$, $v \in X$ and $j \in I_v$, and Corollary 4.4 which imply that the average of $f_{v,j}$ on each sphere centered in o is a multiple of $f_{v,j}(o) = 0$. Indeed

$$\langle f_{v,j}, f_0 \rangle_{\mathcal{A}^2(\sigma)} = \sum_{x \in X} f_{v,j}(x)\sigma(x) = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} f_{v,j}(x) = 0.$$

Let us consider $v, w \in X$ with $v \neq w$. Without loss of generality we can consider two situations: either $T_v \cap T_w = \varnothing$ or $T_v \subsetneq T_w$. In the first case $f_{v,j} \perp f_{w,k}$ for every $j \in I_v$ and $k \in I_w$, because their supports are disjoint. If $T_v \subsetneq T_w$, then we can suppose that $|w| \leqslant |v|$. By the fact that $f_{v,j}|_{B(o,|w|+1)} = 0$, from Lemma 4.6 we have

$$\langle f_{v,j}, f_{w,k} \rangle_{\mathcal{A}^2(\sigma)} = \sum_{|y|=|w|+1} (b_{|w|} f_{v,j}(y) - B'_{|w|} f_{v,j}(p(y)))\overline{E_{w,k}(y)} = 0$$

It remains to prove orthogonality in the case $v = w$. Let $j, k \in I_v$ $(= I_w)$ be such that $j \neq k$. We know that $f_{v,k}|_{B(o,|v|)} = 0$, then by Lemma 4.6 we have

$$\langle f_{v,j}, f_{v,k} \rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{|y|=|v|+1} E_{v,j}(y)\overline{E_{v,k}(y)} = b_{|v|} \sum_{y \in s(v)} e_{v,j}(y)\overline{e_{v,k}(y)} = 0,$$

where we used the fact that $\mathrm{supp}(E_{v,k}), \mathrm{supp}(E_{v,j}) \subseteq s(v)$ and the orthogonality of $e_{v,j}$ and $e_{v,k}$ in W_v.

We show now that $\mathcal{F}$ is complete. Take $g \in \mathcal{A}^2(\sigma)$ such that $\langle g, f\rangle_{\mathcal{A}^2(\sigma)} = 0$ for every $f \in \mathcal{F}$. We will show that g is the null function in $\mathcal{A}^2(\sigma)$. In particular we prove by induction that $g = 0$ on every $B(o, m)$, $m \in \mathbb{N}$.
We start by observing that $\langle g, f_0\rangle_{\mathcal{A}^2(\sigma)} = 0$ implies $g(o) = 0$. Indeed by (4.4)

$$0 = \langle g, f_0\rangle_{\mathcal{A}^2(\sigma)} = \sum_{n=0}^{+\infty} \sigma_n \sum_{|x|=n} g(x) = \left(1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^n \sigma_n\right) g(o) = B_\sigma g(o). \tag{4.8}$$

We assume now $g = 0$ on $B(o, m)$ for some $m \in \mathbb{N}$. Let $v \in S(o, m)$. Observe that by the fact that g is harmonic and $g(v) = 0$, we have $g|_{s(v)} \in W_v$. Hence for every $j \in I_v$

$$0 = \langle g, f_{v,j}\rangle_{\mathcal{A}^2(\sigma)} = b_m \sum_{y \in s(v)} \overline{e_{v,j}(y)} g(y) \tag{4.9}$$

and this implies that $g(y) = 0$ for every $y \in s(v)$ and so for every $y \in S(o, m+1)$, that is g vanishes on $B(o, m+1)$. The thesis follows by induction. □

We have proved that, for every reference measure σ, $\mathcal{F} \subseteq \mathcal{A}^2(\sigma)$ is a complete orthogonal system. We now fix a measure σ and compute the norm of the functions of the family $\mathcal{F}$ in $\mathcal{A}^2(\sigma)$. It is immediate to see that $\|f_0\|^2_{\mathcal{A}^2(\sigma)} = B_\sigma$ Let $v \in X$ and $j \in I_v$. By (4.9), we have

$$\|f_{v,j}\|^2_{\mathcal{A}^2(\sigma)} = \langle f_{v,j}, f_{v,j}\rangle_{\mathcal{A}^2(\sigma)} = b_{|v|} \sum_{y \in s(v)} \overline{e_{v,j}(y)} e_{v,j}(y) = b_{|v|}. \tag{4.10}$$

Hence the norm of $f_{v,j}$ does not depend on j and coincides with the constant in (4.5). Hence

$$\mathcal{F}_\sigma = \{B_\sigma^{-\frac{1}{2}} f_0\} \cup \{b_{|v|}^{-\frac{1}{2}} f_{v,j} \colon v \in X,\, j \in I_v\} \tag{4.11}$$

is an orthonormal basis of $\mathcal{A}^2(\sigma)$.

4.2 The reproducing kernel of $\mathcal{A}^2(\sigma)$

In this section we analyze an important aspect of the Bergman spaces $\mathcal{A}^2(\sigma)$: they are reproducing kernel Hilbert spaces. In the following we present a recursive formula for the kernel and we find its explicit formula. Observe that the main ingredient used in the proofs of the formulae are the harmonic extension and the orthonormal basis defined in the previous section together with the fact that W_v, $v \in X$, are reproducing kernel Hilbert spaces, too.

Let $z \in X$. We consider the evaluation functional $\Phi_z \colon \mathcal{A}^2(\sigma) \to \mathbb{C}$ defined by $\Phi_z g = g(z)$. Observe that Φ_z is a bounded operator, indeed by the Cauchy-Schwarz inequality

$$|g(z)| = \frac{1}{q+1} \left|\sum_{x \sim z} g(x)\right| \leqslant \frac{1}{q+1} \sum_{x \sim z} |g(x)| \leqslant \frac{1}{q+1} \|g\|_{L^2(\sigma)} \left\| \frac{\mathbb{1}_{S(z,1)}}{\sigma} \right\|_{L^2(\sigma)},$$

where $\mathbb{1}_{S(z,1)}$ is the characteristic function of the sphere $S(z,1)$. Let $x \in S(z,1)$, observe that $\sigma(x) \geqslant \sigma_{|z|+1}$ since $|x| \leqslant |z|+1$. Hence

$$\left\| \frac{\mathbb{1}_{S(z,1)}}{\sigma} \right\|^2_{L^2(\sigma)} = \sum_{d(z,x)=1} \frac{1}{\sigma(x)^2} \leqslant \sum_{d(z,x)=1} \frac{1}{\sigma^2_{|z|+1}} = \frac{q+1}{\sigma^2_{|z|+1}}.$$

Hence

$$|g(z)| \leqslant (q+1)^{-\frac{1}{2}} \sigma^{-1}_{|z|+1} \|g\|_{L^2(\sigma)}.$$

Thus $\mathcal{A}^2(\sigma)$ is a reproducing kernel Hilbert space (RKHS), that is for every $z \in X$ there exists $K_z \in \mathcal{A}^2(\sigma)$ such that

$$\langle g, K_z \rangle_{\mathcal{A}^2(\sigma)} = g(z), \qquad g \in \mathcal{A}^2(\sigma).$$

Since $\mathcal{F}_\sigma$ defined in (4.11) is an orthonormal basis of $\mathcal{A}^2_\sigma$, for every $z \in X$ we can write

$$K_z = \sum_{f \in \mathcal{F}_\sigma} \langle K_z, f \rangle_{\mathcal{A}^2(\sigma)} f = \sum_{f \in \mathcal{F}_\sigma} f(z) f = \frac{1}{B_\sigma} + \sum_{v \in X} \sum_{j \in I_v} \frac{f_{v,j}(z) f_{v,j}}{b_{|v|}}. \tag{4.12}$$

We recall that by (4.6), for every $z \in X$

$$\{v \in X \colon f_{v,j}(z) \neq 0 \text{ for some } j \in I_v\} \subseteq B(o, |z|-1).$$

Hence for every $z \in X$ the sum in (4.12) is finite and the decomposition of K_z holds true pointwisely.

Our goal is to compute K_z. We introduce an auxiliary function $\Gamma \colon X^3 \to \mathbb{R}$. For every $(v,z,x) \in X^3$ we set

$$\Gamma(v,z,x) = \begin{cases} 0, & \text{if } \{z,x\} \nsubseteq T_v \backslash \{v\}; \\ \dfrac{|s(v)|-1}{|s(v)|}, & \text{if } \{z,x\} \subseteq T_y \text{ for some } y \in s(v); \\ -\dfrac{1}{|s(v)|}, & \text{otherwise.} \end{cases}$$

Observe that Γ is symmetric in the second and third component, that is $\Gamma(v,z,x) = \Gamma(v,x,z)$. Furthermore, $\Gamma(v,z,\cdot)$ is the null function if $z \notin T_v \backslash \{v\}$ and whenever $z \in T_v \backslash \{v\}$ we have $\operatorname{supp}(\Gamma(v,z,\cdot)) = T_v \backslash \{v\}$. Finally, we have that the value of $\Gamma(v,z,\cdot)$ on $T_v \backslash \{v\}$ is completely determined by the values on $s(v)$, indeed the value of $\Gamma(v,z,\cdot)$ at $x \in T_v \backslash \{v\}$ is equal to the value at $p^{|x|-|v|-1}(x) \in s(v)$.

Since W_v is a RKHS for every $v \in X$, for $z \in s(v)$ we have

$$\varphi(z) = \langle \varphi, \Gamma(v,z,\cdot) \rangle_{W_v}, \qquad \varphi \in W_v.$$

Indeed $\Gamma(v,z,\cdot) \in W_v$ because

$$\sum_{y \in s(v)} \Gamma(v,z,y) = -(|s(v)|-1)\frac{1}{|s(v)|} + \frac{|s(v)|-1}{|s(v)|} = 0.$$

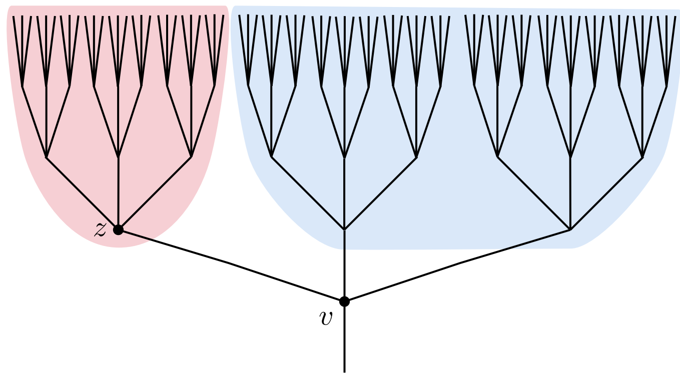

Figure 4.2: Partial representations of the function $\Gamma(v,z,\cdot)$ on T_v. The value of $\Gamma(v,z,\cdot)$ at vertices in the red area is $\frac{|s(v)|-1}{|s(v)|}$ while in the blue area is $-\frac{1}{|s(v)|}$. Clearly, $\Gamma(v,z,v)=0$. Observe that the values of $\Gamma(v,z,\cdot)$ do not change as z moves in the red area.

Furthermore,

$$\begin{aligned}\langle\varphi,\Gamma(v,z,\cdot)\rangle_{W_v} &= \frac{|s(v)|-1}{|s(v)|}\varphi(z)-\frac{1}{|s(v)|}\sum_{\substack{y\in s(v)\\ y\neq z}}\varphi(y)\\ &= \frac{|s(v)|-1}{|s(v)|}\varphi(z)+\frac{1}{|s(v)|}\varphi(z)=\varphi(z),\end{aligned}$$

by the fact that $\varphi\in W_v$.

It is easy to see that $\Gamma(v,z,\cdot)$ is harmonic on $B(o,|v|)$ and then we can consider the harmonic extension $(\Gamma(v,z,\cdot))^H_{|v|}$, which is bounded by construction. Indeed it is easy to observe that from the definition of harmonic extension we have for every $x\in T_v\backslash\{v\}$

$$(\Gamma(v,z,\cdot))^H_{|v|}(x)=\left(\sum_{j=0}^{|x|-|v|-1}q^{-j}\right)\Gamma(v,z,p^{|x|-|v|-1}(x))=a_{|x|-|v|-1}\Gamma(v,z,x), \quad (4.13)$$

and it vanishes elsewhere. We recall that if $z\notin T_v$, then $\Gamma(v,z,\cdot)=(\Gamma(v,z,\cdot))^H_{|v|})$ is the null function.

Proposition 4.8. *Let $z\in X$ and $[o,z]=\{v_t\}_{t=0}^{|z|}$. The kernel K_z is*

$$K_z=\begin{cases}\dfrac{1}{B_\sigma}, & \text{if } z=o,\\ K_o+\dfrac{1}{b_0}(\Gamma(o,z,\cdot))^H_0, & \text{if } |z|=1,\\ -\dfrac{1}{q}K_{v_{m-2}}+\dfrac{q+1}{q}K_{v_{m-1}}+\dfrac{1}{b_{m-1}}(\Gamma(v_{m-1},z,\cdot))^H_{m-1}, & \text{if } |z|=m>1.\end{cases}$$

Proof. Since the measure σ is finite and the constant functions are harmonic, $K_o=\frac{1}{B_\sigma}\in\mathcal{A}^2(\sigma)$. The reproducing property follows from (4.8).
We prove the case $|z|=1$. The function $K_z\in\mathcal{A}^2(\sigma)$ because it is sum of functions in

$\mathcal{A}^2(\sigma)$. We prove the reproducing property. For $g \in \mathcal{A}^2(\sigma)$, by the reproducing formula of K_o and Lemma 4.6

$$\begin{aligned}\langle g, K_z\rangle_{\mathcal{A}^2(\sigma)} &= g(o) + \frac{1}{b_1}\langle g, (\Gamma(o,z,\cdot))_0^H\rangle_{\mathcal{A}^2(\sigma)}\\ &= g(o) + \sum_{|y|=1} g(y)\Gamma(o,z,y)\\ &= g(o) + \frac{q}{q+1}g(z) - \frac{1}{q+1}\sum_{\substack{|y|=1\\ y\neq z}} g(y) = g(z),\end{aligned}$$

where we used that g is harmonic at o. This proved the case $|z| = 1$.
It remains to prove the case $|z| = m > 1$. We have $K_z \in \mathcal{A}^2(\sigma)$ since it is sum of bounded and harmonic functions. It remains to prove the reproducing formula, for $g \in \mathcal{A}^2(\sigma)$ by Lemma 4.6 we have

$$\begin{aligned}\langle g, K_z\rangle_{\mathcal{A}^2(\sigma)} &= -\frac{1}{q}g(v_{m-2}) + \frac{q+1}{q}g(v_{m-1}) + \frac{1}{b_m}\langle g, (\Gamma(v_{m-1},z,\cdot))_{m-1}^H\rangle_{\mathcal{A}^2(\sigma)}\\ &= -\frac{1}{q}g(v_{m-2}) + \frac{q+1}{q}g(v_{m-1}) + \sum_{y\in s(v_{m-1})}\Gamma(v_{m-1},z,y)g(y)\\ &= -\frac{1}{q}g(v_{m-2}) + \frac{1}{q}\sum_{y\sim v_{m-1}} g(y) + \frac{q-1}{q}g(z) - \frac{1}{q}\sum_{\substack{y\in s(v_{m-1})\\ y\neq z}} g(y) = g(z),\end{aligned}$$

where we used the fact that g is harmonic at v_{m-1}. □

Hence we expressed the kernel K_z through a two-step recursive formula. We aim to find an explicit formula for K_z.

Theorem 4.9. *For every* $(z,x) \in X \times X$

$$K(z,x) = \frac{1}{B_\sigma} + \frac{q^2}{(q-1)^2}\sum_{v\in X}\frac{1}{b_{|v|}}\Gamma(v,z,x)(1-q^{|v|-|z|})(1-q^{|v|-|x|}). \tag{4.14}$$

Proof. Let $z \in X$ and $[o,z] = \{v_t\}_{t=0}^{|z|}$. We start by proving that

$$K_z = \frac{1}{B_\sigma} + \sum_{t=0}^{|z|-1}\left(\sum_{j=0}^{|z|-t-1} q^{-j}\right)\frac{1}{b_t}(\Gamma(v_t,v_{t+1},\cdot))_t^H. \tag{4.15}$$

The case $z = o$ trivially follows from Proposition 4.8. We prove (4.15) by induction on $m = |z| \geqslant 1$. The case $m = 1$ directly follows from Proposition 4.8, too. Let $m \in \mathbb{N}$, $m > 1$ and $z \in X$, $|z| = m$. Suppose that (4.15) holds for every vertex in $B(o, m-1)$.

Hence by (4.8) we have

$$
\begin{aligned}
K_z =& -\frac{1}{q}K_{v_{m-2}} + \frac{q+1}{q}K_{v_{m-1}} + \frac{1}{B_m}(\Gamma(v_{m-1},z,\cdot))^H_{m-1} \\
=& -\frac{1}{q}\left[\frac{1}{B_\sigma} + \sum_{t=0}^{m-3}\left(\sum_{j=0}^{m-t-3} q^{-j}\right)\frac{1}{b_t}(\Gamma(v_t,v_{t+1},\cdot))^H_t\right] \\
&+ \frac{q+1}{q}\left[\frac{1}{B_\sigma} + \sum_{t=0}^{m-2}\left(\sum_{j=0}^{m-t-2} q^{-j}\right)\frac{1}{b_t}(\Gamma(v_t,v_{t+1},\cdot))^H_t\right] \\
&+ \frac{1}{b_m}(\Gamma(v_{m-1},z,\cdot))^H_{m-1} \\
=& \frac{1}{B_\sigma} + \sum_{t=0}^{m-1}\left(\frac{q+1}{q}\left(\sum_{j=0}^{m-t-2} q^{-j}\right) - \frac{1}{q}\left(\sum_{j=0}^{m-t-3} q^{-j}\right)\right)\frac{1}{b_t}(\Gamma(v_t,v_{t+1},\cdot))^H_t \\
=& \frac{1}{B_\sigma} + \sum_{t=0}^{m-1}\left(\sum_{j=0}^{m-t-1} q^{-j}\right)\frac{1}{b_t}(\Gamma(v_t,v_{t+1},\cdot))^H_t .
\end{aligned}
$$

Hence we proved (4.15) by induction. Since $\operatorname{supp}((\Gamma(v_t,v_{t+1},\cdot))^H_t) = T_{v_t}\backslash\{v_t\}$, we have that the t-th term of the sum in (4.15) does not vanish if and only if $x \in T_{v_t}\backslash\{v_t\}$ and hence by (4.13), we have

$$
\begin{aligned}
K(z,x) = K_z(x) &= \frac{1}{B} + \sum_{v\in X}\frac{1}{b_{|v|}}\left(\sum_{j=0}^{|z|-|v|-1} q^{-j}\right)\left(\sum_{j=0}^{|x|-|v|-1} q^{-j}\right)\Gamma(v,z,x) \\
&= \frac{1}{B} + \frac{q^2}{(q-1)^2}\sum_{v\in X}\frac{1}{b_t}(1-q^{|v|-|z|})(1-q^{|v|-|x|})\Gamma(v,z,x).
\end{aligned}
$$

□

We call the confluent of two vertices of the tree z, $x \in X$ the element $z \wedge x \in X$ defined by

$$
z \wedge x := \operatorname*{argmax}_{t\in[o,m]}\{v_t \colon v_t \in [o,x]\}, \qquad \{v_t\}_{t=0}^{|z|} = [o,z].
$$

The confluent $z \wedge x$ is the common vertex of $[o,x]$ and $[o,z]$ farthest from o. It is possible to see that the value of the kernel K at $(z,x) \in X \times X$ depends only on the values of $|x|$, $|z|$ and $|z \wedge x|$. Furthermore, from (4.14) it is clear that K is symmetric, that is $K(z,x) = K(x,z)$.

4.3 Boundedness of the Bergman projector

In this section we study the boundedness properties of the extension of the Bergman projector to L^p spaces.

We restrict our attention to a family of reference measures for which we are able to prove that the extension of the Bergman projector to $L^p(X)$ is bounded if and only if $p > 1$, see Theorem 4.15.

In [17], authors introduce the definition of optimal measure, that is a reference measure σ for which

$$\sup_{n\in\mathbb{N}} \frac{1}{\sigma_n q^n} \sum_{m=n}^{+\infty} \sigma_m q^m < +\infty.$$

It is easy to see that the measures that we study below are optimal. We know that the boundedness of the operator on $L^1(\sigma)$ is false for every optimal measure. On the other hand, we are aware that for some optimal measures the boundedness of the projector on $L^p(\sigma)$ holds for every $p > 1$. The characterization of the family of measure for which the Bergman projector is bounded if and only if $p > 1$ is part of the work that we are completing.

We focus our attention on kernels associated to reference measures σ of the form

$$\mu_\alpha(x) = q^{-\alpha|x|}, \qquad \alpha > 1.$$

This restriction allows us to prove two intermediate results: Theorems 4.10 and 4.11, from which the results regarding the extension of Bergman projectors follows. Since the proofs of Theorems 4.10 and 4.11 needs some technicalities, we present them in Section 4.4. The last result is Theorem 4.15 which states that the extension of the projector P_α associated to the measure μ_α to $L^p(\mu_\alpha)$ is bounded if and only if $p > 1$.

We shall use the notation L^p_α and $\mathcal{A}^p_\alpha$ for the Lebesgue and Bergman spaces w.r.t. μ_α, respectively. Furthermore, we denote by $K_\alpha \colon X \times X \to \mathbb{R}$ the reproducing kernel of $\mathcal{A}^2_\alpha$. It will be useful to keep track of the weight in the constants introduced in (4.5), so we denote them by $b_{\alpha,n}$. In particular observe that in this case there is a relation between the constants: if $n \in \mathbb{N}$

$$\begin{aligned} b_{\alpha,n} &= \sum_{m=n+1}^{+\infty} \left[q^{-\alpha m} \left(\sum_{k=0}^{m-n-1} q^k \right) \left(\sum_{j=0}^{m-n-1} q^{-j} \right) \right] \\ &= \sum_{\ell=1}^{+\infty} \left[q^{-\alpha(\ell+n)} \left(\sum_{k=0}^{\ell-1} q^k \right) \left(\sum_{j=0}^{\ell-1} q^{-j} \right) \right] = q^{-\alpha n} b_{\alpha,0}. \end{aligned} \tag{4.16}$$

Furthermore we put $B_\alpha = \mu_\alpha(X)$.

For any real parameters a, b and for $c > 1$, we define the following integral operators

$$S_{a,b,c} f(z) = q^{-a|z|} \sum_{x\in X} |K_c(z,x)| f(x) q^{-b|x|},$$

and

$$T_{a,b,c} f(z) = q^{-a|z|} \sum_{x\in X} K_c(z,x) f(x) q^{-b|x|}.$$

We are now in a position to state two results, which will imply as a corollary the boundedness properties of the Bergman projectors. Theorem 4.10 is devoted to the study of the boundedness of the operators $S_{a,b,c}$ and $T_{a,b,c}$ on weighted L^p-spaces for $p > 1$; the case $p = 1$ needs different arguments and for this reason is treated apart in Theorem 4.11. The proofs of both theorems are postponed to Section 4.4.

Theorem 4.10. *Let $\alpha \in \mathbb{R}$, $c > 1$ and $1 < p < \infty$. The following conditions are equivalent:*

(i) the operator $S = S_{a,b,c}$ *is bounded on* L^p_α*;*

(ii) the operator $T = T_{a,b,c}$ *is bounded on* L^p_α*;*

(iii) the parameters satisfy

$$c \leqslant a+b, \qquad -pa < \alpha - 1 < p\,(b-1)\,.$$

Theorem 4.11. *Let* $\alpha \in \mathbb{R}$ *and* $c > 1$*. The following conditions are equivalent:*

(i) the operator $S = S_{a,b,c}$ *is bounded on* L^1_α*;*

(ii) the operator $T = T_{a,b,c}$ *is bounded on* L^1_α*;*

(iii) the parameters either satisfy

$$c = a+b, \qquad -a < \alpha - 1 < b-1,$$

or satisfy

$$c < a+b, \qquad -a < \alpha - 1 \leqslant b-1.$$

We state a corollary which is simply a reformulation of the previous theorems when $c = a+b$.

Corollary 4.12. *Let* $1 \leqslant p < \infty$ *and* $\alpha \in \mathbb{R}$*. If* $a, b \in \mathbb{R}$ *are such that* $a+b > 1$*, then the following conditions are equivalent:*

(i) the operator $S = S_{a,b,a+b}$ *is bounded on* L^p_α*;*

(ii) the operator $T = T_{a,b,a+b}$ *is bounded on* L^p_α*;*

(iii) the parameters satisfy

$$-pa < \alpha - 1 < p\,(b-1)\,.$$

Let $\beta > 1$. Since $\mathcal{A}^2_\beta \subseteq L^2_\beta$ is a closed subspace of a Hilbert space, there exists an orthogonal projection $P_\beta \colon L^2_\beta \to \mathcal{A}^2_\beta$. Observe that by the reproducing property of $K_{\beta,z} = K_\beta(z,\cdot)$, $z \in X$, we can write the projection $P_\beta f$ of $f \in L^2_\beta$ as follows

$$P_\beta f(z) = \langle P_\beta f, K_{\beta,z}\rangle_{\mathcal{A}^2_\beta} = \langle f, P_\beta K_{\beta,z}\rangle_{L^2_\beta} = \langle f, K_{\beta,z}\rangle_{L^2_\beta},$$

where we used the orthogonality of P_β. Hence we can rewrite P_β as the integral operator on L^2_β induced by the reproducing kernel K_β, that is

$$P_\beta f(z) = \sum_{x \in X} K_\beta(z,x) f(x) q^{-\beta|x|}, \qquad f \in L^2_\beta,\ z \in X. \tag{4.17}$$

Now we state a preliminary result which, as well as being useful in the proofs in the next section, shows that there is a natural family $\{f_n\}_{n \in N} \subseteq L^1_\alpha$ such that $P_\alpha f_n$ diverges in L^1_α, and then that P_α is not bounded on L^1_α.

Lemma 4.13. *Let $\alpha > 1$. Then:*

$$\sum_{z\in X} |K_\alpha(x,z)|q^{-\alpha|z|} \gtrsim |x|, \qquad x \in X.$$

Proof. For every $x \in X\backslash\{o\}$, we put $\{v_t\}_{t=0}^{|x|} = [o,x]$, then by (4.14) and $\alpha > 1$

$$\begin{aligned}
\sum_{z\in X} |K_\alpha(x,z)|q^{-\alpha|z|} &\geqslant \sum_{t=1}^{|x|} |K_\alpha(v_t,x)|q^{-\alpha t} \\
&= \sum_{t=1}^{|x|} \left(\frac{1}{B_\alpha} + \frac{q^2}{(q+1)^2} \sum_{v\in X} \frac{1}{b_{\alpha,|v|}} \Gamma(v,v_t,x)(1-q^{|v|-t})(1-q^{|v|-|x|}) \right) q^{-\alpha t} \\
&\gtrsim b_{\alpha,o}^{-1} \sum_{t=1}^{|x|} \sum_{v\in X} q^{\alpha(|v|-t)} \Gamma(v,v_t,x)(1-q^{|v|-t})(1-q^{|v|-|x|}) \\
&= \sum_{t=1}^{|x|} \sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)} \Gamma(v_\ell,v_t,x)(1-q^{\ell-t})(1-q^{\ell-|x|}) \\
&\gtrsim \sum_{t=1}^{|x|} \sum_{\ell=0}^{t-1} q^{\alpha(\ell-t)} \simeq \sum_{t=1}^{|x|} q^{-\alpha t} q^{\alpha t} = |x|,
\end{aligned}$$

where we used the fact that $\operatorname{supp}(\Gamma(\cdot,v_t,x)) = [o,v_{t-1}] = [v_0,v_{t-1}]$ and the function is greater than or equal to $\frac{q-1}{q}$ there. □

Corollary 4.14. *Let $\alpha > 1$. Then P_α is unbounded on $L^1_\alpha(X)$.*

Proof. For every $n \in \mathbb{N}$, we fix a vertex $v_n \in X$, $|v_n| = n$, and define

$$f_n(x) = \delta_{v_n}(x) q^{\alpha|x|}, \qquad x \in X.$$

It is easy to show that $\|f_n\|_{L^1_\alpha} = 1$. On the other hand, we have $P_\alpha f_n(z) = K_\alpha(z,v_n)$, and then by Lemma 4.13

$$\|P_\alpha f_n\|_{L^1_\alpha} = \sum_{z\in X} |K_\alpha(z,v_n)|q^{-\alpha|z|} \gtrsim |v_n| = n,$$

which tends to $+\infty$ as $n \to +\infty$. □

The case $p = 1$ is actually the only value of $1 \leqslant p < \infty$ for which P_α is not bounded on L^p_α. This follows from Corollary 4.12.

Theorem 4.15. *Let $1 \leqslant p < \infty$, $\alpha,\beta > 1$. The operator P_β is bounded from L^p_α to $\mathcal{A}^p_\alpha$ if and only if*

$$p(\beta-1) > \alpha - 1.$$

In particular, P_α is bounded from L^p_α to $\mathcal{A}^p_\alpha$ if and only if $p > 1$.

Proof. It is sufficient to observe that from (4.17), $P_\beta = T_{0,\beta,\beta}$. Hence, from Corollary 4.12, the boundedness of P_β on $L^p_\alpha(X)$ is equivalent to $p(\beta-1) > \alpha - 1 (> 0)$. □

As a direct application of Theorem 4.15, we deduce the following result on the dual of Bergman spaces.

Corollary 4.16. *Let $1 < p < \infty$ and $\alpha > 1$. Then*

$$(\mathcal{A}^p_\alpha)^* = \mathcal{A}^{p'}_\alpha,$$

with equivalent norms under the pairing

$$\langle f, g \rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha} = \sum_{z \in X} f(z) g(z) q^{-\alpha|z|} \qquad f \in \mathcal{A}^p_\alpha, g \in \mathcal{A}^{p'}_\alpha. \tag{4.18}$$

Proof. Let $g \in \mathcal{A}^{p'}_\alpha$. By Hölder inequality we have that

$$|\langle f, g \rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha}| \leqslant \|g\|_{\mathcal{A}^{p'}_\alpha} \|f\|_{\mathcal{A}^p_\alpha},$$

for every $f \in \mathcal{A}^p_\alpha$ and then g defines an operator in $(\mathcal{A}^p_\alpha)^*$. Conversely, for $\Phi \in (\mathcal{A}^p_\alpha)^*$, then by Hahn-Banach extension theorem, there exists $\tilde{\Phi} \in (L^p_\alpha)^*$ such that $\tilde{\Phi}|_{\mathcal{A}^p_\alpha} = \Phi$ and $\|\Phi\|_{(\mathcal{A}^p_\alpha)^*} \geqslant \|\tilde{\Phi}\|_{(L^p_\alpha)^*}$. Then by the duality on L^p spaces there exists $h \in L^{p'}$ such that

$$\Phi(f) = \tilde{\Phi}(f) = \langle f, h \rangle_{L^p_\alpha \times L^{p'}_\alpha},$$

for every $f \in \mathcal{A}^p_\alpha$. By the orthogonality of P_α and Theorem 4.15,

$$\Phi(f) = \langle P_\alpha f, P_\alpha h \rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha} = \langle f, P_\alpha h \rangle_{\mathcal{A}^p_\alpha \times \mathcal{A}^{p'}_\alpha}.$$

Hence Φ corresponds to $P_\alpha h \in \mathcal{A}^2_\alpha$ under the pairing (4.18). □

4.4 Proof of Theorems 4.10 and 4.11

This section is devoted to the proofs of Theorems 4.10 and 4.11, splitting up the proofs in various results. In both statements it is straightforward to see that (i) implies (ii). For the rest of the section α, a, b, c are real parameters with $c > 1$.

4.4.1 Proof of (ii) implies (iii)

In this subsection we suppose that the operator $T_{a,b,c}$ is bounded on L^p_α and we deduce necessary conditions on the parameters a, b, c, α in various lemmas.

Lemma 4.17. *Let $1 \leqslant p < \infty$. If $T_{a,b,c} f \in L^p_\alpha$ for every $f \in L^p_\alpha$, then $-pa < \alpha - 1$.*

Proof. Consider $f(x) = q^{-N|x|}$ with $N \in \mathbb{R}$ such that

$$N > \max\left\{\frac{1-\alpha}{p}, 1-b\right\}.$$

Since $N > \frac{1-\alpha}{p}$ we have that $f \in L^p_\alpha$ and

$$\begin{aligned} T_{a,b,c}f(z) &= q^{-a|z|} \sum_{x \in X} K_c(z,x) q^{-(b+N)|x|} \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+N)n} \sum_{|x|=n} K_c(z,x) \\ &= q^{-a|z|} \sum_{n=0}^{+\infty} q^{-(b+N)n} |S(o,n)| K_c(z,o) \end{aligned}$$

by Corollary 4.3 applied to the harmonic function $K_c(z,\cdot)$. Hence, since $N > 1-b$

$$T_{a,b,c}f(z) = q^{-a|z|} \frac{1}{B_c} \left[1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(-b-N+1)n} \right] = \frac{B_{b+N}}{B_c} q^{-a|z|}.$$

Now observe that $T_{a,b,c}f \in L^p_\alpha$ implies

$$\sum_{z \in X} q^{-(ap+\alpha)|z|} = 1 + \frac{q+1}{q} \sum_{n=1}^{+\infty} q^{(1-ap-\alpha)n} < +\infty,$$

which holds if and only if $-pa < \alpha - 1$, as required. □

From now on we put

$$\|e_{v,j}\|_p = \left(\sum_{y \in s(v)} |e_{v,j}(y)|^p \right)^{1/p}, \qquad v \in X,\ j \in I_v,\ 1 \leqslant p < \infty.$$

Lemma 4.18. *Let $1 \leqslant p < \infty$. If $T_{a,b,c}$ is bounded on L^p_α, then $a + b \geqslant c$.*

Proof. Fix a positive integer N such that

$$N > \max\left\{ \frac{1-\alpha}{p}, c - b \right\}.$$

For every $v \in X \backslash \{o\}$ and $j \in I_v$, we define $g_{v,j}(x) = f_{v,j}(x) q^{-N|x|}$, where $f_{v,j} \in \mathcal{F}$ are defined in (4.7). By the condition $N > \frac{1-\alpha}{p}$, we have that $g_{v,j} \in L^p_\alpha$; then

$$\begin{aligned} T_{a,b,c}g_{v,j}(z) &= q^{-a|z|} \sum_{x \in X} K_c(z,x) f_{v,j}(x) q^{-(b+N)|x|} \\ &= q^{-a|z|} \langle f_{v,j}, K_{c,z} \rangle_{L^2_{b+N}} \end{aligned}$$

since $N > c - b$ implies $K_{c,z} \in L^2_c \subseteq L^2_{b+N}$. Now we use the decomposition (4.12) of $K_{c,z}$ on the basis of $\mathcal{A}^2_c$ and obtain

$$\begin{aligned} \langle K_{c,z}, f_{v,j} \rangle_{L^2_{b+N}} &= \langle \frac{1}{B_c} + \sum_{u \in X} \sum_{k \in I_u} \frac{f_{u,k}(z) f_{u,k}}{b_{c,|u|}}, f_{v,j} \rangle_{L^2_{b+N}} \\ &= \frac{f_{v,j}(z)}{b_{c,|v|}} \langle f_{v,j}, f_{v,j} \rangle_{L^2_{b+N}} \\ &= \frac{b_{b+N,|v|}}{b_{c,|v|}} f_{v,j}(z), \end{aligned}$$

where we use the orthogonality of $\mathcal{F}$ and (4.10). We calculate the norm of $T_{a,b,c}g_{v,j}$ in L^p_α

$$\begin{aligned}\|T_{a,b,c}g_{v,j}\|^p_{L^p_\alpha} &= \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p \sum_{z\in X} |f_{v,j}(z)|^p q^{-(ap+\alpha)|z|} \\ &= \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p \sum_{n=0}^{+\infty} q^{-(ap+\alpha)n} \sum_{|z|=n} |f_{v,j}(z)|^p.\end{aligned}$$

Since $\operatorname{supp}(f_{v,j}) \subseteq T_v\backslash\{v\}$, the integral of $|f_{v,j}|^p$ on the sphere $S(o,n)$ vanishes for every $n \leqslant |v|$. If $n > |v|$, then $p^{|z|-n}(z)$ is the unique vertex in $s(v)$ in whose sector z lies. Hence

$$\begin{aligned}\sum_{|z|=n} |f_{v,j}(z)|^p &= \sum_{\substack{|z|=n\\ z\in T_v}} |e_{v,j}(p^{|z|-n}(z))|^p a^p_{n-|p^{|z|-n}(z)|} \\ &= a^p_{n-|v|-1} \sum_{\substack{|z|=n\\ z\in T_v}} |e_{v,j}(p^{|z|-n}(z))|^p \\ &= a^p_{n-|v|-1} q^{n-|v|-1} \sum_{y\in s(v)} |e_{v,j}(y)|^p \\ &= a^p_{n-|v|-1} q^{n-|v|-1} \|e_{v,j}\|^p_p.\end{aligned}$$

Hence we have

$$\begin{aligned}\|T_{a,b,c}g_{v,j}\|^p_{L^p_\alpha} &= \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p \sum_{n=|v|+1}^{+\infty} q^{-(ap+\alpha)n} a^p_{n-|v|-1} q^{n-|v|-1} \|e_{v,j}\|^p_p \\ &= \|e_{v,j}\|^p_p \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p \sum_{m=1}^{+\infty} q^{-(ap+\alpha)(m+|v|)} a^p_{m-1} q^{m-1} \\ &= \|e_{v,j}\|^p_p \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-2(ap+\alpha))m-1} a^p_{m-1} \\ &= \|e_{v,j}\|^p_p k(ap+\alpha) \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha)|v|},\end{aligned}$$

where the existence of the constant $k(ap+\alpha) > 0$ is guaranteed by $ap+\alpha > 1$, through

Lemma 4.17, and the fact that $1 \leqslant a_m < \frac{q}{q-1}$, for every $m \in \mathbb{N}$. On the other hand,

$$\begin{aligned}
\|g_{v,j}\|_{L^p_\alpha}^p &= \sum_{x\in X} |f_{v,j}(x)|^p q^{-(Np+\alpha)|x|} \\
&= \sum_{n=0}^{+\infty} q^{-(Np+\alpha)n} \sum_{|x|=n} |f_{v,j}(x)|^p \\
&= \sum_{n=|v|+1}^{+\infty} q^{-(Np+\alpha)n} a^p_{n-|v|-1} q^{n-|v|-1} \|e_{v,j}\|_p^p \\
&= \|e_{v,j}\|_p^p q^{-(Np+\alpha)|v|} \sum_{m=1}^{+\infty} q^{(1-(Np+\alpha))m-1} a^p_{m-1} \\
&= \|e_{v,j}\|_p^p k(Np+\alpha) q^{-(Np+\alpha)|v|},
\end{aligned}$$

with $k(Np+\alpha) \to 1$ when $N \to +\infty$. From the boundedness of $T_{a,b,c}$ we have that there exists a positive constant C such that for every $v \in X\backslash\{o\}$

$$C \geqslant \frac{\|T_{a,b,c}g_{v,j}\|^p_{L^p_\alpha}}{\|g_{v,j}\|^p_{L^p_\alpha}} \simeq \left(\frac{b_{b+N,|v|}}{b_{c,|v|}}\right)^p q^{-(ap+\alpha-Np-\alpha)|v|} \simeq q^{-p(N+b-c)|v|} q^{-(ap-Np)|v|},$$

from (4.16). Hence we have that $c \leqslant a+b$. □

Lemma 4.19. *Let $1 < p < \infty$. If $T_{a,b,c}$ is bounded on L^p_α, then $\alpha - 1 < p(\beta - 1)$.*

Proof. From Theorem 1.9 in [54], the boundedness of $T_{a,b,c}$ on L^p_α is equivalent to the boundedness of the adjoint operator $T^*_{a,b,c}$ on $L^{p'}_\alpha$. It is easy to see that

$$T^*_{a,b,c}g(x) = q^{-(b-\alpha)|x|} \sum_{z\in X} K_c(x,z) g(z) q^{-(a+\alpha)|z|} = T_{b-\alpha,a+\alpha,c}\, g(x) \qquad g \in L^{p'}_\alpha.$$

Hence, the fact that $T^*_{a,b,c}$ is bounded on $L^{p'}_\alpha$ implies, through Lemma 4.17, that $-p'(b-\alpha) < \alpha - 1$, that is $\alpha - 1 < p(\beta - 1)$. □

Lemmas 4.17, 4.18, 4.19 show that condition (ii) implies condition (iii) in Theorem 4.10. Now we focus on the same implication in the case $p = 1$.

Lemma 4.20. *If $T_{a,b,c}$ is bounded on L^1_α, then*

$$\begin{aligned}
&\alpha < b, \quad \textit{when } \ c = a+b; \\
&\alpha \leqslant b, \quad \textit{when } \ c < a+b.
\end{aligned}$$

Proof. From Lemma 4.18, if $T_{a,b,c}$ is bounded on L^1_α, then $c \leqslant a+b$. From Theorem 1.9 in [54], the boundedness of $T_{a,b,c}$ on L^1_α implies the the boundedness of the adjoint operator $T^*_{a,b,c}$ on L^∞_α defined by

$$T^*g(x) = q^{-(b-\alpha)|x|} \sum_{z\in X} K_c(x,z) g(z) q^{-(a+\alpha)|z|}, \quad g \in L^\infty_\alpha.$$

In particular, for $\mathbb{1}_X \in L^\infty_\alpha$, we have

$$\begin{aligned} T^*_{a,b,c}\mathbb{1}_X(x) &= q^{-(b-\alpha)|x|}\sum_{z\in X} K_c(x,z)q^{-(a+\alpha)|z|} \\ &= q^{-(b-\alpha)|x|}\frac{1}{b_{c,0}}\sum_{n=0}^{+\infty}|S(o,n)|q^{-(a+\alpha)n} = k(c,a+\alpha)q^{-(b-\alpha)|x|}, \end{aligned}$$

which belongs to L^∞_α if and only if $\alpha \leqslant b$.

Suppose now that $a+b=c$. We know that $\alpha \leqslant b$ and we want to prove that $\alpha < b$. Assume by contradiction that $\alpha = b$. Theorem 3.6 in [54] states that $T^*_{a,b,c}$ is bounded on L^∞_b, where

$$T^*g(x) = \sum_{z\in X} K_c(x,z)g(z)q^{-c|z|}, \quad g \in L^\infty_b.$$

The boundedness of $T^*_{a,b,c}$ on L^∞_b implies that

$$\sup_{x\in X}\sum_{z\in X}|K_c(x,z)|q^{-c|z|} < +\infty.$$

Which is a contradiction by Lemma 4.13. Hence $T_{a,b,c}$ is unbounded.

□

Lemmas 4.17, 4.18, 4.20 show that condition (ii) implies condition (iii) in Theorem 4.11.

4.4.2 Proof of (iii) implies (i)

We start by stating a technical lemma, which will be useful in both Propositions 4.22 and 4.23, that are devoted to prove that (iii) implies (i) in the case $p > 1$ and $p = 1$, respectively.

Lemma 4.21. *Let $\beta, \gamma > 1$. Then there exist $C_1, C_2, C_2' > 0$ depending only on β and γ such that*

$$\sum_{x\in X}|K_\gamma(z,x)|q^{-\beta|x|} \leqslant \begin{cases} C_1 + C_2 q^{-(\beta-\gamma)|z|}, & \text{if } \gamma \neq \beta, \\ C_1 + C_2'|z|, & \text{if } \gamma = \beta. \end{cases}$$

Proof. We start by observing that orthogonal basis $\{e_{v,j}\}_{j\in I_v}$ of W_v, $v \in X$, involved in the construction of functions in $\mathcal{F}$ are such that their 1-norms in W_v are bounded from above, namely

$$\|e_{v,j}\|_1 \leqslant \sqrt{|s(v)|}\|e_{v,j}\|_2 = \sqrt{|s(v)|} \leqslant \sqrt{q+1}, \qquad v \in X,\ j \in I_v.$$

Hence

$$\begin{aligned}\sum_{x\in X}|K_\gamma(z,x)|q^{-\beta|x|} &= \sum_{x\in X}\left|\frac{1}{B_\gamma}+\sum_{v\in X}\sum_{j\in I_v}\frac{f_{v,j}(z)f_{v,j}(x)}{b_{\gamma,|v|}}\right|q^{-\beta|x|}\\ &\leqslant C_1+\sum_{x\in X}\sum_{v\in X}\sum_{j\in I_v}\left|\frac{f_{v,j}(z)f_{v,j}(x)}{b_{\gamma,|v|}}\right|q^{-\beta|x|}\\ &= C_1+\sum_{v\in X}\frac{1}{b_{\gamma,|v|}}\sum_{j\in I_v}|f_{v,j}(z)|\sum_{x\in X}|f_{v,j}(x)|q^{-\beta|x|}\\ &\leqslant C_1+\sum_{v\in X}\frac{1}{b_{\gamma,|v|}}\sum_{j\in I_v}|f_{v,j}(z)|B_\beta^{-\frac12}\|f_{v,j}\|_{L^2_\beta}\\ &= C_1+B_\beta^{-\frac12}\sum_{v\in X}\frac{b_{\beta,|v|}}{b_{\gamma,|v|}}\sum_{j\in I_v}|f_{v,j}(z)|,\end{aligned}$$

where we use the fact that the measure μ_β is finite on X and thus $\|f\|_{L^1_\beta(X)}\leqslant B_\beta^{-\frac12}\|f\|_{L^2_\beta(X)}$, by Cauchy-Schwarz inequality. Now observe that from (4.6), we have that $f_{v,j}(z)=0$ if $z\notin T_v\backslash\{v\}$. Hence, if we denote by $\{v_\ell\}_{\ell=0}^{|z|}$ the path $[o,z]$, then

$$\sum_{j\in I_v}|f_{v,j}(z)|=\begin{cases}a_{|z|-\ell-1}\displaystyle\sum_{j\in I_v}|e_{v,j}(v_{\ell+1})|, & \text{if } v=v_\ell,\ 0\leqslant\ell<|z|;\\ 0, & \text{otherwise.}\end{cases}$$

Therefore,

$$\begin{aligned}\sum_{x\in X}|K_\gamma(z,x)|q^{-\beta|x|} &\leqslant C_1+B_\beta^{-\frac12}\sum_{\ell=0}^{|z|-1}\frac{b_{\beta,\ell}}{b_{\gamma,\ell}}\sum_{j\in I_v}|e_{v_\ell,j}(v_{\ell+1})|\\ &\leqslant C_1+B_\beta^{-\frac12}\frac{b_{\beta,0}}{b_{\gamma,0}}\sum_{\ell=0}^{|z|-1}q^{-(\beta-\gamma)\ell}\sup_{\substack{v\in X\\ j\in I_v}}\|e_{v,j}\|_1\\ &\leqslant\begin{cases}C_1+C_2q^{-(\beta-\gamma)|z|}, & \text{if } \gamma\neq\beta,\\ C_1+C_2'|z|, & \text{if } \gamma=\beta.\end{cases}\end{aligned}$$

□

Proposition 4.22. *Let $1<p<\infty$. If $a+b\geqslant c>1$ and $-pa<\alpha-1<p(b-1)$, then $S_{a,b,c}$ is bounded on L^p_α.*

Proof. We set

$$H(z,x)=|K_c(z,x)|q^{-a|z|}q^{-(b-\alpha)|x|},$$

then we write the operator $S_{a,b,c}$ as

$$S_{a,b,c}f(z)=\sum_{x\in X}H(z,x)f(x)q^{-\alpha|x|}.$$

Our purpose is to apply Schur's test (see Theorem 3.6 in [54]) to the integral operator with positive kernel $H\colon X \times X \to [0, +\infty)$. To do so, we have to show that there exists a positive function h on X such that

$$\sum_{z\in X} H(z,x)h(z)^{p'}q^{-\alpha|z|} \lesssim h(x)^{p'}, \qquad \sum_{x\in X} H(z,x)h(x)^{p}q^{-\alpha|x|} \lesssim h(z)^{p}. \tag{4.19}$$

Observe that the two inequalities assumed for α are equivalent to

$$-\frac{a+\alpha-1}{p} < \frac{a}{p'}, \qquad -\frac{b-1}{p'} < \frac{b-\alpha}{p}.$$

Hence, since $a+b>1$, it is possible to choose an element

$$\gamma \in \left(-\frac{b-1}{p'}, \frac{a}{p'}\right) \cap \left(-\frac{a+\alpha-1}{p}, \frac{b-\alpha}{p}\right) \neq \varnothing. \tag{4.20}$$

We want to show that $h(x) = q^{-\gamma|x|}$ satisfies conditions (4.19). Let $z \in X$. We can apply Lemma 4.21 since $b+\gamma p' > 1$ by (4.20), obtaining

$$\begin{aligned}\sum_{x\in X} H(z,x)h(x)^{p'}q^{-\alpha|x|} &= q^{-a|z|}\sum_{x\in X}|K_c(z,x)|q^{-(b+\gamma p')|x|}\\ &\leqslant \begin{cases} q^{-a|z|}(C_1 + C_2 q^{-(b+\gamma p'-c)|z|}), & \text{if } \gamma \neq \frac{c-b}{p'},\\ q^{-a|z|}(C_1 + C_2'|z|), & \text{if } \gamma = \frac{c-b}{p'}\end{cases}\\ &\lesssim q^{-\gamma p'|z|} = h(z)^{p'},\end{aligned}$$

where we used $a+b-c \geqslant 0$ and $a > \gamma p'$. Similarly, we have that, by $a+\gamma p+\alpha > 0$ and by Lemma 4.21,

$$\begin{aligned}\sum_{z\in X} H(z,x)h(z)^{p}q^{-\alpha|z|} &= q^{-(b-\alpha)|x|}\sum_{z\in X}|K_c(z,x)|q^{-(a+\gamma p+\alpha)|z|}\\ &\leqslant \begin{cases} q^{-(b-\alpha)|x|}(C_1 + C_2 q^{-(a+\gamma p+\alpha-c)|z|}), & \text{if } \gamma \neq \frac{c-a-\alpha}{p},\\ q^{-(b-\alpha)|x|}(C_1 + C_2'|z|), & \text{if } \gamma = \frac{c-a-\alpha}{p}\end{cases}\\ &\lesssim q^{-\gamma p|z|} = h(z)^{p},\end{aligned}$$

since $a+b \geqslant c$ and, by (4.20), $b-\alpha > \gamma p$.

In conclusion, (4.19) holds and by Schur's test the operator $S_{a,b,c}$ is bounded on $L^p_\alpha(X)$. ☐

Notice that Proposition 4.22 shows that condition (iii) implies condition (i) in Theorem 4.10.

Proposition 4.23. *If $a+b \geqslant c$ and*

$$\begin{aligned} -a < \alpha - \frac{1}{2} < b-1, &\quad \textit{when } c = a+b;\\ -a < \alpha - \frac{1}{2} \leqslant b-1, &\quad \textit{when } c < a+b,\end{aligned}$$

then $S_{a,b,c}$ is bounded on L^1_α.

Proof. Let $f \in L^1_\alpha$. We observe that, since $a + \alpha > 1$, by Lemma 4.21

$$
\begin{aligned}
\|S_{a,b,c}f\|_{L^1_\alpha} &= \sum_{z\in X} \left| \sum_{x\in X} |K_c(z,x)| f(x) q^{-b|x|} \right| q^{-(a+\alpha)|z|} \\
&\leqslant \sum_{x\in X} |f(x)| q^{-b|x|} \sum_{z\in X} |K_c(z,x)| q^{-(a+\alpha)|z|} \\
&\leqslant \begin{cases} \displaystyle\sum_{x\in X} |f(x)| q^{-b|x|}(C_1 + C_2 q^{-(a+\alpha-c)|x|}), & \text{if } c \neq a+\alpha, \\ \displaystyle\sum_{x\in X} |f(x)| q^{-b|x|}(C_1 + C_2'|x|), & \text{if } c = a+\alpha \end{cases} \\
&\lesssim \sum_{x\in X} |f(x)| q^{-\alpha|x|} = \|f\|_{L^1_\alpha},
\end{aligned}
$$

where we used the fact that $a + b - c \geqslant 0$ and $b \geqslant \alpha$ in the case $c \neq a + \alpha$, and $b > \alpha$ in the case $c = a + \alpha$. Hence, $S_{a,b,c}$ is bounded on L^1_α. □

Proposition 4.23 shows that condition (iii) implies condition (i) in Theorem 4.11.

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