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**CYCLE XXX**

# Reconstruction of the physics of Inflation from next generation cosmological experiments

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# Contents

# Chapter 1

# Standard Cosmology

## 1.1 Introduction

In the modern cosmology, it is common believed that the universe comes out from a quantum-gravity state known as "initial singularity" that characterizes the so called Planck energy scale $E_p \sim 10^{19}$ GeV. Therefore, we need a complete quantum gravity theory to explain the properties of this initial (?) phase of the universe: such a theory has to describe the four fundamental forces (gravitational interaction, weak and strong interaction and the electromagnetic one) as a single interaction. From this point of view, this theory has to combine Quantum Field Theory (QFT) and General Relativity. Nowadays, we have some proposals such as string theory or superstring theory. However, at low energy scales the gravitational interactions decouples from the other ones: the expansion and geometry of the universe can be described by the standard General Relativity. In this introductory chapter we summarize the basics of General Relativity, its applications to cosmology and the so called standard Big Bang Model. In this phd thesis, we will use the particle "natural units" where $c = \hbar = 1$, otherwise indicated.

## 1.2 Basics of General Relativity

The General Relativity is the relativistic theory of the gravitational field (see references for academic discussions [1]). The fundamental assumption is the so called equivalence principle that states that a gravitational field corresponds to a curved spacetime or otherwise a gravitational field is a deformation of the Minkowski spacetime. The curvature is induced by the presence of masses or energies by using the well known Einstein energy-mass relation. In this way we pass from a standard flat spacetime defined by the constant metric tensor $\eta_{\mu\nu}$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad \eta_{\mu\nu} = diag\{-1, +1, +1, +1\} \tag{1.1}$$

to an "Einstenian" spacetime where the metric tensor is a function of the four-dimensional coordinate $x^\mu$

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad g_{\mu\nu} = diag\{-,+,+,+\}. \tag{1.2}$$

The free motion of a particle in this manifold is a geodesic described by the equation

$$\frac{d^2x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{1.3}$$

Here, the parameter $s$ is the worldline element and $\Gamma^\rho_{\mu\nu}$ are the so called Christoffel symbols or affine connections. It is possible to show that these quantities are related to the metric tensor by the following expression

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(-\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha}\right). \tag{1.4}$$

The curvature of the spacetime is described by the rank-4 Riemann tensor given by

$$R^\alpha_{\mu\nu\rho} = \partial_\nu\Gamma^\alpha_{\mu\rho} - \partial_\rho\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\rho}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\rho}. \tag{1.5}$$

The contraction of the Riemann tensor leads to the Ricci tensor

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{1.6}$$

and this describes how the volume (locally) changes wrt the volume in a Euclidean space. A second contraction gives us the Ricci scalar or scalar curvature

$$R = g_{\mu\nu}R^{\mu\nu}. \tag{1.7}$$

In particular, the sign of the scalar curvature indicates the relation between the volume $V_R$ of a $x$-centered sphere of the riemannian manifold and the volume of a equal-radius sphere $V_E$ in a hypothetical euclidean space

- If $R > 0$ then $V_R < V_E$.
- If $R < 0$ then $V_R > V_E$.

The relativistic action of a gravitational field in a empty spacetime (vacuum) is the Einstein-Hilbert action that comes out to be

$$S_{EH} = \int d^4x\sqrt{-g}\frac{1}{2}M_p^2 R \tag{1.8}$$

where $M_p = 1/\sqrt{8\pi G}$ is the reduced Planck mass and $G$ the Newton constant. However, the relativistic action in the presence of matter or some form of energy is characterized by the Einstein-Hilbert lagrangian density with corrections describing the source fields coupled to gravity

$$S = \int d^4x\sqrt{-g}\frac{1}{2}M_p^2 R + S_m\left[\phi_m\right]. \tag{1.9}$$

The principle of least action leads to the well known Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.10}$$

or in more compact way

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.11}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress energy tensor coming from the variation of the $S_m$ integral wrt the metric tensor $g_{\mu\nu}$. Anyway, we can reword the equations as

$$R_{\mu\nu} = 8\pi G S_{\mu\nu}, \quad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \tag{1.12}$$

where $S_{\mu\nu}$ is the source tensor and $T$ is the contraction of $T_{\mu\nu}$. We should note the Einstein-Hilbert action could be written introducing the lagrangian density of a constant scalar field we call $\Lambda$: such operation does not modify the properties of the integral. As a result, the Einstein field equations are naturally characterized by an extra terms and they sometimes are called equations with the cosmological constant $\Lambda$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.13}$$

The history of the cosmological constant is quite known and interesting and we do not linger on it.

## 1.3 Basics of FLRW Universe

In cosmology we typically assume that the universe enjoys the cosmological principle: the universe is homogeneous and isotropic on largest scales (in average). This allows us to consider the time constant hyper-surfaces as maximally symmetric spaces. Because of this assumption, there is a simple way to write the metric of the universe. This metric is the FLRW (Fridmann-Lemaitre-Robertson-Walker) or FRW metric and in spherical coordinates we have

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1.14}$$

The quantities $(r, \theta, \varphi)$ are known as comoving coordinates and follow the motion of the observer in the universe. Then, a particle or an observer in the comoving point $P(r, \theta, \varphi)$ is at rest as the universe evolves, unless a peculiar motion occurs. In particular, $r$ is called radial comoving coordinate and it is important to determine the distances in the FRW universe while $\theta$ and $\varphi$ are the polar and azimuthal angles, respectively. The quantity $t$ is the cosmic time and gives us the time measured in the rest reference system of a comoving clock of a comoving observer. The scalar $a(t)$ is the cosmic scale factor of the universe. modulating the time evolution of the spatial part of the metric elements $ds^2$. In the literature the value of the scale factor at the current epoch (the age of the universe), $t_0$, is fixed to unity: $a(t_0) = a_0 = 1$. Finally, the parameter $k$ describes the allowed topology of our FRW universe. We have three scenarios

- If $k = 1$ the curvature is positive and the universe has a spherical spatial geometry.
- If $k = 0$ the curvature is null and the universe has an asymptotic flat geometry.
- If $k = -1$ the curvature is negative and the universe as an hyperbolic spatial geometry.

These three cases are reported in Fig.(1.1). The physical distances are given

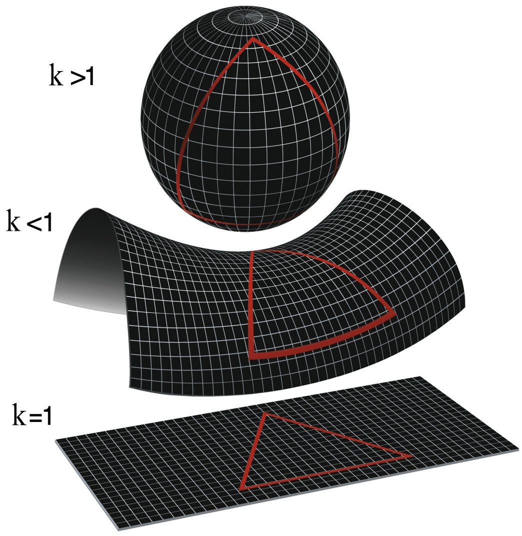


Figure 1.1: Curvature scenarios for the universe.

by the expression

$$R(t) = a(t)r \tag{1.15}$$

An important quantities is the Hubble rate or Hubble parameter (sometimes Hubble constant)

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{1.16}$$

This variable was firstly introduced by Edwin Hubble to parameterize the recession velocity of distant galaxies

$$v = H(t)d \tag{1.17}$$

The Hubble constant has time dimension and it assumes the same value everywhere in the universe at a given time $t$. As we will see in the next section, the Hubble rate is important to determine some fundamental physical scales in the universe.

### 1.3.1 Conformal Time

The casual structure of the universe is provided bu the propagation of the photon in the FLRW universe and the related null geodesics are described by the formula

$$ds^2 = 0 \tag{1.18}$$

The photon paths can be studied in a very simple way introducing the conformal time variable defined as

$$d\tau = \frac{dt}{a(t)} \quad \rightarrow \quad \tau = \int \frac{dt}{a(t)} \tag{1.19}$$

In this formalism, the FLRW metric is factorized in a Minkowski spacetime in the variable $\tau$

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)dl^2 \\ &= a^2(t)\left(-\frac{dt^2}{a^2(t)} + dl^2\right) = a^2(\tau)\left(-d\tau^2 + dl^2\right) \end{aligned} \tag{1.20}$$

We can observe that Eq.(1.18) provides bisector lines in the conformal plane $(l, \tau)$

$$\tau = \pm l + cost. \tag{1.21}$$

In this way we can recover the standard special relativistic framework for the light propagations.

### 1.3.2 Photons propagation

Once we define the metric structure of the universe we can ask what is the relative propagation of the light. Let us imagine photons are emitted at some initial time $t_i$ from the comoving position $r$ and reach us at the time $t > t_i$ at the position $r_0 = 0$. Since the motion of a photon is radial with $ds^2 = 0$, we have

$$ds^2 = -dt^2 + a^2(t)\frac{dr^2}{1 - kr^2} = 0 \tag{1.22}$$

and so

$$\frac{dr}{\sqrt{1 - kr^2}} = \pm\frac{dt}{a(t)}. \tag{1.23}$$

Anyhow, photons reach us ($dr < 0$) as time passes ($dt > 0$), therefore we take the negative sign of the equation. Integrating and changing the sign of the integration extremes, we conclude:

$$\int_0^r dr \frac{1}{\sqrt{1-kr^2}} = \int_{t_i}^t dt' \frac{1}{a(t')} \tag{1.24}$$

and we should note this is a comoving quantity we call comoving distance: it is invariant during the evolution. In literature we often find the following convention

$$\chi(r) = \int_0^r dr \frac{1}{\sqrt{1-kr^2}} = \int_{t_i}^t dt' \frac{1}{a(t')} \tag{1.25}$$

and corresponding proper distance or a physical distance is given by

$$d(r,t) = a(t) \int_{t_i}^t dt' \frac{1}{a(t')}. \tag{1.26}$$

### 1.3.3 Horizons

In cosmology, we can define particular kind of distances called "cosmological horizons" that help us to understand the casual structure of the universe. Let us suppose to consider the initial time $t = 0$. Then we can define the comoving particle horizon as the greatest comoving distance that a photon emitted at the time $t = 0$ has travelled to reach us at the time $t > 0$

$$\chi_p = \int_0^{R_{max}} dr \frac{1}{\sqrt{1-kr^2}} = \int_0^t dt' \frac{1}{a(t')}. \tag{1.27}$$

Instead, the proper particle horizon or physical particle horizon is given by

$$d_p(t) = a(t) \int_0^t dt' \frac{1}{a(t')} \tag{1.28}$$

Note that, the time $t$ could be fixed to the current epoch $t_0$. The particle horizon represents the larger distance from which we can receive information from the past, and so it defines the dimension of our casual past, i.e., our observable universe. But we can also define the comoving event horizon as the greatest comoving distance that a photon emitted at the time $t$ can ever travel before the end of the universe placed at $t = T$

$$\chi_E = \int_0^{r_{max}} dr \frac{1}{\sqrt{1-kr^2}} = \int_t^T dt' \frac{1}{a(t')} \tag{1.29}$$

The corresponding proper event horizon or physical event horizon

$$d_E(t) = a(t) \int_t^T dt' \frac{1}{a(t')}. \tag{1.30}$$

Actually we can also have $T \to \infty$ so in that case we have an event horizon if the integral converge at $\infty$. In addition, we can set the time of emission as the current epoch $t_0$. The event horizon describes the larger distance from which we will ever able to receive information (from the future) before the end of the universe. Then it defines the dimension of our casual future that we acquire as the time passes up to the final moment $t = T$. It is important to note that $T$ can assume a finite or an infinite value, depending on the evolution of the universe. Furthermore we can define the Hubble horizon or Hubble radius representing the dimension of the "observable universe"

$$R_H(t) = \frac{c}{H} \quad \text{in natural units} \rightarrow \quad R_H(t) = \frac{1}{H}. \tag{1.31}$$

The Hubble horizon defines a sphere with radius $R_H$ parameterizing the points of the sky that recede from an observer $O$ (us for instance) with the speed of light.

### 1.3.4 Fridmann Equations

The dynamics of the universe is described by the cosmic factor $a(t)$. We can get the equation of motion for the cosmic factor once we know the physical content of the FLRW universe. This content is conventionally codified by a perfect fluid with some energy density $\rho$ and pressure $p$. In the perfect fluid formalism the stress energy tensor takes the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.32}$$

with components

$$T_{00} = \rho, \quad T_{0i} = T_{i0} = 0, \quad T_{ii} = p \tag{1.33}$$

and $u_\mu$ is the four-velocity

$$u_\mu = \frac{dx_\mu}{d\tau} \quad \text{or} \quad u_\mu = \frac{dx_\mu}{ds} \tag{1.34}$$

where $s$ is the coordinate for the line element (world line). If we use the continuity equation we land to the thermodynamics law

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{1.35}$$

The field equation for $a(t)$ coming from the FLRW metric under the cosmic perfect fluid condition are

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left[\rho + \frac{\Lambda}{8\pi G}\right] \tag{1.36}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.37}$$

In general the cosmic fluid is characterized but different components: non relativistic matters and relativistic matters $\rho_m$, radiation $\rho_r$ and vacuum energy $\rho_\Lambda$

$$\rho = \rho_m + \rho_r + \rho_\Lambda, \quad \rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{1.38}$$

and each of these contributions dominates in different cosmological epoch and it is characterized by a proper equation of state given by the pressure-density ratio

$$w = p/\rho \tag{1.39}$$

In particolar, for the non relativistic matter we have $w = 0$, for the radiation $w = 1/3$ and for the vacuum $w = -1$. The solution of the continuity equation expressed in terms of $w$, for the matter/radiation case results to be

$$\rho(t) = \rho_0 a(t)^{-3(1+w)}, \quad w \neq -1 \tag{1.40}$$

therefore

$$\rho_m = \frac{\rho_m^0}{a^3}, \quad \rho_r = \frac{\rho_r^0}{a^4} \tag{1.41}$$

Otherwise we just get $\dot{\rho} = 0$ and so $\rho = const$. The dynamics of the energy density is shown in Fig.(1.2). Let us consider a universe very close to the

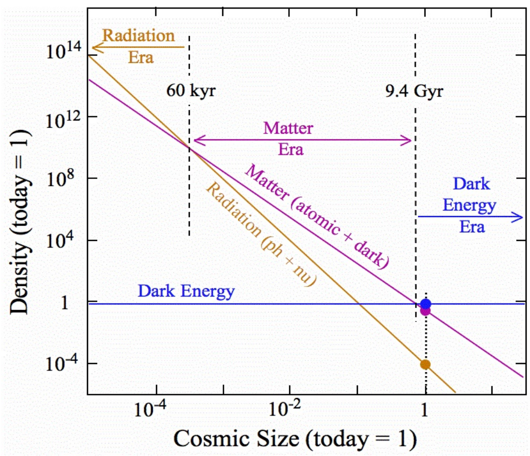


Figure 1.2: Evolution of the energy densities

flatness with $k = 0$ and a negligible $\Lambda$. The the solution of Friedamnn equation is

$$a(t) \sim t^{2/3(1+w)} \tag{1.42}$$

and we land to $a(t) \sim t^{1/2}$ for a matter dominated epoch while we land to $a(t) = t^{2/3}$ for a radiation dominated era. In the case of vacuum dominated era with $k = 0$, we have an exponential solution

$$a(t) = e^{Ht} \tag{1.43}$$

We can use a more compact way to express the previous relations. For example we can write

$$\rho = \sum_i \rho_i \quad p = \sum_i p_i \tag{1.44}$$

and we can introduce the density parameters, defined at the current time as

$$\Omega_i = \frac{\rho_0^i}{\rho_c} \tag{1.45}$$

where $\rho_0^i$ represents the energy density of the i-th component and $\rho_c^{(0)}$ is the critical value of the energy density defined as follows:

$$\rho_c^{(0)} = 3H_0^2/8\pi G \tag{1.46}$$

where $H_0$ is the current value of the Hubble rate. The equation of state parameters follow as

$$w_i = \frac{p_i}{\rho_i} \tag{1.47}$$

By using these quantities it is possible to reword the previous Fridmann equation in a very useful form wrt the Hubble rate

$$H^2(t) = H_0^2 \left[ \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_k a^{-2} \right], \quad \Omega_k = -\frac{k^2}{a_0^2 H_0^2} \tag{1.48}$$

where $\Omega_k$ is the density parameter related to the curvature at the current time $t_0$. As a consequence we have the advantage to outline the behavior of the Hubble rate in terms of the cosmic epoches. The sum of all density parameters (computed at $t_0$) naturally satisfies the consistency relation

$$\sum_i \Omega_i + \Omega_k = 1 \tag{1.49}$$

## 1.4 Hot Big Bang Model and cosmic epoches

The cosmological observations allows us to build a very good model for the expansion and the thermal evolution of the universe and the formation of the large scale structures (LSS). This model is commonly called Hot Big Bang Model (HBB). The standard BB model states universe emerges from an hypothetical “initial singularity” predicted by the Einstein theory. In the following, we have the so called Planck era where the energy scale is of the order of $E_p \sim 10^{19}$ GeV and where all the four fundamental interactions result to be unified in a single interaction. However, the universe expandes and cools and at low energy scales the gravitational interactions decouples from the other ones and the universe enters in an hypothetical

Grand Unified energy scales (GUT scale) where strong, weak and electromagnetic interactions are unified and described by an hypothetical gauge group with a specific symmetry. It is reasonable to think that at scales of the order of $E \sim 10^{16}$ GeV the GUT symmetry is spontaneously broken by an Higgs field in the standard gauge groups $SU(3) \times SU(2) \times U(1)$. The properties and the detailed evolution of the universe just below this scale is almost unknown. We just know the electroweak interaction is decoupled from the strong interaction and at a given time $t$ the baryogenesis mechanism takes place. As the universe reach the TeV scale, the physics become understandable. In particular, below TeV scale we have the spontaneous symmetry breaking of the electroweak symmetry $SU(2) \times U(1)$, ruled by the Higgs field $\Phi$. Here, weak and electromagnetic interactions decouple, bosons $W_{\pm}$ and $Z$ takes a mass while the photon remain massless. In the following, we have a quark-gluon plasma up to the scale of $\sim 100/150$ MeV where quarks bound in hadrons, forming protons and neutrons for instance. This is the well-known quark confinement epoch. In this phase a primordial matter-antimatter asymmetry provides the elimination of the anti-hadrons. When the energy scales reaches the MeV scale a proton-neutron equilibrium establish thanks to weak processes. Anyhow, neutrinos get out from the equilibrium and the weak interactions reduce efficiency: now we only have a lepton-antilepton equilibrium. As the energy scale is close to 100 KeV (3 minutes after the initial singularity), strong interactions become relevant: protons and neutrons start to bound in atomic nuclei. This mechanism is called Primordial Nucleosynthesis and we have the production of light elements such as (H,d,He-3...). However, in this phase the temperature is too high for the binding of electrons in the nuclei: we have a primordial plasma of electrons, nuclei and photons. Finally, when $E \sim 0.1$ eV photons comes out from the equilibrium forming a diffuse background known as cosmic macrowave background (CMB) with a temperature of $T = 2.7$ Kelvin. As a result the universe become transparent and there is the formation of stable hydrogen atoms: we have the recombination. Therefore, the observation of the CMB suggests the thermal evolution of the universe.

## 1.5 The Concordance Model

The cosmological missions of the last 25 years (COBE,WMAP,PLANCK) lead us to define the so called "Concordance Model" for the general features of our universe [2]. This recipe is based on three fundamental results

- The universe is very close to flatness (see Fig.1.4):

$$\Omega_k \sim 0$$

- The universe is composed for the 4% by ordinary matter, that is, barions. For the 23% by a form of matter that do not interact electromagnetically and it is called dark matter (DM). The remaining 73% is associated by a form of mysterious energy called dark energy (DE) parameterized by $\Lambda$. It is common believed that dark energy represents the vacuum energy but there are a lot of open problems about that (Fig.1.3). Furthermore, $H_0 \simeq 70$ h km/s Mpc and we are entering in the almost exponential $\Lambda$-dominated era (or vacuum dominated era).

- The universe is characterized by adiabatic, gaussian and almost scale invariant perturbations responsible for small temperature anisotropies in the CMB ($\Delta T/T \sim 10^{-5}$) and for the formation of the LSS.

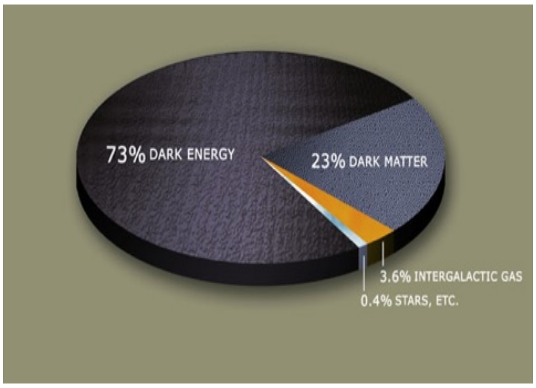


Figure 1.3: Estimated omposition of the universe.

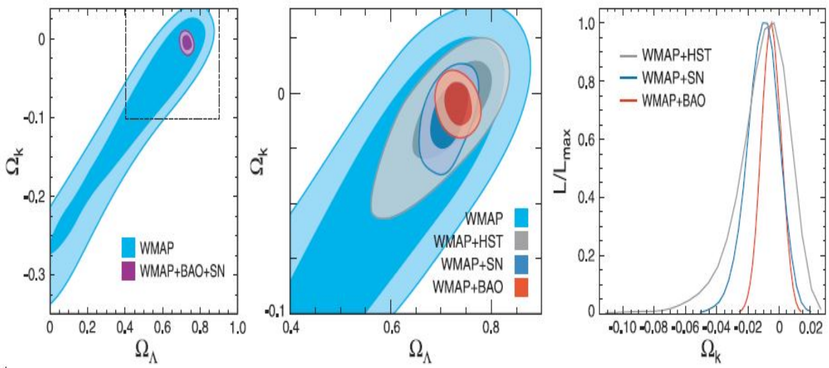


Figure 1.4: Joint analysis of CMB fluctuations, Supernovae Ia and BAO suggest a flat universe.

# Chapter 2

# The Inflationary Universe

## 2.1 Introduction

In the second part of '60 years the Standard Big Bang Model (BB model) was widely accepted by virtue of its power to predict some important features of our universe: the expansion of the universe, called Hubble expansion; the presence of the cosmic macrowave background radiation (CMB) that suggests how the universe was "hot" at early times; the abundances of light elements (like deuterium or helium-4) produced by the primordial big bang nucleo-synthesis (BBN). In addition, the theory of the evolution of the cosmological perturbations, i.e., the formation of the large scale structures (LSS) on an expanding background, was studied in details [3] (although people did not understand the production mechanism of such perturbations). At the same time, there were some features of the universe that resulted unexplained, in particular, the isotropy and homogeneity of the universe as well as the almost flatness of the universe. It was clear that these details probably depended by the physics of the early universe, i.e., by the physical condition near the gravitational singularity. In any case such physics was substantially uncontrolled because of the unknown high energy processes that come into play. In this respect, the progress in the theory of elementary particle physics was fundamental [4] especially in the context of the phase transitions and spontaneous symmetry breaking (SSB) in grand unified theory (GUT) [5]. All of these studies were applied to cosmology and soon people realized that the standard process coming out from this physics was not able to induce an "isotropization" of the universe [6] and they also (unfortunately) provided an additional problem: the production of a large number of magnetic monopoles [7]. These monopoles were unobserved just like now. The solution of all of these "puzzles" of the Standard Hot Big Bang Model was initially proposed by Alan Guth [8]. He imagined that an high temperature phase transition of the vacuum of a scalar field $\phi$ (of some GUT theory) could be induce a short period of accelerated expansion, conventionally called inflation. The accelerated expansion gives an isotropization of the universe and can dilutes monopoles on astronomical scales. However, this first proposal had some problems, it was called "old inflation" and was replaced by the "new inflation" proposed by Albrecht, Steinhardt and Andrei Linde [9]. In this case inflation takes place by a first order phase transition (starting from an unstable point) characterized by a slow roll of the scalar field wrt its potential with some temperature constraints. Unfortunately, also this version of inflation presented some defects and was substituted by the starling chaotic inflation, introduced by Linde [10]. Andrei Linde suggested as the universe could come from a quantum gravity era with stochastic or chaotic initial conditions by the Heisenberg principle. At this point, the universe could be dominated by a scalar field $\phi$. Now, in each domains of the universe where the scalar potential $V(\phi)$ dominates the kinetic and gradient terms (i.e., the scalar field is almost homogeneous) slow roll inflation takes place generating a "mini-universe". In this respect we now live in a "bubble-universe" and chaotic inflation replaces the standard BB model as the cosmological paradigm. It was soon realized inflation is also responsible for the production of quantum fluctuations that turn into classical metric perturbation by a magnification process. In the next section we briefly review the general features of the inflationary dynamics.

## 2.2 Slow roll Inflationary Dynamics

The fundamental mechanism regarding the birth of the inflationary universe (or of a chaotic inflationary universe) is an open issue as well as the nature of the field (scalar ?) or the relation between inflation and the grand unified energy. However it is widely accepted inflation is the low energy effective field theory of a more general quantum gravity theory. In this section we do not want to discuss such aspects of the inflationary universe but the most familiar version of its dynamics. The simplest version of the dynamics of inflation is known as "slow roll inflation" and it is based on the evolution of a very simple scalar field, (i.e neutral, homogeneous, minimally coupled to gravity and canonically normalized) on almost early flat FLRW universe [11]. This scalar field is commonly called inflaton $\phi$ and it is characterized by an effective auto-interaction potential $V(\phi)$ called inflationary potential. In particular, the inflationary potential is characterized by a global minimum. Therefore, the cosmological action at early times is

$$S[\phi, g_{\mu\nu}] = \int dx^4 \sqrt{-g} \left( \frac{1}{2} M_p^2 R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right). \tag{2.1}$$

Here, $R$ is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, $g$ its determinant and $M_p$ is the reduced Planck mass. Let us assume a FLRW flat universe which metric, in spherical coordinate with a spacelike convention $\{-,+,+,+\}$ for the metric tensor, comes out to be

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &= -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \end{aligned} \tag{2.2}$$

where $t$ is the cosmic time, $a(t)$ is the scale factor of the universe while $r$, $\theta$ and $\varphi$ are the coordinate of a comoving observer, i.e. the comoving coordinates. Then, the standard Einstein-Klein Gordon set of field equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\rho(\phi), \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_p}(\rho + 3p) \tag{2.3}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.4}$$

where $\rho$ and $p$ are energy density and pressure of the inflaton field, respectively

$$\rho(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.5}$$

while $'$ labels derivative wrt $\phi$ and $H$ is the common Hubble rate defined as $H = \dot{a}/a$. Let us note that this is a system of a coupled system of differential equations. Alternatively we can use the Hamilton-Jacobi formalism where the scalar field is the fundamental variable instead of cosmic time. In the case, the basic inflationary equations could be written as

$$V(\phi) = 3M_p^2 H^2(\phi) - 2M_p^2 H'(\phi) \tag{2.6}$$

$$\dot{\phi} = -2M_p^2 H'(\phi). \tag{2.7}$$

Here assume the sign of $\dot{\phi}$ remains constant (positive or negative) and so we have a monotonic evolution of the field. For instance, $\dot{\phi} < 0$ implies $H' > 0$. The general slow roll dynamics is quite simple and can be described as follows. The basic principle is that inflaton field $\phi$ slowly explores its potential $V(\phi)$ satisfying the condition

$$V(\phi) \ll \partial_\mu\phi\partial^\mu\phi. \tag{2.8}$$

At the same time we can state the inertial term in the Klein Gordon equation is negligible wrt the friction and potential terms

$$3H\dot{\phi} + V'(\phi) \simeq 0 \tag{2.9}$$

Such a condition could be realized if the inflationary potential is characterized by an almost flat region. This situation is equivalent to have a "false vacuum" likewise the presence (for a short period of time) of a cosmological constant. Indeed, Eq.(2.3) and Eq.(2.6) suggests:

$$H^2(\phi) \sim \frac{1}{3M_p^2} V(\phi) \tag{2.10}$$

As a result, we have an inflationary expansion of the Universe, i.e a quasi-exponential evolution of the scale factor

$$a(t) \sim e^N \quad N \sim H\Delta t \tag{2.11}$$

with slow varying Hubble rate and Hubble horizon $R_H = 1/H$. At the same time we can think the inflaton field itself is slowly varying and it assumes, at first order a constant value $\phi = \phi_*$. Here, $N$ is the number of exponential growth during the expansion called number of $e$-foldings typically. Inflation stops once the field reaches a certain value $\phi = \phi_{end}$.

## 2.3 Slow roll parameters and scalar field values

The slow roll dynamics can be described by a set of quantities called slow-roll parameters. In general, we have different definitions of these parameters. For example we can define the slow roll parameters by a field-dependent Hubble rate. In the case, the first two slow roll parameters (Hubble Slow Roll Parameters or HSRP) are

$$\epsilon(\phi) = 2M_p^2 \left(\frac{H'}{H}\right)^2, \quad \eta(\phi) = 2M_p^2 \left(\frac{H''}{H}\right). \tag{2.12}$$

Alternatively, we can use the potential function so we have (Potential Slow Roll Parameters or PSRP)

$$\epsilon_V(\phi) = \frac{M_p^2}{2} \left(\frac{V'}{V}\right)^2, \quad \eta_V(\phi) = M_p^2 \left(\frac{V''}{V}\right). \tag{2.13}$$

and at first order we get

$$\epsilon \sim \epsilon_V \quad \eta \sim \eta_V - \epsilon_V \tag{2.14}$$

However we can define a complete hierarchy of slow roll parameters (for both formalism) and a complete discussion can found in the third of references [11]. What about the value of the scalar field? The end of inflation can be codified by

$$\epsilon_V(\phi_{end}) \sim 1 \tag{2.15}$$

and this tells us the value of the field at the end of inflation. Further, we can compute the number of $e$-foldings before the end of inflation as

$$N(\phi, \phi_{end}, \beta_i) = \frac{1}{M_p} \int_{\Delta\phi} d\phi \frac{1}{\sqrt{2\epsilon_V(\phi)}} \tag{2.16}$$

where $\Delta\phi = \phi - \phi_{end}$ is the related variation of the field, while $\beta_i$ are the parameters of the chosen potential $V(\phi)$. At this point we can invert the solution and find the vev (vacuum expectation value) of the scalar field for each value of $N$ as

$$\phi = \phi(\phi_{end}, \beta_i, N) \tag{2.17}$$

Anyway, if $\phi \gg \phi_{end}$ we simply get

$$\phi = \phi(\beta_i, N) \tag{2.18}$$

## 2.4 Reheating Phase

Inflation comes to end when the inflaton field reaches the value $\phi_{end}$ and the potential starts to steepen. After that, the inflaton field falls in vacuum state (the minimum of the potential $V(\phi)$), acquires a mass $m_\phi$, oscillates, and decays producing a plasma of relativistic particles. In other words, the energy density stored in the scalar field is converted into relativistic particles producing the observable entropy in the Universe. In this way we enters in the well-known radiation dominated era. This mechanism could proceed in different way. For instance, the simplest one was proposed by Albrecht et al. in [12]. In this particular case, the field coherently oscillates around the minimum of a quadratic potential $\sim m^2\phi^2$. with the production of a cold gas of inflaton-bosons. At this point, the $\phi$-bosons decay into relativistic particles that interact, reach a thermal equilibrium and give rise to the graceful exit toward the radiation-dominated epoch. In other cases (as we can also expect) the physics of reheating can involve far more complicated processes, as suggested by a lot of literature and by several authors [13]. For instance, it is customary to have different decay rates for different particles and to expect that non-equilibrium phenomena, nonlinearities and turbulences could play a role. A complete review is given by [14]. In a complete general way, we can follow the reheating mechanism considering the equation of motion of the "inflaton fluid" that decays in radiation by the equations

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_{reh}) + \Gamma_\phi\rho_\phi = 0 \tag{2.19}$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = \Gamma_\phi\rho_\phi \tag{2.20}$$

Here $\rho_\phi$ is the energy density of the scalar field, $\rho_m$ is the relativistic energy density while $\Gamma_\phi$ is the decay rate of the inflaton field. If we consider an instantaneous reheating phase, we can directly write

$$\rho(\phi_{end}) = \rho_m, \quad \rho_m = \frac{\pi^2}{30} g_{reh} T_{reh}^4 \tag{2.21}$$

At this point is quite easy to show that the reheating temperature is

$$T_{reh} = \left(\frac{90}{\pi^2 g_{reh}}\right)^{1/4} \sqrt{M_p \Gamma_\phi}, \quad \Gamma_\phi = H(\phi_{end}) \tag{2.22}$$

where it is assumed that the dacay rate of the inflaton field matches the value of the Hubble rate at the end of inflation. Otherwise, we have to solve the equation for $\rho_\phi$ and then solve the equation for $\rho_m$. In these cases, it will be not a surprise if in the limit of $t \to t_{end}$ we recover the result Eq.(2.22). Anyhow, we can also model the reheating phase using an effective and conserved cosmic fluid described in terms of an equation-of-state parameter $w_{reh}$. In this formalism, we can define the energy density of the reheating fluid $\rho_{reh}$ and state that it satisfies the equation of motion

$$\dot{\rho}_{reh} + 3H(\rho_{reh} + p_{reh}) = 0, \quad w_{reh} = \frac{p_{reh}}{\rho_{reh}} \tag{2.23}$$

Here, we think $w_{reh}$ as a "mean value". Furthermore, we can always define two quantities that characterize the reheating mechanism in a completely general way. The first one is the number of $e$-foldings during reheating, $N_{reh}$, which parametrizes the duration of the reheating stage itself. The second is the reheating temperature, $T_{reh}$, which represents the energy scale of the Universe at which the reheating is completely realized. In particular, the inflationary number of $e$-foldings depends on the complete cosmic history of the Universe as

$$N_* = -\ln\left(\frac{k_*}{a_0 H_0}\right) - N_{reh} - N_{pr} + \ln\left(\frac{H_*}{H_0}\right), \tag{2.24}$$

where $k_*$ is the pivot scale stretched by the expansion at early times; $a_0$ and $H_0$ are the scale factor and the Hubble rate at current epoch, respectively; $H_*$ is the Hubble rate during inflation; $N_{reh}$ is the number of $e$-foldings during the reheating stage and $N_{pr}$ the number of $e$-foldings during the subsequent postreheating phases. Eq.(2.24) shows how the number $N_*$ is strongly dependent on $N_{reh}$. Inverting the relation in Eq.(2.24) with respect to $N_{reh}$ one gets

$$N_{reh} = \frac{4}{1-3w_{reh}} f(\beta_i, O_i, N_*), \tag{2.25}$$

where the function $f$ comes out to be

$$\begin{aligned} f(\beta_i, O_i, N_*) &= \left[-N_* - \ln\left(\frac{k_*}{a_0 H_0}\right) + \ln\left(\frac{T_0}{H_0}\right)\right] \\ &+ \left[\frac{1}{4}\ln\left(\frac{V_*^2}{M_p^4 \rho_{end}}\right) - \frac{1}{12}\ln(g_{reh})\right] \\ &+ \left[\frac{1}{4}\ln\left(\frac{1}{9}\right) + \ln\left(\frac{43}{11}\right)^{\frac{1}{3}}\left(\frac{\pi^2}{30}\right)^{\frac{1}{4}}\right]. \end{aligned} \tag{2.26}$$

The quantities $\beta_i$ are the parameters of the considered model of inflation while $O_i$ represents the known cosmological parameters. In particular, $T_0$

is the cosmic macrowave background (CMB) photon temperature, $V_*$ is the vacuum energy density at the horizon exit, $\rho_{end}$ is the energy density when inflation stops and $g_{reh}$ indicates the number of degrees of freedom of relativistic species when reheating comes to the end. Otherwise, we can extract the contribution of the inflationary number of e-foldings to get

$$N_{reh} = \frac{4}{1 - 3w_{reh}} \left[ -N_* + \mathcal{F}(\beta_i, O_i) \right] \tag{2.27}$$

In addition, one can show that the reheating temperature is defined as

$$T_{reh} = \left( \frac{40 V_{end}}{\pi^2 g_{reh}} \right)^{1/4} \exp \left[ -\frac{3}{4}(1 + w_{reh}) N_{reh} \right]. \tag{2.28}$$

Thus, information on $N_*$ (and consequently on cosmological observables as we will see) can be translated into information on $N_{reh}$ and $T_{reh}$.

# Chapter 3

# Inflation and Quantum Fluctuations

## 3.1 Introduction

In the previous section we described the general dynamics of the slow roll inflation. In this picture inflaton is a homogeneous scalar field and depends only by cosmic time

$$\phi = \phi(t) \tag{3.1}$$

However, we should notice the inflaton field is a quantum degree of freedom and it is not a surprise it can be characterized by little quantum fluctuations. Therefore we should write

$$\phi(t) \to \hat{\phi}(x) = \bar{\phi}(t) + \hat{\delta\phi}(x) \tag{3.2}$$

where $x$ is the four-dimensional space time coordinate. At this point we can recognize $\phi(t)$ as the mean value of the inflaton field driving the accelerated expansion and quantum fluctuations have a zero mean value in a macro time scale $T$

$$< 0|\hat{\phi}(x)|0 >= \bar{\phi}(t), \quad < 0|\hat{\delta\phi}(x)|0 >_T = 0 \tag{3.3}$$

As we will see later, this condition does not imply the variance of $\delta\phi(x)$ is also zero. Now, The presence of quantum fluctuations on the inflaton field become crucial to determine cosmological perturbations. The reason is very simple and is the following. The stress-energy tensor at the inflationary epoch receives the most contribution from the dominant inflaton field. If the inflaton field is characterized by fluctuations, it is natural to conclude the stress-energy tensor itself is also characterized by quantum fluctuations. However, as we know the structure of the whole inflationary spacetime is ruled by the standard Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.4}$$

Then, fluctuations on the stress-energy tensor naturally induce fluctuations on the Einstein tensor and therefore on the Ricci scalar and on spacetime itself

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \quad T_{\mu\nu} \rightarrow \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \tag{3.5}$$

As a result we have metric fluctuations! Surely, these metric fluctuations influence, in turn, the inflaton field that is naturally coupled to gravity via the cosmological action. In this way we have a perturbed FLRW universe! Now the inflationary expansion comes into play stretching the wavelengths $\lambda$ of quantum fluctuations over the almost-constant Hubble horizon [cfr Eq.(2.11)]. At this point quantum fluctuations freeze out and their mean value on a macro time scale is not more zero. Then, we have a state with a non zero number of particles. In this way quantum fluctuations turn into classical metric perturbations. In general we have fluctuations imprinted on all physical scales/wavelengths. The first fluctuations to be stretched are those of small $\lambda$ and as inflation proceeds all the others fluctuations cross the Hubble horizon. As the scalar field assumes the value $\phi_{end}$ inflation stops. At this point the Hubble horizon starts to growth faster then the physical scales $\lambda$. As a result, as time passes all the scales progressively re-enter inside the Hubble sphere. Clearly metric perturbations on small physical scales could be re-enter during the oncoming radiation era while other scales re-enter during the following matter dominated era or cosmological constant epoch. Anyway the produced metric perturbations are transmitted to photons and matter leading to the temperature fluctuations in the cosmic macrowave background radiation (CMB) and to local matter density fluctuations, i.e. to the formation of the large scale structures (LSS). Now if we want to calculate theoretical predictions of an inflationary slow roll model with scalar potential $V(\phi)$ we need two important tools. The first one is good statistical description for the distribution of the perturbations in the sky or in some sense a way to quantify the correlation between perturbations in two different points of the space in a given time. The second is the evolution of the quantum fluctuations and in particular the freezing-amplitude in superhorizon limit, i.e the amplitude of the "primordial" generated classical perturbations. Historically, the development of a theory of the cosmological perturbations from inflation required a lot of enforces by a lot of people [15]. In the next sections, we just outline the main steps and a useful reference can be [16].

## 3.2 Correlation function and power spectrum

Let us suppose we find a general relativistic solution, e.g. a given classical perturbation field we call $\delta(x, t)$ we can thought realized in a volume $\mathcal{V}_u$ of the universe. Now a good idea is to decompose this field in plane waves in the Fourier space

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\delta_{\vec{k}}(t)e^{i\vec{k}\vec{x}} \tag{3.6}$$

where the time-dependent function $\delta_k$ are the so-called "mode functions" and the integration is performed over the entire $k$-space, generally. Consequently

$$\delta_{\vec{k}}(t) = \int d^3x\delta(x)e^{-i\vec{k}\vec{x}} \tag{3.7}$$

Note that, the classical field $\delta(\vec{x},t)$ is a real field therefore

$$\delta_k^* = \delta_{-k} \tag{3.8}$$

At this point we can define the correlation function as the quantity useful to describe the presence of two perturbations in two different points of the spacetime

$$\xi(r) =< \delta(\vec{x},t)\delta(\vec{x}+\vec{r},t) > \tag{3.9}$$

where the notation "$< ... >$" indicates the average on the ensamble, i.e over all the possible realizations of the region $\mathcal{V}_u$. If we assume the isotropy of the space the correlation function is only function of $|r|$. Substituting the Fourier decomposition into Eq.(3.9) we find

$$\xi(r) = \frac{1}{(2\pi)^3}\int d^3k|\delta_{\vec{k}}|^2 e^{-i\vec{k}\vec{x}} \tag{3.10}$$

At the same time using the spherical coordinates we find an expression of $\xi(r)$ only depending on the module of $k$ and $r$

$$\xi(r) = \frac{1}{2\pi^2}\int_{I(k)} dk k^2|\delta_{\vec{k}}|^2\frac{\sin(kr)}{kr} \tag{3.11}$$

where $I(k)$ is the range of integration of $k$. Now, we can define the dimensionless power spectrum of the perturbation as

$$P(k) = \frac{k^3}{2\pi^2}|\delta_k|^2 \tag{3.12}$$

This means power spectrum defines the same tools but in the Fourier space: it describes the correlation for the presence of two perturbations on two different scales. From this point of view the correlation function in the real space results to be

$$\xi(r) = \int_{I(k)}\frac{dk}{k}P(k)\frac{\sin(kr)}{kr} \tag{3.13}$$

The variance of the perturbation field is given assuming $r = 0$ and so

$$\sigma_\delta^2 =< \delta^2(x) >= \xi(0) = \int_{I(k)}\frac{dk}{k}P(k) \tag{3.14}$$

In particular we can write down the correlation function in the Fourier space using the expansion Eq.(3.7)

$$< \delta_k \delta_{k'} >= (2\pi)^3 \frac{2\pi^2}{k^3} P(k)\delta(k-k') \tag{3.15}$$

or in units of $(2\pi)^3$

$$\frac{< \delta_k \delta_{k'} >}{(2\pi)^3} = \frac{2\pi^2}{k^3} P(k)\delta(k-k') \tag{3.16}$$

In literature there are a couple of ways to define the power spectrum, for instance a way in which $P(k)$ is exactly the Fourier transform of the correlation function. We briefly describe the alternative formalism in Appendix A. However, our problem now is to compute the amplitude of the inflationary fluctuations in the general relativistic framework. At that point we will are able to write down the power spectrum and the correlation function, as well as other useful tools. In this respect, in the next sections we present a useful decomposition of the metric tensor and of the stress energy tensor in order to better approach the computation.

## 3.3 Perturbed Einstein field equations

The Einstein field equations for the description of the inflationary perturbations can be written as

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{3.17}$$

or equivalently,

$$\delta R_{\mu\nu} = 8\pi G \delta S_{\mu\nu} \tag{3.18}$$

where $S_{\mu\nu}$ is the so called "source tensor". But what is the explicit form of these quantities?. This problem can be resolved in a simple way introducing a suitably decomposition for both quantities sometimes called scalar-vector-tensor decomposition or SVT decomposition.

### 3.3.1 Perturbations on metric tensor

Let us suppose the spacetime metric of the inflationary universe is a standard FLRW flat metric

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.19}$$

where $\bar{g}_{\mu\nu}$ is the background metric tensor that depends on the cosmic time by the scale factor $a(t)$. It is defined as usual

$$\bar{g}_{00} = -1, \quad \bar{g}_{0i} = \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = a^2(t)\delta_{ij} \tag{3.20}$$

with

$$\bar{g}^{00} = -1, \quad \bar{g}^{0i} = \bar{g}^{0i} = 0, \quad \bar{g}^{ij} = \frac{1}{a^2(t)}\delta_{ij} \tag{3.21}$$

Now we can introduce a general perturbation

$$\bar{g}_{\mu\nu}(t) \to g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x), \quad \delta g_{\mu\nu} \ll |\bar{g}_{\mu\nu}| \tag{3.22}$$

The tensor perturbation is a rank-2 tensor, symmetric wrt indices and it depends on the spacetime coordinate $x$. In addition we have

$$\delta g^{00} = -\delta g_{00}, \quad \delta g^{0j} = \frac{1}{a^2(t)}, \delta g_{0j} \quad \delta g^{ij} = \frac{1}{a^4(t)} \tag{3.23}$$

Therefore the general spacetime elements is

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= (\bar{g}_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu \\ &= (\bar{g}_{00} + \delta g_{00})dt^2 + 2(\bar{g}_{0j} + \delta g_{0j})dx^0 dx^j + (\bar{g}_{ij} + \delta g_{ij})dx^i dx^j \end{aligned} \tag{3.24}$$

Now we can introduce a smart way to reword these generic perturbations: the so called scalar-vector-decomposition. In this formalism (see Appendix B for details) we can decompose each component of the metric $\delta g_{\mu\nu}$ into a scalar component, a vector components and a tensor components

$$\delta g_{\mu\nu} = \delta g^S_{\mu\nu} + \delta g^V_{\mu\nu} + \delta g^T_{\mu\nu} \tag{3.25}$$

In this respect we can write

$$\delta g_{00} = -2\Phi \tag{3.26}$$

$$\delta g_{i0} = a(t)\left(\partial_i B + G_i\right) \tag{3.27}$$

$$\delta g_{ij} = a^2(t)\left[-2\Psi\delta_{ij} + 2\partial_i\partial_j E + \frac{1}{2}\left(\partial_i C_j + \partial_j C_i\right) + h_{ij}\right] \tag{3.28}$$

Here, $\Phi$,$\Psi$,$B$,$E$ are scalar functions, $G_i$ and $C_i$ are divergenceless vectors while $h_{ij}$ is a $3 \times 3$ traceless and divergenceless tensor

$$\partial^i G_i = \partial^i C_i = 0, \quad h^{ii} = 0, \quad \partial^i h_{ij} = 0 \tag{3.29}$$

We may notice that $\Phi$ represents the standard Newton gravitational field in the weak field limit while $\Psi$ is the spatial curvature perturbation or simply curvature perturbation. The quantity $B$ is called shift while the tensor $\partial_i\partial_j E$ is called shear and results to be a $3\times3$ symmetric and traceless tensor. Then, the spacetime metric for the scalar perturbations is

$$ds^2 = (-1+2\Phi)dt^2 + 2a(t)\partial_i B dx^i dx^0 + a^2(t)\left[(1-2\Psi)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j \tag{3.30}$$

Instead the spacetime metric for the vector modes is

$$ds^2 = -dt^2 + 2a(t)G_i dx^i dx^0 + a^2(t)\left[\delta_{ij} + \frac{1}{2}\left(\partial_i C_j + \partial_j C_i\right)\right] dx^i dx^j \quad (3.31)$$

Finally, the metric for the tensor perturbation is

$$ds^2 = -dt^2 + a^2(t)\left[\delta_{ij} + h_{ij}\right] dx^i dx^j \quad (3.32)$$

At linear order scalar, vector and tensor perturbations are completely decoupled to each others. Therefore, we can simply solve the Einstein field equations for each sector. However, in the following we will take into account only the scalar and tensor perturbations. Indeed vector modes decrease as the universe expands and they result to be not interesting. In the next section we study the perturbations on stress energy tensor and the relative SVT decomposition.

### 3.3.2 Perturbations on stress energy tensor

On the other hand we need a perturbed form for the stress-energy tensor. A possibility is to compute the standard stress energy tensor for a scalar field in a curved spacetime and then perturb its expression. Otherwise we can suppose the inflaton field as a perfect fluid with small fluctuations. This second scenario implies a more compact form of the Einstein field equations and allows us to directly recognize the form of the perturbed energy density, pressure and velocity of the scalar field. In general, we can write

$$\bar{T}_{\mu\nu} \to T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \quad (3.33)$$

Since

$$T_{\mu\nu}(\phi) = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} \quad (3.34)$$

and being

$$\begin{aligned} \rho(\phi) &= \bar{\rho}(t) + \delta\rho & (3.35)\\ p(\phi) &= \bar{p}(t) + \delta p & (3.36)\\ u_\mu(\phi) &= \bar{u_\mu}(t) + \delta u_{\mu\nu} & (3.37) \end{aligned}$$

we get

$$\begin{aligned} T_{\mu\nu} &= \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \\ &= (\bar{\rho} + \bar{p})\, \bar{u}_\mu \bar{u}_\nu + \bar{p}\bar{g}_{\mu\nu} \\ &+ (\delta\rho + \delta p)\, \bar{u}_\mu \bar{u}_\nu + (\bar{\rho} + \bar{p})\left(\bar{u}_\mu \delta u_\nu + \delta u_\mu \bar{u}_\nu\right) + \bar{g}_{\mu\nu}\delta p + \delta g_{\mu\nu}\bar{p} \quad (3.38) \end{aligned}$$

where

$$\delta T_{00} = -\bar{\rho} h_{00} + \delta\rho \quad (3.39)$$

$$\delta T_{0j} = \bar{p}\delta g_{0j} - (\bar{\rho} + \bar{p})\,\delta u_j \tag{3.40}$$

$$\delta T_{ij} = \bar{p}\delta g_{ij} + a^2(t)\delta p \delta_{ij} \tag{3.41}$$

where we can define $\delta u_i$ as the "velocity perturbation" while

$$\delta q_i = (\bar{\rho} + \bar{p})\,\delta u_i \tag{3.42}$$

as the "momentum density". What are the form of $\delta\rho$,$\delta p$ and $\delta u^\mu$? It is possible to show that energy density, pressure and velocity of a perfect fluid results to be

$$\rho(\phi) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) \tag{3.43}$$

$$p(\phi) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{3.44}$$

$$u^\mu(\phi) = \frac{-g^{\mu\sigma}\partial_\sigma\phi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}} \tag{3.45}$$

so that

$$\delta\rho(\phi) = -\dot{\phi}\dot{\delta\phi}\bar{g}^{00} - \frac{1}{2}\dot{\phi}^2\delta g^{00} + V_\phi(\phi)\delta\phi \tag{3.46}$$

$$\delta p(\phi) = -\dot{\phi}\dot{\delta\phi}\bar{g}^{00} - \frac{1}{2}\dot{\phi}^2\delta g^{00} - V_\phi(\phi)\delta\phi \tag{3.47}$$

$$\delta u^i = -\frac{\delta\phi}{\dot{\phi}} \tag{3.48}$$

In this way we can completely express the components of the stress energy tensor in terms of the metric tensor (mean and perturbed contributions). Moreover, we ought to perform the SVT decomposition of these new functions to get the SVT decomposition of the stress energy tensor. In fact these are the only two decomposable quantities because they are vectors. In particular we can write

$$\delta u_i = \partial_i\delta u + \delta u_i^V \tag{3.49}$$

where $\delta u$ is the scalar part know as "scalar velocity potential" while $\delta u_i^V$ is the divergenceless vector contribution

$$\partial^i\delta u_i^V = 0 \tag{3.50}$$

At this point we can perform the decomposition of the momentum density as

$$\delta q_i = \partial_i\delta q + \delta q_i^V \tag{3.51}$$

Here, $\delta q$ is the scalar part we can call "scalar momentum density potential"

$$\delta q = (\bar{\rho} + \bar{p})\,\delta u \tag{3.52}$$

while $\delta q_i^V$ is the divergenceless vector part

$$\partial^i\delta q_i^V = 0 \tag{3.53}$$

We should outline that we can have a deviation from the perfect fluid condition, i.e. dissipative corrections. Therefore it is essential to introduce opportune quantities to take into account this scenario. In this respect we can define an "anisotropic tensor" $\Sigma_{\mu\nu}$. Assuming that

$$\Sigma^{\mu\nu}u_\nu = 0 \tag{3.54}$$

we find the tensor is only characterized by the space-space part so that $\Sigma_{\mu\nu} = \Sigma_{ij}$. In addition it is symmetric and traceless. We can decompose the anisotropic tensor using the SVT formalism as follows

$$\Sigma_{ij} = \partial_i\partial_j\pi^s + \left(\partial_i\pi_j^V + \partial_j\pi_i^V\right) + \partial\pi_{ij}^T \tag{3.55}$$

However we will perform the calculation of the amplitude of the perturbations neglecting these "extra-source" contributions.

### 3.3.3 Equations for metric and scalar field perturbations

Let us consider the spacetime metric with scalar perturbations defined by Eq.(3.30). In this case we find four equations corresponding to $\delta G_{0j}$, $\delta G_{ij}$, $\delta G_{00}$ and $\delta G_{ij}$. In particular, the time-time equation relative to $\delta G_{00}$ is

$$3\alpha - 3\beta + \frac{\nabla^2\Psi}{a^2} - \nabla^2\left[\ddot{E} + 2H\dot{E} - \frac{\dot{B}}{a} - \frac{\dot{a}}{a^2}B\right] = 4\pi G\left(\delta\rho + 3\delta p + \nabla^2\pi^s\right) \tag{3.56}$$

The space-time equation defined by $\delta G_{0j}$ comes out to be

$$\dot{\Psi} + H\Phi = -4\pi G\delta q \tag{3.57}$$

The space-space equation can be splitted into two equations. The first one is the off-diagonal equation relative to the $\partial_i\partial_j$-term. It is

$$\partial_i\partial_j\left[\frac{\Psi-\Phi}{a^2} + (\partial_t + 3H)\left(\dot{E} - \frac{B}{a}\right) - 8\pi G\pi^s\right] = 0 \tag{3.58}$$

The second is the diagonal equation relative to kronecker term $\delta_{ij}$

$$\alpha + 3\beta - \frac{\nabla^2\Psi}{a^2} - H\nabla^2\dot{E} + \frac{\dot{a}}{a^2}\nabla^2 B = -4\pi G\left(\delta\rho - \delta p - \nabla^2\pi^s\right) \tag{3.59}$$

In the first and last equations the contributions $\alpha$ and $\beta$ are respectively

$$\alpha = \ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + (3H^2 + 2H')\Phi \tag{3.60}$$

$$\beta = H\left(\dot{\Psi} + H\Phi\right) \tag{3.61}$$

Summing the first and the last equations we get the following constraint

$$\alpha = 4\pi G \left(\delta p + \nabla^2 \pi^s\right) \tag{3.62}$$

that we can use to write down a simplified versions of those equations. To conclude we have two constrain equation, one for the energy density and the second for the momentum, and two evolution equations

$$3H\left(\dot{\Psi} + H\Phi\right) - \frac{\nabla^2}{a^2}\left[\Psi + a^2 H\left(\dot{E} - \frac{B}{a}\right)\right] = -4\pi G\delta\rho \tag{3.63}$$

$$\dot{\Psi} + H\Phi = -4\pi G\delta q \tag{3.64}$$

$$\frac{\Psi - \Phi}{a^2} + (\partial_t + 3H)\left(\dot{E} - \frac{B}{a}\right) = 8\pi G\pi^s \tag{3.65}$$

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{3.66}$$

Let us consider now the spacetime metric with tensor perturbation given by Eq.(3.32). In this case we have a single equation, i.e. the equation for gravitational waves in an expanding spacetime

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 0 \tag{3.67}$$

This means tensor perturbation is a set of independent 4 massless scalar field. We can decompose the perturbation into two modes $h_+$ e $h_\times$ by appropriate polarization tensor

$$h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij} \tag{3.68}$$

where

$$e_{ij} = e_{ji} \tag{3.69}$$

$$e_{ii} = 0 \tag{3.70}$$

$$k^i e_{ij} = 0 \tag{3.71}$$

$$e^*_{ij}(k,\lambda) = e_{ij}(-k,\lambda) \tag{3.72}$$

In addition the completness relation is

$$\sum_{\lambda=+,\times} e^*_{ij}(k,\lambda)e^{ij}(k,\lambda) = 4 \tag{3.73}$$

The Klein-Gordon equation for the inflaton field in a general curved spacetime is

$$-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + V_\phi(\phi) = 0 \tag{3.74}$$

while perturbed version of this equation is given by

$$-\delta\left(\frac{1}{\sqrt{-g}}\right)\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)$$
$$-\frac{1}{\sqrt{-g}}\partial_\mu\left[(\delta\sqrt{-g})g^{\mu\nu}\partial_\nu\phi+\sqrt{-g}(\delta g^{\mu\nu})\partial_\nu\phi+\sqrt{-g}g^{\mu\nu}(\partial_\nu\delta\phi)\right]+\delta V_\phi = 0 \tag{3.75}$$

Now, if we consider only scalar perturbations, (at linear order) we get

$$\delta\ddot{\phi}+3H\delta\dot{\phi}+\dot{\phi}\dot{\Phi}-3\dot{\Psi}\dot{\phi}-\frac{1}{a}\dot{\phi}\partial_i\partial^i B-\frac{1}{a^2}\partial_i\partial^i\delta\phi-2\Phi V_\phi+V_{\phi\phi}\delta\phi=0 \tag{3.76}$$

## 3.4 Gauge transformations of perturbations

A fundamental features in perturbation theory regard the existence of a whole family of solutions for the Einstein field equations. This family owns the inflationary background. We can lable this family by a parameter $\lambda$

$$F_\lambda = \text{ family }, \quad \lambda \text{ real} \tag{3.77}$$

For instance, it is possible to label with $F = 0$ the background solution and with $F = 1$ the physical spacetime characterized by fluctuations. Now, we have tensors quantities $T_\lambda$ on each spacetime $F_\lambda$. These tensors represents the geometrical and physical quantities (for example, the metric tensor, the ricci tensor, the stress energy tensor...). Each correspondence $1:1$ between points on $F_0$ and points on the other solutions $F_\lambda$, could be viewed as a path or an orbit $\lambda$-dependent. We can call this orbit $\theta$

$$\lambda \to \theta_\lambda \tag{3.78}$$

Let us consider now a point we call $x^0_\mu$ on $F_0$ and an orbit $\psi_0$ arising from the spacetime point $x^0_\mu$. This orbit crosses the other solution $F_\lambda$ and it defines a collection of points $x^\lambda_\mu$ connected to $x^0_\mu$. However we can consider a different orbit $\psi_1$ arising from $x^0_\mu$ and we can define a different collection of points, say $\hat{x}^\lambda_\mu$ again connected to $x^0_\mu$. In this respect, we can state there is not a unique choice for the orbit function: we can connect different relativistic solutions in different ways. This means we have a gauge freedom and passing from an orbit to another one means passing to a gauge choice to another gauge choice. As a result, different gauge choice imply different definitions of the physical quantities $T_\lambda$ and in particular, of the perturbations. Let us suppose to consider two orbits or gauge choice $\theta_\lambda$ e $\gamma_\lambda$ and a physical quantity $T_\lambda$. We have two representations for $T_\lambda$

$$\theta_\lambda \to T^\theta_\lambda \tag{3.79}$$
$$\gamma_\lambda \to T^\gamma_\lambda \tag{3.80}$$

In addition let us consider the background value of $T_\lambda = T_0$. It is clear we can compute two perturbations

$$\Delta T_\lambda^\theta = T_\lambda^\theta - T_0 \tag{3.81}$$
$$\Delta T_\lambda^\gamma = T_\lambda^\gamma - T_0 \tag{3.82}$$

Therefore perturbations have a gauge dependency. Such a situation implies an ambiguity and as a consequence we have

- The Einstein field equation are very complicated due to the many involved variables.
- Some solutions of the Einstein field equations are not physical and they correspond to a change of the coordinate system related to the gauge transformations.

The simplest method to resolve this problem is fixing a coordinate system, i.e. a gauge. The choice should be done to have more simple equations. From a geometrical point of view a gauge fixing corresponds to assume a slicing and threading of the spacetime. However, we should also know how the perturbations change under a gauge transformation. Let us introduce a generic transformation:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \tag{3.83}$$

Here $\epsilon^\mu$ is a arbitrary shift of the same order of the metric perturbations, i.e. the generator of the transformation

$$\epsilon^\mu = (\epsilon^0, \epsilon^i) \quad \text{con} \quad \epsilon^0 = -\epsilon_0 \quad \text{e} \quad \epsilon^i = \frac{1}{a^2}\epsilon_i \tag{3.84}$$

It is possible to show that the corresponding transformation of the metric perturbations is given by the following Lie derivative

$$\Delta\delta g_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) - \frac{\partial g_{\mu\nu}(x)}{\partial x^\rho}\epsilon^\rho - g_{\mu\nu}(x) \tag{3.85}$$

Using Eq.(3.83) we conclude

$$\Delta\delta g_{\mu\nu} = -\bar{g}_{\mu\rho}\frac{\partial \epsilon^\rho}{\partial x^\nu} - \bar{g}_{\nu\rho}\frac{\partial \epsilon^\rho}{\partial x^\mu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\rho}\epsilon^\rho \tag{3.86}$$

so that

$$\Delta\delta g_{00} = -2\frac{\partial \epsilon_0}{\partial t} \tag{3.87}$$

$$\Delta\delta g_{0i} = -\frac{\partial \epsilon_j}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_j \tag{3.88}$$

$$\Delta\delta g_{ij} = -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0 \tag{3.89}$$

We can perform the same operation for the stress energy tensor finding

$$\Delta\delta T_{\mu\nu} = -\bar{T}_{\mu\rho}\frac{\partial\epsilon^\rho}{\partial x^\nu} - \bar{T}_{\nu\rho}\frac{\partial\epsilon^\rho}{\partial x^\mu} - \frac{\partial\bar{T}_{\mu\nu}}{\partial x^\rho}\epsilon^\rho \tag{3.90}$$

and so

$$\Delta\delta T_{00} = 2\bar{\rho}\frac{\partial\epsilon_0}{\partial t} + \dot{\bar{\rho}}\epsilon_0 \tag{3.91}$$

$$\Delta\delta T_{0i} = -\bar{p}\frac{\partial\epsilon_i}{\partial t} + \bar{\rho}\frac{\partial\epsilon_0}{\partial x^i} + 2\bar{p}\frac{\dot{a}}{a}\epsilon_i \tag{3.92}$$

$$\Delta\delta T_{ij} = -\bar{p}\left(\frac{\partial\epsilon_i}{\partial x^j} + \frac{\partial\epsilon_j}{\partial x^i}\right) + \frac{\partial\left(a^2\bar{p}\right)}{\partial t}\delta_{ij}\epsilon_0 \tag{3.93}$$

In our framework we have a 3-vector $\epsilon_i$ we can decompose as usual

$$\epsilon_i = \partial\epsilon_i^S + \epsilon_i^V, \quad \partial_i\epsilon_i = 0 \tag{3.94}$$

In this way we can write down the explicit form of the transformations for our perturbative field. For the scalar sector we have

$$\Delta\Psi = -H\epsilon_0, \quad \Delta E = -\frac{1}{a^2}\epsilon^S, \quad \Delta\Phi = \dot{\epsilon_0} \tag{3.95}$$

and

$$\Delta F = \frac{1}{a}\left(-\epsilon_0 - \dot{\epsilon}^S + 2H\epsilon^S\right). \tag{3.96}$$

For the vector sector

$$\Delta C = -\frac{1}{a^2}\epsilon_i^V, \quad \Delta G = \frac{1}{a}\left(-\dot{\epsilon}_i^V + \frac{2\dot{a}}{a}\epsilon_i^V\right) \tag{3.97}$$

while for the tensor sector

$$\Delta h_{ij} = 0 \tag{3.98}$$

The stress energy tensor quantities satisfy

$$\Delta\delta\rho = \dot{\bar{\rho}}\epsilon_0, \quad \Delta\delta p = \dot{\bar{p}}\epsilon_0, \quad \Delta\delta u = -\epsilon_0, \quad \Delta\delta q = -(\bar{\rho}+\bar{p})\epsilon_0 \tag{3.99}$$

while the fluctuations on the scalar field transforms as the other scalars

$$\Delta\delta\phi = \dot{\phi}\epsilon_0 \tag{3.100}$$

The equation for the scalar perturbations are those that receive more benefits from this situation. The vector modes involve only two functions and it is always possible to choose $\epsilon_\mu$ to eliminate both. Finally, the tensor mode is gauge independent. It should be outline the vector irreducible contribution sof $\epsilon_i$, introduces two extra degrees of freedom as we can see above. Anyway we can neglect these contributions because we are neglecting vector sector as the universe expands. The complete picture about the problem of the gauge transformations has been discussed by Matarrese et. al [17].

### 3.4.1 Gauge choice

In the context of gauge fixing we are able to define a collection of possible gauge choice starting from the general one

$$ds^2 = (-1+2\Phi)dt^2 + 2a(t)\partial_i B dx^i dx^0 + a^2(t)\left[(1-2\Psi)\,\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j \tag{3.101}$$

The first one is the Newtonian gauge or longitudinal gauge where we choose the generator parameter such that $B = E$=0

$$ds^2 = (-1+2\Phi)dt^2 + a^2(t)\,(1-2\Psi)\,\delta_{ij}dx^i dx^j \tag{3.102}$$

This gauge is very useful to perform analytic calculations and it does not have further degrees of freedom. Sometimes it is also called conformal gauge. Another important gauge is the synchronous gauge or unitarity gauge where the generator parameter is chosen such that $\Phi = B = 0$

$$ds^2 = -dt^2 + a^2(t)\left[(1-2\Psi)\,\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j \tag{3.103}$$

This gauge was one of the first to be introduced but it presents some gauge freedom. The uniformity density gauge is obtained imposing $\delta\rho = E = 0$ so that

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)\partial_i B dx^i dx^0 + a^2(t)\,(1-2\Psi)\,\delta_{ij}dx^i dx^j \tag{3.104}$$

We can also define a comoving gauge assuiming a generator parameter such that $\delta u = 0$ which also implies $E = 0$

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)\partial_i B dx^i dx^0 + a^2(t)\,(1-2\Psi)\,\delta_{ij}dx^i dx^j \tag{3.105}$$

so it comes out to be equal to the uniformity density gauge but with a different dynamics assumption. The last important gauge is the spatially flat gauge in which we assume a generator such that the perturbations to the spatial curvature are null $E = \Psi = 0$

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)\partial_i B dx^i dx^0 + a^2(t)\delta_{ij}dx^i dx^j \tag{3.106}$$

so we get the opposite situation compared to the Eq.(3.103). This gauge result to be particularly useful if we want to focus on the scalar field fluctuation. Using the definitions for the transformation of the perturbations we can pass from a gauge to another one. In Appendix C we report the Einstein field equations for the discussed gauge.

### 3.4.2 Gauge invariant potential and Bardeen potentials

Another interesting procedure is to exploit the transformation relations to construct gauge invariant quantities, i.e. quantities which are invariant under gauge transformations as well as their related differential equations. In this respect, these quantities are those we should be computed in our context. Let us see what kind of gauge invariant potential we can define. Firstly, we can imagine a slicing of the spacetime in hypersurfices where the energy density is uniform so $\delta\rho = 0$. In this case we can define a new curvature potential from $\Psi$, called uniformity curvature perturbation

$$\zeta = \Psi + H\frac{\delta\rho}{\dot{\rho}} \tag{3.107}$$

It should be noticed that if we assume uniformity gauge then uniformity curvature perturbation match the standard curvature potential. Otherwise we can consider a slicing of the spacetime in hypersurfices where we are a comoving observer such that $\delta u = 0$. In the same way as the above case, we can define the so called comoving curvature perturbation

$$\mathcal{R} = \Psi + H\frac{\delta\phi}{\dot{\phi}} \tag{3.108}$$

Now, it is clear if we consider the comoving gauge the general gauge invariant comoving curvature mathces $\Psi$. Another interesting choice is consider a slicing of the spacetime where $\Psi = 0$ and define the spatially flat potential from the scalar field fluctuations.

$$\mathcal{Q} = \delta\phi + \dot{\phi}\frac{\Psi}{H} \tag{3.109}$$

Note that there is an exact relation between the comoving potential and the spatially flat potential

$$\mathcal{Q} = \frac{\dot{\phi}}{H}\mathcal{R} \tag{3.110}$$

therefore the ratio of the two potentials is given by the variation of the scalar field in units of an Hubble time $H^{-1}$. Other two important gauge invariant quantities are the so called Bardeen potential. These function was introduced by Bardeen and are particularly important from an hystorical point of view. To derive Bardeen potential we can note the following fact. The expression

$$f(E,B) = \dot{E} - \frac{B}{a} \tag{3.111}$$

that compares in the Einstein equation for scalar perturbations, results to be non gauge invariant due to an $\epsilon_0$-proportional term

$$f(E,B) \to \tilde{f}(E,B) = f(E,B) + \Delta f(E,B), \quad \Delta f(E,B) = \frac{\epsilon_0}{a^2} \tag{3.112}$$

This piece of evidence suggests we can use $f(E,B)$ to construct the related gauge invariant quantities of a given quantity $Q$:

$$Q_B = Q + \gamma f(E,B) \tag{3.113}$$

where $\gamma$ is a parameter we can choose to annihilate the $\Delta Q$ contribution after a gauge transformation

$$\gamma = -\frac{a^2}{\epsilon_0}\Delta Q \tag{3.114}$$

In this way we can define the Bardeen potential from $\Psi$ and $\Phi$

$$\Psi_B = \Psi + a^2 H\left(\dot{E} + \frac{B}{a}\right), \quad \Phi_B = \Phi - \frac{d}{dt}\left[a^2\left(\dot{E} - \frac{B}{a}\right)\right] \tag{3.115}$$

At the same time we can define a collection of other Bardeen variables. The Bardeen energy density perturbation and Bardeen pressure perturbation are

$$\delta\rho_B = \delta\rho - a^2\dot{\bar{\rho}}\left(\dot{E} + \frac{B}{a}\right), \quad \delta p_B = \delta p - a^2\dot{\bar{\rho}}\left(\dot{E} + \frac{B}{a}\right) \tag{3.116}$$

the Bardeen perturbation on scalar field is

$$\delta\phi_B = \delta\phi - a^2\dot{\bar{\rho}}\left(\dot{E} + \frac{B}{a}\right) \tag{3.117}$$

while the Bardeen velocity perturbation and the Bardeen momentum perturbation are

$$\delta u_B = \delta u + a^2\delta u\left(\dot{E} + \frac{B}{a}\right), \quad \delta q_B = -\dot{\phi}\delta\phi_B \tag{3.118}$$

## 3.5 Mukhanov-Sasaki equation for perturbations

### 3.5.1 The case of scalar perturbation

In inflationary perturbation theory it is quite common solving the Einstein field equation wrt the curvature perturbation $\Psi$. This can be done in a very different way. For example we can use the Newtonian gauge and then calculate $\Psi$ and $\delta\phi$ to get the gauge invariant quantities $\mathcal{R}$ or alternatively we can calculat $\delta\rho$ in place of $\delta\phi$ to get $\zeta$. Another possibilities is to perform the calculation using the set of Einstein field equation in the Bardeen formalism and then focusing on $\Psi_B$. Herein, we approach the computation of the amplitude of the curvature perturbation in the comoving gauge. This is very useful because we can directly coompute the amplitude for $\Psi = \mathcal{R}$. In this case the system of equations is

$$3H\left(\dot{\Psi} + H\Phi\right) - \frac{\nabla^2}{a^2}(\Psi - aHB) = -4\pi G\delta\rho \tag{3.119}$$

$$\dot{\Psi} + H\Phi = 0 \tag{3.120}$$

$$\frac{\Psi-\Phi}{a^2}-(\partial_t+3H)\,\frac{B}{a}=0 \tag{3.121}$$

$$\ddot{\Psi}+3H\dot{\Psi}+H\dot{\Phi}+\left(3H^2+2\dot{H}\right)\Phi=4\pi G\delta p \tag{3.122}$$

where $\delta q=0$, $\delta\rho-\delta p=0$ and $\pi^S=0$. At the same time the equation of the scalar field fluctuations is

$$\dot{\phi}\dot{\Phi}+3\dot{\Psi}\dot{\phi}+a\dot{\phi}\partial_i\partial^i B-2\Phi V_\phi=0, \quad \partial_i\partial^i=\frac{1}{a^2}\partial^i\partial^i \tag{3.123}$$

In particular we can use this equation instead of the third equation of the above system and reword $\Psi=\mathcal{R}$. Straightforward algebraic steps provide us the equation

$$\dot{\mathcal{R}}(x)+\left(3H-2\frac{\dot{H}}{H}+\frac{\ddot{H}}{\dot{H}}\right)\dot{\mathcal{R}}(x)-\frac{\nabla^2}{a^2}\mathcal{R}(x)=0 \tag{3.124}$$

At this point we have to remember we are working with quantum fluctuations. This means we should quantize the field $\mathcal{R}$ as usual

$$\hat{\mathcal{R}}(x)=\frac{1}{2\pi^3}\int d^3k\left\{\mathcal{R}_k(t)\hat{a_k}e^{ikx}+\mathcal{R}_k^*(t)\hat{a}_k^\dagger e^{-ikx}\right\} \tag{3.125}$$

Here the creation and annihilation operators satisfy the standard commutation rules for the scalar field

$$[a_k,a_{k'}]=0 \quad \left[a_k,a_{k'}^\dagger\right]=\delta(k-k') \tag{3.126}$$

while the modes $R_k$ of the field operator are normalized as follow

$$\mathcal{R}_k^*\dot{\mathcal{R}}_k-\mathcal{R}_k\dot{\mathcal{R}}_k{}_k^*=-i \tag{3.127}$$

Therefore, the equation for the mode $R_k$ comes out to be

$$\ddot{\mathcal{R}}_k+\left(3H-2\frac{\dot{H}}{H}+\frac{\ddot{H}}{H}\right)\dot{\mathcal{R}}_k+\frac{k^2}{a^2}\mathcal{R}_k=0 \quad \forall k \tag{3.128}$$

Now, this differential equation appears quite difficult to solve in this form! From this point of view, may be more convenient reword it in a different way. In this regard we can introduce the auxiliary field $u_k$ defined as follow

$$u_k=z\mathcal{R}, \quad z=a\frac{\dot{\phi}}{H} \tag{3.129}$$

where $z$ is the Mukhanov variable. Afterwards we rewrite the resulting equation wrt conformal time $\tau$ in place of cosmic time $t$. As a result we get the following equation for the mode function $\mathcal{R}_k$

$$u_k''(\tau)+\left[k^2-\frac{z''}{z}\right]u_k(\tau)=0, \quad z=a\frac{\phi'}{H}, \quad u_k=z\mathcal{R}_k \tag{3.130}$$

where $'$ indicates a derivative wrt conformal time. This equation is often called Mukhanov-Sasaki equation and can be obtained also by exchanging the two described steps: firstly we can pass to $\tau$ and then introduce the field $u_k$. Let us consider constant the slow roll parameters: $\dot{\beta}_n(\phi) = 0$. Then the ratio in the Mukhanov variable is given by

$$\frac{z''}{z} = 2a^2H^2\left[1 + \epsilon - \frac{3}{2}\eta + \frac{1}{2}\eta^2 - \frac{1}{2}\epsilon\eta\right] \tag{3.131}$$

The last problem is to find a solution for this equation! In particular we aim to find the so called superhorizon limit of related solution. This is because this limit tells us the freezing-value of quantum fluctuation, i.e. the classical value of the perturbation field, as mentioned in the introduction.

### 3.5.2 The case of tensor perturbation

The problem for the tensor perturbation is simpler then the scalar one because tensor sector is described by a single differential equation, i.e. the equation for the inflationary gravitational waves Eq.(3.67)

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 0$$

and the tensor field $h_{ij}$ comes out to be gauge invariant as we saw before. Again, we deal with quantum fluctuations so that we should quantize $h_{ij}$. In this case the expansion in Fourier space of the tensor field must be performed on two possible polarizations (see Eq.(3.68))

$$\hat{h}_{ij}(x) = \sum_{\lambda=+,\times} \frac{1}{(2\pi)^3}\int d^3k \left\{h_k(t)e_{ij}\hat{a}_k e^{ikx} + h_k^*(t)e_{ij}^*\hat{a}_k^\dagger e^{-ikx}\right\} \tag{3.132}$$

where the creator and annihilation operator satisfy the usual commutation rules

$$[a_k, a_{k'}] = 0 \quad \left[a_k, a_{k'}^\dagger\right] = \delta(k - k') \tag{3.133}$$

while the mode functions $h_k$ of the quantum operator satisfy the expression

$$h_k^*\dot{h}_k - h_k\dot{h}_k^* = -i \tag{3.134}$$

Then, the equation for the mode $h_k$ is completely analogous to the one of a massless scalar field

$$\ddot{h}_k + 3Hh_k + \frac{k^2}{a^2}h_k = 0 \tag{3.135}$$

Now if we introduce the auxiliar field $v_k(t) = a(t)h_k$ and we pass to the conformal time we have the Mukhanov-Sasaki equation for the tensor modes

$$v_k''(\tau) + \left(k^2 - \frac{a''}{a}\right)v_k(\tau) = 0 \tag{3.136}$$

where the ratio in the scale factor is given by

$$\frac{a''}{a} = a^2 H^2 (2 - \epsilon) \tag{3.137}$$

In the next sections we solve the differential equation for scalar and tensor sector.

## 3.6 Computation of the Power Spectrum

### 3.6.1 Power spectrum for scalar perturbations

The Mukhanov-Sasaki equation for the inflationary scalar fluctuations assumes the form

$$u_k''(\tau) + \left[k^2 - \frac{z''}{z}\right] u_k(\tau) = 0, \quad z = a\frac{\phi'}{H}, \quad u_k = z\mathcal{R}_k \tag{3.138}$$

and we remember $\tau$ is the conformal time, $z$ is the Mukhanov variable and $R_k$ is the Fourier-mode of the curvature perturbation and

$$\frac{z''}{z} = 2a^2 H^2 \left[1 + \epsilon - \frac{3}{2}\eta + \frac{1}{2}\eta^2 - \frac{1}{2}\epsilon\eta\right] \tag{3.139}$$

as remarked in the previous section. We can reword the differential equation as a Bessel differential equation, imposing

$$\frac{z''}{z} = \frac{1}{\tau^2} g(\tau) \quad \rightarrow \quad \frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) \tag{3.140}$$

where $g$ is a function of the slow roll parameter and can be written in terms of the Bessel parameter $\nu$. At this point we must impose initial conditions for our Bessel equation. Which initial conditions? We can assume that in the limit inside the Hubble horizon where $k \gg aH$, quantum fluctuations recover wave planes of the standard QFT. This condition is commonly called Bunch-Davies vacuum condition. Therefore the Cauchy problem for the scalar fluctuations is

$$u_k''(\tau) + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] u_k(\tau) = 0 \tag{3.141}$$

$$u_k(k \gg aH) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad \tau < 0 \tag{3.142}$$

The general solution of the Bessel equation in the conformal time $\tau$ is

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right] \tag{3.143}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Henkel functions of first and second type, respectively:

$$H_\nu^{(1)} = \sqrt{\frac{2}{\pi(-k\tau)}}e^{i(-k\tau-\pi\nu/2-\pi/4)} \tag{3.144}$$

$$H_\nu^{(2)} = \sqrt{\frac{2}{\pi(-k\tau)}}e^{-i(-k\tau-\pi\nu/2-\pi/4)} \tag{3.145}$$

Remembering that, $\tau < 0$, we get a plane wave assuming $c_2 = 0$. As a result

$$c_1(k) = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{2}(\nu+1/2)} \tag{3.146}$$

Then the solution of the Bessel equation is

$$u_k(\tau) = \frac{1}{2}\sqrt{\pi}e^{i\frac{\pi}{2}(\nu+\frac{1}{2})}\sqrt{-\tau}H_\nu^{(1)}(-k\tau) \tag{3.147}$$

We are interested in the superhorizon limit of this solution to understand how fluctuations freez out. In the limit of $\tau \to 0$ or equivalently $k \ll aH$, the Henkel function becomes

$$H_\nu^{(1)}(k \ll aH) \simeq \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\tau)^{-\nu} \tag{3.148}$$

Substituting in Eq.(3.147) we have

$$u_k(\tau) = e^{i\frac{\pi}{2}(\nu-\frac{1}{2})}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{-\nu+\frac{1}{2}} \tag{3.149}$$

and

$$|u_k(\tau)| = 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{-\nu+\frac{1}{2}} \tag{3.150}$$

The solution for the original curvature perturbation is

$$\begin{aligned}\left|\mathcal{R}_k\right| &= \left|\frac{u_k}{z}\right| \\ &= \frac{H}{\phi'}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{k}{aH}\right)\frac{H}{\sqrt{2k^3}}(-k\tau)^{-\nu+\frac{1}{2}}\end{aligned} \tag{3.151}$$

At this point we have to make an assumption about the evolution of the inflationary background to reword the conformal time $\tau$ in terms of the scale factor. If we consider a general quasi-deSitter evolution (not a pure deSitter case) we have

$$\tau = -\frac{1}{aH(1-\epsilon)} \quad \to \quad -k\tau = \frac{1}{1-\epsilon}\left(\frac{k}{aH}\right) \tag{3.152}$$

and so

$$\begin{aligned}\Big|\mathcal{R}_k\Big| &= \frac{H}{\phi'}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{k}{aH}\right)\frac{H}{\sqrt{2k^3}}\left(\frac{1}{1-\epsilon}\frac{k}{aH}\right)^{-\nu+\frac{1}{2}} \\ &= \frac{H}{\phi'}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon)^{-\nu+\frac{1}{2}}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{-\nu+\frac{3}{2}}\end{aligned}$$

Now we are ready to compute the general form of the adimensional power spectrum of the scalar perturbations. In particular, we have the following power-law function in $k$

$$P_S(k) = \frac{k^3}{2\pi^2}\Big|\mathcal{R}_k\Big|^2 \tag{3.153}$$

$$= \left[2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon)^{\frac{1}{2}-\nu}\right]^2\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.154}$$

Note that, we can also reword the power spectra as

$$P_S(k) = \left[2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon)^{\frac{1}{2}-\nu}\right]^2\left(\frac{H^2}{8\pi^2M_p^2\epsilon}\right)\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.155}$$

by using the Hubble definition of $\epsilon$ Eq.(2.12). In more compact way

$$P_S(k) = f(\nu)\left(\frac{H^2}{8\pi^2M_p^2\epsilon}\right)\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.156}$$

or

$$P_S(k) = f(\nu)\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.157}$$

with

$$f(\nu) = 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon)^{\frac{1}{2}-\nu} \tag{3.158}$$

It is possible to show that the Bessel coefficients at first order in the slow roll approximation comes out to be

$$\nu \simeq \frac{3}{2} + 2\epsilon - \eta \tag{3.159}$$

so that

$$3 - 2\nu = 4\epsilon - 2\eta \tag{3.160}$$

We can reword the scale $aH$ of the Hubble horizon during inflation introducing an appropriate label

$$k_* = a_*H_* = aH \tag{3.161}$$

The Hubble horizon during inflation is slowly varying, therefore the above wording does not correspond to a fixed scale: we have "many" Hubble horizon, i.e. we have many $k_* = a_* H_*$ as inflation proceeds. When we compute the power spectra of a primordial perturbation relative to a generic scale $k$, we would like to express it in terms of the horizon crossing moment. In this respect we can introduce the horizon crossing condition

$$k = k_* = a_* H_* \tag{3.162}$$

Therefore we can write

$$P_S(k) = f(\nu) \left( \frac{H^2}{8\pi^2 M_p^2 \epsilon} \right) \Big|_{k=aH} \tag{3.163}$$

Clearly, this condition does not tell we are computing the power spectrum for a fixed scale $k$! Indeed each considered $k$ is different from the others and they cross different Hubble horizons at different times. In particular, the horizon crossing condition transfers the scale dependence into $H$. At first order, in the limit of $\nu = 3/2$ one has $f(\nu) \simeq 1$ and we get the common result

$$P_S(k) = \frac{1}{8\pi^2 M_p^2} \frac{H^2}{\epsilon} \Big|_{k=aH} \tag{3.164}$$

or

$$P_S(k) = \left( \frac{H^2}{2\pi\dot{\phi}} \right)^2 \Big|_{k=aH} \tag{3.165}$$

We can conclude the power spectrum of scalar perturbations depends in general on the variation of the scalar field $\phi$ or in some sense, on $\epsilon$. This is a specific result for the scalar sector and it tells us the scalar perturbations are particularly sensitive to the slow roll dynamics. In the following we discuss what it is the dependency of the spectrum wrt scale $k$.

### 3.6.2 Power spectrum for tensor perturbations

The Mukhanov-Sasaki for the primordial gravitational waves takes on the form

$$v_k''(\tau) + \left( k^2 - \frac{a''}{a} \right) v_k(\tau) = 0 \quad v_k = a h_k \tag{3.166}$$

where $a$ is the scale factor and $v$ the auxiliar field. We should note that in this case we do not have a Mukhanov variable but a simple ratio in the scale factor

$$\frac{a''}{a} = a^2 H^2 (2 - \epsilon) \tag{3.167}$$

therefore the problem is more simple to treat. Also in this case we can impose a Cauchy problem adopting the Bunch-Davies condition as initial condition of the gravitational waves. So that

$$v_k''(\tau) + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] v_k(\tau) = 0 \tag{3.168}$$

$$v_k(k \gg aH) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad \tau < 0 \tag{3.169}$$

The general solution is the same of the scalar sector

$$v_k(\tau) = \sqrt{-\tau}\left[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\right]$$

Applying the initial condition we have

$$v_k(\tau) = c_1(k)\sqrt{-\tau}H_\nu^{(1)}(-k\tau) = \frac{1}{2}\sqrt{\pi}e^{i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)}\sqrt{-\tau}H_\nu^{(1)}(-k\tau)$$

while in the superhorizon limit

$$\begin{aligned} v_k(\tau) &= \frac{1}{2}\sqrt{\pi}e^{i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)}\sqrt{-\tau}\sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\tau)^{-\nu} \\ &= e^{i\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{-\nu+\frac{1}{2}} \end{aligned}$$

Therefore, amplitude of the auxiliar field results to be

$$|v_k(\tau)| = 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{-\nu+\frac{1}{2}} \tag{3.170}$$

The amplitude of the original gravitational modes is provided by the equation of the previous section. Here we can apply the same argument of the scalar sector imposing a quasi-deSitter evolution of the background. As a consequence we get

$$\begin{aligned} |h_k| &= \left|\frac{v_k}{a}\right| \\ &= 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\frac{1}{1-\epsilon}\right)^{-\nu+\frac{1}{2}}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{-\nu+\frac{3}{2}} \end{aligned} \tag{3.171}$$

So the amplitude seems to be the same of the scalar perturbations. However, we have to take into account the two possible polarization $\times$ e $+$ for the computation of the power spectrum:

$$P_T(k) = \frac{k^3}{2\pi^2}\sum_{\lambda=+,\times}|h_k|^2 \tag{3.172}$$

Then we have

$$P_T(k) = \frac{k^3}{2\pi^2} 4 \times 2 \frac{1}{M_p^2} |h_k|^2 \tag{3.173}$$

$$= \left[ 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1-\epsilon)^{-\nu+\frac{1}{2}} \right]^2 \left( \frac{2}{\pi^2} \frac{H^2}{M_p^2} \right) \left( \frac{k}{aH} \right)^{3-2\nu} \tag{3.174}$$

where we normalized wrt the reduced Planck mass. Using the formalism of the previous section, we can rewritten this result as

$$P_T(k) = f(\nu) \left( \frac{2}{\pi^2} \frac{H^2}{M_p^2} \right) \left( \frac{k}{aH} \right)^{3-2\nu} \tag{3.175}$$

In this case the Bessel coefficient at first order slow roll approximation result

$$\nu \simeq \frac{3}{2} + \epsilon \tag{3.176}$$

At first order, in the limit of $\nu = 3/2$ the function $f(\nu) = 1$ so that (using the horizon crossing formalism)

$$P_T(k) = \frac{2}{\pi^2} \frac{H^2}{M_p^2} \Big|_{k=aH} \tag{3.177}$$

that is the most used expression in literature. In this case we can see that gravitational waves do not depend on the slow roll parameters, at first order. This is due to the fact GW have no coupling with matter fields and so they travel freely after their production.

### 3.6.3 Spectral indices and tensor-to-scalar ratio

The power spectrum of cosmological perturbations show a scale dependency. Therefore, once we define a definition for $P_S(k)$ or $P_T(k)$ we need a way to quantify this scale dependence. In this respect we introduce the scalar spectral index as the log-derivative of $P_S$ wrt $k$

$$n_S(k) - 1 = \frac{dP_s(k)}{d\ln k} \tag{3.178}$$

The computation of this quantity provides

$$n_S = 1 - 4\epsilon + 2\eta \tag{3.179}$$

As a result, if $\epsilon = \eta = 0$, then we have exactly $n_S = 1$ that corresponds to the scale invariance condition or the so called Harrison-Zel'dovich power spectrum. However $\epsilon$ and $\eta$ are different from zero so we conclude inflation naturally produce a quasi scale invariant power spectrum. At the same time we have

$$n_T(k) = \frac{dP_s(k)}{d\ln k} \tag{3.180}$$

so that

$$n_T = -2\epsilon \tag{3.181}$$

In addition we can define the tensor-to-scalar amplitude as the ratio between the two power spectra

$$r = \frac{P_T}{P_S} = 16\epsilon \tag{3.182}$$

This result tells us the following consistency relation

$$r = -8n_T \tag{3.183}$$

Now it is important to underline that all of these quantities can be easily calculated once we have a specific inflationary potential $V(\phi)$. For this purpose we can use Eq.(2.14) and write

$$n_S = 1 - 6\epsilon_V + 2\eta_V, \quad r = 16\epsilon_V \tag{3.184}$$

These quantities are field dependent because the potential function itself depends on $\phi$. However we have to calculate them at the horizon crossing moment. In this phase, we are in the pure accelerated phase and the scalar field takes on a value we call $\phi = \phi_*$ (approximatively constant). In such a way

$$n_S = n_S(\phi_*), \quad r = r(\phi_*) \tag{3.185}$$

What is $\phi_*$? We compute the vacuum expectation value (vev) of the scalar field at horizon crossing by the relation Eq.(2.16) and imposing here $\phi = \phi_*$. Once we resolved the integral we can express the vev at horizon crossing as

$$\phi_* = \phi_*(\phi_{end}, \beta_i, N_*) \tag{3.186}$$

Again, if $\phi_* \gg \phi_{end}$ we get

$$\phi_* = \phi_*(\beta_i, N_*) \tag{3.187}$$

As a result we can write the inflationary predictions wrt the number of $e$-foldings

$$n_S = n_S(N_*), \quad r = r(N_*) \tag{3.188}$$

Generally, we can impose $N_* \sim 50$ or $N_* \sim 60$.

### 3.6.4 Power spectrum for scalar field fluctuation

It is particular interesting computing the fluctuations in the scalar field. This could be done using the spatially flat gauge where

$$\mathcal{R} = H\frac{\delta\phi}{\dot{\phi}} \tag{3.189}$$

Then, in the Fourier space we have

$$|\delta\phi_k|^2 = \left(\frac{\dot{\phi}}{H}\right)|\mathcal{R}_k|^2 \tag{3.190}$$

so it is natural to write

$$P_{\delta\phi}(k) = \left(\frac{\dot{\phi}}{H}\right)^2 P_S(k) \tag{3.191}$$

that is

$$P_{\delta\phi}(k) = f(\nu)\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.192}$$

Using the horizon crossing condition and that $f(\nu) \simeq 1$ for $\nu \simeq 3/2$, we can conclude

$$P_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \tag{3.193}$$

This result is particular important as we will see in the next section because tells us fluctuations in the scalar field is of the order of Gibbons-Hawking temperature.

### 3.6.5 Correlation functions and variances

Once we found the amplitude in the superhorizon limit of the mode $\mathcal{R}_k$ we can write the correlation function in the $k$-space of the classical scalar perturbations. It results to be

$$< \mathcal{R}_k\mathcal{R}_{k'} > \;=\; (2\pi)^3 P_S(k)\delta(k-k') \tag{3.194}$$

and so

$$< \mathcal{R}_k\mathcal{R}_{k'} >= (2\pi)^3 f(\nu)\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}\delta(k-k') \tag{3.195}$$

or equivalently

$$< \mathcal{R}_k\mathcal{R}_{k'} >= (2\pi)^3 f(\nu)\left(\frac{H^2}{8\pi^2 M_p^2\epsilon}\right)\left(\frac{k}{aH}\right)^{3-2\nu}\delta(k-k') \tag{3.196}$$

However, we can also write down the correlation function in the real space that is

$$\begin{aligned} < 0|\mathcal{R}(x)\mathcal{R}(x+r)|0 > \;&=\; \int_{I(k)}\frac{dk}{k}P_S(k)e^{-ikr} \\ &=\; \int_{I(k)}\frac{dk}{k}f(\nu)\left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}e^{-ikr} \end{aligned} \tag{3.197}$$

where $I(k)$ is the range of integration in the $k$-space. In general the integration should be done in the range $(0, \infty)$. However, we do not observe all the scales in the universe rather just a little part we can write as $(k_{min}, k_{max})$. The variance in the real space is given by the integral

$$\sigma_S^2 =< \mathcal{R}^2(x) >= \int_{I(k)} \frac{dk}{k} f(\nu) \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.198}$$

We can perform the same steps for the tensor sector. Let us remember the definition of spatially flat gauge and the related gauge invariant potential $\mathcal{Q}$. Then, the correlation function in the Fourier space for the scalar field fluctuation is related to the one of the curvature perturbation by the relation

$$< \delta\phi_k \delta\phi_{k'} >= \left(\frac{\dot{\phi}}{H}\right)^2 < \mathcal{R}_k \mathcal{R}'_k > \tag{3.199}$$

or by definition

$$< \delta\phi_k \delta\phi_{k'} >= (2\pi)^3 P_{\delta\phi}(k) \delta(k - k') \tag{3.200}$$

that means

$$< \delta\phi_k \delta\phi_{k'} >= (2\pi)^3 f(\nu) \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} \delta(k - k') \tag{3.201}$$

Once we know the amplitude of the inflaton Fourier mode and the related power spectrum we can write down the relative correlation function in the real space that comes out to be

$$\begin{aligned} < 0|\delta\phi(x)\delta\phi(x + r)|0 > &= \int_{I(k)} \frac{dk}{k} P_{\delta\phi}(k) e^{-ikr} \\ &= \int_{I(k)} \frac{dk}{k} f(\nu) \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} e^{-ikr} \end{aligned} \tag{3.202}$$

and consequently, the variance

$$< \delta\phi^2(x) >= \int_{I(k)} \frac{dk}{k} f(\nu) \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.203}$$

Therefore the standard deviation of the distribution of the $\delta\phi$-values is proportional to the so called Gibbons-Hawking temperature $T_{GH}$. This means the scalar field is in a thermal bath of magnitude $T_{GH}$. In addition we should also note

$$|\delta\phi(x)| \sim \frac{H}{2\pi} \tag{3.204}$$

Both of these results were underlined by different people during '80 years after the introduction of the inflationary paradigm. In particualar Eq.(3.204) is strongly related to the structure of the inflationary universe. As outlined by Gibbons and Hawking each general relativistic solution with a finite event horizon is characterized by a Gibbons-Hawking temperature. The inflationary universe is a deSitter universe at first order and so we have a finite event horizon (that at a given time $t$ coincide with the dimension of the Hubble horizon $R_H$) inducing a finite temperature just like happen at the event horizon of a black hole.

### 3.6.6 Comments on power spectrum

In cosmology it was common to describe the power spectrum of cosmological perturbations by a power-law in the observable regions of the universe wrt to a reference fixed scale $k_*$. In the following, the introduction of the inflationary paradigm justified such a procedure as well as perturbation with a quasi scale invariant spectrum. In general we can write

$$P_s(k) = A_s \left( \frac{k}{k_*} \right)^{f(k,k_*)} \tag{3.205}$$

where $A_s$ is a reference amplitude, $k_*$ is the pivot scale we use to probe the inflationary observables while the function $f(k, k_*)$ is given by

$$f(k, k_*) = (n_s - 1) + \alpha_s \ln \left( \frac{k}{k_*} \right) + \beta_s \ln^2 \left( \frac{k}{k_*} \right) + ... \tag{3.206}$$

where $\alpha_s$ is the running of the scalar spectral index ($\alpha_s = dn_s/d\ln k$) while $\beta_s$ are the higher order derivatives. In the case where the higher order terms are negligible we simply get

$$P_s(k) = A_s \left( \frac{k}{k_*} \right)^{n_s - 1} \tag{3.207}$$

Naturally, if we detect an almost scale invariance with $n_s \sim 1$ then $P_s \sim A_s$. Each experiment use a different pivot scale. For example, PLANCK mission uses $k_* = 0.002\text{Mpc}^{-1}$. In any case, this assumption is strongly related to the set up of the experiment. From a general point of view the main result is the scale invariance of the power spectra: this means that measures of that quantity on $k_* = 0.002\text{Mpc}^{-1}$ rather on $k_* = 0.001\text{Mpc}^{-1}$ get little difference. Another important fact to stress is that we treated with gaussian perturbations up to now, described by the two point correlation function. However, the inflationary mechanism naturally predicts some level of "non-gaussianity", for the cosmological perturbations, described by a three point correlation function (see the last refs of [15] for a complete introduction and summary). In some sense, this property of the cosmological modes represents one of the "smoking gun" of inflation together with the detection of primordial gravitational waves.

## 3.7 Inflationary energy scales and Lyth Bound

The Planck and BICEP2 mission derived the following bounds on the main inflationary parameters, i.e., the order of magnitude of the scalar spectra, the scalar spectral index and the tensor-to-scalar ratio [18].

$$P_s = 2.3 \times 10^{-9}, \quad n_s = 0.9680 \pm 0.006, \quad r < 0.07 \text{ at } 95\% \text{ C.L.} \tag{3.208}$$

At the same time we have the following bound on the running of the scalar perturbations

$$\alpha_s = \frac{dn_s}{d\ln k} = -0.0033 \pm 0.0074 \tag{3.209}$$

These constraints allow us to write down some important results. Let us start from the definition of the scalar spectra in the form

$$P_s = \frac{H^2}{8\pi^2 M_p^2 \epsilon}\Big|_{k=k_*} \tag{3.210}$$

Now, rewording the Hubble rate and the first slow roll parameters in terms of the scalar potential, imposing $\phi = \phi_*$ and using the above estimation for $P_s$ we land to the following expressions

$$M_p^{-6}\frac{V^3}{V'^2} \simeq 10^{-7}, \quad \text{or} \quad m_p^{-6}\frac{V^3}{V'^2} \simeq 10^{-11} \tag{3.211}$$

These equations allow us to evaluate the order of magnitude of free parameters (like a mass $m$ or a coupling $\lambda$) appearing in a potential function. At the same time we can also reword $\epsilon$ in terms of the tensor-to-scalar ratio. In this way we get an estimation of the order of magnitude of the inflationary energy density (in Planck units) as function of the observables $P_s$ and r

$$\frac{V(\phi_*)}{M_p^4} = \frac{3}{2}\pi^2 P_s r \tag{3.212}$$

In particular, we can provide an estimation for the energy scale of inflation

$$\Lambda \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\text{GeV}, \quad V_* = \Lambda^4 \tag{3.213}$$

This relation is very important because tells us the scale of inflation depends on the amplitude of the produced inflationary tensor modes. In other words the energy scales gets higher as $r$ gets higher. In particular, if $r > 0.01$ inflation could be strongly related to an hypothetical GUT physics (see the first of references [19]). However, the importance of the ratio $r$ is also related to the variation of the scalar field during inflation. How? The excursion of the scalar field $\Delta\phi$ over the inflationary phase shows a lower limit thanks to

Lyth (see the second ref in [19]). This limit can be derived using different approach. One way is to consider the Eq.(2.7) and rewrite the derivative with respect to time in a derivative with respect to the number of $e$-foldings $N$. As a result, we get:

$$\frac{d\phi}{dN} = \sqrt{2}M_p\sqrt{\epsilon(\phi)}. \tag{3.214}$$

This equation is particularly important because allows to reword a $\phi$-type derivative in a $N$-type derivative. Note that this quantity is modulated by the first slow roll parameter $\epsilon$. The slow roll condition for the inflaton dynamics implies $r \sim 16\epsilon$ and both of these parameters are slowly varying. Then, the variation of the field along the horizon crossing period can be written as:

$$\frac{\Delta\phi_{h.c.}}{M_p} = \sqrt{\frac{r}{8}}|\Delta N| \quad \text{or} \quad \frac{\Delta\phi_{h.c}}{m_p} = \sqrt{\frac{r}{64\pi}}|\Delta N|. \tag{3.215}$$

Nevertheless, the observable scales from which we deduce the value of or the bounds for the inflationary parameters like $n_s, r, \alpha_s, ...$, correspond to multipoles $2 < l < 100$. These scales leave the Hubble horizon $R_H = 1/H$ along a period of $\Delta N \sim 4$ $e$-foldings. Therefore, we can state that the total excursion of the scalar field is larger than:

$$\frac{\Delta\phi}{m_p} \geq \sqrt{\frac{r}{4\pi}}. \tag{3.216}$$

This is the so called Lyth Bound. Note that is a model dependent expression in the sense that, each model of inflation is characterized by a minimal excursion modulated by the amplitude $r$. Let us assume that the number of $e$-foldings between the horizon crossing epoch and the end of inflation epoch is $N_* \sim 60$. Then, for a quadratic potential $V(\phi) \sim m^2\phi^2$ we have:

$$\frac{\Delta\phi_{h.c.}}{m_p} \sim 0.1 \tag{3.217}$$

while for a quartic potential $V(\phi) \sim \lambda\phi^4$ the corresponding variation is

$$\frac{\Delta\phi_{h.c.}}{m_p} \sim 0.14. \tag{3.218}$$

Naturally, different potentials can produce the same $r$ and so the same $\Delta\phi$ at horizon crossing. On the other hand, a precise detection of the tensor-to-scalar ratio plays an important role to determine the minimal displacement of the field beyond the specific model. We should observe the Lyth Bound is a first order result. In the future, different cosmological missions aim to improve the current knowledge about the inflationary parameters. In this respect, in Chapter VI we generalize the Lyth Bound in order to include the $n_s$ and $\alpha_s$ contributions and exploring so possible constraints on $\Delta\phi_{H.c.}$ at next orders.

# Chapter 4

# Inflationary Models

## 4.1 Introduction

The fundamental origin of the inflationary mechanism itself as well as the nature of the inflaton field are compelling open issues. In Chapter II we saw that inflationary mechanism allows for two different "initial frameworks" for entering the inflationary phase: the new inflationary scenario (phase transition with slow roll starting from an unstable point) and the chaotic inflationary scenario (no particular phase transition, just stochastic initial conditions and slow roll). At the beginning, these two frameworks were introduced with two different kind of inflationary potentials. In the following, some studies showed new inflation needed several appropriate constraints and so chaotic inflation became (definitively) the most plausible framework for entering in the inflationary phase. At this point, the basic problem is to find and derive consistent inflationary models. But which models? We believe inflaton field could be an effective light scalar degree of freedom of some more fundamental quantum gravity cosmological theory (although we ignore its fundamental nature as said above) and we do not know the form of the related effective lagrangian density or better, of the inflationary potential $V(\phi)$. We just know that the lagrangians good to describe inflation is a little subset of all possible lagrangians and that the slow roll scenario could be the simplest and promising version of inflation. Now, starting from these considerations, we can still formulate a large a number of models. It could be useful to have a general classification for all of these proposals. The most familiar solution is based on the value assumed by the scalar field during inflation. In particular

- If $\phi < M_p$ : small field model (SFM) and $\Delta\phi < M_p$
- If $\phi > M_p$ : large field model (LFM) and $\Delta\phi > M_p$

As we have seen in the previous chapter, small field models imply a little amount of gravitational waves. On the other side, large field models provide a gravitational waves with a relevant amplitude. Regardless, the number of small field models is still very large as well as the number of large field models. Furthermore we can have models that depend on one or more parameters. In these scenarios, we can have a small field model or a large field model as we change the order of magnitude of such parameter/parameters. Moreover, it is important to underline that we can also consider models with a shape of the inflationary potential mimicking the form of the potentials of the new inflation. Indeed, what is really important is just the initial chaotic conditions. In the following, we provide a sketch of the main inflationary models and their general properties.

## 4.2 General single field models

The simplest potential we can consider is a monomial expression in some integer power $n$ of $\phi$ with a self coupling constant $\lambda_n$:

$$V(\phi) = \lambda_n \phi^n, \quad \lambda_n = \Lambda^4 {M_p}^{-n} \tag{4.1}$$

This class of models represents a typical large field model because the scalar field naturally assumes large value in Planck units. Historically, they are improperly called chaotic inflationary models just because Andrei Linde considered these class of functions to discuss the issue of the chaotic initial conditions, [10]. However, as we saw before, chaotic inflation is a general framework for entering in the inflationary stage and it does not label a specific class of inflationary potentials. The theoretical predictions for the scalar spectral index and the tensor-to-scalar ratio are

$$n_s - 1 = -n(n+1)\frac{M_p^2}{\phi_*^2}, \quad r = 8n^2\frac{M_p^2}{\phi_*^2} \tag{4.2}$$

If we express the scalar field $\phi_*$ in terms of the number of e-foldings $N_*$, as discussed in the previous chapter, we can get very simple expression for the predictions that result to be

$$n_s - 1 = -2\frac{n+2}{4N_* + n}, \quad r = \frac{16n}{4N_* + n} \tag{4.3}$$

In the most of cases, $N_*$ is large compared to $n$ so we can conclude

$$n_s - 1 \simeq -\frac{n+2}{2N_*}, \quad r \simeq \frac{4n}{N_*} \tag{4.4}$$

For example for $n = 2$ we get

$$n_s \sim 0.967, \quad r \sim 0.13 \tag{4.5}$$

while for $n = 4$ we have

$$n_s \sim 0.950, \quad r \sim 0.26 \tag{4.6}$$

A second class of models is the so called Hilltop Inflation [20] where the inflationary potential comes out to be

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p + ...\right] \tag{4.7}$$

where $p$ is an integer value and $\mu$ is a free parameter and the higher order terms guarantee the positiveness of the potential. Here, as the amplitude of the ratio $\phi/\mu$ changes we can have small field models or large field models. In particular, if $\phi < \mu \ll M_p$ during inflation then we have a small field model otherwise if $\phi \sim \mu \geq M_p$ throughout inflation, then we have a large field model. It may useful to outline that in this kind of potential the inflaton field rolls from an unstable point, therefore they represent the prototype function that appear in the new inflation framework. The computation of the theoretical predictions is not so easy as in the monomial inflation. In general we find

$$n_s - 1 = -2p(p-1)\left(\frac{M_p}{\mu}\right)^2 \frac{x^{p-2}}{1-x^p} - \frac{3r}{8}, \quad r = 8p^2 \left(\frac{M_p}{\mu}\right)^2 \frac{x^{2p-2}}{(1-x^p)^2} \tag{4.8}$$

where $x = \phi/\mu$. At this point one should set $\phi = \phi_*$ and solve the related equation of motion for $\phi_*$ in terms of $N_*$. Naturally, the specific results will depend on the value of the mass parameter $\mu$. A third important model of inflation is the called Hybrid Inflation. Historically, it was first introduced and studied by Linde, Lyth, Liddle and Stewart [21] in some context. In Hybrid Inflation we have a phase transition driven by a second additional field occurring either before or after the breaking of slow-roll conditions $\epsilon \ll 1, |\eta| \ll 1$. It should be important to underline the second field does not play a particular role during inflation: it takes no part in the accelerated evolution or in the production of the cosmological perturbations. It just plays a role in the breakdown of inflation. From this point of view, Hybrid Inflation is one field model and not a multifield model. There are a couple of different "hybrid mechanism" but we will not enter in the details. The most familiar effective potential for hybrid inflation comes out to be (also called valley hybrid inflation)

$$V(\phi) = V_0 \left[1 + \left(\frac{\phi}{\mu}\right)^p\right] \tag{4.9}$$

where $p$ is an integer. The corresponding theoretical predictions are quite similar to the ones of the Hilltop case

$$n_s - 1 = 2p(p-1)\left(\frac{M_p}{\mu}\right)^2 \frac{x^{p-2}}{1+x^p} - \frac{3r}{8}, \quad r = 8p^2 \left(\frac{M_p}{\mu}\right)^2 \frac{x^{2p-2}}{(1+x^p)^2} \tag{4.10}$$

with $x = \phi/\mu$. Typical cases are the hybrid inflation with a small quadratic term, i.e. $p = 2$

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2, \quad m^2 = \frac{2V_0}{\mu^2} \tag{4.11}$$

where $m$ is the effective mass of the inflaton field or the hybrid inflation with a small quartic term, i.e. $p = 4$

$$V(\phi) = V_0 + \frac{\lambda}{4}\phi^4, \quad \lambda = \frac{4V_0}{\mu^4} \tag{4.12}$$

where $\lambda$ is the dimensionless self-coupling. In other cases we can find logarithmic function

$$V(\phi) = V_0 \left[1 + \alpha \log(\frac{\phi}{\phi_{end}})\right] \tag{4.13}$$

very similar to the ones emerging from the spontaneous symmetry breaking. Actually, this piece of evidence suggests a general result: the most general form of an Hybrid-potential is given by

$$V(\phi) = V_0 \left[1 + f(\phi)\right], f(\phi) > 0 \tag{4.14}$$

where $\phi_{end}$ is such that the corresponding values of the slow roll parameters are less the unity (in modulus). What is really important to stress is the following situation. Let us assume a potential $V(\phi)$. Then, if little values of the scalar field lead to a phase transition then $V(\phi)$ could be a candidate for hybrid inflation. One of the most elegant models is the so called natural inflation where the inflaton field is a pseudo Nambu-Goldstone boson [22]. The corresponding inflationary potential is a trigonometric function in the field

$$V(\phi) = V_0 \left[1 + \cos\left(\frac{\phi}{f}\right)\right] \tag{4.15}$$

where $f$ is the parameter of the model. In this scenario inflation occurs if the scale $f$ is of the order of $E_p$, the Planck scale, and so we need a little ratio $V_0/f$

$$f \sim m_p \sim 10^{18}\text{GeV} \quad V_0 \sim 10^{15}\text{GeV} \quad \rightarrow \quad \frac{V_0}{f} \sim 10^{-3} \tag{4.16}$$

Natural inflation allow us to get both small field scenario and large field scenario. However it is particularly interest having large periods

$$2\pi f > m_p \tag{4.17}$$

for two simple reasons

- Because the model supports a large amount of GW.
- Because shift symmetry preserves the potential from higher order planck corrections Furthermore, a Taylor expansion around the minimum provides the Linde's model $m^2\phi^2$.

## 4.3 Non minimally coupled models

In the previous section we discussed minimally coupled to gravity models. This is just a simplification because we can also have non minimally coupled model with an extra interaction term between the field and the Ricci scalar [23]. An example might be

$$\mathcal{L}_{coupled} \simeq \xi\phi^2 R. \tag{4.18}$$

Here, $\xi$ is the fundamental constant describing the non-trivial inflaton-graviton interaction. Naturally, this interaction term is negligible as $\xi$ is very small. The general form of the inflationary lagrangian is therefore

$$S = \int d^4x\sqrt{g}\left\{\frac{1}{2}M_p^2 R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \frac{1}{2}\xi\phi^2 R\right\}. \tag{4.19}$$

At this point, the computation of the prediction can be result difficult because of the presence of the extra term. By using a conformal Weyl transformation we can reword the relativistic action in the so called Einstein frame where we have the standard Einstein-Hilbert action plus an action associated to a minimally coupled and canonically normalized scalar field that result to be different from $\phi$:

$$S = \int d^4x\sqrt{g}\left\{\frac{1}{2}M_p^2 R - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V(\psi)\right\}. \tag{4.20}$$

It is quite common to have an exponential form for $V(\psi)$. This piece of evidence suggests an equivalence between minimally coupled and non minimally coupled models and, in conclusion that non minimally coupled models are just particular (negative) exponential potential in the canonical field $\psi$. The typical example could be the non minimally coupled $m^2\phi^2$ model. In this case the action is

$$S = \int d^4x\sqrt{g}\left\{\frac{1}{2}M_p^2 R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\xi\phi^2 R\right\}. \tag{4.21}$$

Putting in evidence the Ricci scalar,using the Weyl transformation

$$\begin{aligned} g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) &= \Omega^2(x) g_{\mu\nu}(x) \\ &= e^{-2\omega(x)} g_{\mu\nu}(x) = \left(1 + \frac{\xi\phi^2}{M_p^2}\right)^{-1} g_{\mu\nu}(x) \end{aligned} \tag{4.22}$$

and introducing the canonical normalized field $\psi$, we can express the integral as

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2 R - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{m^2 M_p^2}{2\xi}\frac{e^{-\sqrt{2/3}\psi/M_p}}{(1+e^{\sqrt{2/3}\psi/M_p})^2}\right\} \tag{4.23}$$

Therefore the form of the potential depends on the coupling $\xi$ as one can expect at the beginning.

## 4.4 Higgs Inflation

Most of the inflationary models need the introduction of an additional scalar field, the inflaton field. Nevertheless one can imagine the standard models of particle physics contains the inflaton field itself as the Higgs field as proposed by Bezrukov and Shaposhnikov in [24]. Then, the inflationary action for the Higgs inflation comes out to be

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M^2R - \partial_\mu H^\dagger\partial^\mu H - \lambda\left(H^\dagger H - \nu^2\right)^2 + \xi H^\dagger H R\right\} \tag{4.24}$$

where $H$ represent the Higgs doublet and $\xi$ is the Higgs-graviton coupling. We need $1 \ll \xi \ll 10^{34}$ for inflation to occur. In the unitarity gauge, the Higgs doublet has the first component null while the secondo component takes the form

$$H_2(x) = v + \frac{1}{\sqrt{2}}h(x) \quad < 0|\hat{H}(x)|0 >= v. \tag{4.25}$$

Here $h(x)$ is our Higgs field and $v$ the relative vacuum expectation value. The first step is to rewrite the action in terms of $h(x)$ to get the starting Jordan frame

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2 + \frac{1}{2}\xi h^2 R - \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{4}\lambda\left(h^2 - v^2\right)^2\right\}. \tag{4.26}$$

Then we can put in evidence $h$ and use the conformal transformation

$$g_{\mu\nu} \to \overline{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} = e^{-2\omega(x)}g_{\mu\nu} \quad \Omega^2(x) = \left(1 + \frac{\xi h^2(x)}{M_p^2}\right)^{-1}. \tag{4.27}$$

At this point one can introduce an auxiliary canonically normalized scalar field to land, finally, at the fundamental Einstein frame

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right\} \tag{4.28}$$

where the inflationary "Higgs" potential is

$$U(\chi) = \frac{\lambda M_p^4}{4\xi^4}\left(1 + e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_p}}\right)^{-2}. \tag{4.29}$$

The predictions of this model result in

$$n_S = 0.97, \quad r = 0.003, \quad N_* \sim 60. \tag{4.30}$$

We should stress that the study of the renormalization group equation with bounds on the quark top mass, suggests how the vacuum induced by the standard Higgs field is unstable on scales of the order of ($H \approx 10^{14} - 10^{15}$GeV). This piece of evidence lead lot people to neglect the Higgs inflation as a good inflationary scenario.

## 4.5 Modified gravity motivated models

Another interesting class of inflationary models are those coming from a modification of the standard general relativity by means of additional terms to the Einstein-Hilbert action. This framework is commonly known as modified gravity-motivated models. In the simplest scenario we can have an action of the form

$$S = \int d^4x\sqrt{-g}f(R) \tag{4.31}$$

where $f(R)$ is function of the Ricci scalar. An example might be a polynomial function in the scalar curvature

$$f(R) = R + \alpha R^2 + \beta R^3 + ... \tag{4.32}$$

where $\alpha$,$\beta$,... are the autointeraction coefficients. The most famous case is the so called Starobinsky model with a quadratic correction in $R$ [25]

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2 R + \frac{1}{12M^2}R^2\right\}. \tag{4.33}$$

This is the usual Jordan frame for the Starobinsky model, where $M$ is a mass scale defining the importance of the quadratic term. As in the case of minimally coupled models ($m^2\phi^2$ or Higgs inflation) we can reword the action putting in evidence the Ricci scalar and introducing the auxiliary field

$$\psi = \frac{R}{6M} \tag{4.34}$$

In this way we get the "linear rappresentation" of the Starobinsky model

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2 R\left(1 + \frac{2}{MM_p^2}\psi\right) - 3\psi^2\right\} \tag{4.35}$$

After, we can perform the following conformal transformation

$$g_{\mu\nu} \to \overline{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} = e^{-2\omega(x)}g_{\mu\nu} \quad \Omega^2(x) = \left(1 + \frac{2\psi}{MM_p^2}\right)^{-1} \tag{4.36}$$

As last step, we introduce a canonically normalized scalar field $\phi$, the so called "scalaron" to get an exponential potential

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_p^2 R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{3}{4}M^2M_p^2\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)\right\} \tag{4.37}$$

where

$$V_0 = \frac{3}{4}M^2M_p^2 \tag{4.38}$$

This quantity just represents the asymptotic value of the potential, the plateau region in the limit of large field values. The predictions of the Starobinsky models are

$$n_S = 1 - \frac{2}{N} \quad r = \frac{12}{N^2} \tag{4.39}$$

and for $N_* \sim 60$ they are

$$n_S = 0.97 \quad r = 0.003 \tag{4.40}$$

Therefore we recover the prediction of the Higgs Inflation. This is not a surprise just because Higgs inflation is a "descendant" of the Starobinsky model. However, we have to remind that in literature there are a lot of modified gravity motivated models and they are often derived also considering non integer power of the Ricci scalar (see for instance other refs in [25]).

## 4.6 Sketch on Supergravity and Superstring motivated models

The inflationary mechanism could take place at very high energy scales maybe of the order of $E \simeq 10^{14}$ GeV or even larger. In this sense, it could be very difficult trying to construct inflationary models within the standard physics frameworks. On the other side, it appears quite natural looking at modified version of General Relativity or at extensions of the standard model of particle physics. An important example is supersymmetry. Supersymmetry (SUSY) is a symmetry between fermions and bosons, introduced in order to solve some problems in the standard particle physics, like the unification of gauge couplings for the electroweak and strong interactions, and the magnitude of radiative corrections to the mass of the Higgs boson. Sometimes it is also useful to study dark matter. In general, we can consider supersymmetry as a global symmetry. In this case, supersymmetry is not able to solve and describe the unification of gravity with the other forces. Nevertheless, it was realized at beginning of '90 years that global supersymmetry could be a natural and promising framework for the inflationary model building. This because global supersymmetry allows us to have almost flat potential with little values for the self-couplings (in Planck units). Therefore, global supersymmetry appears to "kill" possible fine-tunings. Anyhow, it is quite preferable to consider a more general framework: the supergravity theory (SUGRA). Indeed, SUGRA includes gravitons (the quanta of the gravitational field), we can address the problem of the unification of gravity with other interactions and supersymmetry is realized locally (just like a gauge symmetry) rather than globally. As we can see in the references given in the previous sections, the most of the discussed models can be realized in the supergravity (and supersymmetry) framework. Nevertheless it is important to underline that the first important supergravity model is the so called Goncharov-Linde model ($\alpha = 1/9$) discussed in [26]. Nowadays, a special role has to be reserved to the so called $\alpha$-attractor models of inflation [27]. In general we have two class of $\alpha$-attractor. The first class is the $T$-class, where $T$ here states of "tangent" and the inflationary potential is defined by

$$V(\phi) = \Lambda^4 \tanh^{2n}\left(\frac{\phi}{\sqrt{6\alpha}M_p}\right). \tag{4.41}$$

The second class is the E-class, where E states for "exponential" and the related potential is given by

$$V(\phi) = V_0\left(1 - e^{-b\frac{\phi}{M_p}}\right)^{2n}, \quad b = \sqrt{\frac{2}{3\alpha}}. \tag{4.42}$$

Note that, imposing $n = 1$ and $\alpha = 1$ we recover the Starobinsky model while specializing in $\alpha = 1/9$ we derive Goncharov-Linde inflation. In any case the $\alpha$ parameter is related to the curvature of the Kälher geometry associated with the inflaton field

$$R_K = -\frac{2}{3\alpha}. \tag{4.43}$$

The $\alpha$-attractor models are important for some reasons. The first one is that they can interpolate a broad range of value for the inflationary variables as $\alpha$ changes. For example, the theoretical predictions for the first type of potential interpolate between those of a monomial models $\phi^{2n}$ for large values of $\alpha$ ($\alpha >> 1$) and those of Starobinsky model for small values of $\alpha$ ($\alpha \ll 1$). In the second case the predictions interpolate between $\phi^{2n}$ for large value $\alpha$ and those below the scale of Starobinsky model with a very small tensor-to-scalar ratio (at least $r \sim 0$). In particular we can show (at first order) that

$$n_s \sim 1 - \frac{2}{N_*}, \quad r \sim \frac{12\alpha}{N_*}. \tag{4.44}$$

The second important feature is the following: the $\alpha$-attractor potential are characterized by an asymptotic flatness not broken by quantum gravity corrections. It is important to underline the $\alpha$-attractors can be derived just invoking conformal symmetry although the most advanced version is realized in supergravity. Now, we should stress an important fact: despite of the great efforts made in the elementary particle physics, we still have no confirmation of supersymmetry and supergravity up to the scale of many TeV. From this point of view, many people criticize the model building in such frameworks. On the other side, other people suppose that supergravity is an effective field theory in $D = 4$ or in other words, the low-energy approximation of a more general cosmological theory such as superstring theory. Thus, it appears natural to analyze realizations of inflation within superstring theory. In this context, the main drawback related to string-derived or string-inspired models of inflation is notoriously connected to the presence of large quantum corrections that could destroy the flatness of the semi-classical scalar potential, obtained using the just low-energy effective supergravity approximation in four-dimension. In the last twenty years a large number of string or D-brane motivated models have been derived. Very useful readings are [28]. Here we focus on a particular class of string models known as Fiber Inflation where quantum corrections are under quite robust control and predictions are in agreement with the experimental data. The first version of Fiber Inflation was developed by Cicoli et al. [29] and the resulting inflationary potential is

$$V(\phi) = V_0 \left( c_0 + c_1 e^{-k\phi/2} + c_2 e^{-2k\phi} + c_3 e^{k\phi} \right), \tag{4.45}$$

where

$$\begin{aligned} c_0 &= 3 - M, \\ c_1 &= -4\left(1 + \frac{M}{6}\right), \\ c_2 &= \left(1 + \frac{2M}{3}\right), \\ c_3 &= M. \end{aligned}$$

The parameter $M$ is a constant depending of some string-assumptions. The predictions for Fibre Inflation are given by

$$n_s \sim 1 - \frac{2}{N_*}, \quad r \sim \frac{24}{N_*^2}. \tag{4.46}$$

Then, Fibre inflation can be considered as the string realization of an $\alpha$-attractor model with $\alpha = 2$. One can also generalize Fibre Inflation in order to include higher order corrections in the superstring framework. This provides us a new class of potential with similar shape and predictions (see [30]). In the most of cases, we deal with a canonically normalized inflaton field with a cosmological action we can also write as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_p^2 R + X - V(\phi) \right\}, \quad X = -\frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi. \tag{4.47}$$

However, in some particular cases the kinetic term of the inflaton field could be non canonically normalized and then we can have an action of the form

$$S = \int d^4x \left\{ \frac{1}{2} M_p^2 R + \mathcal{L}(X, \phi) \right\} \tag{4.48}$$

where lagrangian density is

$$\mathcal{L} = f(X, \phi) + V(\phi), \quad X = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{4.49}$$

Here, $f(\phi, X)$ can be a completely generic function of the standard term $X$. In this case the inflationary mechanism can be driven by the kinetic term and the flatness of the potential is not more required. This condition is completely independent from the form of the potential and the resulting cosmological perturbations show non trivial velocity propagation ($c_S < 1$ ) as well as non gaussianity. Now, D-Brane cosmology can naturally lead to these type of relativistic actions. A typical example arises from the type IIB string compactification in 6-dimensional Calabi-Yau manifolds [16, 31]. In this models inflaton field result to be a deformation coming from the compactification process. In particular, the scalar field parameterizes the motion of a brane/antibrane wrt a stack of antibrane/brane in a curved geometry. The metric elements of this geometry is given by

$$ds_{10}^2 = \frac{1}{\sqrt{h(g)}} g_{\mu\nu} dx^\mu dx^\nu + \sqrt{h(g)} \left( dr^2 + r^2 ds_{X_5}^2 \right) \tag{4.50}$$

where the second contribution is the metric for the inner space that is related to a 5-dimensional Einstein manifolds $X_5$. The radial coordinate of the structure is related to the inflaton field by the relation

$$\phi(x) = \sqrt{T_3} r \rightarrow r^2 = \frac{\phi^2(x)}{T_3} \quad \text{con} \quad T_3 = \frac{1}{(2\pi)^3 g_\alpha \alpha} \tag{4.51}$$

and $T_3$ is the brane tension, $g_\alpha$ is the string coupling while $2\pi\alpha$ is the inverse of the string tension. In general there are two main scenarios

- UV scenario: a brane moves towards a stack of antibrane
- IR scenario: an antibrane moves towards a stack of brane

In the effective field theory framework, the brane dynamics is subjected to the so called "AdS/CFT" correspondence. Now it is just this relation that lead to a bound on the brane velocity in the bulk and as consequence a non canonical kinetic terms arises as well as non trivial values for $c_s$. The resulting action in $D = 4$ is known as Dirac-Born-Infeld action or DBI action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_p^2 R - \frac{1}{f(\phi)} \sqrt{1 + 2f(\phi) X} + \frac{1}{f(\phi)} - V(\phi) \right\}. \tag{4.52}$$

The first term is the Einstein-Hilbert action while the second is the brane action in $D = 4$. It should be important to underline that the analytic form of the function $f(\phi)$ depends on the curvature factor appearing in the metric, $h(g)$ by the expression

$$h^4(g) = T_3 f(\phi) \tag{4.53}$$

Therefore, the theoretical predictions for spectra and spectral index will depend on the motion in the bulk, i.e., on the parameter $c_s$.

# Chapter 5

# Constraining the physics of Inflation

## 5.1 Introduction

In the previous chapters we discussed the general properties of the simplest version of inflation (the slow-roll inflation) like its dynamics or the production of the cosmological perturbations. Moreover, we discussed some inflationary models emerging from some context such as modified or extended theories of gravity and particle physics. For all of these models we can compute theoretical predictions, in particular the scalar spectral index $n_s$, the tensor-to-scalar ratio $r$ as well as the scalar running $\alpha_s$. Now, the natural issue is: how we can test the goodness of these models in detail? And what about their general reheating properties? In the next sections, we discuss just these two fundamental points.

## 5.2 Constraining inflationary models

The process by which we evaluate the goodness of inflationary models wrt observations is typically called "constraining inflationary models". This process is based on our knowledge of the inflationary variables. The current state-of-the art for the values of the main inflationary parameters is given by the bounds we saw in Chapter II

$$n_s = 0.9680 \pm 0.006, \quad r < 0.07 \text{ at } 95\% \text{ C.L.} \tag{5.1}$$

and

$$\alpha_s = \frac{dn_s}{d\ln k} = -0.0033 \pm 0.0074 \tag{5.2}$$

Now, for each models we can compute not just the values $n_s(N_*)$ or $r(N_*)$ for a given $N_*$. Rather, we can generate functions of $n_s$ and $r$ in terms of a complete range of values of $N_*$ ($N_* = [50, 60]$, for instance) and build a $(n_s, r)$-plane. At this point, we can overlap the cosmological estimations. In this way we can see how or when given models are compatible with the experimental data. In this respect, Fig.(5.1) shows the status of some important models of inflation in view of the current Planck data [18]. In the close future, we can have important improvements due to the next generation of CMB and gravitational waves experiments. Some of these foreseen missions aim to reduce the uncertainty on the scalar spectral index better then $\sigma_{n_s} \sim 0.002$ and to probe scales of the tensor-to-scalar ratio of the order of $r \sim 10^{-3}$. As a consequence, the current status will change and a lot of models will be ruled out.

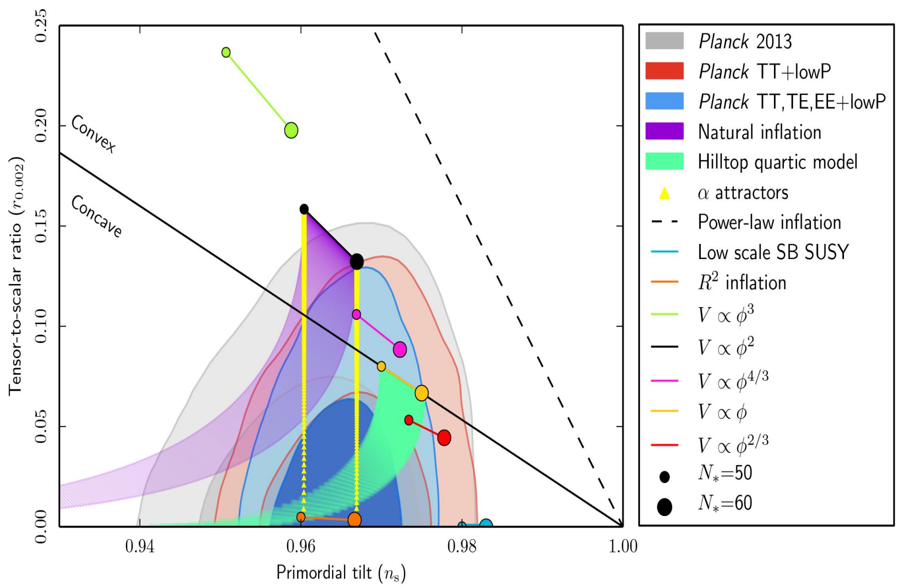


Figure 5.1: Planck 2015 constraints of some important inflationary models.

## 5.3 Constraining reheating phase

Having a robust information on the inflationary variables, in particular on the scalar spectral index $n_s$ allows us to say something about the general reheating properties of a given model and in particular on the mean value of the equation of state of the reheating fluid $w_{reh}$. Indeed, we remember that the number of e-foldings during the reheating era is provided by the following compact expression

$$N_{reh} = \frac{4}{1 - 3w_{reh}} \left[ -N_* + \mathcal{F}(\beta_i, O_i) \right]. \tag{5.3}$$

However, we can reword this equation splitting the terms related to the cosmological observables $O_i$ (Hubble rate, CMB temperature ecc...) to the ones associated to the considered model (and parameterized by $\beta_i$). In such a way we get

$$N_{reh} = \frac{4}{1-3w_{reh}}\left[\xi_0 - N_* + \frac{1}{4}\ln\left(\frac{V_*^2}{M_p^4\rho_{end}}\right)\right], \tag{5.4}$$

where $\xi_0$ depends on the value we provide to the cosmological parameters $O_i$. In general the scalar spectral index $n_s$ is a function of $N_*$. Conversely we can think $N_*$ as function of $n_s$. In this way we can rewrite the number of e-folds $N_{reh}$ in terms of $n_s$ and eventually, of $w_{reh}$

$$N_{reh} = N_{reh}(n_s, w_{reh}). \tag{5.5}$$

Furthermore, we can land to the same conclusions for the reheating temperature

$$T_{reh} = \left(\frac{40V_{end}}{\pi^2 g_{reh}}\right)^{1/4}\exp\left[-\frac{3}{4}(1+w_{reh})N_{reh}\right]. \tag{5.6}$$

to get

$$T_{reh} = T_{reh}(n_s, w_{reh}). \tag{5.7}$$

At this point, it is possible to plot $N_{reh}$ and $T_{reh}$ as function of $n_s$ for a chosen value of $w_{reh}$. The natural consequence is that some reheating phases for some values of $w_{reh}$ result to be favored compared to other cases, within some experimental results (PLANCK data, for example). This procedure have been applied in some works [32] and in particular to $\alpha$-attractor models [33].

## 5.4 Reheating phase in Fibre Inflation

In this section, we want to apply the previous recipe to a superstring motived model, never done before. In particular we consider the simplest example of Fibre Inflation [34]. Let us start by properly rewriting Eq.(2.25). For this, it is convenient to set $k_* = 0.002$ Mpc$^{-1}$, $H_0 = 1.75 \times 10^{-42}$ GeV, $T_0 = 2.3 \times 10^{-13}$ GeV and $g_{reh} = 100$, in such a way that

$$N_{reh} = \frac{4}{1-3w_{reh}}\left[\xi_0 - N_* + \frac{1}{4}\ln\left(\frac{V_*^2}{M_p^4\rho_{end}}\right)\right], \tag{5.8}$$

where $\xi_0 = 64.24$. In this case the inflationary number of $e$-foldings at first order in the observables results to be

$$N_* \sim \frac{2}{1-n_s}. \tag{5.9}$$

In our analysis the "Fibre Potential" is characterized by $M \sim 2.3 \times 10^{-6}$. The quantities $N_{reh}$ and $T_{reh}$ as functions of the scalar spectral index $n_s$ are reported in Fig.(5.2) and Fig.(5.3), respectively. We plot $N_{reh}$ and $T_{reh}$ in terms of the five different values of the equation-of-state parameter,i.e. $w_{reh} = -1/3, 0, 1/6, 2/3, 1$. We notice that the range $w_{reh} < 1/3$ is favored by Quantum Field Theory (QFT), while $w_{reh} > 1/3$ is quite unnatural from the QFT point of view, because it requires a potential that behaves like $\sim \phi^n$, with $n > 6$, around the minimum. In Fig.(5.4) we show the behavior of $T_{reh}$ as a function of $N_{reh}$. This relation is useful to read the energy scale at the end of reheating with respect to its time duration, for each considered $w_{reh}$. In Fig.(5.5) we show the favored reheating phase with respect to a future detection of the scalar spectral index with $n_s = 0.9680$ (consistently with PLANCK current results) and $\sigma_{n_s} = 0.002$. It appears clear that large value of the equation of state are favored. Therefore a larger $n_s$ lead to an "exotic scenario" for the reheating fluid requiring new theoretical developments. On the other hand, in Fig.(5.6) we report the same investigation, assuming that a future cosmological mission [35] will displace the mean value of the scalar index to $n_s = 0.9650$ with the same $\sigma_{n_s}$. In this case, large values of the equation of state are particularly disfavored.

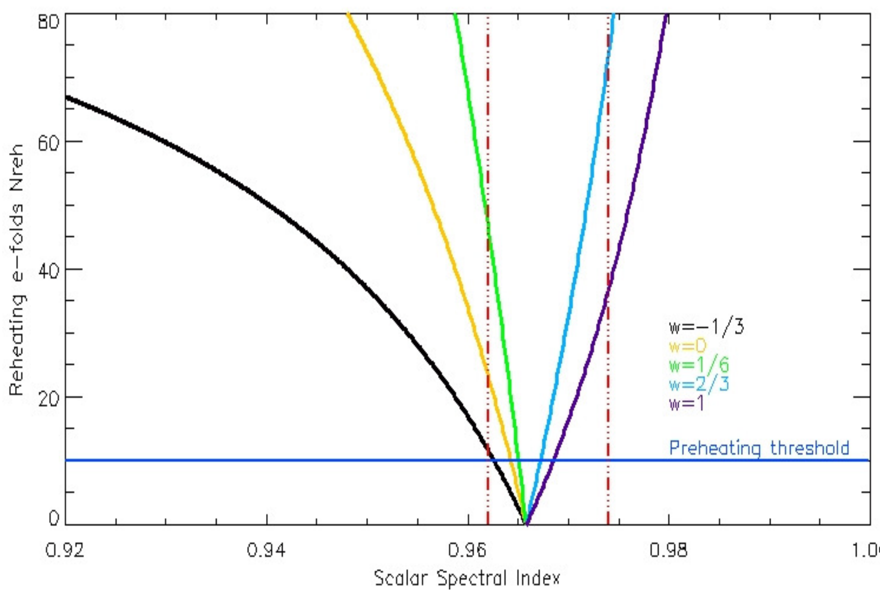


Figure 5.2: Beheaviour of the $N_{reh}(n_s)$ function in the case of fibre inflation for different value of EoS. The red vertical dashed and dotted lines represent the current 1-$\sigma$ value on the scalar spectral index given by the PLANCK mission, $\sigma_{n_s} = 0.006$

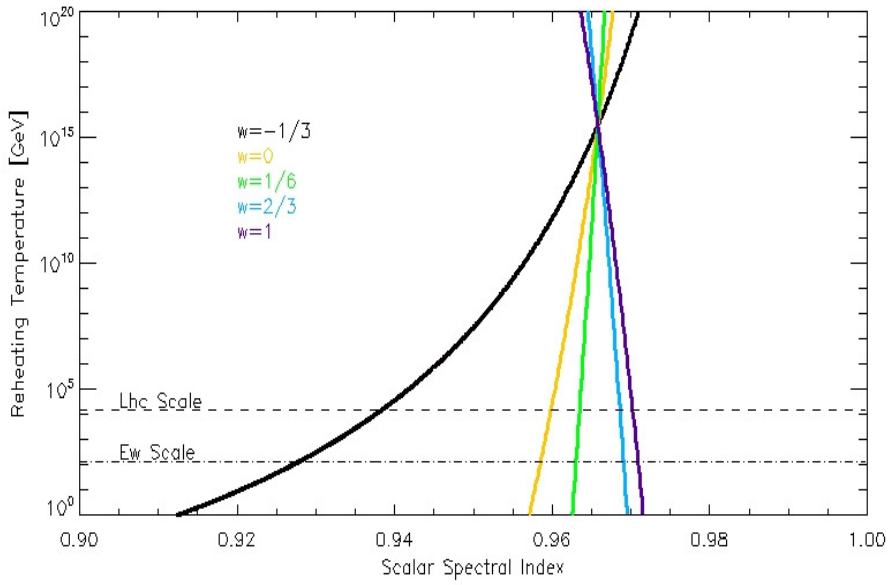


Figure 5.3: Beheaviour of the $T_{reh}$ function in the case of fibre inflation for different values of EoS. The black horizontal dashed lines represent two fundamental physical scales: the energy scale at which LHC currently works and the electroweak scale.

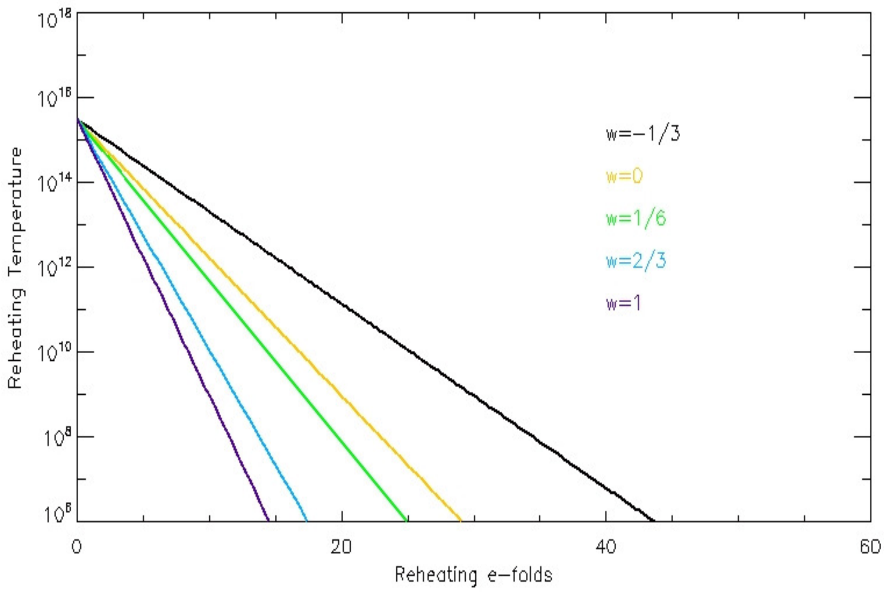


Figure 5.4: Reheating temperature (log scale) as function of the number of $e$-foldings of the reheating epoch for five different values of the equation of state parameter.

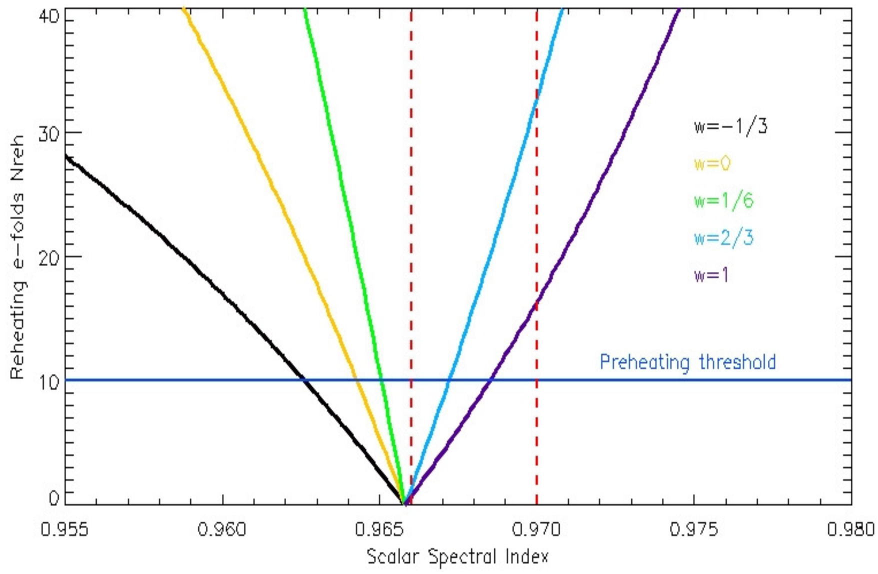


Figure 5.5: Sketch on the $N_{reh}(n_s)$ function around the mean value of a future detection with a mean value $n_s = 0.9680$ with 1-$\sigma_{n_s} \sim 0.002$. In this case, low values of the EoS ($w_{reh} < 1/3$) lead to short reheating phase. On the contrary, larger values ($w_{reh} > 1/3$) can allow a pre-heating phase.

## 5.5 Reheating phase in Fibre Inflation and $\alpha$-attractors

It is widely known that $\alpha$ attractors [27] reproduce the inflationary shape of several models of inflation like the Goncharov-Linde model ($\alpha = 1/9$) the Starobinsky model as well as Higgs Inflation. In addition, it should be noticed that the Fiber Inflation itself behaves in similar way to the $\alpha$-attractor model with $\alpha = 2$, at least in the "plateau" regime as we discussed in Chapter IV. The general properties of the reheating stage of $\alpha$ attractors have been very well studied in literature. In Fig.(5.7) we report quantitative results about the reheating temperature in both the Fiber Inflation and the corresponding $\alpha = 2$-attractor model, for the values $w_{reh} = -1/3$ and $w_{reh} = 0$ of the equation of state parameter. The temperature curves are very similar in the two cases, suggesting a very close postinflationary cosmic history with analogous predictions about the values of $n_s$ and $r$. A natural question is how these two models can be discriminated. In order to distinguish between the two models, we should focus on regions of the potentials different from the inflationary one. In particular they have to be

- cosmologically relevant
- significally different

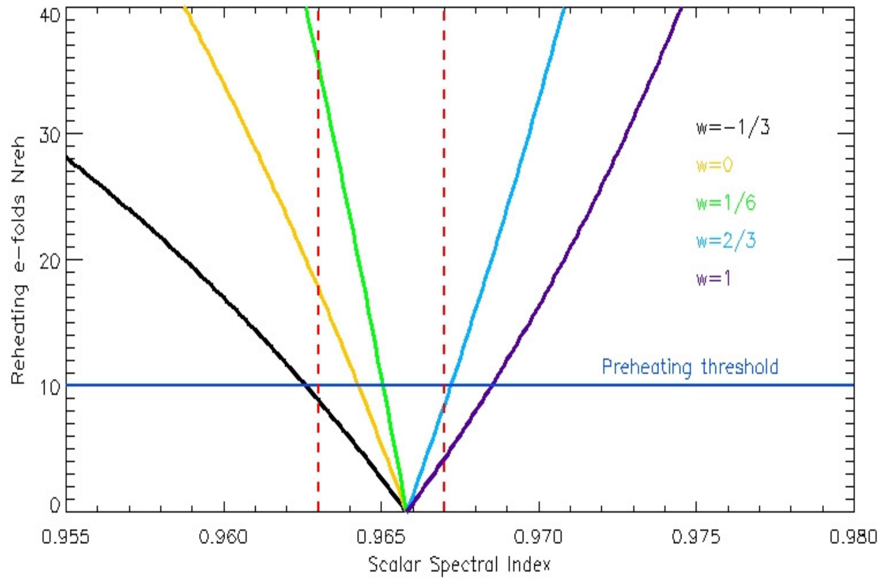


Figure 5.6: Sketch on the $N_{reh}(n_s)$ function around the mean value of a future detection $n_s = 0.9650$ with 1-$\sigma_{n_s} \sim 0.002$. In this case, low values of the EoS allow prolonged reheating phase while larger values lead to a short reheating phase.

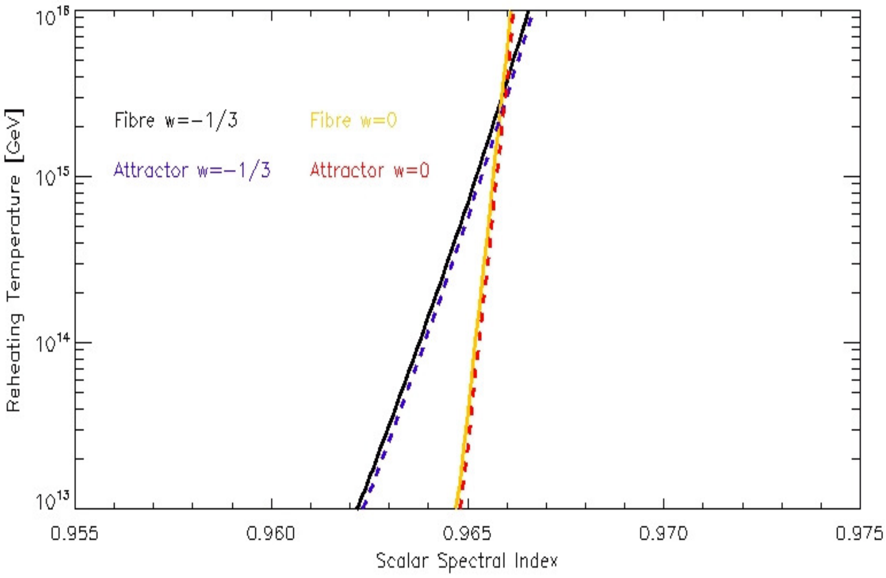


Figure 5.7: Sketch on the reheating temperature as function of $n_s$ for Fibre Inflation and $\alpha = 2$-attractor model. As one can see, the postinflationary reheating phases are quit similar.

One way could be to choose regions around the minimum of the potentials where the functions do not coincide (except for very tiny values of $\phi$). The form of the potential about the minimum is crucial to determine the production of an additional diffuse background of GW due to possible preheating effects. This background would be characterized by sharp peaks at very high frequencies. Symmetric models with $\phi^2$ or $\phi^4$ potentials were studied especially by Kofmann et al. [13]. More recently, analysis about non symmetric models have been proposed where it is shown how the peaks in the GW spectrum of these non symmetic models should exceed the standard preheating spectrum [36]. Scalar potentials like those of $\alpha$-attractors or Fiber Inflation models are not symmetric around the minimum in $\phi = 0$, also for $\phi < M_p$. The expansion around $\phi = 0$ of the $\alpha$-attractor can be written as

$$V(\varphi) \simeq \frac{1}{2} m^2_{(\alpha)} \varphi^2 - \frac{1}{3!} g_{(\alpha)} \varphi^3 + \frac{1}{4!} \lambda_{(\alpha)} \varphi^4 \tag{5.10}$$

where

$$m^2_{(\alpha)} = 2b^2 \Lambda^4, \quad g_{(\alpha)} = 6b^3 \Lambda^2, \quad \lambda_{(\alpha)} = 14 b^4 \Lambda^4, \tag{5.11}$$

while the corresponding result for the Fibre model is

$$V(\varphi) \simeq \frac{1}{2} m^2_{(f)} \varphi^2 - \frac{1}{3!} g_{(f)} \varphi^3 + \frac{1}{4!} \lambda_{(f)} \varphi^4, \tag{5.12}$$

with

$$m^2_{(f)} = V_0 k^2 \left( \frac{c_1}{4} + 4c_2 + c_3 \right), \tag{5.13}$$

$$g_{(f)} = V_0 k^3 \left( \frac{c_1}{8} + 8c_2 - c_3 \right), \tag{5.14}$$

$$\lambda_{(f)} = V_0 k^4 \left( \frac{c_1}{16} + 16c_2 + c_3 \right). \tag{5.15}$$

In both cases, the third-order term measures the antisymmetry of the potentials. Because of

$$V_0 = \Lambda^4 \quad \text{and} \quad k = 2b, \tag{5.16}$$

the ratio between the third-order terms results in

$$\frac{g_{(f)}}{g_{(\alpha)}} = \frac{4}{3} \left( \frac{c_1}{8} + 8c_2 - c_3 \right) \simeq 10. \tag{5.17}$$

Therefore, one can expect that this level of discrepancy could induce appreciable differences in the structure of GW preheating peaks and consequently resolve the degeneracy.

# Chapter 6

# Reconstruction of the physics of Inflation

## 6.1 Introduction

The standard recipe to understand the properties of the inflationary universe is based on two points. The first one is to derive physical consistent theoretical models, i.e. proper lagrangian densities with inflationary potentials characterized by one or more parameters. The second point is to compute the theoretical predictions, the scalar spectral index $n_s$ and the tensor to scalar ratio $r$, generally. At this point we can compare the predictions of all of these models with the cosmological data and see that there models compatible with the data (maybe for a given range of their parameters) and other models that are not compatible with the data (even for all possible values of their parameters). In this way we can deduce which models better reproduce the experimental results. As a consequence, we can get an idea about how inflation (probably) occurs or proceeds. However, we can also use the pure cosmological data to literally reconstruct some aspects of inflation. This procedure is typically known as "reconstructing the physics of inflation". In this Chapter, we will give some examples of reconstructing. In particular, we will see how reconstruct from the data

- The blind shape of the inflationary potential up to some order in the scalar field around the horizon crossing moment.
- The shape of the inflationary potential in some classes of models.
- The reheating phase of a given model.
- The minimum variation of the scalar field during inflation.

## 6.2 Reconstruction of the Inflationary Potential

### 6.2.1 The model independent framework

The simplest version of the potential reconstruction technique is based on a local constraint on the shape of the inflationary potential [37]. In fact, during inflation:

- The value of the inflaton field is approximately constant, since $\dot{\phi}^2 \ll V(\phi)$.
- The observable modes are stretched out over the Hubble radius, $R_H$, when $N_* \sim 60$.

Therefore, it is possible to expand the potential around $\phi_*$, the value of the inflaton field at the horizon crossing:

$$V(\phi) = V(\phi_*) + V'(\phi_*)(\phi - \phi_*) + \frac{1}{2}V''(\phi_*)(\phi - \phi_*)^2 + ....$$

At this point, one can write the coefficients of this expansion in terms of the slow roll parameters and, then, with respect to the observable quantities, $n_s$ and $r$. The weights of the polynomial form are given by the Hamilton-Jacobi equation [cf. Eq.(2.6)]. The expansion up to the second order in $\Delta\phi$ is given by

$$V(\phi) = \Lambda^4 \left[1 + d_1 \left(\frac{\Delta\phi}{M_p}\right) + \frac{1}{2}d_2 \left(\frac{\Delta\phi}{M_p}\right)^2 + ...\right] \tag{6.1}$$

where, to first order in $n_s$ and $r$, one has

$$\Lambda^4 = \frac{3}{2}\pi^2 M_p^4 P_s(k) r \tag{6.2}$$

and

$$d_1 = \frac{1}{2}\sqrt{\frac{r}{2}}, \quad d_2 = \frac{1}{3}\left[9\frac{r}{16} - \frac{3}{2}(1 - n_s)\right] \tag{6.3}$$

Note that, the $d_i$ are dimensionless quantities, as well as the ratio $\Delta\phi/M_p$. These definitions provide a model-independent constraint on the shape of the inflaton potential, as they are directly connected with the first and second order derivatives of the potential. The set of plots Fig.6.1,Fig.6.2 show some examples of the relation between the first and second order coefficients in some simulated conditions, involving the role of the correlation parameter $\rho$. Anyway, further approaches for the reconstruction problem have been discussed in the literature [38].

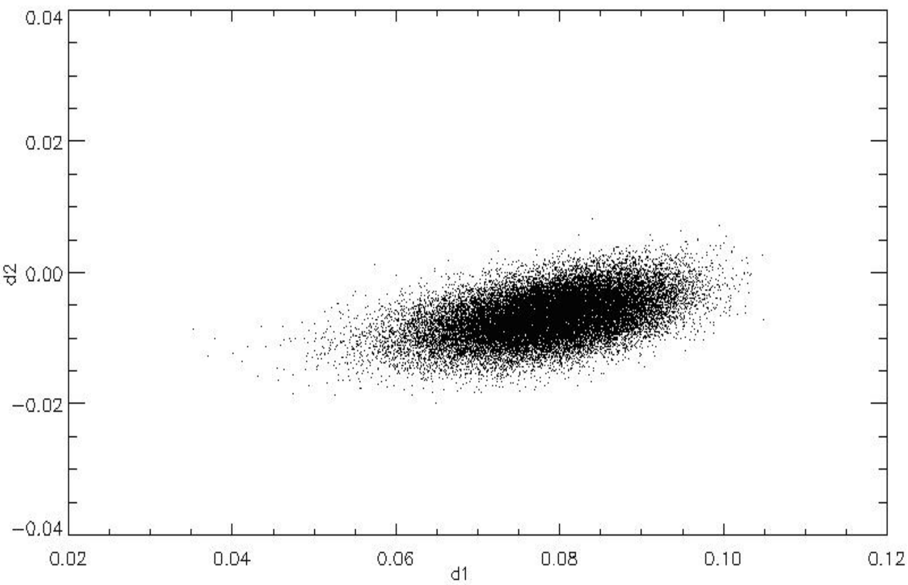


Figure 6.1: Distribution of $d_2$ Vs $d_1$ at 1st order with $\rho = 0.1$.

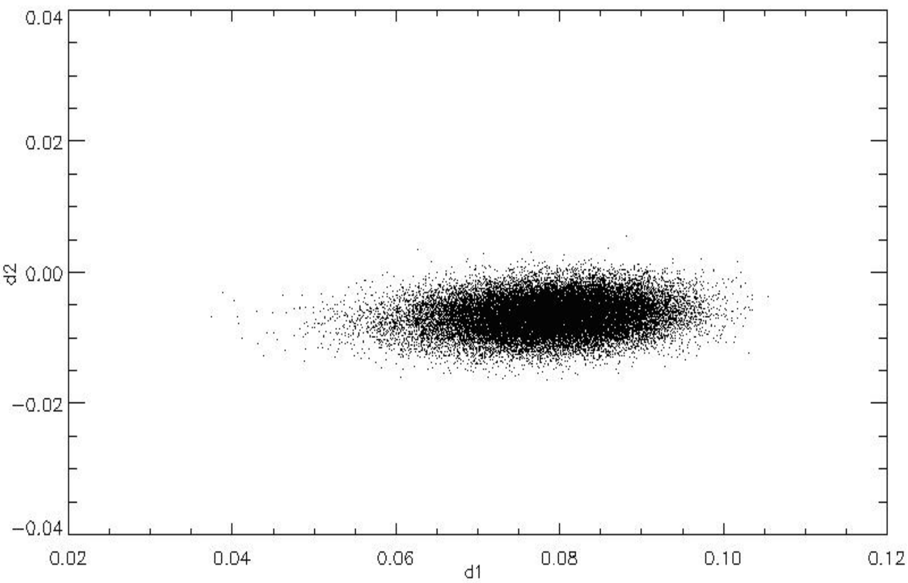


Figure 6.2: Distribution of $d_2$ Vs $d_1$ at 1st order with $\rho = 0.5$: in this case the cloud of samples is denser with respect to the previous case.

### 6.2.2 The model dependent framework

The reconstruction technique can be also applied to a specific class of models. A simple example is based on the equation of the previous section. The general recipe is the following [39]. Let us consider a specific model of inflation with a potential $V_{\beta_i}(\phi)$, where $\beta_i$ is the set of parameters that modulates the potential function. We can expand this potential up to second order around $\phi_*$:

$$V(\phi) = \Lambda^4 \left[ 1 + c_1 \left( \frac{\Delta\phi}{M_p} \right) + \frac{1}{2} c_2 \left( \frac{\Delta\phi}{M_p} \right)^2 + ... \right] \tag{6.4}$$

The coefficients $c_1$ and $c_2$ are both functions of the inflaton value, $\phi_*$, and of the free parameters, $\beta_i$: $c_1 = c_1(\phi_*, \beta_i)$ and $c_2 = c_2(\phi_*, \beta_i)$. By comparing the model-independent and the model-dependent expansion, we have

$$c_1 = d_1, \qquad c_2 = d_2. \tag{6.5}$$

Using these relations, we can derive predictions for $\phi_*$ and $\beta_i$ in the form: $\phi_* = \phi_*(n_s, r)$, $\beta_i = \beta_i(n_s, r)$. The functional dependency of $d_1$ and $d_2$ on $n_s$ and $r$ is strongly model dependent. So, there may be cases in which it is not possible to write both $\phi_*$ and $\beta_i$ in terms of the cosmological observables. Herein, we apply this recipe to the the E-model attractors whose potential is

$$V(\phi) = \Lambda^4 \left( 1 - e^{-b\phi/M_p} \right)^2, \quad b = \sqrt{\frac{2}{3\alpha}} \tag{6.6}$$

In the supergravity framework, the parameter $\alpha$ is related to the Kälher curvature of the inflaton's scalar manifold:

$$R_K = -\frac{2}{3\alpha} \tag{6.7}$$

Let us now compute the quadratic Taylor expansion of $V$:

$$V(\phi) \simeq \Lambda^4 \left[ c_0 + c_1 \left( \frac{\Delta\phi}{M_p} \right) + \frac{1}{2} c_2 \left( \frac{\Delta\phi}{M_p} \right)^2 \right]. \tag{6.8}$$

where

$$c_0 = 1 - 2e^{-b\phi_*/M_p} \sim 1 \tag{6.9}$$

$$c_1 = 2be^{-b\phi_*/M_p} \tag{6.10}$$

$$c_2 = -2b^2 e^{-b\phi_*/M_p} \tag{6.11}$$

with $\phi_*/M_p \gg 1$. We can use these relations to evaluate $\phi_*$ and $\alpha$ from a given CMB experiment. From Eq.(6.5) it follows that

$$\frac{d_2}{d_1} = -b \tag{6.12}$$

Moreover, from Eq.(6.10), we get

$$\frac{\phi_*}{M_p}(b, d_1) = -\frac{1}{b}\ln\left(\frac{d_1}{2b}\right). \tag{6.13}$$

These equations provide information on the inflationary models, given $n_s$ and $r$ from CMB data. Since, current CMB experiments still do not provide a measurement on $r$, this approach is by now not very effective. However, the situation should rapidly change in the near future, with a strong improvement on the knowledge of the tensor-to-scalar ratio. Having this in mind, we discuss what kind of constraints we may have on $\alpha$-attractor models, assuming that next-generation of CMB experiments will be able to probe a tensor-to-scalar ratio of the order of $r \sim 10^{-3}$ [35]. From this purpose, we simulate values of $n_s$ and $r$, randomly extracted from a gaussian multivariate distribution of the form

$$\mathcal{G}(n_s, r) = \frac{1}{\sqrt{4\pi^2\sigma_{n_s}^2\sigma_r^2(1-\rho^2)}}\exp\left(-\frac{Q^2}{2}\right). \tag{6.14}$$

Here

$$Q^2 = \frac{1}{1-\rho^2}\hat{Q}^2 \tag{6.15}$$

where

$$\hat{Q} = \left[\frac{(n_s-\mu_{n_s})^2}{\sigma_{n_s}^2} - 2\rho\frac{(n_s-\mu_{n_s})(r-\mu_r)}{\sigma_{n_s}\sigma_r} + \frac{(r-\mu_r)^2}{\sigma_r^2}\right] \tag{6.16}$$

where $\mu_{n_s}, \mu_r$ and $\sigma_{n_s}, \sigma_r$ are mean and rms values of the scalar spectral-index and the tensor-to-scalar ratio, respectively, while $\rho$ is, again, the correlation coefficient. In particular, we use (consistentely with current the PLANCK data) the values $\mu_{n_s} = 0.968$ and $\sigma_{n_s} = 0.006$, while for $r$ we choose three different values ($\mu_r = 0.001, 0.002, 0.003$) with $\sigma_r = 0.0001$. The correlation coefficient between $n_s$ and $r$ is fixed to be $\rho = 0.1$. We extract from the distribution of Eq.(6.14) pairs of values for $n_s$ and $r$. For each extraction, we reconstruct the coefficients $d_1$ and $d_2$ from Eq.(6.3).

| $r$ | $d_1$ mean value | $d_1$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | 0.01117 | 0.00056 |
| 0.002 | 0.01581 | 0.00039 |
| 0.003 | 0.01936 | 0.00032 |

Table 6.1: Simulation results for the coefficients $d_1$ of the Taylor expansion. As we can see, the 1-$\sigma$ value increases as the mean value of $r$ gets larger.

Then, we use Eq.(6.12) and Eq.(6.13) to estimate $\alpha$, $\phi_*$ and $R_K$, from a sample of $\simeq 10^4$ draws. We note that, increasing $r$, the shape of the

| $r$ | $d_2$ mean value | $d_2$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | -0.01583 | 0.00302 |
| 0.002 | -0.01564 | 0.00302 |
| 0.003 | -0.01545 | 0.00302 |

Table 6.2: Simulation results for the coefficients $d_2$ of the Taylor expansion: the resulting 1-$\sigma$ converges to the same value up to the 5th decimal place. The negative sign suggests that the shape of the inflationary potential about the horizon crossing moment is locally described by a parabola which opens downward.

inflaton potential gets smoother, pushing $\phi_*$ to larger values, as shown in Fig.6.3. The resulting distribution functions for $\phi_*$ are shown in Fig.6.4, Fig.6.5 and Fig.6.6 while those of $p = \ln \alpha$ are shown in Fig.6.7, Fig.6.8 and Fig.6.9. The mean and standard deviation values for the distribution of the coefficients $d_1$ and $d_2$ are summarized in Tab.6.1 and in Tab.6.2. The resulting mean and standard deviation values for the parameters $\phi_*$, $p$ and $R_K$ are summarized in Tab.6.3, Tab.6.4 and in Tab.6.5. The Fig.6.4, Fig.6.5 and Fig.6.6 and the Fig.6.7, Fig.6.8 and Fig.6.9 show the constraining power of a hypothetical CMB experiments of new generation. As expected in this inflationary scenario, when the mean values of the tensor-to-scalar ratio increases, the mean values of $p$ and $\phi_*$ increase as well. Indeed, the distribution functions of the constrained parameters move to high values in the frequency plots. The mean value of $R_K$ decreases (in modulus) in complete agreement with its definition Eq.(6.7).

| $r$ | $\phi_*$ mean value | $\phi_*$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | 4.02 | 0.70 |
| 0.002 | 5.02 | 0.85 |
| 0.003 | 5.68 | 0.95 |

Table 6.3: Simulation results for the vacuum expectation value of the scalar field in the (supergravity) $\alpha$-attractor models. The table shows an increasing on the uncertainty $\sigma$ as the mean value of $r$ increases.

Moreover, we can also constrain the particular shape of the related effective potential $V(\phi)$ around the global minimum of $V_\alpha(\phi)$, providing constraints on the coupling constants. The minimum of the $\alpha$-attractor models is localized in $\phi = 0$, so that:

$$V(\phi) \simeq \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4 + \mathcal{O}(\phi^5)... \quad . \tag{6.17}$$

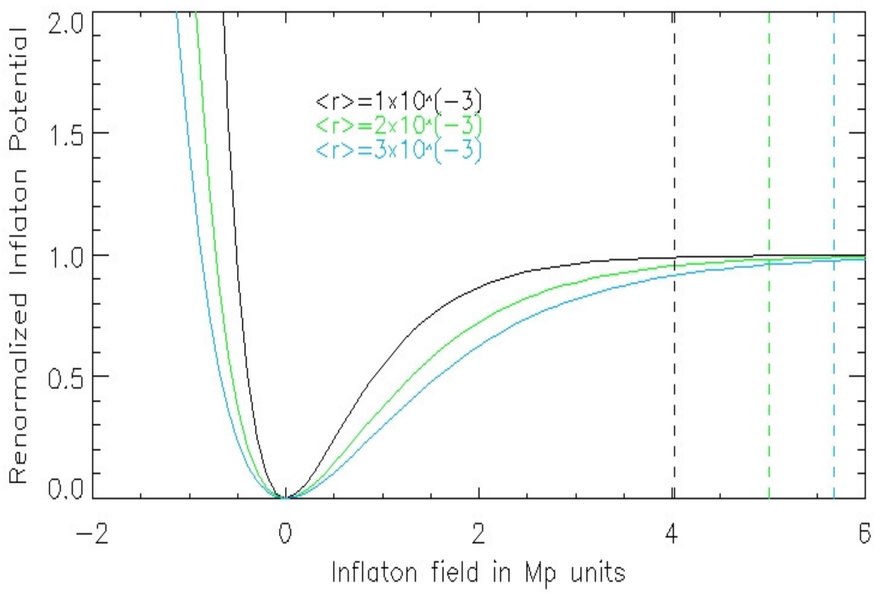


Figure 6.3: Shape of the potentials normalized to the energy density $\Lambda^4$ and the relative inflaton value at horizon exit: the solid lines represent the three "mean" potential curves for the computed simulations. When the mean value of $r$ increases, the curve is less steep. The dashed lines represent the three "mean" value of $\phi_*$ at horizon crossing. When $r$ increases $\phi_*$ is always moved farther.

| $r$ | $p$ mean value | $p$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | -1.07 | 0.40 |
| 0.002 | -0.34 | 0.40 |
| 0.003 | 0.08 | 0.41 |

Table 6.4: Simulation results for the $p = \ln \alpha$ parameter. In this case the uncertainty on the parameter is (by an large) the same up to the 2th decimal place, for all the three cases.

| $r$ | $R_k$ mean value | $R_k$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | -2.09 | 0.79 |
| 0.002 | -1.01 | 0.38 |
| 0.003 | -0.66 | 0.25 |

Table 6.5: Simulation results for the scalar Kälher curvature. Here, the 1-$\sigma$ value gets larger as $r$ increases. In particular, it gets smaller as the value of $R_K$ become larger.

Here:

$$\frac{m_\phi^2}{M_p^2} = 3\pi^2 P_s r b^2, \quad \frac{g}{M_p} = 9\pi^2 P_s r b^3, \quad \lambda = 21\pi^2 P_s r b^4 \tag{6.18}$$

where $m_\phi$ is the mass of the inflaton field while $g$ and $\lambda$ are the coupling constants of the third and fourth order terms of the expansion, respectively. From a quantum field theory point of view, the three different terms correspond to the standard auto-interaction term, the three-legs interaction and fourth-legs interaction. Further details can be found in the second of refs [39].

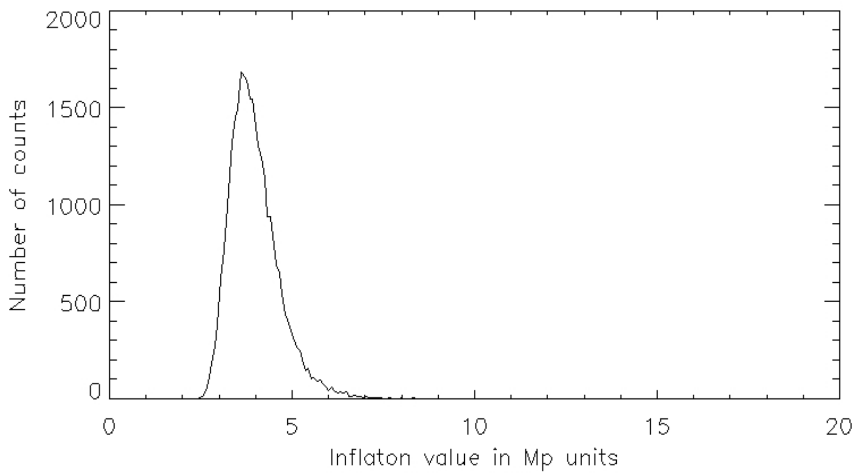


Figure 6.4: Distribution of the estimated $\phi_*$ values for $r = 0.001$.

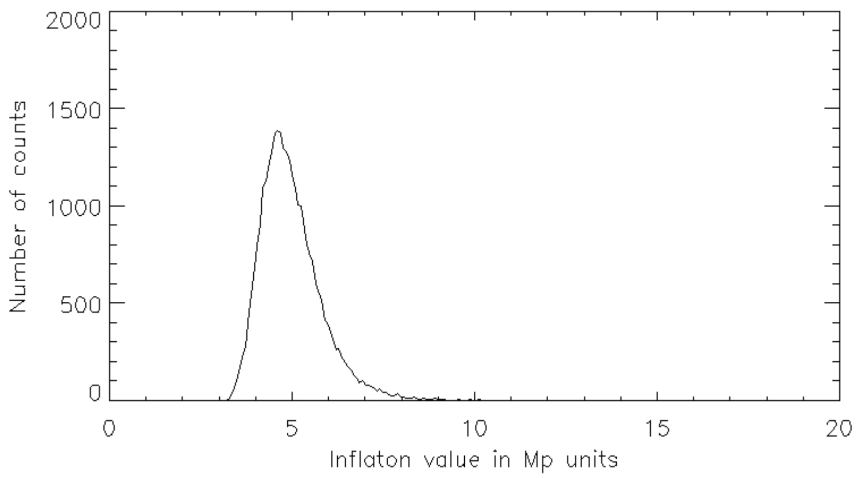


Figure 6.5: Distribution of the estimated $\phi_*$ values for $r = 0.002$.

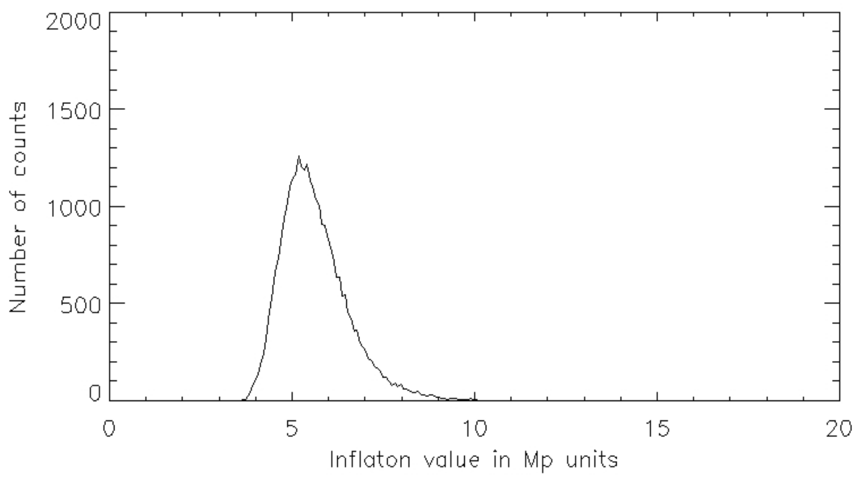


Figure 6.6: Distribution of the estimated $\phi_*$ values for $r = 0.003$. As one can expect, the vacuum expectation value of the scalar field increases as $r$ increase.

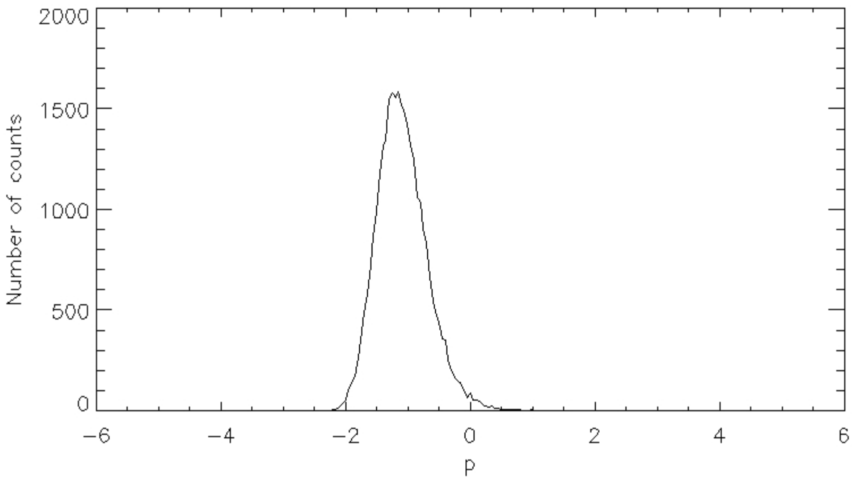


Figure 6.7: Distribution of the $p = \ln\alpha$ values for $r = 0.001$

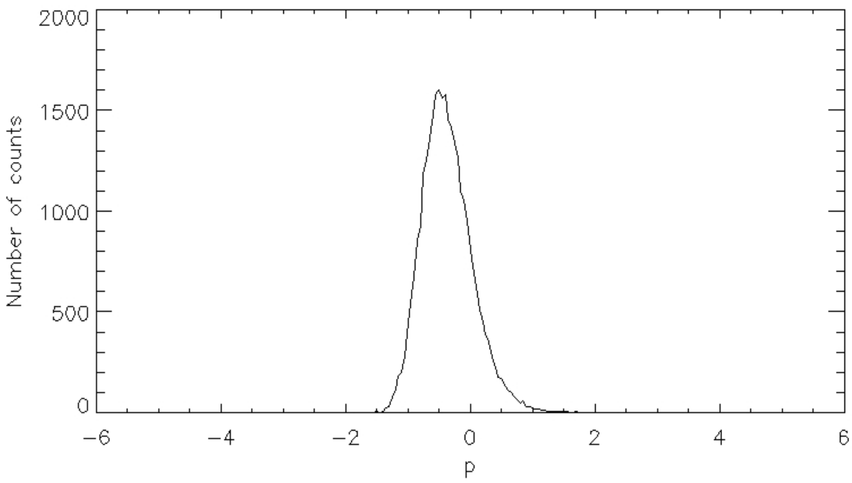


Figure 6.8: Distribution of the $p = \ln\alpha$ values for $r = 0.002$

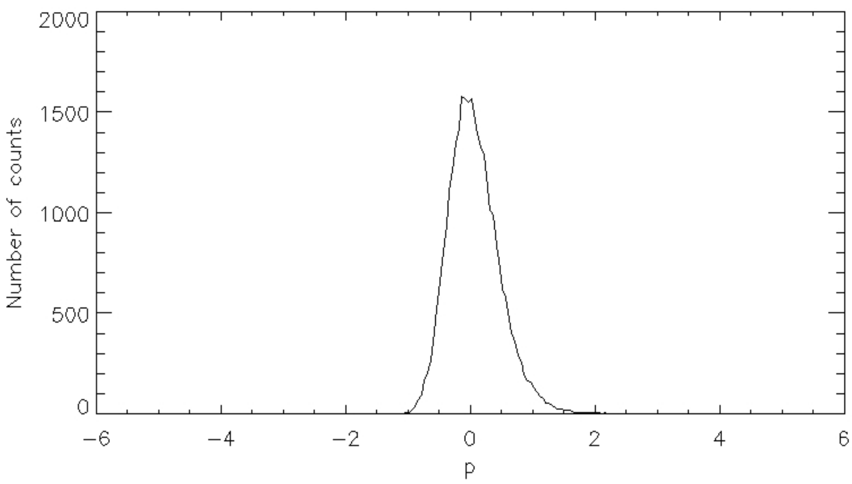


Figure 6.9: Distribution of the $p = \ln\alpha$ values for $r = 0.003$. The vacuum expectation value of $\alpha$ itself increases as $r$ increases, recovering Starobinsky inflation $\alpha \sim 1$ for $r = 0.003$.

### 6.2.3 Reheating reconstruction

Starting from the previous results on $\phi_*$ and $b$ we can reconstruct the postaccelerated epoch (see also [39]),i.e., reconstruct the value of the number of e-foldings during the reheating stage and the reheating temperature. For simplicity, we use the same statistical recipe of the previous section ($\mu_{n_s} = 0.968$, $\sigma_{n_s} = 0.006$, $\mu_r = 0.001, 0.002, 0.003$, $\sigma_r = 0.0001$). The fundamental relation is

$$N_{reh} = \frac{4}{1-3w_{reh}}\left[\xi_0 - N_* + \frac{1}{4}\ln\left(\frac{V_*^2}{M_p^4\rho_{end}}\right)\right], \quad \xi_0 = 64.24 \tag{6.19}$$

where we have three distribution functions: $V(\phi_*)$, $N_*$ and $\rho_{end}$. Therefore we should also know $\phi_{end}$. The vacuum expectation value of the scalar field at the end of inflation can be reconstructed through $\alpha$ (or $b$) by the condition

$$\epsilon_V(\phi_{end}) \sim 1, \quad \rightarrow \quad \frac{\phi_{end}}{M_p} = \frac{1}{b}\ln\left(1+\sqrt{2}b\right). \tag{6.20}$$

In Tab.6.6 we report the results for the three chosen values of $r$. Note that, these results represent the extremal values that $\phi_{end}$ assumes in the considered cases: in principle inflation can stops before, in general when $\epsilon_V < 1$ and this induces larger values for $\phi_{end}$. Now, the distribution function for $V(\phi_*)$ directly follows from the information on $\phi_*$. The energy density $\rho_{end}$ is given by the relation

$$\rho_{end} \simeq \frac{4}{3}V_{end}. \tag{6.21}$$

Finally, the distribution function for $N_*$ is given by the solution of the equation of motion

$$N_* = \frac{1}{2b^2}\left(e^{b\frac{\phi_*}{M_p}} - e^{b\frac{\phi_{end}}{M_p}}\right) - \frac{1}{2b}\left(\frac{\phi_*}{M_p} - \frac{\phi_{end}}{M_p}\right) \tag{6.22}$$

where $\phi_*$, $\phi_{end}$ and $b$ are distribution functions in the variable $n_s, r$. Alternatively, one can also use the approximate result

$$N_* \simeq \frac{1}{2b^2}e^{b\frac{\phi_*}{M_p}} \tag{6.23}$$

if the condition $\phi_{end} \ll \phi_*$ is taken into account. In Tab.6.7 we report the related Monte Carlo results. These kinds of 1-$\sigma$ values on $N_*$ implies a great uncertainty on the number of $e$-foldings $N_{reh}$ and especially, on the reheating temperature $T_{reh}$ due to Eq.(2.25) and Eq.(5.6).

This piece of evidence suggests that improvements of the estimation on $n_s$ could be strongly reduce the uncertainty on the inflationary number $e$-folds $N_*$. Nowadays, foreseen cosmological experiments [35] aim to have an

| $\mu_r$ | $\phi_{end}/M_p$ mean value | $\phi_{end}/M_p$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | 0.781 | 0.061 |
| 0.002 | 0.889 | 0.058 |
| 0.003 | 0.950 | 0.056 |

Table 6.6: Simulation results for the vacuum expectation value of the inflaton field when inflation stops.

| $\mu_r$ | $N_*$ mean value | $N_*$ 1-$\sigma$ value |
|---|---|---|
| 0.001 | 63.7 | 13.5 |
| 0.002 | 63.0 | 13.3 |
| 0.003 | 62.5 | 13.1 |

Table 6.7: Simulation results for the number of $e$-foldings $N_*$ before the end of inflation when $\sigma_{n_s} = 0.006$.

uncertainty $\sigma_{n_s} \sim 0.002$ or even better. Herein, we try to be optimistic and we set $\sigma_{n_s} = 0.0006$. We should observe that it is hard to get such a sensitivity from a single experiment but rather by combining future several experiments. We assume two possible scenarios for the scalar spectral index $n_s$

- $\mu_{n_s} = 0.9650$, $\sigma_{n_s} = 0.0006$
- $\mu_{n_s} = 0.9680$, $\sigma_{n_s} = 0.0006$

and for both cases we consider two possible realizations for the tensor-to-scalar-ratio

- $\mu_r = 0.001, 0.002, 0.003$
- $\mu_r = 0.010, 0.013, 0.016$

with $\sigma_r = 0.0001$ and $\rho = 0.1$.

As we can see from Tab.6.8, in the first case $n_s = 0.9650$ there is an independence on the order of magnitude that the mean value of $r$ takes on. In all considered cases of $r$, the function $f$ results to be positive and low values of the EoS, $w_{reh} < 1/3$, are allowed. Once the parameter called $\mu_r$ results to be setted then, the mean value of $N_{reh}$ increases as $w_{reh}$ gets larger. Consequently, the final reheating temperature gets lower. When we increase the value of the scalar spectral index up to $n_s = 0.9680$, we deal with two different situations. In the range of $r \sim 10^{-3}$, we have $f < 0$ so the reheating is characterized by a large mean value of the EoS, $w_{reh} > 1/3$ (Tab.6.9). In this case, once $\mu_r$ results are fixed, $N_{reh}$ gets lower as $w_{reh}$

| | $w_{reh} = -1/3$ | | $w_{reh} = 0$ | |
|---|---|---|---|---|
| $\mu_r$ | $N_{reh}$ | $\ln T_{reh}/M_p$ | $N_{reh}$ | $\ln T_{reh}/M_p$ |
| 0.001 | $3.70 \pm 1.95$ | $-8.89 \pm 0.97$ | $7.40 \pm 3.90$ | $-12.59 \pm 2.91$ |
| 0.002 | $5.27 \pm 1.90$ | $-9.57 \pm 0.94$ | $10.55 \pm 3.81$ | $-14.84 \pm 2.85$ |
| 0.003 | $6.46 \pm 1.87$ | $-10.11 \pm 0.93$ | $12.93 \pm 3.75$ | $-16.57 \pm 2.81$ |
| 0.010 | $12.02 \pm 1.73$ | $-12.76 \pm 0.86$ | $24.02 \pm 3.47$ | $-24.77 \pm 2.59$ |
| 0.013 | $13.95 \pm 1.67$ | $-13.72 \pm 0.83$ | $27.89 \pm 3.35$ | $-27.67 \pm 2.51$ |
| 0.016 | $15.86 \pm 1.61$ | $-14.67 \pm 0.78$ | $31.73 \pm 3.22$ | $-30.54 \pm 2.41$ |

Table 6.8: Simulation results for the duration of the reheating phase and the reheating temperature related to a detection of the scalar spectral index $n_s = 0.9650$ for detection of the tensor-to-scalar ratio of different orders. The sign of the function $f$ is positive so are allowed mean value of the EoS $w_{reh} < 1/3$. We report the results for two different mean value of EoS: $w_{reh} = -1/3$ and $w_{reh} = 0$.

| | $w_{reh} = 2/3$ | | $w_{reh} = 1$ | |
|---|---|---|---|---|
| $\mu_r$ | $N_{reh}$ | $\ln T_{reh}/M_p$ | $N_{reh}$ | $\ln T_{reh}/M_p$ |
| 0.001 | $13.43 \pm 4.65$ | $-23.84 \pm 5.81$ | $6.71 \pm 2.32$ | $-17.13 \pm 3.49$ |
| 0.002 | $9.95 \pm 4.55$ | $-19.37 \pm 5.70$ | $4.97 \pm 2.27$ | $-14.40 \pm 3.42$ |
| 0.003 | $7.27 \pm 4.47$ | $-15.98 \pm 5.60$ | $3.63 \pm 2.24$ | $-12.35 \pm 3.36$ |

Table 6.9: Simulation results for the duration of the reheating phase and the reheating temperature related to a detection of the scalar spectral index $n_s = 0.9680$ and a tensor-to-scalar ratio $r \sim 10^{-3}$. The sign of the function $f$ is negative so are allowed large value of the EoS $w_{reh} > 1/3$. We report the results for: $w_{reh} = 2/3$ and $w_{reh} = 1$.

| | $w_{reh} = -1/3$ | | $w_{reh} = 0$ | |
|---|---|---|---|---|
| $\mu_r$ | $N_{reh}$ | $\ln T_{reh}/M_p$ | $N_{reh}$ | $\ln T_{reh}/M_p$ |
| 0.010 | $2.68 \pm 2.05$ | $-8.13 \pm 1.02$ | $5.36 \pm 4.10$ | $-10.81 \pm 3.01$ |
| 0.013 | $4.96 \pm 1.97$ | $-9.23 \pm 0.98$ | $9.92 \pm 3.95$ | $-14.21 \pm 2.95$ |
| 0.016 | $7.26 \pm 1.88$ | $-10.40 \pm 0.93$ | $14.52 \pm 3.76$ | $-17.67 \pm 2.82$ |

Table 6.10: Simulation results for the duration of the reheating phase and the reheating temperature related to a detection of the scalar spectral index $n_s = 0.9680$ for a detection of the variable $r$ of the order of $10^{-2}$. In this case, the sign of the function $f$ is still positive, therefore $w_{reh} < 1/3$ are allowed. We report the results: $w_{reh} = -1/3$ and $w_{reh} = 0$.

increases. Therefore, the reheating temperature becomes larger. In the range of $r \sim 10^{-2}$, instead, we have $f > 0$ and so the required values of the EoS turns to be $w_{reh} < 1/3$. Consequently, we recover the same behavior seen when $n_s = 0.9650$ (see Tab.6.10). Anyhow, we should stress that $w_{reh} > 1/3$ is not favored from the standpoint of QFT but it still remain a possibility and in this sense it represents a compelling challenge in the context of early fundamental physics, as mentioned in Chapter V.

## 6.3 Lyth Bound Generalization

In this section we aim to derive a higher order expression of the Lyth Bound [40]. To reach this goal, we need to introduce some important tools. The Hubble slow roll parameters are defined by the following hierarchy of variables (see the last of refs [11]):

$$\epsilon(\phi) = 2M_p^2 \left(\frac{H'}{H}\right)^2 \tag{6.24}$$

$$\beta^{(n)}(\phi) = (2M_p^2)^n \left[\frac{H'^{n-1} H^{(n+1)}}{H^n}\right] \tag{6.25}$$

where $\eta = \beta^{(1)}$, $\xi^2 = \beta^{(2)}$, $\sigma^3 = \beta^{(3)}$ ... with $\sqrt{\epsilon} > 0$ for $\dot{\phi} < 0$. The slow roll parameters are not constant in general but they change as the scalar field evolves. Mathematically, the evolution of the $\beta_i(\phi)$ are well-described by the system of inflationary flow equations [41]. Here, for sake of convenience, the slow roll parameters are suitably expressed as functions of the number of $e$-foldings. In particular, we can write

$$\frac{d\epsilon}{dN} = 2\epsilon(\eta - \epsilon) \tag{6.26}$$

$$\frac{d\eta}{dN} = \xi^2 - \epsilon\eta \tag{6.27}$$

$$\frac{d\beta^{(n)}}{dN} = \left[(n-1)\eta - n\epsilon\right]\beta^{(n)} + \beta^{(n+1)}. \tag{6.28}$$

Now we can write down a Taylor expansion of the scalar field (as a function of the number of $e$-foldings) around the horizon crossing of the observable scales

$$\phi(N) = \phi(N_*) + \phi'(N_*)\Delta N + \frac{1}{2!}\phi''(N_*)\Delta N^2 + ... \tag{6.29}$$

where $'$ indicates a derivative with respect to $N$ and $\Delta N = N - N_*$. Assuming again that $\phi$ is decreasing along the inflationary evolution and that $N_* > N$, we can rewrite the Taylor expansion as follows:

$$\Delta\phi = +\phi'(N_*)|\Delta N| - \frac{1}{2!}\phi''(N_*)|\Delta N|^2 + ... \tag{6.30}$$

where $\Delta\phi = \phi_* - \phi$ and $|\Delta N| = -(\Delta N)$. Generalizing, we have a sum of infinite terms

$$\Delta\phi = \sum_{n=1}^{\infty} c_n |\Delta N|^n, \quad c_n = (-1)^{n+1}\phi^{(n)}(N_*). \tag{6.31}$$

The first order derivative of the expansion is the equation we met in the last section of Chapter III

$$\frac{d\phi}{dN} = \sqrt{2}M_p\sqrt{\epsilon(\phi)}. \tag{6.32}$$

Using the flow equations we can derive the higher order derivatives of the field in terms of the slow roll parameters. For example, next orders result in

$$\frac{d^2\phi}{dN^2} = \sqrt{2}M_p\sqrt{\epsilon}\left(\eta - \epsilon\right), \tag{6.33}$$

$$\frac{d\phi^3}{dN^3} = \sqrt{2}M_p\sqrt{\epsilon}\left(3\epsilon^2 + \eta^2 - 5\epsilon\eta + \xi^2\right) \tag{6.34}$$

and

$$\frac{d\phi^4}{dN^4} = \sqrt{2}M_p\sqrt{\epsilon}\left[f_1(\epsilon,\eta,\xi^2) + f_2(\epsilon,\eta,\xi^2,\sigma^3)\right] \tag{6.35}$$

where

$$f_1(\epsilon,\eta,\xi^2) = (\eta - \epsilon)(15\epsilon^2 + \eta^2 - 15\epsilon\eta + \xi^2), \tag{6.36}$$

$$\begin{aligned} f_2(\epsilon,\eta,\xi^2,\sigma^3) = (\xi^2 - \epsilon\eta)(2\eta - 5\epsilon) \\ +\xi^2(\eta - 2\epsilon) + \sigma^3 \quad . \end{aligned} \tag{6.37}$$

Here $\sigma^3$ is the fourth slow roll parameter. At this point, we can calculate the derivatives at horizon crossing moment $t_*$ (or $N_*$, let us say). In doing so, we have to reword the slow roll parameters in terms of the inflationary observables. At first order, we have

$$\epsilon \sim \frac{r}{16}, \quad \eta \sim \frac{1}{2}(n_s - 1) + \frac{r}{8} \tag{6.38}$$

and

$$\xi^2 \sim 5\epsilon\eta - 4\epsilon^2 - \frac{1}{2}\alpha_s. \tag{6.39}$$

Substituting these information in the expansion Eq.(6.31) with $|\Delta N| \sim 4$ $e$-folds, we have (at the third order, for instance)

$$\frac{\Delta\phi_{h.c}}{m_p} \sim \sqrt{\frac{r}{4\pi}}\Big\{\Pi_0 - \Pi_1 - \Pi_2 + ...\Big\} \tag{6.40}$$

where:

$$\Pi_0 = 1 \quad (6.41)$$

$$\Pi_1 = (n_s - 1) + \frac{r}{8} \quad (6.42)$$

$$\Pi_2 = \frac{8}{3}\left(\frac{r}{16}\right)^2 - \frac{8}{3}\left[\frac{1}{2}(n_s - 1) + \frac{r}{8}\right]^2 + \frac{4}{3}\alpha_s \quad (6.43)$$

where $\Pi_1 = \Pi_1(r, n_s)$, $\Pi_2 = \Pi_2(r, n_s, \alpha_s)$. The first term of the expansion is the standard Lyth Bound. The second order term is a linear correction in $n_s$ and $r$. Finally, the last term is a quadratic correction in $n_s$ and $r$ and takes into account the contribution of the running of the scalar spectral index. Note that, the contribution of $\alpha_s$ in our formula appears with a negative sign

$$\frac{\Delta\phi_{h.c}}{m_p} \sim -\frac{4}{3}\alpha_s. \quad (6.44)$$

Therefore a negative value for $\alpha_s$ pushes the local variation $\Delta\phi$ to larger values. On the contrary, if $\alpha_s$ results to be positive, the excursion receives a negative contribution. In the previous chapter we mentioned the current state-of-the art for the values of the main inflationary parameters. However, such a limits could rapidly improve in the close future due to the upcoming cosmological experiments, like polarization missions of the cosmic macrowave background (CMB) or gravitational waves (GW) detection missions as seen previously, [35]. In this regard, it might be interesting to investigate the variation of the field over the first 4 $e$-folds (more or less) of inflationary expansion. To do this, we consider a multivariate Gaussian $\mathcal{G}(r, n_s, \alpha_s)$ distribution for the parameters $(r, n_s, \alpha_s)$ with

1. $\mu_r = 0.003, 0.004, 0.005$
2. $\mu_{n_s} = 0.9620, 0.9650, 0.9680$
3. $\mu_{\alpha_s} = -0.0005, -0.0015, -0.0025$

with uncertainties $\sigma_r = 0.0001$, $\sigma_{n_s} = 0.002$, $\sigma_{\alpha_s} = 0.0023$, respectively. Furthermore, we set to zero (for simplicity) all the correlation coefficients between the variables.

Fig.(6.10) and in Fig.(6.11) show the resulting distribution functions for the minimum value of $\Delta\phi$ supposing a detection of primordial tensor modes given by a forthcoming CMB baloon polarization mission or GW mission while Fig.(6.12) and Fig.(6.13) report the associated dispersion relations

- $\Delta\phi_{h.c}(\Pi_1) Vs \Delta\phi_{h.c}(\Pi_0)$
- $\Delta\phi_{h.c}(\Pi_2) Vs \Delta\phi_{h.c}(\Pi_0)$

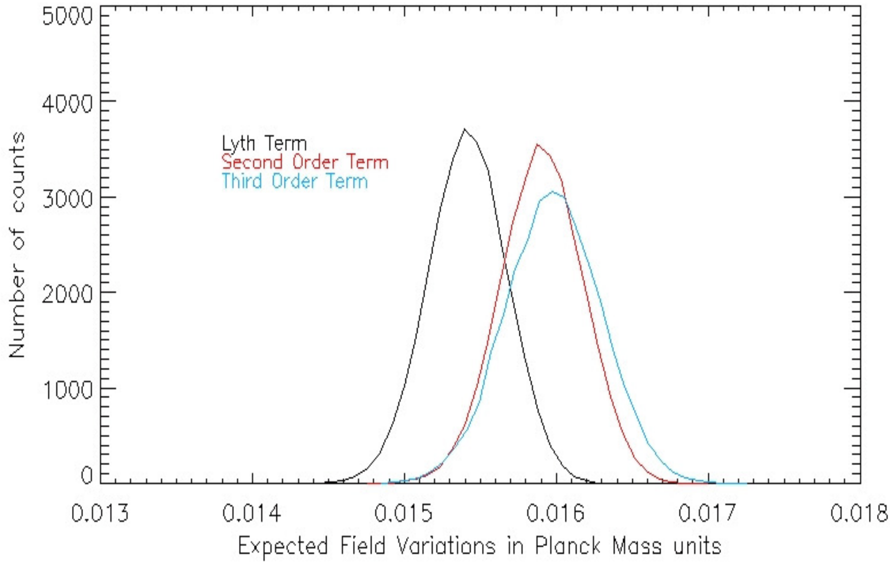


Figure 6.10: Distribution functions for the minimum excursion of the scalar field during inflation, $\Delta\phi_{h.c}$, for the three different slow roll expansions. Here, we have supposed a tensor-to-scalar ratio amplitude of the order of $\mu_r \sim 3 \times 10^{-3}$ with $n_s = 0.9680$, $\alpha_s = -0.0033$

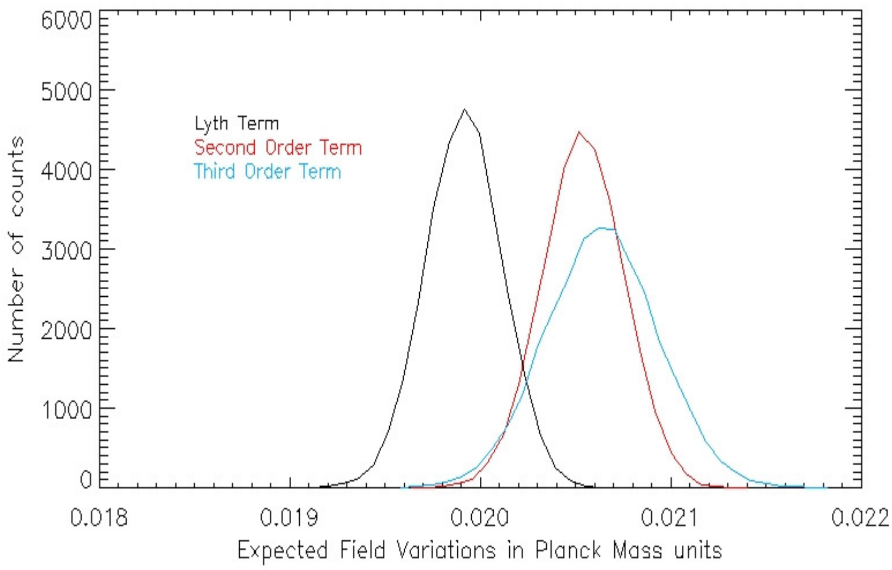


Figure 6.11: Distribution functions for the variable $\Delta\phi_{h.c}$ for the three different slow roll expansions. In this case, we have supposed a detection of primordial GW of the order of $\mu_r \sim 5\times10^{-3}$ with $n_s = 0.9680$, $\alpha_s = -0.0033$

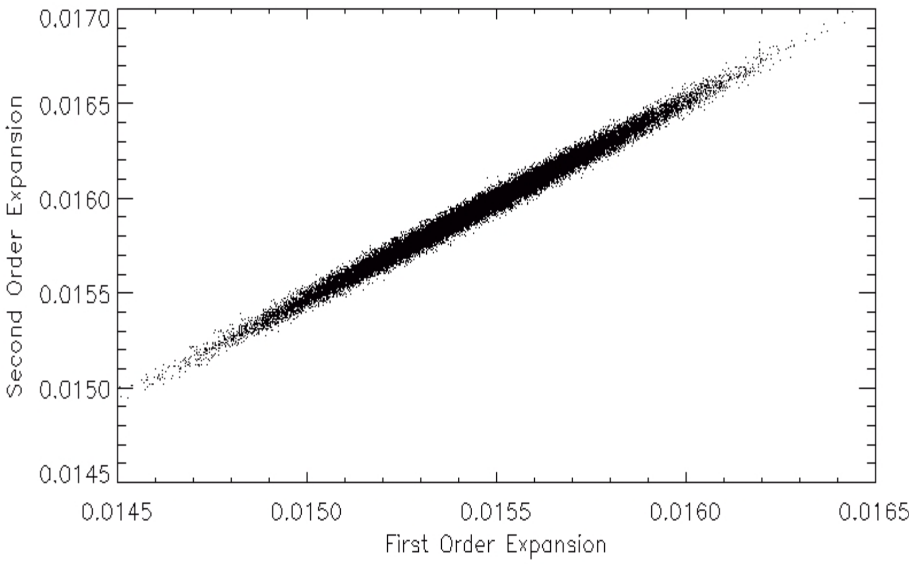


Figure 6.12: Dispersion plot for the ratio $\Delta\phi(\Pi_1)/\Delta\phi(\Pi_0)$ in the case of a detection of $r \sim 3 \times 10^{-3}$ with $n_s = 0.9680$, $\alpha_s = -0.0033$.

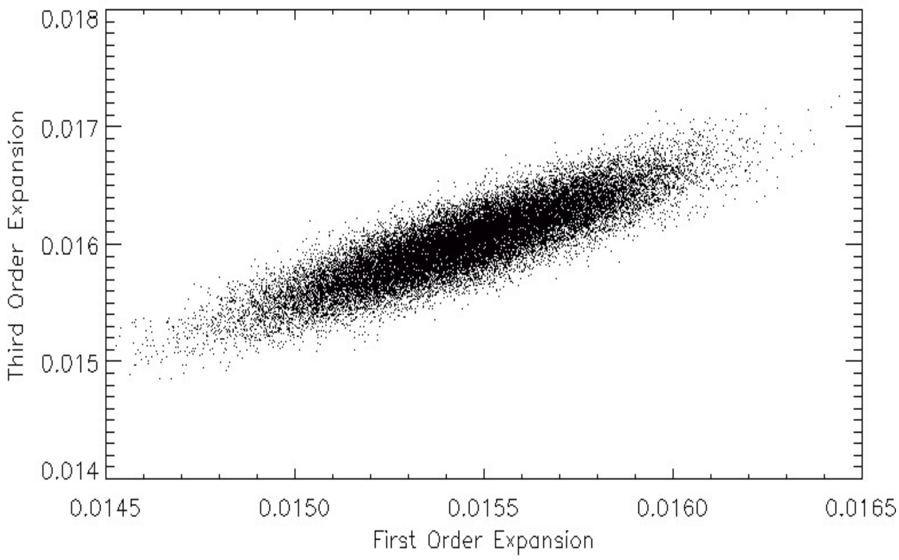


Figure 6.13: Dispersion plot for the ratio $\Delta\phi(\Pi_2)/\Delta\phi(\Pi_0)$ in the case of a detection of $r \sim 3 \times 10^{-3}$ with $n_s = 0.9680$, $\alpha_s = -0.0033$.

In both cases, we assume the current data for the sampling of $n_s$ and $\alpha_s$. Now, these Monte-Carlo simulations show a couple of properties. First of all, we can see that the inclusion of the scalar spectral index $n_s$ and the running $\alpha_s$ produces a larger $\Delta\phi_{h.c}$. Second, the 1-$\sigma_{\Delta\phi_{h.c}}$-value turns out to be larger as the slow roll reconstruction become deeper. Somehow, this is an expected result and it is originated by the introduction of more uncertainty due to the new parameters. On the other hand, both plots show that the variation of the field become smaller as $n_s$ increases. In fact, the scalar spectral index, always appears in the $\Delta\phi_{h.c}$ expression like $1 - n_s$. Therefore, when $n_s$ gets larger values, the previous term approaches to zero so it is a subdominant contribution. Furthermore, a bigger running index (in modulus) implies a larger excursion because of Eq.(6.44). In the next section, we use our third order result to explore the relation between $\Delta\phi_{h.c}$ and an hypothetical eternal inflation stage.

## 6.4 Generalized Lyth Bound and Eternal Inflation

The inflationary mechanism could lead to a very interesting and spectacular consequence: an infinitely self-reproducing state commonly called eternal inflation. The eternal inflation process was firstly outlined by A.Linde, P.Steinhardt and A.Vilenkin (see [42]) in the framework of new inflation. Afterwards, A.Linde realized that such a mechanism could be also possible in the chaotic inflationary scenario [43]. In this second case, the eternal inflation stage is laid down when the inflaton field explores a region $\Sigma$ of the inflationary potential in which quantum fluctuations of the field $\delta\phi$ are larger then the classical variation ones, $\Delta\phi$. This leads to a very interesting implications about the amplitude of the metric perturbations as pointed out by A.Linde in his works [43]: the power spectrum of the scalar metric perturbations must exceed the unity on those physical scales related to $\Sigma$

$$P_s(k) = A_s \left(\frac{k}{k_*}\right)^{f(k,k_*)} > 1 \text{ for all } k \in \mathcal{I}_\Sigma(k) \tag{6.45}$$

where $\mathcal{I}_\Sigma(k)$ is the set of scales $k$ related to the portion $\Sigma$ of inflationary potential, $k_*$ is the pivot scale for probing the cosmological parameters and the function $f(k, k_*)$ is given by

$$f(k, k_*) = (n_s - 1) + \alpha_s \ln\left(\frac{k}{k_*}\right) + \beta_s \ln^2\left(\frac{k}{k_*}\right) + ... \tag{6.46}$$

in which $\beta_s$ and the higher order terms represent the running-of-the-running, the running-of-the-running-of-the-running and so on. Who is $I_\Sigma(k)$? We should note that the current cosmological results show a very low $P_s$ (and $A_s$), with a negligible dependence on the scale ($P_s \simeq 2.2 \times 10^{-9}$, see Chapter III). Therefore, we can be fairly confident the observable and accessible region of $V(\phi)$ does not provide an eternal inflationary epoch. Then, we can expect an increase (at least) of the order of $10^9$ on $P_s$ only at extremely large scales $k \ll k_*$, beyond our horizon in principle. How can we get $P_s(k) \geq 1$? In general, we can get such a condition by proper choices of the function Eq.(6.46), i.e., by proper choice of the cosmological parameters. Then, if we truncate the series at higher orders, the condition Eq.(6.45) is provided by a large number of choices, generally. However, we may imagine that the power spectrum is modulated only by the scalar spectral index and by a non null and constant $\alpha_s$

$$\alpha_s \neq 0, \quad \text{and} \quad \frac{d\alpha_s(k)}{d\ln k} = 0. \tag{6.47}$$

From this point of view, one can use the running as the only degree of freedom or "temperature parameter" to study the transition to a blue spectrum $P(k) > 1$. We can call the threshold value of the running preventing the eternal inflation, $\alpha_s^*$ and find [44]

$$\alpha_s^* = \frac{(1-n_s)^2}{4\ln A_s}. \tag{6.48}$$

Therefore, we may argue that

$$\alpha_s < \alpha_s^* \text{ eternal inflation does not occur} \tag{6.49}$$
$$\alpha_s > \alpha_s^* \text{ eternal inflation occurs} \tag{6.50}$$

For instance,

$$n_s \sim 0.9620, \quad \alpha_s^* \sim -1.8109 \times 10^{-5} \tag{6.51}$$
$$n_s \sim 0.9650, \quad \alpha_s^* \sim -1.5363 \times 10^{-5} \tag{6.52}$$
$$n_s \sim 0.9680, \quad \alpha_s^* \sim -1.2842 \times 10^{-5} \tag{6.53}$$

In the previous section we derived an expression of the Lyth Bound in terms of $\alpha_s$. Then, we can distinguish which are the minimum variations of the field related to an eternal inflation (EI) stage with respect to those are not

$$\Delta\phi_{h.c}(\alpha_s < \alpha_s^*) \text{ no EI-related variations} \tag{6.54}$$
$$\Delta\phi_{h.c}(\alpha_s > \alpha_s^*) \text{ EI-related variations} \tag{6.55}$$

In Fig.(6.14) we report the third order expression of $\Delta\phi_{h.c.}$ as a function of $\alpha_s$ for some detections of primordial gravitational waves. In addition, we outline the range of the possible EI-related $\Delta\phi$ at horizon crossing of quantum modes. In the next section we will discuss the reasonableness of these discussions.

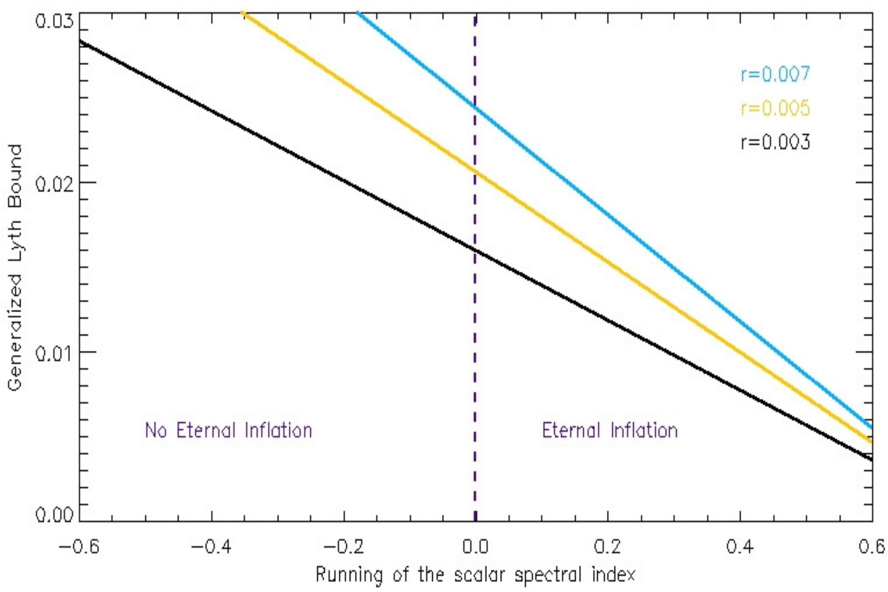


Figure 6.14: Third order Lyth Bound as a function of the running $\alpha_s$. In this case we consider $n_s \sim 0.9680$ with a threshold value $\alpha_s \sim -1.2842 \times 10^{-5}$. Possible eternal inflation phases are associated with smaller variations of the field at horizon crossing of quantum fluctuations.

## 6.5 Conclusions and Prospects

The performed analysis is based on a generalization of the standard Lyth formula for the variation of the field about the horizon crossing epoch of quantum modes. The computation of the expansion retraces the steps of the local reconstruction of the effective inflationary potential around $\phi_*$. The first step is to introduce higher order terms in the slow roll picture and then, reword them in terms of the cosmological variables. In particular we derived a third order expansion for $\Delta\phi_{h.c}$. The Eq.(6.44) shows a linear dependence on $\alpha_s$ and it turns out to be decoupled from the other inflationary parameters. This is because there is not coupling between the third slow roll parameter, $\xi^2$, and the first two slow roll parameters [cfn. Eq.(6.34)]. It is straightforward to imagine that a hypothetical fourth order term will introduce a coupling between the running $\alpha_s$ and $r$ and $n_s$ and in addition it will introduce the higher order variables $d^2\alpha_s/d\ln k^2$. The performed Monte-Carlo simulations assume possible results of a single foreseen cosmological experiment. Nevertheless, we could combine the results of more cosmological missions in order to get more accurate estimation on the main cosmological parameters or on the macro reheating variable (see Chapeter V) as well as on $\Delta\phi_{h.c}$. Mathematically, constraining the variation of the scalar field at horizon crossing is particularly important because provides a lower limit for the total excursion of the field. At the same time, we can observe the contribution of the observable quantities to $\Delta\phi$. Constraining little variations of $\Delta\phi_{h.c}$ in terms of $M_p$ (or $m_p$), does not imply a sub-Planckian total variation of field. This is because the last $e$-folds could provide a large contribution on the total variation. The size of the minimum variation of the scalar field can be related to a hypothetical eternal inflation. In the simplest scenario, following W.H.Kinney and K.Freese [44], one can argue that a constant negative running $\alpha_s$ can prevent an eternal inflationary phase to occur. Since, at third order in the slow roll expansion, the variation of the field around the horizon crossing gets dependence in $\alpha_s$ we find that is possible to split $\Delta\phi_{h.c.}$ in two classes: the first one associated with the occurrence of the eternal inflation phase while the second not. However, we should underline a couple of important questions about these results. First of all we do not have any guarantee that the function Eq.(6.46) is well described only by constant $\alpha_s$ (and $n_s$). In other words, we cannot be sure that a dramatic change (or not) of the order of magnitude about $P_s$ is only related to the single running $\alpha_s$ parameter. In principle, the higher order terms may play an important role and so we should generalize the procedure. On the other side, we cannot measure the running as well as any other cosmological variables outside our horizon, so it is also plausible that there is no observation can tell us if eternal inflation takes place or not. From this point of view, it is important to stress that a discussion about the relation between $\Delta\phi_{h.c.}$ and eternal inflation is still speculative. Surely, one can think that eternal inflation can be occur inside our horizon scale. In that case we should have observable consequences such as a natural and strong production of black holes. In our analysis we adopted a single field slow roll version of inflation as the paradigm of the early universe. This scenario could be confirmed and strengthened in the close future. Nevertheless there is still room (although little) for non-trivial signature of inflation. In this respect, the standard definition of Lyth Bound for the minimum variation of the scalar field is not more consistent as well as any kind of its slow roll generalizations. Then we need a new definition of $\Delta\phi$ during the horizon crossing of quantum fluctuations. This problem has been approached for example by Baumann and Green in [45], in the context of general single field model ($P(X)$-model) using the Goldstone picture of effective field theory of inflation.

# Appendix A

# Dimensional Power spectrum

The standard definition of the correlation function is

$$\xi(r) =< \delta(\vec{x}, t)\delta(\vec{x} + \vec{r}, t) > \tag{A.1}$$

If we assume to decompose the perturbation field in plane waves by a Fourier transform, we get

$$\xi(r) = \frac{1}{(2\pi)^3} \int d^3k |\delta_{\vec{k}}|^2 e^{-i\vec{k}\vec{x}} \tag{A.2}$$

Now we can introduce the power spectrum as the squared of the amplitude of the perturbations

$$P(k) = |\delta_{\vec{k}}|^2 \tag{A.3}$$

In this way the power spectrum (as well as the quantity $|\delta_{\vec{k}}|^2$) is found to be exactly as the Fourier transform of the correlation function! That is

$$P(k) = \int d^3r \xi(r) e^{-i\vec{k}\vec{r}} \tag{A.4}$$

In particular, the integration over the $k$-space provides

$$\xi(r) = \frac{1}{2\pi^2} \int_{I(k)} dk k^2 P(k) \frac{\sin(kr)}{kr} \tag{A.5}$$

where $I(k)$ is again the range of integration of $k$. As a result, the variance of the perturbation field is given by

$$\xi(0) = \sigma_\delta^2 = \frac{1}{2\pi^2} \int_{I(k)} dk k^2 P(k) \tag{A.6}$$

while the correlation function in the Fourier space comes out to be

$$< \delta_k \delta_{k'} >= (2\pi)^3 P(k) \delta(k - k') \tag{A.7}$$

that is an usual definition we can find in literature. Only a this point we can introduce an adimensional quantities often defined as $\Delta(k)$

$$P(k) = \frac{k^3}{2\pi^2}\Delta(k) \tag{A.8}$$

Therefore when we deal with perturbations we can use two different formalisms about the definitions of the power spectrum. In literature, especially in the inflationary cosmology, we do not have a "most used" framework rather different authors prefer to use different notations. What we should remember is that only the quantity

$$\frac{k^3}{2\pi^2}|\delta_k|^2 \tag{A.9}$$

is the one used to show the cosmological results! Independently if we call it $P(k)$ or $\Delta(k)$.

# Appendix B

# Scalar-Vector-Tensor decomposition

The SVT decomposition aims to reword a generic quantities as sum of a scalar contribution, a vector contribution and a tensor contribution and it is based on the standard Helmoholtz theorem a particular case of the Hodge decomposition. Then, let us consider a generic physical quantities $\omega(x)$. We can write

$$\omega(x) := \left(\omega^S, \omega^V, \omega^T\right) \tag{B.1}$$

Naturally, this decomposition strongly depends on the rank of our quantity $\omega(x)$! Let us start considering $\omega$ as a pure scalar. It is evident that in this case we can simply write

$$\omega(x) = \omega^S = \omega \tag{B.2}$$

Let us now consider $\omega$ as a vector. In this case we can introduce two contributions: the first one obtained by a scalar, in particular as the gradient of a scalar, the second as a pure vector

$$\omega_i = \omega_i^S + \omega_i^V = \partial_i \omega + \omega_i^V \tag{B.3}$$

Here the scalar contribution is called longitudinal part or potential flux while the vector contribution is called transverse part or vorticity. In addition the scalar part is irrotational while the vector is divergenceless

$$\partial^i \omega_i^V = 0 \tag{B.4}$$

Let us now imagine $\omega$ as a tensor with two Lorentz indices that we want to be symmetric, divergenceless and traceless. In this case we can introduce three contributions: the first obtained by a scalar, the second by a vector and the third one as a pure tensor

$$\omega_{ij} = \omega_{ij}^S + \omega_{ij}^V + \omega_{ij}^T \tag{B.5}$$

It is simply to realize that the scalar part may be written as the Hessian of a pure scalar function plus a $\delta_{ij}$-term

$$\omega_{ij}^S = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\omega \tag{B.6}$$

while the vector one as the Jacobian of a divergenceless vector

$$\omega_{ij}^V = \frac{1}{2}\left(\partial_i\omega_j + \partial_j\omega_i\right), \quad \partial^i\omega_i = 0 \tag{B.7}$$

with $\omega_{ij}^T$ is divergenceless and traceless

$$\partial^i\omega_{ij}^T = 0, \quad \omega_{ii} = 0 \tag{B.8}$$

As a result we can write $\omega_{ij}$ as

$$\begin{aligned}
\omega_{ij} &= \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\omega + \frac{1}{2}\left(\partial_i\omega_j + \partial_j\omega_i\right) + \omega_{ij}^T \\
&= -\frac{1}{3}\delta_{ij}\nabla^2\omega + \partial_i\partial_j\omega + \frac{1}{2}\left(\partial_i\omega_j + \partial_j\omega_i\right) + \omega_{ij}^T \\
&= A\delta_{ij} + \partial_i\partial_j B + \frac{1}{2}\left(\partial_i C_j + \partial_j C_i\right) + D_{ij}
\end{aligned} \tag{B.9}$$

where we reworded for sake of convenience

$$-\frac{1}{3}\delta_{ij}\nabla^2\omega = A\delta_{ij} \tag{B.10}$$

$$\omega = B \tag{B.11}$$

$$\omega_i = C_i \tag{B.12}$$

$$\omega_{ij}^T = D_{ij} \tag{B.13}$$

This is the procedure we applied in Sec.V to write down the SVT decomposition of the metric tensor. It is quite interesting to analysis what happen in the Fourier space. To start, let us consider the $z$-axis as the direction of propagation

$$\vec{k} = (0, 0, k_z) \tag{B.14}$$

and the condition $\partial_i \to -ik_i$. Then the scalar contribution of a vector results to be

$$\omega_i^S(k) = -ik_i\omega \tag{B.15}$$

Introducing $k$ we have

$$\omega_i^S(k) = -i\frac{k_i}{k}\tilde{\omega}(k), \quad \tilde{\omega} = k\omega(k) \tag{B.16}$$

while the divergenceless condition for the pure vector contribution is

$$\partial^i\omega_i^V = -ik_i\omega_i^V = 0, \quad k_i\omega_i^V = 0 \tag{B.17}$$

This means the divergenceless corresponds to an orthogonality between $k_i$ and $\omega_i$ in the Fourier space. The scalar contribution of a tensor is

$$\omega_{ij}^S(k) = \left[-k_i k_j + \frac{1}{3}\delta_{ij}k^2\right]\omega(k) = \left[-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right]\tilde{\omega}(k) \tag{B.18}$$

and this relation shows that $\omega_{ij}^S$ has component different from 0 only on the diagonal. The vector contribution is

$$\omega_{ij}^V(k) = \frac{-i}{2}\left(k_i\omega_j + k_j\omega_i\right) = \frac{-i}{2k}\left(k_i\tilde{\omega}_j + k_j\tilde{\omega}_i\right) \tag{B.19}$$

and in this case we have components different from zero only off diagonal. The last contribution can be written as

$$\omega_{ij}^T(k) = \omega_\times \epsilon_{ij}^\times(k) + \omega_+\epsilon_{ij}^+(k) \tag{B.20}$$

where $\epsilon_{ij}^\times$ and $\epsilon_{ij}^+$ are the polarizations defined as

$$\epsilon_{ij}^\times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_{ij}^+ = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.21}$$

Therefore the matrix representation for the scalar and vector contributions are

$$\omega_{ij}^S(k) = \frac{1}{3}\begin{pmatrix} \tilde{\omega} & 0 & 0 \\ 0 & \tilde{\omega} & 0 \\ 0 & 0 & -2\tilde{\omega} \end{pmatrix}, \quad \omega_{ij}^V(k) = \frac{-i}{2}\begin{pmatrix} 0 & 1 & \tilde{\omega}_1 \\ 1 & 0 & \tilde{\omega}_2 \\ \tilde{\omega}_1 & \tilde{\omega}_2 & 0 \end{pmatrix} \tag{B.22}$$

while the one of the pure tensor is

$$\omega_{ij}^T = \begin{pmatrix} \omega^\times & \omega^+ & 0 \\ \omega^+ & \omega^\times & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{B.23}$$

# Appendix C

# Gauge Frameworks

In this appendix we report the system of perturbed Einstein-Klein Gordon field equations for the some gauge choices. Let us start with the Newtonian/conformal gauge framework. The Einstein field equations are defined by $B = E = 0$

$$3H\left(\dot{\Psi} + H\Phi\right) - \frac{\nabla^2}{a^2}\Psi = -4\pi G\delta\rho \tag{C.1}$$

$$\dot{\Psi} + H\Phi = -4\pi G\delta q \tag{C.2}$$

$$\frac{\Psi - \Phi}{a^2} = 8\pi G\pi^s \tag{C.3}$$

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{C.4}$$

The perturbed Klein-Gordon equation in this gauge is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \dot{\phi}\dot{\Phi} - 3\dot{\Psi}\dot{\phi} - \frac{1}{a^2}\partial_i\partial^i\delta\phi - 2\Phi V_\phi + V_{\phi\phi}\delta\phi = 0 \tag{C.5}$$

The Einstein field equations are defined by $\Phi = B = 0$

$$3H\dot{\Psi} - \frac{\nabla^2}{a^2}\left(\Psi + aH\dot{E}\right) = -4\pi G\delta\rho \tag{C.6}$$

$$\dot{\Psi} = -4\pi G\delta q \tag{C.7}$$

$$\frac{\Psi}{a^2} + \left(\partial_t + 3H\right)\dot{E} = 8\pi G\pi^s \tag{C.8}$$

$$\ddot{\Psi} + 3H\dot{\Psi} = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{C.9}$$

and related perturbed Klein-Gordon equation is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - 3\dot{\Psi}\dot{\phi} - \frac{1}{a^2}\partial_i\partial^i\delta\phi + V_{\phi\phi}\delta\phi = 0 \tag{C.10}$$

The uniformity density gauge is defined by $E = \delta\rho = 0$ and we have $\Psi = \zeta$ so that

$$3H\left(\dot{\Psi} + H\Phi\right) - \frac{\nabla^2}{a^2}\left(\Psi - aHB\right) = 0 \tag{C.11}$$

$$\dot{\Psi} + H\Phi = -4\pi G\delta q \tag{C.12}$$

$$\frac{\Psi - \Phi}{a^2} - \left(\partial_t + 3H\right)\frac{B}{a} = 8\pi G\pi^s \tag{C.13}$$

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{C.14}$$

where the perturbed Klein Gordon equation coincides with the general one

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \dot{\phi}\dot{\Phi} - 3\dot{\Psi}\dot{\phi} - \frac{1}{a}\dot{\phi}\partial_i\partial^i B - \frac{1}{a^2}\partial_i\partial^i\delta\phi - 2\Phi V_\phi + V_{\phi\phi}\delta\phi = 0 \tag{C.15}$$

The comoving gauge is defined setting $\delta u = \delta\phi = 0$ and $E = 0$ and $\mathcal{R} = \Psi$ then

$$3H\left(\dot{\Psi} + H\Phi\right) - \frac{\nabla^2}{a^2}\left(\Psi - aHB\right) = -4\pi G\delta\rho \tag{C.16}$$

$$\dot{\Psi} + H\Phi = 0 \tag{C.17}$$

$$\frac{\Psi - \Phi}{a^2} - \left(\partial_t + 3H\right)\frac{B}{a} = 8\pi G\pi^s \tag{C.18}$$

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{C.19}$$

and the perturbed Klein Gordon equation comes out to be

$$\dot{\phi}\dot{\Phi} - 3\dot{\Psi}\dot{\phi} - \frac{1}{a}\dot{\phi}\partial_i\partial^i B - 2\Phi V_\phi = 0 \tag{C.20}$$

Finally, the spatially flat gauge provides $\Psi = E = 0$ and system of equations results to be

$$3H^2\Phi + H\frac{\nabla^2}{a}B = -4\pi G\delta\rho \tag{C.21}$$

$$H\Phi = -4\pi G\delta q \tag{C.22}$$

$$-\frac{\Phi}{a^2} - \left(\partial_t + 3H\right)\frac{B}{a} = 8\pi G\pi^s \tag{C.23}$$

$$H\dot{\Phi} + \left(3H^2 + 2\dot{H}\right)\Phi = 4\pi G\left(\delta p + \nabla^2\pi^s\right) \tag{C.24}$$

while the perturbed Klein Gordon equation is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \dot{\phi}\dot{\Phi} - \frac{1}{a}\dot{\phi}\partial_i\partial^i B - \frac{1}{a^2}\partial_i\partial^i\delta\phi - 2\Phi V_\phi + V_{\phi\phi}\delta\phi = 0 \tag{C.25}$$

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