SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Dipartimento di / Department of

Mathematics

Dottorato di Ricerca in / PhD program Mathematics Ciclo / Cycle 36

Curriculum in (se presente / if it is)

Reflection groups and flat structures

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ANNO ACCADEMICO / ACADEMIC YEAR 2022-2023

Abstract

In this thesis, by means of a suitable generalization of the construction proposed in [3], we show that the orbit space of B_2 can be equipped with two Frobenius manifold structures related to the defocusing and focusing NLS (nonlinear Schrodinger) equation respectively.
Motivated by this example, we generalize this construction for any B_n, with $n > 2$. Such a construction is based on the existence of a homogeneous flat pencil of cometrics (defined as in [15]) defined on the orbit space of B_n. The proof of the existence of a homogeneous flat pencil relies on the Dubrovin-Saito procedure (see [18] and [46]), modified in a suitable way.
Starting from this pencil, one can reconstruct a unique Frobenius manifold structure $\mathcal{M}_{B_n}$ on the orbit space of B_n, for any $n > 2$, by following an alternative procedure with respect to the standard one presented by Dubrovin in [18]; this technical obstacle is due to the non-regularity of the pencil of cometrics.
Remarkably, $\mathcal{M}_{B_n}$ is isomorphic to the Hurwitz-Frobenius manifold structure on $M_{0;n-2,0}$ (as evidenced in [41]). This is related to the constrained KP hierarchy (see [35]). Such an identification makes it possible to compute explicitly the structure constants corresponding to the dual product of $\mathcal{M}_{B_n}$, for any integer n.

Introduction

The fulcrum of the thesis is a remarkable system of partial differential equations that appeared at the beginning of the 90s of the previous century in two papers ([14] and [56]). The general problem is to find a quasi-homogeneous function $F(t)$, of the variable $(t^1, ..., t^n)$, such that its the third derivatives

$$c_{ijk}(t) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}$$

are the structure constants (with a lowered index) of an associative and unital algebra for any t, whose unity doesn't depend on t.
Such an algebra is automatically commutative. The associativity condition of the algebra, written in terms of F, reads as an overdetermined system of non-linear PDEs for F; we call them WDVV (Witten-Dijkgraaf-E.Verlinde-H.Verlinde) equation.
Physically, it was first derived as an equation for the primary free energy for a two-dimensional topological field theory.

Later on, it was discovered to be an efficient tool to join together many areas of mathematics, such as Gromov-Witten invariants, reflection groups, singularity theory, Painlevé equations, and integrable systems, in a remarkable geometric framework.
In particular, Boris Dubrovin introduced the notion of the Dubrovin-Frobenius manifold (see [17]), i.e. a free-coordinate formulation of WDVV equations and 2-D topological field theories. Essentially, a Frobenius manifold is a manifold endowed with a flat metric and two special vector fields (the unity and the Euler field). Furthermore, at any point, the tangent space has a structure of a Frobenius algebra. The structure constants of the product satisfy certain conditions of invariance and compatibility with respect to the metric and the associated Levi-Civita connection.

There exists a natural approach to the theory of Frobenius manifold by means of the geometry of flat pencil of cometrics (see [15]). In particular, it was proven that under certain homogeneity and regularity assumptions, these structures coincide.
The notion of the flat pencil of cometrics appears in the theory of Dubrovin and Novikov of Poisson bracket of hydrodynamic type on the loop space of a manifold. This fact highlights the connection between Frobenius manifold and integrable hierarchies.

In December 1992 during a talk of Dubrovin at I.Newton institute Arnol'd immediately recognized in the weighted degrees of certain polynomial solutions of WDVV equation the Coxeter numbers (plus one) of the three Coxeter groups in the three-dimensional space. Motivated by this observation Dubrovin showed, in 1993, that the orbit space of all Coxeter groups (i.e. finite group generated by real reflection) has a structure of Frobenius manifold, moreover, the corresponding prepotential is a polynomial function.

Dubrovin's work relies on a reinterpretation of Saito's procedure, proposed in [46] (see also [47]), in terms of bihamiltonian geometry.
According to Chavalley theorem, the subring of invariant polynomials, for the action of the finite group generated by (pseudo-)reflection, is generated by a set of homogeneous polynomials, called basic invariants. In general, such invariants aren't uniquely defined, while their degrees are uniquely defined by the choice of the group.
In the case of the Coxeter group, Saito, Yano, and Sekiguchi propose a strategy to select uniquely the corresponding basic invariants of the group. In particular, these polynomial invariants are defined as the flat coordinates for a specific metric and are called Saito flat coordinates.
Moreover, they exhibit explicit formulae for the invariants for any group (with the exception of the case of the group E_7 and E_8).
Exploiting Saito's results, Dubrovin proved that, starting from the Euclidean metric, one endows the orbit space of any Coxeter group with a flat pencil of cometrics; furthermore, such a pencil yields a Frobenius manifold structure on the orbit space.
Dubrovin's construction relies heavily on the assumption that the unity field has the form

$$e = \frac{\partial}{\partial u^n} \quad (♋)$$

here u^n is the basic invariant of the highest degree.
Dubrovin also conjectured that, under some technical assumptions, all polynomial solutions of the WDVV equation can be obtained by means of this procedure applied to a suitable Coxeter group. Later, Hertling proved the conjecture (see [31]).
In [55] Zuo observed that in the case of B_n and D_n, the non-standard choice of the unity yields a Frobenius manifold structure different from the standard Coxeter case.

In 2004, Dubrovin introduced the notion of almost-duality [16]. In the Coxeter case, the Frobenius potential of almost-dual structure has a universal form given by Veselov in [53]; this expression is related to the notion of check-system. Moreover, in [16] Dubrovin generalizes Saito's construction in the case of the Shephard group (i.e. the symmetry groups of regular complex polytopes). In this case, the role of the Euclidean metric is played by the Hessian of the lowest degree basic invariant. The flatness of such a metric relies on a result of Orlik and Solomon (see [43]).
It turned out that the Frobenius structure obtained in this way on the orbit space of a Shephard group is isomorphic to the Dubrovin-Frobenius structure defined on the orbit space of the corresponding Coxeter group.

Analogously to the Coxeter case, finite complex reflection groups have a standard representation in terms of linear endomorphism acting on vector space of dimension n. There exists a family of complex reflection groups, called well-generated, whose minimal set of generators has cardinality n.

In 2015 Kato, Mano and Sekiguchi proposed a further generalization of Dubrovin–Saito construction for the case of well-generated complex reflection groups (see [32]). In this case, this construction doesn't yield a Frobenius manifold structure but only a flat $F-$manifiold. The notion of flat $F-$manifiold structure was introduced by Manin in [42]; such structures in the literature of meromorphic connection (see [45]) are called Saito structures without a metric.

In 2013, Arsie and Lorenzoni introduced the notion of bi-flat $F-$manifold in [1], which generalizes Dubrovin's almost-duality for the case of Dubrovin manifold without metric. Roughly speaking, such a structure consists of two flat $F-$structures intertwined with some compatibility conditions.
In 2017, Arsie and Lorenzoni proposed in [3] a new construction of bi-flat F-manifolds on the orbit space of a well-generated complex reflection group. In particular, the dual structure is defined by means of a family of flat connections defined in terms of collections of reflecting (hyper-)planes. These connections appear in the literature in [27], [33], and [36].
This family of connections is parameterized by a collection of invariant functions defined on the set of reflecting (hyper-)planes (i.e. for each (hyper-)plane one assigns a "weight" and the weights chosen for distinct (hyper-)planes must coincide if the (hyper-)planes belong to the same orbit with respect the action of the group).
The dual product generalizes the notion of the Veselov $\vee$-system (check-system). A standard choice of the weights of the product is to consider each weight proportional to the order of the corresponding (pseudo-)reflection. Such a prescription leads to a family of bi-flat $F-$structure defined on the orbit space of the chosen reflection group. Conjecturally, in the well-generated case, the number of parameters on which the family of bi-flat $F-$structures depends coincide with $N-1$, where N is the number of orbits for the action of the reflection group on the collection of reflecting hyperplanes. In the case of well-generated complex reflection groups N is equal to 1 or 2. In the first case there is no freedom and the natural structure should coincide with Kato-Mano-Sekiguchi structure while in the second case one should obtain Kato-Mano-Sekiguchi structure for a particular value of the parameter.
This conjecture has been verified for Weyl groups of rank 2, 3 and 4, for the dihedral groups $I_2(m)$, for any of the exceptional well-generated complex reflection groups of rank 2 and 3, and for any of the groups series $G(m,1,2)$ and $G(m,1,3)$.
In analogy with Dubrovin's construction one chooses (♋) as the unity field. The removal of this hypothesis is the cornerstone of this thesis, which is based on [8].

In [8] Arsie, Lorenzoni, Mencattini, and Moroni proposed a further generalization of the Duvorin-Saito's procedure for the Coxeter group B_n.
The first step is to apply Arsie-Lorenzoni's procedure (of [3]) and to equip the orbit space of B_2, B_3, and B_4 with a bi-flat $F-$structure taking the non-standard choice of the unity field

$$e = \frac{\partial}{\partial u^{n-1}} \qquad (♌)$$

and prescribing suitable choices for the dual product and the dual connection. In particular, in the case of B_2 two choices of the weights are admissible:

I. assign weights zero to the coordinate axes and non-vanishing weights to the remaining mirrors

II. assign non-vanishing weight to the coordinate axes and zero weights to the remaining mirrors

while in the case of B_3 and B_4 only the first choice is allowable.
It turns out that these structures are uniquely defined (up to the rescaling of the basic invariants) and admit underlying Frobenius manifold structures respectively. The corresponding solutions of WDVV equations are no longer polynomial (as in Dubrovin's construction) due to the appearance of a logarithmic term. In the case of B_2 the choices I and II of the weights yield the Frobenius manifolds associated with the defocusing and focusing NLS equation respectively.
These solutions of WDVV coincide (for arbitrary n) with the prepotentials of the Frobenius manifolds associated with constrained KP hierarchies (see [35]).
Now, the key observation is that the corresponding intersection form has always the same form.

$$g^{ij} = \frac{(1-\delta^{ij})}{p^i p^j} \tag{♏}$$

Thus, in the second step, motivated by the expression for the intersection form in the case of B_2, B_3, and B_4, we show that starting from the B_n-invariant cometric $g = (g^{ij})$, we get a flat pencil of cometrics (g, η), where $\eta = \mathcal{L}_e g$ and e is the vector field (♌), defined on the orbit space of B_n.
In the third part, in order to prove the existence of a Frobenius manifold structure $\mathcal{M}_{B_n}$, for arbitrary n, one has to follow an alternative procedure with respect to the standard one proposed by Dubrovin in [15]. This technical obstacle is due to the non-regularity of the pencil. In particular, it turns out that it is not possible to define all the structure constants of the natural product in terms of the Christoffel symbols of the intersection form.

This procedure relies on dealing with the cometric (♏) instead of dealing with explicit formulas for the dual product and dual connection. Thus one possible open issue was to prove that the dual product of $\mathcal{M}_{B_n}$ coincides with the product given by the choice I of the weights, for arbitrary n.
In 2023 Ma and Zuo showed that $\mathcal{M}_{B_n}$ is isomorphic to the Hurwitz-Frobenius manifold structure on $M_{0;n-2,0}$ (see [41] for details, which is related to constrained KP hierarchy (see [35]). Frobenius manifold structure on Hurwitz space has been defined by Dubrovin in [17]. This notion highlights connections between the theory of Frobenius manifold and the singularity theory.
In general, given the LG (Landau-Ginzburg) superpotential one can recover the Frobenius manifold data by means of residue formulas. Hence, using the LG superpotential of [41] we have computed the structure constants associated with the dual product of $\mathcal{M}_{B_n}$, for any n. As conjectured, the outcome coincides with the product associated with the choice I of the weights.

Scheme of the Thesis

- *First chapter:* we will recall some basic notions of differential geometry.
- *Second chapter:* we will introduce the concepts of WDVV equations and Frobenius manifold. Moreover, we will highlight the connection between these structures.
- *Third chapter:* we will expose the notion of flat pencil of metric on a manifold M. We will show that, under certain assumptions, a flat pencil of metric endows M with a Frobenius manifold structure.
- *Fourth chapter:* we will introduce the notion of Coxeter group and expose a fundamental result of the invariant theory due to Chevallay. So we will show that the orbit space for a Coxeter group is endowed with a Frobenius manifold structure.
- *Fifth chapter:* we will introduce the notion of complex reflection group. We will outline the concept of flat F-manifold; then, by exploiting the notion of Dubrovin's almost-duality we will define the notion of bi-flat F-manifold. We will show how to equip the orbit space for a complex reflection group with a bi-flat F-manifold structure.
- *Sixth chapter:* by applying the results of the fifth chapter, we will construct explicitly a bi-flat F-manifold on the orbit space for B_2 taking a standard choice of the unity field. Moreover, we will recover the underlying polynomial Frobenius manifold structure.
  Remarkably, by a non-standard choice of the unity field, we will recover a non-polynomial Frobenius structure on the orbit space for B_2.
  Generalizing the B_2 case, taking a non-standard choice of the unity, we will equip the orbit space for B_3 and B_4 with a Frobenius manifold structure. We will observe that in all these cases the intersection form has always the same form. We will assume that this expression for the intersection form holds true for an arbitrary dimension.
- *Seventh chapter:* starting from the expression of the intersection form for the case B_2, B_3, and B_4, we will equip the orbit space of B_n, with arbitrary n, with a structure of flat pencil of cometrics.
- *Eighth chapter:* we will show that the flat pencil defined in the seventh chapter is quasi-homogenous. Unfortunately, it doesn't automatically yield a Frobenius manifold structure because of the non-regularity of the pencil. Then we will apply a non-standard procedure to prove the existence of a Frobenius manifold $\mathcal{M}_{B_n}$ structure on the orbit space for B_n. We will observe that such a structure is related to the constrained KP hierarchy.
- *Ninth chapter:* we will recall the notion of constrained KP hierarchy and central invariants. Following the work [41], we will see that $\mathcal{M}_{B_n}$ is isorphic

to the Hurwitz-Frobenius manifold on $M_{0;n-2,0}$. Then, we will recover the structure constant associated with the dual product of $\mathcal{M}_{B_n}$.

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1 Differential-geometric preliminaries

In this section, we summarize some results of differential geometry.
In this manuscript, we will consider as ambient space a real or complex manifold M, of dimension n, equipped with a local coordinates system (x^i).
In the first case, we will assume that all the geometric data are smooth, while in the second case, we will assume that all geometric data are holomorphic (we will denote by TM the holomorphic tangent bundle).
The Einstein summation convention is assumed (we sum over repeated indexes), if not stated otherwise. In the case of free indexes, we assume that they range from 1 to n.
Let $\langle\cdot,\cdot\rangle^*$ be a symmetric and non-degenerate bilinear form on the cotangent bundle T^*M.

Definition 1.1 *We define the contravariant metric, or briefly cometric,* $g = (g^{ij})$ *to be the* $(2,0)$ *tensor field on* M *defined by*

$$g^{ij}(x) := \langle dx^i, dx^j\rangle^* \tag{1.1}$$

where $\big(g^{ij}(x)\big)$ *is a symmetric and non-degenerate matrix for any* $x \in M$ *and* (dx^i) *are differential* 1*-forms.*

Denote by $(g_{ij}) := (g^{ij})^{-1}$ the inverse matrix of (g^{ij}).

Definition 1.2 *We define a symmetric non-degenerate bilinear form on the tangent bundle* TM *by*

$$\langle \partial_i, \partial_j\rangle := g_{ij}(x) \tag{1.2}$$

where $\partial_i := \frac{\partial}{\partial x^i}$.

Recall a well-known notion of differential geometry.

Definition 1.3 *We define the Levi-Civita connection* ∇ *of the metric* (g_{ij}) *to be the (unique) linear connection with vanishing torsion and compatible with the metric (i.e.* $\nabla g = 0$*). In local coordinates, these conditions read*

$$\begin{cases}\Gamma^s_{ki}g_{sj} + \Gamma^s_{kj}g_{is} = \partial_k g_{ij}\\ \Gamma^s_{ij} = \Gamma^s_{ji}\end{cases} \tag{1.3}$$

Proposition 1.4 *The system of equations (1.3) has unique solution*

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{ks}\big(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\big) \tag{1.4}$$

Remark 1.5 *Under the change of coordinates* $x \mapsto \tilde{x}$ *the Christoffel symbols transform as*

$$\tilde{\Gamma}^k_{ij} = \frac{\partial \tilde{x}^k}{\partial x^s}\frac{\partial x^r}{\partial \tilde{x}^i}\frac{\partial x^q}{\partial \tilde{x}^j}\Gamma^s_{rq} + \frac{\partial \tilde{x}^k}{\partial x^s}\frac{\partial^2 x^s}{\partial \tilde{x}^i\partial \tilde{x}^j} \tag{1.5}$$

Similarly, the inverse transformation $\tilde{x} \mapsto x$ *is obtained by interchanging* x *and* $\tilde{x}$

$$\Gamma^k_{ij} = \frac{\partial x^k}{\partial \tilde{x}^s}\frac{\partial \tilde{x}^r}{\partial x^i}\frac{\partial \tilde{x}^q}{\partial x^j}\tilde{\Gamma}^s_{rq} + \frac{\partial x^k}{\partial \tilde{x}^s}\frac{\partial^2 \tilde{x}^s}{\partial x^i\partial x^j} \tag{1.6}$$

For our purpose is convenient to work with a modified version of the Christoffel symbols.

Definition 1.6 *We define the contravariant Christoffel symbols corresponding to the connection ∇ the functions*

$$\Gamma_k^{ij} := \langle dx^i, \nabla_k dx^j \rangle^* = -g^{is}\Gamma_{sk}^j \tag{1.7}$$

The system of equations (1.3) written in terms of the contravariant Christoffel symbols reads

$$\begin{cases} \Gamma_k^{ij} + \Gamma_k^{ji} = \partial_k g^{ij} \\ g^{is}\Gamma_s^{jk} = g^{js}\Gamma_s^{ik} \end{cases} \tag{1.8}$$

Definition 1.7 *Given a connection ∇, we define the Riemann curvature tensor $R = (R_{slq}^k)$ to be the $(1,3)$ tensor field on M of components*

$$R_{slq}^k := \partial_s \Gamma_{lq}^k - \partial_l \Gamma_{sq}^k + \Gamma_{sr}^k \Gamma_{lq}^r - \Gamma_{lr}^k \Gamma_{sq}^r \tag{1.9}$$

It's useful to define a modified version of the curvature tensor.

Lemma 1.8 *The following formula holds true*

$$R_l^{ijk} := g^{is}g^{jq}R_{slq}^k = g^{is}(\partial_s \Gamma_l^{jk} - \partial_l \Gamma_s^{jk}) + \Gamma_s^{ij}\Gamma_l^{sk} - \Gamma_s^{ik}\Gamma_l^{sj} \tag{1.10}$$

Proof: Multiplying (1.9) by $g^{is}g^{jq}$ and using (1.8) and (1.7) one has the thesis. ∎

Recall the notion of flatness.

Definition 1.9 *Let Γ_{ij}^k be the Christoffel symbols corresponding to the connection ∇. The connection ∇ is said flat if there exists a coordinate system (t^i) such that the Christoffel symbols Γ_{ij}^k vanish in these coordinates, i.e.*

$$\Gamma_{ij}^k(t) = 0 \tag{1.11}$$

The coordinates (t^i) are called flat.

Definition 1.10 *A tensor field T on M is said to be covariantly constant if*

$$\nabla T = 0 \tag{1.12}$$

Proposition 1.11 *A connection is flat if and only if the corresponding curvature tensor (1.9) and torsion tensor vanish.*

Let ∇ be the Levi-Civita connection associated with the metric (g_{ij}). The following proposition gives an alternative characterization of the flat coordinates.

Proposition 1.12 *Given a flat connection, the corresponding flat coordinates system (t^i) reduces the metric (g_{ij}) to a constant matrix. Conversely, a coordinates system (t^i) that reduces (g_{ij}) to a constant matrix is a flat system.*

Remark 1.13 *Let ∇ be the Levi-Civita connection associated with the metric (g_{ij}). If ∇ is flat we call the metric (g_{ij}) flat.*

Remark 1.14 *The flat coordinates (t^i) are determined uniquely up to affine transformation with constant coefficients. They can be found by the following fundamental system*

$$\nabla^i (dt)_j = g^{is}\partial_s (dt)_j + \Gamma^{is}_j (dt)_s = 0 \tag{1.13}$$

for $i, j = 1, ..., n$, where $(dt)_j = \partial_j t$.

Definition 1.15 *Assume the bilinear form on the cotangent bundle $\langle \cdot, \cdot \rangle^*$ to be positive definite. The flat coordinates (p^i) are said to be orthonormalized if*

$$\tilde{g}^{ij}(p) := \langle dp^i, dp^j \rangle^* = \delta^{ij} \tag{1.14}$$

Lemma 1.16 *Let (x^i) be a coordinate system and (p^i) be an orthonormal coordinate system for the Levi-Civita connection ∇ corresponding to the cometric (g^{ij}). For the components of the cometric (g^{ij}) and contravariant Christofell symbols of the corresponding Levi-Civita connection the following formulas hold true:*

$$g^{ij}(x) = \frac{\partial x^i}{\partial p^a}\frac{\partial x^j}{\partial p^a} \tag{1.15}$$

$$\Gamma^{ij}_k(x) dx^k = \frac{\partial x^i}{\partial p^a}\frac{\partial^2 x^j}{\partial p^a \partial p^b} dp^b \tag{1.16}$$

Proof: (1.15) follows from the transformation formula for $(2, 0)$ tensor induced by the change of coordinates $p \mapsto x$, indeed

$$g^{ij}(x) = \frac{\partial x^i}{\partial p^s}\frac{\partial x^j}{\partial p^r}\underbrace{\tilde{g}^{sr}(p)}_{=\delta^{sr}} = \frac{\partial x^i}{\partial p^s}\frac{\partial x^j}{\partial p^s}$$

Since (p^i) are flat coordinates, the transformation formula (1.5) induced by the change of coordinates $x \mapsto p$ reduces to

$$\underbrace{\tilde{\Gamma}^k_{ij}(p)}_{=0} = \frac{\partial p^k}{\partial x^s}\frac{\partial x^r}{\partial p^i}\frac{\partial x^q}{\partial p^j}\Gamma^s_{rq}(x) + \frac{\partial p^k}{\partial x^s}\frac{\partial^2 x^s}{\partial p^i \partial p^j}$$

Multiplying by $\frac{\partial x^b}{\partial p^k}\frac{\partial p^i}{\partial x^p}\frac{\partial p^j}{\partial x^m}$ one has

$$\Gamma^b_{pm}(x) = -\frac{\partial p^i}{\partial x^p}\frac{\partial p^j}{\partial x^m}\frac{\partial^2 x^b}{\partial p^i \partial p^j}$$

Multiplying by (1.15) one yields

$$\underbrace{g^{dp}(x)\Gamma^b_{pm}(x)}_{\overset{(1.7)}{=} -\Gamma^{db}_m(x)} = -\frac{\partial x^d}{\partial p^a}\underbrace{\frac{\partial x^p}{\partial p^a}\frac{\partial p^i}{\partial x^p}}_{\delta^i_a}\frac{\partial p^j}{\partial x^m}\frac{\partial^2 x^b}{\partial p^i \partial p^j}$$

$$\Gamma^{db}_m(x) = \frac{\partial x^d}{\partial p^a}\frac{\partial p^j}{\partial x^m}\frac{\partial^2 x^b}{\partial p^a \partial p^j}$$

or equivalently in terms of 1- form

$$\Gamma^{db}_{m}(x)dx^m = \frac{\partial x^d}{\partial p^a}\frac{\partial^2 x^b}{\partial p^a \partial p^j}dp^j$$

which coincides with (1.16). ■

By similar arguments, one gets the following:

Lemma 1.17 *Let (x^i) be a coordinate system and (t^i) be a flat coordinate system (not necessarily orthonormal) for the connection ∇. For the components of the Christofell symbols, the following formula holds true:*

$$\Gamma^k_{ij}(x) = \frac{\partial x^k}{\partial t^s}\frac{\partial^2 t^s}{\partial x^i \partial x^j} \tag{1.17}$$

2 WDVV equations and Frobenius manifolds

The main reference of this section is [17].

2.1 WDVV equations

This subsection is devoted to introducing the theory of WDVVV equations. This remarkable system of non-linear PDE was discovered by E.Witten, R.Dijkgraaf, E.Verlinde and H.Verlinde ([56] and [14]) at the beginning of the 90′. There was derived as equations for the so-called primary free energy of a family of two-dimensional topological field theories. Later, WDVV equations have been shown to be an efficient tool in the theory of Gromov-Witten invariants, reflection groups, singularities, and integrable systems.
We introduce the main subject of this section: we are looking for a function $F = F(t)$, where $t = (t^1, ..., t^n)$, such that the third derivatives

$$c_{ijk}(t) := \frac{\partial F(t)}{\partial t^i \partial t^j \partial t^k}$$

satisfy the following equations:

1. *Normalization:*

   $$\eta_{ij} := c_{1ij}(t)$$

   is a constant nondegenerate matrix. Let $(\eta^{ij}) := (\eta_{ij})^{-1}$.

2. *Associativity:* the functions

   $$c^k_{ij}(t) := \eta^{ks} c_{sij}(t)$$

   define, for any t, in a n-dimensional space with a basis $e_1, ..., e_n$ a structure of associative algebra A_t, defined by

   $$e_i \circ e_j := c^k_{ij}(t) e_k$$

   The vector e_1 will be chosen as the unity of the algebra, i.e.

   $$c^j_{1i} = \delta^j_i$$

3. $F(t)$ is a quasi-homogeneous function of its variable, i.e.

   $$F(c^{d_1} t^1, ..., c^{d_n} t^n) = c^{d_F} F(t^1, ..., t^n) \tag{2.1}$$

   for a non-zero constant and for some numbers $d_1, ..., d_n, d_F$.

It will be convenient to rewrite the quasi-homogeneous condition (2.1) in infinitesimal form, introducing the Euler vector field $E = E^i \partial_i$ (where $\partial_i = \frac{\partial}{\partial t^i}$) as

$$\mathcal{L}_E F(t) := E^i(t) \partial_i F(t) = d_F F(t) \tag{2.2}$$

In view of the quasi-homogeneity condition (2.2), it turns out that $E(t)$ reads

$$E(t) = d_i t^i \partial_i$$

Observe that for the Lie derivative of the unity field $e = \partial_1$ one has

$$\mathcal{L}_E e = -d_1 e$$

Remark 2.1 *We will consider a generalization of the quasi-homogeneity condition. Since (2.1) is defined as the third-derivatives of F, we will consider the functions $F(t^1, ..., t^n)$ up to adding a (non-homogeneous) quadratic function in $t^1, ..., t^n$. Thus the algebra A_t will remain unchanged. Hence the quasi-homogeneity condition (2.2) can be modified as follows:*

$$\mathcal{L}_E F(t) = d_F F(t) + A_{ij} t^i t^j + B_i t^i + C$$

where A_{ij}, B_i and C are constants.

Summarising, we give a precise formulation of WDVV equations.

Definition 2.2 *Let $\eta = (\eta_{ij})$ be a symmetric and nondegenerate $n \times n$ matrix. We are looking for functions $F(t)$ such that*

I.

$$\boxed{\partial_1 \partial_i \partial_j F = \eta_{ij}} \tag{2.3}$$

II.

$$\boxed{(\partial_s \partial_i \partial_j F)\eta^{sk}(\partial_t \partial_k \partial_l F) = (\partial_s \partial_i \partial_l F)\eta^{sk}(\partial_t \partial_k \partial_j F)} \tag{2.4}$$

III.

$$\boxed{\mathcal{L}_E F = d_F F + \tfrac{1}{2} A_{ij} t^i t^j + B_i t^i + C} \tag{2.5}$$

where $E^i = E^i(t)$ are linear functions of (t^i), moreover, d_F, A_{ij}, B_i and C are real constants such that $A_{ij} = A_{ji}$.
Equations (2.4),(2.4) and (2.5) are called normalization condition, associativity equation, and homogeneity condition respectively. Furthermore, the function F will be called free energy.

We have defined an overdetermined system of non-linear PDEs. The next step is to construct a geometrical framework where these equations naturally arise.

2.2 Frobenius manifolds

In this subsection, we present the notion of Frobenius manifold introduced by Dubrovin to give a free-coordinate formulation of WDVV equations and two-dimensional topological field theories.
We define a complex Frobenius algebra as a finite-dimensional vector space equipped with a multiplication and a bilinear form.

Definition 2.3 *An algebra $(A, \circ)$ over $\mathbb{C}$ is a (commutative) complex Frobenius algebra if the following axioms are fulfilled:*

- *$\circ$ is a commutative and associative $\mathbb{C}$-algebra on A.*
- *There exists a element $e \in A$, called the unity of the algebra, such that*

$$a \circ e = e \circ a = a \tag{2.6}$$

for any $a \in A$.

- *A is equipped with a nondegenerate $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle$, invariant in the following sense:*

$$\langle a \circ b, c\rangle = \langle a, b \circ c\rangle \tag{2.7}$$

  for any $a, b, c \in A$.

Consider a family of Frobenius algebras depending on the parameters $p = (p^1, ..., p^n)$. We denote the space of parameters by M. Let M and N be a smooth (or complex) manifolds. Consider the fiber bundle

$$\pi : N \to M \tag{2.8}$$

with fiber $A_p := \pi^{-1}(p)$.
The basic idea is to identify such a fiber bundle with the tangent bundle ($N = TM$) so that each tangent space of M is equipped with a Frobenius algebra structure

$$\big(A_p, \circ_p, e_p, \langle\cdot,\cdot\rangle_p\big) \tag{2.9}$$

for any $p \in M$, where

- $A_p := T_pM$.
- $\circ_p$ is the product defined by

$$\partial_i \circ_p \partial_j := c^k_{ij}(p)\, \partial_k \tag{2.10}$$

  here $\partial_i = \frac{\partial}{\partial p^i} \in T_pM$ and c^k_{ij} are the structure constants of the product.
- $e : M \to TM$ is the unity vector field, i.e.

$$X_p \circ_p e_p = e_p \circ_p X_p = e_p \tag{2.11}$$

  for any vector field X and any $p \in M$.
- $\langle\cdot,\cdot\rangle_p$ is the nondegenerate bilinear form defined by

$$\langle\partial_i, \partial_j\rangle_p := \eta_{ij}(p) \tag{2.12}$$

  where $\eta = (\eta_{ij})$ is a pseudo-Riemannian metric tensor on M.

For brevity, from here on, we omit the subscript p.

Remark 2.4 *In view of (2.10) one has that c^k_{ij} are the component of a $(1, 2)$ tensor field.*

Remark 2.5 *Let X, Y and Z be arbitrary vector fields. The commutativity and associativity condition of the product, i.e.*

$$X \circ Y = Y \circ X \tag{2.13}$$

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) \tag{2.14}$$

in terms of the structure constants (2.10) read

$$c^k_{ij} = c^k_{ji} \tag{2.15}$$

$$c^k_{is}c^s_{jq} = c^k_{js}c^s_{iq} \tag{2.16}$$

respectively.

We give the precise definition of Frobenius manifold proposed by Dubrovin in [17].

Definition 2.6 *A smooth (or complex) Frobenius manifold* $(M, \eta, \nabla, \circ, e, E)$ *of charge* d*, is a manifold* M *equipped with a structure of Frobenius algebra on each tangent space* T_pM *smoothly depending on* p*, such that the following axioms are fulfilled:*

I. ∇ *is the (flat) Levi-Civita connection corresponding to the flat metric* η*.*

II. η *is invariant with respect the product* $\circ$

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z) \tag{2.17}$$

for any vector fields X, Y *and* Z*, or equivalently in components*

$$\eta_{is} c^s_{jk} = \eta_{js} c^s_{ik} \tag{2.18}$$

III. *The unity vector field* e *is covariantly constant (or briefly flat), i.e.*

$$\nabla e = 0 \tag{2.19}$$

IV. *Let* c *be the* $(0, 3)$ *symmetric tensor field on* M *(i.e. a symmetric trilinear form on* TM*) defined by*

$$c(X, Y, Z) := \eta(X \circ Y, Z) \tag{2.20}$$

or equivalently in components

$$c_{ijk} := c^s_{ij} \eta_{sk} \tag{2.21}$$

We require the $(0, 4)$ *tensor field* ∇c *also to be symmetric.*

V. *There exists a linear vector field* E*, called Euler field, such that the corresponding one-parameter group of diffeomorphism acts by conformal transformation of the metric* η *and by rescaling on the Frobenius algebra* T_pM*, for any* p*. In formulas one has*

$$\nabla\nabla E = 0 \tag{2.22}$$

$$\mathcal{L}_E c^k_{ij} = c^k_{ij} \tag{2.23}$$

$$\mathcal{L}_E e = -e \tag{2.24}$$

$$\mathcal{L}_E \eta_{ij} = (2 - d)\eta_{ij} \tag{2.25}$$

Remark 2.7 *The linearity condition of the Euler field (2.22) is redundant. Indeed, it can be proven that it follows from the other axioms.*

Denote by $\eta^{-1} = (\eta^{ij})$ the inverse of the metric η.

Remark 2.8 *Contracting the formula (2.18) by* $\eta^{ri}\eta^{qj}$ *one obtains*

$$\eta^{qj} c^s_{jk} = \eta^{ri} c^q_{ik} \tag{2.26}$$

Remark 2.9 *In view of the flatness of the metric η there exists a flat coordinate system (t^i) such that the entries $\eta_{ij} = \eta_{ij}(t)$ are constants. In such coordinates (2.19) and (2.22) read*

$$\partial_i e^j = 0 \tag{2.27}$$
$$\partial_i \partial_j E^k = 0 \tag{2.28}$$

where $\partial_i = \frac{\partial}{\partial t^i}$. Then integrating one has

$$e^i = a^i \tag{2.29}$$
$$E^i = b^i_j t^j + c^i \tag{2.30}$$

where a^i, b^i_j and c^i are constants.
Recall that the flat coordinates (t^i) are defined up to an affine transformation. Thus, from here on, we will choose flat coordinate so that the unity field has the form

$$\boxed{e^i = \delta^i_1} \tag{2.31}$$

From here on, following Dubrovin, we will consider only the case

$$b^i_1 = \delta^i_1 \tag{2.32}$$
$$c^1 = 0 \tag{2.33}$$

Remark 2.10 *The symmetry condition $IV.$ of the tensor ∇c, written in the flat coordinates (t^i), reads*

$$\partial_i c_{jks} = \partial_j c_{iks} \tag{2.34}$$

or equivalently, raising one index, one has

$$\partial_i c^q_{jk} = \partial_j c^q_{ik} \tag{2.35}$$

Remark 2.11 *Consider e of the form (2.31). The formula $e \circ X = e$, written in terms of structure constants, reads*

$$c^s_{1j} X^j = X^s$$

then

$$c^k_{1j} = \delta^k_j \tag{2.36}$$

Definition 2.12 *We define the grading operator $Q = (Q^i_j)$ to be the $(1,1)$ tensor field on M defined by*

$$Q^i_j = \nabla_i E^j \tag{2.37}$$

Remark 2.13 *In view of (2.28) one observes that the matrix $\left(Q^i_j(t)\right)$, where $Q^i_j(t) = \partial_j E^i$, has constant entries on M.*

Under certain assumptions on the operator Q and the metric η can be reduced to a simpler form; in particular, the following lemma holds true:

Lemma 2.14 *If $\eta_{11} = 0$ and all roots of Q are simple then by a linear change of coordinates (t^i) the matrix η can be reduced to the anti-diagonal form*

$$\eta_{ij} = \delta_{i+j,n+1} \tag{2.38}$$

Proof: See [17] for details. ∎

Remark 2.15 *In view of the constancy of the metric η in the coordinates (t^i) one has*

$$\mathcal{L}_e \eta_{ij} \overset{(2.31)}{=} \partial_1 \eta_{ij} = 0 \tag{2.39}$$

Similarly, the following formula holds true:

$$\mathcal{L}_e c^k_{ij} = \partial_1 c^k_{ij} = 0 \tag{2.40}$$

Indeed

$$\mathcal{L}_e c^k_{ij} \overset{(2.31)}{=} \partial_1 c^k_{ij} \overset{(2.35)}{=} \partial_i c^k_{1j} \overset{(2.36)}{=} \partial_i \delta^k_j = 0$$

The structure constants (2.10) induce a product on the cotangent bundle by raising one index, more precisely the following lemma holds true:

Lemma 2.16 *The $\mathcal{C}^\infty(M)$-bilinear (or $\mathcal{O}(M)$-bilinear in the complex case) application defined by*

$$dx^i \,\tilde{\circ}\, dx^j := c^{ij}_k dx^k \tag{2.41}$$

where

$$c^{ij}_k := \eta^{is} c^j_{sk} = \eta^{is}\eta^{jq} c_{qsk} \tag{2.42}$$

establish a commutative and associative product on T^*M; i.e.

$$c^{qs}_k = c^{sq}_k \tag{2.43}$$

$$c^{bk}_s c^{rs}_q = c^{rk}_s c^{sb}_q \tag{2.44}$$

respectively.

Proof: In view of the definition (2.42), formula (2.26) reads

$$c^{qs}_k = c^{sq}_k \tag{2.45}$$

which coincides with the commutativity of the product.
Recall the associativity condition (2.16)

$$c^k_{is} c^s_{jq} = c^k_{js} c^s_{iq}$$

Multiplying by $\eta^{bi}\eta^{rj}$ one has

$$\underbrace{\eta^{bi} c^k_{is}}_{=c^{bk}_s} \underbrace{\eta^{rj} c^s_{jq}}_{=c^{rs}_q} = \underbrace{\eta^{rj} c^k_{js}}_{=c^{rk}_s} \underbrace{\eta^{bi} c^s_{iq}}_{=c^{sb}_q}$$

which coincides with the associativity of the product.

∎

Remark 2.17 *Assume the operator* ∇E *to be diagonalizable, then*

$$E^i = d_i t^i + c_i \tag{2.46}$$

here there is no summation over repeated indexes, moreover, d_i *and* c_i *are constants. If* $d_i \neq 0$*, for some* i*,* c_i *may be killed by performing a shift in* $\frac{\partial}{\partial t^i}$ *direction.*

The following proposition elucidates the connection between the notion of the Frobenius manifold and WDVV equations.

Proposition 2.18 *Let* $(M, \eta, \nabla, \circ, e, E)$ *be Frobenius manifold with* ∇E *diagonalizable and* $c_i = 0$ *for any* i*. In flat coordinates* (t^i) *for* ∇ *there exists, at least locally, a function* F*, called Frobenius potential, such that*

$$c_{ijk} = \eta_{il} c^l_{jk} = \partial_i \partial_j \partial_k F. \tag{2.47}$$

Furthermore, F *fulfills the WDVV eqauations (2.3), (2.4) and (2.5) taking*

$$d_F = 3 - d \tag{2.48}$$

Conversely, any solutions of WDVV equation F*, such that such that* $d_1 \neq 0$*, defines a Frobenius manifold with structure constants given by (2.47).*

Proof: See [17] for details. ■

2.3 Intersection form

A new metric play an important role in the theory of Frobenius manifold. This metric was found by Dubrovin in [20], see also [17].

Definition 2.19 *Given a Frobenius manifold* $(M, \eta, \nabla, \circ, e, E)$ *we define* $\langle \cdot, \cdot \rangle^*_{(g)}$ *to be the bilinear form on* T^*M *defined by*

$$\langle \omega, \lambda \rangle^*_{(g)} := i_E(\omega \circ \lambda) \tag{2.49}$$

for any differential 1*-form* ω *and* λ*. The product of* 1*-form has been defined in (2.41) while* i_E *is defined as the operator of the inner contraction of a* 1*-form with respect to the Euler vector field, i.e.* $i_E(\omega) := E^j \omega_j$*.*

Remark 2.20 *The formula (2.49) explicitly reads*

$$i_E(\omega \circ \lambda) = E^s c^{ij}_s \omega_i \lambda_j \tag{2.50}$$

where E^s*,* ω_i *and* λ_j *are the components of* E*,* ω *and* λ *respectively.*

Let (t^i) be a flat coordinate system for η.

Definition 2.21 *The bilinear form (2.49) defines a cometric* $g = (g^{ij})$ *on the manifold* M*, where*

$$g^{ij}(t) := \langle dt^i, dt^j \rangle^*_{(g)} = E^s(t) c^{ij}_s(t) = E^s(t) c^j_{ks}(t) \eta^{ki} \tag{2.51}$$

where $\eta^{ij} = \eta^{ij}(t)$*. We call* g *the intersection form of the Frobenius manifold.*

Lemma 2.22 *(2.51) doesn't degenerate on a dense set of M. Thus the definition (2.49) is well-posed.*

Proof: One has

$$g^{ij}(t) = E^s(t)c_s^{ij}(t) = E^1(t)\underbrace{c_1^{ij}(t)}_{=\eta^{ir}c^j_{1r}=\eta^{ij}} + \sum_{s=2}^{n} E^s(t)c_s^{ij}(t)$$

Recalling that $E^i = b^i_r t^r + c^i$ one gets

$$E^1(t)\eta^{ij} \overset{(2.30)}{=} (b^1_r t^r + \underbrace{c^1}_{\overset{(2.33)}{=}0})\eta^{ij} \overset{(2.32)}{=} t^1\eta^{ij} + \underbrace{\left(\sum_{r=2}^{n} b^1_r t^r\right)}_{\overset{(2.32)}{=}0}\eta^{ij} = t^1\eta^{ij}$$

Furthermore, since $b^1_j = \delta^1_j$ and $\partial_1 c^k_{ij} = 0$, one has that $\sum_{s=2}^n E^s(t)c_s^{ij}(t)$ is independent on t^1. Then

$$g^{ij}(t) = t^1\eta^{ij} + f(t^2, ..., t^n) \tag{2.52}$$

where $f(t^2, ..., t^n)$ is a smooth function of $(t^2, ..., t^n)$.
In the limit of large t^1 one obtains the asymptotic expansion

$$g^{ij}(t) \sim t^1\eta^{ij} \tag{2.53}$$

Therefore, being η^{ij} a non-degenerate matrix, $g^{ij}(t)$ doesn't degenerate on a dense subset of M. ■

Remark 2.23 *In view of (2.52) one has that*

$$\partial_1 g^{ij} = \eta^{ij} \tag{2.54}$$

Lemma 2.24 *The following identity holds true*

$$g^{ij}(t) = R^i_s F^{sj}(t) + R^j_s F^{si}(t) + A^{ij} \tag{2.55}$$

where

$$R^i_j = R^i_j(t) := \frac{d-1}{2}\delta^i_j + \partial_j E^i \tag{2.56}$$

$$F^{ij}(t) := \eta^{is}\eta^{jk}\partial_s\partial_k F(t) \tag{2.57}$$

$$A^{ij} := \eta^{is}\eta^{jk}A_{sk} \tag{2.58}$$

where F and A_{ij} are defined by (2.3), (2.4) and (2.5), while (R^i_j) are the components of a $(1,1)$ tensor field R on M written in the coordinates (t^i). In particular, (R^i_j) are constant functions.

Proof: Recall that, given a Frobenius manifold, one has the homogeneity condition (2.5)

$$E^i\partial_i F = (3-d)F + \tfrac{A_{ij}}{2}t^i t^j + B_i t^i + C$$

Differentiate with respect t^k

$$\partial_k(E^i\partial_i F) = (3-d)\partial_k F + \tfrac{A_{kj}}{2}t^j + \tfrac{A_{ik}}{2}t^i + B_k$$

Differentiate with respect t^m

$$\partial_m\partial_k(E^i\partial_i F) = (3-d)\partial_m\partial_k F + \underbrace{\tfrac{A_{km}}{2} + \tfrac{A_{mk}}{2}}_{=A_{mk}}$$

One observes that

$$\begin{aligned}\partial_m\partial_k(E^i\partial_i F) &= \partial_m(\partial_k E^i\partial_i F + E^i\partial_k\partial_i F)\\ &= \underbrace{\partial_m\partial_k E^i}_{\overset{(2.28)}{=}0}\partial_i F + \partial_k E^i\partial_m\partial_i F + \partial_m E^i\partial_k\partial_i F + E^i\underbrace{\partial_m\partial_k\partial_i F}_{\overset{(2.47)}{=}c_{mki}}\end{aligned}$$

then

$$\partial_k E^i\partial_m\partial_i F + \partial_m E^i\partial_k\partial_i F + E^i c_{mki} = (3-d)\partial_m\partial_k F + A_{mk}$$

Multiply by $\eta^{pm}\eta^{sk}$

$$\eta^{pm}\eta^{sk}\partial_k E^i\partial_m\partial_i F+\eta^{pm}\eta^{sk}\partial_m E^i\partial_k\partial_i F+E^i\underbrace{\eta^{sk}\eta^{pm}c_{mki}}_{\overset{(2.42)}{=}c_i^{sp}} = (3-d)\eta^{pm}\eta^{sk}\partial_m\partial_k F+\eta^{pm}\eta^{sk}A_{mk}$$

Formula (2.25) in flat coordinates (t^i) reads

$$E^s\underbrace{\partial_s\eta^{ij}}_{=0} - \partial_s E^i\eta^{sj} - \partial_s E^j\eta^{is} = (d-2)\eta^{ij}$$

which yields

$$\begin{aligned}&\eta^{pm}\partial_m\partial_i F\big((2-d)\eta^{is} - \eta^{ik}\partial_k E^s\big) + \eta^{sk}\partial_k\partial_i F\big((2-d)\eta^{ip} - \eta^{im}\partial_m E^p\big) + E^i c_i^{sp} = (3-d)F^{ps} + A^{ps}\\ &(2-d)F^{ps} - F^{pk}\partial_k E^s + (2-d)F^{ps} - F^{sk}\partial_k E^p + E^i c_i^{sp} = (3-d)F^{sp} + A^{sp}\\ &E^i c_i^{sp} = \underbrace{(d-1)F^{sp}}_{=\frac{d-1}{2}\delta_k^s F^{kp}+\frac{d-1}{2}\delta_k^p F^{sk}} + \partial_k E^s F^{pk} + \partial_k E^p F^{sk} + A^{sp}\end{aligned}$$

Then, in view of the definition of R_j^i, one has

$$g^{ij} = E^i c_i^{sp} = R_k^s F^{pk} + R_k^p F^{sk} + A^{sp}$$

The constancy of the matrix (R_j^i) follows from the observation (2.13). ■

Definition 2.25 *We define the operator of multiplication by Euler vector field*

$$L := E\circ \tag{2.59}$$

to be the $(1,1)$ *tensor field of components*

$$L_i^j := c_{js}^i E^s \tag{2.60}$$

Lemma 2.26 *The following formulas hold true*

$$L^i_j = g^{is}\eta_{sj} \tag{2.61}$$
$$(L^{-1})^i_j = \eta^{is}g_{sj} \tag{2.62}$$

Proof: One has

$$g^{is}\eta_{sj} \overset{(2.51)}{=} E^k c^i_{kq} \underbrace{\eta^{qs}\eta_{sj}}_{=\delta^q_j} = E^k c^i_{kj}$$

similarly

$$L^i_q (L^{-1})^q_j = g^{is} \underbrace{\eta_{sq}\eta^{qp}}_{=\delta^p_s} g_{pj} = g^{is}g_{sj} = \delta^i_j$$

■

2.4 Semisimple Frobenius manifolds

First, we recall the notion of semisimple Frobenius algebra.

Definition 2.27 *A commutative and associative* $\mathbb{C}$*-algebra* A *with unity is called semisimple if there is no nonzero nilpotent element, i.e. there is no element* $a \in A - \{0\}$ *such that* $a^k = 0$*, for some* $k > 0$*.*

Thus we give the following:

Proposition 2.28 *Let* $(A, \circ, \langle\cdot,\cdot\rangle, e)$ *be a Frobenius algebra over* $\mathbb{C}$ *of dimension* n*. The following statements are equivalent:*

1. *A is semisimple.*
2. *A is isomorphic to* $\bigoplus_{i=1}^n \mathbb{C}$*.*
3. *A has a basis of idempotents, i.e.* n *elements* $\pi_1, ..., \pi_n$ *such that*

   $$\pi_i \circ \pi_j = \delta_{ij}\pi_i$$
   $$\langle \pi_i, \pi_j\rangle = \eta_{ii}\delta_{ij}$$

   where $\eta_{ii} = \langle e, \pi_i\rangle$*.*
4. *There is a vector* $E \in A$ *such that the multiplication operator* $E\circ : A \to A$ *has* n *pairwise distinct eigenvalues.*

Definition 2.29 *A point* p *of a Frobenius manifold* M *is semisimple if the corresponding Frobenius algebra* T_pM *is semisimple. If there exists an open dense subset of* M *of semisimple points, then* M *is called a semisimple Frobenius manifold.*

Remark 2.30 *It's clear that semisimplicity is an open property: if* p *is a semisimple point, then all points in a neighborhood of* p *are semisimple.*

Denote by M_{ss} the set of semisimple points of M.

Definition 2.31 *Let M be a semisimple Frobenius manifold. We define the caustic as the set*

$$K_M := M - M_{ss} = \{p \in M : T_pM \text{ is not a semisimple Frobenius algebra}\}$$

A basis of idempotents of T_pM can be prolonged in a neighborhood of p, in particular the following theorem holds true:

Theorem 2.32 *Let $p \in M_{ss}$ be a semisimple point, and $\pi_1(p), ..., \pi_n(p)$ a basis of idempotents of T_pM. Then*

$$[\pi_i, \pi_j] = 0$$

for any $i, j = 1, ..., n$. Thus there exist a local coordinate system $(u^1, ..., u^n)$, defined in a neighborhood of p, such that

$$\pi_i(p) = \left.\frac{\partial}{\partial u^i}\right|_p$$

for any $i = 1, ..., n$.

Definition 2.33 *Let M be a Frobenius manifold and $p \in M$ be a semisimple point. The coordinates $(u^1, ..., u^n)$ defined in a neighborhood of p, given by the latter theorem, are called canonical coordinates.*

Theorem 2.34 *Let $(u^1, ..., u^n)$ be canonical coordinates, defined in a neighborhood U of $p \in M$. Then the following formulas hold true in U:*

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i} \tag{2.63}$$

$$e = \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \tag{2.64}$$

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \tag{2.65}$$

2.5 Symmetries of WDVV equations and Legendre-type transformations

Roughly speaking, we are interested in transformations that map solutions to solutions of WDVV equations.

Definition 2.35 *We define a symmetry of $WDVV$ equations as the transformation*

$$t^i \mapsto \hat{t}^i$$
$$\eta_{ij} \mapsto \hat{\eta}_{ij}$$
$$F \mapsto \hat{F}$$

preserving the WDVV equations (2.3), (2.4) *and* (2.5).

One family of symmetries of WDVV equations are Legendre-type transformations.

Definition 2.36 *We define the Legendre-type transformation S_ϵ, for $\epsilon = 1, ..., n$, by the following formulas:*

$$\hat{t}_i = \partial_i \partial_\epsilon F(t) \tag{2.66}$$

$$\frac{\partial^2 \hat{F}}{\partial \hat{t}^i \partial \hat{t}^j} = \frac{\partial^2 F}{\partial t^i \partial t^j} \tag{2.67}$$

$$\hat{\eta}_{ij} = \eta_{ij} \tag{2.68}$$

Remark 2.37 *Using the above axioms it turns out that*

$$\partial_i = \partial_\epsilon \circ \hat{\partial}_i = (\partial_\epsilon \circ) \hat{\partial}_i \tag{2.69}$$

for $i = 1, ..., n$, where $\partial_i = \frac{\partial}{\partial t^i}$ and $\hat{\partial}_i = \frac{\partial}{\partial \hat{t}^i}$.
Recall that the flat coordinates (t^i) have been chosen so that the unit of the algebra reads $e = \partial_1$ (see (2.31)). Thus taking $i = 1$ in (2.69) one gets

$$e = \partial_\epsilon \circ \hat{\partial}_1$$

so ∂_ϵ is an invertible element of the Frobenius algebra of vector fields, with inverse $\hat{\partial}_1$ (i.e. $\partial_\epsilon^{-1} = \hat{\partial}_1$).

This remark yields the following:

Corollary 2.38 *The transformation (2.66) is invertible (at least locally).*

Proof: Applying the transformation rule for a vector, (2.69) yields

$$\partial_i = \frac{\partial t^s}{\partial \hat{t}^i} (\partial_\epsilon \circ \partial_s) = \frac{\partial t^s}{\partial \hat{t}^i} c^k_{\epsilon s} \partial_k$$

Thus

$$\frac{\partial t^s}{\partial \hat{t}^i} c^k_{\epsilon s} = \delta^k_i$$

so

$$\left(\frac{\partial t^s}{\partial \hat{t}^i} \right) = (c^k_{\epsilon s})^{-1} = (\partial_\epsilon^{-1} \circ)^k_s$$

Then the Jacobian of the transformation $\hat{t} \mapsto t$ is defined at least locally. This concludes the proof. ∎

Remark 2.39 *Note that in transformed coordinates $(\hat{t}^i)$ the unity field reads*

$$e = \frac{\partial}{\partial \hat{t}^\epsilon}$$

Moreover, observe that S_1 is the identity transformation (in view of the choice of the unity $e = \partial_1$).

Remark 2.40 *To prove that (2.66), (2.67) and (2.68) provide symmetry of WDVV equations one introduces a new metric $\langle \cdot, \cdot \rangle_\epsilon$ on the Frobenius manifold M, given by*

$$\langle X, Y \rangle_\epsilon := \langle \partial_\epsilon^2, X \circ Y \rangle$$

for arbitrary vector fields X and Y, with $\hat{\eta}_{ij} = \langle \partial_i, \partial_j \rangle_\epsilon$.

Proposition 2.41 *$(\hat{t}^i)$ are flat coordinates for the metric $(\hat{\eta}_{ij})$ and*

$$\langle \partial_i \circ \partial_j, \partial_k \rangle_\epsilon = \hat{\partial}_i \hat{\partial}_j \hat{\partial}_k \hat{F}(\hat{t})$$

Proof: See [17] for details. ■

Example 2.42 *Let's consider the Frobenius potential*

$$F = \tfrac{1}{2}(t^1)^2 t^2 + e^{t^2}$$

F is related to the Toda-chain hierarchy. Consider the transformation S_2, i.e.

$$\hat{t}_1 = \partial_1 \partial_2 F = t^1$$
$$\hat{t}_2 = \partial_2 \partial_2 F = e^{t^2}$$

Raising the index by $(\eta^{ij} = \delta^{i+j,3})$ one has

$$\hat{t}^1 = e^{t^2}$$
$$\hat{t}^2 = t^1$$

Now, using (2.67) one obtains

$$\frac{\partial^2 \hat{F}}{\partial \hat{t}^1 \partial \hat{t}^1} = \frac{\partial^2 F}{\partial t^1 \partial t^1} = t^2 = log(\hat{t}^1)$$
$$\frac{\partial^2 \hat{F}}{\partial \hat{t}^1 \partial \hat{t}^2} = \frac{\partial^2 F}{\partial t^1 \partial t^2} = t^1 = \hat{t}^2$$
$$\frac{\partial^2 \hat{F}}{\partial \hat{t}^2 \partial \hat{t}^2} = \frac{\partial^2 F}{\partial t^2 \partial t^2} = e^{t^2} = \hat{t}^1$$

Thus

$$\hat{F} = \tfrac{1}{2}(\hat{t}^2)^2 t^1 + \tfrac{1}{2}(t^1)^2 \big(log(\hat{t}^1) - \tfrac{3}{2}\big)$$

3 Flat pencil of cometrics

The main references of this section are [15] and [5].
Dubrovin in [15] proposed an approach to Frobenius manifold that relied on the geometry of flat pencil of contravariant metrics. It was shown that, under certain assumptions, these two objects are identical. The flat pencil of cometrics arises naturally in the classification of bi-Hamiltonian structures of hydrodynamic type on the loop of space ([21],[22],[23]). This highlights the relations between the theory of Frobenius manifold and integrable hierarchies.

Definition 3.1 *The cometrics $g_{(1)} = (g^{ij}_{(1)})$ and $g_{(2)} = (g^{ij}_{(2)})$ on a smooth (or complex) manifold M form a linear flat pencil, denoted by $(g_{(1)}, g_{(2)})$, if for any $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the following axioms are fulfilled:*

I. The pencil (i.e. the linear combination)

$$g^{ij}_{(\lambda)} := g^{ij}_{(1)} - \lambda g^{ij}_{(2)} \tag{3.1}$$

defines a flat cometric on a dense subset on M.

II. The functions defined by

$$\Gamma^{ij}_{k(\lambda)} := \Gamma^{ij}_{k(1)} - \lambda\Gamma^{ij}_{k(2)} \tag{3.2}$$

are the contravariant Christoffel symbols associated with the Levi-Civita connections of (3.1), where $\Gamma^{ij}_{k(1)}$ and $\Gamma^{ij}_{k(2)}$ are the contravariant Christoffel symbols corresponding to the Levi-Civita connections $\nabla_{(1)}$ and $\nabla_{(2)}$ of $g_{(1)}$ and $g_{(2)}$ respectively.

Lemma 3.2 *Let $(g_{(1)}, g_{(2)})$ be a flat pencil, then the cometrics $g_{(1)}$ and $g_{(2)}$ are flat.*

Proof: In view of the definition, the cometric $g^{ij}_{(\lambda)} = g^{ij}_{(1)} - \lambda g^{ij}_{(2)}$ is flat for any λ and has contravariant Christoffel symbols $\Gamma^{ij}_{k(\lambda)} = \Gamma^{ij}_{k(1)} - \lambda\Gamma^{ij}_{k(2)}$ Therefore the vanishing of the corresponding (modified) curvature tensor

$$R^{ijk}_{l(\lambda)} = g^{is}_{(\lambda)}(\partial_s\Gamma^{jk}_{l(\lambda)} - \partial_l\Gamma^{jk}_{s(\lambda)}) + \Gamma^{ij}_{s(\lambda)}\Gamma^{sk}_{l(\lambda)} - \Gamma^{ik}_{s(\lambda)}\Gamma^{sj}_{l(\lambda)}$$

reads

$$\begin{aligned}&\underbrace{g^{is}_{(1)}(\partial_s\Gamma^{jk}_{l(1)} - \partial_l\Gamma^{jk}_{s(1)}) + \Gamma^{ij}_{s(1)}\Gamma^{sk}_{l(1)} - \Gamma^{ik}_{s(1)}\Gamma^{sj}_{l(1)}}_{=R^{ijk}_{l(1)}} + O(\lambda)\\ &+ \lambda^2\big(\underbrace{g^{is}_{(2)}(\partial_s\Gamma^{jk}_{l(2)} - \partial_l\Gamma^{jk}_{s(2)}) + \Gamma^{ij}_{s(2)}\Gamma^{sk}_{l(2)} - \Gamma^{ik}_{s(2)}\Gamma^{sj}_{l(2)}}_{=R^{ijk}_{l(2)}}\big) = 0\end{aligned}$$

This formula can be regarded as a polynomial in λ. Then the coefficient of each power in λ vanishes (as λ is arbitrary). In particular, the vanishing of the constant term in λ corresponds to the flatness of the metric $g_{(1)}$ (i.e. $R^{ijk}_{l(1)} = 0$), while the vanishing of the quadratic term in λ corresponds to the flatness of the metric $g_{(2)}$ (i.e. $R^{ijk}_{l(2)} = 0$). ■

Definition 3.3 *We say that the flat pencil $(g_{(1)}, g_{(2)})$ is quasi-homogeneous of degree d if there exists a function τ on M such that the vector fields*

$$E := \nabla_{(1)}\tau, \; E^i = g^{is}_{(1)}\partial_s\tau \tag{3.3}$$

$$e := \nabla_{(2)}\tau, \; e^i = g^{is}_{(2)}\partial_s\tau \tag{3.4}$$

satisfy the following conditions

$$[e, E] = e \tag{3.5}$$
$$\mathcal{L}_E g_{(1)} = (d-1)g_{(1)} \tag{3.6}$$
$$\mathcal{L}_e g_{(1)} = g_{(2)} \tag{3.7}$$
$$\mathcal{L}_e g_{(2)} = 0 \tag{3.8}$$

We call (3.3) and (3.4) the Egorov conditions and (3.6) the homogeneity condition of the flat pencil. Furthermore, if (3.7) and (3.8) are fulfilled the flat pencil is said to be exact.

Remark 3.4 *If the formula (3.5), (3.6), (3.7) and (3.8) hold true then one has*

$$\mathcal{L}_E g_{(2)} = \mathcal{L}_E(\mathcal{L}_e g_{(1)}) = \mathcal{L}_e \underbrace{(\mathcal{L}_E g_{(1)})}_{=(d-1)g_{(1)}} - \underbrace{\mathcal{L}_{[E,e]} g_{(1)}}_{=\mathcal{L}_e g_{(2)}} = (d-2)g_{(2)} \tag{3.9}$$

Remark 3.5 *If the formula (3.3) and (3.4) hold true then one has*

$$E^i = g^{is}_{(1)} g^{(2)}_{sk} e^k \tag{3.10}$$

3.1 From Frobenius manifolds to flat pencils

In this section, we will see that any Frobenius manifold yields a flat pencil of cometrics.
Let $\mathcal{F} = (M, \eta, \nabla, \circ, e, E)$ be a Frobenius manifold with intersection form g. We show that $\mathcal{F}$ admits a natural structure of quasi-homogeneous linear flat pencil $(g_{(1)}, g_{(2)})$, where $g_{(1)} = g = (g^{ij})$ and $g_{(2)} = \eta^{-1} = (\eta^{ij})$.

Proposition 3.6 *The cometrics g and η^{-1} define a quasi-homogeneous linear flat pencil*

$$g^{ij}_{(\lambda)} := g^{ij} - \lambda\eta^{ij} \tag{3.11}$$
$$\Gamma^{ij}_{k(\lambda)} := \Gamma^{ij}_{k(g)} - \lambda\Gamma^{ij}_{k(\eta)} \tag{3.12}$$

of degree d.

Definition 3.7 *Let R be the linear operator on the tangent bundle TM (i.e a $(1,1)$ tensor field) defined by*

$$R^j_i := \frac{d-1}{2}\delta^j_i + \nabla^{(2)}_i E^j \tag{3.13}$$

A quasi-homogeneous linear flat pencil is said regular if R doesn't degenerate on M.

We prove the following preliminary lemma.

Lemma 3.8 *The contravariant Christoffel symbols (3.12) associated with the Levi-Civita connection of the cometric (3.11) are given by*

$$\Gamma^{ij}_{k(\lambda)}(t) = \Gamma^{ij}_{k(g)}(t) = c^{is}_k(t) R^j_s \tag{3.14}$$

where the structure constants $c^{ij}_s(t)$ are defined in (2.42), (t^i) is a flat coordinates system for η and $R^i_j = R^i_j(t) = \frac{d-1}{2}\delta^j_i + \partial_i E^j$.
In particular, the functions (3.14) doesn't depend on the parameter λ.

Proof: Recall that the Christoffel symbols $\Gamma^{ij}_{k(\eta)}$ evaluated in the flat coordinates (t^i) vanishes, then $\Gamma^{ij}_{k(\lambda)}(t) = \Gamma^{ij}_{k(g)}(t)$. For simplicity we denote $\Gamma^{ij}_{k(g)} = \Gamma^{ij}_k$. We prove that the functions (3.14) are the contravariant Christoffel symbols associated with the Levi-Civita connection of the cometric g. Differentiating (2.55) with respect t^k one has

$$\partial_k g^{ij}(t) = R^i_s \partial_k F^{sj}(t) + R^j_s \partial_k F^{si}(t) + \underbrace{\partial_k A^{ij}}_{=0}$$

Using the formula (2.57) one gets

$$\partial_k g^{ij}(t) = R^i_s \eta^{sq}\eta^{jp} \underbrace{\partial_k\partial_q\partial_p F(t)}_{\overset{(2.47)}{=} c_{kqp}(t)} + R^j_s \eta^{sp}\eta^{iq} \underbrace{\partial_k\partial_q\partial_p F(t)}_{\overset{(2.47)}{=} c_{kqp}(t)}$$

where $\eta^{ij} = \eta^{ij}(t)$ are constants. Raising two indexes of c_{kqp} one obtains

$$\partial_k g^{ij}(t) = R^i_s c^{sj}_k(t) + R^j_s c^{si}_k(t) \overset{(3.14)}{=} \Gamma^{ij}_k(t) + \Gamma^{ji}_k(t)$$

which coincides with the compatibility condition for the the connection $\nabla_{(g)}$ with the metric g. The condition of vanishing torsion follows immediately from the associativity of the product induced on the cotangent bundle T^*M, indeed

$$g^{sk}\Gamma^{ij}_k = g^{sk} c^{iq}_k R^j_q \overset{(3.14)}{=} E^p c^{sk}_p c^{iq}_k R^j_q \overset{(2.44)}{=} \underbrace{E^p c^{ik}_p}_{=g^{ik}} \underbrace{c^{sq}_k R^j_q}_{=\Gamma^{sj}_k} = g^{ik}\Gamma^{sj}_k \tag{3.15}$$

In the end, we have to show that the functions (3.14) are the contravariant Christoffel symbols associated with the Levi-Civita connection of the cometric (3.11), i.e. they fulfill the system of equations

$$\begin{cases} \partial_k g^{ij}_{(\lambda)}(t) = \Gamma^{ij}_k(t) + \Gamma^{ji}_k(t) \\ g^{sk}_{(\lambda)}(t)\Gamma^{ij}_k(t) = g^{ik}_{(\lambda)}(t)\Gamma^{sj}_k(t) \end{cases} \tag{3.16}$$

Taking e in the form (2.31), the formula (2.54) holds true, i.e. $\partial_1 g^{ij}(t) = \eta^{ij}(t) = \eta^{ij}$. Then expanding the functions $g^{ij}(t^1-\lambda, t^2, ..., t^n)$ in Taylor series about $\lambda = 0$ one has

$$g^{ij}(t^1 - \lambda, t^2, ..., t^n) = g^{ij}(t) - \lambda\eta^{ij} = g^{ij}_\lambda(t)$$

Hence $g_\lambda(t)$ coincides with $g(t)$ shifted by a quantity $-\lambda$ in the first entry t^1.
The first equation of (3.16) is satisfies since $\partial_k g^{ij}_{(\lambda)}(t) = \partial_k g^{ij}(t)$.
Therefore shifting $t^1 \mapsto t^1 - \lambda$ in the formula (3.15) one has

$$g^{sk}_{(\lambda)}(t)\Gamma^{ij}_k(t^1-\lambda, t^2, ..., t^n) = g^{ik}_{(\lambda)}(t)\Gamma^{sj}_k(t^1-\lambda, t^2, ..., t^n) \tag{3.17}$$

In view of the remark (2.13), the tensor R written in (t^i) has constant components. Therefore

$$\partial_1\Gamma^{ij}_k(t) = \mathcal{L}_e(c^{ij}_s(t)R^s_k) = \mathcal{L}_e(c^{ij}_s(t))R^s_k + c^{ij}_s \underbrace{\mathcal{L}_e(R^s_k)}_{=\partial_1 R^s_k = 0} = \mathcal{L}_e(\eta^{ik}c^j_{ks}(t))R^s_k$$
$$= \underbrace{\mathcal{L}_e(\eta^{ik})}_{\overset{(2.39)}{=}0} c^j_{ks}(t)R^s_k + \eta^{ik}\underbrace{\mathcal{L}_e(c^j_{ks}(t))}_{\overset{(2.40)}{=}0} R^s_k = 0$$

So $\Gamma^{ij}_k(t)$ doesn't depend on t^1 and, consequently, $\Gamma^{ij}_k(t)$ aren't affected by the shifting. Then (3.17) coincides with the second of (3.16). This concludes the proof. ■

Now, we can prove the proposition (3.6).

Proof: Recall the asymptotic expansion (2.53) for large t^1

$$g^{ij}(t) \sim t^1\eta^{ij}$$

Then for any λ one has

$$g^{ij}(t) - \lambda\eta^{ij} \sim (t^1-\lambda)\eta^{ij}$$

Hence $g^{ij}_{(\lambda)}$ doesn't degenerate on a dense subset of M, for any λ.
The curvature tensor associated to the Levi-Civita connection of the cometric (3.11) written in (t^i) reads

$$R^{ijk}_{s(\lambda)}(t) = g^{ip}_{(\lambda)}(\partial_p\Gamma^{jk}_{s(\lambda)} - \partial_s\Gamma^{jk}_{p(\lambda)}) + \Gamma^{ij}_{p(\lambda)}\Gamma^{pk}_{s(\lambda)} - \Gamma^{ik}_{p(\lambda)}\Gamma^{pj}_{s(\lambda)}$$
$$\overset{(3.14)}{=} (g^{ij}(t)-\lambda\eta^{ij})\underbrace{(\partial_p(c^{jq}_s R^k_q) - \partial_s(c^{jq}_p R^k_q))}_{(*)} + \underbrace{c^{iq}_p R^j_q c^{pm}_s R^k_m - c^{iq}_p R^k_q c^{pm}_s R^j_m}_{(**)}$$

One has

$$(*) = \partial_p(\eta^{jb}c^q_{bs}R^k_q) - \partial_s(\eta^{jb}c^q_{bp}R^k_q) = \eta^{jb}R^k_q(\partial_p c^q_{bs} - \partial_s c^q_{bp}) \overset{(2.35)}{=} 0$$

Exchanging $q \leftrightarrow m$ one obtains

$$(**) = \underbrace{c^{iq}_p R^j_q c^{pm}_s R^k_m}_{=c^{im}_p R^j_m c^{pq}_s R^k_q} - c^{iq}_p R^k_q c^{pm}_s R^j_m = R_q R^j_m(c^{im}_p c^{pq}_s - c^{iq}_p c^{pm}_s) \overset{(2.44)}{=} 0$$

It remains to show the Egorov condition, the condition (3.5), the homogeneity and the exactness of the pencil. We define $\tau := \eta_{is}t^s$ and take $e^k = \delta^k_1$, then

$$g^{im}\partial_m\tau = g^{im}\eta_{1s}\underbrace{\partial_m t^s}_{=\delta^s_m} \overset{(2.51)}{=} E^b c^i_{ba}\underbrace{\eta^{am}\eta_{1m}}_{=\delta^a_1} = E^b c^i_{b1} \overset{(2.36)}{=} E^b\delta^i_b = E^i$$
$$\eta^{im}\partial_m\tau = \eta^{im}\eta_{1m} = \delta^i_1$$

therefore (3.3) and (3.4) are fulfilled.
The axiom (2.24) of Frobenius manifold coincides with (3.5).
By direct computation

$$\mathcal{L}_E g^{ij} = \mathcal{L}_E(E^q c^i_{qs}\eta^{sj}) = \underbrace{\mathcal{L}_E(E^q)}_{=[E,E]^q=0} c^i_{qs}\eta^{sj} + E^q \underbrace{\mathcal{L}_E(c^i_{qs})}_{\overset{(2.23)}{=} c^i_{qs}} \eta^{sj} + E^q c^i_{qs} \underbrace{\mathcal{L}_E(\eta^{sj})}_{\overset{(2.25)}{=}(d-2)\eta^{sj}}$$

$$\overset{(2.51)}{=} g^{ij} + (d-2)g^{ij} = (d-1)g^{ij}$$

and

$$\mathcal{L}_e g^{ij} = \mathcal{L}_e(E^q c^i_{qs}\eta^{sj}) = \underbrace{\mathcal{L}_e(E^q)}_{\substack{=-\mathcal{L}_E e^q \\ \overset{(2.24)}{=} e^q=\delta^q_1}} c^i_{qs}\eta^{sj} + E^q \underbrace{\mathcal{L}_e(c^i_{qs})}_{\overset{(2.40)}{=}0} \eta^{sj} + E^q c^i_{qs} \underbrace{\mathcal{L}_e(\eta^{sj})}_{\overset{(2.39)}{=}0}$$

$$= c^i_{1s}\eta^{sj} \overset{(2.36)}{=} \delta^i_s\eta^{sj} = \eta^{ij}$$

then (3.6) and (3.7) are satisfied.
(3.8) coincides with (2.39), this concludes the proof. ■

3.2 From flat pencils to Frobenius manifolds

In this section we will show that under suitable condition on the operator R a quasi-homogeneous linear flat pencil defines a Frobenius manifold.
The following technical lemma will be useful later.

Lemma 3.9 *The functions defined by*

$$\Delta^{ijk}(x) := g^{js}_{(2)}\Gamma^{ik}_{s(1)} - g^{is}_{(1)}\Gamma^{jk}_{s(2)} \tag{3.18}$$

are the components of a $(3,0)$ tensor field on M. Furthermore, the connections $\nabla_{(1)}$ and $\nabla_{(2)}$ have a common system of flat coordinates if and only if $\Delta^{ijk} = 0$ at least locally.

Proof: First, we prove that the functions $\Delta^k_{ij}(x) = \Gamma^{k(1)}_{ij} - \Gamma^{k(2)}_{ji}$ are the components of a $(1,2)$ tensor field, where $\Gamma^{k(\alpha)}_{ji} = -g^{(\alpha)}_{js}\Gamma^{sk}_{i(\alpha)}$ $(\alpha = 0,1)$.
Recall the transformation law of the Christoffel symbols induced by the coordinate transformation $x \mapsto \tilde{x}$

$$\tilde{\Gamma}^i_{jk} = \frac{\partial \tilde{x}^i}{\partial x^s}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial x^b}{\partial \tilde{x}^k}\Gamma^s_{pb} + \frac{\partial^2 x^s}{\partial \tilde{x}^j \partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^s}$$

Therefore Δ^k_{ij} transforms as a $(1,2)$ tensor field, indeed

$$\tilde{\Delta}^i_{jk} = \tilde{\Gamma}^{i(1)}_{jk} - \tilde{\Gamma}^{i(2)}_{kj} = \frac{\partial \tilde{x}^i}{\partial x^s}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial x^b}{\partial \tilde{x}^k}\Gamma^{s(1)}_{pb} + \frac{\partial^2 x^s}{\partial \tilde{x}^j \partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^s} - \frac{\partial \tilde{x}^i}{\partial x^s}\frac{\partial x^p}{\partial \tilde{x}^k}\frac{\partial x^b}{\partial \tilde{x}^j}\Gamma^{s(2)}_{pb} - \frac{\partial^2 x^s}{\partial \tilde{x}^k \partial \tilde{x}^j}\frac{\partial \tilde{x}^i}{\partial x^s}$$
$$= \frac{\partial \tilde{x}^i}{\partial x^s}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial x^b}{\partial \tilde{x}^k}\Delta^s_{pb}$$

Suppose that the connections $\nabla_{(1)}$ and $\nabla_{(2)}$ have a common system of flat coordinates (p^i) defined in a neighborhood U, i.e. $\Gamma^{i(1)}_{jk}(p) = \Gamma^{i(2)}_{jk}(p) = 0$ for any $p \in U$.

Then $\Delta^i_{jk} = 0$ for any $p \in M$.
We prove the converse statement. If $\tilde{\Delta}^i_{jk}(\tilde{p}) = 0$ for any $\tilde{p} \in U$ one has that $\tilde{\Gamma}^{i(1)}_{jk}(\tilde{p}) = \tilde{\Gamma}^{i(2)}_{jk}(\tilde{p}) = 0$.
Let (p^i) be a flat system for $\nabla_{(1)}$ (i.e. $\Gamma^{i(1)}_{jk}(p) = 0$). Since Δ^i_{jk} is a tensor $\Delta^i_{jk}(p) = \frac{\partial p^i}{\partial \tilde{p}^s}\frac{\partial \tilde{p}^p}{\partial p^j}\frac{\partial \tilde{p}^b}{\partial p^k}\tilde{\Delta}^s_{pb}(\tilde{p}) = 0$, then $\Gamma^{i(1)}_{jk}(p) = \Gamma^{i(2)}_{jk}(p) = 0$ and the two connections have a common system of flat coordinates.
Ultimately, we prove the vanishing of Δ^{ijk}. Rising the indexes of Δ^i_{jk} ,contracting by $g^{sj}_{(1)}g^{qk}_{(2)}$, one obtains the $(3,0)$ tensor

$$\Delta^{sqi} := g^{sj}_{(1)}g^{qk}_{(2)}\Delta^i_{jk} = g^{sj}_{(1)}g^{qk}_{(2)}\tilde{\Gamma}^{i(1)}_{jk} - g^{sj}_{(1)}g^{qk}_{(2)}\tilde{\Gamma}^{i(2)}_{jk} = g^{qk}_{(2)}\Gamma^{si}_{k(1)} - g^{sj}_{(1)}\Gamma^{qi}_{j(2)}$$

Being the metrics $g_{(1)}$ and $g_{(2)}$ non-degenerate, $\Delta^{ijk}(x) = 0$ if and only if $\Delta^i_{jk}(x) = 0$ for any $x \in U$. ■

Let's consider the metric $g^{(2)} = (g^{(2)}_{ij})$ where $g^{(2)}_{ij} = (g^{ij}_{(2)})^{-1}$.
In particular, the metric $g^{(2)}$ is defined only on $M_0 := M - \Sigma$, where $\Sigma := \{p \in M : det(g^{ij}_{(2)}) = 0\}$.
Now, we can give the following:

Definition 3.10 *Let's consider the $(2,1)$ tensor field on M_0 of components*

$$\Delta^{ij}_k := g^{(2)}_{ks}\Delta^{sij} \tag{3.19}$$

The tensor Δ^{ij}_k defines the bilinear application Δ on T^*M_0 by

$$\Delta : \Gamma(T^*M_0) \times \Gamma(T^*M_0) \to \Gamma(T^*M_0) \tag{3.20}$$
$$(u,v) \mapsto \Delta(u,v) \tag{3.21}$$

where the action of Δ is obtained by extending by linearity the actions on a base, i.e.

$$\Delta(u,v) = u_i v_j \Delta(dx^i, dx^j) := u_i v_j \Delta^{ij}_k dx^k \tag{3.22}$$

here u_i and v_i are the components of the 1-form u and v with respect to a local frame of cotengent vetctors (dx^i).

Remark 3.11 *(3.19) can be rearranged as*

$$\Delta^{ij}_k = g^{(2)}_{ks}\Delta^{sij} = g^{(2)}_{ks}(g^{ip}_{(2)}\Gamma^{sj}_{p(1)} - g^{sp}_{(1)}\Gamma^{ij}_{p(2)}) \tag{3.23}$$

Furthermore, in flat coordinates (t^i) for $g_{(2)}$ it reads

$$\Delta^{ij}_k(t) = \Gamma^{ij}_{k(1)}(t) \tag{3.24}$$

Lemma 3.12 *Given a linear flat pencil $(g_{(1)}, g_{(2)})$, the following identities hold true:*

$$g^{sk}_{(1)}\Delta^{ij}_k = g^{ik}_{(1)}\Delta^{sj}_k \tag{3.25}$$
$$g^{sk}_{(2)}\Delta^{ij}_k = g^{ik}_{(2)}\Delta^{sj}_k \tag{3.26}$$
$$\Delta^{ij}_k\Delta^{ks}_p = \Delta^{ij}_k\Delta^{ks}_p \tag{3.27}$$
$$\partial_s\Delta^{jk}_l = \partial_l\Delta^{jk}_s \tag{3.28}$$

Proof: Quasi-homogeneity of the pencil isn't a necessary condition.
Recall that, in view of the definition of flat pencil, we have that the curvature tensor corresponding to the cometric $g_{(1)} - \lambda g_{(2)}$ vanishes for any λ. In flat coordinates (t^i) it reads

$$R^{ijk}_l(t) \overset{(3.24)}{=} (g^{is}_{(1)} - \lambda g^{is}_{(2)})(\partial_s \Delta^{jk}_l - \partial_l \Delta^{jk}_s) + \Delta^{ij}_s \Delta^{sk}_l - \Delta^{ik}_s \Delta^{sj}_l = 0$$

The vanishing of the linear term in λ yields (3.28). While the vanishing of the constant term in λ yields (3.27).
The torsionless condition of the Levi-Civita connection corresponding to the cometric $g^{is}_{(1)} - \lambda g^{is}_{(2)}$, written in (t^i), reads

$$(g^{is}_{(1)} - \lambda g^{is}_{(2)})\Delta^{jk}_s = (g^{js}_{(1)} - \lambda g^{js}_{(2)})\Delta^{ik}_s$$

The vanishing in any order in λ yield (3.25) and (3.26). ∎

Corollary 3.13 *(3.25) and (3.26) can be rearranged as follows:*

$$g^{(1)}_{is} \Delta^{sj}_k = g^{(1)}_{ks} \Delta^{sj}_i \tag{3.29}$$

$$g^{(2)}_{is} \Delta^{sj}_k = g^{(2)}_{ks} \Delta^{sj}_i \tag{3.30}$$

Proof: Contracting (3.25) by $g^{(1)}_{ps} g^{(1)}_{pi}$ one gets (3.29). Similarly one gets (3.30). ∎

Lemma 3.14 *Given a quasi-homogeneous linear flat pencil $(g_{(1)}, g_{(2)})$, the following formula hold true:*

$$\nabla_{(2)} \nabla_{(2)} \tau = 0 \tag{3.31}$$

$$\nabla_{(2)} \nabla_{(2)} E = 0 \tag{3.32}$$

Proof: For simplicity we denote $g^{ij}_{(2)} = \eta^{ij}$.
Observe that, in flat coordinates (t^i) for $\nabla_{(2)}$, one has

$$\nabla_{(2)} \nabla_{(2)} \tau = \partial_i \partial_j \tau$$

The condition (3.8) written in (t^i) reads

$$e^s \underbrace{\partial_s \eta^{ij}}_{=0} + \partial_s e^i \eta^{sj} + \partial_s e^j \eta^{is} = 0 \tag{3.33}$$

Differentiating (3.4) with respect t^s one obtains

$$\partial_s e^i = \underbrace{\partial_s \eta^{ik}}_{=0} \partial_k \tau + \eta^{ik} \partial_s \partial_k \tau$$

The latter substituted in (3.33) yields

$$\eta^{sj} \eta^{ik} \partial_s \partial_k \tau + \eta^{is} \eta^{jk} \partial_s \partial_k \tau = 0$$

Exchanging $s \leftrightarrow k$ one has

$$2\eta^{is}\eta^{jk}\partial_s\partial_k\tau = 0$$

Contracting by $\eta_{rs}\eta_{qk}$ one gets

$$\partial_r\partial_q\tau = 0$$

for any r and q.
Similarly, differentiating

$$\mathcal{L}_E\eta^{ij} = (d-2)\eta^{ij}$$

one obtains $\partial_r\partial_q E^s = 0$. ■

Remark 3.15 *From here on, for simplicity, we denote $g_{(1)} = g$, $g_{(2)} = \eta^{-1}$, we denote by Γ^{ij}_k the contravariant Christoffel symbols corresponding to the Levi-Civita connection of g and by (t^i) a flat coordinate system for g_2.*

Lemma 3.16 *The vector field e (3.4) has constant components in the flat coordinates (t^i). Furthermore, the following formula holds true*

$$Q^n_q = (1-d)\delta^n_q$$

Proof: Recall the formulas (3.4) and (3.32)

$$e = \nabla_{(2)}\tau$$
$$\nabla_{(2)}\nabla_{(2)}\tau = 0$$

Then e has constant components in the coordinates (t^i).
Recall that the flat coordinates are defined up to an affine transformation. Then we choose the flat system (t^i) so that

$$e^i = e^i(t) = \eta^{in} \tag{3.34}$$

Since $e^i = \eta^{is}\partial_s\tau$, one has $\eta^{in} = \eta^{is}\partial_s\tau$. Hence $\partial_s\tau = \eta_{si}\eta^{in} = \delta^n_s$. Then $\tau = t^n + c$, where c is a constant.
As $E^i = g^{is}\partial_s\tau$, we have that E written in (t^i) has components

$$E^i = E^i(t) = g^{is}\partial_s t^n = g^{is}\delta^n_s = g^{in} \tag{3.35}$$

The formula (3.5) written in (t^i) reads

$$e^i\,\underbrace{\partial_i E^j}_{=Q^j_i} - \underbrace{\partial_i e^j}_{=0}\,E^i = e^j$$

Then, in view of (3.34), one obtains $\eta^{in}Q^j_i = \eta^{jn}$.
Since η is symmetric, we have that

$$\eta^{ni}Q^j_i = \eta^{jn} = \eta^{nj} = \eta^{ji}Q^n_i$$

The formula (3.9) written explicitly reads

$$E^s\,\underbrace{\partial_s\eta^{ij}}_{=0} - \underbrace{\partial_s E^i}_{=Q^i_s}\,\eta^{sj} - \underbrace{\partial_s E^j}_{=Q^j_s}\,\eta^{is} = (d-2)\eta^{ij}$$

Take $j=n$

$$\underbrace{Q^i_s\eta^{sn}}_{=\eta^{in}} + Q^n_s\eta^{is} = (2-d)\eta^{in}$$
$$Q^n_s\eta^{is} = (1-d)\eta^{in}$$

Multiplying by η_{iq} one concludes the proof. ■

We will assume the choice (3.34) of the flat coordinates (t^i) also below.

Lemma 3.17 *In the coordinates (t^i), the following formulas hold true:*

$$\Delta^{in}_j = \frac{1-d}{2}\delta^i_j \tag{3.36}$$
$$\Delta^{ni}_j = \frac{d-1}{2}\delta^i_j + Q^i_j \tag{3.37}$$

Proof: Using the explicit formula of the Christoffel symbols corresponding to the Levi-Civita connection of g one gets

$$\Delta^n_{ij} = \Gamma^n_{ij} = \frac{1}{2}g^{ns}\big(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\big) = \tfrac{1}{2}\big(\partial_i(\underbrace{g^{ns}g_{sj}}_{=\delta^n_j}) - g_{sj}g^{ns} + \partial_j(\underbrace{g^{ns}g_{si}}_{=\delta^n_i}) - g_{si}g^{ns} - g^{ns}\partial_s g_{ij}\big)$$
$$\overset{(3.35)}{=} \big(-\partial_i E^s g_{sj} - \partial_j E^s g_{si} - E^s\partial_s g_{ij}\big) = -\frac{1}{2}\mathcal{L}_E g_{ij} \overset{(3.6)}{=} \frac{d-1}{2}g_{ij}$$

Multiplying by g^{is} the one has

$$\Delta^{in}_j = \Gamma^{in}_j := -g^{is}\Gamma^n_{sj} = \frac{1-d}{2}g^{is}g_{ij} = \frac{1-d}{2}\delta^s_j$$

which coincides with (3.36).
Recall the condition of compatibility of the connection with the metric

$$\partial_k g^{ij} = \Gamma^{ij}_k + \Gamma^{ji}_k$$

Take $j=n$

$$\underbrace{\partial_k g^{in}}_{=\partial_k E^i = Q^i_k} = \underbrace{\Gamma^{in}_k}_{=\frac{1-d}{2}\delta^i_k} + \Gamma^{ni}_k$$

Then one obtains

$$\Delta^{ni}_k = \Gamma^{ni}_k = \frac{d-1}{2}\delta^i_k + Q^i_k$$

which concludes the proof. ■

Lemma 3.18 *In the coordinates (t^i) the following formulas hold true:*

$$\mathcal{L}_E\Delta^{ij}_k = (d-1)\Delta^{ij}_k \tag{3.38}$$
$$\mathcal{L}_e\Delta^{ij}_k = 0 \tag{3.39}$$

Proof: Define the functions

$$\begin{aligned}\tilde{\Gamma}^{ij}_k : &= \mathcal{L}_E \Delta^{ij}_k + (1-d)\Delta^{ij}_k \\ &= E^s\partial_s\Delta^{ij}_k + \partial_k E^s \Delta^{ij}_s - \partial_s E^i \Delta^{sj}_k - \partial_s E^j \Delta^{is}_k + (1-d)\Delta^{ij}_k \\ &\overset{(3.24)}{=} E^s\partial_s\Gamma^{ij}_k + Q^s_k\Gamma^{ij}_s - Q^i_s\Gamma^{sj}_k - Q^j_s\Gamma^{is}_k + (1-d)\Gamma^{ij}_k\end{aligned}$$

Thus differentiating the system of equations

$$\begin{cases}\partial_k g^{ij} = \Gamma^{ij}_k + \Gamma^{ji}_k \\ g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s\end{cases} \tag{3.40}$$

along the vector field $E = E^i\partial_i$ one obtains the system

$$\begin{cases}\tilde{\Gamma}^{ij}_k + \tilde{\Gamma}^{ji}_k = 0 \\ g^{is}\tilde{\Gamma}^{jk}_s = g^{js}\tilde{\Gamma}^{ik}_s\end{cases} \tag{3.41}$$

Recall that the system (3.40) has a unique solution Γ^{ij}_k.
We observe that (3.41) is the linear homogeneous system corresponding to (3.40). Since (3.40) has a unique solution the corresponding linear homogeneous system has only the trivial solution $\tilde{\Gamma}^{ij}_k = 0$. Then $\mathcal{L}_E\Delta^{ji}_k = (d-1)\Delta^{ji}_k$.
Similarly, it can be proven that $\mathcal{L}_e\Delta^{ji}_k = 0$. ■

Lemma 3.19 *Following [5], (3.13) and (3.23) can be written as follows:*

$$R^j_i = \nabla^{(2)}_j E^i - \nabla^{(1)}_j E^i \tag{3.42}$$

$$\Delta^{jk}_m = L^s_m \eta^{jt}\left(\Gamma^{k(2)}_{st} - \Gamma^{k(1)}_{st}\right) \tag{3.43}$$

where $L^s_h = g^{sm}\eta_{mh}$.

Proof: One has

$$\begin{aligned}\nabla^{(1)}_i E^k &= \partial_i E^k + \Gamma^k_{ij}E^j = \partial_i E^k + \tfrac{1}{2}g^{ks}\left(\partial_i g^{ks}E^j + \partial_j g^{si}E^j - \partial_s g^{ij}E^j\right) \\ &\overset{(3.6)}{=} \partial_i E^k + \tfrac{1}{2}g^{ks}\left(\partial_i g_{js}E^j - g_{sj}\partial_i E^j - g_{ij}\partial_s E^j + (1-d)g_{si} - \partial_s g_{ij}E^j\right) \\ &= \partial_i E^k + \tfrac{1}{2}\left(g^{ks}\partial_i g_{js}E^j - \delta^k_j\partial_i E^j - g^{ks}\partial_s(g_{ij}E^j) + (1-d)\delta^k_i\right)\end{aligned}$$

Denote by $\theta_i := g_{ij}E^j$ the components of the 1-form θ. Then

$$\begin{aligned}\nabla^{(1)}_i E^k &= \tfrac{1}{2}\partial_i E^k + \tfrac{1}{2}\left(g^{ks}\partial_i g_{js}E^j - g^{ks}\partial_s\theta_i\right) + \tfrac{1-d}{2}\delta^k_i \\ &= \tfrac{1}{2}\partial_i E^k + \tfrac{1}{2}\Big(g^{ks}\partial_i\underbrace{(g_{js}E^j)}_{=\theta_s} - \underbrace{g^{ks}g_{js}}_{=\delta^k_j}\partial_i E^j - g^{ks}\partial_s\theta_i\Big) + \tfrac{1-d}{2}\delta^k_i \\ &= \tfrac{1}{2}g^{ks}\left(\partial_i\theta_s - \partial_s\theta_i\right) + \tfrac{1-d}{2}\delta^k_i\end{aligned}$$

Using the Egorov condition (3.4) one obtains that $\theta_s = \partial_s\tau$. Then

$$\partial_i\theta_s - \partial_s\theta_i = \partial_i\partial_s\tau - \partial_s\partial_i\tau = 0$$

since τ is a smooth function. Therefore

$$\nabla_i^{(2)} E^k - \nabla_i^{(1)} E^k = \nabla_i^{(2)} E^k + \tfrac{d-1}{2}\delta_i^k \overset{(3.13)}{=} R_i^k$$

(3.42) is proven. By a straightforward computation, one has

$$L_m^s \eta^{jt}\left(\Gamma_{st}^{k(2)} - \Gamma_{st}^{k(1)}\right) = g^{sq}\eta_{qm}\eta^{jt}\left(\Gamma_{st}^{k(2)} - \Gamma_{st}^{k(1)}\right) = \eta_{qm}\left(g^{sq}\underbrace{\eta^{jt}\Gamma_{st}^{k(2)}}_{=-\Gamma_{s(2)}^{jk}} - \eta^{jt}\underbrace{g^{sq}\Gamma_{st}^{k(1)}}_{=-\Gamma_{t(1)}^{qk}}\right)$$

$$= \eta_{qm}\left(\eta^{jt}\Gamma_{t(1)}^{qk} - g^{sq}\Gamma_{s(2)}^{jk}\right)$$

which coincides with (3.23). ■

Proposition 3.20 *Let (g, η^{-1}) be a quasi-homogeneous linear flat pencil on M of degree d. If the operator R is invertible on M, then the data $(M, \eta, \nabla^{(2)}, \circ, e, E)$, where $\circ$ is the product defined by the structure constants*

$$c_{hk}^j := L_h^s\left(\Gamma_{sk}^{l(2)} - \Gamma_{sk}^{l(1)}\right)(R^{-1})_l^j \tag{3.44}$$

defines a Frobenius manifold on M.

Remark 3.21 *Using the tensor (3.43), the tensor (3.42) and (3.43) can be rearagend as follows:*

$$R_s^m = \left(\Gamma_{sl}^{m(2)} - \Gamma_{sl}^{m(1)}\right)E^l \overset{(3.43)}{=} g_{sr}\eta^{rq}\Delta_q^{pm}\eta_{pl}E^l \overset{(3.31)}{=} g_{sr}\eta^{rq}\Delta_l^{pm}\eta_{pq}E^l = g_{sp}\Delta_l^{pm}E^l \tag{3.45}$$

$$c_{hk}^j \overset{(3.43)}{=} \Delta_h^{ml}\eta_{mk}(R^{-1})_l^j \tag{3.46}$$

We give the preliminary lemma.

Lemma 3.22 *The following identity holds true:*

$$\Delta_k^{tl}(R^{-1})_l^s = \Delta_k^{sl}(R^{-1})_l^t \tag{3.47}$$

Proof: Equivalently, we have to prove that $\Delta_k^{sh}R_s^m = \Delta_k^{sm}R_s^h$. Suddenly, (3.47) is obtained by contracting by $(R^{-1})_m^q(R^{-1})_h^p$. We have

$$\Delta_k^{sh}R_s^m \overset{(3.45)}{=} \Delta_k^{sh}g_{sp}\Delta_l^{pm}E^l \overset{(3.29)}{=} \Delta_p^{sh}g_{sk}\Delta_l^{pm}E^l \overset{(3.27)}{=} \Delta_p^{sm}g_{sk}\Delta_l^{ph}E^l \overset{(3.29)}{=} \Delta_k^{sm}g_{sp}\Delta_l^{ph}E^l \overset{(3.45)}{=} \Delta_k^{sm}R_s^h$$

■

Now, we can prove the proposition 3.20.
Proof: In order to have a Frobenius manifold we have to prove the following points 1. The product is commutative

$$c_{hk}^j \overset{(3.46)}{=} \Delta_h^{ml}\eta_{mk}(R^{-1})_l^j \overset{(3.30)}{=} \Delta_k^{ml}\eta_{mh}(R^{-1})_l^j \overset{(3.46)}{=} c_{kh}^j$$

2. The product is associative

$$R_i^l c_{sj}^i c_{hk}^s \overset{(3.45),(3.46)}{=} \Delta_s^{ql}\eta_{qj}(R^{-1})_r^s\Delta_h^{mr}\eta_{mk} \overset{(3.47)}{=} \Delta_s^{ql}\eta_{qj}(R^{-1})_r^m\Delta_h^{sr}\eta_{mk} \overset{(3.27)}{=} \Delta_s^{qr}\eta_{qj}(R^{-1})_r^m\Delta_h^{sl}\eta_{mk}$$

$$\stackrel{(3.47)}{=} \Delta_s^{mr}\eta_{qj}(R^{-1})^q_r\Delta_h^{sl}\eta_{mk} \stackrel{(3.27)}{=} \Delta_s^{ml}\eta_{qj}(R^{-1})^q_r\Delta_h^{sr}\eta_{mk} \stackrel{(3.47)}{=} \Delta_s^{ml}\eta_{qj}(R^{-1})^s_r\Delta_h^{qr}\eta_{mk} \stackrel{(3.27)}{=} R^l_i c^i_{sk} c^s_{hj}$$

3. The vector field e is the unity of the product

$$c^j_{hk}e^h \stackrel{(3.44)}{=} L^s_h\left(\Gamma^{(2)l}_{sk} - \Gamma^{(1)l}_{sk}\right)(R^{-1})^j_l e^h \stackrel{(2.61)}{=} \underbrace{g^{sq}\eta_{ph}e^h}_{\stackrel{(3.10)}{=} E^s}\left(\Gamma^{(2)l}_{sk} - \Gamma^{(1)l}_{sk}\right)(R^{-1})^j_l \stackrel{(3.42)}{=} R^l_k(R^{-1})^j_l = \delta^j_k$$

4. The metric η is invariant with respect to the product $\circ$

$$\begin{aligned} \eta_{sj}c^j_{hl} &\stackrel{(3.46)}{=} \eta_{sj}\Delta_h^{ql}\eta_{qk}(R^{-1})^j_l \stackrel{(3.45)}{=} \eta_{sj}\Delta_h^{jl}\eta_{qk}(R^{-1})^q_l \stackrel{(3.30)}{=} \eta_{hj}\Delta_s^{jl}\eta_{qk}(R^{-1})^q_l \stackrel{(3.45)}{=} \eta_{hj}\Delta_s^{ql}\eta_{qk}(R^{-1})^j_l \\ &\stackrel{(3.46)}{=} \eta_{hj}c^j_{sl} \end{aligned}$$

5. $\mathcal{L}_E c^k_{ij} = c^k_{ij}$. Observe that

$$\mathcal{L}_E R^i_j \stackrel{(3.45)}{=} \mathcal{L}_E(g_{jp}\Delta_l^{pi}E^l) = \underbrace{\mathcal{L}_E g_{jp}}_{\stackrel{(3.6)}{=}(1-d)g_{jp}}\Delta_l^{pi}E^l + g_{jp}\underbrace{\mathcal{L}_E\Delta_l^{pi}}_{\stackrel{(3.38)}{=}(d-1)\Delta_l^{pi}}E^l + g_{jp}\Delta_l^{pi}\underbrace{\mathcal{L}_E E^l}_{=[E,E]^l=0} = 0$$

Then $\mathcal{L}_E(R^{-1})^i_j = 0$. Hence

$$\begin{aligned} \mathcal{L}_E c^k_{ij} &\stackrel{(3.46)}{=} \mathcal{L}_E\left(\Delta_i^{ml}\eta_{mj}(R^{-1})^k_l\right) \\ &= \underbrace{\mathcal{L}_E\Delta_l^{pi}}_{\stackrel{(3.38)}{=}(d-1)\Delta_l^{pi}}\eta_{mj}(R^{-1})^k_l + \Delta_i^{ml}\underbrace{\mathcal{L}_E\eta_{mj}}_{\stackrel{(3.9)}{=}(2-d)\eta_{mj}}(R^{-1})^k_l + \Delta_i^{ml}\eta_{mj}\underbrace{\mathcal{L}_E(R^{-1})^k_l}_{=0} \\ &\stackrel{(3.46)}{=} c^k_{ij} \end{aligned}$$

This concludes the proof. ■

4 Coxeter groups and Frobenius manifolds

The main references of this section are [18] and [46].
In this section we will recall Dubrovin's procedure, following [18], aimed at constructing a Frobenius manifold on the orbit space of a Coxeter group. The realization of a Frobenius structure relies on the notion of flat pencil of cometrics and on the existence of a distinguished set of polynomial basic invariants for the Coxeter group, called Saito flat coordinates (see [46]).
Dubrovin conjectured that any polynomial Frobenius manifold may be obtained following this scheme. Later, Hertling proved this statement (see [31]).

In the following subsections, we recall some facts concerning the theory of Coxeter group. For a comprehensive survey see for example [Humpryes].

4.1 Coxeter groups

Recall what is meant by a reflection acting on a n-dimensional real vector space V equipped with a positive-definite symmetric bilinear form $\langle\cdot,\cdot\rangle$.

Definition 4.1 *A real reflection is a linear operator on V which sends some non-zero vector α to its negative while fixing pointwise the (hyper-)plane H_α orthogonal to α. Explicitly, the action of a reflection, with respect to the vector α, is defined by the formula*

$$S_\alpha\lambda := \lambda - 2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \tag{4.1}$$

for any $\lambda \in V$.

Remark 4.2 *The formula (4.1) defines actually a reflection, indeed*

1. *taking $\lambda = \alpha$ one has $S_\alpha\alpha = \alpha - 2\frac{\langle\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha = -\alpha$*
2. *taking $\lambda \in H_\alpha$, i.e. $\langle\lambda,\alpha\rangle = 0$, one has $S_\alpha\lambda = \lambda - \underbrace{2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha}_{=0} = \lambda$*

Definition 4.3 *We define a Coxeter group to be a finite group generated by real reflection. A generic Coxeter group will be denoted by W.*

Remark 4.4 *Any element of a Coxeter group has order two, i.e. $S_\alpha^2 = id$.*

Remark 4.5 *Any real reflection is an orthogonal transformation, or equivalently*

$$W \subset O(n,\mathbb{R})$$

Here $O(n,\mathbb{R})$ is the group of the orthogonal transformations on a n-dimensional real vector space. Indeed

$$\langle S_\alpha\lambda, S_\alpha\mu\rangle = \langle\lambda - 2\tfrac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha, \mu - 2\tfrac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\rangle = \langle\lambda,\mu\rangle - 2\tfrac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\langle\alpha,\mu\rangle + 4\tfrac{\langle\lambda,\alpha\rangle\langle\mu,\alpha\rangle\langle\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} - 2\tfrac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\langle\alpha,\lambda\rangle = \langle\lambda,\mu\rangle$$

Definition 4.6 *A Coxeter group is called reducible if it can be decomposed if it can be decomposed as*

$$W = W_1 \times W_2$$

where both W_1 and W_2 are nontrivial subgroups generated by reflection of W.

Remark 4.7 *The complete classification of Coxeter group was obtained by Coxetr in [13]. The complete classification consists of the following groups:*

- *The Weyl group $A_n, B_n, D_n, E_6, E_7, E_8, F_4, G_2$.*
- *The group H_3, H_4 of symmetries of the regular icosahedron and of the regular 600-cell in 4-dimensional space respectively.*
- *The group $I_2(k)$ of symmetries of the regular k-gone, i.e. the dihedral group.*

4.2 Polynomial invariants of a finite group

Before considering reflection groups, recall some facts about the polynomial invariants of an arbitrary finite subgroup of $GL(V)$. Here $GL(V)$ is the group of linear invertible transformation on the n-dimensional vector space V over a field of characteristic 0.
Denote by S the symmetric algebra $S(V^*)$ of the dual space V^*, which coincides with the algebra of polynomial functions on V.
Fixed a basis of V, S may be identified with the polynomials ring $K[x_1, ..., x_n]$, where x_i are (linear) coordinate functions on V. There is a natural action of G on S induced by the contragredient action of G on V^*, defined by

$$(g \cdot f)(v) := f(g^{-1}v) \tag{4.2}$$

where $g \in G$, $v \in V$ and $f \in V^*$.
We observe that this action preserves the natural grading of S.

Definition 4.8 *We say that $f \in S$ is G-invariant if*

$$g \cdot f = f \tag{4.3}$$

for any $g \in G$.

Definition 4.9 *We define S^G to be the subalgebra of S generated by the G-invariant element of S.*

Definition 4.10 *We say that the polynomials $\{f_1, ..., f_k\}$ in $K[x_1, ..., x_n]$ are algebraically independent if, for any nonzero polynomial $h \in K[x_1, ..., x_k]$, we have $h(f_1, ..., f_k) \neq 0$.*

Let W be a Coxeter group. We present a version of the Chevalley theorem for the invariant ring, taking as base field $\mathbb{R}$. This result can be generalized to the complex case.

Theorem 4.11 *Let $B = S^G$ be the subalgebra of $\mathbb{R}[x_1, ..., x_n]$ of W-invariant polynomials. Then B is generated as an $\mathbb{R}$-algebra by n homogeneous and algebraically independent polynomials of positive degree together with 1.*

Remark 4.12 *B has the natural decomposition*

$$B = B_+ \oplus \mathbb{R}$$

where B_+ is the subalgebra of W-invariant polynomials with strictly positive degree.

Definition 4.13 *We define a set of algebraically independent homogeneous generators of B, with strictly positive degree, a set of (polynomial) basic invariants of B.*

Remark 4.14 *The algebraically independent generators of B are not uniquely defined in general. However, the degrees corresponding to the basic invariants turn out to be uniquely defined by the choice of the Coxeter group. More precisely, the following proposition holds true:*

Proposition 4.15 *Suppose that $\{f_1, ..., f_n\}$ and $\{g_1, ..., g_n\}$ are two sets of basic invariants of B and denote by $\{d_1, ..., d_n\}$ and $\{e_1, ..., e_n\}$ the respective degrees. Then, up to a renumbering of the degrees, we have $d_i = e_i$ for all i.*

For any irreducible Coxeter group, the degrees of the basic invariants are known. We give the list of the degrees corresponding to the basic invariant incrementally ordered.

W	$d_n, ..., d_1$
A_n	$d_i = n + 2 - i$
B_n	$d_i = 2(n + 2 - i)$
$D_n(n = 2k)$	$d_i = 2(n - i)(i \leq k)$
	$d_i = 2(n - i + 1)(i > k)$
$D_n(n = 2k + 1)$	$d_i = 2(n - i)(i \leq k)$
	$d_{k+1} = 2k + 1$
	$d_i = 2(n - i + 1)(i > k + 1)$
E_6	$12, 9, 8, 6, 5, 2$
E_7	$18, 14, 12, 10, 8, 6, 5, 2$
E_8	$30, 24, 20, 18, 14, 12, 8, 2$
F_4	$12, 8, 6, 2$
G_2	$6, 2$
H_3	$10, 6, 2$
H_4	$30, 20, 12, 2$
$I_2(k)$	$k, 2$

4.3 Frobenius structure on the orbit space of a Coxeter group

Let V be a n-dimensional real vector space. V is endowed with an Euclidean inner product $\langle\cdot,\cdot\rangle_{(g)}$ of components

$$g_{ij}(p) := \left\langle \frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^i} \right\rangle_{(g)} = \delta_{ij} \tag{4.4}$$

here $(p^1, ..., p^n)$ is a system of linear and orthogonal coordinates.
Let W be a Coxeter group. We denote by $R = \mathbb{R}[p^1, ..., p^n]$ the polynomial ring in $(p^1, ..., p^n)$ with real coefficients.
The group W acts in a natural way on the ring R. Let R^W to be the subring of R of W-invariant polynomials. In view of the Chevallay theorem, there exists n algebraically independent homogeneous polynomials $\{u^1, ..., u^n\}$ of degrees $d_i := deg(u^i)$ ordered so that

$$d_1 = 2 < d_2 \leq d_3 \leq .... \leq d_{n-1} < d_n = h$$

such that $R^W = \langle u^1, ..., u^n\rangle_{\mathbb{R}}$. h is called the Coxeter number of W.
We highlight that the basic invariant $\{u^1, ..., u^n\}$ aren't uniquely defined. While the degrees are uniquely determined by the Coxeter group W.
We denote by $\Omega(V)$ the R-module of differential forms on V with polynomial coefficients. Let $\Omega(V)^W$ be the submodule of W-invariant differential forms with polynomial coefficients.

Remark 4.16 *Let be C be an element in W.The eigenvalues of C have the form (see [9])*

$$\lambda_i = exp\left(\frac{2\pi i(d_i - 1)}{h}\right) \tag{4.5}$$

Moreover, the degrees d_i satisfy the duality condition (see [9])

$$d_i + d_{n-i+1} = h + 2 \tag{4.6}$$

for any $i = 1, ..., n$.

Any Coxeter transformation $A \in W$ induces a transformation on any metric by the transformation rule of a $(0, 2)$ tensor.
Similarly, one gets the induced transformation for any cometric g, i.e.

$$g \mapsto A^T g A \tag{4.7}$$

Remark 4.17 *Recall that, in view of remark (4.5), any transformation in W is orthogonal. Since the orthogonal group $O(n, \mathbb{R})$ is the group of transformations that leaves invariant the Euclidean metric, one has that the latter is left invariant by any Coxeter transformation as well. In particular, the Euclidean norm $(p^1)^2 + ... + (p^n)^2$ on V is left invariant by any Coxeter transformation. Observe that for any irreducible Coxeter group, the lowest polynomial basic invariant degree turns out to be 2. Therefore for any irreducible Coxeter group the basic invariant polynomial of lowest degree coincides, up to*

rescaling, with the Euclidean norm.
We normalize u^1 so that

$$u^1 = \frac{1}{2h}\left((p^1)^2 + ... + (p^n)^2\right) \tag{4.8}$$

Remark 4.18 *One may extend the action of the group W to the complexified vector space*

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \tag{4.9}$$

Since V is a real vector space, the subscript $\mathbb{R}$ may be omitted. According to the result of Coxeter and Chevalley, the orbit space

$$M := (V \otimes \mathbb{C})/W \tag{4.10}$$

has a natural structure of affine algebraic variety with coordinate ring

$$R^W \otimes \mathbb{C} \tag{4.11}$$

Therefore $(u^1, ..., u^n)$ may be used as local coordinates of M.

Remark 4.19 *As the basic invariants aren't uniquely defined, the coordinates $(u^1, ..., u^n)$ of M are defined up to an invertible transformation*

$$u^j \longmapsto \tilde{u}^i = \tilde{u}^i(u^1, ..., u^n) \tag{4.12}$$

where $\tilde{u}^i(u^1, ..., u^n)$ is a quasi-homogeneous polynomial in $(u^1, ..., u^n)$, with $deg(\tilde{u}^i) = deg(u^i) = d_i$

Lemma 4.20 *The transformation (4.12) leaves invariant the vector field*

$$e := \partial_n = \frac{\partial}{\partial u^n} \tag{4.13}$$

up to a constant factor.

Proof: The transformation law of e induced by (4.12) has the form

$$\tilde{e}^i = \frac{\partial \tilde{u}^i}{\partial u^j} e^j \overset{(2.31)}{=} \frac{\partial \tilde{u}^i}{\partial u^n}$$

Since the basic invariants are ordered so that

$$deg(\tilde{u}^n) = deg(u^n) > deg(u^i)$$

one has

$$\frac{\partial \tilde{u}^i}{\partial u^n} = 0 \tag{4.14}$$

for any $i \neq n$. Since $\tilde{u}^n = \tilde{u}^n(u^1, ..., u^n)$ is a quasi-homogeneous polynomial of degree $d_n = deg(u^n)$, $\tilde{u}^n$ must has the form

$$\tilde{u}^n = cu^n + g(u^1, ..., u^{n-1})$$

where c is a constant and $g(u^1, ..., u^{n-1})$ is a quasi-homogeneous polynomial of degree d_n. Therefore

$$\frac{\partial \tilde{u}^n}{\partial u^n} = c$$

Then

$$\tilde{e} = c\frac{\partial}{\partial u^i}$$

which concludes the proof. ■

Definition 4.21 *Let $(u^1, ..., u^n)$ be a set of basic invariants of degrees $d_1, ..., d_n$ respectively. We define*

$$E := d_i u^i \frac{\partial}{\partial u^i}$$

to be the vector field generating the scaling transformation.

Lemma 4.22 *The vector E is well-defined on M. Furthermore, the following identity holds true*

$$E = d_i u^i \frac{\partial}{\partial u^i} = p^i \frac{\partial}{\partial p^i} \tag{4.15}$$

Proof: This formula is a consequence of Euler's theorem for homogeneous functions. ■

We extend $\langle \cdot, \cdot \rangle_{(g)}$ to the complexified space $V \otimes \mathbb{C}$ as a complex bilinear form.

Remark 4.23 *The factorization map*

$$\pi : V \otimes \mathbb{C} \to M \tag{4.16}$$

is a local diffeomorphism of an open set of $V \otimes \mathbb{C}$ onto its image. The image of this subset in M consists of regular orbits (i.e. the orbit whose cardinality coincides with the order of the Coxeter group W). We define its complement as the discriminant of W; we denoted it by $Discr(W)$. In particular, for a Coxeter group W, it coincides with the union of the reflecting (hyper-)planes associated with W. Notice that the linear coordinates $(p^1, ..., p^n)$ of V can serve also as local coordinates in a neighborhood in $M - Discr(W)$.

We want to define a new cometric on $M - Discr(W)$. In particular, the Euclidean metric can be extended onto M as follows. Let $\langle \cdot, \cdot \rangle^*_{(g)}$ be the bilinear form on the dual space V^* defined by the cometric $g^{-1} = (g^{ij})$, where g^{ij} is the matrix inverse of (4.4), i.e.

$$g^{ij}(p) := \big(g_{ij}(p)\big)^{-1} = \delta^{ij} \tag{4.17}$$

Remark 4.24 *The bilinear form $\langle \cdot, \cdot \rangle_{(g)}$ coincides with the Euclidean inner product on $T^*V \cong V^*$. Then, in view of the remark (4.17), one has that the cometric g^{-1} is well-defined on the orbit space M. Let*

$$g^{ij}(u) := \langle du^i, du^j \rangle^*_{(g)} = \frac{\partial u^i}{\partial p^s}\frac{\partial u^j}{\partial p^k}\delta^{sk} = \frac{\partial u^i}{\partial p^s}\frac{\partial u^j}{\partial p^s}$$

be the components of the cometric g^{-1} written in the basic invariants, here the latter quantity is evaluated in $p = p(u)$ (these functions are defined at least locally outside $Discr(W)$). Since the partial derivative of any polynomial invariant is a homogeneous polynomial, observe that any entry of $\big(g^{ij}(u)\big)$ is a homogenous polynomial.

Lemma 4.25 *The Euclidean metric on V induces a W-invariant cometric with quasi-polynomial entries, defined on the orbit space M, of components*

$$g^{ij}(u) = \frac{\partial u^i}{\partial p^s}\frac{\partial u^j}{\partial p^s} \tag{4.18}$$

Furthermore, the corresponding contravariant Christoffel symbols of the Levi-Civita connection, written in basic invariants $(u^1, ..., u^n)$, defined by

$$\Gamma^{ij}_k(u) = \frac{\partial u^i}{\partial p^a}\frac{\partial^2 u^j}{\partial p^a \partial p^b}\frac{dp^b}{du^k} \tag{4.19}$$

are also quasi-homogeneous polynomial functions on the orbit space M.

Proof: The first part of the thesis coincides with the previous remark. Using Solomon's result (see [50] for details), one has that the Christofell symbols associated with the Levi-Civita connection of (g^{ij}) are W-invariant functions. In view of (4.17), one has that (p^i) is a flat and orthonormal coordinates system for (g^{ij}). Therefore applying the formula (1.16), one obtains the formula

$$\Gamma^{ij}_k(u)du^k = \frac{\partial u^i}{\partial p^a}\frac{\partial^2 u^j}{\partial p^a \partial p^b}dp^b \tag{4.20}$$

Since the functions $\frac{\partial u^i}{\partial p^a}$ and $\frac{\partial^2 u^j}{\partial p^a \partial p^b}$ are homogeneous polynomials, the right-hand side of (4.20) is a differential form with homogeneous coefficients. Then the functions $\Gamma^{ij}_k(u)$ are quasi-homogeneous polynomials for any choice of the indexes. ∎

Remark 4.26 *The matrix $\big(g^{ij}(u)\big)$ doesn't degenerate on $M - Discr(W)$ where the projection (4.16) is a local diffeomorphism. Thus the polynomial (also called the discriminant of W)*

$$D(u) := det\big(g^{ij}(u)\big) \tag{4.21}$$

vanish identically where the linear function $(p^1, ..., p^n)$ fail to be local coordinates of M. So $\big(g^{ij}(u)\big)$ is often called the discriminant matrix of W.

Corollary 4.27 *The polynomial functions $g^{ij}(u)$ and $\Gamma^{ij}_k(u)$, defined by (4.18) and (4.19), depend at most linearly on u^n.*

Proof: Recall the formula (4.18) and (4.19);

$$g^{ij}(u) = \frac{\partial u^i}{\partial p^s}\frac{\partial u^j}{\partial p^s} \tag{4.22}$$

$$\Gamma^{ij}_k(u)\frac{\partial u^k}{\partial p^b} = \frac{\partial u^i}{\partial p^a}\frac{\partial^2 u^j}{\partial p^a \partial p^b} \tag{4.23}$$

Observe that the partial derivative of a homogeneous polynomial coincides with a homogeneous polynomial of degree lowered by one. Therefore by comparing the degrees of both sides of (4.22) and (4.23) one gets

$$deg\big(g^{ij}(u)\big) = \big(deg(u^i) - 1\big) + \big(deg(u^j) - 1\big) = \underbrace{d_i + d_j}_{\leq 2d_n} - 2 < 2d_n \tag{4.24}$$

Similarly

$$deg\big(\Gamma^{ij}_k(u)\big) + \big(deg(u^k) - 1\big) = \big(deg(u^i) - 1\big) + \big(deg(u^j) - 2\big)$$

Then

$$deg\big(\Gamma^{ij}_k(u)\big) = \underbrace{d_i + d_j}_{\leq 2d_n} - d_k - 2 < 2d_n \tag{4.25}$$

Since d_n is the degree of u^n, both $g^{ij}(u)$ and $\Gamma^{ij}_k(u)$ depend at most linearly on u^n, for any choice of the indexes. ■

Following Saito [46], we give the following:

Corollary 4.28 *The matrix of components*

$$\eta^{ij}(u) := \partial_n g^{ij}(u) = \frac{\partial g^{ij}(u)}{\partial u^n} \tag{4.26}$$

has a triangular form, i.e.

$$\eta^{ij}(u) = 0 \tag{4.27}$$

for $i + j < n + 1$. Furthermore, the anti-diagonal entries

$$\eta^{i,n+1-i}(u) = c_i \tag{4.28}$$

are non-zero constants. In particular, (η^{ij}) is a non-degenerate matrix with

$$c := det(\eta^{ij}) = (-1)^{\frac{n(n-1)}{2}} c_1....c_n \neq 0 \tag{4.29}$$

Proof: Recall that (see (4.24))

$$deg\big(g^{ij}(u)\big) = d_i + d_j - 2$$

In view of the definition (4.26) one obtains

$$deg\big(\eta^{ij}(u)\big) = deg\big(g^{ij}(u)\big) - d_n = d_i + d_j - \underbrace{d_n}_{:=h} - 2 \tag{4.30}$$

where h is the Coxeter number. Using the duality condition

$$d_i + d_{n+1-i} = h + 2 \tag{4.31}$$

one gets that

$$deg\big(\eta^{i,n+1-i}\big) = \underbrace{d_i + d_{n+1-i}}_{=h+2} - h - 2 = 0$$

Then the anti-diagonal terms are constants (polynomials of zero degree).
Recall that, for any Coxeter group, we ordered the basic invariants so that $d_i > d_j$ for $i > j$. Therefore $deg\big(\eta^{ij}\big) < deg\big(\eta^{ik}\big)$ for $j < k$.
Then, for $i + j < n + 1 \iff j < n + 1 - i$, one has

$$deg\big(\eta^{ij}\big) < deg\big(\eta^{i,n+1-i}\big) = 0$$

Hence $\eta^{ij}(u) = 0$ for $i + j < n + 1$. The triangularity is proven.
To prove the non-degenerateness of (η^{ij}) we consider, following Saito, the discriminant (4.21) written as a polynomial in u^n

$$D(u) = c(u^n)^n + a_1(u^n)^{n-1} + ... + a_{n-1}u^n + a_n \tag{4.32}$$

where $a_1, ..., a_n$ are quasi-homogeneous polynomials in $(u^1, ..., u^{n-1})$ respectively of degrees $h, ..., nh$ respectively and the leading coefficient c is defined by (4.29).
Let λ be an eigenvector of any Coxeter transformation with eigenvalue

$$exp\Big(\frac{2\pi i}{h}\Big) = exp\Big(\frac{2\pi i(d_1 - 1)}{h}\Big)$$

as $d_1 = 2$ for any Coxeter group.
Since by definition, the polynomial basic invariants polynomials are constant on the orbit defined by the action of W, for any Coxeter transformation C one has

$$u^k(\lambda) = u^k(C\lambda) = u^k\big(e^{\frac{2\pi i}{h}}\lambda\big) = e^{\frac{2\pi i d_k}{h}} u^k(\lambda)$$

where in the last equality we have exploited the homogeneity of u^k.
Hence $u^k(\lambda) = 0$ for $k = 1, ..., n-1$ (since $d_n = h$).
Evaluating the discriminant in $u(\lambda)$ one gets

$$D\big(u(\lambda)\big) = c(u^n(\lambda))^n + \underbrace{a_1\big(u^1(\lambda), ..., u^{n-1}(\lambda)\big)}_{=0}\big(u^n(\lambda)\big)^{n-1} + ... + \underbrace{a_n\big(u^1(\lambda), ..., u^{n-1}(\lambda)\big)}_{=0}$$

as $a_1, ..., a_n$ are quasi-homogeneous functions they vanish at the origin. Therefore

$$D\big(u(\lambda)\big) = c(u^n(\lambda))^n$$

But $D\big(u(\lambda)\big) \neq 0$ (see [9]), hence $c \neq 0$. ■

Recall briefly the notion of flat pencil of cometrics presented in the previous section. Let's consider a manifold M supplied with two non-proportional cometrics $g_{(1)} = (g^{ij}_{(1)})$ and $g_{(2)} = (g^{ij}_{(2)})$. We denote by $\Gamma^{ij}_{k(1)}$ and $\Gamma^{ij}_{k(2)}$ the corresponding contravariant Christoffel symbols of the Levi-Civita connection.
Recall that the functions

$$\Delta^{ijk} = g^{is}_{(2)}\Gamma^{jk}_{s(1)} - g^{is}_{(1)}\Gamma^{jk}_{s(2)} \tag{4.33}$$

define a $(3, 0)$ tensor field on the manifold M.
The cometrics $(g_{(1)}, g_{(2)})$ define a linear flat pencil if:

1. The cometric
$$g^{ij}_{(\lambda)} := g^{ij}_{(1)} + \lambda g^{ij}_{(2)}$$
is flat for any $\lambda \in \mathbb{R}$.

2. The contravariant Christoffel symbols of the cometric $g_{(\lambda)}$ have the form
$$\Gamma^{ij}_{k(\lambda)} = \Gamma^{ij}_{k(1)} + \lambda\Gamma^{ij}_{k(2)}$$

We give a technical result.

Proposition 4.29 *Given a flat pencil* $(g_{(1)}, g_{(2)})$ *there exists a vector field* $f = f^i\partial_i$ *such that tensor* (4.33) *and the cometric* $g_{(1)}$ *have the forms*

$$\Delta^{ijk} = \nabla^i_{(2)}\nabla^j_{(2)}f^k \tag{4.34}$$
$$g^{ij}_{(1)} = \nabla^i_{(2)}f^j + \nabla^j_{(2)}f^i + bg^{ij}_{(2)} \tag{4.35}$$

where b *is a real constant and* $\nabla^i_{(2)} := g^{is}_{(2)}\nabla^{(2)}_s$.
The $(2,0)$ *tensor field*

$$\Delta^{ij}_k := g^{(2)}_{ks}\Delta^{sij} = g^{(2)}_{ks}\nabla^s_{(2)}\nabla^i_{(2)}f^j \tag{4.36}$$

satisfies the identity

$$\Delta^{ij}_s\Delta^{sk}_l = \Delta^{ik}_s\Delta^{sj}_l \tag{4.37}$$

Moreover

$$\left(g^{is}_{(1)}g^{jq}_{(2)} - g^{is}_{(2)}g^{jq}_{(1)}\right)\nabla^{(2)}_s\nabla^{(2)}_q f^k = 0 \tag{4.38}$$

Conversely, given a flat cometric $g_{(2)}$ *and a solution* (f^k) *of* (4.38)*, the cometrics* $(g_{(1)}, g_{(2)})$*, where* $g_{(1)}$ *is defined by* (4.35)*, form a flat pencil.*

Proof: Let (t^i) be a flat coordinate system for $g_{(2)}$. Recall that, in these coordinates, $g_{(2)}$ reduces to a constant matrix and the corresponding Christoffel symbols vanish.
In these coordinate the formula (3.24) holds true:

$$\Delta^{ij}_s(t) = \Gamma^{ij}_{k(1)}(t) \tag{4.39}$$

Recall that, the vanishing of the curvature tensor associated with the cometric $g_{(\lambda)} = g_{(1)} + \lambda g_{(2)}$ yield the formulas (see the lemma (3.12))

$$g^{sk}_{(1)}\Delta^{ij}_k = g^{ik}_{(1)}\Delta^{sj}_k \tag{4.40}$$
$$g^{sk}_{(2)}\Delta^{ij}_k = g^{ik}_{(2)}\Delta^{sj}_k \tag{4.41}$$
$$\Delta^{ij}_k\Delta^{ks}_p = \Delta^{ij}_k\Delta^{ks}_p \tag{4.42}$$
$$\partial_s\Delta^{jk}_l = \partial_l\Delta^{jk}_s \tag{4.43}$$

Then (4.37) is proven.
(4.43) implies the existence of a tensor f^{ij} such that

$$\Delta^{ij}_k = \partial_k f^{ij} \tag{4.44}$$

at least locally.
By means of (4.44), the formula (4.43) reads

$$g^{is}_{(2)}\partial_s f^{jk} = g^{js}_{(2)}\partial_s f^{ik}$$

or equivalently

$$\partial^i f^{jk} = \partial^j f^{ik}$$

where $\partial^j := g^{js}_{(2)}\partial_s$.
Then there exists a vector of component f^k such that

$$f^{ik} = \partial^i f^k = g^{is}_{(2)}\partial_s f^k \tag{4.45}$$

Then using (4.44) one gets

$$\Delta^{ij}_k = \partial_k f^{ij} = \partial_k\left(g^{is}_{(2)}\partial_s f^j\right) = g^{is}_{(2)}\partial_k\partial_s f^j \tag{4.46}$$

In a general coordinate system we have

$$\Delta^{ij}_k = g^{is}_{(2)}\nabla^{(2)}_k\nabla^{(2)}_s f^j = \nabla^{(2)}_k\nabla^i_{(2)} f^j$$

Contracting by $g^{qk}_{(2)}$ one obtains (4.34). The compatibility condition for the connection $\nabla_{(\lambda)} = \nabla_{(1)} + \lambda\nabla_{(2)}$ with respect the cometric $g_{(\lambda)} = g_{(1)} + \lambda g_{(2)}$ reads

$$\partial_k g^{ij}_{(\lambda)} = \Gamma^{ij}_{k(\lambda)} + \Gamma^{ji}_{k(\lambda)}$$

which written in the flat coordinates (t^i) for $\nabla_{(2)}$ reads

$$\partial_k(g^{ij}_{(1)} + \lambda g^{ij}_{(2)}) = \Gamma^{ij}_{k(1)} + \Gamma^{ji}_{k(1)}$$

Using (4.39) and (4.46) one gets

$$\partial_k(g^{ij}_{(1)} + \lambda g^{ij}_{(2)}) = \partial^i\partial_k f^j + \partial^j\partial_k f^i$$

Integrating in dt^k one has (taking vanishing integration constant)

$$g^{ij}_{(1)} + \lambda g^{ij}_{(2)} = \partial^i f^j + \partial^j f^i$$

which in a general coordinate system reads

$$g^{ij}_{(1)} + \lambda g^{ij}_{(2)} = \nabla^i_{(2)} f^j + \nabla^j_{(2)} f^i$$

Taking $\lambda = b$ we obtain (4.35).
Plugging (4.46) in (4.40) one gets

$$g^{sk}_{(1)}g^{iq}_{(2)}\partial_k\partial_q f^j = g^{ik}_{(1)}g^{sq}_{(2)}\partial_k\partial_q f^j$$

which written in a general coordinate system yields (4.38). ■

The following lemma provides a sufficient condition to have linear flat pencils of cometrics.

Lemma 4.30 *Let $g = (g^{ij})$ be a non-degenerate flat cometric on a manifold and denote by Γ^{ij}_k to be the contravariant Christoffel symbols corresponding to Levi-Civita connection of g. If in some coordinate system (x^i) each functions $g^{ij}(x)$ and $\Gamma^{ij}_k(x)$ depend at most linearly on x^n (for $i, j, k = 1, ..., n$) then the cometrics*

$$g^{ij}_{(1)}(x) = g^{ij}(x) \tag{4.47}$$

$$g^{ij}_{(2)}(x) = \partial_n g^{ij}(x) \tag{4.48}$$

form a flat pencil if $(g^{ij}_{(2)})$ is a non-degenerate matrix.
The corresponding contravariant Christoffel symbols corresponding to the Levi-Civita connection of the cometrics $(g^{ij}_{(1)})$ and $(g^{ij}_{(2)})$ have the forms

$$\Gamma^{ij}_{k(1)}(x) = \Gamma^{ij}_{k}(x) \tag{4.49}$$
$$\Gamma^{ij}_{k(2)}(x) = \partial_n \Gamma^{ij}_{k}(x) \tag{4.50}$$

Proof: Recall that the functions Γ^{ij}_k fulfill the following system of equations:

$$\begin{cases} \Gamma^{ij}_k + \Gamma^{ji}_k = \partial_k g^{ij} \\ g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s \end{cases} \tag{4.51}$$

Furthermore, Γ^{ij}_k satisfy the equation of vanishing curvature

$$g^{is}(\partial_s \Gamma^{jk}_l - \partial_l \Gamma^{jk}_s) + \Gamma^{ij}_s \Gamma^{sk}_l - \Gamma^{ik}_s \Gamma^{sj}_l = 0 \tag{4.52}$$

One observes that (4.51) and (4.52) form a system of first-order differential equations with constant coefficients. Hence the transformation (the shift in the last entry)

$$g^{ij}(x^1, ..., x^n) \longmapsto g^{ij}_{(\lambda)}(x) := g^{ij}(x^1, ..., x^n + \lambda) \tag{4.53}$$
$$\Gamma^{ij}_k(x^1, ..., x^n) \longmapsto \Gamma^{ij}_{k(\lambda)}(x) := \Gamma^{ij}_k(x^1, ..., x^n + \lambda) \tag{4.54}$$

map solutions to solutions of these equations, for any λ.
Therefore $\Gamma^{ij}_{k(\lambda)}(x^1, ..., x^n)$ are the contravariant Christoffel symbols corresponding to the Levi-Civita connection of the flat metric $g^{ij}_{(\lambda)}(x^1, ..., x^n)$.
Using the linearity in x^n one has

$$g^{ij}(x^1, ..., x^n + \lambda) = g^{ij}(x^1, ..., x^n) + \lambda \partial_n g^{ij}(x^1, ..., x^n) \tag{4.55}$$
$$\Gamma^{ij}_k(x^1, ..., x^n + \lambda) = \Gamma^{ij}_k(x^1, ..., x^n) + \lambda \partial_n \Gamma^{ij}_k(x^1, ..., x^n) \tag{4.56}$$

We identify $g^{ij}_{(1)} = g^{ij}$ and $g^{ij}_{(2)} = \partial_n g^{ij}$.
We have to show that $\Gamma^{ij}_{k(1)}(x) = \Gamma^{ij}_k(x)$ and $\Gamma^{ij}_{k(2)}(x) = \partial_1 \Gamma^{ij}_k(x)$. The former it's true by definition. Previously, we have shown that the function $\Gamma^{ij}_{k(\lambda)}$ fulfill the system

$$\begin{cases} \Gamma^{ij}_{k(\lambda)} + \Gamma^{ji}_{k(\lambda)} = \partial_k g^{ij}_{(\lambda)} \\ g^{is}_{(\lambda)}\Gamma^{jk}_{s(\lambda)} = g^{js}_{s(\lambda)}\Gamma^{ik}_{s(\lambda)} \end{cases}$$

Substituting (4.55) and (4.56) one has

$$\begin{cases} \Gamma^{ij}_k + \Gamma^{ji}_k + \lambda(\partial_1 \Gamma^{ij}_k + \partial_1 \Gamma^{ji}_k) = \partial_k g^{ij} + \lambda \partial_k \partial_1 g^{ij} \\ g^{is}\Gamma^{jk}_s + \lambda g^{is}\partial_1 \Gamma^{jk}_s + \lambda \partial_1 g^{is}\Gamma^{jk}_s + \lambda^2 \partial_1 g^{is}\partial_1 \Gamma^{jk}_s = g^{js}\Gamma^{ik}_s + \lambda g^{js}\partial_1 \Gamma^{ik}_s + \lambda \partial_1 g^{js}\Gamma^{ik}_s + \lambda^2 \partial_1 g^{js}\partial_1 \Gamma^{ik}_s \end{cases}$$

We compare in each order in λ; taking the linear term in λ of the first equation and the quadratic term of the second equation we get

$$\begin{cases} \partial_1 \Gamma^{ij}_k + \partial_1 \Gamma^{ji}_k = \partial_k(\partial_1 g^{ij}) \\ (\partial_1 g^{is})(\partial_1 \Gamma^{jk}_s) = (\partial_1 g^{js})(\partial_1 \Gamma^{ik}_s) \end{cases}$$

Therefore $\partial_1 \Gamma^{ij}_k$ are the contravariant Christoffel symbols corresponding to the Levi-Civita connection of the cometric $\partial_1 g^{js}$. ■

Remark 4.31 *Let e be the constant vector defined by*

$$e = \frac{\partial}{\partial u^n} \tag{4.57}$$

Then the matrix (4.26) can be written as

$$\eta^{ij} = \mathcal{L}_e g^{ij} \tag{4.58}$$

where (g^{ij}) is defined by (4.18).
Hence (η^{ij}) automatically defines a $(2,0)$ tensor field, by the definition of Lie derivative. Moreover, the cometric $\eta = (\eta^{ij})$ is well-defined on the quotient space $M = (V \otimes \mathbb{C})/W$, in view of the lemma (4.20). It will be called the Saito metric. Let's denote by

$$\gamma^{ij}_k(u) := \partial_n \Gamma^{ij}_k(u)$$

the contravariant Christoffel symbols associated with the Levi-Civita connection of the metric (η^{ij}). These are polynomial quasi-homogeneous functions of degrees

$$deg\big(\gamma^{ij}_k(u)\big) = d_i + d_j - d_k - h - 2 \tag{4.59}$$

Corollary 4.32 *The cometrics (g^{ij}) and (η^{ij}), defined by (4.18) and (4.58), form a flat pencil of cometrics.*

Proof: Since the metric (4.18) and its corresponding contravariant Christoffel symbols are at most linear in u^n, in view of the corollary (4.27), one can apply the lemma (4.30), as the Euclidean metric g is flat, and get a flat pencil. ■

Remark 4.33 $\big((g^{ij}), (\eta^{ij})\big)$ *is a linear flat pencil. Therefore, in view of the lemma (3.2), η is flat.*

The remarkable result of Saito is to show that among the sets of basic invariants, there exists a unique distinguished one such that it forms a flat coordinate system for η. More precisely one has the following:

Corollary 4.34 *There exists a set of homogeneous polynomials $\{t^1(p), ..., t^n(p)\}$ of degrees $d_1, ..., d_n$ respectively, such that the matrix*

$$\eta^{ij} = \eta^{ij}(t) := \partial_1 \langle dt^i, dt^j \rangle^*_{(g)} \tag{4.60}$$

is constant in each entry. The coordinates $(t^1, ..., t^n)$, defined on the orbit space M, are called Saito flat coordinates.

Proof: For details see corollary 2.4 in [18]. Otherwise, the theorem (7.31) gives an alternative proof of the statement (taking as the Saito metric the Lie derivative of a non-Euclidean metric). ■

Lemma 4.35 *Let the coordinate t^1 be normalized as in (4.8), then the following identities hold true:*

$$g^{1i} = \frac{d_i}{h} t^i \tag{4.61}$$

$$\Gamma^{1i}_j = \frac{(d_i - 1)}{h} \delta^i_j \tag{4.62}$$

where in both formulas there is no summation over repeated indexes.

Proof: Using the Euler's identity applied to the homogeneous function (t^i) (in view of the previous corollary) one has

$$g^{1i}(t) \overset{(4.18)}{=} \frac{\partial t^1}{\partial p^k} \frac{\partial t^i}{\partial p^k} \overset{(4.8)}{=} \frac{p^k}{h} \frac{\partial t^i}{\partial p^k} = \frac{d_i}{h} t^i$$

Similarly

$$\Gamma^{1i}_j dt^j \overset{(4.20)}{=} \frac{\partial t^1}{\partial p^a} \frac{\partial^2 t^i}{\partial p^a \partial p^b} dp^b \overset{(4.8)}{=} \frac{p^a}{h} d\left(\frac{\partial t^i}{\partial p^a}\right) = \frac{1}{h} d\Big(\underbrace{p^a \frac{\partial t^i}{\partial p^a}}_{=d_i t^i}\Big) - \frac{1}{h} \underbrace{\frac{\partial t^i}{\partial p^a} dp^a}_{=dt^i} = \frac{(d_i - 1)}{h} dt^i$$

then $\Gamma^{1i}_j = \frac{(d_i-1)}{h}\delta^i_j$. ■

Remark 4.36 *Applying lemma 2.14, by a linear change of coordinates, η reduces to a anti-diagonal matrix*

$$\eta^{i+j,n+1} = \delta^{i+j,n+1} \tag{4.63}$$

We present the main result of this section, which defines a structure of Frobenius manifold on the orbit space of an irreducible Coxeter group.

Proposition 4.37 *Let (t^i) be Saito flat coordinates on the orbit space M of an irreducible Coxeter group, with Coxeter number h. Denote by*

$$\eta^{ij} = \eta^{ij}(t) := \partial_1 \langle dt^i, dt^j \rangle^*_{(\eta)} \tag{4.64}$$

the corresponding components of the Saito metric.
Then there exists a quasi-homogeneous polynomial $F(t)$, of degree $2h + 2$, such that

$$\boxed{\langle dt^i, dt^j \rangle^*_{(g)} = \tfrac{d_i + d_j - 2}{h} \eta^{is} \eta^{jk} \partial_s \partial_k F(t)} \tag{4.65}$$

where $\partial_i = \frac{\partial}{\partial t^i}$. Furthermore, the function $F(t)$ determines on M a polynomial Frobenius structure, i.e. a Frobenius algebra on each tangent space, with structure constants

$$c^i_{jk}(t) := \eta^{is} \partial_s \partial_j \partial_k F(t) \tag{4.66}$$

with unity

$$e := \partial_n = \frac{\partial}{\partial u^n} \tag{4.67}$$

and invariant metric η.

Proof: Apply the proposition (4.29) taking $g_{(1)} = (\eta^{ij})$ and $g_{(2)} = (g^{ij})$. Thus in flat coordinates (t^i) for $\nabla_{(2)}$, (4.36) reads

$$\Gamma_k^{ij}(t) = \Delta_k^{ij}(t) = \partial_k \partial^i f^j(t) = \eta^{is}\partial_k \partial_s f^j(t) \tag{4.68}$$

The torsionless condition for the connection $\nabla_{(g)}$ written in (t^i) reads

$$g^{is}\Gamma_s^{jk} = g^{js}\Gamma_s^{ik}$$

Take $i = 1$

$$g^{1s}\Gamma_s^{jk} = g^{is}\Gamma_s^{1k}$$

Using the identities (4.61) and (4.62) one gets

$$d_s t^s \Gamma_s^{jk} = g^{js}(d_k - 1)\delta_s^k$$

Using (4.68) one obtains

$$d_s t^s \eta^{jq}\partial_q \partial_s f^k = (d_k - 1)g^{jk}$$

or equivalently

$$\eta^{jq}\sum_{s=1}^{n} d_s t^s \partial_s(\partial_q f^k) = (d_k - 1)g^{jk}$$

Working in a coordinates system such that (4.63) holds true one has

$$\sum_{s=1}^{n} d_s t^s \partial_s(\partial_{n+1-j} f^k) = (d_k - 1)g^{jk} \tag{4.69}$$

Recall that the flat coordinates (t^i) are homogeneous polynomials. Then, in view of the corollary (4.34), the Christoffel symbols written in the homogeneous coordinates (t^i) are quasi-homogeneous polynomials.
Moreover, in view of (4.68) the functions $\partial_q f^k$ are also quasi-homogeneous polynomials. Comparing the degrees of both sides of (4.69) one has

$$deg(\partial_{n+1-j} f^k) = deg(g^{jk}) \overset{(4.24)}{=} d_j + d_k - 2$$

Now, using the Euler's identity to (4.69) one has

$$(d_j + d_k - 2)\partial_{n+1-j} f^k = (d_k - 1)g^{jk}$$

Therefore applying (4.63) one obtains

$$(d_k + d_j - 2)\eta^{jq}\partial_q f^k = (d_k - 1)g^{jk}$$

here there is no summation over j. Then

$$\frac{\eta^{jq}\partial_q f^k}{d_k - 1} = \frac{g^{jk}}{d_k + d_j - 2} \tag{4.70}$$

The latter formula implies the symmetry condition

$$\frac{\eta^{jq}\partial_q f^k}{d_k - 1} = \frac{\eta^{kq}\partial_q f^j}{d_j - 1} \tag{4.71}$$

then it's natural to define the functions F^i by the formula

$$\frac{F^i}{h} := \frac{f^i}{d_i - 1} \tag{4.72}$$

So (4.71) reads

$$\eta^{jq}\partial_q F^k = \eta^{kq}\partial_q F^j$$

or equivalently

$$\partial^j F^k = \partial^k F^j$$

Hence there exists a function F such that

$$F^i = \partial^i F = \eta^{is}\partial_s F$$

which, substituted in (4.72), yields

$$\frac{\eta^{is}\partial_s F}{h} = \frac{f^i}{d_i - 1} \tag{4.73}$$

Differentiating with respect t^q one has

$$\tfrac{d_i-1}{h}\eta^{is}\partial_s\partial_q F = \partial_q f^i \tag{4.74}$$

which, substituted in (4.70), yields

$$\tfrac{d_k+d_j-2}{h}\eta^{ks}\eta^{jq}\partial_s\partial_q F(t) = g^{jk}(t) \tag{4.75}$$

which coincides with (4.65). Furthermore, since f^i are quasi-homogeneous polynomials and in view of (4.73), one has that F is a quasi-homogeneous polynomial. By direct computation we have $deg(F) = 2h + 2$.
By substituting (4.74) in (4.68) one has

$$\Gamma^{ij}_k = \eta^{is}\partial_s\partial_k f^j = \tfrac{d_j-1}{h}\eta^{is}\eta^{ja}\partial_s\partial_a\partial_k F$$

Raising one index of (4.66) one obtains

$$c^{ij}_k = \eta^{is}\eta^{ja}\partial_s\partial_a\partial_k F$$

Then comparing the last two formulas one has

$$\Gamma^{ij}_k = \frac{d_j - 1}{h}c^{ij}_k \tag{4.76}$$

Hence, in Saito coordinates, one gets

$$\Delta^{ij}_k = \Gamma^{ij}_k = \frac{d_j - 1}{h}c^{ij}_k \tag{4.77}$$

The latter formula plugged in (4.37), i.e.

$$\Delta_s^{ij}\Delta_l^{sk} = \Delta_s^{ik}\Delta_l^{sj}$$

implies the associativity of the product defined by the structure constants (4.66), indeed

$$\cancel{\frac{d_j - 1}{h}}\cancel{\frac{d_k - 1}{h}} c_s^{ij} c_l^{sk} = \cancel{\frac{d_k - 1}{h}}\cancel{\frac{d_j - 1}{h}} c_s^{ik} c_l^{sj}$$

Recall (4.62):

$$\Gamma_j^{1i} = \frac{(d_i - 1)}{h}\delta_j^i$$

Comparing with (4.76) one obtains

$$c_j^{1i} = \delta_j^i$$

Recall that

$$\eta^{ij} = \delta^{i+j,1+n}$$

then

$$c_j^{1i} = \eta^{1s} c_{sj}^i = c_{nj}^i$$

so

$$c_{nj}^i = \delta_j^i$$

which implies that the vector field $e = \partial_n = \frac{\partial}{\partial t^n}$ is the unity field of the Frobenius structure. In view of the formula (4.77), (4.41) reads as

$$\eta^{sk} c_k^{ij} = \eta^{ik} c_k^{sj}$$

which yields the invariance of the metric η with respect the product $\circ$. ■

Theorem 4.38 *Let W be an irreducible Coxeter group. The following data:*

- *The Saito invariant cometric $\eta = (\eta^{ij})$ defined by*

$$\eta = \mathcal{L}_e g \tag{4.78}$$

  where g is the W-invariant cometric (4.18) induced by the Euclidean metric on the vector space V.

- *The unity field*

$$e := \partial_n = \frac{\partial}{\partial u^n} \tag{4.79}$$

- *The Euler vector field*

$$E := \frac{1}{h}\left(d_i u^i \frac{\partial}{\partial u^i}\right) \tag{4.80}$$

- *The product on the tangent bundle $\circ$ defined by the structure constants (4.66)* *define a unique polynomial Frobenius structure on the orbit space $M := (V \otimes \mathbb{C})/W$, up to a rescaling of the Saito flat coordinates, of charge*

$$d = 1 - \frac{2}{h} \tag{4.81}$$

The cometric g coincides with the intersection form of the Frobenius manifold, defined by

$$\langle u, v \rangle^* := i_E(u \,\tilde{\circ}\, v) \tag{4.82}$$

for any u and v differential 1-form on M. Where $\tilde{\circ}$ is the product on the cotangent bundle T^*M induced by $\circ$.

Proof: The existence of such a structure follows from the previous lemma. We have to prove the uniqueness.
Let's consider a polynomial Frobenius manifold with invariant metric (4.64), intersection form (4.82), unity field (4.67) and structure constants

$$c_k^{ij}(t) = \eta^{is}\eta^{jq}\partial_s\partial_q\partial_k F(t)$$

where $F(t)$ quasi-homogeneous polynomial of degree $2h+2$.
In Saito coordinates, the right-hand side of (4.82), taking $u = dt^i$ and $v = dt^j$, reads

$$i_E(dt^i \,\tilde{\circ}\, dt^j) = \frac{d_s}{h} t^s c_s^{ij}(t) = \frac{d_s}{h} t^s \eta^{ik}\eta^{jq}\partial_k\partial_q\partial_s F(t) = \frac{1}{h}\eta^{ik}\eta^{jq} d_s t^s \partial_s\big(\partial_k\partial_q F(t)\big)$$

Observe that $\partial_k\partial_q F$ is a quasi-homogeneous polynomial of degree $2h+2-d_k-d_q$, then applying the Euler's identity one obtains

$$i_E(dt^i \,\tilde{\circ}\, dt^j) = \tfrac{2h+2-d_k-d_q}{h}\eta^{ik}\eta^{jq}\partial_k\partial_q F(t)$$

Working in coordinates such that (4.63) holds true one has

$$i_E(dt^i \,\tilde{\circ}\, dt^j) = \tfrac{2h+2-d_{n+1-i}-d_{n+1-j}}{h}\eta^{ik}\eta^{jq}\partial_k\partial_q F(t)$$

Using the duality condition (4.31) one gets

$$i_E(dt^i \,\tilde{\circ}\, dt^j) = \tfrac{\cancel{2h}+\cancel{2}+d_i-\cancel{h}-\cancel{2}+d_j-\cancel{h}-2}{h}\eta^{ik}\eta^{jq}\partial_k\partial_q F(t)$$

Taking $u = dt^i$ and $v = dt^j$, (4.82) reads

$$\langle dt^i, dt^j \rangle^* = i_E(dt^i \,\tilde{\circ}\, dt^j) = \tfrac{d_i+d_j-2}{h}\eta^{ik}\eta^{jq}\partial_k\partial_q F(t) \overset{(4.65)}{=} \langle dt^i, dt^j \rangle^*_{(g)}$$

Then the cometric defined by the formula (4.65) coincides with the intersection form associated with the considered Frobenius manifold structure. The uniqueness is proven. ∎

Remark 4.39 *The Frobenius manifold structure defined by Theorem 4.38 is semisimple. For details see [18].*

Consider some examples of polynomial Frobenius manifolds.

Example 4.40 *It's worth mentioning the Frobenius potential associated with some rank three Coxeter groups.*

$$F_{A_3} = \tfrac{1}{2}(t^1)^2t^3 + \tfrac{1}{2}t^1(t^2)^2 + \tfrac{1}{4}(t^2)^2(t^3)^2 + \tfrac{1}{60}(t^3)^5$$
$$F_{B_3} = \tfrac{1}{2}(t^1)^2t^3 + \tfrac{1}{2}t^1(t^2)^2 + \tfrac{1}{6}(t^2)^3t^3 + \tfrac{1}{6}(t^2)^2(t^3)^3 + \tfrac{1}{210}(t^3)^7$$
$$F_{H_3} = \tfrac{1}{2}(t^1)^2t^3 + \tfrac{1}{2}t^1(t^2)^2 + \tfrac{1}{6}(t^2)^3(t^3)^2 + \tfrac{1}{20}(t^2)^2(t^3)^5 + \tfrac{1}{3960}(t^3)^{11}$$

Example 4.41 *Let $W = I_2(k)$, here $k > 2$, be the dihedral group of order $2k$. The action of W on the complex z-plane is generated by the transformations*

$$z \mapsto e^{\frac{2\pi i}{k}} z$$
$$z \mapsto \bar{z}$$

i.e. the anticlockwise rotation of $\frac{2\pi}{k}$ radiant and the reflection with respect the $Re(z)$-axis, respectively. The $I_2(k)$-invariant metric $(\cdot, \cdot)$ on $\mathbb{C}$ is given by the line element

$$ds^2 = 4dzd\bar{z}$$

A basis of homogeneous polynomial invariants is given by

$$u^1 = \tfrac{1}{2k} z\bar{z}$$
$$u^2 = z^k + \bar{z}^k$$

One has $deg(u^1) = 2$ and $deg(u^2) = k$. One obtains

$$g^{11}(u) = (du^1, du^1)^* = 4\frac{\partial u^1}{\partial z}\frac{\partial u^1}{\partial \bar{z}} = 2u^1$$
$$g^{12}(u) = (du^1, du^2)^* = 2\left(\frac{\partial u^1}{\partial z}\frac{\partial u^2}{\partial \bar{z}} + \frac{\partial u^2}{\partial z}\frac{\partial u^1}{\partial \bar{z}}\right) = u^2$$
$$g^{22}(u) = (du^2, du^2)^* = 4\frac{\partial u^2}{\partial z}\frac{\partial u^2}{\partial \bar{z}} = (2k)^{k+1}(u^1)^{k-1}$$

Where $(\cdot, \cdot)^*$ is the metric induced on the cotangent space by ds^2.
Hence the Saito metric $\left(\frac{\partial g^{ij}}{\partial u^2}\right)$ is constant in (u^1, u^2). Thus (u^1, u^2) are Saito flat coordinates for $I_2(k)$. Now, using the formula (4.65) one obtains the Frobenius potential

$$F_{I_2(k)} = \tfrac{1}{2}(u^2)^2u^1 + \tfrac{(2k)^{k+1}}{2(k^2-1)}(u^1)^{k+1}$$

In particular

- *for $k = 3$, F gives the polynomial Frobenius structure on $\mathbb{C}^2/A_2$*
- *for $k = 4$, F gives the polynomial Frobenius structure on $\mathbb{C}^2/B_2$*
- *for $k = 6$, F gives the polynomial Frobenius structure on $\mathbb{C}^2/G_2$*

Example 4.42 *Following [17], we mention a remarkable example of Frobenius structure. Moreover, this structure coincides with the polynomials Frobenius structure associated with A_n. Let M be the space of all polynomials of the form*

$$M = \{\lambda(p) = p^{n+1} + a_n p^{n-1} + ... + a_2 p + a_1 |\, a_1, ..., a_n \in \mathbb{C}\} \tag{4.83}$$

with a nonstandard affine structure.
We identify the tangent plane to M the space of all polynomials of degree less than n. Moreover, the Frobenius algebra A_λ on $T_\lambda M$ coincides with the algebra of truncated polynomials

$$A_\lambda = \mathbb{C}[p]/\big(\lambda'(p)\big) \tag{4.84}$$

The invariant metric is given by the following formula:

$$\langle f, g\rangle_\lambda = \underset{p=\infty}{res} \frac{f(p)g(p)}{\lambda'(p)} \tag{4.85}$$

Furthermore, the unity and the Euler vector field are defined by

$$e = \frac{\partial}{\partial a_1} \tag{4.86}$$

$$E = \sum_{i=1}^{n} (n - i + 1) a_i \frac{\partial}{\partial a_i} \tag{4.87}$$

Let's consider the Frobenius manifold structure defined on the orbit space of the Coxeter group A_n (defined by theorem 4.38).
Recall that A_n acts on $\mathbb{R}^{n+1} = \{\xi^0, ..., \xi^n\}$ by permutations (restricting on the hyperplane $\xi^0 + ... + \xi^n = 0$).
The elementary invariant polynomial can be taken as a homogeneous basis. Thus, by taking as $\{a_1, ..., a_n\}$ the elementary symmetric polynomials, the complexified orbit space $\mathbb{C}/A_n$ can be identified with the space of polynomials (4.83) of an auxiliary variable p. Moreover, the polynomial Frobenius structure associated with A_n coincides with the structure exposed in Example 4.42.
The formula for the metric (4.85) may be written in a peculiar way. In particular, the following statement holds true:

Lemma 4.43 *1. The metric (4.85) and the 3-rank tensor $c(\partial, \partial', \partial'') := \langle \partial \circ_\lambda \partial', \partial''\rangle_\lambda$ (where $\circ_\lambda$ is the product corresponding to A_λ) have the form*

$$\langle X, Y\rangle_\lambda = -\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(p)dp\big) Y\big(\lambda(p)dp\big)}{d\lambda(p)} \tag{4.88}$$

$$c(X, Y, Z) = -\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(p)dp\big) Y\big(\lambda(p)dp\big) Z\big(\lambda(p)dp\big)}{d\lambda(p)} \tag{4.89}$$

here X, Y and Z are arbitrary vector fields.
2. Let $q^1, ..., q^n$ be the critical points of $\lambda(p)$, i.e. $\lambda'(q^i) = 0$ for $i = 1, ..., n$. Let $u^i = \lambda(q^i)$ be the corresponding critical values. Hence the variables $(u^1, ..., u^n)$ are local

coordinates near the point λ, where the polynomial $\lambda(p)$ has no multiple roots. These are canonical coordinates for the multiplication $\circ_\lambda$. In these coordinates the metric (4.85) reduces to the diagonal form

$$\langle \cdot, \cdot \rangle_\lambda = \sum_{i=1}^{n} \eta_{ii}(u)(du^i)^2$$

where $\eta_{ii}(u) = \frac{1}{\lambda''(q^i)}$.

We expose the notion of Hurwitz-Frobenius manifold (see [17] and [41] for additional details).

4.4 Landau-Ginzburg superpotentials

First, recall the notion of Hurwitz space.

Definition 4.44 *A Hurwitz space $M_{g;n_1,....,n_m}$ is a moduli space (C, λ), where C is a genus g Riemann surface with m distinct ordered marked point $\infty_1, ..., \infty_m$ and λ is a meromorphic function on C with poles at ∞_i of order $n_i + 1$.*

Motivated by lemma 4.43, we will introduce a class of semisimple Frobenius manifold structures defined on a Hurwitz space $M_{g;n_1,....,n_m}$.
We will construct a family of functions $\lambda(z; u)$, where $u = (u^1, ..., u^n)$, of the complex variable z defined in a domain D of a Riemann surface R realized as a branched covering of the complex plane with a finite number of sheets. The Riemann surface may depend on u. We fix the projection of the domain D on the complex plane.
The functions $\lambda(z; u)$ depend on complex pairwise distinct parameters $u^1, ..., u^n$ belonging to a sufficiently small domain $\Omega \subset \mathbb{C}^n$. We require $\lambda(z; u)$, as a function of z, to fulfills the following properties:

1. $\lambda(z; u)$ has critical values $u^1, ..., u^n$, moreover, the corresponding critical points must be non-degenerate.

2. For any two points $z_i^{(1,2)} \in D$ with the same critical value, we require that

$$\lambda''(z_i^{(1)}; u) = \lambda''(z_i^{(2)}; u)$$

Definition 4.45 *The function $\lambda(z; u)$ on $D \times \Omega$ satisfies the properties 1. and 2. is called LG (Landau-Ginzburg) superpotential of some domain M_Ω of the Frobenius manifold structure $(M, \eta, \nabla, \circ, e, E)$, with intersection form g, if for any critical points $z^1, ..., z^n \in$*

D of $\lambda(z;u)$, with critical values $u^1, ..., u^n$, the following expression hold true:

$$\eta(X,Y) = -\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(z)dz\big)Y\big(\lambda(z)dz\big)}{d\lambda(z)} \tag{4.90}$$

$$g(X,Y) = -\sum \underset{d\lambda=0}{res} \frac{X\big(log\lambda(z)dz\big)Y\big(log\lambda(z)dz\big)}{d\ log\lambda(z)} \tag{4.91}$$

$$c(X,Y,Z) = -\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(z)dz\big)Y\big(\lambda(z)dz\big)Z\big(\lambda(z)dz\big)}{d\lambda(z)} \tag{4.92}$$

$$c^*(X,Y,Z) = -\sum \underset{d\lambda=0}{res} \frac{X\big(log\lambda(z)dz\big)Y\big(log\lambda(z)dz\big)Z\big(log\lambda(z)dz\big)}{d\ log\lambda(z)} \tag{4.93}$$

for X,Y,Z arbitrary vector fields. Here $c(X,Y,Z) := \eta(X \circ Y, Z)$ and $c^*(X,Y,Z) := g(X * Y, Z)$, where $X * Y = E^{-1} \circ X \circ Y$ is the dual product of $\circ$.
Moreover, $X\big(\lambda(z)dz\big)$, $Y\big(\lambda(z)dz\big)$,.... are calculated keeping z constant. We omit u in the argument of λ.

5 Bi-flat F-manifolds and complex reflection groups

The main references of this section are [1], [3], [4], and [52].
Following [3] and [4] we will see that the orbit space of certain complex reflection groups may be endowed with a structure of bi-flat F-manifold. In some cases this structure appears in family depending on parameters; in [3] it has been proposed a conjecture relating the number of the orbits corresponding to the action of the group on the collection of reflecting hyperplanes.
First of all, we recall some definitions and facts concerning the theory of complex reflection groups. In particular, we will consider groups acting on a complex n-dimensional vector space V via their matrix representation.

5.1 Complex reflection groups

Definition 5.1 *A pseudo-reflection is a unitary transformation on V that leaves invariant a (hyper-)plane.*

Remark 5.2 *A pseudo-reflection is characterized by the property that all the eigenvalues of the corresponding matrix representation are equal to 1, except for one that coincides with the k-root of the unity, where k is the order (or period) of the transformation.*

Definition 5.3 *A finite complex reflection group is a finite subgroup of unitary transformation generated by pseudo-reflections.*

Remark 5.4 *Irreducible and finite complex reflection groups were classified by Shephard and Todd in [48]. The classification consists of an infinite family $G(n, p, m)$ parameterized by 3 positive integers and 34 exceptional cases.*

Shephard and Todd proved a Chavalley-type theorem for complex reflection groups. Let $(p^1, ..., p^n)$ be a system of coordinates for V.

Proposition 5.5 *Let G be a complex reflection group, then the subring of invariant polynomials $\mathbb{C}[V]^G$ is generated by n algebraically independent polynomials $\{u^1, ..., u^n\}$ of degrees $d_1, ..., d_n$ respectively. They are called basic invariants. Moreover, the choice of the basic invariants is in general not unique, while the corresponding degrees are positive integers uniquely defined by the group.*

Remark 5.6 *Similarly to the case of Coxeter groups, $(u^1, ..., u^n)$ may be used as local coordinates of the orbit space $M := V/G$.*
Although $(p^1, ..., p^n)$ and $(u^1, ..., u^n)$ are coordinates of different spaces, since the quotient map $\pi : V \to V/G$ is a local diffeomorphism, we will treat (p^i) and (u^i) as different local coordinate systems.

Definition 5.7 *Well-generated irreducible complex reflection groups are irreducible complex reflection groups, whose minimal set of generators consists of n pseudo-reflections.*

Remark 5.8 *Recall that any Coxeter group is generated by n reflections corresponding to simple roots. Then any finite group generated by (real) reflection, i.e. a Coxeter group, is automatically well-generated.*

5.2 Frobenius manifolds and Shephard groups

Now, we review the Dubrovin's construction (see [16]) which equips the orbit space for the action of a Shephard group with a Frobenius manifold structure.

Definition 5.9 *A Shephard group is a well-generated complex reflection group which consists in the symmetry transformations of a regular complex polytopes. In particular, a Shephard group includes the symmetry group of a regular real polyhedra.*

Remark 5.10 *Any Shephard group admits a Coxeter-like representation. Let G be a Shephard group, then there exists two sets positive integers $p_1, ..., p_n$ and $q_1, ..., q_{n-1}$ such that the generating system $\{s_1, ..., s_n\}$ of G satisfy the following condition:*

- $s_i^{p_i} = id$
  for $i = 1, ..., n$
- $s_i s_j = s_j s_i$
  for $|i - j| > 2$
- $\underbrace{s_i s_{i+1} s_i ....}_{q_i\ terms} = \underbrace{s_{i+1} s_i s_{i+1} ....}_{q_i\ terms}$

In particular, s_i are pseudo-reflections of order p_i.

Definition 5.11 *The Coxeter group obtained taking $p_i = 2$,for any i, in the above representation, is called the Coxeter group associated with G (or underlying G).*

Remark 5.12 *Among the families of complex reflection groups $G(m, p, n)$, the family $G(m, 1, n)$ is constituted by Shephard groups. Moreover, there are also 18 exceptional Shephard group, whose 2 are real.*

Remark 5.13 *Given any Shephard group, being a complex reflection group, there exists a set of polynomial basic invariants $\{u^1, ..., u^n\}$ (due to proposition 5.5).*

The crucial point is the following:

Proposition 5.14 *Let $(u^1, ..., u^n)$ be a set of basic invariant of degrees $d_1, ..., d_n$ respectively ordered so that u^1 is the lowest degree polynomial and u^n is highest degree polynomial. The inverse (h^{ij}) of the Hessian matrix $(h_{ij}) = Hess(u^1)$ defines a flat metric which depends linearly on u^n.*

Proof: The proof is a consequence of the results of Orlik and Solomon (see [43] for details). ■

Remark 5.15 *Using this proposition and applying the Dubrovin's argument for the case of Coxeter group, exposed in the previous section, the flat pencil (g, η) defined by*

$$g^{ij} := h^{ij} \tag{5.1}$$

$$\eta^{ij} := \partial_n h^{ij} = \frac{\partial}{\partial u^n} h^{ij} \tag{5.2}$$

one obtains a Frobenius manifold structure on the orbit space of any irreducible Shephard group. Exploiting the well-known formula

$$(d_i + d_j - 2)\eta^{il}\eta^{jm}\partial_l\partial_m F = g^{ij} \tag{5.3}$$

where $\partial_i = \frac{\partial}{\partial t^i}$, (t^i) *is a flat coordinate system for* η, $g^{ij} = g^{ij}(t)$ *and* $\eta^{ij} = \eta^{ij}(t)$, *one reconstructs the Frobenius potential* F.

Remark 5.16 *Since the Frobenius manifold structure on the orbit space of a Shephard group is a polynomial Frobenius manifold with strictly positive invariant degree, it must be isomorphic to a polynomial Frobenius manifold structure associated with a Coxeter group (in view of Hertling's theorem (see [31])).*
In particular, for any Shephard G, *this Coxeter group is exactly the Coxeter group associated to* G. *Then the potential defined by (5.3) coincides, up to a rescaling of the flat coordinates* (t^i), *with the potential corresponding to the underlying Coxeter group. For instance, the prepotential for the Shephard group* $G(m,1,2)$ *and* $G(m,1,3)$ *don't depend on* m *and coincides with the prepotential for the associated Coxeter group* B_2 *and* B_3, *respectively.*

5.3 $\vee$-systems

$\vee-$system were introduced by Veselov in [52] to construct solutions of WDVV associativity equation given a set of covectors. In particular, the conditions defining a $\vee$-system guarantees that a function, constructed using a special set of covectors, satisfies the WDVV associativity equation.
Let V be a finite-dimensional real vector space (the construction can be generalized to the complex case), with linear coordinates $(p^1, ..., p^n)$, and denote by $\mathcal{V}$ a finite set of non-collinear covectors $\{\alpha\}$ defined by the linear functionals $\alpha(\cdot) \in V^*$.
Let

$$g := \sum_{\alpha\in\mathcal{V}} \alpha \otimes \alpha \tag{5.4}$$

be a non-degenerate metric tensor on V.
We denote by $\check{\alpha}$ the vector uniquely defined by

$$\langle \check{\alpha}, \cdot \rangle = \alpha(\cdot) \tag{5.5}$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form defined by g.

Remark 5.17 *Let* $\alpha = \alpha_i e^i$ *be a linear functional on* V, *where* $\{e^j\}$ *is a dual basis corresponding to* $\{e_j\}$, *i.e.* $e^s(e_k) = \delta^s_k$. *Then*

$$\langle \check{\alpha}, e_i \rangle = \alpha(e_i)$$

reads as

$$g_{ji}\check{\alpha}^j = \alpha_i$$

or equivalently

$$\check{\alpha}^j = g^{ji}\alpha_i \tag{5.6}$$

Recall the definition of $\vee$-system. Let's consider the following function on V:

$$F_{\mathcal{V}}(p) = \frac{1}{2} \sum_{\alpha \in \mathcal{V}} \alpha^2(p)\, log(\alpha(p)) \tag{5.7}$$

here $\alpha(p) := \alpha_i p^i$. Imposing that (5.7) is a solution of the WDVV associativity equation one gets some prescriptions on the collection of covectors $\mathcal{V}$:

Definition 5.18 *We say that $\mathcal{V}$ is a $\vee$-system if for any two-dimensional plane $\Pi \subset V^*$ we have*

$$\sum_{\beta \in \Pi \cap \mathcal{V}} \beta(\check{\alpha})\check{\beta} = \mu \check{\alpha} \tag{5.8}$$

for any $\alpha \in \Pi \cap \mathcal{V}$, where μ is a real constant which depends on Π and α.

Remark 5.19 *Observe that the condition (5.8) is independent by the normalization of the covectors $\alpha \in \mathcal{V}$. Then the collection $\widetilde{\mathcal{V}} = \{\sigma_\alpha \alpha\}$, where $\sigma_\alpha \in \mathbb{R}$ for any α, also defines a $\vee$-system.*

Given a set $\mathcal{V}$ we construct a collection of (hyper-)planes $\mathcal{H}$ associated to the covectors contained in $\mathcal{V}$.

Definition 5.20 *By definition, a (hyper-)planes H belongs to $\mathcal{H}$ if and only if $Ker(\alpha) = H$ for some covectors $\alpha \in \mathcal{V}$. Furthermore, we denote by α_H the covectors corresponding to H and by*

$$\pi_H : V \to H^{\perp} \tag{5.9}$$

the linear application having kernel H and image $H^{\perp}$.

In view of the above definition, the linear map π_H can be written as

$$\pi_H = \check{\alpha}_H \otimes \alpha_H$$

or in component

$$(\pi_H)^i_j = g^{is}(\alpha_H)_s(\alpha_H)_j \tag{5.10}$$

Proposition 5.21 *The definition of $\vee$-system is equivalent to the requirement that the one-parameter family of connections*

$$\nabla^0 - \lambda \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \pi_H \tag{5.11}$$

is flat for any real λ. Here ∇^0 is the trivial flat linear connection with flat coordinates $(p^1, ..., p^n)$.

Proof: See [2] and [29] for details. ■

Remark 5.22 *Given a set of covectors $\mathcal{V}$ defines a $\vee$-system with $\mathcal{H}$ it corresponding collections of reflecting (hyper-)planes, the corresponding solution of the WDVV associativity equation is given by the formula*

$$F_{\mathcal{V}}(p) = \frac{1}{2} \sum_{H \in \mathcal{H}} \alpha_H^2(p)\, log(\alpha_H(p)) \tag{5.12}$$

here $\alpha_H(p) := (\alpha_H)_i p^i$.
One remarkable example of $\vee-$system is given by a Coxeter system of (hyper-)planes (see [52]). Furthermore, in [16] it was proven that the Veselov's solution (5.12) of WDVV associativity equation, constructed from any Coxeter group, coincides with the Frobenius potential associated with the almost-dual structure of the Frobenius manifold defined by the theorem (4.38). In this case, the dual product has the form

$$* = \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \pi_H \tag{5.13}$$

Remark 5.23 *Let's consider the following deformation of the function (5.12):*

$$\widetilde{F}_{\mathcal{V}}(p) = \frac{1}{2} \sum_{H \in \mathcal{H}} \sigma_H (\alpha_H(p))^2\, log(\alpha_H(p))$$

here $\sigma_H \in \mathbb{R}$. One observes that

$$\begin{aligned} \widetilde{F}_{\mathcal{V}}(p) &= \frac{1}{2} \sum_{H \in \mathcal{H}} (\sqrt{\sigma_H} \alpha_H(p))^2\, log\left(\frac{\sqrt{\sigma_H}}{\sqrt{\sigma_H}} \alpha_H(p) \right) \\ &= \underbrace{\frac{1}{2} \sum_{H \in \mathcal{H}} (\widetilde{\alpha}_H(p))^2\, log(\widetilde{\alpha}_H)}_{:= F_{\widetilde{\mathcal{V}}}(p)} - \underbrace{\frac{1}{2} log(\sqrt{\sigma_H}) \sum_{H \in \mathcal{H}} (\widetilde{\alpha}_H(p))^2}_{:= G(p)} \end{aligned}$$

where $\widetilde{\mathcal{V}} = \{\sqrt{\sigma}_H \alpha_H\}$ is a rescaled set of covectors and $G(p)$ is a homogeneous polynomial of degree 2. Recall that (in view of the remark (5.19)) the collection $\widetilde{\mathcal{V}}$ is also a $\vee$-system, then $F_{\widetilde{\mathcal{V}}}(p)$ is a solution of the $WDVV$ associativity equation. Moreover, since the solutions of the WDVV associativity equation are defined up to a quadratic polynomial, $\widetilde{F}_{\mathcal{V}}(p)$ is also a solution of WDVV associativity equation.

5.4 Flat and bi-flat F-manifolds

F-manifold with a compatible flat structure, or briefly flat F-manifold, have been introduced by Manin as a generalization of the notion of Frobenius manifold. Such structures are a particular example of a more general class of structure, introduced by Hertling and Manin, called F-manifold.
Following [42], we recall the general definition.

Definition 5.24 *A flat F-manifold $(M, \circ, \nabla, e)$ is a complex manifold M equipped with the following data:*

I. *A commutative and associative product* $\circ$ *on the sheaf of holomorphic vector fields of* M

$$\circ : \chi_M \times \chi_M \to \chi_M \tag{5.14}$$

with unity flat vector field e*, i.e.*

$$\nabla e = 0 \tag{5.15}$$

II. *A flat and torsionless linear connection compatible with the product* $\circ$*, i.e.*

$$\nabla_i c^k_{js} = \nabla_j c^k_{is} \tag{5.16}$$

where c^k_{ij} *are the structure constants corresponding to the product* $\circ$*.*

Remark 5.25 *The notion of flat* F*-manifold makes sense in the smooth category as well. In this case,* M *is a smooth manifold,* TM *its tangent bundle, and* χ_M *the sheaf of smooth vector fields on* M*.*

The definition of flat F-manifold can be rephrased in a more compact manner.

Definition 5.26 *A flat* F*-manifold* $(M, \circ, \nabla, e)$*, where* M *is a complex manifold,* $\circ : \chi_M \times \chi_M \to \chi_M$ *is a product on the sheaf of holomorphic vector bundle* χ_M*,* ∇ *is a linear connection on the holomorphic tangent bundle* TM*, and* e *is a holomorphic vector field, satisfying the following axioms:*

1. *For any* $\lambda \in \mathbb{C}$*,* $\nabla^{(\lambda)} := \nabla + \lambda\circ$ *is a flat and torsionless connection.*
2. e *is the unity of the product.*
3. e *is a flat vector field.*

A manifold equipped with a product $\circ$*, a connection* ∇*, and a vector field* e *satisfying 1. and 2. are called an almost-flat* F*-manifold.*

Let $(u^1, ..., u^n)$ be a coordinate system for M. Denote by c^k_{ij} the structure constants of the product $\circ$. Denote by Γ^k_{ij} and $R = (R^s_{ijk})$ the Christoffel symbols and the Riemann tensor, respectively, of the connection ∇. Then the condition of vanishing torsion for the connection $\nabla^{(\lambda)}$ reads (let $\Gamma^{k(\lambda)}_{ij} = \Gamma^k_{ij} + \lambda c^k_{ij}$ be the Christoffel symbols of the connection $\nabla^{(\lambda)}$)

$$T^{k(\lambda)}_{ij} = \Gamma^{k(\lambda)}_{ij} - \Gamma^{k(\lambda)}_{ji} = \Gamma^k_{ij} - \Gamma^k_{ji} + \lambda(c^k_{ij} - c^k_{ji}) = 0$$

Since the latter formula holds true for any λ, one gets the following conditions:

- The connection ∇ is torsionless.
- The product $\circ$ is commutative.

Analogously, the zero-curvature condition reads

$$\begin{aligned} R^{s(\lambda)}_{ijk} &= \partial_j\Gamma^{s(\lambda)}_{ik} - \partial_k\Gamma^{s(\lambda)}_{ij} + \Gamma^{q(\lambda)}_{ik}\Gamma^{s(\lambda)}_{qj} - \Gamma^{q(\lambda)}_{ij}\Gamma^{s(\lambda)}_{qk} \\ &= \underbrace{\partial_j\Gamma^s_{ik} - \partial_k\Gamma^s_{ij} + \Gamma^q_{ik}\Gamma^s_{qj} - \Gamma^q_{ij}\Gamma^s_{qk}}_{=R^s_{ijk}} + \lambda(\nabla_j c^s_{ik} - \nabla_k c^s_{ij}) + \lambda^2(c^q_{ik}c^s_{qj} - c^q_{ij}c^s_{qk}) \end{aligned}$$

which yields the following conditions:

- The connection ∇ is flat.
- ∇ is compatible with $\circ$.
- The product $\circ$ is associative.

Remark 5.27 *By definition, it turns out that any flat F-manifold defines a one-parameter family of flat and torsionless connections.*

Remark 5.28 *In analogy with the case of Frobenius manifold, in flat coordinated (t^i) for ∇, as a consequence of the axioms of flat $F-$manifold, one has that the structure constants can be expressed in terms of the second derivatives of a vector potential, i.e.*

$$c_{ij}^k = \partial_i \partial_j A^k \tag{5.17}$$

where $\partial_i = \frac{\partial}{\partial t^i}$. Furthermore, taking the unity field $e = \partial_1$, (A^i) satisfies the following equations:

$$\partial_j \partial_l A^i \partial_k \partial_m A^l = \partial_k \partial_l A^i \partial_j \partial_m A^l \tag{5.18}$$

$$\partial_1 \partial_i A^j = \delta_i^j \tag{5.19}$$

(5.18) and (5.19) are called oriented associativity equations.

The notion of flat F-manifold shares several properties with the notion of Frobenius manifold. For instance, Dubrovin's duality and Dubrovin's principal hierarchy are well-defined for flat F-manifold.
By comparing it with the definition of Frobenius manifold, the missing data are the invariant metric and the Euler field.
Replacing the flat matric η, in the definition of Frobenius manifold, with a flat connection ∇ one obtains the following:

Definition 5.29 *A Frobenius manifold without a metric $(M, \nabla, \circ, e, E)$ is a smooth (or complex) manifold equipped with the following data:*

I. *A flat and torsionless linear connection ∇ compatible with the product $\circ$.*

II. *A commutative and associative product $\circ$ on the tangent bundle of M*

III. *A flat unity vector field e.*

IV. *A linear Euler field E, i.e.*

$$\nabla \nabla E = 0 \tag{5.20}$$

$$[e, E] = e \tag{5.21}$$

$$\mathcal{L}_E c_{ij}^k = c_{ij}^k \tag{5.22}$$

Frobenius manifolds without a metric are also called Saito structures.

Adding a suitable (pseudo-)Riemannian metric we can reconstruct the definition of Frobenius manifold.

Definition 5.30 *A (flat) metric η is invariant for a Frobenius manifold $(M, \circ, \nabla, e, E)$ if the following conditions are fulfilled:*

$$\nabla\eta = 0 \tag{5.23}$$

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z) \tag{5.24}$$

for any vector field X, Y, Z.

Definition 5.31 *A Frobenius manifold $(M, \eta, \nabla, \circ, e, E)$ is a Frobenius manifold without a metric endowed with an invariant metric η, where the linear Euler field acts as a conformal Killing vector field for η, i.e.*

$$\mathcal{L}_E \eta_{ij} = (2-d)\eta_{ij} \tag{5.25}$$

here d is the charge of the Frobenius manifold.

5.5 Frobenius manifolds and almost-duality

Recall the notion of almost-duality, introduced by Dubrovin in [16].
Consider a Frobenius manifold structure on M. It turns out that M may be equipped with an almost-dual Frobenius manifold structure, since the unity field isn't flat in general. In particular, the following theorem holds true:

Theorem 5.32 *Given a Frobenius manifold $(M, \eta, \nabla, \circ, e, E)$, let U be an open subset of M such that the linear operator on the tangent bundle $E\circ$ is invertible. Then the following data:*

- *the intersection form $g = (g^{ij})$ defined by*

  $$g^{ij} := E^s c^i_{sk} \eta^{kj}$$

  where (η^{ij}) are the components of the inverse matrix of η,

- *the Levi-Civita connection $\tilde{\nabla}$ corresponding to g,*

- *a dual product $*$ defined by*

  $$X * Y := E^{-1} \circ X \circ Y = (E\circ)^{-1} X \circ Y$$

  for any vector field X and Y, and

- *the vector field E*

define an almost-dual Frobenius manifold structure on M, with invariant metric g^{-1}, flat connection $\tilde{\nabla}$, product $*$ and unity field E.

Remark 5.33 *The structure defined above is called almost-dual since $\tilde{\nabla}E \neq 0$ in general (since for a Frobenius manifold we require the unity to be covariantly constant with respect to the flat connection).*
However, replacing $\tilde{\nabla}$ with $\nabla^* := \tilde{\nabla} + \bar{\lambda}*$, for a suitable $\bar{\lambda} \in \mathbb{R}$, one obtains a flat connection such that $\nabla^* E = 0$. Moreover, by definition of flat F-manifold, gauging the connection $\tilde{\nabla}$ by a multiple of $*$ one gets a flat and torsionless connection.

Lemma 5.34 *$\nabla^* E = 0$, where ∇^* is the gauged connection $\nabla^* = \tilde{\nabla} + \frac{d-1}{2}*$.*

Proof: Recall that the contravariant Christoffel symbols corresponding to Levi-Civita connection of the intersection form, written in a flat coordinate system (t^i) for η, are given by the formula (3.14), i.e.

$$\tilde{\Gamma}^{ij}_k = c^{is}_k R^j_s = \eta^{iq} c^s_{qk} R^j_s$$

where $R^i_j = \frac{d-1}{2}\delta^i_j + \partial_j E^i$. Therefore

$$\tilde{\nabla}_i E^j = \partial_i E^j - g_{iq}\tilde{\Gamma}^{qj}_s E^s = \partial_i E^j - g_{iq}\eta^{qb} c^l_{bs} R^j_l E^s$$

In view of the formulas (2.61) and (2.62) one observes that

$$g_{iq}\eta^{qb} c^l_{bs} E^s = g_{iq}\eta^{qb} g^{ls}\eta_{sb} = \delta^l_i$$

Hence

$$\tilde{\nabla}_i E^j = \partial_i E^j - \underbrace{\delta^l_i R^j_l}_{=R^j_i} = \cancel{\partial_i E^j} - \frac{d-1}{2}\delta^j_i - \cancel{\partial_i E^j} = \frac{1-d}{2}\delta^j_i$$

Taking $\bar{\lambda} = \frac{d-1}{2}$, being E the unity of $*$, one has

$$\nabla^*_i E^j = \left((\tilde{\nabla} + \bar{\lambda}*)E\right)^j_i = \underbrace{\tilde{\nabla}_i E^j}_{=\frac{1-d}{2}\delta^j_i} + \frac{d-1}{2}\underbrace{c^{j*}_{is}E^s}_{=\delta^j_i} = 0$$

■

Remark 5.35 *In view of the previous lemma, one has that for any Frobenius manifold $(M, \eta, \nabla, \circ, e, E)$, the open set U where the linear operator is invertible is equipped with two flat F-manifold:*

1. *the flat structure $(\nabla, \circ, e)$,*
2. *the flat structure $(\nabla^*, *, E)$.*

The two structures are intertwined in a special way.
For an arbitrary vector field X, it turns out that

$$(d_\nabla - d_{\nabla^*})(X\circ) = 0 \qquad (5.26)$$

where $d_\nabla : \Omega^k(M, TM) \to \Omega^{k+1}(M, TM)$ (here $\Omega^k(M, TM)$ denotes the $\mathcal{O}(M)$-module of TM-valued differential k-forms) is the exterior covariant derivative. In this case, ∇ and ∇^* are called almost hydrodynamically equivalent (see [6]).

5.6 Bi-flat F-manifolds

In order to generalize Dubrovin's duality for Frobenius manifold without a metric and motivated by the theory of integrable system of hydrodynamic type (not Hamiltonian and bi-Hamiltonian necessarily), Arsie and Lorenzoni proposed the notion of bi-flat F-manifold (see [1] and [3]).
In the semisimple case, Dubrovin's almost duality can be extended to the Frobenius manifold without a metric prescribing the following data:

- A dual product $*$ defined by

$$X * Y := E^{-1} \circ X \circ Y \tag{5.27}$$

  for any vector field X and Y.

- A dual connection ∇^* satisfying the following properties:
  1. $\nabla^* E = 0$
  2. ∇^* is compatible with $*$
  3. ∇ and ∇^* are almost hydrodynamically equivalent, i.e. the following formulas hold true:

$$(d_\nabla - d_{\nabla^*})(X\circ) = 0 \tag{5.28}$$
$$(d_\nabla - d_{\nabla^*})(X*) = 0 \tag{5.29}$$

     for any vector field X, here d_∇ and d_{∇^*} denotes the exterior covariant derivative corresponding to the connection ∇ and ∇^* respectively.

Remark 5.36 *In the case of Frobenius manifold, the dual connection, in general, doesn't coincide with the Levi-Civita connection of the intersection form. In view of the remark 5.33, the difference between these two connections is proportional to the dual product, i.e.*

$$\nabla^* - \tilde{\nabla} = \nu * \tag{5.30}$$

where ν is a real number and $\tilde{\nabla}$ is the Levi-Civita connection of the intersection form.

Definition 5.37 *From here on, ∇ and ∇^* will be called natural connection and dual connection respectively.*

In the semisimple case, the natural and the dual connection are related in the following way:

Theorem 5.38 *In the semisimple case, the dual connection defined above is uniquely defined in terms of the natural connection. Furthermore, the flatness of the dual connection is equivalent to the linearity of the Euler field.*

Proof: See [1] for details. ■

Lemma 5.39 *The formulas (5.29) and (5.28) in components read*

$$\Gamma^k_{lj}(X*)^l_i - \Gamma^k_{li}(X*)^l_j = \Gamma^{k*}_{lj}(X*)^l_i - \Gamma^{k*}_{li}(X*)^l_j \tag{5.31}$$
$$\Gamma^k_{lj}(X\circ)^l_i - \Gamma^k_{li}(X\circ)^l_j = \Gamma^{k*}_{lj}(X\circ)^l_i - \Gamma^{k*}_{li}(X\circ)^l_j \tag{5.32}$$

respectively, where Γ^k_{ij} and Γ^{k*}_{ij} are the Christoffel symbols corresponding to the connection ∇ and ∇^* respectively.

Proof: Recall that, the exterior derivative of a $(1,p)$ tensor field coincides with its covariant derivative antisymmetrized with respect the lower indexes, explicitly

$$(d_\nabla T)^j_{i_1,\dots,i_{p+1}} = \nabla_{[i_1} T^j_{i_2,\dots,i_{p+1}]}$$

Then $(d_\nabla - d_{\nabla^*})(X*)$ explicitly reads

$$A\big(\cancel{\partial_i(X*)^s_j} + \Gamma^s_{ik}(X*)^k_j - \Gamma^k_{ij}(X*)^s_k\big) = A\big(\cancel{\partial_i(X*)^s_j} + \Gamma^{s*}_{ik}(X*)^k_j - \Gamma^{k*}_{ij}(X*)^s_k\big)$$

where A denotes the operator of antisymmetrization with respect the lower indexes. Then one has

$$\Gamma^s_{ik}(X*)^k_j - \Gamma^s_{jk}(X*)^k_i - \Gamma^k_{ij}(X*)^s_k + \Gamma^k_{ji}(X*)^s_k = \Gamma^{s*}_{ik}(X*)^k_j - \Gamma^{s*}_{jk}(X*)^k_i - \Gamma^{k*}_{ij}(X*)^s_k + \Gamma^{k*}_{ji}(X*)^s_k$$

being ∇ and ∇^* torsionless, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $\Gamma^{k*}_{ij} = \Gamma^{k*}_{ji}$, one obtains

$$\Gamma^s_{ik}(X*)^k_j - \Gamma^s_{jk}(X*)^k_i = \Gamma^{s*}_{ik}(X*)^l_i - \Gamma^{s*}_{jk}(X*)^k_i$$

which coincides with (5.31). Similarly, one can prove (5.32). ■

Following [3], we give the definition of bi-flat F-manifold.

Definition 5.40 *A bi-flat $F-$manifold is a Frobenius manifold without a metric equipped with an almost-dual structure defined as above.*

More explicitly one has

Definition 5.41 *A bi-flat $F-$manifold $(M, \nabla, \nabla^*, \circ, *, e, E)$ is a smooth (or complex) manifold equipped with a pair of flat connection ∇ and ∇^*, a pair of product $\circ$ and $*$ on the holomorphic tangent bundle of M and a pair of vector field e and E satisfying the following axioms:*

- *E behaves like a Euler field, i.e.*

$$[e, E] = e \tag{5.33}$$
$$\mathcal{L}_E c^k_{ij} = c^k_{ij} \tag{5.34}$$

- *The product $\circ$ is commutative, associative and with unity e. Moreover*

$$\nabla e = 0 \tag{5.35}$$

- *The product $*$ is commutative, associative and with unity e. Moreover*

$$\nabla^* E = 0 \tag{5.36}$$

- *∇ and ∇^* are compatible with $\circ$ and $*$ in the following sense:*

$$\nabla^{(l)}_X \circ_{(l)} (Y, Z) = \nabla^{(l)}_Y \circ_{(l)} (X, Z) \tag{5.37}$$

for any vector field X, Y, Z. Where $\nabla^{(l)} = \nabla$, $\nabla^{(2)} = \nabla^*$, $\circ_{(l)} = \circ$, $\circ_{(2)} = *$.

Remark 5.42 *Frobenius manifolds equipped with an almost-dual structure, such that the dual connection is gauged so that $\nabla^* E = 0$, are bi-flat F-manifold equipped with an invariant metric and such that the Euler field acts as a conformal Killing vector field for the metric.*

Lemma 5.43 *Given a bi-flat F-manifold $(M, \nabla, \nabla^*, \circ, *, e, E)$, the Christoffel symbols corresponding to the dual connection ∇^* can be written uniquely in terms of the Christoffel symbols corresponding to the natural connection ∇, the structure constants of the dual product $*$ and the Euler field E, i.e.*

$$\Gamma^{k*}_{ij} = \Gamma^k_{ij} - c^{s*}_{ij} \nabla_s E^k \tag{5.38}$$

Furthermore, a "dual" formula for the Christoffel symbols corresponding to the natural connection holds true, i.e.

$$\Gamma^k_{ij} = \Gamma^{k*}_{ij} - c^s_{ij} \nabla^*_s e^k \tag{5.39}$$

Proof: Recall the formula (5.31)

$$\Gamma^k_{lj} \underbrace{(X*)^l_i}_{=c^{l*}_{si} X^s} - \Gamma^k_{li} \underbrace{(X*)^l_j}_{=c^{l*}_{js} X^s} = \Gamma^{k*}_{lj} \underbrace{(X*)^l_i}_{=c^{l*}_{is} X^s} - \Gamma^{k*}_{li} \underbrace{(X*)^l_j}_{=c^{l*}_{js} X^s}$$

Being X an arbitrary vector field one has

$$\Gamma^k_{lj} c^{l*}_{si} - \Gamma^k_{li} c^{l*}_{js} = \Gamma^{k*}_{lj} c^{l*}_{is} - \Gamma^{k*}_{li} c^{l*}_{js} \tag{5.40}$$

Recall that E is the unity of the product $*$, i.e. $Y * E = Y$ for any vector field Y, or equivalently $c^{i*}_{jk} E^k = \delta^i_j$. Contracting (5.40) by E^i one gets

$$\Gamma^k_{lj} \underbrace{c^{l*}_{si} E^i}_{=\delta^l_s} - \Gamma^k_{li} c^{l*}_{js} E^i = \Gamma^{k*}_{lj} \underbrace{c^{l*}_{is} E^i}_{=\delta^l_s} - \Gamma^{k*}_{li} c^{l*}_{js} E^i$$

then

$$\Gamma^k_{sj} - \Gamma^k_{li} c^{l*}_{js} E^i = \Gamma^{k*}_{sj} - \Gamma^{k*}_{li} c^{l*}_{js} E^i \tag{5.41}$$

Recall that E is covariantly constant with respect ∇^*, i.e. $\nabla^* E = 0$, or equivalently

$$\partial_l E^k + \Gamma^{k*}_{li} E^i = 0$$

Then substituting the above formula in (5.41) one has

$$\Gamma^k_{sj} - \Gamma^k_{li} c^{l*}_{js} E^i = \Gamma^{k*}_{sj} + c^{l*}_{js} \partial_l E^k$$
$$\Gamma^k_{sj} = \Gamma^{k*}_{sj} + c^{l*}_{js} \underbrace{(\partial_l E^k + \Gamma^k_{li} E^i)}_{=\nabla_l E^k}$$

which coincides with (5.38). Similarly, one can prove (5.39), using that the field e is flat and is the unity of the product $\circ$. ∎

5.7 Bi-flat F-manifold and principal hierarchies

In this subsection, we will show how to construct commuting PDEs of hydrodynamic type starting from a bi-flat F-manifold structure (see [37] for details).

5.7.1 The principal hierarchy

Let's consider a flat F-manifold structure $(\nabla, e, \circ)$ on a manifold M. Let $(v^1, ..., v^n)$ be a system of local coordinates for M.
Given such a structure, we associate with it a collection of systems of quasi-linear evolutionary PDEs (or briefly hydrodynamic system) given by

$$v^i_{t_{(p,l+1)}} = c^i_{jk} X^j_{(p,l)} v^k_x \tag{5.42}$$

where $p = 1, ..., n$ and $l = -1, 0, 1, ...$, $c^i_{jk} = c^i_{jk}(v)$ are the structure constants of the product $\circ$ and $X^j_{(p,l)} = X^j_{(p,l)}(v)$ are the components of the vector fields $X_{(p,l)}$.
For each value of l, which defines the level of the hierarchy, there are n systems of quasi-linear evolutionary PDEs.

Definition 5.44 *The flows given by (5.42) define the so-called principal hierarchy associated with the flat F-manifold $(M, \nabla, \circ, e)$, where the vector fields $X_{(p,l)}$ are the coefficients of the formal expansion in λ of the (flat) section with respect the deformed flat connection $\nabla^{(\lambda)} := \nabla - \lambda\circ$.*

The minus sign is due to avoid minus in the recurrence relation.

Remark 5.45 *Explicitly, $X_{(p,l)}$ are defined by the formal expansion*

$$(\nabla - \lambda\circ)\big(X_{(p,-1)} + \lambda X_{(p,0)} + \lambda^2 X_{(p,1)} + ...\big) = 0 \tag{5.43}$$

Comparing each term containing equal power in λ one obtains the following conditions:

- *The vector fields $X_{(p,-1)}$, defining the primary flows, are covariantly constant with respect ∇, i.e.*
$$\nabla X_{(p,-1)} = 0 \tag{5.44}$$
for any $p = 1, ..., n$.

- *The remaining vector fields are obtained by the recurrence relation*
$$\nabla X_{(p,l+1)} = X_{(p,l)}\circ \tag{5.45}$$
for $l = -1, 0, 1, ...$

The integrable hierarchy (5.42) is a generalization of Dubrovin's principal hierarchy associated with a Frobenius manifold, introduced in [17].

Remark 5.46 *The consistency of the recurrence relation (5.45), and the proof of commutativity of the flows defined by (5.42) are given in [38].*

We have obtained

- n primary flows, corresponding to $t_{(p,0)}$, defined by

$$v^i_{t_{(p,0)}} = c^i_{jk} X^j_{(p,-1)} v^k_x \tag{5.46}$$

- infinitely commutating flows, corresponding to $t_{(p,l)}$ with $l > 0$, defined by

$$v^i_{t_{(p,l)}} = c^i_{jk} X^j_{(p,l-1)} v^k_x \tag{5.47}$$

  where $p = 1, ..., n$, $l = 1, 2, ...$ and $X^j_{(p,l)}$ are the components of a collection of a vector defined by the recursion formula

$$\nabla_j X^i_{(p,l+1)} = c^i_{jk} X^k_{(p,l)} \tag{5.48}$$

  where $l = -1, 0, 1, ...$

Remark 5.47 *Let $(u^1, ..., u^n)$ be a flat coordinate system for the connection ∇, then one has $\nabla_i X_{(p,l)} = \partial_i X_{(p,l)}$ for any p and l. Then the primary fields has constant components, i.e. $\partial_i X^j_{(p,-1)} = 0$, while the recursion relation reduces to $\partial_j X^i_{t_{(p,l+1)}} = c^i_{jk} X^k_{(p,l)}$.*
Moreover, in flat coordinates (u^i), the flows (5.42) turn out to be systems of conservation laws, indeed

$$u^i_{t_{(p,l)}} = \underbrace{c^i_{jk} X^j_{(p,l-1)}}_{=\partial_k X^i_{(p,l)}} u^k_x = \partial_x X^i_{(p,l)} \tag{5.49}$$

where in the last equality we have exploited the chain rule.

Remark 5.48 *Assuming the product $\circ$ semisimple, there exists a distinguished coordinate system $(r^1, ..., r^n)$ such that the structure constants reduces to $\tilde{c}^k_{ij} = \delta^k_i \delta^k_j$. Furthermore, in canonical coordinates (r^i) the flows (5.42) read*

$$r^i_{t_{(p,l)}} = \tilde{c}^i_{jk} \tilde{X}^j_{(p,l-1)} r^k_x = \delta^i_j \delta^i_k \tilde{X}^j_{(p,l-1)} r^k_x = \tilde{X}^i_{(p,l-1)} r^i_x \tag{5.50}$$

which implies that the canonical coordinates (r^i) are Riemann invariant for the conservation law (5.49), with generalized velocities $\tilde{X}^i_{(p,l-1)}$.

Remark 5.49 *In the case of flat F-manifold with invariant metric η, the principal hierarchy becomes Hamiltonian with respect to the Poisson bracket of hydrodynamic type corresponding to the cometric η^{-1} (for details see [7]).*
Furthermore, in the case of the Frobenius manifold, the principal hierarchy becomes bi-Hamiltonian, where the further Poisson bracket of hydrodynamic is given by the intersection form.

5.8 Bi-flat F-manifold and complex reflection groups

We will see how the orbit space with respect to the action of an irreducible well-generated complex reflection group may be endowed with a structure of bi-flat F-manifold. In some cases, such a structure appears in family, as we will see in a simple example.
First, we recall some results concerning flat structure associated with Coxeter group, exposed previously.

5.8.1 Flat structures associated with Coxeter group

Recall the Dubrovin's result.

Theorem 5.50 *The orbit space of an irreducible Coxeter group is endowed with a Frobenius manifold structure* $(\eta, \nabla, \circ, e, E)$ *where*

- *the flat coordinates* $(u^1, ..., u^n)$ *for the metric* η*, called Saito flat coordinates, are a unique set of basic invariants of the the group*
- *in Saito flat coordinates we have*

$$e = \frac{\partial}{\partial u^n}$$
$$E = \frac{1}{h}\left(d_i \frac{\partial}{\partial u^n}\right)$$

  where d_i *are the degrees of the invariant polynomials with*

$$2 = d_1 < d_2 \leq d_3 ..... \leq d_{n-1} < d_n = h$$

  where h *is the Coxeter number.*

Dubrovin's construction relies on the existence of flat pencil of cometric. One is the Euclidean metric and one is the Saito metric.
The orbit space of these groups can be equipped with another structure, constructed by means of a collection of reflecting (hyper-)planes. In [16] Dubrovin highlighted that the Frobenius potential corresponding to the almost-dual structure of a polynomial Frobenius structure, associated with any Coxeter group, turns out to be an expression in terms of reflecting (hyper-)planes (associated with the group). These solutions of the WDVV equation have already been discovered by Veselov in [52]. We have the following:

Theorem 5.51 *Let* G *be an irreducible Coxeter group acting on an Euclidean space* $\mathbb{R}^n$ *with Euclidean coordinates* $(p^1, ..., p^n)$*. Let* g *be the Euclidean metric and* $\tilde{\nabla}$ *the corresponding Levi-Civita connection. Then the following data:*

- $\tilde{\nabla}$
- $* := \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \sigma_H \alpha_H$
- $E := p^i \frac{\partial}{\partial p^i}$

where

- $\mathcal{H}$ *is a collection of reflecting (hyper-)planes*
- α_H *is a covector defining a reflecting (hyper-)plane*
- π_H *is the orthogonal projection onto the orthogonal complement of* H

- *the collection of weights* $\{\sigma_H\}_{H\in\mathcal{H}}$ *are* G*-invariant (i.e* $\sigma_H = \sigma_{H'}$ *iff* H *and* H' *belong to the same orbit with respect to the action of* G *on the collection of reflecting (hyper-)plane) and satisfy the normalization condition*

$$\sum_{H\in\mathcal{H}} \sigma_H \pi_H = id_{\mathbb{R}^n}$$

define an almost-flat (i.e. $\tilde{\nabla} E \neq 0$ *in general) Frobenius structure with invariant metric* g*.*

Proof: This is an equivalent reformulation of a result of Veselov (see [52] and [53]). The equivalence between the flatness condition and the definition of $\vee$-system is discussed in [2] and [29]. ■

5.8.2 Flat structures associated with complex reflection groups

Similarly to the case of Coxeter group, it's possible to define two flat structures on the orbit space of some classes of finite complex reflection groups.
The first structure generalizes the Dubrovin-Saito construction relying on Coxeter groups to irreducible well-generated finite complex reflection groups. The second one is defined by a Dunkle-Kohno-type connection associated with a complex reflection group, which can be thought of as a generalization of Veselov's $\vee$-system. In general, the two F-manifold flat structures don't admit invariant metrics, therefore they don't come from a Frobenius and its almost-dual structure.
Following the work of Sato, Kato, and Sekiguci [32], we get the first flat structure.

Theorem 5.52 *The orbit space of an irreducible well-generated finite complex reflection group is equipped with a flat* F*-structure* $(\nabla, \circ, e)$ *with linear Euler vector field* E*, defined by the following prescriptions:*

- *The flat coordinates* $(u^1, ..., u^n)$ *for* ∇*, called generalized Saito flat coordinates are a distinguished set of basic invariants of the group.*

- *In Saito flat coordinates we have*

$$e = \frac{\partial}{\partial u^n} \tag{5.51}$$

$$E = \frac{1}{d_n}\left(d_i \frac{\partial}{\partial u^i}\right) \tag{5.52}$$

where d_i *are the degrees of the polynomial invariants.*

The second flat structure (more precisely a family of flat structures), is given by the following (we refer to see [4]):

Theorem 5.53 *Let $(p^1, ..., p^n)$ be a system of standard coordinates for $\mathbb{C}^n$ and let G be an irreducible well-generated finite complex reflection group. Then the following data:*

$$\nabla^* := \nabla^0 - \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \tau_H \alpha_H \tag{5.53}$$

$$* := \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \sigma_H \alpha_H \tag{5.54}$$

$$E := p^i \frac{\partial}{\partial p^i} \tag{5.55}$$

where

- *$\mathcal{H}$ is a collection of reflecting (hyper-)planes*
- *α_H is a covector defining a reflecting (hyper-)plane*
- *π_H is the unitary projection onto the unitary complement of H*
- *the collections of weights $\{\tau_H\}_{H\in\mathcal{H}}$ and $\{\sigma_H\}_{H\in\mathcal{H}}$ are G-invariant and satisfy the normalization conditions*

$$\sum_{H\in\mathcal{H}} \tau_H \pi_h = id_{\mathbb{C}^n} \tag{5.56}$$

$$\sum_{H\in\mathcal{H}} \sigma_H \pi_h = id_{\mathbb{C}^n} \tag{5.57}$$

- *∇^0 is the trivial flat connection on $\mathbb{C}^n$, with flat coordinates $(p^1, ..., p^n)$*

define a flat F-structure on the orbit space with respect to the action of G on $\mathbb{C}^n$.

Proof: First, we prove the connection $\nabla^0 + \lambda *$ is flat and torsionless for any λ. Using the definitions (5.53) and (5.54) one has

$$\nabla^* + \lambda * = \nabla^* - \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \underbrace{(\tau_H - \lambda\sigma_H)}_{:=\omega_H} \alpha_H$$

Observe that the collection $\{\omega_H\}_{H\in\mathcal{H}}$ is also G-invariant. Then, applying Looijenga's result of [36], one gets the flatness of $\nabla^0 + \lambda *$. The zero torsion condition is obvious. Moreover

$$\nabla^* E \overset{(5.55)}{=} \nabla^0 E - \sum_{H\in\mathcal{H}} \frac{d\alpha_H(E)}{\alpha_H(p)} \tau_H \alpha_H \overset{(5.55)}{=} id_{\mathbb{C}^n} - \sum_{H\in\mathcal{H}} \tau_H \alpha_H \overset{(5.56)}{=} 0$$

Similarly, one proves that $*E = id_{\mathbb{C}^n}$ (i.e. E is the unity of $*$). ■

Remark 5.54 *Recall that any Coxeter group is automatically well-generated. Therefore the orbit space of any Coxeter group can be equipped with two flat structures defined by the theorems (5.52) and (5.53).*

We have seen that the orbit space of any well-generated complex reflection group can be endowed with structures:

1. The first structure generalizes the notion of Saito coordinates without the necessity of a metric
2. The second structure generalizes the notion of $\vee$-system

One question arises naturally: it's possible to define a bi-flat F-manifold structure on the orbit space of a well-generated complex reflection group having the natural structure of the form given by theorem (5.52) and dual structure of the form given by theorem (5.53). If this question has an affirmative answer:

- Which choices of weights are allowed?
- What is the relation (if exists) between the flat structure and Dubrovin's polynomial structure given by proposition (4.37)?

Remark 5.55 *Let G be Weyl group of rank 2, 3 and 4, or the dihedral group $I_2(m)$, or any of the exceptional well-generated complex reflection groups of rank 2 and 3, or any of the group $G(m,1,2)$ and $G(m,1,3)$, then the corresponding orbit spaces admit a bi-flat F-structure. In all these cases the number of parameters appearing coincides with the number of orbits for the action of the group on the collection of reflecting (hyper-)planes minus one. For details see [3].*
For a well-generated complex reflection group, the number of orbits is 1 or 2 (this integer is related to the length of the roots corresponding to the reflection group). Moreover, in all these cases the weights $\{\sigma_H\}$ coincide, up to a multiplicative constant, to the order of the (pseudo-)reflection corresponding to the (hyper-)planes H.
In particular, if $\sigma_H = \tau_H$ for any H, the flat structure coincides with Dubrovin's structure defined by proposition (4.37) associated with the considered Coxeter group.
It's worth highlighting that, even in the Coxeter case, the corresponding bi-flat F-structure might be not uniquely defined. In order to elucidate such a theory, in the next section we will study the bi-flat F-structures on the orbit space of B_2.

6 Flat structures on B_2

Recall that B_2 is the symmetry group of the square, in particular, $B_2 \cong D_4$ (where D_n denotes the dihedral group of order $2n$).
Moreover, since B_2 is a Coxeter group, it's automatically well-generated. Therefore we can equip the orbit space of B_2, following [3], with a bi-flat F-structure, where the two underlying flat structures are given by the theorem (5.52) and (5.53). Let (p^1, p^2) be a system of Euclidean coordinates of $\mathbb{R}^2$. Recall that the general basic invariants for B_2 (up to rescaling factors) are

$$\begin{aligned} u^1 &= (p^1)^2 + (p^2)^2 \\ u^2 &= (p^1)^2(p^2)^2 + c\big((p^1)^2 + (p^2)^2\big)^2 \end{aligned}$$

where c is a real constant.
First, applying Dubrovin-Saito's procedure we reconstruct the unique polynomial Frobenius manifold structure on the orbit space of B_2. The contravariant Euclidean metric, written in (u^1, u^2), reads

$$g^{ij}(u) = \begin{pmatrix} 4u^1 & (8c+1)(u^1)^2 + 8u^2 \\ (8c+1)(u^1)^2 + 8u^2 & \frac{(32c^2+8c+1)}{2}(u^1)^3 + \frac{(64c+8)}{2}u^1u^2 \end{pmatrix}$$

where $g^{ij}(u) = \sum_{s,q=1}^{2} \frac{\partial u^i}{\partial p^q}\frac{\partial u^j}{\partial p^s} g^{ij}(p)$.
The unity field has the form $e = \frac{\partial}{\partial u^2}$.
We define the Saito flat coordinates (t^1, t^2) as the basic invariants such that Saito cometric

$$\eta^{ij} = \mathcal{L}_e g^{ij} = \frac{\partial g^{ij}}{\partial u^2}$$

reduces to a constant non-degenerate matrix. Thus we have

$$\eta^{ij}(u) = \begin{pmatrix} 0 & 8 \\ 8 & \frac{(64c+8)}{2}u^1 \end{pmatrix}$$

Hence (u^1, u^2) are the Saito flat basic invariants (t^1, t^2) for $c = -\frac{1}{8}$.
We reconstruct the Frobenius potential by the equation system

$$\tfrac{d_i+d_j-2}{h}\eta^{is}\eta^{jk}\partial_s\partial_k F(t) = g^{ij}(t) \tag{6.1}$$

for $i, j = 1, 2$, where $d_1 = 2, d_2 = 4$ and $\partial_i = \frac{\partial}{\partial t^i}$.
(6.1) reads explicitly

$$\begin{pmatrix} 32\partial_2\partial_2 F & 64\partial_1\partial_2 F \\ 64\partial_1\partial_2 F & 96\partial_1\partial_1 F \end{pmatrix} = \begin{pmatrix} 4t^1 & 4t^2 \\ 4t^2 & \frac{(t^1)^3}{4} \end{pmatrix}$$

Thus by integrating one gets

$$F(t) = \tfrac{1}{7680}(t^1)^5 + \tfrac{1}{16}t^1(t^2)^2 \tag{6.2}$$

By rescaling $t^1 \mapsto \frac{1}{8}t^1$, F takes the form

$$F(t) = \tfrac{64}{15}(t^1)^5 + \tfrac{1}{2}t^1(t^2)^2 \tag{6.3}$$

6.1 Bi-flat $F-$structures on B_2

6.1.1 The dual product $*$

Let's consider the dual product

$$* = \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \sigma_H \pi_H \tag{6.4}$$

where

- $\alpha_H(p)$ are the following linear forms:
  $\alpha_1(p) = p^1$
  $\alpha_2(p) = p^2$
  $\alpha_3(p) = p^1 - p^2$
  $\alpha_4(p) = p^1 + p^2$
- $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ is the collection of reflecting lines, with $H_i := Ker(\alpha_i)$
- π is the orthogonal projection obtained via the Euclidean cometric $g^{ij} = \delta^{ij}$, with components

$$(\pi_H)^i_j = g^{is}\frac{(\alpha_H)_s}{||\alpha_H||}\frac{(\alpha_H)_j}{||\alpha_H||} = \frac{(\alpha_H)_i(\alpha_H)_j}{||\alpha_H||^2}$$

- $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a collection of B_2-invariant weights such that

$$\sum_{H\in\mathcal{H}} \sigma_H \pi_H = id_{\mathbb{R}^2}$$

  then

$$\sigma_1 = \sigma_2 = \frac{x}{x+y}$$
$$\sigma_3 = \sigma_4 = \frac{y}{x+y}$$

  indeed

$$\sum_{H\in\mathcal{H}} \sigma_H \pi_H = \frac{x}{x+y}\underbrace{(\check{\alpha}_{H_1}\otimes\alpha_{H_1} + \check{\alpha}_{H_2}\otimes\alpha_{H_2})}_{id_{\mathbb{R}^2}} + \frac{y}{x+y}\underbrace{(\check{\alpha}_{H_3}\otimes\alpha_{H_3} + \check{\alpha}_{H_4}\otimes\alpha_{H_4})}_{id_{\mathbb{R}^2}} = id_{\mathbb{R}^2}$$

Thus the structure constants associated with the product (6.4) reads

$$c^{i*}_{jk}(p) = \sum_{H=1,2,3,4} \frac{\sigma_H(\alpha_H)_i(\alpha_H)_j(\alpha_H)_k}{\alpha_H(p)||\alpha_H||^2}$$

In particular, one has the rational functions of (p^1, p^2)

$$c^{1*}_{11}(p) = \frac{(x+y)(p^1)^2 - x(p^2)^2}{(x+y)p^1\big((p^1)^2-(p^2)^2\big)}$$

$$c^{1*}_{12}(p) = c^{1*}_{21}(p) = c^{2*}_{11}(p) = \frac{-yp^2}{(x+y)\big((p^1)^2-(p^2)^2\big)}$$

$$c^{1*}_{22}(p) = c^{2*}_{21}(p) = c^{2*}_{12}(p) = \frac{-yp^1}{(x+y)\big((p^1)^2-(p^2)^2\big)}$$

$$c^{2*}_{22}(p) = \frac{x(p^1)^2 - (x+y)(p^2)^2}{(x+y)p^2\big((p^1)^2-(p^2)^2\big)}$$

6.1.2 The natural connection ∇

Let (u^1, u^2), defined by

$$u^1 = (p^1)^2 + (p^2)^2$$
$$u^2 = (p^1)^2(p^2)^2 + c\big((p^1)^2 + (p^2)^2\big)^2$$

where c is a real constant, be a set of general basic invariants for B_2. In particular, $d_1 = deg(u^1) = 2$ and $d_2 = deg(u^2) = 4$.
We assume (u^1, u^2) to be a system of flat coordinates for ∇ (i.e (u^1, u^2) are generalized Saito flat coordinates).
Using formula (1.16), i.e.

$$\Gamma^k_{ij}(p) = \frac{\partial p^k}{\partial u^s}\frac{\partial^2 u^s}{\partial p^i \partial p^j} \tag{6.5}$$

one obtains the following rational functions of (p^1, p^2) for the Christoffel symbols corresponding to the connection ∇:

$$\Gamma^1_{11} = \frac{(-4c+1)(p^1)^2 - (p^2)^2}{(p^1)^3 - p^1(p^2)^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{(4c+2)p^1}{(p^1)^2 - (p^2)^2}$$

$$\Gamma^2_{11} = \frac{4c(p^1)^2}{p^2(p^1)^2 - (p^2)^3}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{(4c+2)p^1}{(p^1)^2 - (p^2)^2}$$

$$\Gamma^1_{22} = \frac{(4c-1)(p^2)^2 + (p^1)^2}{p^2(p^1)^2 - (p^2)^3}$$

$$\Gamma^2_{22} = \frac{(4c-1)(p^2)^2 - (p^1)^2}{(p^1)^2p^2 - (p^2)^3}$$

6.1.3 The unity field

We define the unity vector field

$$e = \frac{\partial}{\partial u^2} \tag{6.6}$$

6.1.4 The Euler field

We define the Euler vector field

$$E = p^1 \frac{\partial}{\partial p^1} + p^2 \frac{\partial}{\partial p^2} = d_1 u^1 \frac{\partial}{\partial u^1} + d_2 u^2 \frac{\partial}{\partial u^2} \tag{6.7}$$

E is normalized so that (5.21) reads

$$[e, E] = d_2 e$$

6.1.5 The natural product $\circ$

The natural product $\circ$ is given, in terms of the dual product $*$, by

$$X \circ Y = (e*)^{-1} X * Y$$

for any vector field X and Y, where $(e*)^{-1}$ denotes the inverse of the linear endomorphism $e*$.
Denote $M := (e*)^{-1}$. Hence the structure constants of $*$ are given by the formula

$$c^i_{jk} = M^i_s c^{s*}_{jk}$$

for $i, j, k = 1, 2$. We want explicit expressions for c^i_{jk} in terms of (p^1, p^2), thus we have to write the vector e in the same coordinates.
By applying the vector transformation rule one has

$$e = \frac{\partial}{\partial u^2} = \frac{\partial p^1}{\partial u^2}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial u^2}\frac{\partial}{\partial p^2}$$

Then e in the coordinate system (p^1, p^2) has components

$$e(p) = \begin{pmatrix} \frac{\partial p^1}{\partial u^2} \\ \frac{\partial p^2}{\partial u^2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \frac{1}{p^1\left((p^1)^2 - (p^2)^2\right)} \\ -\frac{1}{p^2\left((p^1)^2 - (p^2)^2\right)} \end{pmatrix}$$

We obtain the rational functions of (p^1, p^2)

$$c^1_{11} = \frac{4p^1\left(x(p^1)^2 - x(p^2)^2 - y(p^2)^2\right)}{x + y}$$

$$c^1_{12} = c^1_{21} = c^2_{11} = -\frac{4yp^2(p^1)^2}{x + y}$$

$$c^2_{12} = c^2_{21} = c^1_{22} = -\frac{yp^1(p^2)^2}{x + y}$$

$$c^2_{22} = -\frac{4p^2\left(x(p^1)^2 - x(p^2)^2 - y(p^2)^2\right)}{x + y}$$

6.1.6 The compatibility condition and the constraint on the weights

Recall that the compatibility condition between ∇ and $\circ$ is given by

$$\nabla_i c^s_{jk} = \nabla_j c^s_{ik}$$

for $i, j, k, s = 1, 2$. In particular, $\nabla_1 c^1_{21} = \nabla_2 c^1_{11}$ and $\nabla_1 c^2_{21} = \nabla_2 c^1_{11}$ read

$$\frac{4p^1p^2(x-y)}{x+y} = 0$$
$$\frac{4(p^1)^2(x-y)}{x+y} = 0$$

which are true if and only if $x = y$ and for any real c.
Hence one has $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \frac{1}{2}$.

6.1.7 The dual connection ∇^*

Recall that the Christoffel symbols corresponding to the dual connection ∇^* are given by the formula

$$\Gamma^{k*}_{ij} = \Gamma^k_{ij} - c^{s*}_{ij}\nabla_s E^k$$

for $i, j, k = 1, 2$. Then we have the rational functions of (p^1, p^2)

$$\Gamma^{1*}_{11} = \frac{(4c+1)(p^2)^2 - (p^1)^2}{(p^1)^3 - p^1(p^2)^2}$$
$$\Gamma^{1*}_{12} = \Gamma^{1*}_{21} = \Gamma^{2*}_{11} = -\frac{4cp^2}{(p^1)^2 - (p^2)^2}$$
$$\Gamma^{1*}_{22} = \Gamma^{2*}_{12} = \Gamma^{2*}_{21} = \frac{4cp^1}{(p^1)^2 - (p^2)^2}$$
$$\Gamma^{2*}_{22} = \frac{-(4c+1)(p^1)^2 + (p^2)^2}{(p^1)^2p^2 - (p^2)^3}$$

One observes that ∇^* can be written in terms of a Dunkle-Kohno connection

$$\widetilde{\nabla} := \nabla^0 - \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \tau_H \pi_H$$

with suitable weights $\{\tau_H\}$. Let $\widetilde{\Gamma}^k_{ij}$ be the Christoffel symbols associated with $\widetilde{\nabla}$. It turns out that

$$\widetilde{\Gamma}^1_{11} = \frac{(2\tau_1 + \tau_3 + \tau_4)(p^1)^2 + (\tau_3 - \tau_4)p^1 - 2\tau_1(p^2)^2}{2((p^1)^3 - p^1(p^2)^2)}$$
$$\widetilde{\Gamma}^1_{12} = \widetilde{\Gamma}^1_{21} = \widetilde{\Gamma}^2_{11} = \frac{(-\tau_3 + \tau_4)p^1 - (\tau_3 + \tau_4)p^2}{2((p^1)^2 - (p^2)^2)}$$
$$\widetilde{\Gamma}^1_{22} = \widetilde{\Gamma}^2_{12} = \widetilde{\Gamma}^2_{21} = \frac{(\tau_3 + \tau_4)p^1 + (\tau_3 - \tau_4)p^2}{2((p^1)^2 - (p^2)^2)}$$
$$\widetilde{\Gamma}^2_{22} = \frac{(-2\tau_2 - \tau_3 - \tau_4)(p^2)^2 - (\tau_3 - \tau_4)p^2 + 2\tau_2(p^1)^2}{2((p^1)^2p^2 - (p^2)^3)}$$

Then

$$\nabla^* = \nabla^0 - \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \tau_H \pi_H$$

for $\tau_1 = \tau_2 = -4c - 1$ and $\tau_3 = \tau_4 = 4c$.
In particular, for $c = -\frac{1}{8}$ one observes that $\Gamma^{k*}_{ij} = -c^{k*}_{ij}$, i.e. $\nabla^* = \nabla^0 - *$.

Remark 6.1 *The collection of weights $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is B_2-invariant. This fact ensures the flatness of the connection ∇^* (see [36]).*

6.1.8 The vector potential A

It turns out that the above data define a bi-flat F-manifold structure $(\nabla, \nabla^*, \circ, *, e, E)$ on the orbit space $\mathbb{C}^2/B_2$, for any choice of the parameter c.
Recall that the structure constants corresponding to the natural product $\circ$, written in generalized coordinates (u^1, u^2), define the vector potential $A_{B_2} = A_{B_2}(u)$ (up to a polynomial of degree one) by the formula

$$\widetilde{c}^k_{ij}(u) = \partial_i \partial_j A^k_{B_2}$$

for $i, j, k = 1, 2$ and where $\partial_i = \frac{\partial}{\partial u^i}$.
Using the transformation law of a $(1, 2)$ tensor field

$$\widetilde{c}^k_{ij}(u) = \frac{\partial u^k}{\partial p^s} \frac{\partial p^r}{\partial u^i} \frac{\partial p^q}{\partial u^j} c^s_{rq}(p)$$

one gets

$$\begin{aligned} \widetilde{c}^1_{11} &= -(4c + \tfrac{1}{2})u^1 \\ \widetilde{c}^1_{12} &= \widetilde{c}^1_{21} = \widetilde{c}^2_{22} = 1 \\ \widetilde{c}^2_{12} &= \widetilde{c}^2_{21} = \widetilde{c}^1_{22} = 0 \\ \widetilde{c}^2_{11} &= -c(4c + 1)(u^1)^2 \end{aligned}$$

Therefore we get the equation system for $A^1_{B_2}$

$$\begin{cases} \partial_1 \partial_1 A^1 = -\frac{1}{2}(8c + 1)u^1 \\ \partial_1 \partial_2 A^1 = 1 \\ \partial_2 \partial_2 A^1 = 0 \end{cases}$$

By integrating one obtains

$$A^1_{B_2}(u) = -\tfrac{1}{12}(8c + 1)(u^1)^3 + u^1 u^2 \tag{6.8}$$

Similarly for $A^1_{B_2}$ we have the system

$$\begin{cases} \partial_1 \partial_1 A^2 = -c(4c + 1)(u^1)^2 \\ \partial_1 \partial_2 A^2 = 0 \\ \partial_2 \partial_2 A^2 = 0 \end{cases}$$

Therefore one has

$$A^2_{B_2}(u) = -\tfrac{1}{12}c(4c + 1)(u^1)^4 + \frac{1}{2}(u^2)^2 \tag{6.9}$$

Remark 6.2 *Summarizing, we got a bi-flat F-structure, assuming $e = \frac{\partial}{\partial u^2}$, for any value of c, for a unique choice of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ (which coincide, up to a multiplicative constant, to the order of the reflection) and for certain values of $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ (which depend only on c). Then we have a family of bi-flat F-structures parameterized by a single real parameter. Furthermore, the number of parameters coincides with the number of orbits for the action of B_2 on the collection of reflecting (hyper-)planes minus one.*

One question arises naturally: does this bi-flat F-manfiold admit a Frobenius structure with its dual structure gauged so that $\nabla^* E = 0$?
It's natural to guess that if such a structure exists it coincides with the polynomial Frobenius manifold on the orbit space of B_2 with prepotential (6.2).
Recall that (see propositions (5.30) and (5.31)), in order to obtain a Frobenius structure, one has to check if there exists a flat metric η invariant with respect to the product $\circ$, compatible with the connection ∇ and such that the Euler field acts a conformal Killing vector field for η.
The flatness of η implies that, in generalized Saito flat coordinates, it reduces to a constant matrix

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$$

The compatibility condition between η and $\circ$

$$\eta_{is} c^s_{jk} = \eta_{js} c^s_{ik}$$

(where $i, j, k = 1, 2$) written in (u^1, u^2), yields the equation system for the components of η and the constant c

$$\begin{cases} \eta_{11} = -4u^1 \left(\eta_{22}(c^2 + \frac{c}{4}) u^1 + \eta_{21}(c + \frac{1}{8}) \right) \\ \eta_{12} = \eta_{21} \end{cases}$$

which has solutions

$$\begin{cases} \eta_{11} = \eta_{22} = 0 \\ \eta_{12} = \eta_{21} \\ c = -\frac{1}{8} \end{cases}$$

In general, the invariant metric η is defined up to rescaling. Rescale η so that

$$\eta = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}$$

Moreover, $\frac{1}{d_2} E$ acts on η as a conformal Killing vector field, indeed

$$\begin{aligned} \mathcal{L}_E \eta_{ij} &= E^s \underbrace{\partial_s \eta_{ij}}_{=0} + \underbrace{\partial_i E^s}_{=\frac{d_i}{d_2}\delta^s_i} \eta_{sj} + \underbrace{\partial_j E^s}_{=\frac{d_j}{d_2}\delta^s_i} \eta_{is} = \frac{(d_i + d_j)}{d_2} \eta_{ij} = 8 \begin{pmatrix} 0 & \frac{(d_1+d_2)}{d_2} \\ \frac{(d_1+d_2)}{d_2} & 0 \end{pmatrix} \\ &= \frac{(d_1+d_2)}{d_2} \eta_{ij} = (2 - d) \eta_{ij} \end{aligned}$$

where $d = \frac{1}{2}$. Hence the existence of the metric η endows the orbit space of B_2 with a Frobenius manifold structure of charge $d = \frac{1}{2}$. The corresponding Frobenius potential is defined by

$$\eta_{is} A^s_{B_2} = \partial_i F_{B_2}$$

for $i = 1, 2$, where $\partial_i = \frac{\partial}{\partial u^i}$. Then we have the system

$$\begin{cases} \partial_2 F_{B_2} = \frac{1}{8} u^1 u^2 \\ \partial_1 F_{B_2} = \frac{1}{1536} (u^1)^4 + \frac{1}{16} (u^2)^2 \end{cases}$$

By integrating one obtains the polynomial prepotenital

$$F_{B_2}(u) = \frac{1}{7680} (u^1)^5 + \frac{1}{16} u^1 (u^2)^2$$

The existence of a unique structure of polynomial Frobenius manifolds of charge $d = \frac{1}{2}$ on the orbit space for B_2 is also guaranteed by the Theorem (4.38).

Remark 6.3 *Observe that this prepotential coincides with (6.2), which was obtained by applying the Dubrovin-Saito procedure. Furthermore, for $c = -\frac{1}{8}$ the generalized Saito flat coordinates coincide with the standard Saito flat coordinates.*

6.2 A modified construction for B_2

In view of the definition of flat F-manifold, the components of the unity vector field, written in flat coordinates, are constants.
Following Dubrovin-Saito and Kato-Mano-Sekiguchi constructions, in the previous example for the orbit space of B_2, we assumed that the flat coordinates are polynomial basic invariants for B_2 and that the unity field has the form

$$e = \frac{\partial}{\partial u^2} \tag{6.10}$$

where u^2 is the basic invariant of the highest degree. The last assumption is natural since the vector field e is not affected by any change in the choice of the basic invariants and for homogeneity reasons.
Now, following [8], we will study the bi-flat F-structures on B_2, defined as previously, by removing this hypothesis (6.10) on e.
The compatibility condition between $\circ$ and ∇

$$\nabla_i c^s_{jk} = \nabla_j c^s_{ik}$$

for $i, j, k, s = 1, 2$, is fulfilled by the following set of solutions:

1. $y = x$, $e^1 = 0$
2. $c = 0$, $x = 0$, $e^2 = 0$
3. $c = -\frac{1}{4}$, $y = 0$, $e^2 = 0$

Solution 1. corresponds to the one-parameter family of bi-flat F-manifold structures with vector potential of components (6.8) and (6.9).
Following the same steps outlined before, solutions 2. and 3. lead to the following Frobenius potentials

$$F_{\pm}(u) = \frac{1}{2}(u^1)^2u^2 \pm \frac{1}{2}(u^2)^2\big(log(u^2) - \frac{3}{2}\big) \tag{6.11}$$

respectively, where the unity field has been normalized so that

$$e = \frac{\partial}{\partial u^1} \tag{6.12}$$

and the invariant metric has been rescaled so that

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.13}$$

Observe that these prepotentials are related to focusing/defocusing NLS equation

$$iq_t + q_{xx} + 2\lambda|q|^2 q = 0$$

where $\lambda = \pm 1$, respectively, as follows.
Recall that the evolution equations

$$u^i_{t_{l+1}} = c^{i\pm}_{jk} X^j_{(l)} u^k_x$$

where $l = -1, 0, 1, ...$, define two chains of commutating flows, where

$$c^{i\pm}_{jk} = \eta^{is}\partial_s\partial_i\partial_j F_{\pm}$$

denote the structure constants corresponding to the Frobenius manifolds related to the prepotentials $F_{\pm}$ respectively.
Recall that (6.11) is a tensorial expression, thus it holds true in the Saito flat coordinates (u^1, u^2) corresponding to the Frobenius manifolds associated with $F_{\pm}$.
We take as the primary field $X_{(-1)} = \frac{\partial}{\partial u^1}$. Then the primary flow ($l = -1$)

$$u^i_{t_0} = c^{i\pm}_{jk} \underbrace{X^j_{(-1)}}_{=\delta^j_1} u^k_x = c^{i\pm}_{1k} u^k_x = c^{i\pm}_{11} u^1_x + c^{i\pm}_{12} u^2_x$$

reads

$$\begin{pmatrix} u^1_{t_0} \\ u^2_{t_0} \end{pmatrix} = \begin{pmatrix} \underbrace{c^{1\pm}_{11}}_{=1} u^1_x + \underbrace{c^{1\pm}_{12}}_{=0} u^2_x \\ \underbrace{c^{2\pm}_{11}}_{=0} u^1_x + \underbrace{c^{2\pm}_{12}}_{=1} u^2_x \end{pmatrix} = \begin{pmatrix} u^1_x \\ u^2_x \end{pmatrix}$$

i.e. the wave equation $u^i_{t_0} = u^i_x$ (for $i = 1, 2$). Observe that the primary flows don't depend on the choice of the prepotential F_+ or F_-.
Recall that the components $X^j_{(p)}$, of the higher vector field, are given by the recurrence relation

$$\partial_j X^i_{(p)} = c^{i\pm}_{jk} X^k_{(p-1)}$$

for $i = 1, 2$, which for $p = 0$ reduces to

$$\partial_j X^i_{(0)} = c^{i\pm}_{j1}$$

since $X^j_{(-1)} = \delta^j_1$.
Then $X^1_{(0)}$ and $X^2_{(0)}$ are obtained by solving the system

$$\begin{cases} \partial_1 X^1_{(0)} = 1 \\ \partial_2 X^1_{(0)} = 0 \\ \partial_1 X^2_{(0)} = 0 \\ \partial_2 X^2_{(0)} = 1 \end{cases}$$

It turns out that $X^1_{(0)} = u^1$ and $X^2_{(0)} = u^2$.
Let's consider the flow corresponding to t_1

$$u^i_{t_1} = c^{i\pm}_{jk} X^j_{(0)} u^k_x$$

which explicitly reads

$$\begin{cases} u^1_{t_1} = c^{1\pm}_{jk} X^j_{(0)} u^k_x = \underbrace{c^{1\pm}_{11}}_{=1} \underbrace{X^1_{(0)}}_{=u^1} u^1_x + \underbrace{c^{1\pm}_{21}}_{=0} X^2_{(0)} u^1_x + \underbrace{c^{1\pm}_{12}}_{=0} X^1_{(0)} u^2_x + \underbrace{c^{1\pm}_{22}}_{=\pm\frac{1}{u^2}} \underbrace{X^2_{(0)}}_{=u^2} u^2_x \\ u^2_{t_1} = c^{2\pm}_{jk} X^j_{(0)} u^k_x = \underbrace{c^{2\pm}_{11}}_{=0} X^1_{(0)} u^1_x + \underbrace{c^{2\pm}_{21}}_{=1} \underbrace{X^2_{(0)}}_{=u^2} u^1_x + \underbrace{c^{2\pm}_{12}}_{=1} \underbrace{X^1_{(0)}}_{=u^1} u^2_x + \underbrace{c^{2\pm}_{22}}_{=0} X^2_{(0)} u^2_x \end{cases}$$

then

$$\begin{cases} u^1_{t_1} = u^1 u^1_x \pm u^2_x \\ u^2_{t_1} = u^2 u^1_x + u^1 u^2_x = (u^1 u^2)_x \end{cases} \tag{6.14}$$

This system coincides with the dispersionless limit of the evolutionary PDEs associated with the defocusing/focusing NLS equation respectively. Indeed the focusing/defocusing rescaled NLS equation (with $\frac{\partial}{\partial t} \mapsto \epsilon\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x} \mapsto \epsilon\frac{\partial}{\partial x}$)

$$i\epsilon q_t + \tfrac{1}{2}\epsilon^2 q_{xx} \pm \lambda |q|^2 q = 0$$

can be reduced by the substitution

$$\begin{cases} w = |q|^2 \\ v = \frac{\epsilon}{2i}\left(\frac{q_x}{q} - \frac{\bar{q}_x}{\bar{q}}\right) \end{cases}$$

to the equation system

$$\begin{cases} w_t = (wv)_x \\ v_t = v v_x \mp w_x + \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\frac{u_x^2}{u^2}\right)_x \end{cases}$$

which in the dispersionless limit $\epsilon \to 0$, and with the identifications $w = -u^2$ and $v = u^1$, coincides with the system (6.14).

6.3 The case of B_3 and B_4

We want to study the bi-flat F-structures, following the scheme exposed before, for the group B_n, with $n > 2$.
The computation turns out to be very cumbersome and it seems hopeless to carry out all the steps without some additional assumption.
Motivated by the solutions 2. and 3. of the compatibility condition (6.10) it's natural to study bi-flat F-structures associated with two prescriptions of weights $\{\sigma_H\}_{H\in\mathcal{H}}$ as follows.
Recall that the number of orbits for the action of B_n on the collection of reflecting (hyper-)planes coincides with 2 for any n.
Let (p^i) be an Euclidean coordinate system for $\mathbb{R}^n$.
Let I be the orbit containing the (hyper-)planes $\{p^j = 0\}$, where $j = 1, ..., n$.
Let II be the orbit containing the remaining (hyper-)planes, i.e. $\{p^s - p^r = 0\}$ and $\{p^s + p^r = 0\}$, where $s, r = 1, ..., n$ with $r \neq r$.
So we consider the following choice of weights:

- $\sigma_H = 0$ if H is one of the (hyper-)planes belonging to the orbit I, otherwise $\sigma_H = 1$.
- $\sigma_H = 0$ if H is one of the (hyper-)planes belonging to the orbit II, otherwise $\sigma_H = 1$.

It turns out that the first choice of weights leads to a Frobenius manifold structure for $n = 3, 4$. Moreover, the corresponding prepotentials have the form

$$\boxed{F_{B_3}(u) = \frac{1}{6}(u^2)^3 + u^1u^2u^3 + \frac{1}{12}(u^1)^3u^3 - \frac{3}{2}(u^3)^2 + (u^3)^2log(u^3)}$$

$$\boxed{F_{B_4} = \frac{1}{108}(u^1)^4u^4 + \frac{1}{6}(u^1)^2u^2u^4 - \frac{1}{72}(u^2)^4 + u^1u^3u^4 + \frac{1}{2}(u^2)^2u^4 + \frac{1}{2}u^2(u^3)^2 - \frac{9}{4}(u^4)^2 + \frac{3}{2}(u^4)^2log(u^4)}$$

While the second choice does not produce any bi-flat structure, since the compatibility condition between ∇ and $\circ$ isn't fulfilled.

Remark 6.4 *The above solutions of WDVV can be obtained also from solutions of WDDV equations associated with extended affine Weyl groups of type A_n by a Legendre transformation. For instance, the details of the Legendre transformation between F_{B_2} and the prepotential associated with $A_1^{(1)}$ can be found in [44] while for details of the Legendre transformation between F_{B_3} and the prepotential associated with $A_2^{(1)}$ we refer to [17] and [51].*

In order to prove the existence of a Frobenius manifold structure on the orbit space of B_n for any $n > 0$, we will use a different strategy. The key observation is that in all the above examples ($n = 2, 3, 4$) the intersection form has always the same form, i.e.

$$g_{B_2} = \begin{pmatrix} 0 & \frac{1}{p^1p^2} \\ \frac{1}{p^1p^2} & 0 \end{pmatrix}$$

$$g_{B_3} = \begin{pmatrix} 0 & \frac{1}{p^1p^2} & \frac{1}{p^1p^3} \\ \frac{1}{p^1p^2} & 0 & \frac{1}{p^2p^3} \\ \frac{1}{p^1p^3} & \frac{1}{p^2p^3} & 0 \end{pmatrix}$$

$$g_{B_4} = \begin{pmatrix} 0 & \frac{1}{p^2p^1} & \frac{1}{p^1p^3} & \frac{1}{p^1p^4} \\ \frac{1}{p^1p^2} & 0 & \frac{1}{p^2p^3} & \frac{1}{p^2p^4} \\ \frac{1}{p^1p^3} & \frac{1}{p^2p^3} & 0 & \frac{1}{p^3p^4} \\ \frac{1}{p^1p^4} & \frac{1}{p^2p^4} & \frac{1}{p^3p^4} & 0 \end{pmatrix}$$

Then it's natural to consider the intersection form $g = (g^{ij})$ defined by

$$\boxed{g^{ij} := \frac{1-\delta^{ij}}{p^ip^j}}$$

Starting from g we will prove the existence of a flat pencil of cometrics and the existence of Frobenius manifold structure on the orbit space of B_n, for any $n > 0$. The proof relies on a suitable generalization of Dubrovin-Saito's construction. In particular, the proof of the existence of the Saito metric follows the scheme of [46], proposed by Saito-Yano-Sekiguchi, while the reconstruction of the Frobenius manifold structure requires overcoming some additional technical difficulties with respect to the standard procedure exposed in [15], by Dubrovin, due to the non-regularity of the corresponding (homogeneous) flat pencil.

7 A flat pencil of cometrics associated with B_n

The main reference of this section in [8].
Starting from the previous examples (B_2, B_3, B_4), whose prepotentials contain logarithmic terms, we generalize the construction of a Frobenius structure for any B_n.
In this section, following [8], we will see that, taking the Lie derivative of $g = (g^{ij})$ with respect the second highest degree invariant polynomial u_{n-1} we obtain a new cometric $\eta = (\eta^{ij})$ so that the pair (g, η) form a linear flat pencil of cometrics, which is also exact, satisfies the Egorov property and the homogeneity condition. First, we will prove some preliminary lemmas concerning g, which, in this setup, plays the role played by the Euclidean cometric in the standard case. Analogously to the Euclidean case, we will observe that g turns out to be B_n invariant and flat.

7.1 Invariance of g with respect to the action of B_n

Let (p^i) be a system of Euclidean coordinates for a n-dimensional real vector space V. First, we observe that

Lemma 7.1 *The metric of components*

$$g_{ij} := \left(\frac{1}{n-1} - \delta_{ij}\right) p^i p^j \tag{7.1}$$

and the cometric of components

$$g^{ij} := \frac{(1-\delta^{ij})}{p^i p^j} \tag{7.2}$$

are inverse to each other.

Proof: First we consider $g^{ki}g_{ik}$. Then one gets

$$g^{ki}g_{ik} = \sum_{i=1}^{n}\left(\frac{1}{n-1} - \delta_{ki}\right)(1-\delta^{ik})\frac{p^ip^k}{p^ip^k} = \sum_{i=1}^{n}\left(\frac{1}{n-1} - \delta_{ki}\right)(1-\delta^{ik}) =$$

$$\sum_{i,i\neq k}\left(\frac{1}{n-1} - \delta_{ki}\right) = \underbrace{\sum_{i,i\neq k}\frac{1}{n-1}}_{=\frac{1}{n-1}\sum_{i,i\neq k}1=1} - \underbrace{\sum_{i,i\neq k}\delta_{ki}}_{=0} = 1$$

Next we consider $g^{ki}g_{il}$, it turns out that

$$g^{ki}g_{il} = \sum_{i=1}^{n}\left(\frac{1}{n-1} - \delta_{ki}\right)\frac{p^l}{p^k}(1-\delta^{il}) = \sum_{i,i\neq l}\left(\frac{1}{n-1} - \delta_{ki}\right)\frac{p^l}{p^k} =$$

$$= \left(\sum_{i,i\neq l,k}\frac{1}{n-1}\frac{p^l}{p^k}\right) + \left(\frac{1}{n-1} - 1\right)\frac{p^l}{p^k} = \left(\frac{n-2}{n-1} - \frac{n-2}{n-1}\right)\frac{p^l}{p^k} = 0.$$

■

The next lemma shows that the metric introduced above is invariant under the action of B_n. Recall that the action induced by a Coxeter transformation on a cometric is given by the tensor transformation law of a $(2,0)$. Moreover, the invariance of the corresponding cometric follows immediately.

Proposition 7.2 *The metric $g = (g_{ij})$, where $g_{ij}(p) := \left(\frac{1}{n-1} - \delta_{ij}\right) p^i p^j$, is invariant under the action of B_n on $V \cong \mathbb{R}^n$.*

Proof: Recall that the action of B_n on V is generated by reflections with respect to the (hyper-)planes

$$\{p^j = 0\} \tag{7.3}$$

where $j = 1, ..., n$ and

$$\{p^i \pm p^j = 0\} \tag{7.4}$$

where $i, j = 1, \dots, n$ with $i < j$. We denote by A_{p^j} the Jacobian of the transformation associated with the reflection with respect to the hyperplane $\{p^j = 0\}$, and analogously for $A_{p^i \pm p^j}$.
We observe that the matrix A_{p^j} is a constant diagonal matrix with 1s on the main diagonal except in position (j, j) where there is -1. Under the action of the reflection with respect to the (hyper-)plane $\{p^j = 0\}$, the metric transforms as $A^T_{p^j} g A_{p^j} (p = \tilde{p})$ where g is the matrix associated to the metric, T denotes transposition and $p = \tilde{p}$ means that after the matrix operations have been completed, the metric is rewritten in terms of the new coordinates $p^i = \tilde{p}^i$ for $i \neq j$ and $p^j = -\tilde{p}^j$. Now, it is immediate to see that the action of A_{p^j} on g is to change the sign of all terms that contain p^j except the diagonal term $(\frac{1}{n-1} - 1)(p^j)^2$. Then once it is rewritten in terms of the coordinates $\tilde{p}$, the metric coincides with the original one.
As for the reflections with respect to the (hyper-)planes $\{p^i - p^j = 0\}$ we argue as follows. The matrix $A_{p^i - p^j}$ is a constant matrix with 1s on the main diagonal, except in position (i, i) and (j, j) where there is zero and it has 1 in position (i, j) and (j, i), while all the other entries are zero. Notice that $A^T_{p^i - p^j} = A_{p^i - p^j}$ and that $A_{p^i - p^j}$ is the matrix representation of a transposition. Therefore, when $A_{p^i - p^j}$ acts on the left on a column vector, it exchanges the positions of i-th and j-th components of the column vector but it leaves the other unchanged. Similarly, when $A_{p^i - p^j}$ acts on the right on a row vector, it exchanges the positions of i-th and j-th components of the row vector but it leaves the other unchanged. Thus, $A^T_{p^i - p^j} g A_{p^i - p^j} = A_{p^i - p^j} g A_{p^i - p^j}$ is obtained from g first exchanging the i-th and j-th rows and then exchanging the i-th and j-th columns (or first working with the columns and then with the rows) and leaving the rest unchanged. By the form of the columns and rows of g, after performing the change of variables $p^k = \tilde{p}^k$ when $k \neq i, j$, $p^i = \tilde{p}^j$ and $p^j = \tilde{p}^i$, $A^T_{p^i - p^j} g A_{p^i - p^j}$ coincides with g.
Reflections with respect to the hyperplane $\{p^i + p^j = 0\}$ are obtained as composition of reflections with respect to the hyperplanes $\{p^i = 0\}$, $\{p^j = 0\}$ and $\{p^i - p^j = 0\}$. To see this, just observe that the matrix $A_{p^i + p^j}$ is a constant matrix with 1s on the main diagonal except in positions (j, j) and (i, i) where there is 0, and it has -1 in positions (i, j) and (j, i). Therefore $A_{p^i + p^j} = A_{p^i} A_{p^j} A_{p^i - p^j}$. Now invariance follows from the previous paragraphs. The proposition is proved. ■

Definition 7.3 *We define the elementary symmetric polynomials* $\{f_1, ..., f_n\}$ *to be the functions, in the variables* $y^1, ..., y^n$*, define by*

$$f_k(y) := \sum_{1 \leqslant i_1 < .... < i_k \leqslant n} y^{i_1} .... y^{i_k} \tag{7.5}$$

where $k = 1, ..., n$*.*

Remark 7.4 *Let* u^i *be the polynomials defined by*

$$u^0 := 1 \tag{7.6}$$

$$u^i := f_i(p_1^2, \ldots, p_n^2) \tag{7.7}$$

where $i = 1, ..., n$ *and*

$$u^k := 0 \tag{7.8}$$

where $k > n$*.*
We can take $\{u_1, ..., u_n\}$ *as a set of basic invariant polynomials for* B_n*.*

Lemma 7.5 *The cometric* $g = (g^{ij})$*, defined by (7.2), can be written in terms of the invariant polynomials*

$$\tilde{g}^{ij}(u) = \frac{\partial u^i}{\partial p^k} \frac{\partial u^j}{\partial p^l} g^{kl}(p) \tag{7.9}$$

and it is well-defined on the quotient. Moreover, for any i *and* j*,* $g^{ij}(u)$ *is a homogeneous polynomial in the* p*-variables of degree* $2i + 2j - 4$*, which depends at most linearly on* u^{n-1}*. In particular,*

$$g^{11}(u) = 4(n^2 - n) \tag{7.10}$$

Proof: The homogeneity of the functions $g^{ij}(u)$, as functions of the p-variables, is clear.
Since all invariant polynomials are really polynomials in $(p^1)^2, \ldots, (p^n)^2$ no matter which ones we choose, then $\frac{\partial u^i}{\partial p^k}$ contains a factor p^k that cancels the factor p^k in the denominator of $g^{kl}(p)$ and similarly for $\frac{\partial u^j}{\partial p^l}$. Thus $g^{ij}(u)$ are polynomials in the p-variables, and since it is invariant by Proposition 7.2, it can be written in terms of the invariant polynomials, and thus it is well-defined on the quotient.
As u^i is a homogeneous polynomial in the p-variables of degree $deg(u^i) = 2i$, see (7.7). Furthermore, $deg(p^k p^l g^{kl}(p)) = 0$ for any k and l, see (7.2).
Then

$$deg(\tilde{g}^{ij}(u)) = 2i - 1 + 2j - 1 - 2 = 2i + 2j - 4, \tag{7.11}$$

as a function of the p-variables.
It turns out that $deg(u^{n-1}) = 2n - 2$.
Consider the entries $\tilde{g}^{ij}(u)$ above the anti-diagonal, i.e. for $i + j < n + 1$. We have

$$deg(\tilde{g}^{ij}(u)) = 2(i + j) - 4 < 2(n + 1) - 4 = 2(n - 1) \tag{7.12}$$

so those entries can not depend on u^{n-1}.
All the entries with (i, j) such that $n + 1 \leq i + j < 2n$ depend at most linearly

on u^{n-1}, since in this range we have $2n-2 \leq \deg(\tilde{g}^{ij}(u)) < 4n-4$. Finally, since $u^n = (p^1 \cdots p^n)^2$, it is immediate to see that each term in the sum (over k and l) $\tilde{g}^{nn}(u) = g^{kl}(p)\frac{\partial u^n}{\partial p^l}\frac{\partial u^n}{\partial p^k}$ contains u^n. Since $\deg(u^n) = 2n$ and $\deg(\tilde{g}^{nn}(u)) = 4n-4$, we can write $\tilde{g}^{nn}(u) = u^n f$, where f is polynomial in p of degree $2n-4$, so f can not contain u^{n-1}. This proves the claim. Now

$$\tilde{g}^{11}(u) = g^{kl}(p)\frac{\partial u^1}{\partial p^k}\frac{\partial u^1}{\partial p^l} = \sum_{k,l=1,\dots,n} \frac{(1-\delta^{kl})}{p^k p^l} 2p^k 2p^l =$$

$$4\sum_{k,l=1,\dots,n}(1-\delta^{kl}) = 4(n^2-n),$$

thus proving (7.10). ■

7.2 Flatness of g

Recall that the Christoffel symbols, written in the local coordinates (p^i), corresponding to the Levi-Civita connection ∇ defined by the metric g are the functions

$$\Gamma^k_{ij}(p) = \frac{1}{2}\sum_{m=1}^{n} g^{mk}\left(\frac{\partial g_{im}}{\partial p^j} + \frac{\partial g_{jm}}{\partial p^i} - \frac{\partial g_{ij}}{\partial p^m}\right), \tag{7.13}$$

On the other hand, the contravariant Christoffel symbols associated with ∇ are defined by

$$\Gamma^{ij}_k(p) := -\sum_{s=1}^{n} g^{is}(p)\Gamma^j_{sk}(p) \tag{7.14}$$

We have the following:

Lemma 7.6 *Let g the metric defined by (7.1), then the corresponding Christoffel symbols read*

$$\Gamma^i_{ii}(p) = \frac{1}{p^i} \quad \text{and} \quad \Gamma^k_{ij}(p) = 0 \quad \text{otherwise.} \tag{7.15}$$

Proof: In the following proof all the metric coefficients and all Christoffel symbols depend only on the p-variables. To prove (7.15), first one computes

$$\begin{aligned}\frac{\partial g_{im}}{\partial p^j} &\overset{(7.1)}{=} \frac{\partial}{\partial p^j}\left[\left(\frac{1}{n-1} - \delta_{im}\right)p^i p^m\right] \\ &= \left(\frac{1}{n-1} - \delta_{im}\right)(\delta_{ji}p^m + \delta_{jm}p^i) \\ &= g_{im}\left(\frac{\delta_{ij}}{p^i} + \frac{\delta_{mj}}{p^m}\right)\end{aligned}$$

This yields

$$g^{mk}\left(\frac{\partial g_{im}}{\partial p^j} + \frac{\partial g_{jm}}{\partial p^i} - \frac{\partial g_{ij}}{\partial p^m}\right) =$$

$$g^{mk}\left[g_{im}\left(\frac{\delta_{ij}}{p^i} + \frac{\delta_{mj}}{p^m}\right) + g_{jm}\left(\frac{\delta_{ij}}{p^j} + \frac{\delta_{mi}}{p^m}\right) - g_{ij}\left(\frac{\delta_{im}}{p^i} + \frac{\delta_{mj}}{p^j}\right)\right]$$

which inserted in (7.13) gives

$$\begin{aligned}\Gamma^k_{ij} &= \frac{\delta_{ij}}{2}\left[\frac{1}{p^i}\sum_{m=1}^{n} g^{mk}g_{im} + \frac{1}{p^j}\sum_{m=1}^{n} g^{mk}g_{jm}\right]\\ &+ \frac{1}{2}\left[\sum_{m=1}^{n} g^{mk}g_{im}\frac{\delta_{mj}}{p^m} + \sum_{m=1}^{n} g^{mk}g_{jm}\frac{\delta_{mi}}{p^m}\right]\\ &- \frac{g_{ij}}{2}\left[\frac{1}{p^i}\sum_{m=1}^{n} g^{mk}\delta_{im} + \frac{1}{p^j}\sum_{k=1}^{n} g^{mk}\delta_{mj}\right]\\ &= \frac{\delta_{ij}}{2}\left(\frac{\delta_{ik}}{p^i}+\frac{\delta_{kj}}{p^j}\right) + \frac{1}{2}\left(\cancel{\frac{g^{jk}g_{ij}}{p^j}} + \cancel{\frac{g^{ik}g_{ij}}{p^i}}\right) - \frac{1}{2}\left(\cancel{\frac{g_{ij}g^{jk}}{p^i}} + \cancel{\frac{g_{ij}g^{jk}}{p^j}}\right)\end{aligned}$$

i.e.

$$\Gamma^k_{ij} = \frac{\delta_{ij}}{2}\left(\frac{\delta_{ik}}{p^i}+\frac{\delta_{jk}}{p^j}\right)$$

which entails the thesis. ■

Proposition 7.7 *The metric (7.1) is flat.*

Proof: This can be proved by direct computation of the curvature tensor using the Christoffel symbols (7.15). A quicker way to do this is to introduce the connection 1-form

$$\omega^i_j := \Gamma^i_{jk}dp^k$$

and the corresponding curvature 2-form

$$\Omega^i_j := d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

In view of (7.15) one has that

$$\omega^i_i = \Gamma^i_{ik}dp^k \overset{(7.15)}{=} \frac{dp^i}{p^i} = d(log(p^i)) \tag{7.16}$$

and

$$\omega^i_j = \Gamma^i_{jk}dp^k \overset{(7.15)}{=} 0 \tag{7.17}$$

for $i \neq j$. Then

$$\Omega^i_i = d\omega^i_i + \underbrace{\omega^i_k \wedge \omega^k_i}_{\overset{(7.17)}{=}\omega^i_i\wedge\omega^i_i} \overset{(7.16)}{=} \underbrace{d^2(log(p^i))}_{=0} + \underbrace{d(log(p^i)) \wedge d(log(p^i))}_{=0} = 0$$

and

$$\Omega^i_j = \underbrace{d\omega^i_j}_{\overset{(7.17)}{=}0} + \underbrace{\omega^i_k \wedge \omega^k_j}_{\overset{(7.17)}{=}\omega^i_i\wedge\omega^i_j\overset{(7.17)}{=}0} = 0$$

for $i \neq j$.
Recall that the following formula holds true

$$\big(R(X,Y)\big)^j_i = \Omega^j_i(X,Y)$$

where X and Y are arbitrary vector fields and $R = (R^s_{ijk})$ is the curvature tensor corresponding to the Christoffel symbols Γ^k_{ij}.
Thus the vanishing of the curvature two-form entails the vanishing of the curvature tensor. A third way to prove the flatness of g is to observe that the connection defined by (7.15) is a logarithmic connection with weights that are invariant under the action of B_n (see Example 2.5 in [12]). ■

Remark 7.8 *In the flat local coordinates $y^i = \frac{(p^i)^2}{2}$ the cometric (7.2) reads*

$$\tilde{g}^{ij}(y) = \frac{\partial y^i}{\partial p^s}\frac{\partial y^j}{\partial p^k} g^{sk}(p) = 1 - \delta^{sk} \tag{7.18}$$

Lemma 7.9 *The cometric (7.18) is invariant under the action of A_n on V.*

7.3 Definition of η

In this section, we introduce a new cometric η defined as the Lie derivative of the cometric g with respect to the second highest degree basic invariant polynomial for B_n. First, recall the following:

Lemma 7.10 *Let*

$$f(z) := \sum_{i=1}^{n} f_i z^i \tag{7.19}$$

$$g(z) := \sum_{i=1}^{n} g_i z^i \tag{7.20}$$

be two complex polynomials of degree at most n, then the following identity holds true:

$$\frac{f(x)g(y) - f(x)g(y)}{x - y} = \sum_{i,j=0}^{n} b_{ij} x^i y^j \tag{7.21}$$

$$b_{ij} := \sum_{k=0}^{m_{ij}} (u_{j+k+1} v_{i-k} - u_{i-k} v_{j+k+1}) \tag{7.22}$$

here $m_{ij} := min\{i, n-1-j\}$.

Proposition 7.11 *The partial derivative with respect to the vector field $\frac{\partial}{\partial u^{n-1}}$ of the intersection form $\left(g^{ij}(u)\right)$ is given by the formula*

$$\eta^{ij}(u) := \frac{\partial g^{ij}(u)}{\partial u^{n-1}} = 4(2n - i - j)u^{i+j-n-1}. \tag{7.23}$$

Hence $\left(\eta^{ij}(u)\right)$ is a non-degenerate Hankel matrix with all vanishing entries above the anti-diagonal. In particular, the entries of the anti-diagonal, i.e. $i + j = n + 1$, are

$$\eta^{i,n-i+1}(u) = 4(n-1).$$

Proof: Let's define

$$h(x) := \sum_{k=0}^{n} u^k x^{n-k} = \prod_{l=1}^{n} (x + (p^l)^2) \tag{7.24}$$

where the former equality holds true in view of the Vieta's formula.
One has

$$g^{ij}(u) = \sum_{s,k=1}^{n} \frac{(1-\delta^{sk})}{p^s p^k} \frac{\partial u^i}{\partial p^s} \frac{\partial u^j}{\partial p^k},$$

one has

$$\begin{aligned}
\frac{1}{4}\sum_{i,j=1}^{n} g^{ij}(u) x^{n-i} y^{n-j} &= \frac{1}{4}\sum_{i,j=1}^{n}\sum_{s,k=1}^{n} \frac{(1-\delta^{sk})}{p^s p^k} \frac{\partial u^i}{\partial p^s} \frac{\partial u^j}{\partial p^k} x^{n-i} y^{n-j} \\
&= \frac{1}{4}\sum_{s,k=1}^{n} \frac{(1-\delta^{sk})}{p^s p^k} \frac{\partial}{\partial p^s}\left(\sum_{i=1}^{n} u^i x^{n-i}\right) \frac{\partial}{\partial p^k}\left(\sum_{j=1}^{n} u^j y^{n-j}\right) \\
&\overset{u^0=1}{=} \frac{1}{4}\sum_{s,k=1}^{n} \frac{(1-\delta^{sk})}{p^s p^k} \frac{\partial}{\partial p^s}\left(\sum_{i=0}^{n} u^i x^{n-i}\right) \frac{\partial}{\partial p^k}\left(\sum_{j=0}^{n} u^j y^{n-j}\right) \\
&\overset{(7.24)}{=} \frac{1}{4}\sum_{s,k=1}^{n} \frac{(1-\delta^{sk})}{p^s p^k} \frac{\partial h(x)}{\partial p^s} \frac{\partial h(y)}{\partial p^k}.
\end{aligned}$$

Since

$$\frac{\partial h(x)}{\partial p^s} = \frac{\partial}{\partial p^s} \prod_{l=1}^{n} (x + (p^l)^2) = 2p_s \prod_{l \neq s}^{n} (x + (p^l)^2),$$

then

$$\begin{aligned}
\frac{1}{4}\sum_{s,k=1}^{n} \frac{1}{p^s p^k} \frac{\partial h(x)}{\partial p^s} \frac{\partial h(y)}{\partial p^k} &= \sum_{s,k=1}^{n} \prod_{l \neq s}^{n} (x + (p^l)^2) \prod_{q \neq k}^{n} (y + (p^q)^2) \\
&= \sum_{s=1}^{n} \prod_{l \neq s}^{n} (x + (p^l)^2) \left(\sum_{k=1}^{n} \prod_{q \neq k}^{n} (y + (p^q)^2) \right) \\
&= h'(x) h'(y)
\end{aligned}$$

and

$$
\begin{aligned}
-\frac{1}{4}\sum_{s,k=1}^{n}\frac{\delta^{sk}}{p^sp^k}\frac{\partial h(x)}{\partial p^s}\frac{\partial h(y)}{\partial p^k} &= -\frac{1}{4}\sum_{k=1}^{n}\frac{1}{(p^k)^2}\frac{\partial h(x)}{\partial p^k}\frac{\partial h(y)}{\partial p^k}\\
&= -\sum_{k=1}^{n}\prod_{l\neq k}^{n}(x+(p^l)^2)\prod_{q\neq k}^{n}(x+(p^q)^2)\\
&= -\sum_{k=1}^{n}\frac{h(x)h(y)}{(x+(p^k)^2)(y+(p^k)^2)}\\
&= -\sum_{k=1}^{n}\Big[\frac{-h(x)h(y)}{(x-y)(x+(p^k)^2)}+\frac{h(x)h(y)}{(x-y)(y+(p_k)^2)}\Big]\\
&= \frac{1}{x-y}\left(\left(\sum_{k=1}^{n}\frac{h(x)}{x+(p^k)^2}\right)h(y)-\left(\sum_{k=1}^{n}\frac{h(y)}{y+(p^k)^2}\right)h(x)\right)\\
&= -\frac{h'(y)h(x)-h'(x)h(y)}{x-y},
\end{aligned}
$$

which yields

$$
\frac{1}{4}\sum_{i,j=1}^{n}g^{ij}(u)x^{n-i}y^{n-j} = h'(x)h'(y)-\frac{h'(y)h(x)-h'(x)h(y)}{x-y}.
$$

Since $h'(x)=\sum_{k=0}^{n-1}(n-k)u^kx^{n-k-1}$ (see (7.24)) deriving both sides of the previous identity with respect to u^{n-1} we obtain

$$
\begin{aligned}
\frac{1}{4}\sum_{i,j=1}^{n}\underbrace{\frac{\partial g^{ij}(u)}{\partial u^{n-1}}}_{:=\eta^{ij}(u)}x^{n-i}y^{n-j} &= \frac{\partial}{\partial u^{n-1}}\left(h'(x)h'(y)-\frac{h'(y)h(x)-h'(x)h(y)}{x-y}\right)\\
&= h'(y)+h'(x)-\frac{1}{x-y}\Big(-h(y)-yh'(x)+h(x)+xh'(y)\Big)\\
&= \frac{h(y)-h(x)+xh'(x)-yh'(y)}{x-y}.
\end{aligned}
$$

Now, applying the Lemma (7.10) one gets the thesis. ∎

Remark 7.12 *From now on, since we want $\eta^{i,n-i+1}(u)=1$ for all i, we normalize the cometric g^{ij} dividing it by $4(n-1)$. Thus, using (7.10) we have that*

$$
g^{11}(u)=n \tag{7.25}
$$

Remark 7.13 *We observe that the matrix (η^{ij}), defined by (7.23), is lower anti-triangular. Moreover, in view of the formula (7.23), each entry is a polynomial function in (u^i) and its determinant is a non-vanishing constant. Then we have the following:*

Lemma 7.14 *The metric $\eta^{-1}=(\eta_{ij})$ depends polynomially on the (u^i) as well. Moreover, (η_{ij}) is also lower anti-triangular.*

Proof: Let $p_\eta(\lambda)$ be the characteristic polynomial of the matrix associated with η. It is a polynomial in λ with coefficients that are polynomials in the entries of η and thus they are polynomials in the u's. By Cayley-Hamilton theorem, $p_\eta(\eta) = 0$ identically, then

$$p_\eta(\eta) = \eta^n + c_{n-1}\eta^{n-1} + \cdots + c_1\eta + c_0\mathbb{I}$$

where $c_0 = (-1)^n \det(\eta)$ and $\mathbb{I}$ denotes the identity matrix. From this, we get immediately

$$\eta^{-1} = \frac{(-1)^{n-1}}{\det(\eta)}(\eta^{n-1} + c_{n-1}\eta^{n-2} + \cdots + c_1\mathbb{I}),$$

from which it is clear that the entries of η^{-1} are polynomials in the us, since $\det(\eta)$ is a non-zero constant and all the other terms depend on the us as polynomials. To show that it is also lower anti-triangular, it is enough to observe that every lower anti-triangular matrix can be obtained as a product LA of two matrices, where L is lower triangular and A is the matrix with all ones on the anti-diagonal and zero in the other entries. Furthermore, it is well-known that the inverse of a lower triangular matrix is lower triangular while the inverse of A coincides with A. This immediately shows that η^{-1} is also lower anti-triangular. ∎

7.4 The pair (g, η) is a flat pencil of metrics

Recall the definition of flat pencil of metrics:

Definition 7.15 *A pair of metrics $(g_{(1)}, g_{(2)})$ forms a flat pencil if the following conditions hold true:*

- *The metric*

$$g := g_{(1)} + \lambda g_{(2)}$$

  is a flat metric for all λ.

- *The contravariant Christoffel symbols Γ^{ij}_k of the metric g are of the form*

$$\Gamma^{ij}_k = \Gamma^{ij}_{k(1)} + \lambda\Gamma^{ij}_{k(2)}$$

  for any λ.

In this subsection, we will show that the pair (g, η), where g and η are defined in (7.9) and (7.23), respectively, gives rise to a flat pencil of cometrics on the orbit space $\mathbb{C}^n/B_n$.
Our proof is based on the Lemma 4.30. Recall that

Proposition 7.16 *If for a flat metric g on some coordinate system $(x^1, ..., x^n)$ both the components $\left(g^{ij}(x)\right)$ of the metric g and the contravariant components $\Gamma^{ij}_k(x)$ of the associated Levi-Civita connection depend at most linearly on the variable x^1, then $g_1 := g$ and g_2 defined by*

$$g_2^{ij}(x) := \partial_{x^1} g^{ij}(x)$$

form a flat pencil if $\det(g_2^{ij}(x)) \neq 0$. *Moreover, the contravariant components of the corresponding Levi-Civita connections are*

$$\Gamma^{ij}_{k(1)}(x) := \Gamma^{ij}_k(x)$$

and

$$\Gamma^{ij}_{k(2)}(x) := \partial_{x^1}\Gamma^{ij}_k(x)$$

Remark 7.17 *As a system of coordinates on* $\mathbb{C}^n/B_n$ *we choose the set of basic invariants* $(u^1, \ldots, u^n)$ *defined by* (7.7). *Under this assumption, Lemma 7.5 entails that the cometric defined in* (7.2) *descends to a metric on the quotient space having the properties required in the Proposition 7.16, where the role of* x^1 *is played by* u^{n-1}. *To conclude the proof, we are left to prove that the contravariant components* $\Gamma^{ij}_k(u)$ *of the Levi-Civita connection defined by* g *satisfy the conditions stated in Proposition 7.16. More precisely we will prove that*

Proposition 7.18 *The contravariant components of the Levi-Civita connection defined by* g *are polynomial functions of* $(u^1, \ldots, u^n)$, *moreover, they depend at most linearly on* u^{n-1}.

We will split the proof of this proposition into two lemmas.

Lemma 7.19 *The contravariant components of the Levi-Civita connections defined by* g, *written in the coordinates* $(u^1, ..., u^n)$, *are polynomial functions of* $(u^1, ..., u^n)$.

Proof: Denote by $\Gamma^i_{jk}(p)$ the Christoffel symbols in the p-variables and by $\Gamma^i_{jk}(u)$ the Christoffel symbols in the u-variables. One has that the transformation law for the Christoffel symbols induced by the chance of coordinates $p \mapsto u$ is given by the formula

$$\Gamma^l_{ij}(p) = \frac{\partial p^l}{\partial u^c}\frac{\partial^2 u^c}{\partial p^i \partial p^j} + \frac{\partial p^l}{\partial u^c}\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u). \tag{7.26}$$

Multiplying both sides of (7.26) by $g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}dp^j$, we obtain

$$g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\Gamma^l_{ij}(p)dp^j = g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\partial p^l}{\partial u^c}\frac{\partial^2 u^c}{\partial p^i \partial p^j}dp^j + g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\partial p^l}{\partial u^c}\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u)dp^j$$

Now, observe that in the two terms of the right-hand side of the above expression $\frac{\partial u^d}{\partial p^l}\frac{\partial p^l}{\partial u^c} = \delta^d_c$, so it simplifies to

$$g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\Gamma^l_{ij}(p)dp^j = g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j + \underbrace{g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial u^a}{\partial p^i}}_{=g^{fa}(u)}\Gamma^d_{ab}(u)du^b.$$

Using the definition of contravariant Christoffel symbols one gets

$$-\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\Gamma^{kl}_j(p)dp^j = g^{ki}(p)\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j - \Gamma^{fd}_b(u)du^b \tag{7.27}$$

The contravariant Christoffel symbols $\Gamma_l^{ik}(p) = -g^{im}(p)\Gamma_{ml}^k(p)$, using the formulas (7.15), read

$$\Gamma_l^{ik}(p) \overset{(7.15)}{=} -\frac{(1-\delta^{im})}{p^i p^k p^m}\delta_{km}\delta_{kl} = \frac{(\delta^{ki}-1)\delta_{kl}}{p^i (p^k)^2}$$

which, inserted in (7.27), yields

$$\Gamma_b^{fd}(u)du^b = \frac{(1-\delta^{ki})}{p^i p^k}\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j + \frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{(\delta^{kl}-1)\delta_{lj}}{p^k (p^l)^2}dp^j. \tag{7.28}$$

Rearranging the right-hand side of (7.28) one obtains

$$\sum_{k,i,j,k\neq i}\frac{1}{p^i p^k}\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j - \sum_{k,l,j,k\neq l}\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^l}\frac{\delta_{lj}}{p^k (p^l)^2}dp^j$$

Splitting the first sum in j in $j = i$ and $j \neq i$ one has

$$\sum_{k,j,k\neq j}\frac{1}{p^j p^k}\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{(\partial p^j)^2}dp^j + \sum_{k,i,j,k\neq i,j\neq i}\frac{1}{p^i p^k}\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j - \sum_{k,j,k\neq j}\frac{\partial u^f}{\partial p^k}\frac{\partial u^d}{\partial p^j}\frac{1}{p^k (p^j)^2}dp^j$$

which can be written as

$$\sum_{k,j,k\neq j}\frac{1}{p^j p^k}\frac{\partial u^f}{\partial p^k}\left(\frac{\partial^2 u^d}{(\partial p^j)^2} - \frac{\partial u^d}{\partial p^j}\frac{1}{p^j}\right)dp^j + \sum_{k,i,j,k\neq i,j\neq i}\frac{1}{p^i p^k}\frac{\partial u^f}{\partial p^k}\frac{\partial^2 u^d}{\partial p^i \partial p^j}dp^j \tag{7.29}$$

Recall that

$$u^k = \sum_{1\leq i_1<\cdots<i_k\leq n}(p^{i_1}\cdots p^{i_k})^2$$

It is immediate to check that first term of (7.29) vanishes identically, since $u^1, \dots, u^n$ are polynomials of degree 1 in each of the $(p^i)^2$ (i.e. each monomial has degree 1 or 0 in $(p^i)^2$). Furthermore, the second term does not contain any denominator, since they are simplified (unless $d = 1$ in which case the second term is identically zero). Hence one obtains that

$$\Gamma_b^{rs}(u)du^b = \sum_{k,i,j,k\neq i,j\neq i}\frac{1}{p^i p^k}\frac{\partial u^r}{\partial p^k}\frac{\partial^2 u^s}{\partial p^i \partial p^j}dp^j \tag{7.30}$$

whose right-hand side is a 1-form with polynomial coefficients in the p-variables. We conclude the proof as follows. Since the left-hand side of (7.30) is B_n-invariant (see [50, Theorem page 3]), the right-hand side is so. Now, as the latter is a 1-form with polynomial coefficients, the coefficients of the left-hand side are necessarily polynomial functions in $(u^1, \dots, u^n)$. ■

Remark 7.20 *The previous argument is the same used in the proof Lemma 2.1 in [18]. However, while it is evident that the left-hand side of Formula (2.8) in [18] is a 1-form with polynomial coefficients, the polynomiality of the coefficients of the right-hand side of (7.30) was not so and it needed to be shown.*

To complete the proof of Proposition 7.18, we are left to show that the contravariant components of the Levi-Civita connection of g depend at most linearly on u^{n-1}. This result follows from the following:

Lemma 7.21 *For any choice of indexes, the following inequality holds true*

$$deg(\Gamma^{si}_{k}(u)) < 4n-4. \tag{7.31}$$

Proof: First we will show that for every choice of the indices

$$deg(\Gamma^{c}_{ab}(u)) = deg(u^c) - deg(u^a) - deg(u^b) \tag{7.32}$$

To this end, we start noticing that, if not all the indixes in the left-hand side of (7.26) are equal, (7.15) implies

$$\frac{\partial p^l}{\partial u^c}\frac{\partial^2 u^c}{\partial p^i \partial p^j} + \frac{\partial p^l}{\partial u^c}\frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u) = 0,$$

which yields

$$\frac{\partial^2 u^c}{\partial p^i \partial p^j} + \frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^j}\Gamma^c_{ab}(u) = 0.$$

This identity, together with the definition of the invariants $(u^1, ..., u^n)$, implies that $\Gamma^c_{ab}(u)$ is a homogeneous polynomial of degree

$$deg(u^c) - 2 + deg(u^a) - 1 + deg(u^b) - 1 + deg(\Gamma^c_{ab}(u))$$

or, equivalently, that

$$deg(\Gamma^c_{ab}(u)) = deg(u^c) - deg(u^a) - deg(u^b)$$

On the other hand, if in (7.26) $i = j = l$, (7.15) entails

$$\frac{\partial u^c}{\partial p^i}\frac{1}{p^i} = \frac{\partial^2 u^c}{\partial^2 p^i} + \frac{\partial u^a}{\partial p^i}\frac{\partial u^b}{\partial p^i}\Gamma^c_{ab}(u)$$

which implies that

$$deg(u^c) - 2 = deg(\Gamma^c_{ab}(u)) + deg(u^a) + deg(u^b) - 2$$

or, equivalently, that

$$deg(\Gamma^c_{ab}(u)) = deg(u^c) - deg(u^a) - deg(u^b)$$

proving (7.32). To conclude the proof of the lemma, it suffices to note since $\Gamma^{si}_k(u) = -g^{sj}(u)\Gamma^i_{jk}(u)$ and $deg(u^i) = 2i$ (for all $i = 1, \dots, n$), one has

$$\begin{aligned} deg\big(\Gamma^{si}_k(u)\big) &= deg\big(g^{sj}(u)\big) + deg\big(\Gamma^i_{jk}(u)\big) \\ &\overset{(7.11)}{=} \deg(u^s) + deg(u^j) - 4 + deg(\Gamma^i_{jk}(u)) \\ &\overset{(7.32)}{=} deg(u^s) - 4 + deg(u^i) - deg(u^k) \\ &= 2s + 2i - 2k - 4 \leqslant 4n - 6 < 4n - 4 = 2(2n-2) \end{aligned}$$

∎

Remark 7.22 *One has that*

$$deg\big(\Gamma_k^{si}(u)\big) = 2s + 2i - 2k - 4 \tag{7.33}$$

for any choice of indexes.

Corollary 7.23 *Since $deg(u^{n-1}) = 2n-2$, it follows from Lemma 7.21 that the functions $\Gamma_k^{si}(u)$ depend at most linearly on u^{n-1}, for any choice of the indexes.*

Summarizing, we get the following:

Theorem 7.24 *The pair (g, η) gives rise to a flat pencil of metrics.*

Proof: The metric $\big(g^{ij}(u)\big)$ is well-defined on the quotient, it depends at most linearly on u^{n-1} by Lemma 7.5 and it is flat by Proposition 7.7. Furthermore, its contravariant Christoffel symbols are also polynomial functions that depend at most linearly on u^{n-1} by Proposition 7.18. Therefore, since $\eta^{ij}(u) := \frac{\partial g^{ij}}{\partial u^{n-1}}(u)$ has non-zero constant determinant by Proposition 7.11, (g, η) forms a flat pencil of metrics by Proposition 7.16. ■

Corollary 7.25 *The cometric (η^{ij}) is flat.*

Proof: Since (g, η) form a flat pencil, by applying the Lemma 3.2 one gets that η is flat. ■

We close this subsection with a result that will play a crucial role to prove the existence of a Dubrovin-Frobenius structure on the orbit space $\mathbb{C}^n/B_n$.
First, recall some notions regarding the linear Pfaffian systems (see [30] for details). Denote by $(x^1, ..., x^n)$ a standard coordinate system of $\mathbb{C}^n$.
Let $X \subset \mathbb{C}^n$ be an open domain of $\mathbb{C}^n$ and $a_{ij}^k(x)$ be holomorphic functions on X, where $i, j = 1, ..., N$ and $j = 1, ..., n$ with N positive integer. The first-order (overdetermined) linear system

$$\frac{\partial u_i}{\partial x^j} = \sum_{k=1}^{N} a_{ij}^k(x) u_k \tag{7.34}$$

of partial differential equation, with unknown functions $\{u_1(x), ..., u_N(x)\}$, is called linear Pfaffian system.
Let u to be the unknown column vector

$$u = \begin{pmatrix} u_1 \\ . \\ . \\ . \\ . \\ u_N \end{pmatrix} \tag{7.35}$$

and let $\{A_j(x)\}_{j=1,....,n}$ to be the matrices of functions $A_j(x) := \big(a_{ij}^k(x)\big)$. So the system (7.34) can be written in the form

$$\frac{\partial u}{\partial x^j} = A_j(x) u$$

Introduce the matrix $\Omega = (\Omega^i_j)$ of differential 1-form defined by

$$\Omega := \sum_{k=1}^{n} A_k(x)dx^k$$

Then the system (7.4) can be written in terms of 1-forms as

$$du = \Omega u \tag{7.36}$$

Thus a linear Pfaffian system is also called a total differential equation.
For arbitrarily matrices $\{A_j(x)\}_{j=1,....,n}$ there may not exist solutions in general.

Remark 7.26 *We observe that a holomorphic solution $u(x)$ of (7.36) is C^∞ in $\mathbb{C}^n$, then in particular is C^2. Thus we have*

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \frac{\partial^2 u}{\partial x^j \partial x^i} \tag{7.37}$$

for any choice of indexes.

Definition 7.27 *For a linear Pfaffian system (7.34), (7.37) is called the integrability (or compatibility) condition. Moreover, if (7.37) holds true, the system (7.34) is called completely integrable.*

Remark 7.28 *The integrability condition (7.37) can be written in terms of Ω as a flatness condition*

$$d\Omega = \Omega \wedge \Omega \tag{7.38}$$

Indeed, using the properties of exterior derivative one has that

$$0 = d^2u = d(\Omega u) = d\Omega u - \underbrace{\Omega \wedge du}_{=(\Omega\wedge\Omega)u} = (d\Omega - \Omega \wedge \Omega)u$$

Theorem 7.29 *Assume that the linear Pfaffian system (7.34) satisfies the integrability condition (7.37). Then, for any $a \in X$ and any $u_0 \in \mathbb{C}^N$, there exists a unique solution $u(x)$ such that $u(0) = u_0$. Moreover, the power series expansion of the solution $u(x)$ converges in any polydisk centered at a and contained in X.*

Remark 7.30 *By interpreting Ω as a curvature 1-form, equation (7.38) reads as a zero-curvature (flatness) condition.*

We state a modified version of the Corollary 2.4 in [18].

Proposition 7.31 *There exists a set of B_n-invariant homogeneous polynomials*

$$\{t^1(p), ..., t^n(p)\}$$

with $deg(t^k(p)) = 2k$, such that (η^{ij}) reduces to a constant matrix in the coordinates $(t^1, ..., t^n)$.

Proof: In [18] the existence of a flat and homogeneous coordinate system was proven, for all Coxeter groups, taking as g the inverse of the standard Euclidean metric on $\mathbb{R}^n$.
Let ∇ be the Levi-Civita connection of (η^{ij}).
Recall that a flat function t is determined by the following fundamental (linear) system of equations

$$\partial_k \xi_j - \gamma^s_{kj} \xi_s = 0 \tag{7.39}$$

where $\partial_s = \frac{\partial}{\partial u^s}$, $\xi_j = \partial_j t$ and $\gamma^s_{kj} = \gamma^s_{kj}(u)$ are the contravariant Christoffel symbols corresponding to ∇. Recall that $\gamma^s_{kj}(u) = -\eta_{kq}(u)\gamma^{qj}_s(u)$. Now, because of the lemmmas (7.14) and (7.19), the functions $\gamma^s_{kj}(u)$ are polynomials in $(u^1, ..., u^n)$.
The flatness of (η^{ij}) yields the integrability condition $\partial_i \partial_k \xi_l = \partial_k \partial_i \xi_l$. Now since the linear (7.39) is completely integrable, one may apply Darboux's theorem: the space of solutions of (7.39) is a linear vector space of dimension n, i.e. a general solution is of the form

$$\xi_i = \sum_{\alpha=1}^{n} c_\alpha \xi_i^{(\alpha)}$$

where $\{c_\alpha\}$ are constant and $\{\xi^{(\alpha)} = (\xi_i^{(\alpha)})\}$ are linearly independent fundamental solutions of (7.39), i.e. $\xi_i^{(\alpha)}(0) = \delta_i^\alpha$. Since γ^s_{kj} are polynomials in $(u^1, ..., u^n)$, they are also analytic functions in $(u^1, ..., u^n)$. Then applying the Theorem (7.29), one concludes that $\{\xi^{(\alpha)}\}$ are analytic functions on a polydisk. We define the flat coordinate system $(t^1, ..., t^n)$ by

$$dt^\alpha = \xi^{(\alpha)}$$

where $\alpha = 1, ..., n$ and so that $t^\alpha(0) = 0$. Here the integrability of the system is ensured by the exactness of the 1-forms $\{\xi^{(1)}, ..., \xi^{(n)}\}$ (that follows from the vanishing of the torsion of ∇). Moreover, the functional independence of $(t^1, ..., t^n)$ follows, by definition, from the linear independence of the 1-forms $\{\xi^{(1)}, ..., \xi^{(n)}\}$.
It's clear that the functions (t^i) are analytic as well.
We have to show that the system of solutions $\big(t^i(u)\big)$ is invariant with respect to the scaling transformations

$$u^i \mapsto c^{d_i} u^i$$

where $i = 1, ..., n$, $d_i := deg(u^i)$ and c is a constant.
Denote

$$\tilde{t}(u^1, ..., u^n) := t(c^{d_1}u^1, ..., c^{d_n}u^n)$$

Using the chain rule one has

$$\partial_i\partial_j\tilde{t}(u^1, ..., u^n)+\gamma^k_{ij}\partial_k\tilde{t}(u^1, ..., u^n) = c^{d_i}c^{d_j}\tilde{\partial}_i\tilde{\partial}_j t(\tilde{u}^1, ..., \tilde{u}^n)+c^{d_k}\gamma^k_{ij}(u^1, ..., u^n)\tilde{\partial}_k t(\tilde{u}^1, ..., \tilde{u}^n) \tag{7.40}$$

where $\tilde{u^i} := c^{d_i}u^i$ and $\tilde{\partial}_i = \frac{\partial}{\partial \tilde{u}^i}$. Recall that the functions $\gamma^k_{ij}(u)$ are quasi-homogeneous polynomials in (u^i) of degrees $deg\big(\gamma^k_{ij}(u)\big) = d_k - d_i - d_j$, then

$$c^{d_k - d_i - d_j}\gamma^k_{ij}(u^1, ..., u^n) = \gamma^k_{ij}(c^{d_i}u^1, ..., c^{d_n}u^n) \tag{7.41}$$

therefore (7.40) vanish identically. So $(t^i(u))$ are quasi-homogeneous functions in $(u^1, ..., u^n)$.
Since $\partial_i t^j(0) = \delta_i^j$ and $t(0) = 0$ the flat coordinates have the form

$$t^i = u^i + f(u^1, ..., \hat{u}^i, ..., u^n)$$

where f is a quasi-homogeneous function of the same degree of u^i. Then $(t^1, ..., t^n)$ has positive degrees $(d^1, ..., d^n)$ respectively.
Since all the degrees are positive, the power series expansion of $t^i(u)$ must be a quasi-homogeneous polynomial in $(u^1, ..., u^n)$. Now, in view of the invertibility of the transformation $u^i \mapsto p^j$ we conclude that $t^i(u(p))$ are homogeneous polynomials in $(p^1, ..., p^n)$ of degrees $deg(t^i(p)) = deg(u^i) = 2i$. ■

Lemma 7.32 *In our case, the flat coordinates of Proposition 7.31 can be further chosen so that:*

$$\eta^{ij}(t) = \delta_{i,n+1-j} \tag{7.42}$$

The coordinates so defined are called Dubrovin-Saito flat coordinates.

Proof: By Proposition 7.31 flat coordinates for (η^{ij}) are homogenous invariant polynomials with distinct degrees. Therefore, in order to prove the claim of the Lemma, by Corollary 1.1 in [17] it is enough to show that there exists a system of flat coordinates $(t^1, ..., t^n)$ such that $\eta_{nn}(t) = 0$. Consider the contravariant metric η written in the u-variables, see (7.23). Observe that $\eta^{nn}(u) = 4(2n-n-n)u^{n-1} = 0$. Recall that $\eta_{nn}(u) = \frac{1}{\det(\eta)}\mathrm{adj}(\eta)_{\mathrm{nn}}$, where $\mathrm{adj}(\eta) = C^T$ and where C is the cofactor matrix of η, whose entry (i, j) is $(-1)^{i+j}$ times the (i, j) minor of η.
Since η is lower anti-triangular, its (n, n) minor is zero, therefore $\eta_{nn}(u) = 0$. Rewriting η^{-1} in a flat coordinate system $(t^1, \dots, t^n)$ we have $\eta_{kl}(t) = \eta_{ij}(u(t))\frac{\partial u^i}{\partial t^k}\frac{\partial u^j}{\partial t^l}$. Then

$$\eta_{nn}(t) = \eta_{ij}(u(t))\frac{\partial u^i}{\partial t^n}\frac{\partial u^j}{\partial t^n}.$$

Observe that $\frac{\partial u^i}{\partial t^n} = 0$ for any i unless $i = n$, for degree reasons. Then

$$\eta_{nn}(t) = \eta_{nn}(u(t))\left(\frac{\partial u^n}{\partial t^n}\right)^2 = 0,$$

(no sum over n) since $\eta_{nn}(u) = 0$. ■

Proposition 7.33 *The contravariant Christoffel γ_k^{ij} of the Saito metric η, written in the coordinates $(u^1, ..., u^n)$, are given by*

$$\gamma_k^{ij}(u) = 4(n-j)\delta_{k,i+j-n-1} \tag{7.43}$$

Moreover, using the above formula one can verify that the polynomial invariants u^1, u^n and the function τ, see Formula (8.3) in Section 8 below, are flat functions.

Remark 7.34 *It is also immediate to verify that, in the Dubrovin-Saito flat coordinates, $g^{11}(t) = n$, up to a possible rescaling by a constant, see Remark 7.12.*

Remark 7.35 *The same results of this section can be obtained writing the A_n-invariant cometric (7.18) in a suitable set of A_n-invariant polynomials of degrees $1, 2, ..., n$ obtained combining the elementary symmetric polynomials*

$$f_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} y^{i_1} \cdots y^{i_k}$$

where $k = 1, ..., n$, in a suitable way (like in the case of B_n with $(p^i)^2$ replaced by $2y^i$). The drawback of this "interpretation" is that the dual product does not seem to admit a natural explanation in this context in terms of reflecting (hyper-)planes.

8 Dubrovin-Frobenius structure of NLS type on $\mathbb{C}^n/B_n$

In this section, we will show that the flat pencil of metrics, given by Theorem (7.24), defines a Frobenius manifold structure on the orbit space $\mathbb{C}^n/B_n$.

8.1 From flat pencils of metrics to Dubrovin-Frobenius manifolds

Recall some notions regarding the flat pencil of cometrics. One can prove that any Dubrovin-Frobenius structure defines a flat pencil of contravariant metrics, and, conversely, that a Dubrovin-Frobenius structure can be defined starting from a flat pencil of metrics satisfying the following three additional properties (see chapter 3).

- *Exactness*: there exists a vector field e such that

$$\mathcal{L}_e g = \eta \qquad \mathcal{L}_e \eta = 0 \tag{8.1}$$

  where $\mathcal{L}_e$ denotes Lie derivative with respect to e. This condition play an important role in the theory of evolutionary bihamiltonian PDEs both in the dispersionless and in the dispersive cases (see for instance [28]).

- *Homogeneity*: the following condition holds true:

$$\mathcal{L}_E g = (d-1)g \tag{8.2}$$

  where $E^i := g^{il}\eta_{lj}e^j$.

- *Egorov property*: locally there exists a function τ such that

$$e^i = \eta^{is}\partial_s\tau \qquad E^i = g^{is}\partial_s\tau \tag{8.3}$$

Exactness implies that $[e, E] = e$ and by combining this with the homogeneity condition one obtains

$$\mathcal{L}_E\eta = \mathcal{L}_E\mathcal{L}_e g = \mathcal{L}_e\mathcal{L}_E g - \mathcal{L}_{[E,e]}g = (d-2)\eta \tag{8.4}$$

Moreover, for Dubrovin-Frobenius manifolds the vector fields e and E coincide with the unity vector field and the Euler vector field respectively.
To prove that the flat pencil (g, η) induces a Dubrovin-Frobenius structure on $\mathbb{C}^n/B_n$, we will start to show that the (g, η) is exact, homogeneous and that it satisfies the Egorov property or, using Dubrovin's terminology, that it is quasi-homogeneous.
Let $e = \frac{\partial}{\partial u^{n-1}}$. Denote $\partial_k = \frac{\partial}{\partial u^k}$ for all k. Recall that

$$\eta^{ij} = 4(2n-i-j)u^{i+j-n-1} \tag{8.5}$$

Then

Lemma 8.1 *The pair (g, η) form an exact pencil.*

Proof: The first of (8.1) is true by definition and the second follows from the fact that η does not depend on u^{n-1} as it can be inferred from formula (8.5). ∎
Moreover

Lemma 8.2 *If τ is given by*

$$\tau := \frac{1}{4(n-1)}\left(u^2 - \frac{(n-2)}{2(n-1)}(u^1)^2\right) \tag{8.6}$$

then

$$e^i = \eta^{ij}\partial_j \tau, \tag{8.7}$$

so the first of (8.3) *is fulfilled.*

Proof: The proof is by a direct computation. Using (8.6) and (8.5), one obtains

$$\begin{aligned} e^i &= \frac{1}{4(n-1)}\sum_{j=1}^{n}\left(\eta^{ij}\delta_{j2} - \frac{(n-2)}{(n-1)}\eta^{ij}\delta_{j1}u^1\right) \\ &= \frac{1}{4(n-1)}\eta^{i2} - \frac{(n-2)}{4(n-1)^2}\eta^{i1}u^1 \\ &\overset{(8.5)}{=} \frac{(2n-i-2)}{(n-1)}u^{i+1-n} - \frac{(n-2)(2n-i-1)}{(n-1)^2}u^{i-n}u^1 \end{aligned} \tag{8.8}$$

Since $u^k = 0$ for all $k < 0$, if $i < n-1$ both summands in (8.8) are zero. If $i = n$, (8.8) becomes

$$\frac{(2n-n-2)}{(n-1)}u^{n+1-n} - \frac{(n-2)(2n-n-1)}{(n-1)^2}u^{n-n}u^1 = \frac{(n-2)}{(n-1)}u^1 - \frac{(n-2)}{(n-1)}u^0u^1 = 0,$$

since $u^0 = 1$. Finally, if $i = n-1$, one obtains

$$\frac{(2n-(n-1)-2)}{(n-1)}u^{n-1+1-n} - \frac{(n-2)(2n-(n-1)-1)}{(n-1)^2}u^{(n-1)-n}u^1 = 1,$$

which proves our statement since $e^i = \delta^i_{n-1}$. ∎

Lemma 8.3 *Defining*

$$E^i := g^{ij}\partial_j\tau \tag{8.9}$$

one has that

$$E^i = g^{il}\eta_{lj}e^j, \tag{8.10}$$

so that the second of (8.3) *is fulfilled.*

Proof: This follows from (8.7) and from (8.9), recalling that $\eta^{ij}\eta_{jl} = \delta^i_l$. ∎

Similarly, one can prove that

Lemma 8.4 *The vector field E has the following form*

$$E = \frac{1}{2(n-1)}\sum_{k=1}^{n} p^k \frac{\partial}{\partial p^k} \tag{8.11}$$

Proof: The proof follows at once from (7.7), (8.6) and (8.9). First one computes

$$\frac{\partial (u^1)^2}{\partial p^j} = 4p^j u^1$$

One observes that u^2 can be written as

$$u^2 = (p^j)^2 \sum_{i=1}^{n} (p^i)^2 + f(p^1, ..., \hat{p}^j, ..., p^n)$$

where f is a homogeneous function that doesn't depend on p^j. Therefore

$$\frac{\partial u^2}{\partial p^j} = 2p^j u^1 - 2(p^j)^3$$

Which yield

$$\frac{\partial \tau}{\partial p^j} = \frac{1}{2(n-1)} \left[\frac{p^j u^1}{n-1} - (p^j)^3 \right].$$

Then

$$\begin{aligned}
E^i = g^{ij}(p) \frac{\partial \tau}{\partial p^j} &\overset{(7.2)}{=} \frac{1}{2(n-1)} \sum_{j=1}^{n} \frac{(1-\delta_{ij})}{p^i p^j} \left[\frac{p^j u^1}{n-1} - (p^j)^3 \right] \\
&= \frac{1}{2p^i(n-1)} \sum_{j \neq i} \left[\frac{u^1}{n-1} - (p^j)^2 \right] \\
&= \frac{1}{2p^i(n-1)} \left[\frac{(n-1)u^1}{n-1} - u^1 + (p^i)^2 \right] \\
&= \frac{p^i}{2(n-1)}.
\end{aligned}$$

since $\sum_{j \neq i} (p^j)^2 = u^1 - (p^i)^2$. ■

Recall that $deg(u^k) = 2k$, and that g^{lk} is a homogeneous polynomial of degree $2k + 2l - 4$ (in the us). From this it follows:

Proposition 8.5 *We have that*

$$\mathcal{L}_E g = (d-1)g, \tag{8.12}$$

where $d = 1 - \frac{2}{(n-1)}$*, therefore condition (8.2) is fulfilled.*

Proof: First one observes that

$$\mathcal{L}_E(du^k) = \frac{k}{(n-1)} du^k \tag{8.13}$$

where $(du^k)_i := \partial_i u^k$. Indeed, in component, one has that

$$\begin{aligned}
\mathcal{L}_E(du^k)_i = E^j \partial_j \partial_i u^k + \partial_i E^j \partial_j u^k &\overset{(8.11)}{=} \frac{1}{2(n-1)} \left(p^j \partial_i \partial_j u^k + \partial_i p^j \partial_j u^k \right) \\
= \frac{1}{2(n-1)} \Big(\partial_i (\underbrace{p^j \partial_j u^k}_{=2ku^k}) - \partial_i p^j \partial_j u^k + \partial_i p^j \partial_j u^k \Big) &= \frac{k}{(n-1)} \partial_i u^k \\
= \frac{k}{(n-1)} (du^k)_i
\end{aligned}$$

here we have exploited the homogeneity of the function u^k.
Using the homogeneity of the entries of (g^{ij}) (see Lemma (7.5)), one gets

$$\begin{aligned}(\mathcal{L}_E g)(du^l, du^k) &= \mathcal{L}_E(g(du^l, du^k)) - g(\mathcal{L}_E du^l, du^k) - g(du^l, \mathcal{L}_E du^k)\\ &= \mathcal{L}_E g^{kl} - \frac{l}{(n-1)} g^{lk} - \frac{k}{(n-1)} g^{lk}\\ &= \mathcal{L}_E g^{kl} - \frac{l+k}{(n-1)} g^{lk}\\ &= \frac{l+k-2}{(n-1)} g^{lk} - \frac{l+k}{(n+1)} g^{lk}\\ &= -\frac{2}{(n-1)} g^{lk}\\ &= -\frac{2}{(n-1)} g(du^l, du^k).\end{aligned}$$

■

Before moving on, we observe that

Remark 8.6 *If* $(f^1, \ldots, f^n)$ *is any system of* homogeneous *coordinates in the* p*-variables*

$$\begin{aligned}E = \frac{1}{2(n-1)} \sum_{k=1}^{n} p^k \frac{\partial}{\partial p^k} &= \frac{1}{2(n-1)} \sum_{k=1}^{n} p^k \sum_{j=1}^{n} \frac{\partial f^j}{\partial p^k} \frac{\partial}{\partial f^j}\\ &= \frac{1}{2(n-1)} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} p^k \frac{\partial f^j}{\partial p^k} \right) \frac{\partial}{\partial f^j}\\ &= \frac{1}{2(n-1)} \sum_{j=1}^{n} deg(f^j) f^j \frac{\partial}{\partial f^j}\end{aligned}$$

here we have exploited the Euler's identity.
In particular, in our case we have

$$E = \sum_{k=1}^{n} p^k \frac{\partial}{\partial p^k} = \sum_{j=1}^{n} d_j u^j \frac{\partial}{\partial u^j} \tag{8.14}$$

where (u^i) are defined in (7.7).
Our next step in the construction of the Dubrovin-Frobenius structure on $M := \mathbb{C}^n/B_n$, will be the introduction of the structure constants defining the relevant product. To this end, recall that a homogeneous flat pencil (g, η) on M is called regular if the endomorphism of TM defined by

$$R^i_j = \nabla^{(\eta)}_j E^i - \nabla^{(g)}_j E^i, \tag{8.15}$$

is invertible, where, in the previous formula, $\nabla^{(\eta)}$ and $\nabla^{(g)}$ denote the covariant derivative operators corresponding to the Levi-Civita connections of the metrics η and g respectively.

Under the regularity assumption, the flat pencil defines a structure of a Dubrovin-Frobenius manifold on M whose structure constants are defined by

$$c^j_{hk} = L^s_h(\Gamma^{l(\eta)}_{sk} - \Gamma^{l(g)}_{sk})(R^{-1})^j_l \tag{8.16}$$

where $L^s_h = g^{si}\eta_{ih}$, $\Gamma^{l(\eta)}_{sk}$ and $\Gamma^{l(g)}_{sk}$ are the Christoffel's symbols of the metrics η and g respectively.
From now on, unless explicitly stated, all the tensors will be written in the flat Dubrovin-Saito coordinates, see Proposition 7.31 and Lemma 7.32 above. Since in these coordinates $\Gamma^{l(\eta)}_{sk} = 0$ for any choice of the indexes, in order to keep the notation more readable, we use directly the notation Γ^i_{jk} to denote the Christoffel symbols associated to g (as we did in Section 5.4). Under these assumptions, Formula (8.16) becomes

$$c^j_{hk} = -L^s_h\Gamma^l_{sk}(R^{-1})^j_l = -g^{si}\eta_{ih}\Gamma^l_{sk}(R^{-1})^j_l \overset{(7.14)}{=} \eta_{hi}\Gamma^{il}_k(R^{-1})^j_l, \tag{8.17}$$

see [5] and references therein.

Remark 8.7 *One can prove that the flat pencil of metrics (g, η) defined above is not regular. To this end it suffices to recall that (see formula (3.42))*

$$R^i_j = \frac{d-1}{2}\delta^i_j + \nabla^\eta_j E^i, \tag{8.18}$$

which, in our case, entails

$$R^i_j = \frac{(j-1)}{n-1}\delta^i_j. \tag{8.19}$$

In fact, since $d = 1 - \frac{2}{n-1}$, using the Dubrovin-Saito flat coordinates, which are invariant homogeneous polynomials in view of Proposition (7.31), one has that

$$R^i_j = \frac{d-1}{2}\delta^i_j + \nabla^{(\eta)}_j E^i \overset{(8.14)}{=} -\frac{1}{n-1}\delta^i_j + \frac{j}{n-1}\delta^i_j = \frac{(j-1)}{n-1}\delta^i_j$$

which has a vanishing eigenvalue.

Despite our flat pencil of metrics is not regular, we will be able to prove the following:

Theorem 8.8 *The flat pencil of metrics (g, η) gives rise to a Dubrovin-Frobenius structure on $\mathbb{C}^n/B_n$ generalizing those computed explicitly for the cases $n = 2, 3, 4$.*

The proof of this result will consist of the following steps:

(i) Definition of the structure constants of the product.

(ii) Proof of the commutativity of the product.

(iii) Existence of a flat unity vector field.

(iv) Identification of the metric η with the invariant metric of the Dubrovin-Frobenius manifold.

(v) Identification of the cometric g with the intersection form of the Dubrovin-Frobenius manifold.

(vi) Symmetry of the tensor ∇c.

(vii) Associativity of the product.

In all steps of the proof we will work in Saito flat coordinates. In order to prove the last step we will preliminarily prove that the functions

$$b^{ij}_k = \left(1 + d_j - \frac{d_F}{2}\right) c^{ij}_k, \tag{8.20}$$

coincide with the contravariant Christoffel symbols of the cometric g. This will allow us to obtain part of the associativity conditions as a consequence of the vanishing of the curvature.
We start with a preliminary lemma:

Lemma 8.9 *In Saito flat coordinates the contravariant symbols of the Levi-Civita of the metric g satisfy*

$$\begin{aligned}
\Gamma^{n+1-h,k}_m &= \Gamma^{n+1-m,k}_h & (8.21)\\
g^{is}\Gamma^{jk}_s &= g^{js}\Gamma^{ik}_s & (8.22)\\
\Gamma^{ij}_s\Gamma^{sk}_l &= \Gamma^{ik}_s\Gamma^{sj}_l & (8.23)\\
\frac{\Gamma^{mh}_k}{R^h_h} &= \frac{\Gamma^{hm}_k}{R^m_m} \qquad (h,m)\neq(1,1) & (8.24)
\end{aligned}$$

where Γ^{jk}_i are the contravariant Christoffel of g in Saito flat coordinates.

Proof: Recall that the following identities hold true (see (3.25), (3.26), (3.27) and (3.47)):

$$\begin{aligned}
\eta_{hs}\Delta^{sk}_m &= \eta_{ms}\Delta^{sk}_h, & (8.25)\\
g^{is}\Delta^{jk}_s &= g^{js}\Delta^{ik}_s, & (8.26)\\
\Delta^{ij}_s\Delta^{sk}_l &= \Delta^{ik}_s\Delta^{sj}_l, & (8.27)\\
\Delta^{tl}_k(R^{-1})^s_l &= \Delta^{sl}_k(R^{-1})^t_l. & (8.28)
\end{aligned}$$

where the tensor Δ^{jk}_m is given in terms of the Levi-Civita connections $\nabla^{(\eta)}$ and $\nabla^{(g)}$ by the formula

$$\Delta^{jk}_m = \eta_{lm}\left(\eta^{js}\Gamma^{lk}_{s(g)} - g^{sl}\Gamma^{jk}_{s(\eta)}\right) = \eta_{lm}\left(\eta^{ls}\Gamma^{jk}_{s(g)} - g^{js}\Gamma^{lk}_{s(\eta)}\right).$$

In Saito flat coordinates $\Gamma^{jk}_{i(g)} = \Gamma^{jk}_i$, $\Gamma^{jk}_{i(\eta)} = 0$, $\Delta^{jk}_i = \Gamma^{jk}_i$ and $\eta_{ij} = \delta_{i,n+1-j}$. Then the above identities reduce to identities (8.21), (8.22), (8.23) and (8.24). ■

8.2 Step 1: Definition of the c^i_{jk}s

As we have already mentioned, the definition of the Dubrovin-Frobenius structure on $\mathbb{C}^n/B_n$ cannot *completely* rely on (8.16) since the endomorphism R defined in (8.19) in not invertible. On the other hand, the loss of information is restricted to the case $R^i_j = 0$, i.e. $i = j = 1$, see Formula (8.19).
In this way, Formula (8.17) permits to define all the c^i_{jk} except the ones with $i = 1$. In other words

$$\boxed{c^i_{jk} := \frac{\eta_{jh}\Gamma^{hi}_k}{R^i_i}} \tag{8.29}$$

for all $i \neq 1$. We observe that

$$c^i_{jk} \overset{(7.42)}{=} \frac{\Gamma^{n+1-j,i}_k}{R^i_i} \tag{8.30}$$

Furthermore, the commutativity of the product entails

$$\frac{\Gamma^{n+1-j,i}_k}{R^i_i} \overset{(8.21)}{=} \frac{\Gamma^{n+1-k,i}_j}{R^i_i} \tag{8.31}$$

We highlight that the above equality holds true only in Dubrovin-Saito coordinates.
The remaining c^i_{jk} will be defined by the following formulas:

$$\boxed{c^1_{ij} := c^{n+1-j}_{ni}} \tag{8.32}$$

for any $(i, j) \neq (n, n)$, and

$$\boxed{c^1_{nn} := \frac{(n-1)}{t^n}} \tag{8.33}$$

Since R^i_i are constants in Dubrovin-Saito coordinates, the structure constants c^k_{ij}, defined in (8.29) and (8.32), are homogeneous polynomials of the p-variables of degree

$$deg(c^k_{ij}) \overset{(8.30)}{=} deg(\Gamma^{n+1-i,k}_j) \overset{(7.33)}{=} 2(n+1-i)+2k-2j-4 = 2(n-1-i-j+k) \tag{8.34}$$

for $k \neq 1$, and

$$deg(c^1_{ij}) = deg(c^{n+1-j}_{ni}) \overset{(8.34)}{=} 2(n-1-n-i+n+1-j) = 2(n-i-j) \tag{8.35}$$

In particular, note that, with the exception of c^1_{nn},

$$c^k_{ij} = 0 \tag{8.36}$$

for all i, j, k such that $i + j > n + k - 1$.
Notice that due to (8.33) the corresponding prepotential cannot be defined when $t^n = 0$. As a consequence the Dubrovin-Frobenius manifold structure, we are going to study, is defined on the orbit space of B_n less the image of the coordinate (hyper-)planes under the quotient map.

Remark 8.10 *Hereafter, we will normalize the degree of the* p*-homogeneous polynomials by* $\frac{1}{2(n-1)}$ *accordingly with the expression of Euler vector field, see (8.11). In other words, we will set*

$$d_k := deg\ (f_k) = \frac{k}{n-1} \tag{8.37}$$

where f_k *is any degree* $2k$*, homogeneous polynomial in the* p*-variables. For example*

$$d_{n-1+k-i-j} := deg\ (c^k_{ij}) = \frac{n-1+k-i-j}{n-1} \tag{8.38}$$

and

$$d_{i+j-2} := deg\ (g^{ij}) = \frac{i+j-2}{n-1} \tag{8.39}$$

see (7.11) and (8.34).

8.3 Step 2: Commutativity of the product

We have to prove that

$$c^i_{jk} = c^i_{kj}. \tag{8.40}$$

for any choice of the indexes.
For $i \neq 1$ this follows automatically from (8.31), indeed

$$c^i_{jk} = \frac{\Gamma^{n+1-j,i}_k}{R^i_i} = \frac{\Gamma^{n+1-k,i}_j}{R^i_i} = c^i_{kj}$$

For $i = 1, k = n, j \neq n$ we have

$$c^1_{jn} \overset{(8.32)}{=} c^{n+1-n}_{nj} = c^1_{nj}$$

For $i = 1, j = n, k \neq n$ it's enough to read the above line backward.
For $i = 1, k \neq n, j \neq n$ we have

$$c^1_{jk} \overset{(8.32)}{=} c^{n+1-k}_{nj} \overset{(8.30)}{=} \frac{\Gamma^{1,n+1-k}_j}{R^{n+1-k}_{n+1-k}} \overset{(8.21)}{=} \frac{\Gamma^{n+1-j,n+1-k}_n}{R^{n+1-k}_{n+1-k}} \overset{(8.24)}{=} \frac{\Gamma^{n+1-k,n+1-j}_n}{R^{n+1-j}_{n+1-j}} = \dots$$

Since $\Gamma^{n+1-f,g}_l = \Gamma^{n+1-l,g}_f$ for any choice of the indexes, taking $f = n$ one has that $\Gamma^{1,g}_l = \Gamma^{n+1-l,g}_k$, then

$$\dots = \frac{\Gamma^{1,n+1-j}_k}{R^{n+1-j}_{n+1-j}} \overset{(8.30)}{=} c^{n+1-j}_{nk} \overset{(8.32)}{=} c^1_{kj}$$

8.4 Step 3: Existence of a flat unity vector field

We now prove that the unity of the product defined above is the vector field $e = \frac{\partial}{\partial u^{n-1}}$, such that

$$c^i_{jk} e^k = \delta^i_j,$$

For $i \neq 1$ this follows from the results for regular quasi-homogeneous pencil (see [15]). For $i = 1$, since $e^k = \delta^k_{n-1}$, we have

$$c^1_{jk} e^k = c^1_{j,n-1}$$

This means that we have to prove the identities

$$\begin{aligned} c^1_{1,n-1} &= 1 \\ c^1_{j,n-1} &= 0 \end{aligned}$$

for $j = 2, ..., n$. Recall that (see (8.38)) c^k_{ij} are polynomial functions in the (p^i) of degrees $deg(c^k_{ij}) = \frac{n-1-i-j+k}{n-1}$, then $deg(c^1_{j,n-1}) = \frac{1-j}{n-1}$, therefore for $j \neq 1$ they vanish identically.
For $j = 1$ we have

$$c^1_{1,n-1} = c^1_{n-1,1} \overset{(8.32)}{=} c^n_{n,n-1} = c^n_{nk} e^k = \delta^n_n,$$

where the last equality follows from the fact that $c^i_{jk} e^k = \delta^i_j$ for $i \neq 1$.
It is immediate to check that $\nabla^{(\eta)} e = 0$. Indeed, since u^n is flat, the passage from the coordinates $(u^1, , , , , , u^n)$ to the flat basic invariants $(t^1, , , , , , t^n)$ does not affect the form of e that remains constant in the new coordinates.

8.5 Step 4: Identification of the metric η with the invariant metric.

We need a preliminary lemma:

Lemma 8.11 *For any choice of the indexes we have*

$$c^i_{jk} = c^{n+1-k}_{n+1-i,j} = c^{n+1-j}_{n+1-i,k} \tag{8.41}$$

Proof: The case $i = 1$, and $(j, k) = (n, n)$ is trivial. If $i = 1$ and $(j, k) \neq (n, n)$, then

$$c^1_{jk} \overset{(8.32)}{=} c^{n+1-k}_{nj}$$

which coincides with the first of the (8.41). The second one holds true because of the symmetry of the lower indices of the c^i_{jk} (formula (8.40)), indeed

$$\underbrace{c^{n+1-k}_{nj}}_{\overset{(8.32)}{=} c^1_{jk}} = \underbrace{c^{n+1-j}_{nj}}_{\overset{(8.32)}{=} c^1_{kj}}$$

If $i \neq 1$ and $k \neq n$ (so $n + 1 - k \neq 0$) we have

$$c^i_{jk} \overset{(8.30)}{=} \frac{\Gamma^{n+1-j,i}_k}{R^i_i},$$

and

$$c^{n+1-k}_{n+1-i,j} \overset{(8.30)}{=} \frac{\Gamma^{i,n+1-k}_j}{R^{n+1-k}_{n+1-k}} \overset{(8.24)}{=} \frac{\Gamma^{n+1-k,i}_j}{R^i_i} \overset{(8.21)}{=} \frac{\Gamma^{n+1-j,i}_k}{R^i_i} \overset{(8.30)}{=} c^i_{jk}$$

and

$$c^{n+1-k}_{n+1-i,j} \overset{(8.30)}{=} \frac{\Gamma^{i,n+1-k}_{j}}{R^{n+1-k}_{n+1-k}} \overset{(8.21)}{=} \frac{\Gamma^{n+1-j,n+1-k}_{n+1-i}}{R^{n+1-k}_{n+1-k}} \overset{(8.24)}{=} \frac{\Gamma^{n+1-k,n+1-j}_{n+1-i}}{R^{n+1-k}_{n+1-k}} \overset{(8.30)}{=} c^{n+1-j}_{k,n+1-i} \overset{(8.40)}{=} c^{n+1-j}_{n+1-i,k}$$

On the other hand, if $i \neq 1$, $k = n$ and $j \neq n$

$$c^{n+1-k}_{n+1-i,j} = c^{1}_{n+1-i,j} \overset{(8.32)}{=} \underbrace{c^{n+1-j}_{n,n+1-i}}_{=c^{n+1-j}_{n+1-i,n}} \overset{(8.30)}{=} \frac{\Gamma^{i,n+1-j}_{n}}{R^{n+1-j}_{n+1-j}} \overset{(8.24)}{=} \frac{\Gamma^{n+1-j,i}_{n}}{R^{i}_{i}} \overset{(8.30)}{=} c^{i}_{jn}$$

Finally if $i \neq 1$ and $(j,k) = (n,n)$, then the three terms of the identity are zero, see (8.36). Indeed

$$deg(c^{i}_{nn}) = \frac{2}{n-1}(n-1-i-n-n) = \frac{2}{n-1}(-1-i-n) < 0$$
$$deg(c^{n}_{n+1-i,n}) = \frac{2}{n-1}(n-1-n-n-1+i-n) = \frac{2}{n-1}(i-2n-2) < 0$$

for any i. ∎

We have now all the ingredients to prove that

$$\eta_{is}c^{s}_{jk} = \eta_{js}c^{s}_{ik}. \tag{8.42}$$

This follows at once from (8.41) and from $\eta_{ij} = \delta_{i,n+1-j}$. In fact, substituting $i \mapsto n-1+i$ in (8.41), one has

$$\eta_{is}c^{s}_{jk} = c^{n+1-i}_{jk} = c^{n+1-j}_{ik} = \eta_{js}c^{s}_{ik}$$

8.6 Step 5: Identification of the cometric g with the intersection form.

We will now prove the identity

$$c^{i}_{jk}E^{k} = g^{il}\eta_{lj}; \tag{8.43}$$

which amounts to say that the operator of multiplication by the Euler vector field E, defined via the (8.30), (8.32) and (8.33) is the affinor, i.e. a tensor field of type $(1,1)$, defined composing (the covariant metric) η with (the contravariant metric) g. To prove (8.43), we write $E = E^{i}\partial_i$ and first we observe that (8.15) written in Dubrovin-Saito coordinates reduces to

$$R^{i}_{j} = \nabla^{(\eta)}_{j}E^{i} - \nabla^{(g)}_{j}E^{i} = -\Gamma^{i}_{jl}E^{l}, \tag{8.44}$$

which, for $i \neq 1$, yields

$$c^{i}_{jl}E^{l} \overset{(8.29)}{=} \frac{1}{R^{i}_{i}}\eta_{jl}\Gamma^{li}_{k}E^{k} = -\frac{1}{R^{i}_{i}}\eta_{jl}g^{ls}\Gamma^{i}_{sk}E^{k} \overset{(8.44)}{=} \frac{1}{R^{i}_{i}}\eta_{jl}g^{ls}\underbrace{R^{i}_{s}}_{=\delta^{i}_{s}R^{i}_{i}} \overset{(8.19)}{=} \eta_{jl}g^{li}$$

On the other hand, the case $i=1$ and $j\neq n$ can be reduced to the previous one. In fact

$$c^1_{jl}E^l \overset{(8.32)}{=} c^{n+1-j}_{nl}E^l = g^{n+1-j,l}\eta_{ln} = g^{n+1-j,1} = g^{1l}\eta_{lj}$$

where the other equalities follow from the case $i\neq 1$ and from the explicit form of η. Finally, if $i=1$ and $j=n$:

$$c^1_{nl}E^l \overset{(8.36)}{=} c^1_{nn}E^n \overset{(8.14)}{=} c^1_{nn}d_n t^n \overset{(8.33),(8.37)}{=} \frac{n-1}{t^n}\frac{n}{n-1}t^n = n \overset{(7.25)}{=} g^{11} \overset{(7.42)}{=} g^{1l}\eta_{ln}$$

Note that in the first equality we used the explicit form of the Euler vector field, in the fifth the normalization of g (see Remark 7.34) and in the last the explicit form of η.

This identity (8.43), multiplied by the inverse of (η_{ij}), implies that

$$g^{ih} = c^i_{jk}E^k\eta^{jh} = c^h_{jk}E^k\eta^{ji}. \tag{8.45}$$

In other words the cometric g can be identified with the intersection form.

We prove now an useful identity that we will use later.

Lemma 8.12

$$g^{is}c^l_{sm} = g^{ls}c^i_{sm}, \tag{8.46}$$

for any choice of the indexes.

Proof: If $m\neq n$, $l\neq 1$ and for any i

$$\begin{aligned} g^{is}c^l_{sm} &\overset{(8.30)}{=} g^{is}\frac{\Gamma^{n+1-m,l}_s}{R^l_l} \overset{(8.24)}{=} g^{is}\frac{\Gamma^{l,n+1-m}_s}{R^{n+1-m}_{n+1-m}} \\ &\overset{(8.22)}{=} g^{ls}\frac{\Gamma^{i,n+1-m}_s}{R^{n+1-m}_{n+1-m}} \overset{(8.30)}{=} g^{ls}c^{n+1-m}_{n+1-i,s} \overset{(8.41)}{=} g^{ls}c^i_{ms}. \end{aligned}$$

If $m\neq n$, $l=1$ and $i=1$ is trivially true.
If $m\neq n$, $l=1$ and $i\neq 1$

$$\begin{aligned} g^{is}c^1_{sm} &\overset{(8.32)}{=} g^{is}c^{n+1-m}_{ns} \overset{(8.30)}{=} g^{is}\frac{\Gamma^{1,n+1-m}_s}{R^{n+1-m}_{n+1-m}} \\ &\overset{(8.22)}{=} g^{1s}\frac{\Gamma^{i,n+1-m}_s}{R^{n+1-m}_{n+1-m}} \overset{(8.30)}{=} g^{1s}c^{n+1-m}_{n+1-i,s} \overset{(8.41)}{=} g^{1s}c^i_{sm}. \end{aligned}$$

If $m=n$, $l=1$ and $i=1$ is trivially true.
On the other hand, if $m=n$, $l=1$ and $i\neq 1$ we have

$$\begin{aligned} (g^{1s}c^i_{sn} - g^{is}c^1_{sn})E^n &= (g^{1s}c^i_{sk} - g^{is}c^1_{sk})E^k \\ &= g^{1s}g^{ir}\eta_{rs} - g^{is}g^{1r}\eta_{rs} \\ &= 0, \end{aligned}$$

and this implies $g^{1s}c^i_{sn} - g^{is}c^1_{sn} = 0$ since $E^n = d_n u^n$. The first equality follows from (8.43) and from the fact that (8.46) holds true if $m\neq n$, $l=1$ and $i\neq 1$, see the previous computation. On the other hand, the last equality is obtained trading r with s in (for example) the second summand. Finally, since i and l appear symmetrically in (8.46), the case $m=n$, $i=1$ and $l\neq 1$ follows from the previous computation simply exchanging the role of i and l. ∎

8.7 Step 6: Symmetry of ∇c

In Saito flat coordinates the vanishing of the curvature of the pencil implies

$$\partial_s \Gamma^{jk}_l = \partial_l \Gamma^{jk}_s, \tag{8.47}$$

for any choice of the indexes, where Γ^{ij}_k denote the contravariant Christoffel symbols of the metric g (see formula (3.28) written in Saito coordinates). This observation entails that

Proposition 8.13

$$\partial_s c^k_{jl} = \partial_l c^k_{js} \tag{8.48}$$

for any choice of the indexes.

Proof: If $k \neq 1$, then (8.48) follows from the definition of the structure constants. In this case we have $c^k_{jl} = \frac{\eta_{jr}\Gamma^{rk}_l}{R^k_k}$, where $\frac{\eta_{jr}}{R^k_k}$ are constants. Then

$$\partial_s c^k_{jl} = \frac{\eta_{jr}}{R^k_k}\partial_s \Gamma^{rk}_l \overset{(8.47)}{=} \frac{\eta_{jr}}{R^k_k}\partial_l \Gamma^{rk}_s = \partial_l c^k_{js} \tag{8.49}$$

If $k = 1$ we want to prove that

$$\partial_s c^1_{jl} = \partial_l c^1_{js}$$

Observing that $deg(\partial_s c^1_{ij}) = 2(n - i - j - s)$, see (8.35), one has that $deg(\partial_n c^1_{ns}) = 2(n - s)$. Then $c^1_{ns} = 0$ for any $s \neq n$. Therefore, if $k = 1$ and $(j, l) = (n, n)$, the right-hand side of (8.48) is zero unless $s = n$ when this identity is trivially true. Moreover, if $s \neq n$, then also the left-hand side of (8.48) is zero since c^1_{nn} depends on u^n only. Finally, if $k = 1$ and $(j, n) \neq (n, n)$, then

$$\partial_s c^1_{jl} = \partial_s c^1_{lj} = \partial_s c^{n+1-j}_{nl} \overset{(8.49)}{=} \partial_l c^{n+1-j}_{ns} = \partial_l c^1_{sj} = \partial_l c^1_{js}.$$

■

8.8 Interlude: Structure constants of the product and Christoffel symbols

Let

$$d_F = 3 - d = 2 + \frac{2}{n-1}$$

and let

$$c^{ij}_k := \eta^{is} c^j_{sk} \tag{8.50}$$

for any choice of the indexes, where c^j_{sk} were defined in (8.29), (8.32) and (8.33). Let

$$b^{ij}_k := \left(1 + d_j - \frac{d_F}{2}\right) c^{ij}_k \tag{8.51}$$

Remark 8.14 *Note that*

$$1 + d_j - \frac{d_F}{2} = \frac{j-1}{n-1}.$$

for all $j = 1, \ldots, n$.

We will prove that

Theorem 8.15 *The functions b_k^{ij}, defined in (8.51), satisfy the following system of equations*

$$\partial_k g^{ij} = b_k^{ij} + b_k^{ji} \tag{8.52}$$
$$g^{is} b_s^{jk} = g^{js} b_s^{ik} \tag{8.53}$$

where $i, j, k = 1, \ldots, n$.

To prove this statement we need to prove the following preliminary lemma.

Lemma 8.16 *Let $c = (c_k^{ij})$ the $(1,2)$-tensor field defined by (8.29), (8.32) and (8.33). Then*

$$\mathcal{L}_E c = c \tag{8.54}$$

Proof: Let $c = c_{jk}^i \partial_i \otimes dt^j \otimes dt^k$. Recall that

$$\mathcal{L}_E dt^i = \tfrac{i}{n-1} dt^i$$
$$\mathcal{L}_E \partial_i = -\tfrac{i}{n-1} \partial_i$$
$$deg(c_{jk}^i) = \tfrac{n-1+i-j-k}{n-1}$$

see (8.38) and (8.13).
Therefore, we observe that

$$\mathcal{L}_E c_{jk}^i = \underbrace{E^s \partial_s c_{jk}^i}_{=\frac{n-1+i-j-k}{n-1} c_{jk}^i} - \underbrace{\partial_s E^i}_{=\frac{i}{n-1}} c_{jk}^s + \underbrace{\partial_j E^s}_{=\frac{j}{n-1}} c_{sk}^i + \underbrace{\partial_k E^s}_{=\frac{k}{n-1}} c_{js}^i = c_{jk}^i$$

Then, in view of the Leibniz's rule, one has

$$\begin{aligned}\mathcal{L}_E c &= (\mathcal{L}_E c_{jk}^i) \partial_i \otimes dt^j \otimes dt^k + c_{jk}^i (\mathcal{L}_E \partial_i) \otimes dt^j \otimes dt^k \\ &\quad + c_{jk}^i \partial_i \otimes (\mathcal{L}_E dt^j) \otimes dt^k + c_{jk}^i \partial_i \otimes dt^j \otimes (\mathcal{L}_E dt^k) \\ &= c.\end{aligned}$$

For later use, we observe that from the very last equality, solving for $(\mathcal{L}_E c_{jk}^i) \partial_i \otimes dt^j \otimes dt^k$ one obtains

$$E^m \partial_m c_{lk}^j = c_{lk}^j + d_j c_{lk}^j - d_l c_{lk}^j - d_k c_{lk}^j, \tag{8.55}$$

where d_j were defined in (8.37). ■

Once these preliminary results are settled, one can prove Theorem 8.15.
Proof: First note that (8.43) implies

$$g^{hk} = \eta^{ki} c^h_{is} E^s. \tag{8.56}$$

Then we compute

$$\partial_k(g^{ij}) = \partial_k(\eta^{il} c^j_{lm} E^m) = \eta^{il}(\partial_k c^j_{lm}) E^m + \eta^{il} c^j_{lm} \partial_k E^m \overset{(8.48)}{=} \eta^{il}(E^m \partial_m c^j_{lk}) + d_k \eta^{il} c^j_{lk}. \tag{8.57}$$

Using (8.55) to substitute $E^m \partial_m c^j_{lk}$ in (8.57), we obtain

$$\partial_k(g^{ij}) = \eta^{il}(c^j_{lk} + d_j c^j_{lk} - d_l c^j_{lk}). \tag{8.58}$$

Since the pencil (g, η) is homogeneous and exact,

$$\mathcal{L}_E \eta = (d-2)\eta = (1-d_F)\eta,$$

see (8.4) (here η denotes the contravariant metric). On the other hand, since η is constant when written in the Saito flat coordinates, working with the covariant metric, one has

$$\begin{aligned} 0 = \mathcal{L}_E\big(\eta(\partial_i, \partial_l)\big) &= (\mathcal{L}_E \eta)(\partial_i, \partial_l) + \eta(\mathcal{L}_E \partial_i, \partial_l) + \eta(\partial_i, \mathcal{L}_E \partial_l) \\ &= (d_F - 1)\eta(\partial_i, \partial_l) - \partial_i E^m \eta(\partial_m, \partial_l) - \partial_l E^m \eta(\partial_i, \partial_m) \\ &= (d_F - 1)\eta^{il} - d_i \eta^{il} - d_l \eta^{il}, \end{aligned}$$

which entails

$$-\eta^{il} d_l = \eta^{il}(-d_F + 1 + d_i)$$

Inserting this identity in (8.58), one gets

$$\partial_k(g^{ij}) = \eta^{il}(2 + d_i + d_j - d_F) c^j_{lk} = (2 + d_i + d_j - d_F) c^{ij}_k$$

This should be compared with

$$b^{ij}_k + b^{ji}_k = \left(1 + d_j - \frac{d_F}{2}\right) c^{ij}_k + \left(1 + d_i - \frac{d_F}{2}\right) c^{ji}_k$$

To this end, first one observes that the invariance of the metric η w.r.t. the product implies

$$c^{mh}_k = c^{hm}_k \tag{8.59}$$

for any $h, m, k = 1, ..., n$. In fact, in view of the invariance of the metric w.r.t. product, one gets

$$c^{mh}_k \overset{(8.50)}{=} \delta^h_l \eta^{mj} c^l_{jk} = \eta^{hi} \eta^{mj} \eta_{il} c^l_{jk} \overset{(8.42)}{=} \eta^{hi} \eta^{mj} \eta_{jl} c^l_{ik} = \delta^m_l \eta^{hi} c^l_{ik} \overset{(8.50)}{=} c^{hm}_k$$

From this one concludes that

$$b^{ij}_k + b^{ji}_k = (2 + d_i + d_j - d_F)\, c^{ij}_k = \partial_k g^{ij}$$

To prove (8.53) we use (8.50), (8.51), (8.56) and we compute

$$\begin{aligned} g^{is}b^{jk}_s &\overset{(8.51)}{=} \eta^{im}c^s_{mh}E^h\left(1+d_k-\frac{d_F}{2}\right)\eta^{jl}c^k_{ls} \\ &\overset{(8.43)}{=} \eta^{im}g^{sh}\eta_{hm}\left(1+d_k-\frac{d_F}{2}\right)\eta^{jl}c^k_{ls} \\ &\overset{(8.46)}{=} \eta^{im}g^{sk}\eta_{hm}\left(1+d_k-\frac{d_F}{2}\right)\eta^{jl}c^h_{ls} \\ &\overset{(8.42)}{=} \eta^{im}g^{sk}\eta_{hl}\left(1+d_k-\frac{d_F}{2}\right)\eta^{jl}c^h_{ms} \\ &\overset{(8.46)}{=} \eta^{im}g^{sh}\eta_{hl}\left(1+d_k-\frac{d_F}{2}\right)\eta^{jl}c^k_{ms} \\ &\overset{(8.43)}{=} \eta^{jl}c^s_{lh}E^h\left(1+d_k-\frac{d_F}{2}\right)\eta^{im}c^k_{ms} \overset{(8.51)}{=} g^{js}b^{ik}_s. \end{aligned}$$

■

Theorem 8.15 implies that

Proposition 8.17 *The functions b^{ij}_k defined in (8.51) are the contravariant Christoffel symbols of the metric g in the Saito flat coordinates, i.e.*

$$b^{ij}_k = \Gamma^{ij}_k \tag{8.60}$$

for any choice of the indexes.

8.9 Step 7: Associativity of the product.

We start noticing that since (g, η) is a flat pencil, expressing the conditions of zero-curvature for the Levi-Civita connection defined by $g_{(\lambda)} := g - \lambda\eta$ in the Saito flat coordinates, one obtains the following set of equations

$$\partial_s b^{jk}_l - \partial_l b^{jk}_s = 0 \tag{8.61}$$

$$b^{ij}_s b^{sk}_l - b^{ik}_s b^{sj}_l = 0 \tag{8.62}$$

The first set of conditions (8.61) does not provide additional information since it follow from the symmetry (in the lower indices) of $\nabla^{(\eta)}c$. Indeed

$$\left(1+d_k-\frac{d_F}{2}\right)\left(\partial_s c^{jk}_l - \partial_l c^{jk}_s\right) \overset{(8.51)}{=} R^k_k\eta^{jh}\left(\partial_s c^k_{hl} - \partial_l c^k_{hs}\right) \overset{(8.48)}{=} 0 \tag{8.63}$$

Let us consider the second set of conditions (8.62). First we note that using the (8.51) and recalling that $R^k_k = (1 + d_k - \frac{d_F}{2})$ for all k, these conditions can be rewritten as follows

$$R^k_k R^j_j(c^{ij}_s c^{sk}_l - c^{ik}_s c^{sj}_l) \overset{(8.59)}{=} R^k_k R^j_j(c^{ji}_s c^{ks}_l - c^{ki}_s c^{js}_l) = R^k_k R^j_j \eta^{jh}\eta^{km}(c^i_{hs}c^s_{ml} - c^i_{ms}c^s_{hl}) = 0 \tag{8.64}$$

(no summation over j and k). The quadratic conditions (8.64) entail the associativity of the product defined by the c^i_{jk}, that is

$$c^i_{hs}c^s_{ml} = c^i_{ms}c^s_{hl},$$

but when one of the index m, h is equal to n (of course, if both indices are equal to n the statement is trivially true), For this reason, to conclude the proof we are left to show that

$$c^i_{nl}c^l_{km} = c^i_{kl}c^l_{nm}, \tag{8.65}$$

for all possible values of i, k, m. It is worth noticing that if $k = n$ the previous identity is trivially satisfied. We start checking that

$$c^i_{nl}c^l_{km} - c^i_{kl}c^l_{nm} = 0 \tag{8.66}$$

for all $(m, k, i) \neq (n, n, 1)$.
First recall that, since $b^{ij}_k = \Gamma^{ij}_k$, we have $c^i_{jk} = \frac{b_k^{n+1-j,i}}{R^i_i} = \frac{b_j^{n+1-k,i}}{R^i_i}$ by (8.30). By a direct computation

$$
\begin{aligned}
&c^i_{nl}c^l_{km} - c^i_{kl}c^l_{nm} = c^i_{n1}c^1_{km} - c^i_{k1}c^1_{nm} + \sum_{l\neq 1}\left(c^i_{nl}c^l_{km} - c^i_{kl}c^l_{nm}\right) \overset{(8.32),(8.30)}{=}\\
&c^i_{n1}c^{n+1-m}_{nk} - c^i_{k1}c^{n+1-m}_{nn} + \sum_{l\neq 1}\left(\frac{b^{1i}_l}{R^i_i}\frac{b_k^{n+1-m,l}}{R^l_l} - \frac{b_l^{n+1-k,i}}{R^i_i}\frac{b_n^{n+1-m,l}}{R^l_l}\right) \overset{(8.30),(8.24)}{=}\\
&\frac{b^{1i}_1}{R^i_i}\frac{b_k^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b^{ni}_k}{R^i_i}\frac{b_n^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} + \sum_{l\neq 1}\left(\frac{b^{1i}_l}{R^i_i}\frac{b_k^{l,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b_l^{n+1-k,i}}{R^i_i}\frac{b_n^{l,n+1-m}}{R^{n+1-m}_{n+1-m}}\right) =\\
&\frac{b^{1i}_1}{R^i_i}\frac{b_k^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b^{ni}_k}{R^i_i}\frac{b_n^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} + \sum_{l\neq 1}\left(\frac{b^{1i}_l}{R^i_i}\frac{b_k^{l,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b_k^{n+1-l,i}}{R^i_i}\frac{b_{n+1-l}^{1,n+1-m}}{R^{n+1-m}_{n+1-m}}\right) =\\
&\frac{b^{1i}_1}{R^i_i}\frac{b_k^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b^{ni}_k}{R^i_i}\frac{b_n^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} + \sum_{l\neq 1}\frac{b^{1i}_l}{R^i_i}\frac{b_k^{l,n+1-m}}{R^{n+1-m}_{n+1-m}} - \sum_{l\neq n}\frac{b^{li}_k}{R^i_i}\frac{b_l^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} =\\
&\frac{b^{1i}_l}{R^i_i}\frac{b_k^{l,n+1-m}}{R^{n+1-m}_{n+1-m}} - \frac{b^{li}_k}{R^i_i}\frac{b_l^{1,n+1-m}}{R^{n+1-m}_{n+1-m}} = 0.
\end{aligned}
$$

Remark 8.18 *In the previous computation, the fourth line follows from the third one, applying (8.21) to both $b_l^{n+1-k,i}$ and $b_n^{l,n+1-m}$. In the fifth line, the second summation stems after declaring $s = n + 1 - l$ (and then $s = l$) in the second summand of the summation of the fourth line.*

If $(m, k) \neq (n, n)$ and $i = 1$, (8.65) becomes

$$c^1_{nl}c^l_{km} = c^1_{kl}c^l_{nm}. \tag{8.67}$$

By (8.64), we know that

$$c^1_{il}c^l_{km} = c^1_{kl}c^l_{im} \tag{8.68}$$

for $i = 1, \dots n-1$, since we are also assuming $k \neq n$ and $m \neq n$. Therefore (8.67) can be rewritten in the following equivalent form

$$(c^1_{il} c^l_{km} - c^1_{kl} c^l_{im}) E^i = 0,$$

since, for what already proven, the only non-zero contribution in this sum is the one with $i = n$.
Using (8.43) one gets

$$\begin{aligned}(c^1_{il} c^l_{km} - c^1_{kl} c^l_{im}) E^i &= c^1_{il} E^i c^l_{km} - c^1_{kl} c^l_{im} E^i \\ &= g^{1s} \eta_{sl} c^l_{km} - c^1_{kl} g^{ls} \eta_{sm} \\ &\overset{(8.46)}{=} g^{1s} \eta_{ml} c^l_{ks} - c^s_{kl} g^{l1} \eta_{sm} \\ &= 0,\end{aligned} \tag{8.69}$$

whose last equality is obtained changing s with l in the second summand of (8.69). Therefore (8.67) holds true.

As already observed above, if $m = k = n$ and for any i, (8.65) becomes

$$c^i_{nl} c^l_{nn} - c^i_{nl} c^l_{nn} = 0.$$

We are left to consider the case $m = n$, $k \neq n$ and any i; that is we need to prove

$$c^i_{nl} c^l_{kn} - c^i_{kl} c^l_{nn} = 0, \tag{8.70}$$

for $k \neq n$ and for any i.
We first observe that $c^i_{nl} c^l_{ks} - c^i_{kl} c^l_{ns} = 0$ for $s = 1, \dots, n-1$ and any i, since for $i \neq 1$ this is (8.66), while for $i = 1$ this is (8.67).
Therefore we can rewrite (8.70) in the equivalent form

$$c^i_{nl} c^l_{ks} E^s - c^i_{kl} c^l_{ns} E^s = 0,$$

which, together with (8.43), yields

$$c^i_{nl} g^{ls} \eta_{sk} - c^i_{kl} g^{ls} \eta_{sn} \overset{(8.46)}{=} c^s_{nl} g^{li} \eta_{sk} - c^s_{kl} g^{li} \eta_{sn} \overset{(8.42)}{=} (c^s_{nl} \eta_{sk} - c^s_{kl} \eta_{sn}) g^{li} = 0.$$

This concludes the proof of Theorem 8.8. ■

8.10 Conclusions and Open problems

In this section, combining the following:

- the procedure presented in [3] for complex reflection groups, which relies on explicit formula for the multiplication and the connection of the dual structure and a yields a possible expression for the intersection form
- a generalization of the classical Dubrovin-Saito procedure presented in [18] and [46], which shows that the candidate intersection form produces a homogeneous flat pencil of cometrics

we have obtained a non-standard Dubrovin-Frobenius structure on the orbit space of B_n. More precisely, such a structure is defined on the orbit space of B_n less the image of coordinate (hyper-)planes under the quotient map (where the intersection form and the dual structure constants are not defined).
In other words, the procedure of [3] allowed us to get explicit formulas in the cases $n = 2, 3, 4$ while the generalized Dubrovin-Saito procedure allowed us to prove the existence of this structure for arbitrary integer n.
Two main questions are open:

1. For $n = 2, 3, 4$ the dual product is defined by

$$* = \frac{1}{N} \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \sigma_H \pi_H$$

   with $\sigma_H = 0$ for all the mirrors in the Orbit I and $\sigma_H = 1$ for all the mirrors in the Orbit II. Is it true for arbitrary n?

2. For $n = 2, 3, 4$ the Dubrovin-Frobenius prepotentials

$$\begin{aligned} F_{B_2} &= \frac{1}{2}(t^1)^2 t^2 \pm \frac{1}{2}(t^2)^2 \left(\ln t^2 - \frac{3}{2}\right). \\ F_{B_3} &= \frac{1}{6}(t^2)^3 + t^1 t^2 t^3 + \frac{1}{12}(t^1)^3 t^3 - \frac{3}{2}(t^3)^2 + (t^3)^2 \ln t^3, \\ F_{B_4} &= \frac{1}{108}(t^1)^4 t^4 + \frac{1}{6}(t^1)^2 t^2 t^4 - \frac{1}{72}(t^2)^4 + t^1 t^3 t^4 + \frac{1}{2}(t^2)^2 t^4 + \frac{1}{2} t^2 (t^3)^2 \\ &- \frac{9}{4}(t^4)^2 + \frac{3}{2}(t^4)^2 \ln t^4, \end{aligned}$$

   coincide with the solutions of WDVV equations associated with constrained KP equation (see [35]) and enumeration of hypermaps (see [26]), in particular the case $n = 2$ is related to the defocusing NLS equation and higher genera Catalan numbers. Is it true for arbitrary n?

Both these questions, as we will see in the next section, have affirmative answers.

9 Constrained KP hierarchies and central invariants

The main references of this section are [35] and [41].
In [35] Liu, Zhang and Zhou computed the central invariants of the bi-Hamiltonian structure associated with constrained KP hierarchies. In particular, they all coincide with $\frac{1}{24}$.
The notion of central invariant of a semi-simple bi-Hamiltonian structure, which possesses a hydrodynamic limit, was introduced by Dubrovin, Liu and Zhang in [24]. Moreover, in [24] it was also proven that such invariants completely characterize the equivalence classes of infinitesimal deformation of a semi-simple bi-Hamiltonian structure of hydrodynamic type modulo Miura type transformation. In [24] and [39] it was also conjectured that for any given semi-simple bihamiltonian structure of hydrodynamic type and a set of central invariants there exists such a deformation of the bi-Hamiltonian structure. In particular, one has to verify the triviality of the associated third bihamiltonian cohomology, introduced in [24]. In [40] the conjecture has been verified in the scalar case, while in [10] the conjecture has been verified in the general case.
Given a bi-Hamiltonian structure, the notion of central invariant is an efficient tool in order to characterize the associated integrable hierarchy of evolutionary PDEs. For example, for the bi-Hamiltonian integrable hierarchy that controls a cohomological field theory, associated with a semi-simple Frobenius manifold, all the central invariants coincide with $\frac{1}{24}$ (see [25] for details). This bi-Hamiltonian structure is called topological deformation of its hydrodynamic limit.
Following [35], we will expose the notion of constrained KP hierarchy and its corresponding bi-hamiltonian structure. In particular, we will consider the associated central invariants. Then, we will recall the Frobenius manifold structure associated with the constrained KP hierarchy. Following [41], this manifold has a structure of Frobebius-Hurwitz manifold; furthermore, it is isomorphic to the Frobenius manifold $\mathcal{M}_{B_n}$, exposed in chapter eight (thus conjecture 2 of page 124 is confirmed).
Such identification allows us to compute explicitly the structure constants associated with the dual product of $\mathcal{M}_{B_n}$, confirming conjecture 1 of page 124.

9.1 Constrained KP and their bi-Hamiltonian structure

Let D be the ring of pseudo-differential operator of the form

$$\sum_{k\leq m} f_k\partial^k$$

where $\partial=\frac{\partial}{\partial x}$, $f_k : x\mapsto f_k(x)$ are smooth functions on $\mathbb{R}$ and m is an integer.
For any two pseudo-differential operator $A=\sum_{k\leq m_1} f_k\partial^k$ and $B=\sum_{k\leq m_2} g_k\partial^k$ of D, their product is given by

$$AB := \sum_{k\leq m_1}\sum_{j\leq m_2}\binom{k}{l} f_k\frac{\partial^l g_j}{\partial x^l}\partial^{k+j-l}$$

For a fixed non-negative integer m, we define the pseudo-differential operator L, depending on $v_n, v_{n-1}, ..., v_1, w, u$, as follows:

$$L := \partial^{n+1} + v_n \partial^{n-1} + ... + v_2 \partial + v_1 + (\partial - w)^{-1} u \tag{9.1}$$

where the operator

$$(\partial - w)^{-1} = a_1 \partial^{-1} + a_2 \partial^{-2} + ...$$

is uniquely defined by the identity

$$(\partial - w)(a_1 \partial^{-1} + a_2 \partial^{-2} + ...) = 1$$

We define the $(n+1)-$constrained KP hierarchy by the following Lax equation:

$$\frac{\partial L}{\partial t^k} = [\left(L^{\frac{k}{n+1}}\right)_+, L] \tag{9.2}$$

for $k = 1, 2, ...$, where $[A, B] : AB - BA$ (A and B pseudo-differential operator). The constrained KP hierarchy has a bihamiltonian structure defined as follows. First, denote

$$B := L_+ = \partial^{n+1} + v_n \partial^{n-1} + ... + v_2 \partial + v_1$$

the differential part of L. Given a functional

$$F = \int f(\mathbf{v}, \mathbf{v}_x, ...) dx$$

with a suitable domain of integration, where $\mathbf{v} = (v_1, ..., v_n)$, we define the variational derivative with respect the pseudo-differential operator L by

$$\frac{\delta F}{\delta L} := \frac{\delta F}{\delta B} + \frac{\delta F}{\delta u} + \frac{\delta F}{\delta w} \frac{1}{w} (\partial - u)$$

The variational derivative of F with respect to the differential operator B is defined by

$$\frac{\delta F}{\delta B} := \sum_{i=1}^{n} \partial^{-i} \frac{\delta F}{\delta v_i}$$

while $\frac{\delta}{\delta u}$ denotes the variational derivative.
Recall that the residue of a pseudo-differential operator is defined by

$$res\left(\sum_{k \leq m} a_k \partial^k\right) := a_{-1}$$

It can be proven that the variation of the functional F, defined by

$$\delta F := \int \left(\sum_{i=1}^{n} \frac{\delta F}{\delta v_i(x)} \delta v_i(x) + \frac{\delta F}{\delta u(x)} \delta u(x) + \frac{\delta F}{\delta w(x)} \delta w(x)\right) dx$$

can be represented as

$$\delta F = \int res\left(\frac{\delta F}{\delta L} \delta L\right) dx$$

For two functionals $F = \int f(\mathbf{v}, \mathbf{v}_x, ...)dx$ and $G = \int g(\mathbf{v}, \mathbf{v}_x, ...)dx$, denote their variational derivative with respect L by $X := \frac{\delta F}{\delta L}$ and $Y := \frac{\delta G}{\delta L}$. Then the compatible pair of Poisson brackets for constrained KP hierarchy are given by

$$\{F, G\}_{(1)} = \int res\big([L, X_+]Y - [L, X]_+ Y\big)dx \tag{9.3}$$

$$\{F, G\}_{(2)} = \int res\big((LY)_+ LX - (YL)_+ XL + \tfrac{1}{n+1} X[L, K_Y]\big)dx \tag{9.4}$$

where $K_Y := \partial^{-1} res\big([L, Y]\big)$. Define the Hamiltonians

$$H_k := \int h_k(\mathbf{v}, \mathbf{v}_x, ...)dx$$

for $k \geq -n$, with densities

$$h_k := \tfrac{n+1}{k+n+1} res\big(L^{\frac{k+n+1}{n+1}}\big)$$

Thus it follows that the constrained KP hierarchy (9.2) has the following bihamiltonian representation:

$$\frac{\partial \mathbf{v}}{\partial t_k} = \{\mathbf{v}(x), H_k\}_{(1)} = \{\mathbf{v}(x), H_{k-n+1}\}_{(2)} \tag{9.5}$$

for $k \geq 1$.

9.2 Central invariants

Consider the bihamiltonian structure given by the pair of Poisson brackets

$$\{F, G\}_{(a)} := \int \frac{\delta F}{\delta w^i(x)} P^{ij}_{(a)} \frac{\delta G}{\delta w^j(x)} \tag{9.6}$$

for $a = 1, 2$, where the local functionals F and G are defined on the jet space a $n-$dimensional manifold M with local coordinates $(w^1, ..., w^n)$. The Hamiltonian operators $P^{ij}_{(1)}$ and $P^{ij}_{(2)}$ are given by the formulas

$$P^{ij}_{(a)} := g^{ij}_{(a)}(\mathbf{w})\partial_x + \Gamma^{ij}_{k(a)}(\mathbf{w})w^k_x + \sum_{z \geq 1} \epsilon^k \sum_{l=0}^{k+1} A^{ij}_{k,l,(a)}(\mathbf{w}, \mathbf{w}_x, ..., \partial^l_x \mathbf{w})\partial_x^{(k-l+1)} \tag{9.7}$$

for $a = 1, 2$, where $\mathbf{w} = (w^1, ..., w^n)$. In the above formula the matrices $(g^{ij}_{(a)})$, for $a = 1, 2$, are assumed to be non-degenerate and symmetric with entries smooth functions of $w^1, ..., w^n$. For $\epsilon = 0$ we get a Poisson bracket of hydrodynamic type. The functions $A^{ij}_{k,l(a)}$ are homogeneous polynomials of degree l; here we define the degrees of the jet variables as

$$deg(\partial^k_x w^j) = k$$

for $k \leq 0$ and $j = 1, ..., n$.
Recall that the semisemplicity of a bihamiltonian structure is characterized by the following proposition (see [17] for details):

Proposition 9.1 *The n roots $\lambda^1(\mathbf{w}), ..., \lambda^n(\mathbf{w})$ of the characteristic polynomial*

$$det\left(g^{ij}_{(2)} - \lambda g^{ij}_{(1)}\right) = 0$$

constitute a local coordinate system of the manifold M in a neighborhood of a semi-simple point $p \in M$. We call them canonical coordinates for the bihamiltonian structure on M. Moreover, the leading coefficients of the operators $P^{ij}_{(1)}$ and $P^{ij}_{(2)}$ in canonical coordinates reduce to the diagonal matrices

$$g^{ij}_{(1)}(\lambda) = f(\lambda)\delta^{ij}$$
$$g^{ij}_{(2)}(\lambda) = \lambda^i f(\lambda)\delta^{ij}$$

respectively.

Denote

$$P^{ij}_{(a)}(\lambda) := \frac{\partial\lambda^i}{\partial w^k} A^{kl}_{1,0,(a)} \frac{\partial\lambda^j}{\partial w^l}$$
$$Q^{ij}_{(a)}(\lambda) := \frac{\partial\lambda^i}{\partial w^k} A^{kl}_{2,0,(a)} \frac{\partial\lambda^j}{\partial w^l}$$

for $a = 1, 2$. The central invariants $c_1(\lambda), ..., c_n(\lambda)$ of the bihamiltonian structure (9.6) are defined by

$$c_i(\lambda) := \frac{1}{3(f^i(\lambda))^2}\left(Q^{ij}_{(2)}(\lambda) - \lambda^i Q^{ij}_{(1)} + \sum_{k\neq i} \frac{\left(P^{ij}_{(2)}(\lambda) - \lambda^i P^{ij}_{(1)}\right)^2}{f^k(\lambda)(\lambda^k - \lambda^i)}\right) \tag{9.8}$$

Now, introduce the dispersion parameter ϵ in the constrained KP hierarchy (9.2) and its bihamiltonian structure defined by (9.3) and (9.4), by the following rescalings:

$$\frac{\partial}{\partial t^k} \mapsto \epsilon\frac{\partial}{\partial t^k}$$
$$\frac{\partial}{\partial x^k} \mapsto \epsilon\frac{\partial}{\partial x^k} := D$$

Then, for the two pseudo-differential operator $A = \sum_{k\leq m_1} f_k\partial^k$ and $B = \sum_{k\leq m_2} g_k\partial^k$ of D, their product reads as

$$AB := \sum_{k\leq m_1}\sum_{j\leq m_2} \epsilon^l \binom{k}{l} f_k \frac{\partial^l g_j}{\partial x^l} D^{k+j-l}$$

Similarly, after rescaling, the constrained KP hierarchy (9.2) takes the form

$$\frac{\partial L}{\partial t^k} = [\left(L^{\frac{k}{n+1}}\right)_+, L]$$

for $k \geq 1$, where the Lax operator is given by

$$L := D^{n+1} + v_n D^{n-1} + ... + v_2 D + v_1 + (D - w)^{-1}u$$

The corresponding compatible pair of Poisson bracket (9.3) and (9.4), after rescaling, read as

$$\{F,G\}_{(1)} = \frac{1}{\epsilon}\int res\big([L,X_+]Y - [L,X]_+Y\big)dx \tag{9.9}$$

$$\{F,G\}_{(2)} = \frac{1}{\epsilon}\int res\big((LY)_+LX - (YL)_+XL + \tfrac{1}{n+1}X[L,K_Y]\big)dx \tag{9.10}$$

Remark 9.2 *It can be shown that the Poisson brackets (9.9) and (9.10) for the constrained KP hierarchy have the form (9.6).*
Thus using the formula (9.8), one defines the central invariants associated with constrained KP hierarchy.

Following [35], we recall the following fundamental result:

Theorem 9.3 *The central invariants associated with constrained KP hierarchy coincide with $\frac{1}{24}$.*

9.3 Frobenius manifold underlying constrained KP hierarchy

We know that the central invariants of the bihamiltonian structure associated with constrained KP hierarchy coincide with $\frac{1}{24}$. This characterizes the topological deformation of the principal hierarchy of a semi-simple Frobenius manifold. Indeed there exists a $(n+2)$-dimensional semi-simple Frobenius manifold M underlying the constrained KP hierarchy. Let's consider the LG superpotential

$$\lambda(p) = p^{n+1} + v_n p^{n-1} + ... + v_2 p + v_1 + \frac{u}{p-w} \tag{9.11}$$

The data of the Frobenius manifold M are given by the residue formulas given in section 4. In particular, the invariant flat metric reads

$$\eta\Big(\frac{\partial}{\partial v_i},\frac{\partial}{\partial v_j}\Big) = -\big(\underset{p=\infty}{res} + \underset{p=w}{res}\big)\frac{\frac{\partial\lambda(p)}{\partial v_i}\frac{\partial\lambda(p)}{\partial v_j}}{\lambda'(p)}dp$$

and the multiplication on the tangent bundle reads

$$c\Big(\frac{\partial}{\partial v_i},\frac{\partial}{\partial v_j},\frac{\partial}{\partial v_k}\Big) = -\big(\underset{p=\infty}{res} + \underset{p=w}{res}\big)\frac{\frac{\partial\lambda(p)}{\partial v_i}\frac{\partial\lambda(p)}{\partial v_j}\frac{\partial\lambda(p)}{\partial v_k}}{\lambda'(p)}dp$$

for $i,j = 1,....,n+2$, where $v_{n+1} := w$ and $v_{n+2} := u$. The flat coordinates of the Frobenius manifold can be chosen as

$$\tilde v_i = -\tfrac{n+1}{n+1-i}\underset{p=\infty}{res}\,\lambda(p)^{1-\frac{i}{n+1}} \quad (i=1,....,n)$$
$$\tilde v_{n+1} = v_{n+1}$$
$$\tilde v_{n+2} = v_{n+2}$$

Then the prepotential $F(\tilde{v})$ of the Frobenius manifold has the form

$$F(\tilde{v}) = P(\tilde{v}_1, ..., \tilde{v}_{n+2}) + \frac{1}{2}\tilde{v}_{n+2}^2\left(log(v_{n+2}) - \frac{3}{2}\right)$$

where $P(\tilde{v}_1, ..., \tilde{v}_{n+2})$ is a quasi-homogeneous polynomial.

Remark 9.4 *The flat metric η defines a Poisson bracket of hydrodynamic on the loop space of the Frobenius manifold M associated with constrained KP hierarchy. It can be shown that the Poisson bracket coincides with the Poisson bracket obtained from the dispersionless limit $\epsilon \to 0$ of the Poisson bracket (9.9).*
There exists a second flat metric, the intersection form g (defined outside the discriminant Σ), which is given by the formula

$$g\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = -\left(\underset{p=\infty}{res} + \underset{p=w}{res}\right)\frac{\frac{\partial\lambda(p)}{\partial v_i}\frac{\partial\lambda(p)}{\partial v_j}}{\lambda(p)\lambda'(p)}dp$$

for $i, j = 1, ..., n+2$. Similarly to η, g defines a Poisson bracket of hydrodynamic type on the loop space of M, which coincides with the dispersionless limit $\epsilon \to 0$ of the Poisson bracket (9.10). These two compatible Poisson brackets, associated with η and g, yield a bihamiltonian structure on the loop space of the Frobenius manifold M and a bihamiltonian integrable hierarchy of hydrodynamic type, called the principal hierarchy of M. It can be represented as

$$\frac{\partial\tilde{v}_i}{\partial t^{j,m}} = \eta^{ik}\frac{\partial}{\partial x}\left(\frac{\partial\theta_{j,m+1}}{\partial\tilde{v}_k}\right)$$

for $m > 0$ and $i, j = 1, ..., n+2$ and where the functions $\theta_{j,m+1}$ define the calibration of M. For details about the principal hierarchy associated with Frobenius manifolds and calibration see [17] and [24].

9.4 Equivalence between KP constrained hierarchy and B_n Frobenius manifold

We observed, in the previous section, that the prepotential F_{B_n}, for $n = 2, 3, 4$, coincides with the solution of WDVV equations (9.3) associated with constrained KP hierarchy. Is it true for any n? The answer is affirmative. In particular, following [41], we will show that the Frobenius manifold structure M_{B_n} associated with B_n (exposed in the section 7) is isomorphic to the Hurwitz-Frobenius manifold structure on $M_{0;n-2,0}$, which coincides with the Frobenius manifold structure associated with constrained KP hierarchy exposed previously, with prepotential (9.3).
Let's consider the Hurwitz space $M_{0;n-2,0}$ of a particular class of LG superpotential consisting of Laurent polynomials in one variable with bidegree $(n-1, 1)$. These are the functions of the form

$$\lambda(z) = z^{n-1} + a_1 z^{n-2} + ... + a_{n-1} + a_n z^{-1} \tag{9.12}$$

where any $a_i \in \mathbb{C}$ and $a_n \neq 0$.
We have seen how to define a semi-simple Frobenius manifold structure on $M_{0;n-2,0}$ with coordinates $(a_1, ..., a_n)$.
Recall that the invariant metric η and intersection form g are given by the formulas

$$\eta(X,Y) = \frac{1}{4}\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(z)dz\big)Y\big(\lambda(z)dz\big)}{d\lambda(z)} \tag{9.13}$$

$$g(X,Y) = \frac{1}{4}\sum \underset{d\lambda=0}{res} \frac{X\big(log(\lambda(z))dz\big)Y\big(log(\lambda(z))dz\big)}{dlog\lambda(z)} \tag{9.14}$$

where $-\frac{1}{4}$ is a arbitrary normalization factor and X, Y are vector fields.
Consider the Euler vector field of the form

$$E = \frac{j}{n-1}a_j\frac{\partial}{\partial a_j} \tag{9.15}$$

moreover, we have $\mathcal{L}_E\lambda(z) = \lambda(z) - \frac{z}{n-1}\lambda'(z)$.
The symmetric $(0,3)$ tensor field $c(X,Y,Z) := \eta(X \circ Y, Z)$ is given by

$$c(X,Y,Z) = \frac{1}{4}\sum \underset{d\lambda=0}{res} \frac{X\big(\lambda(z)dz\big)Y\big(\lambda(z)dz\big)Z\big(\lambda(z)dz\big)}{d\lambda(z)} \tag{9.16}$$

where $\circ$ is the multiplication of tangents vectors on $M_{0;n-2,0}$.
Let e be the unity vector field of $\circ$, i.e. $e \circ X = X \circ e = X$ for any X.
Using explicit expression (9.13) and (9.16), $c(e,Y,Z) = \eta(e \circ Y, Z)$ implies $\mathcal{L}_e\lambda(z) = 1$. Thus we have

$$e = \frac{\partial}{\partial a_{n-1}} \tag{9.17}$$

For $z \to \infty$, one inverts (9.12) as follows (Puiseux expansion):

$$z = \lambda(z)^{\frac{1}{n-1}} - \big(t^1 + t^2\lambda(z)^{-\frac{1}{n-1}} + ... + t^n\lambda(z)^{-1}\big) + O\big(\lambda(z)^{-\frac{j-1}{n-1}}\big) \tag{9.18}$$

where $(t^1, ..., t^n)$ are flat coordinates for the metric η.
By using the "thermodynamical" identity

$$\frac{\partial}{\partial t^i}(\lambda dz)_{z=const.} = -\frac{\partial}{\partial t^i}(zd\lambda)_{\lambda=const.} \tag{9.19}$$

one gets (using (9.18))

$$\frac{\partial\lambda}{\partial t^i} = \Big(\lambda(z)^{\frac{i-1}{n-1}}\lambda'(z)\Big)_{\geq -1} \qquad (i = 1, ..., n) \tag{9.20}$$

where $(f)_{\geq -1} = \sum_{j=-1}^{n} f_j z^j$ for the Laurent series $f = \sum_{j=-\infty}^{n} f_j z^j$.
Thus (9.14) and (9.20) yields

$$\eta\Big(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\Big) = \frac{n-1}{4}\delta_{n+1,i+j} \tag{9.21}$$

Using the identities

$$\mathcal{L}_e\lambda(z) = 1$$
$$\mathcal{L}_E\lambda(z) = \lambda(z) - \frac{z}{n-1}\lambda'(z)$$

one gets

$$e = \frac{\partial}{\partial t_{n-1}} \tag{9.22}$$
$$E = \frac{j}{n-1} t_j \frac{\partial}{\partial t_j} \tag{9.23}$$

Recall, briefly, that the Frobenius manifold structure M_{B_n} consist of a flat pencil of cometrics

$$g^{ij}_{B_n}(u) = \frac{\partial u^i}{\partial p^s}\frac{\partial u^j}{\partial p^q}(dp^s, dp^q)$$
$$\eta^{ij}_{B_n}(u) = \frac{\partial \eta^{ij}(u)}{\partial u^{n-1}}$$

where $(dp^s, dp^q) = \frac{1-\delta^{sq}}{p^s p^q}$ is a $B_n$invariant cometric on $\mathbb{R}^n$ and $(u^1, ..., u^n)$ are the elementary symmetric polynomials in $(p^1)^2, ..., (p^n)^2$. Moreover, the unity and Euler vector fields are given by

$$e = \frac{\partial}{\partial u_{n-1}}$$
$$E = \frac{j}{n-1} u_j \frac{\partial}{\partial u_j}$$

Now, we can state the following fundamental result:

Theorem 9.5 *The map $h : M_{0;n-2,0} \to \mathcal{M}_{B_n}$, given by $a_j \mapsto u_j$ with*

$$u_j = a_j \tag{9.24}$$

for $i = 1, ..., n$, is a local isomorphism of Frobenius manifolds.

Since the Hurwitz-Frobenius manifold on $M_{0;n-2,0}$ is semi-simple by definition. By using the isomorphism h one gets the following:

Corollary 9.6 *The Frobenius manifold $\mathcal{M}_{B_n}$ is semi-simple.*

Remark 9.7 *Substituting $z = p - b_n$ in (9.12), we retrieve the LG superpotential (9.11) of the Frobenius manifold structure associated with the constrained KP hierarchy.*

9.5 Structure constants of $\mathcal{M}_{B_n}$

By the identification of $\mathcal{M}_{B_n}$ with Hurwitz-Frobnius manifold $M_{0;n-2,0}$, by exploiting formula (4.93), one can retrieve the structure constants associated with the dual product of $\mathcal{M}_{B_n}$. One has the following:

Proposition 9.8 *The dual product of $\mathcal{M}_{B_n}$ has the form*

$$* = \sum_{H\in\widetilde{\mathcal{H}}} \frac{d\alpha_H}{\alpha_H(p)} \otimes \pi_H \tag{9.25}$$

where $\widetilde{\mathcal{H}} = \{p^i - p^j = 0\}_{i\neq j\in\{1,\dots,n\}} \cup \{p^i + p^j = 0\}_{i\neq j\in\{1,\dots,n\}}$.

Proof: Using formula (4.93) one has

$$c^*_{ijk} := c^*\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}, \frac{\partial}{\partial p^k}\right) = \frac{1}{4}\sum \underset{\lambda'=0}{res} \frac{\frac{\partial log\lambda(z)}{\partial p^i}\frac{\partial log\lambda(z)}{\partial p^j}\frac{\partial log\lambda(z)}{\partial p^k}}{\big(log\lambda(z)\big)'}dz$$

Applying Vieta's formula the rational superpotential (9.12) factorizes as

$$\lambda(z) = z^{-1}\big(z^n + a_1 z^{n-1} + ... + a_{n-1}z + a_n\big) = z^{-1}\prod_{i=1}^{n}\big(z + (p^i)^2\big)$$

Thus

$$log\lambda(z) = -log(z) + \sum_{i=1}^{n} log\big(z + (p^i)^2\big)$$

So

$$\frac{\partial log\lambda(z)}{\partial p^i} = \frac{2p^i}{z + (p^i)^2}$$

$$\big(log\lambda(z)\big)' = \frac{1}{z} + \sum_{i=1}^{n}\frac{1}{z + (p^i)^2}$$

Hence

$$c^*_{ijk} = 2\sum \underset{\lambda'=0}{res} \underbrace{\frac{p^i p^j p^k}{\big(z + (p^i)^2\big)\big(z + (p^j)^2\big)\big(z + (p^k)^2\big)\big(log\lambda(z)\big)'}dz}_{:=f_{ijk}(z)dz}$$

here $f_{ijk}(z)dz$ is meromorphic 1-form on the Riemann sphere $\mathbb{CP}^1$.
Observe that $\big(log\lambda(z)\big)' = \frac{\lambda'(z)}{\lambda(z)}$, thus the critical value of $\lambda(z)$ are poles for $f_{ijk}(z)dz$.
Now, since $\mathbb{CP}^1$ is a complex compact manifold, the sum of the residue of $f(z)dz$ vanishes. Thus, for $i = j = k$, one has

$$c^*_{iii} = -2 \underset{z=-(p^i)^2}{res} f_{iii}(z)dz = -2 \underset{z=-(p^i)^2}{res} \frac{(p^i)^3}{\big(z + (p^i)^2\big)^3\big(-\frac{1}{z} + \sum_{s=1}^{n}\frac{1}{z+(p^s)^2}\big)}dz$$

In the limit $z \mapsto -(p^i)^2$ one has

$$f_{iii}(z) = \frac{(p^i)^3}{\big(z+(p^i)^2\big)^3\big(-\frac{1}{z}+\sum_{s\neq i}\frac{1}{z+(p^s)^2}+\frac{1}{z+(p^i)^2}\big)} \sim \frac{1}{\big(z+(p^i)^2\big)^2}$$

thus $z=-(p^i)^2$ is a second-order pole for $f(z)dz$. Then

$$c^*_{iii} = -2 \underset{z=-(p^i)^2}{res} f_{iii}(z)dz = -2 \lim_{z\mapsto-(p^i)^2} \frac{d}{dz}\Big(\big(z+(p^i)^2\big)^2 f_{iii}(z)\Big) \tag{9.26}$$

$$= -2(p^i)^3\Big(\sum_{s\neq i}\frac{1}{(p^i)^2-(p^s)^2}-\frac{1}{(p^i)^2}\Big) \tag{9.27}$$

For $i=j\neq k$, $f(z)dz$ has a first-order pole at $z=-(p^i)^2$, then

$$c^*_{iik} = -2 \underset{z=-(p^i)^2}{res} f_{iik}(z)dz = -2 \lim_{z\mapsto-(p^i)^2}\Big(\big(z+(p^i)^2\big)f_{iik}(z)\Big) = \frac{(p^i)^2p^k}{(p^i)^2-(p^k)^2} \tag{9.28}$$

While, for $i\neq j\neq k$, $f(z)dz$ has no pole except the critical points for $\lambda(z)$, hence

$$c^*_{ijk} = 0 \tag{9.29}$$

We conjectured that the dual product of $\mathcal{M}_{B_n}$ has the form

$$* = \sum_{H\in\widetilde{\mathcal{H}}}\frac{d\alpha_H}{\alpha_H(p)}\otimes\pi_H \tag{9.30}$$

where $\widetilde{\mathcal{H}} = \{p^i-p^j=0\}_{i\neq j\in\{1,\ldots,n\}}\cup\{p^i+p^j=0\}_{i\neq j\in\{1,\ldots,n\}}$, here the orthogonal projector π_H is obtained via the Euclidean metric $g=(\delta^{ij})$.
Thus the corresponding structure constants read

$$c^{k*}_{ij} = \underbrace{\frac{1}{2}\sum_{s\neq r\in\{1,\ldots,n\}}\frac{(dp^s-dp^r)_i(dp^s-dp^r)_j(dp^s-dp^r)_k}{p^s-p^r}}_{(I)} + \underbrace{\frac{1}{2}\sum_{s\neq r\in\{1,\ldots,n\}}\frac{(dp^s+dp^r)_i(dp^s+dp^r)_j(dp^s+dp^r)_k}{p^s-p^r}}_{(II)}$$

Check that these functions coincide with the actual dual structure constants of $\mathcal{M}_{B_n}$ computed above.
It turns out that

$$(dp^s-dp^r)_i = \delta_{si}-\delta_{ri}$$

then

$$(I) = \sum_{s=1}^n\sum_{r\neq s}\frac{1}{p^s-p^r}\big(\delta_{si}\delta_{sj}\delta_{sk}-\delta_{si}\delta_{sj}\delta_{rk}-\delta_{si}\delta_{rj}\delta_{sk}+\delta_{si}\delta_{rj}\delta_{rk}-\delta_{ri}\delta_{sj}\delta_{sk}+\delta_{ri}\delta_{sj}\delta_{rk} + \delta_{ri}\delta_{rj}\delta_{sk}-\delta_{ri}\delta_{rj}\delta_{rk}\big)$$

Observe that

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{si}\delta_{sj}\delta_{sk}}{p^s-p^r}=\sum_{r\neq s}\sum_{s=1}^{n}\frac{\delta_{si}\delta_{sj}\delta_{sk}}{p^s-p^r}=\sum_{r\neq i}\frac{\delta_{ij}\delta_{jk}\delta_{ik}}{p^s-p^r}$$

Moreover

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{si}\delta_{sj}\delta_{rk}}{p^s-p^r}=\sum_{s=1}^{n}\delta_{si}\delta_{sj}\sum_{r\neq s}\frac{\delta_{rk}}{p^s-p^r}=\sum_{s\neq k}\frac{\delta_{si}\delta_{sj}}{p^s-p^k}=\delta_{ij}f_{ik}$$

where

$$f_{ik}:=\begin{cases}\frac{1}{p^i-p^k} & (i\neq k)\\ 0 & (i=k)\end{cases} \tag{9.31}$$

Similarly

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{si}\delta_{rj}\delta_{sk}}{p^s-p^r}=\delta_{ik}f_{ij}$$

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{si}\delta_{rj}\delta_{rk}}{p^s-p^r}=\delta_{jk}f_{ij}$$

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{si}\delta_{rj}\delta_{rk}}{p^s-p^r}=\delta_{jk}f_{ij}$$

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{ri}\delta_{sj}\delta_{sk}}{p^s-p^r}=\delta_{jk}f_{ji}$$

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{ri}\delta_{rj}\delta_{sk}}{p^s-p^r}=\delta_{ij}f_{ki}$$

$$\sum_{s=1}^{n}\sum_{r\neq s}\frac{\delta_{ri}\delta_{rj}\delta_{rk}}{p^s-p^r}=-\sum_{r\neq i}\frac{\delta_{ij}\delta_{jk}\delta_{ik}}{p^i-p^r}$$

Thus

$$\begin{aligned}(I)&=2\delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\frac{1}{p^i-p^r}-\delta_{ij}f_{ik}-\delta_{ik}f_{ij}+\delta_{jk}f_{ij}-\delta_{jk}f_{ji}+\delta_{ik}f_{ji}+\delta_{ij}f_{ki}\\ &=2\left(\delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\frac{1}{p^i-p^r}-\delta_{ij}f_{ik}-\delta_{ik}f_{ij}-\delta_{jk}f_{ji}\right)\end{aligned}$$

Analogously

$$\begin{aligned}(II)&=2\delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\frac{1}{p^i+p^r}+\frac{\delta_{ij}(1-\delta_{ik})}{p^i+p^k}+\frac{\delta_{ik}(1-\delta_{ij})}{p^i+p^j}+\frac{\delta_{jk}(1-\delta_{ij})}{p^i+p^j}\\ &+\frac{\delta_{jk}(1-\delta_{ji})}{p^i+p^j}+\frac{\delta_{ik}(1-\delta_{ji})}{p^i+p^j}+\frac{\delta_{ij}(1-\delta_{ik})}{p^k+p^i}\\ &=2\left(\delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\frac{1}{p^i+p^r}+\frac{\delta_{ij}(1-\delta_{ik})}{p^i+p^k}+\frac{\delta_{ik}(1-\delta_{ij})}{p^i+p^j}+\frac{\delta_{jk}(1-\delta_{ij})}{p^i+p^j}\right)\end{aligned}$$

Hence

$$\begin{aligned} c_{ij}^{k*} &= \tfrac{1}{2}(I) + \tfrac{1}{2}(II) \\ &= \delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\left(\frac{1}{p^i+p^r}+\frac{1}{p^i-p^r}\right) + \delta_{ij}\left(\frac{1-\delta_{ik}}{p^i+p^k}-f_{ik}\right) \\ &+ \delta_{ik}\left(\frac{1-\delta_{ij}}{p^i+p^k}-f_{ij}\right) + \delta_{jk}\left(\frac{1-\delta_{ij}}{p^i+p^k}-f_{ji}\right) \end{aligned}$$

Using the definition of f_{ij} one gets

$$c_{ij}^{k*} = 2\delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\frac{p^i}{p^i+p^r}f_{ir} - 2\delta_{ij}\left(\frac{p^k}{p^i+p^k}f_{ik}\right) - 2\delta_{ik}\left(\frac{p^j}{p^i+p^j}f_{ij}\right) + 2\delta_{jk}\left(\frac{p^i}{p^i+p^j}f_{ij}\right)$$

Lowering an index by (7.1) one has

$$c_{qij}^{*} := g_{qk}\, c_{ij}^{k*} = (\tfrac{1}{n-1} - \delta_{qk})p^q p^k c_{ij}^{k*}$$

In particular

$$\begin{aligned} &2(\tfrac{1}{n-1} - \delta_{qk})p^q p^k \delta_{ij}\delta_{jk}\delta_{ik}\sum_{r\neq i}\tfrac{p^i}{p^i+p^r}f_{ir} = \tfrac{2}{n-1}p^q\delta_{ij}\underbrace{p^k\delta_{jk}\delta_{ik}}_{=p^i\delta_{ij}}\sum_{r\neq i}\tfrac{p^i}{p^i+p^r}f_{ir} - 2p^q\delta_{ij}\underbrace{p^k\delta_{qk}\delta_{jk}\delta_{ik}}_{=p^q\delta_{qj}\delta_{ji}\delta_{qi}}\sum_{r\neq i}\tfrac{p^i}{p^i+p^r}f_{ir} \\ &= \tfrac{2}{n-1}p^q(p^i)^2\delta_{ij}\sum_{r\neq i}\tfrac{1}{p^i+p^r}f_{ir} - 2(p^q)^2p^i\underbrace{\delta_{qj}\delta_{ji}\delta_{qi}}_{=\delta_{ij}}\sum_{r\neq i}\tfrac{1}{p^i+p^r}f_{ir} \\ &= 2\delta_{ij}\sum_{r\neq i}\tfrac{f_{ir}}{p^i+p^r}\left(\tfrac{p^q(p^i)^2}{n-1} - (p^q)^2p^i\right) \end{aligned}$$

and

$$-2(\tfrac{1}{n-1} - \delta_{qk})p^q p^k\delta_{ij}\left(\tfrac{p^k}{p^i+p^k}f_{ik}\right) = -\tfrac{2}{n-1}\delta_{ij}p^q\tfrac{(p^k)^2}{p^i+p^k}f_{ik} + 2p^q\delta_{ij}\underbrace{\tfrac{(p^k)^2}{p^i+p^k}\delta_{qk}f_{ik}}_{=\tfrac{(p^q)^2}{p^i+p^q}f_{iq}}$$

and

$$-2(\tfrac{1}{n-1} - \delta_{qk})p^q p^k\delta_{ik}\left(\tfrac{p^j}{p^i+p^j}f_{ij}\right) = -\tfrac{2}{n-1}\underbrace{\delta_{ik}p^k}_{=p^i}p^q\tfrac{p^j}{p^i+p^k}f_{ij} + 2\tfrac{p^qp^j}{p^i+p^j}f_{ij}\underbrace{\delta_{qk}\delta_{ik}p^k}_{=\delta_{iq}p^q}$$

and similarly

$$2(\tfrac{1}{n-1} - \delta_{qk})p^q p^k\delta_{jk}\left(\tfrac{p^i}{p^i+p^j}f_{ij}\right) = \tfrac{2}{n-1}p^qp^j\tfrac{p^i}{p^i+p^j}f_{ij} - 2\delta_{qj}(p^q)^2\tfrac{p^i}{p^i+p^j}f_{ij}$$

Hence

$$\begin{aligned} c_{qij} =& 2\delta_{ij}\sum_{r\neq i}\frac{f_{ir}}{p^i+p^r}\left(\frac{p^q(p^i)^2}{n-1} - (p^q)^2p^i\right) + 2\delta_{ij}p^q\left(-\tfrac{1}{n-1}\frac{(p^k)^2}{p^i+p^k}f_{ik} + \frac{(p^q)^2}{p^i+p^q}f_{iq}\right) \\ &+ 2\delta_{iq}(p^q)^2\frac{p^j}{p^i+p^j}f_{ij} - 2\delta_{qj}(p^q)^2\frac{p^i}{p^i+p^j}f_{ij} \end{aligned}$$

Let $q = i = j$, since $f_{iq} = f_{ij} = 0$ by definition, one has

$$
\begin{aligned}
c^*_{iii} &= 2\sum_{r\neq i} \frac{f_{ir}}{p^i + p^r}\left(\frac{(p^i)^3}{n-1} - (p^i)^3\right) + 2p^i\left(-\tfrac{1}{n-1}\frac{(p^k)^2}{p^i+p^k} f_{ik}\right) \\
&= 2\frac{2-n}{n-1}(p^i)^3 \sum_{r\neq i} \frac{f_{ir}}{p^i+p^r} - \frac{2p^i}{n-1}\sum_{r\neq i}\frac{(p^r)^2}{p^i+p^r} f_{ir} \\
&\overset{(9.31)}{=} 2\frac{2-n}{n-1}(p^i)^3 \sum_{r\neq i}\frac{1}{(p^i)^2-(p^r)^2} - \frac{2p^i}{n-1}\sum_{r\neq i}\frac{(p^r)^2}{(p^i)^2-(p^r)^2} \\
&= \frac{2p^i}{n-1}\sum_{r\neq i}\frac{1}{(p^i)^2-(p^r)^2}\big(\underbrace{(2-n)(p^i)^2-(p^r)^2}_{(p^i)^2-(p^r)^2+(1-n)(p^i)^2}\big) \\
&= \frac{2p^i}{n-1}\Bigg(\underbrace{\sum_{r\neq i} 1}_{=n-1} + \sum_{r\neq i}\frac{(1-n)(p^i)^2}{(p^i)^2-(p^r)^2}\Bigg) \\
&= \frac{2(p^i)^3}{n-1}\left(\frac{1}{(p^i)^2} - \sum_{r\neq i}\frac{1}{(p^i)^2-(p^r)^2}\right)
\end{aligned}
$$

which coincides with (9.27).
Let $q = i \neq j$, then

$$
c_{iij} = 2(p^i)^2 \frac{p^j}{p^i+p^j} f_{ij} \overset{(9.31)}{=} \frac{2(p^i)^2 p^j}{(p^i)^2-(p^j)^2}
$$

which coincides with (9.28).
While, for $q \neq i \neq j$, $c^*_{qij} = 0$ as in (9.29). ∎

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