Index Theorems For Pairs Of Holomorphic Self-Maps In The Lehmann-Suwa Framework

TESI DI DOTTORATO

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Anno Accademico 2015-2016

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Introduction

0.1 Motivation

This thesis lies in the wake of the various works about index theorems in holomorphic foliation theory and of the subsequent works about index theorems for holomorphic self-maps.

First of all, what do we mean here for "index theorem"? Let M be a compact oriented real C^∞ manifold of dimension m and let $\omega \in H^m_{dR}(M, \mathbb{C})$ be a non-zero top degree cohomological class of M (which typically is a characteristic class of a C^∞ or virtual complex vector bundle). Roughly speaking, an "index theorem" is a formula relating the integral of ω over M to a finite sum of complex numbers R_λ, that is a formula of the kind

$$\sum_\lambda R_\lambda = \int_M \omega, \qquad\qquad R_\lambda \in \mathbb{C}.$$

The numbers R_λ are called "indices" or "residues" and they keep memory of geometric informations. A first classical and well-known example of index theorem is the *Poincaré-Hopf theorem*, relating the *Euler-Poincaré characteristic* of M (which is equal to the integral on M of the *Euler class* $e_M \in H^m(M)$) to some indices associated to the isolated vanishing points of a C^∞ vector field on M (see for example [13, Thm.11.25] for a precise statement of the theorem). More generally we may also consider possibly singular analytic varieties M or obtain formulas relating homological classes (this point will be clearified in Chapter 1).

Index theorems are interesting for instance because they relate "global informations" with "local informations". Moreover they furnish "primary obstructions" to the existence of certain geometric objects on the manifold. For instance, the Poincaré-Hopf theorem implies that if the Euler-Poincaré characteristic of M is not zero then it does not exist a never vanishing C^∞ vector field on M. This is the case, among the others, of the Riemann surfaces of genus $g \neq 1$ (if $g = 0$ it is the *hairy ball theorem*) and of the complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 1$. Clearly, the non-existence of a never vanishing C^∞ vector field implies the non-existence of a never vanishing holomorphic one.

As we said, this thesis is inspired by the works concerning residues and index theorems coming from holomorphic foliations on complex varieties. The first ones who worked to this kind of theorems were Baum and Bott in the 60's, in particular [11] and the subsequent [10], [8] and [9] can be considered the foundational and starting papers of the matter. In [11] Bott defined some residues associated to the connected components of the zero set of a holomorphic vector field X on a compact complex manifold M of (complex) dimension n. Then he proved an index theorem relating these residues to the characteristic classes $\varphi(TM) \in H^{2n}_{dR}(M, \mathbb{C})$, with φ homogeneous symmetric polynomials of degree n (of course the residues depend on φ). In the following [10] and [8] there are some generalizations of this theorem, in particular in [8] the authors substituted the holomorphic vector field X with a 1-dimensional holomorphic foliation $\mathscr{F}$ on M. Before stating this theorem we need to refresh some definitions.

Definition 0.1.1. Let M be a complex manifold of dimension n. A 1-*dimensional holomorphic possibly singular foliation* on M is a morphism of holomorphic vector bundles $\mathscr{F} : F \to TM$, where F is a holomorphic line bundle over M called the *tangent bundle* of the foliation. The *singular set* of $\mathscr{F}$ is the analytic subset

$$\mathrm{Sing}(\mathscr{F}) = \{q \in M \text{ s.t. } \mathscr{F}|_q = 0\},$$

then set $M^0 = M - \mathrm{Sing}(\mathscr{F})$. The holomorphic vector bundle $N^0_{\mathscr{F}} = TM^0/F|_{M^0}$ is the *normal bundle* of $\mathscr{F}$, while the virtual bundle $TM - F$ is called the *virtual normal bundle* of $\mathscr{F}$.

In [8] there is then a slightly weaker version of the following theorem.

Theorem 0.1.2 (Baum-Bott index theorem). *Let M be a compact complex manifold of dimension* n*, $\mathscr{F} : F \to TM$ a* 1*-dimensional holomorphic possibly singular foliation on M and $Sing(\mathscr{F}) = \sqcup_\lambda \Sigma_\lambda$ the decomposition in connected components of its singular set. Then for any homogeneous symmetric polynomial φ of degree* n *there exist complex numbers $Res_\varphi(\mathscr{F}, TM - F, \Sigma_\lambda)$, one for each connected component Σ_λ, such that*

$$\sum_\lambda Res_\varphi(\mathscr{F}, TM - F, \Sigma_\lambda) = \int_M \varphi(TM - F)$$

See [8, Thm.1] for the original version or [36, Thm.III.7.6] for a more general formulation. The residues $\mathrm{Res}_\varphi(\mathscr{F}, TM - F, \Sigma_\lambda)$ are called *Baum-Bott residues*. Afterwards Baum and Bott generalized Theorem 0.1.2 considering holomorphic foliations $\mathscr{F}$ of any dimension $k \geq 1$. In this way they obtained also formulas relating homological classes instead of complex numbers (see [9, Thm.2] for the original version or [36, Thm.VI.3.7] for a more general formulation). The generalized version of Theorem 0.1.2 is another example of primary obstruction to the existence of geometric objects since it implies that non-singular holomorphic foliations of dimension $n-1$ cannot exist on $\mathbb{P}^n_{\mathbb{C}}$ for $n > 1$ (see [12, Thm.6.5] for a proof of this fact).

In 1982 Camacho and Sad introduced a new type of index theorem in their paper [19], as a key tool in the solution of the problem of existence of a separatrix passing through an isolated vanishing point of a holomorphic vector field in dimension 2. It was an index theorem for invariant leaves of 1-dimensional holomorphic singular foliations on complex surfaces (see [19, Appendix] for the original formulation).

Theorem 0.1.3 (Camacho-Sad index theorem)**.** *Let M be a complex surface, $S \subset M$ a non-singular compact complex curve and $\mathscr{F}$ a 1-dimensional holomorphic possibly singular foliation defined in a neighborhood of S and leaving S invariant. Then one can associate to any singular point $p \in Sing(\mathscr{F}) \cap S$ a complex number $i_p(\mathscr{F}, S)$, the* index of $\mathscr{F}$ along S at p*, in such a way that*

$$\sum_{p \in Sing(\mathscr{F}) \cap S} i_p(\mathscr{F}, S) = \int_S c_1(N_S),$$

where N_S is the normal bundle of S in M and $c_1(N_S)$ is its first Chern class.

This theorem has been subsequently generalized step by step. Early generalizations can be found in [20] and [30], where the authors provided a residue formula similar to the one of Theorem 0.1.3 when M is a complex manifold of dimension n, $S \subset M$ a non-singular compact complex hypersurface and $\mathscr{F}$ a $(n-1)$-dimensional holomorphic foliation leaving S invariant. A further generalization along this line was done by Lehmann in [25] allowing S and $\mathscr{F}$ to be of any dimension (clearly with dimension of $\mathscr{F}$ less than dimension of S). Other generalizations were obtained by allowing $S \subset M$ to be possibly singular. In particular, in [34] Suwa proved Theorem 0.1.3 for possibly singular compact (reduced and irreducible) curves $S \subset M$, getting a residue formula involving the first Chern class $c_1([S])$ of the natural holomorphic line bundle $[S]$ on M (a partial result in this direction was previously obtained in [31] for the particular case $M = \mathbb{P}^2_{\mathbb{C}}$). A stronger generalization was proved in [27, Thm.1] by Lehmann and Suwa (see also [36, Thm.IV.6.6.] for a more general formulation). For the sake of simplicity we state here only the version for hypersurfaces $S \subset M$, actually the most interesting for us.

Theorem 0.1.4 (generalized Camacho-Sad index theorem)**.** *Let M be a complex manifold of dimension* n*, $S \subset M$ a possibly singular compact analytic hypersurface and $\mathscr{F} : F \to TM$ a* 1*-dimensional holomorphic possibly singular foliation defined in a neighborhood of S and leaving $S' = S - Sing(S)$ invariant. Let $Sing(\mathscr{F}|_S) \cup Sing(S) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components, then there exist complex numbers $Res(\mathscr{F}, S, \Sigma_\lambda)$, one for each connected component Σ_λ, such that*

$$\sum_\lambda Res(\mathscr{F}, S, \Sigma_\lambda) = \int_S c_1^{n-1}([S]),$$

where $[S]$ is the holomorphic line bundle canonically induced by S on M and $c_1^{n-1}([S])$ is the $(n-1)$-th power of its first Chern class.

The residues $\mathrm{Res}(\mathscr{F}, S, \Sigma_\lambda)$ are called *Camacho-Sad residues* to honor the ones who introduced this kind of residues and proved the first version of the theorem. Later on Suwa generalized [27, Thm.1] (hence Theorem 0.1.4 and Theorem 0.1.3) considering holomorphic possibly singular foliations $\mathscr{F}$ of any dimension $k \geq 1$ (less than dimension of S) leaving S' invariant, getting in this way formulas relating homological classes (see [35, Thm.2.4] for the original version or [36, Thm.VI.6.9] for a more general formulation).

Within few of these papers, to be precise [25], [27] and [35], Lehmann and Suwa developed a general theory on a cohomological approach to index theorems based on 'localization of characteristic classes', the so-called *Lehmann-Suwa theory.* It is a very powerful machinery for getting index theorems and we will furnish a survey of its basis in Chapter 1 (see [36] or [18] for a systematic and complete exposition). We stress that the Lehmann-Suwa approach is not the one followed by Baum, Bott, Camacho and Sad, who proved their index theorems in different ways, anyway it provides a global theoretical framework also for them.

In [28] Lehmann and Suwa made the machinery clear and even proved a third type of index theorem, regarding again invariant subvarieties of possibly singular holomorphic foliations on complex manifolds but involving a different type of characteristic classes. The following is a reformulation of [28, Thm.3.6].

Theorem 0.1.5 (Lehmann-Suwa-Khanedani index theorem)**.** *Let M be a complex manifold of dimension* n*, $S \subset M$ a possibly singular compact analytic subvariety of any dimension* m *and $\mathscr{F} : F \to TM$ a* 1*-dimensional holomorphic possibly singular foliation defined in a neighborhood of S and leaving $S' = S - Sing(S)$ invariant. Let $Sing(\mathscr{F}|_S) \cup Sing(S) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components, then for any homogeneous symmetric polynomial φ of degree* m *there exist complex numbers $Res_\varphi(\mathscr{F}, TM|_S - F|_S, \Sigma_\lambda)$, one for each connected component Σ_λ, such that*

$$\sum_\lambda Res_\varphi\left(\mathscr{F}|_S, TM|_S - F|_S, \Sigma_\lambda\right) = \int_S \varphi(TM - F)$$

To be precise [28] contains a more general version of this theorem, that is a version for holomorphic foliations of any dimension $k \geq 1$. As usual in this case, one obtains formulas relating homological classes as well as formulas relating complex numbers (see [28, Thm.3.5] for the original version or [36, Thm.VI.6.4] for a more general formulation). We call Theorem 0.1.5 the 'Lehmann-Suwa-Khanedani index theorem' since it had been previously proved in [23] by Khanedani and Suwa for $n = 2$ (in a different way and using a different terminology). The residues $\mathrm{Res}_\varphi(\mathscr{F}, TM|_S - F|_S, \Sigma_\lambda)$ are clearly named *Lehmann-Khanedani-Suwa residues* (in [23] they are called 'variations').

Subsequently, versions for holomorphic self-maps of these three types of index theorems have been proved. Why? The moral link between the two settings (holomorphic foliations and holomorphic self-maps) is represented by Abate's paper [2]. In his work Abate obtained a complete generalization to two complex variables of the classical Leau-Fatou flower theorem for maps tangent to the identity mimicking the strategy of Camacho and Sad in [19]. A key ingredient in his proof was an index theorem for holomorphic self-maps on complex surfaces M fixing pointwise a non-singular compact complex curve $S \subset M$, clearly inspired by the Camacho-Sad index theorem. Here we state a version of the theorem slightly different from the original one [2, Thm.1.1].

Theorem 0.1.6 (Abate index theorem)**.** *Let M be a complex surface, $S \subset M$ a non-singular compact complex curve and $f : M \to M$ a holomorphic self-map such that $f|_S = id_S$. Assume 'f tangential along S' or that M is the total space of a holomorphic line bundle over S. Then one can associate to any $p \in S$ a complex number $i_p(f, S)$, the* index of f along S at p*, vanishing* *at all but a finite number of points, in such a way that*

$$\sum_{p\in S} i_p(f,S) = \int_S c_1(N_S),$$

where N_S is the normal bundle of S in M and $c_1(N_S)$ is its first Chern class.

The meaning of 'f tangential along S' will be clear later on. A first generalization of Theorem 0.1.6 was made soon after by Bracci and Tovena in [16] (see also [14]), where they assumed S to be possibly singular. Hence it became natural to try and generalize the Abate index theorem in a systematical way, as Lehmann and Suwa did starting from the Camacho-Sad index theorem. A large generalization to any dimension of M and codimension of the possibly singular $S \subset M$ was finally made by Abate, Bracci and Tovena in [4] thanks to a systematic use of the Lehmann-Suwa machinery. In this paper, very important source of inspiration for my thesis, they also proved versions for holomorphic self-maps of the two other kinds of index theorems for holomorphic foliations discussed above.

We now do a step further by replacing the single holomorphic self-map $f : M \to M$ pointwise fixing an analytic subvariety $S \subset M$ with a pair of distinct holomorphic self-maps $f, g : M \to M$ coinciding on a complex hypersurface $S \subset M$ (clearly if one of the two maps is the identity we fall back in the [4] case). An interesting example arises by lifting a pair of holomorphic self-maps $\tilde{f}, \tilde{g} : \mathbb{C}^n \to \mathbb{C}^n$ both fixing the origin 0 to the blow-up $\pi : M \to \mathbb{C}^n$ of $\mathbb{C}^n$ at 0. If $\mathrm{d}\tilde{f}|_0 = \mathrm{d}\tilde{g}|_0$ is invertible we obtain in this way a pair of holomorphic self-maps $f, g : M \to M$ coinciding on the exceptional divisor $S = \pi^{-1}(0) \cong \mathbb{P}^{n-1}_{\mathbb{C}}$. We show that under some hypotheses on the pair (f, g), and on the hypersurface S if needed, one can define a 1-dimensional holomorphic possibly singular foliation on $S' = S - \mathrm{Sing}(S)$. In general, this foliation doesn't admit a *global* extension around the whole S' but it admits good *local* extensions (or, using the terminology of [6], an extension to a suitable 'infinitesimal neighborhood' of the subvariety S). What does it mean "good"? It means that we can use these local extensions to define certain partial holomorphic connections over suitable holomorphic vector bundles on S' outside a 'singular set'. As a consequence if S is compact we can use the Lehmann-Suwa machinery to produce index theorems (see also [5] for a general and deeper explanation).

Our index theorems generalize the ones in [4] and in particular they are versions of Theorem 0.1.2, 0.1.4 and 0.1.5 for couples of holomorphic self-maps. We point out that for the last two of them we need to have a foliation defined in a whole neighborhood of S' (leaving it invariant), while we are able to define foliations on S' *only*. Hence our generalizations of Theorem 0.1.4 and Theorem 0.1.5 are not direct consequences of holomorphic foliation theory.

0.2 Plan of the Thesis and statement of the main results

The aim of Chapter 1 is to furnish the basics of the Lehmann-Suwa machinery. In Section 1.1 we briefly review the definition of the characteristic classes of C^∞ complex vector bundles in the Chern-Weil theory. In Section 1.2 we recall the definition of the Grothendieck group of a manifold, whose elements are called 'virtual complex vector bundles'. This is preparatory to Section 1.3, where we extend the Chern-Weil theory of characteristic classes to virtual complex vector bundles. In Section 1.4 we recall the concept of 'partial holomorphic connection' on a C^∞ complex vector bundle and we state the Bott vanishing theorem, which is a key tool in the theory. From Section 1.5 to 1.8 we recall the basic definitions of Čech-de Rham cohomology and we explain how are defined in this framework the characteristic classes of bundles, the integration over a manifold and the Poincaré and Alexander homomorphisms. The core of the chapter is Section 1.9, in which we explain how one can obtain index theorems within the Lehmann-Suwa theory.

The original part of the thesis starts with Chapter 2. Here we show how a pair (f, g) of distinct holomorphic self-maps of a complex manifold M coinciding on an analytic hypersurface $S \subset M$ can induce a 1-dimensional holomorphic possibly singular foliation on $S' = S - \mathrm{Sing}(S)$. First of all we introduce in Section 2.1 the 'order of coincidence of (f, g) along S', which is a positive integer related to the pair (f, g) denoted by $\nu_{f,g}$. In Section 2.2 we define the 'canonical section associated to (f, g)',

$$\mathscr{D}_{f,g} : N_{S'}^{\otimes \nu_{f,g}} \longrightarrow TM|_{S'},$$

assuming that one of the two maps is a local biholomorphism on a whole neighborhood of S' (we will keep this assumption for the rest of the thesis). In Section 2.3 we briefly recall some definitions about 1-dimensional holomorphic foliations, then in the subsequent Section 2.4 and 2.5 we discuss two hypotheses under which the pair (f, g) induces a 1-dimensional holomorphic foliation on S', denoted here by

$$\mathscr{D} : N_{S'}^{\otimes \nu_{f,g}} \longrightarrow TS'.$$

The hypothesis of Section 2.4 is that '(f, g) is tangential along S', while in Section 2.5 we assume a geometric property on S' regarding the way it sits into M. When this property is satisfied we say that 'S' splits into M'.

In Chapter 3 we assume S non-singular and we prove the first index theorem of the thesis. We show in Section 3.1 that if one of the two hypotheses discussed in Chapter 2 is satisfied, then we are able to define in a canonical way a partial holomorphic connection on the normal bundle of $\mathscr{D}$ outside a 'singular set'. As a consequence, in Section 3.2 we get the following *Baum-Bott-type index theorem* applying the Lehmann-Suwa machinery.

Theorem 0.2.1. *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a non-singular compact connected complex hypersurface. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S, with order of coincidence $\nu = \nu_{f,g}$. Assume that*

(i) (f, g) is tangential along S

or

(ii) S splits into M

and let $\mathscr{D}$ be the foliation induced on S by the pair (f, g) depending on the case. Let Sing$(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set of $\mathscr{D}$. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $Res_\varphi(\mathscr{D}, TS - N_S^{\otimes \nu}, \Sigma_\lambda)$, one for each connected component Σ_λ, such that

$$\sum_\lambda Res_\varphi \left(\mathscr{D}, TS - N_S^{\otimes \nu}, \Sigma_\lambda\right) = \int_S \varphi \left(TS - N_S^{\otimes \nu}\right).$$

We conclude the chapter by computing in Section 3.3 the residues of Theorem 0.2.1 at isolated singular points of $\mathscr{D}$.

In Chapter 4 we talk about extensions of $\mathscr{D}$. In Section 4.1 we show how a *global* extension of $\mathscr{D}$ around S' would induce canonically two partial holomorphic connections on the normal bundle of S' in M and on the normal bundle of $\mathscr{D}$ in M. Unfortunately $\mathscr{D}$ does not admit in general such a global extension, anyway we can extend it *locally* in a "good" canonical way. In Section 4.2 we show how to do it when '(f, g) is tangential along S'. Instead in Section 4.3 we treat the non-tangential case, but in this case we need to assume an embedding property on S' stronger than the splitting one. When it is satisfied we say that 'S' is comfortably embedded into M'.

Lastly, in Chapter 5 we use what we have proved in Chapter 4 to obtain the other two index theorems. In Section 5.1 we show that the 'canonical local extensions' of $\mathscr{D}$ are good enough to define a partial holomorphic connection on the normal bundle of S' in M outside a 'singular set', both in tangential and non-tangential case. Then applying the Lehmann-Suwa machinery we get the following *Camacho-Sad-type index theorem* in Section 5.2.

Theorem 0.2.2. *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a globally irreducible compact analytic hypersurface with regular part $S' = S - Sing(S)$. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S', with order of coincidence $\nu = \nu_{f,g}$. Assume that*

(i) (f, g) is tangential along S

or

(ii) S' is comfortably embedded into M

and let $\mathscr{D}$ be the foliation induced on S' by the pair (f, g) in both cases. Let $Sing(S) \cup Sing(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $Sing(S) \cup Sing(\mathscr{D})$. Then there exist complex numbers $Res(\mathscr{D}, S, \Sigma_\lambda)$, one for each connected component Σ_λ, such that

$$\sum_\lambda Res\left(\mathscr{D}, S, \Sigma_\lambda\right) = \int_S c_1^{n-1}\left([S]\right).$$

Here $[S]$ denotes the holomorphic line bundle on M canonically induced by S and $c_1^{n-1}([S])$ is the $(n-1)$-th power of its first Chern class. In the subsequent Section 5.3 we compute the residues appearing in Theorem 0.2.2 at isolated singular points of $\mathscr{D}$ and, in a case, of S.

In Section 5.4 we show that if '(f, g) is tangential along S' and $\nu_{f,g} > 1$ then the 'canonical local extensions' of $\mathscr{D}$ are good enough to define a partial holomorphic connection on the normal bundle of $\mathscr{D}$ in M outside a 'singular set'. Then applying again the Lehmann-Suwa machinery we get in Section 5.5 the following *Lehmann-Suwa-Khanedani-type index theorem.*

Theorem 0.2.3. *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a globally irreducible compact analytic hypersurface with regular part $S' = S - Sing(S)$. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S', with order of coincidence $\nu = \nu_{f,g}$. Assume that (f, g) is tangential along S and that $\nu > 1$, let $\mathscr{D}$ be the foliation induced on S' by the pair (f, g) and let $Sing(S) \cup Sing(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $Sing(S) \cup Sing(\mathscr{D})$. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $Res_\varphi(\mathscr{D}; TM|_S - [S]|_S^{\otimes\nu}; \Sigma_\lambda)$, one for each connected component Σ_λ, such that*

$$\sum_\lambda Res_\varphi \left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, \Sigma_\lambda\right) = \int_S \varphi \left(TM - [S]^{\otimes\nu}\right).$$

We conlcude by computing in Section 5.6 the residues appearing in Theorem 0.2.3 at isolated singular points of $\mathscr{D}$ and, in a case, of S.

Ringraziamenti

Il primo ringraziamento va al mio relatore Filippo Bracci per la disponibilità che ha sempre mostrato nei miei confronti, per gli utili suggerimenti che mi ha dato in questi anni e, ultimo ma non meno importante, per la comprensione di alcune scelte fatte durante il periodo del dottorato. Ringrazio inoltre il referee per le utilissime osservazioni sulla tesi e tutti i dottorandi e post-dottorandi con i quali mi sono confrontato in questi anni, che mi hanno dato consigli e con i quali ho condiviso incertezze e grandi quantità stress.

Vorrei ringraziare di cuore la mia famiglia, in particolare i miei genitori. I vostri consigli sono sempre molto preziosi e la vostra presenza, costante e silenziosa, è alla base di tutto.

Infine ringrazio profondamente Alessandra per il sostegno che mi ha sempre dato in questi anni e per aver sopportato ed assorbito tutti i miei momenti no, sia riguardo il dottorato che la vita in generale. Grazie di tutto, sicuramente senza di te sarebbe stato tutto più difficile.

Paolo Roma, 30 03 2017

Chapter 1

Basics of the Lehmann-Suwa machinery

The goal of this introductory chapter is to provide the essential notions and ideas at the base of the main tool used in this thesis, the so-called 'Lehmann-Suwa machinery'. The reader can refer to [36] and [18] for a complete treatment of this theory, moreover along the chapter we will furnish specific references for each treated topics.

1.1 Chern-Weil theory for complex vector bundles

We start by reviewing the Chern-Weil theory of characteristic classes of smooth complex vector bundles over smooth manifolds. Specific references for this topic are for example [32, Appendix C] and [21, Sec.3.3].

Let M be a real C^∞ manifold and let $T_{\mathbb{C}}M$ be the complexification of its real tangent bundle. We denote by $\mathcal{C}^\infty_{M,\mathbb{C}}$ the sheaf of $\mathbb{C}$-valued C^∞ functions on M and by $\mathcal{T}^\infty_{M,\mathbb{C}}$ the sheaf of C^∞ sections of $T_{\mathbb{C}}M$. Let $E \to M$ be any C^∞ complex vector bundle of rank r and denote by $\mathcal{E}^\infty$ the associated sheaf of C^∞ sections.

Definition 1.1.1. A C^∞ *connection* on E is a $\mathbb{C}$-linear morphism

$$\nabla : \mathcal{E}^\infty \longrightarrow \left(\mathcal{T}^\infty_{M,\mathbb{C}}\right)^* \otimes \mathcal{E}^\infty$$

satisfying the Leibniz rule, that is such that $\nabla(f \cdot s) = \mathrm{d}f \otimes s + f \cdot \nabla(s)$ for any $s \in \mathcal{E}^\infty$ and $f \in \mathcal{C}^\infty_{M,\mathbb{C}}$.

One can prove that there is always a C^∞ connection on a given a C^∞ complex vector bundle.

Now denote by $\bigwedge^j T^*_{\mathbb{C}}M$ the j-th exterior power of the complexified cotangent bundle of M and by Ω^j the associated sheaf of C^∞ sections (that is the sheaf of $\mathbb{C}$-valued C^∞ differential j-forms on M), for any $j \geq 1$. Moreover, let $\Omega^j(\mathcal{E}^\infty)$ denote the sheaf of C^∞ sections of the bundle $\bigwedge^j T^*_{\mathbb{C}}M \otimes E$ for any $j \geq 1$ (these sections can be thought as E-valued C^∞ differential j-forms on M) and observe that $\Omega^1(\mathcal{E}^\infty) = (\mathcal{T}^\infty_{M,\mathbb{C}})^* \otimes \mathcal{E}^\infty$. A C^∞ connection ∇ on E induces for any $j \geq 1$ a $\mathbb{C}$-linear morphism

$$\nabla^{(j)} : \Omega^j(\mathcal{E}^\infty) \longrightarrow \Omega^{j+1}(\mathcal{E}^\infty),$$

satisfying $\nabla^{(j)}(\alpha \otimes s) = \mathrm{d}\alpha \otimes s + (-1)^j \alpha \wedge \nabla(s)$ for any $s \in \mathcal{E}^\infty$ and $\alpha \in \Omega^j$.

Definition 1.1.2. Let ∇ be a C^∞ connection on E. The morphism of $\mathcal{C}^\infty_{M,\mathbb{C}}$-modules

$$K_\nabla = \nabla^{(1)} \circ \nabla : \mathcal{E}^\infty \longrightarrow \Omega^2(\mathcal{E}^\infty)$$

is called the *curvature* of ∇.

Let ∇ be any C^∞ connection on E. Consider an open subset $U \subset M$ trivializing for E and let $\{e_1, \ldots, e_r\}$ be a C^∞ frame for E on U. Clearly we can write

$$\nabla(e_i) = \sum_{j=1}^{r} \theta_{ij} \otimes e_j$$

for any $i = 1, \ldots, r$, with $\theta_{ij} \in \Omega^1(U)$ for any $i, j = 1, \ldots, r$. Let us call the $r \times r$ matrix $\theta = (\theta_{ij})$ the *connection matrix* of ∇ with respect to the frame $\{e_1, \ldots, e_r\}$. Similarly we can write

$$K_\nabla(e_i) = \sum_{j=1}^{r} \kappa_{ij} \otimes e_j$$

for any $i = 1, \ldots, r$, with $\kappa_{ij} \in \Omega^2(U)$ for any $i, j = 1, \ldots, r$. Let us call the $r \times r$ matrix $\kappa = (\kappa_{ij})$ the *curvature matrix* of ∇ with respect to the frame $\{e_1, \ldots, e_r\}$, and recall the relation $\kappa = \mathrm{d}\theta - \theta \wedge \theta$. The matrix κ can be treated as a 'numerical matrix' since differential forms of even degrees commute with one another with respect to the exterior product. As a consequence we can define the C^∞ differential forms $\sigma_i(\kappa) \in \Omega^{2i}(U)$ for $i = 1, \ldots, r$ by setting

$$\det(\mathrm{Id}_r + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_r(\kappa),$$

where Id_r is the $r \times r$ identity matrix.

Remark 1.1.3. (i) Let A be any $r\times r$ matrix whose entries are complex numbers and define the polynomial functions $\sigma_1(A),\dots,\sigma_r(A)$ by setting

$$\det\left(\mathrm{Id}_r + tA\right) = 1 + \sigma_1(A)t + \cdots + \sigma_r(A)t^r.$$

Furthermore, let $\mathfrak{s}_1,\dots,\mathfrak{s}_r$ be the elementary symmetric polynomials in r variables defined by setting

$$\prod_{j=1}^{r}(1+x_j) = 1 + \mathfrak{s}_1(x_1,\dots,x_r) + \cdots + \mathfrak{s}_r(x_1,\dots,x_r),$$

and observe that $\mathfrak{s}_i$ is a homogenous polynomial of degree i for any $i = 1,\dots,r$. If $\lambda_1,\dots,\lambda_r$ are the eigenvalues of A then

$$\sigma_i(A) = \mathfrak{s}_i(\lambda_1,\dots,\lambda_r)$$

for any $i = 1,\dots,r$.

(ii) The elementary symmetric polynomials $\mathfrak{s}_1,\dots,\mathfrak{s}_r$ generate the algebra of symmetric polynomials in r variables, here denoted by $Sym(r)$. Clearly also the symmetric polynomials defined by

$$\mathfrak{c}_i = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \mathfrak{s}_i$$

generate $Sym(r)$. Consequently, for any $\varphi \in Sym(r)$ there is a polynomial $P_\varphi \in \mathbb{C}[x_1,\dots,x_r]$ such that

$$\varphi(x) = P_\varphi\left(\mathfrak{c}_1(x),\dots,\mathfrak{c}_r(x)\right),$$

where $x = (x_1,\dots,x_r)$. Then it is natural to define

$$\varphi(A) = P_\varphi\left(\frac{\sqrt{-1}}{2\pi}\sigma_1(A),\dots,\left(\frac{\sqrt{-1}}{2\pi}\right)^r \sigma_r(A)\right),$$

for any $r\times r$ matrix A and $\varphi\in Sym(r)$.

Let $\hat{U}\subset M$ be another trivializing open set for E such that $U\cap\hat{U}\neq\emptyset$ and let $\{\hat{e}_1,\dots,\hat{e}_r\}$ be a C^∞ frame for E on $\hat{U}$. Again, we can define the curvature matrix $\hat{\kappa} = (\hat{\kappa}_{ij})$ with respect to the frame $\{\hat{e}_1,\dots,\hat{e}_r\}$ and the corresponding C^∞ differential forms $\sigma_i(\hat{\kappa})\in\Omega^{2i}(\hat{U})$. If $g = (g_{ij})$ is the $r\times r$ matrix of $\mathbb{C}$-valued C^∞ functions defined on $U\cap\hat{U}$ such that $\hat{e}_i = g_{ij}e_j$ then $\hat{\kappa} = g\kappa g^{-1}$ on $U\cap\hat{U}$. As

$$\det(\mathrm{Id}_r + \tilde{\kappa}) = \det(g(\mathrm{Id}_r + \kappa)g^{-1}) = \det(\mathrm{Id}_r + \kappa)$$

clearly $\sigma_i(\hat{\kappa})$ and $\sigma_i(\kappa)$ coincide on the overlap for $i = 1, \ldots, r$. Thus these local C^∞ differential $2i$-forms can be glued on M to give rise to *global* C^∞ differential $2i$-forms, denoted by $\sigma_i(\nabla)$. Consequently we can define the C^∞ differential $2i$-forms

$$c_i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla)$$

for $i = 1, \ldots, r$. All these differential forms are closed (see 'Fundamental Lemma' in [32, Appendix C]).

Let now $\varphi \in Sym(r)$ be any symmetric polynomial, we know by Remark 1.1.3 that there is a polynomial $P_\varphi \in \mathbb{C}[x_1, \ldots, x_r]$ such that

$$\varphi(x) = P_\varphi\left(\mathfrak{c}_1(x), \ldots, \mathfrak{c}_r(x)\right),$$

where $x = (x_1, \ldots, x_r)$. Then the local definition of the $c_i(\nabla)$ suggest to associate to φ the C^∞ differential form

$$\varphi(\nabla) = P_\varphi\left(c_1(\nabla), \ldots, c_r(\nabla)\right). \tag{1.1.1}$$

Clearly if φ is homogeneous of degree $d \geq 0$ then $\varphi(\nabla)$ is a differential $2d$-form. Moreover it is closed since the $c_i(\nabla)$ are closed.

Suppose now to have a finite number of C^∞ connections $\nabla_0, \ldots, \nabla_k$ over E. Consider the vector bundle $E \times \mathbb{R}^k \to M \times \mathbb{R}^k$, which is again a C^∞ complex vector bundle of rank r, and define the C^∞ connection

$$\nabla_t = \left(1 - \sum_{\alpha=1}^{k} t_\alpha\right)\nabla_0 + \sum_{\alpha=1}^{k} t_\alpha \nabla_\alpha$$

over it, where $(t_1, \ldots, t_k)$ are the standard coordinates on $\mathbb{R}^k$. For any $\varphi \in Sym(r)$ we have the C^∞ differential form $\varphi(\nabla_t)$ on $M \times \mathbb{R}^k$ defined as above. Let $\Delta^k \subset \mathbb{R}^k$ be the standard k-simplex and let

$$I_* : \Omega^j(M \times \Delta^k) \longrightarrow \Omega^{j-k}(M), \qquad j \geq k,$$

denote the integration along the fiber (see [36, Sec.II.4]). Then we define the C^∞ differential form on M

$$\varphi(\nabla_0, \ldots, \nabla_k) = I_*\left(\varphi\left(\nabla_t\right)\right),$$

which is a $(2d-k)$-form whenever φ is homogeneous of degree $d \geq 0$ (if $2d < k$ it is zero by construction). This form is alternating in its entries and satisfies the property

$$\sum_{i=0}^{k} (-1)^i \varphi\left(\nabla_0, \ldots, \hat{\nabla}_i, \ldots, \nabla_k\right) + (-1)^k \mathrm{d}\varphi(\nabla_0, \ldots, \nabla_k) = 0 \tag{1.1.2}$$

(see [12] or [36, Sec.II.7] for more details). As a consequence of this construction, if ∇ and ∇' are two distinct C^∞ connections on E then for any $\varphi \in Sym(r)$ we have the C^∞ differential form $\varphi(\nabla, \nabla')$, usually called the *Bott difference form*, with the property that

$$\varphi(\nabla') - \varphi(\nabla) = \mathrm{d}\varphi(\nabla, \nabla').$$

Thus $\varphi(\nabla')$ and $\varphi(\nabla)$ give rise to the same cohomological class and we can define the *characteristic class of E for the symmetric polynomial φ* by

$$\varphi(E) = [\varphi(\nabla)] \in H^{2*}_{dR}(M, \mathbb{C}) = \bigoplus_{j \geq 0} H^{2j}_{dR}(M, \mathbb{C}),$$

where ∇ is any C^∞ connection on E. In particular, when $\varphi = \mathfrak{c}_i$ we have the *i-th Chern class of E*

$$c_i(E) \in H^{2i}_{dR}(M, \mathbb{C}),$$

for $i = 1, \dots, r$, and the cohomological class $c(E) = 1 + c_1(E) + \cdots + c_r(E)$ is called the *total Chern class of E*. Note that it is an invertible element in the ring $H^{2*}_{dR}(M, \mathbb{C})$ since $c_1(E) + \cdots + c_r(E)$ is a nilpotent element (similarly, $c(\nabla)$ is an invertible element in the ring $\Omega^{2*}(M) = \oplus_{j \geq 0} \Omega^{2j}(M)$).

Remark 1.1.4. Let M be a real C^∞ manifold and $E \to M$ a C^∞ complex vector bundle of rank r. In practice we have defined an algebra homomorphism

$$Sym(r) \longrightarrow H^{2*}_{dR}(M, \mathbb{C}). \tag{1.1.3}$$

Let $Inv(r)$ be the algebra of invariant polynomial functions on the space of $r \times r$ complex matrices (that is functions Φ which are polynomial in the entries of the matrices and with the property that $\Phi(B^{-1}AB) = \Phi(A)$, for any matrix A and invertible matrix B). Usually the characteristic classes of a C^∞ complex vector bundle $E \to M$ are defined in the Chern-Weil theory via the *Weil homomorphism*,

$$W_E : Inv(r) \longrightarrow H^{2*}_{dR}(M, \mathbb{C}).$$

This is for example what is done in [32] and [21]. Anyway these two algebra homomorphisms are in fact equivalent since there is a natural algebra isomorphism $Sym(r) \cong Inv(r)$ which commutes with (1.1.3) and W_E. This isomorphism is given by the identification of the $\mathfrak{s}_i$, generating $Sym(r)$, with the invariant polynomial functions σ_i defined in Remark 1.1.3, generating $Inv(r)$.

1.2 The Grothendieck group of a manifold

In this thesis we deal also with 'virtual complex vector bundles', which are the elements of the 'Grothendieck group' of a manifold. For this reason we recall in this section its definition. More details can be found in [7, Chap.II] (in which the author works in the continuous setting).

Consider an abelian monoid $(\mathfrak{M}, +)$, that is a set $\mathfrak{M}$ together with an associative and commutative binary operation $+$ and an identity element 0. We can associate to $\mathfrak{M}$ its Grothendieck group, which roughly speaking is the most general abelian group arising from $\mathfrak{M}$ by introducing the inverses respect to the operation. More precisely, let $\mathfrak{M} \times \mathfrak{M}$ be the abelian monoid with the operation

$$(x, y) + (x', y') = (x + x', y + y'),$$

for any $x, y, x', y' \in \mathfrak{M}$, and let $\sim$ be the equivalence relation on $\mathfrak{M} \times \mathfrak{M}$ defined by

$$(x, y) \sim (x', y') \Longleftrightarrow x + y' + m = x' + y + m, \text{ for a } m \in \mathfrak{M}.$$

Then the *Grothendieck group of* $\mathfrak{M}$ is the quotient

$$K(\mathfrak{M}) = (\mathfrak{M} \times \mathfrak{M}) / \sim \quad ,$$

which is an abelian group since it is an abelian monoid and moreover there are the inverses respect to the (induced) operation. In fact, if $\overline{(x, y)} \in K(\mathfrak{M})$ is any element then its inverse is $\overline{(y, x)}$, as

$$\overline{(x, y)} + \overline{(y, x)} = \overline{(x + y, x + y)} = \overline{(0, 0)} = 0.$$

Observe that there is a natural monoid homomorphism $\mathfrak{u} : \mathfrak{M} \to K(\mathfrak{M})$ given by $x \to \overline{(x, 0)}$ for any $x \in \mathfrak{M}$, and that any element of $K(\mathfrak{M})$ can be written as a difference of elements coming from $\mathfrak{M}$,

$$\overline{(x, y)} = \overline{(x, 0)} - \overline{(y, 0)}.$$

If G is any other abelian group and $\gamma : \mathfrak{M} \to G$ is a monoid homomorphism then there is a unique group homomorphism $\gamma' : K(\mathfrak{M}) \to G$ such that $\gamma = \gamma' \circ \mathfrak{u}$, given by $\gamma'(\overline{(x, y)}) = \gamma(x) - \gamma(y)$. This is the so-called 'universal property' of $K(\mathfrak{M})$, which guarantees the uniqueness of $K(\mathfrak{M})$ for any $\mathfrak{M}$.

Let now M be a real C^∞ manifold of dimension n and let $\mathrm{Vect}(M)$ denote the set of C^∞ isomorphism classes of C^∞ complex vector bundles of finite rank over M. This set is an abelian monoid under the operation of direct sum, thus we have the following definition.

Definition 1.2.1. The *Grothendieck group of a real C^∞ manifold M* is the Grothendieck group of $\mathrm{Vect}(M)$, that is $K(M) = K(\mathrm{Vect}(M))$. The elements of $K(M)$ are called the *virtual complex vector bundles* on M.

Let $\mathfrak{u} : \mathrm{Vect}(M) \to K(M)$ be the monoid homomorphism given by the universal property of the Grothendieck group, and denote by $[E]$ the image under $\mathfrak{u}$ of any $E \in \mathrm{Vect}(M)$. By definition

$$[E] + [F] = [E \oplus F]$$

and any virtual complex vector bundle on M can be expressed as a difference $[F] - [E]$ for some $E, F \in \mathrm{Vect}(M)$. To ease the notation, from now on we will simply write E both for a C^∞ complex vector bundle and its image under $\mathfrak{u}$.

1.3 Characteristic classes of virtual complex vector bundles

In this section we recall the definition of the characteristic classes associated to virtual complex vector bundles in the framework of Chern-Weil theory. For this topic we refer to [9, Sec.4] (see also [36, Sec.II.8] or [18, Chap.11]).

Let M be a real C^∞ manifold of dimension n and let $K(M)$ be its Grothendieck group. We define the *total Chern class of a virtual complex vector bundle* $F - E \in K(M)$ as

$$c(F - E) = c(F) \wedge c(E)^{-1} \in H^{2*}_{dR}(M, \mathbb{C}).$$

The *i-th Chern class of $F - E$* is the component of $c(F - E)$ belonging to $H^{2i}_{dR}(M, \mathbb{C})$ and it is denoted by $c_i(F - E)$. Then the *characteristic class of $F - E$ for any symmetric polynomial* φ is defined as

$$\varphi(F - E) = P_\varphi\left(c_1(F - E), \ldots\ldots\right).$$

Clearly if φ is homogeneous of degree $d \geq 0$ then $\varphi(F - E) \in H^{2d}_{dR}(M, \mathbb{C})$.

Remark 1.3.1. Suppose E is a complex vector bundle of rank r. If $m > 2r$ then $c(E)^{-1}$ may have components $0 \neq \gamma_i \in H^{2i}_{dR}(M, \mathbb{C})$ for $i > r$, thus in general we do not know which is the top degree of the components of $c(F - E)$.

How could we represent these cohomological classes? Let ∇_E and ∇_F be any C^∞ connections respectively on E and on F and set $\nabla^\bullet = (\nabla_E, \nabla_F)$, which is a sort of 'virtual C^∞ connection' on $F - E$. Then $c(F-E)$ is clearly the cohomological class defined by the C^∞ differential form

$$c(\nabla^\bullet) = c(\nabla_F) \wedge c(\nabla_E)^{-1} \in \Omega^{2*}(M)$$

and its i-th component $c_i(\nabla^\bullet) \in \Omega^{2i}(M)$ is a representative of the i-th Chern class $c_i(F-E)$, for any $i \geq 1$. Consequently $\varphi(F-E)$ is the cohomological class defined by the C^∞ differential form

$$\varphi(\nabla^\bullet) = P_\varphi\left(c_1(\nabla^\bullet), \ldots\ldots\right),$$

for any symmetric polynomial φ.

Suppose now to have a finite number of C^∞ connections $\nabla_{E,0}, \ldots, \nabla_{E,k}$ on E and $\nabla_{F,0}, \ldots, \nabla_{F,k}$ on F, and set $\nabla_i^\bullet = (\nabla_{E,i}, \nabla_{F,i})$ for $i = 0, \ldots, k$. In a way very similar to Section 1.1 we can construct a C^∞ differential form $\varphi(\nabla_0^\bullet, \ldots, \nabla_k^\bullet)$ for any symmetric polynomial φ (see [36, Sec.II.8] for more details). Consider the complex vector bundles $E \times \mathbb{R}^k \to M \times \mathbb{R}^k$ and $F \times \mathbb{R}^k \to M \times \mathbb{R}^k$, and define the C^∞ connections

$$\nabla_{E,t} = \left(1 - \sum_{\alpha=1}^{k} t_\alpha\right) \nabla_{E,0} + \sum_{\alpha=1}^{k} t_\alpha \nabla_{E,\alpha}$$

and

$$\nabla_{F,t} = \left(1 - \sum_{\alpha=1}^{k} t_\alpha\right) \nabla_{F,0} + \sum_{\alpha=1}^{k} t_\alpha \nabla_{F,\alpha}$$

over them, where $(t_1, \ldots, t_k)$ are the standard coordinates on $\mathbb{R}^k$. Set $\nabla_t^\bullet = (\nabla_{E,t}, \nabla_{F,t})$ and consider for any $\varphi \in Sym(r)$ the C^∞ differential form $\varphi(\nabla_t^\bullet)$ on $M \times \mathbb{R}^k$. Let $\Delta^k \subset \mathbb{R}^k$ be the standard k-simplex and let

$$I_* : \Omega^j(M \times \Delta^k) \longrightarrow \Omega^{j-k}(M), \qquad j \geq k,$$

denote the integration along the fiber, then similarly to Section 1.1 we define

$$\varphi(\nabla_0^\bullet, \ldots, \nabla_k^\bullet) = I_*\left(\varphi\left(\nabla_t^\bullet\right)\right).$$

Clearly it is a $(2d-k)$-form whenever φ is homogeneous of degree $d \geq 0$ (if $2d < k$ it is zero by construction), moreover it is alternating in its entries and satisfies the property

$$\sum_{i=0}^{k} (-1)^i \varphi(\nabla_0^\bullet, \ldots, \hat{\nabla}_i^\bullet, \ldots, \nabla_k^\bullet) + (-1)^k \mathrm{d}\varphi\left(\nabla_0^\bullet, \ldots, \nabla_k^\bullet\right) = 0. \qquad (1.3.1)$$

Let us conclude the section by considering the short exact sequence of C^∞ complex vector bundles over M,

$$0 \longrightarrow E \xrightarrow{\alpha} F \xrightarrow{\beta} G \longrightarrow 0. \tag{1.3.2}$$

Denote by $\mathcal{E}^\infty$, $\mathcal{F}^\infty$ and $\mathcal{G}^\infty$ the sheaves of C^∞ sections respectively of E, F and G. Since (1.3.2) has always a C^∞ splitting we have that $F - E = G$ in $K(M)$. Is $\varphi(F - E) = \varphi(G)$ for any symmetric polynomial φ ? The answer is positive and is given by (i) of the Lemma stated immediately after the following definition.

Definition 1.3.2. Let ∇_E, ∇_F and ∇_G be C^∞ connections respectively on E, F and G. We say that the triple $(\nabla_E, \nabla_F, \nabla_G)$ is *compatible with the sequence (1.3.2)* if the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{E}^\infty \arrow[r, "\alpha"] \arrow[d, "\nabla_E"'] & \mathcal{F}^\infty \arrow[r, "\beta"] \arrow[d, "\nabla_F"'] & \mathcal{G}^\infty \arrow[d, "\nabla_G"'] \\
\Omega^1(\mathcal{E}^\infty) \arrow[r, "\mathrm{id}\otimes\alpha"] & \Omega^1(\mathcal{F}^\infty) \arrow[r, "\mathrm{id}\otimes\beta"] & \Omega^1(\mathcal{G}^\infty).
\end{tikzcd}$$

The following Lemma is a reformulation of [9, Lem.4.17], [9, Lem.4.22] and [36, Prop.II.8.4]. It basically follows by the existence of a C^∞ splitting of (1.3.2) and by the existence of C^∞ connections over C^∞ complex vector bundles.

Lemma 1.3.3. *Consider the short exact sequence (1.3.2). Then*

(i) if ∇_G is a C^∞ connection on G then there exist C^∞ connections ∇_E and ∇_F respectively on E and F such that $(\nabla_E, \nabla_F, \nabla_G)$ is compatible with (1.3.2). For such a compatible triple we have that $\varphi(\nabla^\bullet) = \varphi(\nabla_G)$ for any symmetric polynomial φ, where $\nabla^\bullet = (\nabla_E, \nabla_F)$.

(ii) if ∇'_G is another C^∞ connection on G and $(\nabla'_E, \nabla'_F, \nabla'_G)$ is another triple compatible with (1.3.2) then $\varphi(\nabla^\bullet, \nabla'^\bullet) = \varphi(\nabla_G, \nabla'_G)$ for any symmetric polynomial φ, where $\nabla'^\bullet = (\nabla'_E, \nabla'_F)$.

1.4 The Bott vanishing theorem

The aim of this section is to state the *Bott vanishing theorem*, which plays a key role in the Lehmann-Suwa machinery. Before to do it we need to recall some definitions, in particular the one of 'partial holomorphic connection' on a holomorphic vector bundle. A reference for this topic is [36, Sec.II.9].

Let M be a real C^∞ manifold and let $E \to M$ be any C^∞ complex vector bundle. If $H \subset T_{\mathbb{C}}M$ is any C^∞ complex vector subbundle then we denote by $\mathcal{H}^\infty$ the associated sheaf of C^∞ sections and by $\rho : T^*_{\mathbb{C}}M \to H^*$ the obvious projection.

Definition 1.4.1. A *partial C^∞ connection along $H \subset T_{\mathbb{C}}M$* on E is a $\mathbb{C}$-linear morphism

$$\delta : \mathcal{E}^\infty \longrightarrow (\mathcal{H}^\infty)^* \otimes \mathcal{E}^\infty$$

such that $\delta(f \cdot s) = \rho(\mathrm{d}f) \otimes s + f \cdot \delta(s)$ for any $s \in \mathcal{E}^\infty$ and $f \in \mathcal{C}^\infty_{M,\mathbb{C}}$.

An *extension* of a partial C^∞ connection δ along $H \subset T_{\mathbb{C}}M$ is a C^∞ connection ∇ on E such that

$$\nabla_v(s) = \delta_v(s),$$

for any $s \in \mathcal{E}^\infty$ and $v \in \mathcal{H}^\infty$.

One can prove that there is always a partial C^∞ connection on a given C^∞ complex vector bundle along a given C^∞ complex vector subbundle of $T_{\mathbb{C}}M$. Moreover an extension of a partial C^∞ connection always exists (see [9, Lem.2.5]).

Consider again the short exact sequence (1.3.2), Definition 1.3.2 can be generalized in the following way.

Definition 1.4.2. Let δ_E, δ_F and δ_G be partial C^∞ connections along $H \subset T_{\mathbb{C}}M$ respectively on E, F and G. We say that the triple $(\delta_E, \delta_F, \delta_G)$ is *compatible with the sequence (1.3.2)* if the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{E}^\infty \arrow[r, "\alpha"] \arrow[d, "\delta_E"'] & \mathcal{F}^\infty \arrow[r, "\beta"] \arrow[d, "\delta_F"'] & \mathcal{G}^\infty \arrow[d, "\delta_G"'] \\
(\mathcal{H}^\infty)^* \otimes \mathcal{E}^\infty \arrow[r, "\mathrm{id}\otimes\alpha"] & (\mathcal{H}^\infty)^* \otimes \mathcal{F}^\infty \arrow[r, "\mathrm{id}\otimes\beta"] & (\mathcal{H}^\infty)^* \otimes \mathcal{G}^\infty.
\end{tikzcd}$$

Then we have the following lemma, which basically follows by the existence of a C^∞ splitting of (1.3.2) and by the existence of extensions of partial C^∞ connections.

Lemma 1.4.3. *Consider the short exact sequence (1.3.2) and suppose that δ_E, δ_F and δ_G are partial C^∞ connections along $H \subset T_{\mathbb{C}}M$ respectively on E, F and G such that the triple $(\delta_E, \delta_F, \delta_G)$ is compatible with (1.3.2). Let ∇_G be a C^∞ connection on G which extends δ_G. Then there exist C^∞ connections ∇_E and ∇_F respectively on E and F extending δ_E and δ_F such that the triple $(\nabla_E, \nabla_F, \nabla_G)$ is compatible with (1.3.2).*

If $H \subset T_{\mathbb{C}}M$ is involutive, that is if $[u, v] \in \mathcal{H}^\infty$ for any $u, v \in \mathcal{H}^\infty$, we can introduce the following definition.

Definition 1.4.4. A partial C^∞ connection δ along $H \subset T_{\mathbb{C}}M$ on E is *flat* if

$$\delta_{[u,v]}(s) = \delta_u\left(\delta_v(s)\right) - \delta_v\left(\delta_u(s)\right),$$

for any $s \in \mathcal{E}^\infty$ and $u, v \in \mathcal{H}^\infty$.

Let now M be a complex manifold and let $E \to M$ be a holomorphic vector bundle. Recall that the complexified tangent bundle $T_{\mathbb{C}}M$ of M (seen as a real C^∞ manifold) has a C^∞ direct sum decomposition

$$T_{\mathbb{C}}M \cong TM \oplus \bar{T}M,$$

where TM is the *holomorphic tangent bundle* of M and $\bar{T}M$ is the *anti-holomorphic tangent bundle* of M, which is C^∞ but not holomorphic (see [37, Sec.2.2.1]). Denote by $\bar{\mathcal{T}}_M^\infty$ the sheaf of C^∞ sections of $\bar{T}M$ and by $\mathcal{E}$ the sheaf of holomorphic sections of E, which is a subsheaf of $\mathcal{E}^\infty$. An important example of partial C^∞ connection is given by the natural partial C^∞ connection along $\bar{T}M$ on E

$$\bar{\partial} : \mathcal{E}^\infty \longrightarrow \left(\bar{\mathcal{T}}_M^\infty\right)^* \otimes \mathcal{E}^\infty$$

with the property that $\bar{\partial}(s) = 0$ for any $s \in \mathcal{E} \subset \mathcal{E}^\infty$.

Definition 1.4.5. A $(1, 0)$-*type connection* on E is a C^∞ connection on E extending $\bar{\partial}$.

The rationale behind the name is that a C^∞ connection on E is $(1, 0)$-type if and only if the entries of its connection matrix with respect to a local holomorphic frame of E are differential 1-forms of type $(1, 0)$.

When we deal with complex manifolds M and holomorphic vector bundles $E \to M$ we can introduce a very important kind of partial connection. Let $H \subset TM$ be a holomorphic vector subbundle and let us denote, as usual, by $\mathcal{H}^\infty$ the associated sheaf of C^∞ sections and by $\mathcal{H}$ the associated sheaf of holomorphic sections.

Definition 1.4.6. A *partial holomorphic connection along* $H \subset TM$ on E is a partial C^∞ connection

$$\delta : \mathcal{E}^\infty \longrightarrow (\mathcal{H}^\infty)^* \otimes \mathcal{E}^\infty$$

such that $\delta(s) \in \mathcal{H}^* \otimes \mathcal{E}$ whenever $s \in \mathcal{E} \subset \mathcal{E}^\infty$.

Unlike the C^∞ case, if $E \to M$ is any holomorphic vector bundle and $H \subset TM$ is any holomorphic vector subbundle then there is not in general a partial holomorphic connection along H on E. In fact, the existence of a partial holomorphic connection implies strong consequences, as will be clear in the next lines. In order to better distinguish C^∞ and *holomorphic* partial connections we will write in the following

$$\delta : \mathcal{E} \longrightarrow \mathcal{H}^* \otimes \mathcal{E}$$

whenever δ is a partial holomorphic connection, although in fact δ is defined also on the C^∞ sections of E.

If δ is a partial holomorphic connection along $H \subset TM$ we automatically have the partial C^∞ connection $\delta \oplus \bar{\partial}$ along $H \oplus \bar{T}M$. Then, as there are always extensions of partial C^∞ connections, we also have a $(1,0)$-type connection on E extending δ (which may be not unique).

Definition 1.4.7. Let δ be a partial holomorphic connection along $H \subset TM$ on E. A *δ-connection* on E is a $(1,0)$-type extension of δ.

Remark 1.4.8. The term 'δ-connection' is used also in [36, Sec.II.9] but in that case the author assumes δ flat.

Now we are able to state the above mentioned *Bott vanishing theorem*. A proof can be found at [36, Thm.II.9.11], moreover see [5, Thm.6.1] for a slightly more general version about not necessarily flat partial holomorphic connections.

Theorem 1.4.9 (Bott vanishing theorem)**.** *Let M be a complex manifold of dimension n and $H \subset TM$ an involutive holomorphic vector subbundle of rank $\ell \leq n$. Let $E \to M$ be a holomorphic vector bundle and suppose there is a flat partial holomorphic connection δ along H on E. If $\nabla_0, \ldots, \nabla_k$ are any δ-connections on E then*

$$\varphi\left(\nabla_0, \ldots, \nabla_k\right) = 0$$

for any homogeneous symmetric polynomial φ of degree $d > n - \ell$.

In some cases we will need the following stronger version of the Bott vanishing theorem, which actually is sligthly weaker than [36, Thm.II.9.11].

Theorem 1.4.10 (Bott vanishing theorem, II). *Let M be a complex manifold of dimension n and $H \subset TM$ an involutive holomorphic vector subbundle of rank $\ell \leq n$. Let $E \to M$ and $F \to M$ be two holomorphic vector bundles and suppose there are two flat partial holomorphic connections δ_E and δ_F along H respectively on E and F. If $\nabla_{E,0}, \ldots, \nabla_{E,k}$ are any δ_E-connections on E and $\nabla_{F,0}, \ldots, \nabla_{F,k}$ are any δ_F-connections on E, then setting $\nabla_i^\bullet = (\nabla_{E,i}, \nabla_{F,i})$ for $i = 0, \ldots, k$ we have that*

$$\varphi\left(\nabla_0^\bullet, \ldots, \nabla_k^\bullet\right) = 0$$

for any homogeneous symmetric polynomial φ of degree $d > n - \ell$.

1.5 Čech-de Rham cohomology

In this section we briefly recall definition and basic properties of the Čech-de Rham cohomology of a real C^∞ manifold, since the Lehmann-Suwa machinery needs this framework. A brief reference is [36, Sec.II.3] while a detailed one is [13, Sec.II.8].

Let M be a real C^∞ manifold. To define the Čech-de Rham cohomology of M one needs to consider an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of it, where the index set I may be countable (and ordered). However for our purposes we just need finite open coverings, so let us consider an open covering $\mathcal{U} = \{U_0, \ldots, U_N\}$. Set

$$U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$$

for any indices $i_0, \cdots, i_p = 0, \ldots, N$ and any $p = 0, \ldots, N$, and define

$$K_{\mathcal{U}}^{p,q} = \bigoplus_{0 \leq i_0 < \cdots < i_p \leq N} \Omega^q(U_{i_0 \cdots i_p})$$

for any $p, q \geq 0$, where Ω^q denotes the sheaf of $\mathbb{C}$-valued C^∞ differential q-forms on M as in Section 1.1. Whenever $U_{i_0 \cdots i_p} = \emptyset$ we set $\Omega^q(U_{i_0 \cdots i_p}) = 0$. By definition any element $\omega \in K_{\mathcal{U}}^{p,q}$ has a finite number of components

$\omega_{i_0\cdots i_p} \in \Omega^q(U_{i_0\cdots i_p})$, and we can define the operator $\mathfrak{d} : K_{\mathcal{U}}^{p,q} \to K_{\mathcal{U}}^{p+1,q}$ by setting

$$(\mathfrak{d}\omega)_{i_0\cdots i_{p+1}} = \sum_{\alpha=0}^{p+1}(-1)^{\alpha}\omega_{i_0\cdots\hat{i}_\alpha\cdots i_{p+1}}$$

for any $\omega \in K_{\mathcal{U}}^{p,q}$. It is a coboundary operator, that is $\mathfrak{d}^2 = 0$ (see [13, Prop.8.3]), hence $K_{\mathcal{U}}^{*,*} = \oplus_{p,q\geq 0}K_{\mathcal{U}}^{p,q}$ equipped with $\mathfrak{d}$ and the exterior derivative d is a double cochain complex. Since $\mathfrak{d}$ and d commute we can derive from $(K_{\mathcal{U}}^{*,*}, \mathfrak{d}, d)$ a single cochain complex by setting

$$C_{\mathcal{U}}^j = \bigoplus_{p+q=j} K_{\mathcal{U}}^{p,q}, \qquad\qquad C_{\mathcal{U}}^* = \oplus_{j\geq 0}C_{\mathcal{U}}^j$$

and

$$D = \mathfrak{d} + \mathrm{d}^{\pm} : C_{\mathcal{U}}^j \longrightarrow C_{\mathcal{U}}^{j+1}$$

for any $j \geq 0$, where $\mathrm{d}^{\pm} = (-1)^p\mathrm{d}$ on $K_{\mathcal{U}}^{p,q}$. Note that $D^2 = 0$ because of the commutativity of $\mathfrak{d}$ and d.

Definition 1.5.1. The cochain complex $(C_{\mathcal{U}}^*, D)$ is called the *Čech-de Rham complex with respect to $\mathcal{U}$* and its cohomology is the *Čech-de Rham cohomology with respect to $\mathcal{U}$*. We denote by $H^j(\mathcal{U}, \mathbb{C})$ the j-th Čech-de Rham cohomology group, for $j \geq 0$.

For any $j \geq 0$ there is a natural map given by restriction of C^∞ differential forms, that is

$$\begin{aligned} \Omega^j(M) &\longrightarrow C_{\mathcal{U}}^j = K_{\mathcal{U}}^{0,j} \oplus \cdots \oplus K_{\mathcal{U}}^{j,0} \\ \omega &\longrightarrow (\omega_{\mathcal{U}}, 0, \cdots, 0), \end{aligned} \tag{1.5.1}$$

where $\omega_{\mathcal{U}} = (\omega|_{U_0}, \ldots, \omega|_{U_N}) \in K^{0,j}$. These maps commute with d and $\mathfrak{d}$ for any $j \geq 0$, then they induce a homomorphism between de Rham and Čech-de Rham cohomology which is in fact an isomorphism, as stated in the following theorem (see [13, Prop.8.8] for the proof).

Theorem 1.5.2. *The restriction map (1.5.1) induce an isomorphism*

$$H_{dR}^j(M, \mathbb{C}) \xrightarrow{\cong} H^j(\mathcal{U}, \mathbb{C}),$$

for any $j \geq 0$.

The Čech-de Rham complex can be endowed with an analogous of the wedge product, that is with the *cup product* $C^j_{\mathcal{U}} \times C^k_{\mathcal{U}} \overset{\smile}{\longrightarrow} C^{j+k}_{\mathcal{U}}$ defined by

$$(\omega \smile \tau)_{i_0 \cdots i_p} = \sum_{\alpha=0}^{p} (-1)^{(j-\alpha)(p-\alpha)} \omega_{i_0 \cdots i_\alpha} \wedge \tau_{i_\alpha \cdots i_p},$$

for any $\omega \in C^j_{\mathcal{U}}$, $\tau \in C^k_{\mathcal{U}}$ and $j, k \geq 0$. This operation passes on the cohomology so we have a well-defined cup product

$$H^j(\mathcal{U}, \mathbb{C}) \times H^k(\mathcal{U}, \mathbb{C}) \overset{\smile}{\longrightarrow} H^{j+k}(\mathcal{U}, \mathbb{C}) \tag{1.5.2}$$

for any $j, k \geq 0$, which is obviously compatible with the wedge product $\wedge$ in de Rham cohomology under the isomorphisms of Theorem 1.5.2.

Example 1.5.3. In order to fix the ideas we make explicit the simple case $\mathcal{U} = \{U_0, U_1\}$. In this case

$$C^j_{\mathcal{U}} = \Omega^j(U_0) \oplus \Omega^j(U_1) \oplus \Omega^{j-1}(U_{01})$$

for any $j \geq 0$, then a cochain $\omega \in C^j_{\mathcal{U}}$ is a triple $\omega = (\omega_0, \omega_1, \omega_{01})$ and the Čech-de Rham differential D acts in the following way

$$D(\omega_0, \omega_1, \omega_{01}) = (\mathrm{d}\omega_0, \mathrm{d}\omega_1, \omega_1 - \omega_0 - \mathrm{d}\omega_{01})\,.$$

The cochain map (1.5.1) is given by the restriction

$$\omega \longrightarrow (\omega|_{U_0}, \omega|_{U_1}, 0)$$

for any $\omega \in C^j_{\mathcal{U}}$, and the cup product is given by

$$\begin{aligned} \omega \smile \tau =& (\omega_0, \omega_1, \omega_{01}) \smile (\tau_0, \tau_1, \tau_{01}) = \\ &= \big(\omega_0 \wedge \tau_0, \omega_1 \wedge \tau_1, (-1)^j \omega_0 \wedge \tau_{01} + \omega_{01} \wedge \tau_1\big)\,, \end{aligned}$$

for any $\omega \in C^j_{\mathcal{U}}$, $\tau \in C^k_{\mathcal{U}}$ and $j, k \geq 0$.

Consider now for any $j \geq 0$ the natural projection

$$\pi^j_0 : C^j_{\mathcal{U}} \longrightarrow \Omega^j(U_0)$$

and define $C^j_{\mathcal{U}}(U_0) = \ker(\pi^j_0)$. It is easy to check that $D\omega \in C^{j+1}_{\mathcal{U}}(U_0)$ whenever $\omega \in C^j_{\mathcal{U}}(U_0)$, that is $\Big(C^*_{\mathcal{U}}(U_0) = \oplus_{j\geq 0} C^j_{\mathcal{U}}(U_0), D\Big)$ equipped with D is a cochain (sub)complex.

Definition 1.5.4. The cochain complex $(C^*_{\mathcal{U}}(U_0), D)$ is called the *relative Čech-de Rham complex with respect to $\mathcal{U}$ and U_0* and its cohomology is the *relative Čech-de Rham cohomology with respect to $\mathcal{U}$ and U_0*. We denote by $H^j(\mathcal{U}, U_0, \mathbb{C})$ the j-th relative Čech-de Rham cohomology group, for $j \geq 0$.

Observe that the obvious inclusions and projections induce the following short exact sequence of cochain complexes

$$0 \longrightarrow C^*_{\mathcal{U}}(U_0) \xrightarrow{\imath^*} C^*_{\mathcal{U}} \xrightarrow{\pi_0^*} \Omega^*(U_0) \longrightarrow 0,$$

hence by a general result in homological algebra (see [22, Thm.II.2.1]) we have the long exact sequence in cohomology

$$\cdots \longrightarrow H^{j-1}_{dR}(U_0, \mathbb{C}) \longrightarrow H^j(\mathcal{U}, U_0, \mathbb{C}) \longrightarrow H^j(\mathcal{U}, \mathbb{C}) \longrightarrow H^j_{dR}(U_0, \mathbb{C}) \longrightarrow \cdots$$

Recall the similar long exact cohomology sequence for the pair (M, U_0)

$$\cdots \longrightarrow H^{j-1}(U_0, \mathbb{C}) \longrightarrow H^j(M, U_0, \mathbb{C}) \longrightarrow H^j(M, \mathbb{C}) \longrightarrow H^j(U_0, \mathbb{C}) \longrightarrow \cdots$$

where $H^j(M, \mathbb{C})$ and $H^j(U_0, \mathbb{C})$ denote the j-th singular cohomology groups of respectively M and U_0, and $H^j(M, U_0, \mathbb{C})$ denotes the relative j-th singular cohomology group of M with respect to U_0. Pairing these two long exact sequences, we get by Theorem 1.5.2, by the classic de Rham theorem (see [21, Sec.0.3]) and by the Five lemma (see [22, Ex.1.2]) the following proposition.

Proposition 1.5.5. *There is a natural isomorphism*

$$H^j(M, U_0, \mathbb{C}) \xrightarrow{\cong} H^j(\mathcal{U}, U_0, \mathbb{C}),$$

for any $j \geq 0$.

Let us conclude the section by considering the simple case $\mathcal{U} = \{U_0, U_1\}$. In this case the cup product in relative Čech-de Rham cohomology with respect to $\mathcal{U}$ and U_0 is given by

$$\omega \smile \tau = (0, \omega_1, \omega_{01}) \smile (0, \tau_1, \tau_{01}) = (0, \omega_1 \wedge \tau_1, \omega_{01} \wedge \tau_1),$$

for any $\omega \in C^j_{\mathcal{U}}(U_0)$, $\tau \in C^k_{\mathcal{U}}(U_0)$ and $j, k \geq 0$. The right hand side depends only on ω_1, ω_{01} and τ_1, then we have a product $C^j_{\mathcal{U}}(U_0) \times \Omega^k(U_1) \xrightarrow{\smile} C^{j+k}_{\mathcal{U}}(U_0)$ and, passing on the cohomology, the product

$$H^j(\mathcal{U}, U_0, \mathbb{C}) \times H^k_{dR}(U_1, \mathbb{C}) \xrightarrow{\smile} H^{j+k}(\mathcal{U}, U_0, \mathbb{C}), \tag{1.5.3}$$

for any $j, k \geq 0$. We will need this product in the following sections.

1.6 Characteristic classes in Čech-de Rham cohomology

The Lehmann-Suwa machinery works in the Čech-de Rham framework since it allows to represent characteristic classes in very convenient ways. We dedicate this section to review the topic and refer to [24] and [36, Sec.II.8] for more details.

Let M be a real C^∞ manifold, let $\mathcal{U} = \{U_0, \dots, U_N\}$ be an open covering of M and let $E \to M$ be any C^∞ complex vector bundle of rank r. Recall that in Section 1.1 we have introduced the C^∞ differential form $\varphi(\nabla_0, \dots, \nabla_k)$ associated to a finite number of C^∞ connections $\nabla_0, \dots, \nabla_k$ on E and a symmetric polynomial $\varphi \in Sym(r)$, satisfying property (1.1.2).

Consider any C^∞ connection ∇_i on the bundle $E|_{U_i}$ for $i = 0, \dots, N$ and let $\nabla_* = (\nabla_0, \dots, \nabla_N)$ denote this collection of C^∞ connections. Observe that for any indices $0 \le i_0 < \cdots < i_k \le N$ the restrictions of $\nabla_{i_0}, \dots, \nabla_{i_k}$ to $U_{i_0 \cdots i_k}$ are C^∞ connections over $E|_{U_{i_0 \cdots i_k}}$. Thus we can associate to any symmetric polynomial $\varphi \in Sym(r)$ the Čech-de Rham cochain $\varphi(\nabla_*) \in C^{2*}_{\mathcal{U}} = \oplus_{j \ge 0} C^{2j}_{\mathcal{U}}$ defined by

$$\varphi(\nabla_*)_{i_0 \cdots i_k} = \varphi(\nabla_{i_0}, \cdots, \nabla_{i_k})$$

for any indices $0 \le i_0 < \cdots < i_k \le N$ and any $k = 0, \dots, N$. Clearly if φ is homogeneous of degree $d \ge 0$ then $\varphi(\nabla_*) \in C^{2d}_{\mathcal{U}}$. By the very definition of D and by property (1.1.2) it follows that $D\varphi(\nabla_*) = 0$, moreover if $\nabla'_* = (\nabla'_0, \dots, \nabla'_N)$ is another collection of C^∞ connections on $E|_{U_0}, \dots, E|_{U_N}$ then there exists an element $\tau \in C^{2*}_{\mathcal{U}}$ such that

$$\varphi(\nabla'_*) - \varphi(\nabla_*) = D\tau.$$

As a consequence, $\varphi(\nabla_*)$ and $\varphi(\nabla'_*)$ give rise to the same cohomological class and we define the *characteristic class of E for the symmetric polynomial φ in Čech-de Rham cohomology* by

$$\varphi\,(E,\mathcal{U}) = [\varphi(\nabla_*)] \in H^{2*}(\mathcal{U}, \mathbb{C}) = \bigoplus_{j \ge 0} H^{2j}(\mathcal{U}, \mathbb{C}),$$

where $\nabla_* = (\nabla_0, \dots, \nabla_N)$ is any collection of C^∞ connections on the bundles $E|_{U_0}, \dots, E|_{U_N}$. As we would expect this definition is coherent with the one recalled in Section 1.1 in de Rham cohomology, that is $\varphi(E)$ corresponds to $\varphi(E,\mathcal{U})$ under the isomorphisms of Theorem 1.5.2. For this reason from now on we will use the same notation $\varphi(E)$ both in de Rham and Čech-de Rham context.

Example 1.6.1. Focus again on the simple case $\mathcal{U} = \{U_0, U_1\}$ to better understand the general case. Let ∇_0 and ∇_1 be any C^∞ connections respectively on $E|_{U_0}$ and $E|_{U_1}$ and set $\nabla_* = (\nabla_0, \nabla_1)$. For any $\varphi \in Sym(r)$ homogeneous of degree $d \geq 0$ we have that

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)) \in C^{2d}_{\mathcal{U}}$$

and by definition of D, property (1.1.2) and recalling that $\mathrm{d}\varphi(\nabla_0) = \mathrm{d}\varphi(\nabla_1) = 0$

$$\begin{aligned} D\varphi(\nabla_*) &= (\mathrm{d}\varphi(\nabla_0), \mathrm{d}\varphi(\nabla_1), \varphi(\nabla_1) - \varphi(\nabla_0) - \mathrm{d}\varphi(\nabla_0, \nabla_1)) \\ &= (0, 0, 0). \end{aligned}$$

Let now ∇'_0 and ∇'_1 be other C^∞ connections respectively on $E|_{U_0}$ and $E|_{U_1}$ and set $\nabla'_* = (\nabla'_0, \nabla'_1)$. Then by (1.1.2) again

$$\begin{aligned} &\varphi(\nabla'_*) - \varphi(\nabla_*) = \\ &\quad = \big(\varphi(\nabla'_0) - \varphi(\nabla_0), \varphi(\nabla'_1) - \varphi(\nabla_1), \varphi(\nabla'_0, \nabla'_1) - \varphi(\nabla_0, \nabla_1)\big) \\ &\quad = \big(\mathrm{d}\varphi(\nabla_0, \nabla'_0), \mathrm{d}\varphi(\nabla_1, \nabla'_1), \varphi(\nabla_1, \nabla'_1) - \varphi(\nabla_0, \nabla'_0) \\ &\qquad - \mathrm{d}\varphi(\nabla_1, \nabla_0, \nabla'_1) - \mathrm{d}\varphi(\nabla_1, \nabla_0, \nabla'_0)\big), \end{aligned}$$

hence $\varphi(\nabla'_*) - \varphi(\nabla_*) = D\tau$ where $\tau \in C^{2d-1}_{\mathcal{U}}$ is (for instance)

$$\tau = \big(\varphi(\nabla_0, \nabla'_0), \varphi(\nabla_1, \nabla'_1), \varphi(\nabla_1, \nabla_0, \nabla'_1) + \varphi(\nabla_1, \nabla_0, \nabla'_0)\big).$$

The characteristic classes of a virtual complex vector bundle $F - E \in K(M)$ are represented in Čech-de Rham cohomology in a very similar way.

Suppose to have two collections of C^∞ connections $\nabla_{E,0}, \dots, \nabla_{E,k}$ and $\nabla_{F,0}, \dots, \nabla_{F,k}$ respectively on E and F and consider the associated 'virtual C^∞ connections' $\nabla^\bullet_i = (\nabla_{E,i}, \nabla_{F,i})$ on $F - E$ for $i = 0, \dots, k$. Recall by Section 1.3 that we can define the C^∞ differential form $\varphi(\nabla^\bullet_0, \dots, \nabla^\bullet_k)$ satisfying property (1.3.1). In sight of this we can work exactly as just done above. Hence choose any C^∞ connections $\nabla_{E,i}$ on $E|_{U_i}$ and any C^∞ connections $\nabla_{F,i}$ on $F|_{U_i}$, for $i = 0, \dots, N$. Set $\nabla^\bullet_i = (\nabla_{E,i}, \nabla_{F,i})$ for $i = 0, \dots, N$ and let $\nabla^\bullet_* = (\nabla^\bullet_0, \dots, \nabla^\bullet_N)$ denote this collection of 'virtual C^∞ connections'. We associate to any symmetric polynomial φ the element $\varphi(\nabla^\bullet_*) \in C^{2*}_{\mathcal{U}}$ defined by

$$\varphi\left(\nabla^\bullet_*\right)_{i_0 \cdots i_k} = \varphi\left(\nabla^\bullet_{i_0}, \cdots, \nabla^\bullet_{i_k}\right)$$

for any indices $0 \leq i_0 < \cdots < i_k \leq N$ and any $k = 0, \dots, N$. Note that if φ is homogeneous of degree $d \geq 0$ then clearly $\varphi(\nabla^\bullet_*) \in C^{2d}_{\mathcal{U}}$. Arguing as above we can define the *characteristic class of $F-E$ for the symmetric polynomial φ in Čech-de Rham cohomology* by

$$\varphi\,(F-E,\mathcal{U}) = [\varphi(\nabla^\bullet_*)] \in H^{2*}(\mathcal{U},\mathbb{C}) = \bigoplus_{j\geq 0} H^{2j}(\mathcal{U},\mathbb{C}),$$

where $\nabla^\bullet_* = (\nabla^\bullet_0,\ldots,\nabla^\bullet_N)$ is any collection of 'virtual C^∞ connections' for $F-E$ on $U_0,\ldots,U_N$. As we would expect this definition is coherent with the one recalled in Section 1.3 in de Rham cohomology, that is $\varphi(F-E)$ corresponds to $\varphi(F-E,\mathcal{U})$ under the isomorphisms of Theorem 1.5.2. For this reason from now on we will use the same notation $\varphi(F-E)$ both in de Rham and Čech-de Rham context.

1.7 Integration in Čech-de Rham cohomology

In this section we review the integration in Čech-de Rham cohomology since we need this tool to reformulate the Alexander and Poincaré homomorphisms in Čech-de Rham framework, as we will do in the next section. What follows was developed by Lehmann mainly in [24] and [26] (see also [36, Sec.II.3]).

Let M be an oriented real C^∞ manifold of dimension m and, as usual, refer to the notation already introduced. Let $\mathcal{U} = \{U_0,\ldots,U_N\}$ be an open covering of M and call *maximal* a set of indices $\{i_0,\ldots,i_k\}$ such that $U_{i_0\cdots i_k j} \neq \emptyset$ implies $j \in \{i_0,\ldots,i_k\}$ for any $0 \leq j \leq N$.

Definition 1.7.1. A *system of honey-comb cells adapted to* $\mathcal{U}$ is a collection $\{R_0,\ldots,R_N\}$ of real C^∞ submanifolds of M of dimension m with piecewise C^∞ boundaries in M such that

(a) $M = \cup_{i=0}^N R_i$ and $R_i \subset U_i$ for any $i = 0,\ldots,N$,

(b) $\mathrm{Int}R_i \cap \mathrm{Int}R_j = \emptyset$ if $i \neq j$,

(c) if $U_{i_0\cdots i_k} \neq \emptyset$ then $R_{i_0\cdots i_k} = \cap_{\alpha=0}^k R_{i_\alpha} = \cap_{\alpha=0}^k \partial R_{i_\alpha}$ is a real C^∞ submanifold of M of dimension $m-k$ with piecewise C^∞ boundary in M,

(d) if $\{i_0,\ldots,i_k\}$ is maximal then $R_{i_0\cdots i_k}$ has no boundary,

(e) R_i has the same orientation as M and the boundary ∂R_i has the *orientation induced by* R_i (see [13, Sec.I.3]) for $i = 0,\ldots,N$,

(f) $\partial R_{i_0\cdots i_k} = \sum_{j\neq i_0,\ldots,i_k} R_{i_0\cdots i_k j}$,

(g) if ρ is a permutation then $R_{i_{\rho(0)}\cdots i_{\rho(k)}} = \mathrm{sign}\rho \cdot R_{i_0\cdots i_k}$.

Conditions (e), (f) and (g) define the rules for the orientation of the submanifolds $R_{i_0\cdots i_k}$.

One can prove that there is always a system of honey-comb cells adapted to a given open covering $\mathcal{U}$ (see [26]).

Let $\{R_0,\ldots,R_N\}$ be any system of honey-comb cells adapted to $\mathcal{U}$ and assume M compact, so each $R_{i_0\cdots i_k}$ is also compact. We define the integration $\int_{\mathcal{U}} : C^m_{\mathcal{U}} \to \mathbb{C}$ by setting

$$\int_{\mathcal{U}} \omega = \sum_{k=0}^{m} \sum_{0\leq i_0<\cdots<i_k\leq N} \int_{R_{i_0\cdots i_k}} \omega_{i_0\cdots i_k}$$

for any $\omega \in C^m_{\mathcal{U}}$. One can prove that if $D\omega = 0$ then this definition does not depend on the choice of the system of honey-comb cells, and that if $\omega = D\tau$ for any $\tau \in C^{m-1}_{\mathcal{U}}$ then $\int_{\mathcal{U}} \omega = 0$. Thus the integration passes on the cohomology,

$$\int_{\mathcal{U}} : H^m(\mathcal{U},\mathbb{C}) \longrightarrow \mathbb{C}. \tag{1.7.1}$$

This Čech-de Rham integration is compatible via the isomorphisms of Theorem 1.5.2 with the usual integration $\int_M : H^m_{dR}(M,\mathbb{C}) \to \mathbb{C}$ in de Rham cohomology. For this reason from now on we will use the same symbol $\int_M$ both in de Rham and Čech-de Rham context.

Suppose now that M may be not compact and let $K \subset M$ be a compact subset. Consider the simple covering $\mathcal{U} = \{U_0, U_1\}$ where $U_0 = M - K$ and $U_1 \subset M$ is any open neighborhood of K and let $\{R_0, R_1\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Since M is not compact we do not have integration (1.7.1), but as $\mathrm{Int}R_1 \supset K$ we can assume R_1 compact and then we can define the integration $\int_M : C^m_{\mathcal{U}}(U_0) \to \mathbb{C}$ by setting

$$\int_M \omega = \int_{R_1} \omega_1 - \int_{\partial R_1} \omega_{01}$$

for any $\omega = (0, \omega_1, \omega_{01}) \in C^m_{\mathcal{U}}(U_0)$. As before one can prove that the integration passes on the cohomology, hence in this setting we have the integration operator

$$\int_M : H^m(\mathcal{U}, U_0, \mathbb{C}) \longrightarrow \mathbb{C}. \tag{1.7.2}$$

In the next chapters we will be most interested in complex manifolds of (complex) dimension n, which in particular are orientable real C^∞ manifolds of (real) dimension $m = 2n$. If M is a compact complex manifold we can of course integrate C^∞ differential $2n$-forms over it, but in fact we can do something more. Indeed if $S \subset M$ is a possibly singular compact analytic subvariety of pure dimension $s \leq n$ then it is well defined the integration over S of C^∞ differential $2s$-forms defined in a whole open neighborhood $U \subset M$ of S (see [29]),

$$\int_S : H^{2s}_{dR}(U, \mathbb{C}) \longrightarrow \mathbb{C}. \tag{1.7.3}$$

We can reformulate also this kind of integration in the Čech-de Rham framework in the following way. Let $\mathcal{U} = \{U_0, \ldots, U_N\}$ be an open covering of an open neighborhood $U \subset M$ of S and let $\{R_0, \ldots, R_N\}$ be a system of honey-comb cells adapted to $\mathcal{U}$ such that S is transverse to each $R_{i_0\cdots i_k}$. Since each $\tilde{R}_{i_0\cdots i_k} = R_{i_0\cdots i_k} \cap S$ is compact we can define the integration $\int_{\mathcal{U}\cap S} : C^{2s}_{\mathcal{U}} \to \mathbb{C}$ by setting

$$\int_{\mathcal{U}\cap S} \omega = \sum_{k=0}^{2s} \sum_{0 \leq i_0 < \cdots < i_k \leq N} \int_{\tilde{R}_{i_0\cdots i_k}} \omega_{i_0\cdots i_k}$$

for any $\omega \in C^{2s}_{\mathcal{U}}$. Also in this case the integration passes on the cohomology,

$$\int_{\mathcal{U}\cap S} : H^{2s}(\mathcal{U}, \mathbb{C}) \to \mathbb{C}. \tag{1.7.4}$$

This integration is compatible via the isomorphisms of Theorem 1.5.2 with (1.7.3), for this reason from now on we will use the same symbol $\int_S$ both in de Rham and Čech-de Rham context.

Let now S be possibly not compact and let $K \subset S$ be a compact subset containing the singular set $\mathrm{Sing}(S)$. Let $U_0 \subset M$ be an open neighborhood of $S - K$ (we may choose it to be a tubular neighborhood since $S - K$ is non-singular), let $U_1 \subset M$ be an open neighborhood of K and set $U = U_0 \cup U_1$, which is an open neighborhood of S in M. Consider the open covering $\mathcal{U} = \{U_0, U_1\}$ of U and let $\{R_0, R_1\}$ be a system of honey-comb cells adapted to $\mathcal{U}$ such that $\mathrm{Int} R_1 \supset K$, R_1 is compact and the boundary ∂R_1 is transverse to S. Since $\tilde{R}_1 = R_1 \cap S$ and $\partial \tilde{R}_0 = -\partial \tilde{R}_1$ are compact we can define the integration $\int_S : C^{2s}_{\mathcal{U}}(U_0) \to \mathbb{C}$ by setting

$$\int_S \omega = \int_{\tilde{R}_1} \omega_1 - \int_{\partial \tilde{R}_1} \omega_{01}$$

for any $\omega = (0, \omega_1, \omega_{01}) \in C^{2s}_{\mathcal{U}}(U_0)$. Again it induces the integration on the cohomology level, hence in this setting we have the integration operator

$$\int_S : H^{2s}(\mathcal{U}, U_0, \mathbb{C}) \longrightarrow \mathbb{C}. \tag{1.7.5}$$

1.8 Alexander and Poincaré homomorphisms in Čech-de Rham cohomology

In this section we reformulate the Alexander and Poincaré homomorphisms in the Čech-de Rham framework making use of what we have recalled in Section 1.5 and 1.7. We say *homomorphisms* instead of *dualities* since we will consider also the 'singular case', in which we don't have isomorphisms in general. For the 'non-singular case' the reader can refer to [36, Sec.II.3] (see also [24]), while for the 'singular case' to [36, Sec.VI.4]. Another basic reference is [17].

Let us start with the 'non-singular case'. Let M be a compact oriented real C^∞ manifold (for example a compact complex manifold) of dimension m and let $\mathcal{U} = \{U_0, \ldots, U_N\}$ be any finite open covering of M. By composing the cup product (1.5.2) with the integration (1.7.1) we obtain a bilinear pairing

$$H^j(\mathcal{U}, \mathbb{C}) \times H^{m-j}(\mathcal{U}, \mathbb{C}) \longrightarrow \mathbb{C}$$

for any $j = 0 \ldots, m$. Since it is non-degenerate we have an isomorphism

$$P_{\mathcal{U}} : H^j(\mathcal{U}, \mathbb{C}) \xrightarrow{\cong} H^{m-j}(\mathcal{U}, \mathbb{C})^*$$

for any $j = 0 \ldots, m$, which we call the *Poincaré duality in Čech-de Rham cohomology*. By the isomorphisms of Theorem 1.5.2 and the classic de Rham theorem we can rewrite $P_{\mathcal{U}}$ in a way more useful for our purposes,

$$P_{\mathcal{U}} : H^j(\mathcal{U}, \mathbb{C}) \xrightarrow{\cong} H_{m-j}(M, \mathbb{C}) \tag{1.8.1}$$

for any $j = 0 \ldots, m$, where $H_{m-j}(M, \mathbb{C})$ denotes the $(m-j)$-th singular homology group of M. This form is quite similar to the classic Poincaré duality, in fact composing $P_{\mathcal{U}}$ on the left with the isomorphisms of Theorem 1.5.2 we get exactly the classic Poincaré duality $P_M : H^j_{dR}(M, \mathbb{C}) \to H_{m-j}(M, \mathbb{C})$,

$j = 0, \ldots, m$ (see [13, Sec.I.5]). For this reason from now on we will use the symbol P_M instead of $P_{\mathcal{U}}$ also in the Čech-de Rham context.

Suppose now that M may be not compact and let $K \subset M$ be a compact subset. Set $U_0 = M - K$, let $U_1 \subset M$ be any open neighborhood of K and consider the open covering $\mathcal{U} = \{U_0, U_1\}$. By composition of the cup product (1.5.3) with the integration (1.7.2) we obtain a bilinear pairing

$$H^j(\mathcal{U}, U_0, \mathbb{C}) \times H^{m-j}_{dR}(U_1, \mathbb{C}) \longrightarrow \mathbb{C}$$

for any $j = 0 \ldots, m$. Again, since it is non-degenerate we have an isomorphism

$$A_{\mathcal{U},K} : H^j(\mathcal{U}, U_0, \mathbb{C}) \xrightarrow{\cong} H^{m-j}_{dR}(U_1, \mathbb{C})^* \cong H_{m-j}(U_1, \mathbb{C})$$

for any $j = 0 \ldots, m$, where the isomorphism on the right hand side is due to the classic de Rham isomorphism. We call $A_{\mathcal{U},K}$ the *Alexander duality in Čech-de Rham cohomology.* If K admits a regular neighborhood U_1 (which is, in particular, of the same homotopy type of K) we have that $H_{m-j}(U_1, \mathbb{C}) \cong H_{m-j}(K, \mathbb{C})$ and then we could rewrite the Alexander duality as

$$A_{\mathcal{U},K} : H^j(\mathcal{U}, U_0, \mathbb{C}) \xrightarrow{\cong} H_{m-j}(K, \mathbb{C}) \tag{1.8.2}$$

for any $j = 0 \ldots, m$. Such a regular neighborhood exists, for instance, if M is a complex manifold and $K \subset M$ is a compact analytic subvariety. Composing $A_{\mathcal{U},K}$ on the left with the isomorphisms of Proposition 1.5.5 we get the classic Alexander duality $A_{M,K} : H^j(M, M - K, \mathbb{C}) \to H_{m-j}(K, \mathbb{C})$, $j = 0, \ldots, m$ (see for example [18, Sec.1.5]). For this reason from now on we will use the symbol $A_{M,K}$ instead of $A_{\mathcal{U},K}$ also in the Čech-de Rham context.

Let us now treat the 'singular case'. Let M be a complex manifold of dimension n and let $S \subset M$ be a possibly singular compact analytic subvariety of pure dimension $s \leq n$. Let $U \subset M$ be an open neighborhood of S and consider any open covering $\mathcal{U} = \{U_0, \ldots, U_N\}$ of U. By composition of the cup product (1.5.2) with the integration (1.7.4) we obtain a bilinear pairing

$$H^j(\mathcal{U}, \mathbb{C}) \times H^{2s-j}(\mathcal{U}, \mathbb{C}) \longrightarrow \mathbb{C}$$

for any $j = 0 \ldots, 2s$. In this case we have 'only' a homomorphism

$$P_{\mathcal{U}\cap S} : H^j(\mathcal{U}, \mathbb{C}) \longrightarrow H^{2s-j}(\mathcal{U}, \mathbb{C})^*$$

for any $j = 0 \ldots, 2s$, which we call the *Poincaré homomorphism in Čech-de Rham cohomology.* Observe that $H^{2s-j}(\mathcal{U}, \mathbb{C})^* \cong H^{2s-j}_{dR}(U, \mathbb{C})^* \cong$

$H_{2s-j}(U,\mathbb{C})$ by the isomorphisms of Theorem 1.5.2 and the classic de Rham theorem. Moreover if we assume $U \subset M$ to be a regular neighborhood of S (we can do it as $S \subset M$ compact analytic subvariety) we have that $H_{2s-j}(U,\mathbb{C}) \cong H_{2s-j}(S,\mathbb{C})$, thus we can rewrite $P_{\mathcal{U}\cap S}$ as

$$P_{\mathcal{U}\cap S} : H^j(\mathcal{U},\mathbb{C}) \longrightarrow H_{2s-j}(S,\mathbb{C}) \tag{1.8.3}$$

for any $j = 0\dots,2s$, where $H_{2s-j}(S,\mathbb{C})$ denotes the $(2s-j)$-th singular homology group of S. Composing $P_{\mathcal{U}\cap S}$ on the left with the isomorphisms of Theorem 1.5.2 and observing that $H^j_{dR}(U,\mathbb{C}) \cong H^j(S,\mathbb{C})$ because U is a regular neighborhood of S, we get the classic Poincaré homomorphism $P_S : H^j(S,\mathbb{C}) \to H_{2s-j}(S,\mathbb{C})$, $j = 0,\dots,2s$ (see [17]). For this reason from now on we will use the symbol P_S instead of $P_{\mathcal{U}\cap S}$ also in the Čech-de Rham context.

Suppose now that S may be not compact and let $K \subset S$ be a compact subset such that $\mathrm{Sing}(S) \subset K$. Let $U_0 \subset M$ be an open neighborhood of $S-K$ and let $U_1 \subset M$ be an open neighborhood of K, then set $U = U_0 \cup U_1$ and consider the open covering $\mathcal{U} = \{U_0, U_1\}$ of U. By composition of the cup product (1.5.3) with the integration (1.7.5) we obtain a bilinear pairing

$$H^j(\mathcal{U},U_0,\mathbb{C}) \times H^{2s-j}_{dR}(U_1,\mathbb{C}) \longrightarrow \mathbb{C}$$

for any $j = 0\dots,2s$. As before, also in this case we have 'only' a homomorphism

$$A_{\mathcal{U}\cap S,K} : H^j(\mathcal{U},U_0,\mathbb{C}) \longrightarrow H^{2s-j}_{dR}(U_1,\mathbb{C})^* \cong H_{2s-j}(U_1,\mathbb{C})$$

for any $j = 0\dots,2s$, where the isomorphism on the right hand side is due to the classic de Rham isomorphism. We call $A_{\mathcal{U}\cap S,K}$ the *Alexander homomorphism in Čech-de Rham cohomology.* If K admits a regular neighborhood U_1 then $H_{2s-j}(U_1,\mathbb{C}) \cong H_{2s-j}(K,\mathbb{C})$ and we can rewrite $A_{\mathcal{U}\cap S,K}$ as

$$A_{\mathcal{U}\cap S,K} : H^j(\mathcal{U},U_0,\mathbb{C}) \longrightarrow H_{2s-j}(K,\mathbb{C}) \tag{1.8.4}$$

for any $j = 0\dots,2s$. Suppose then that $U_1 \subset M$ is a regular neighborhood of K, moreover note that we may assume without loss of generality that U_0 is a tubular neighborhood of $S-K$ (since it is non-singular). With these choices $U = U_0 \cup U_1$ is a regular neighborhood of S and one can prove (using the Five lemma) that $H^j(U,U_0) \cong H^j(S,S-K)$ for any $j \geq 0$. By this remarks and composing $A_{\mathcal{U}\cap S,K}$ on the left with the isomorphisms of Proposition 1.5.5 we recover the classic Alexander duality $A_{S,K} : H^j(S,S-K) \to H_{2s-j}(K,\mathbb{C})$, $j = 0,\dots,2s$ (see [17]). For this reason from now on we will use the symbol $A_{S,K}$ instead of $A_{\mathcal{U}\cap S,K}$ also in the Čech-de Rham context.

1.9 How to get index theorems

We conclude this chapter by outlining the basics of the Lehmann-Suwa machinery, the main tool we will use in Chapter 3 and 5. This machinery is based on the representation of characteristic classes in Čech-de Rham cohomology, on the Alexander and Poincaré homomorphisms and on the Bott vanishing theorem (or, more generally, on vanishing theorems about characteristic classes). The reader can refer to [36, Sec.III.3] and [36, Sec.VI.4] for what will be explained in this section (see also [3]).

Let M be a compact complex manifold of dimension n and let $K \subset M$ be a compact analytic subvariety. Set $U_0 = M - K$, let $U_1 \subset M$ be a regular neighborhood of K and consider the open covering $\mathcal{U} = \{U_0, U_1\}$ of M. By Alexander and Poincaré dualities (1.8.2) and (1.8.1) we have the following diagram

$$\begin{tikzcd} H^j\left(\mathcal{U}, U_0, \mathbb{C}\right) \arrow[r, "\imath^*"] \arrow[d, "A_{M,K}"', "\cong"] & H^j\left(\mathcal{U}, \mathbb{C}\right) \arrow[d, "P_M"', "\cong"] \\ H_{2n-j}\left(K, \mathbb{C}\right) \arrow[r, "i_*"] & H_{2n-j}\left(M, \mathbb{C}\right) \end{tikzcd} \tag{1.9.1}$$

for any $j = 0, \ldots, 2n$, where $\imath^*$ is induced by the inclusion of cochain complexes $\imath^* : C^*_{\mathcal{U}}(U_0) \hookrightarrow C^*_{\mathcal{U}}$ and i_* is induced by the topological inclusion $i : K \hookrightarrow M$. The key point is that (1.9.1) turn out to be commutative (see [36, Prop.II.3.11]). As a consequence, if for any reason we are able to lift a $\omega \in H^j(\mathcal{U}, \mathbb{C})$ to a cocycle $\omega_K \in H^j(\mathcal{U}, U_0, \mathbb{C})$ such that $\imath^*(\omega_K) = \omega$ then we get the formula

$$i_*\left(A_{M,K}\left(\omega_K\right)\right) = P_M\left(\omega\right),$$

which relates homological classes of M. In this case we call ω_K the *localization of* ω *at* K and the homological class $A_{M,K}(\omega_K)$ is the *residue of* ω *at* K. If $K = \sqcup_\lambda K_\lambda$ is the decomposition in connected components then clearly $H_{2n-j}(K, \mathbb{C}) = \oplus_\lambda H_{2n-j}(K_\lambda, \mathbb{C})$ and accordingly $A_{M,K}(\omega_K)$ decomposes in a finite sum of components. We denote these components by $\mathrm{Res}(\omega, K_\lambda)$ and we call them the *residues of* ω *at* K_λ for any λ. The previous formula becomes then

$$\sum_\lambda i_* \mathrm{Res}(\omega, K_\lambda) = P_M\left(\omega\right)$$

and we get an *index theorem* (according to the definition given in Introduction). The most interesting case for our aims is when $\omega \in H^{2n}(\mathcal{U}, \mathbb{C})$, that is when we deal with top degree characteristic classes. In this case the previous formula may be rewritten as

$$\sum_{\lambda} \mathrm{Res}(\omega, K_\lambda) = \int_M \omega,$$

where $\mathrm{Res}(\omega, K_\lambda) \in \mathbb{C}$ for any λ.

As we said, we obtain index theorems whenever we are able to lift cohomological classes (typically characteristic classes) from $H^j(\mathcal{U}, \mathbb{C})$ to $H^j(\mathcal{U}, U_0, \mathbb{C})$. However given a $\omega \in H^j(\mathcal{U}, \mathbb{C})$ its localization $\omega_K \in H^j(\mathcal{U}, U_0, \mathbb{C})$ may be not unique in general and clearly any choice of ω_K gives rise to a different index theorem and different residues. Instead an index theorem is useful if we find a "good" lifting which is in particular unique (and if we are able to compute the residues). A typical situation is when there exists a 'geometric object' θ (vector field, connection, foliation, etc.) over the manifold M which guarantees the vanishing of a specific family of characteristic classes outside its 'singular set' $K \subset M$, thanks to a vanishing theorem. In this case any ω belonging to the family can be represented by a unique Čech-de Rham cocycle of the kind

$$[(0, \omega_1, \omega_{01})] \in H^j(\mathcal{U}, \mathbb{C})$$

and then we can lift ω to $H^j(\mathcal{U}, U_0, \mathbb{C})$, where $U_0 = M - K$, gaining an index theorem as just explained. The localization of ω at K and hence the residues will keep memory of the properties of θ, which is the "reason" why we can lift ω. Instead on the right hand side of the formula of the index theorem there will be something indipendent from θ, and this is one of the reasons why index theorems are interesting and useful.

Example 1.9.1. Let M be a compact complex manifold of dimension n and let $E \to M$ be a holomorphic vector bundle of rank r. We know by Section 1.6 that for any $\varphi \in Sym(r)$ we can represent the characteristic class $\varphi(E)$ in Čech-de Rham cohomology by

$$\varphi(E) = [(\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1))] \in H^{2*}(\mathcal{U}, \mathbb{C}),$$

where $\mathcal{U} = \{U_0, U_1\}$ is any open covering of M and ∇_0 and ∇_1 are any C^∞ connections respectively on $E|_{U_0}$ and on $E|_{U_1}$.

Suppose that outside a compact analytic subvariety $K \subset M$, that is on $U_0 = M - K$, there are a holomorphic line subbundle $F_0 \subset TM|_{U_0}$ and a flat partial holomorphic connection δ_0 along F_0 on $E|_{U_0}$ (the 'geometric object'). By Theorem 1.4.9, the Bott vanishing theorem, we have that $\varphi(\tilde{\nabla}_0) = 0$

for any δ_0-connection $\tilde{\nabla}_0$ on $E|_{U_0}$ and for any homogeneous $\varphi \in Sym(r)$ of degree $d = n$. Let $U_1 \subset M$ be a regular neighborhood of K and consider the open covering $\mathcal{U} = \{U_0, U_1\}$. For any $\varphi \in Sym(r)$ of degree $d = n$ we can then localize the characteristic class $\varphi(E) \in H^{2n}(\mathcal{U}, \mathbb{C})$ to

$$\varphi(E, K, \tilde{\nabla}_0) = \left[\left(0, \varphi(\nabla_1), \varphi(\tilde{\nabla}_0, \nabla_1)\right)\right] \in H^{2n}(\mathcal{U}, U_0, \mathbb{C}),$$

where ∇_1 is any C^∞ connection on $E|_{U_1}$. Observe that $\varphi(E, K, \tilde{\nabla}_0)$ keeps memory of δ_0 since it depends on a δ_0-connection. Moreover one can prove that this localization does not depend on the choices of $\tilde{\nabla}_0$ and ∇_1 (see [36, Lem.III.3.1]). Hence we get the index theorem

$$\sum_\lambda \mathrm{Res}_\varphi(E, K_\lambda) = \int_M \varphi(E)$$

for any $\varphi \in Sym(r)$ homogeneous of degree $d = n$, where $K = \cup_\lambda K_\lambda$ is the decomposition in connected components and $\mathrm{Res}_\varphi(E, K_\lambda) \in \mathbb{C}$ is the residue of $\varphi(E)$ at K_λ for any λ. See [36, Thm.III.3.5] for a general index theorem which implies the one just explained.

Assume now that M may be not compact and that there is a possibly singular compact analytic subvariety $S \subset M$ of pure dimension $s \leq n$. Let $K \subset S$ be a compact analytic subvariety of S such that $\mathrm{Sing}(S) \subset K$ (it will be typically the union of $\mathrm{Sing}(S)$ with the 'singular set' of a 'geometric object' defined over $S - \mathrm{Sing}(S)$). Let $U_0 \subset M$ be a tubular neighborhood of $S - K$ and let $U_1 \subset M$ be a regular neighborhood of K, then we can assume that $U = U_0 \cup U_1$ is a regular neighborhood of S. Consider the open covering $\mathcal{U} = \{U_0, U_1\}$ of U. As well as in the 'non-singular case', by Alexander and Poincaré homomorphisms (1.8.4) and (1.8.3) we have the following diagram

$$\begin{tikzcd} H^j(\mathcal{U}, U_0, \mathbb{C}) \arrow[r, "i^*"] \arrow[d, "A_{S,K}"'] & H^j(\mathcal{U}, \mathbb{C}) \arrow[d, "P_S"'] \\ H_{2s-j}(K, \mathbb{C}) \arrow[r, "i_*"] & H_{2s-j}(S, \mathbb{C}) \end{tikzcd} \tag{1.9.2}$$

for any $j = 0, \ldots, 2s$, where i_* is here induced by the topological inclusion $i : K \hookrightarrow S$. Again the diagram turn out to be commutative (see [36, Prop.VI.4.4]) and we obtain an index theorem whenever we are able to localize a cocycle $\omega \in H^j(\mathcal{U}, \mathbb{C})$ at K, that is to lift ω to a $\omega_K \in H^j(\mathcal{U}, U_0, \mathbb{C})$. In particular, if we deal with cohomological classes $\omega \in H^{2s}(\mathcal{U}, \mathbb{C})$ we get formulas as

$$\sum_{\lambda} \operatorname{Res}(\omega, K_\lambda) = \int_S \omega,$$

where $K = \sqcup_\lambda K_\lambda$ is the decomposition in connected components and $\operatorname{Res}(\omega, K_\lambda) \in \mathbb{C}$ is the residue of ω at K_λ for any λ. A slight difference respect to the 'non-singular case' is that in this setting one uses to localize characteristic classes of C^∞ or virtual complex vector bundles defined on a whole neighborhood of the variety, while in the 'non-singular case' one deals with bundles defined on the variety. Examples of general index theorems in the 'singular setting' are [36, Thm.IV.2.4] and [36, Thm.VI.4.8].

Chapter 2

Holomorphic foliation induced by a pair (f, g)

The original part of the thesis starts with this chapter. From now on M is a complex manifold of dimension n and $S \subset M$ is a possibly singular globally irreducible analytic hypersurface with regular part $S' = S - \mathrm{Sing}(S)$. We denote by TM the holomorphic tangent bundle of M and, if S is non-singular, by TS and N_S respectively the holomorphic tangent bundle and the holomorphic normal bundle of S. The corresponding sheaves of holomorphic sections are denoted respectively by $\mathcal{T}_M$, $\mathcal{T}_S$ and $\mathcal{N}_S$. The sheaf of holomorphic functions on M is denoted by $\mathcal{O}_M$ and its subsheaf of holomorphic functions vanishing on S is denoted by $\mathcal{I}_S$. For the sake of simplicity we shall denote a germ of holomorphic function at some point and any local representative of it by the same symbol.

In this chapter we start to consider a pair (f, g) of distinct holomorphic self-maps of M, that is $f, g : M \to M$, with the property that $f|_S \equiv g|_S$ and (from Section 2.2) that one of the two maps is a local biholomorphism over an open neighborhood of S'. We show that assuming a local property on the pair (f, g) or a geometric property on S' we are able to obtain a 1-dimensional holomorphic possibly singular foliation over S'.

2.1 Order of coincidence of a pair (f, g)

Let $p \in S$ be any point. Observe that for any germ $h \in \mathcal{O}_{M,f(p)}$ we have the well-defined germ $h \circ f - h \circ g \in \mathcal{I}_{S,p}$.

Definition 2.1.1. For any $h \in \mathcal{O}_{M,f(p)}$ define

$$\nu^p_{f,g}(h) = \max\left\{\nu \in \mathbb{N} \text{ s.t. } h \circ f - h \circ g \in \mathcal{I}^{\nu}_{S,p}\right\}.$$

We call the number

$$\nu^p_{f,g} = \min\left\{\nu^p_{f,g}(h), \text{ for } h \in \mathcal{O}_{M,f(p)}\right\}$$

the *order of coincidence of* (f, g) *at* p *along* S.

Let $(w^1, \dots, w^n)$ be any local holomorphic coordinates at $f(p) = g(p)$ and set $f^j = w^j \circ f$ and $g^j = w^j \circ g$ for any $j = 1, \dots, n$. For any germ $h \in \mathcal{O}_{M,f(p)}$ we have the fundamental relations

$$\begin{aligned} h \circ f - h \circ g &= \sum_{j=1}^{n} \left(f^j - g^j\right)\left(\frac{\partial h}{\partial w^j} \circ g\right)\left(\operatorname{mod} \mathcal{I}^{2\nu^p_{f,g}}_{S,p}\right) \\ &= \sum_{j=1}^{n} \left(f^j - g^j\right)\left(\frac{\partial h}{\partial w^j} \circ f\right)\left(\operatorname{mod} \mathcal{I}^{2\nu^p_{f,g}}_{S,p}\right). \end{aligned} \tag{2.1.1}$$

In fact by Definition 2.1.1 we have that

$$f^j - g^j = s^j \in \mathcal{I}^{\nu^p_{f,g}}_{S,p}$$

and

$$\frac{\partial h}{\partial w^j} \circ f - \frac{\partial h}{\partial w^j} \circ g \in \mathcal{I}^{\nu^p_{f,g}}_{S,p}$$

for any $j = 1, \dots, n$, hence

$$\begin{aligned} h \circ f - h \circ g &= h(g^1 + s^1, \dots, g^n + s^n) - h(g^1, \dots, g^n) \\ &= \sum_{j=1}^{n} s^j \left.\frac{\partial h}{\partial w^j}\right|_{(g^1,\dots,g^n)} + \frac{1}{2}\sum_{j,k=1}^{n} s^j s^k \left.\frac{\partial^2 h}{\partial w^j \partial w^k}\right|_{(g^1,\dots,g^n)} + \cdots \\ &= \sum_{j=1}^{n} (f^j - g^j)\frac{\partial h}{\partial w^j} \circ g \left(\operatorname{mod} \mathcal{I}^{2\nu^p_{f,g}}_{S,p}\right) \\ &= \sum_{j=1}^{n} (f^j - g^j)\frac{\partial h}{\partial w^j} \circ f \left(\operatorname{mod} \mathcal{I}^{2\nu^p_{f,g}}_{S,p}\right), \end{aligned}$$

where the second row is given by the Taylor expansion of h at $f(p) = g(p)$.

Lemma 2.1.2. *Let* $p \in S$ *be any point. Then* *(i)* *if* $(w^1, \dots, w^n)$ *are any local holomorphic coordinates at* $f(p) = g(p)$ *then*

$$\nu^p_{f,g} = min\left\{\nu^p_{f,g}(w^1), \dots, \nu^p_{f,g}(w^n)\right\}.$$

(ii) *for any* $h \in \mathcal{O}_{M,f(p)}$ *the function*

$$q \to \nu^q_{f,g}(h)$$

is constant in a neighborhood of p *in* S.

(iii) *the function*

$$p \to \nu^p_{f,g}$$

is constant on S.

Proof. (*i*) By Definition 2.1.1 we have $\nu^p_{f,g} \leq \min\left\{\nu^p_{f,g}(w^1), \dots, \nu^p_{f,g}(w^n)\right\}$ and by (2.1.1) we have the reverse inequality.

(*ii*) Let $h \in \mathcal{O}_{M,f(p)}$ be any germ and let $\{\gamma^1, \dots, \gamma^t\}$ be any set of generators of $\mathcal{I}_{S,p}$. Then

$$h \circ f - h \circ g = \sum_{|I| = \nu^p_{f,g}(h)} \gamma^I c_I$$

where $I = (i_1, \dots, i_t) \in \mathbb{N}^t$ is a multi-index, $|I| = i_1 + \cdots + i_t$, $\gamma^I = (\gamma^1)^{i_1} \cdots (\gamma^t)^{i_t}$ and $c_I \in \mathcal{O}_{M,p}$. This equality clearly holds in a neighborhood of p by definition of germ and since $\mathcal{I}_S$ is a coherent $\mathcal{O}_M$-module the γ's are generators of $\mathcal{I}_{S,q}$ for points q close enough to p. Moreover by definition of $\nu^p_{f,g}(h)$ there must be an index I_0 such that $c_{I_0} \notin \mathcal{I}_{S,p}$, thus $c_{I_0} \notin \mathcal{I}_{S,q}$ for all $q \in S$ close enough to p. Then the assertion follows.

(*iii*) By (*i*) and (*ii*) the function $p \to \nu^p_{f,g}$ is locally constant on S. Then it is globally constant since S is connected. □

As a consequence of (*iii*) the following definition makes sense.

Definition 2.1.3. The *order of coincidence of the pair* (f, g) *along* S is the number $\nu_{f,g} = \nu^p_{f,g}$ for any point $p \in S$.

Example 2.1.4. Let $\tilde{f}, \tilde{g} : \mathbb{C}^n \to \mathbb{C}^n$ be a pair of (germs of) holomorphic self-maps both fixing the origin $0 \in \mathbb{C}^n$ and such that $d\tilde{f}|_0 = d\tilde{g}|_0$ is invertible. As mentioned in Introduction, we get an interesting example by lifting $\tilde{f}$ and $\tilde{g}$ to the blow-up $\pi : M \to \mathbb{C}^n$ of $\mathbb{C}^n$ at 0. In fact it is well-known that there exist unique liftings $f, g : M \to M$ such that $\pi \circ f = \tilde{f} \circ \pi$ and $\pi \circ g = \tilde{g} \circ \pi$ (see [1, Prop.2.1] for details), and they coincide on the *exceptional divisor*

$S = \pi^{-1}(0) \cong \mathbb{P}^{n-1}_{\mathbb{C}}$ because they both act on S as $\mathrm{d}\tilde{f}|_0 = \mathrm{d}\tilde{g}|_0$. Moreover $f|_S \equiv g|_S : S \to S$ is a biholomorphism since $\mathrm{d}\tilde{f}|_0 = \mathrm{d}\tilde{g}|_0$ is invertible.

In this example we want to understand how the order of coincidence of f and g along S is related to the constant $\tilde{\nu}$ defined as follows. Let $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^n)$ be the canonical coordinates of $\mathbb{C}^n$, since $\mathrm{d}\tilde{g}|_0$ is invertible there exists an open neighborhood $\tilde{U} \subset \mathbb{C}^n$ of 0 such that $\tilde{g}|_{\tilde{U}}$ is a biholomorphism onto $\tilde{W} = \tilde{g}(\tilde{U})$. Then consider the coordinates $\tilde{w} = \tilde{z} \circ \tilde{g}^{-1} = (\tilde{w}^1, \ldots, \tilde{w}^n)$ over $\tilde{W}$ and set $\tilde{f}^j = \tilde{w}^j \circ \tilde{f}$ and $\tilde{g}^j = \tilde{w}^j \circ \tilde{g}$ for any $j = 1, \ldots, n$. Observe that with this choice of coordinates clearly $\tilde{f}^j = \tilde{z}^j \circ \tilde{g}^{-1} \circ \tilde{f}$ and $\tilde{g}^j = \tilde{z}^j$ for any $j = 1, \ldots, n$. By the equality $\mathrm{d}\tilde{f}|_0 = \mathrm{d}\tilde{g}|_0$ it follows that

$$\tilde{f}^j - \tilde{g}^j = \sum_{d \geq 2} \tilde{P}^j_d(\tilde{z}^1, \ldots, \tilde{z}^n)$$

for any $j = 1, \ldots, n$, where the $\tilde{P}^j_d$ are homogeneous polynomials of degree $d \geq 2$ in the coordinates $\tilde{z}$. Let us define the numbers

$$\tilde{\nu}^j = \min \left\{ d \geq 2 \text{ s.t. } \tilde{P}^j_d \neq 0 \right\}$$

for any $j = 1, \ldots, n$, then $\tilde{\nu}$ is defined as

$$\tilde{\nu} = \min \left\{ \tilde{\nu}^1, \ldots, \tilde{\nu}^n \right\}.$$

Clearly $\tilde{\nu} \geq 2$ and it is a sort of 'order of coincidence' of $\tilde{f}$ and $\tilde{g}$ at 0. Note that

$$g|_{\pi^{-1}(\tilde{U})} : \pi^{-1}(\tilde{U}) \longrightarrow \pi^{-1}(\tilde{W})$$

is a biholomorphism since $\tilde{g}|_{\tilde{U}} : \tilde{U} \to \tilde{W}$ is a biholomorphism and $\pi \circ g = \tilde{g} \circ \pi$, and recall that any system of holomorphic coordinates of $\mathbb{C}^n$ centered at 0 induces canonically n local holomorphic charts over the blow-up M which cover S. We denote by (U_j, z_j) and (W_j, w_j) the local holomorphic charts induced on M respectively by $(\tilde{U}, \tilde{z})$ and $(\tilde{W}, \tilde{w})$, for $j = 1, \ldots, n$. By definition

$$\bigcup_{j=1}^n U_j = \pi^{-1}(\tilde{U}) \qquad \text{and} \qquad \bigcup_{j=1}^n W_j = \pi^{-1}(\tilde{W}),$$

moreover one can easily check that $g(U_j) = W_j$ and $w_j = z_j \circ g^{-1}$ for any $j = 1, \ldots, n$. For the sake of simplicity set now $(U, z) = (U_1, z_1)$ and $(W, w) = (W_1, w_1)$, with $z = (z^1, \ldots, z^n)$ and $w = (w^1, \ldots, w^n)$. We compute $\nu^p_{f,g}$ for one point $p \in U \cap S$ since it is automatically $\nu_{f,g}$ by Lemma 2.1.2. Set

$f^j = w^j \circ f$ and $g^j = w^j \circ g$ for any $j = 1, \ldots, n$, and note again that $f^j = z^j \circ g^{-1} \circ f$ and $g^j = z^j$ by the remarks just done. Recalling that by definition $\tilde{z}^1 = z^1$ and $\tilde{z}^j = z^1 z^j$ for any $j = 2, \ldots, n$, and similarly for the coordinates $\tilde{w}$ and w, one can check that

$$f^1(z^1, \ldots, z^n) = \tilde{f}^1(z^1, z^1 z^2, \ldots, z^1 z^n)$$

and

$$\begin{aligned} f^j(z^1, \ldots, z^n) &= \frac{\tilde{f}^j(z^1, z^1 z^2, \ldots, z^1 z^n)}{\tilde{f}^1(z^1, z^1 z^2, \ldots, z^1 z^n)} \\ &= \frac{z^1 z^j + \sum_{d \geq 2} (z^1)^d \tilde{P}^j_d(1, z^2, \ldots, z^n)}{z^1 + \sum_{d \geq 2} (z^1)^d \tilde{P}^1_d(1, z^2, \ldots, z^n)} \end{aligned}$$

for any $j = 2, \ldots, n$. It follows that

$$f^1 - g^1 = \sum_{d \geq 2} (z^1)^d \tilde{P}^1_d(1, z^2, \ldots, z^n)$$

and

$$\begin{aligned} f^j - g^j = f^j - z^j =& \frac{z^1 z^j + \sum_{d \geq 2} (z^1)^d \tilde{P}^j_d(1, z^2, \ldots, z^n)}{z^1 + \sum_{d \geq 2} (z^1)^d \tilde{P}^1_d(1, z^2, \ldots, z^n)} - z^j \\ =& \frac{z^j + \sum_{d \geq 2} (z^1)^{d-1} \tilde{P}^j_d(1, z^2, \ldots, z^n)}{1 + \sum_{d \geq 2} (z^1)^{d-1} \tilde{P}^1_d(1, z^2, \ldots, z^n)} - z^j \\ =& \frac{\sum_{d \geq 2} (z^1)^{d-1} \left[\tilde{P}^j_d(1, z^2, \ldots, z^n) - z^j \tilde{P}^1_d(1, z^2, \ldots, z^n) \right]}{1 + \sum_{d \geq 2} (z^1)^{d-1} \tilde{P}^1_d(1, z^2, \ldots, z^n)} \end{aligned}$$

for any $j = 2, \ldots, n$. Since z^1 is a local generator of $\mathcal{I}_S$ we have that $\nu^p_{f,g}(w^1) = \tilde{\nu}^1$ and

$$\nu^p_{f,g}(w^j) = \min \left\{ d \geq 2 \text{ s.t. } \tilde{P}^j_d(1, z^2, \ldots, z^n) - z^j \tilde{P}^1_d(1, z^2, \ldots, z^n) \neq 0 \right\} - 1$$

for any $j = 2, \ldots, n$, which implies that $\nu^p_{f,g}(w^j) \geq \min\{\tilde{\nu}^1, \tilde{\nu}^j\} - 1$ for any $j = 2, \ldots, n$. Then we gain the relation $\nu_{f,g} \geq \tilde{\nu} - 1$.

We conclude this example by noting that since $\tilde{g}$ is locally invertible at 0 one may consider also the holomorphic map $\tilde{g}^{-1} \circ \tilde{f}$ over an open neighborhood of 0, which clearly fixes 0. By the choice of the coordinates $\tilde{w} = \tilde{z} \circ \tilde{g}^{-1}$ it follows that

$$\tilde{f}^j - \tilde{g}^j = \tilde{z}^j \circ \tilde{g}^{-1} \circ \tilde{f} - \tilde{z}^j$$

for any $j = 1, \ldots, n$, hence the number $\tilde{\nu}$ coincides with the order of $\tilde{g}^{-1} \circ \tilde{f}$ at 0. Since $\mathrm{d}(\tilde{g}^{-1} \circ \tilde{f})|_0 = \mathrm{Id}$, the lifting of $\tilde{g}^{-1} \circ \tilde{f}$ to the blow-up $\pi : M \to \mathbb{C}^n$ fixes pointwise the exceptional divisor $S \subset M$. Therefore we have the 'order of contact' of this lifting with S (see [4, Def.1.1]) and one might ask how it relates to $\nu_{f,g}$. Observe that the lifting of $\tilde{g}^{-1} \circ \tilde{f}$ is exactly $g^{-1} \circ f$ and if (U, z) and (W, w) are the local holomorphic charts over M introduced just above then

$$f^j - g^j = z^j \circ g^{-1} \circ f - z^j$$

for any $j = 1, \ldots, n$. Then by the very definitions it follows that $\nu_{g^{-1} \circ f} = \nu_{f,g}$, and moreover we recover the relation $\nu_{g^{-1} \circ f} \geq \tilde{\nu} - 1$ already showed in [4, Ex.1.2].

2.2 The canonical section associated to a pair (f, g)

Since $f|_S \equiv g|_S$, there is a well-defined morphism of holomorphic vector bundles

$$\mathrm{d}f - \mathrm{d}g : N_{S'} \longrightarrow (f^*TM)|_{S'} \tag{2.2.1}$$

given on the fibers by $[v] \to \mathrm{d}f|_p(v) - \mathrm{d}g|_p(v)$, for any $[v] \in N_{S',p}$ and $p \in S'$. We point out that in general $f^*TM \neq g^*TM$ but clearly $(f^*TM)|_S = (g^*TM)|_S$. From now on suppose that one of the maps of the pair, say g, is a local biholomorphism over an open neighborhood of S', or equivalently $\mathrm{d}g|_p : T_pM \longrightarrow T_{g(p)}M$ is an isomorphism of complex vector spaces for any $p \in S'$. Under this hypothesis we have also the isomorphism of holomorphic vector bundles

$$\mathrm{d}g|_{S'} : TM|_{S'} \longrightarrow (f^*TM)\,|_{S'}$$

and composing (2.2.1) with the inverse $\mathrm{d}g|_{S'}^{-1}$ we obtain

$$\mathrm{d}g|_{S'}^{-1} \circ (\mathrm{d}f - \mathrm{d}g) : N_{S'} \longrightarrow TM|_{S'}. \tag{2.2.2}$$

Observe that this last morphism can be seen as a holomorphic section of $N_{S'}^* \otimes TM|_{S'}$, then we want to express it respect to a local holomorphic frame of that bundle. To this end, it is useful to consider only certain kinds of local holomorphic charts of M.

Definition 2.2.1. Let M be a complex manifold of dimension n and let $V \subset M$ be a complex submanifold of codimension $0 < k < n$. We say that a local holomorphic chart $(U, z) = (U, z^1, \ldots, z^n)$ of M is *adapted to* V if $U \cap V = \emptyset$ or $U \cap V = \{z^1 = \cdots = z^k = 0\}$. We call an atlas $\mathfrak{U}$ of local holomorphic charts of M adapted to V an *atlas adapted to* V.

Let $W \subset M$ be an open neighborhood of V and suppose there is a local biholomorphism $\Phi : W \to M$. If (U, z) is a local holomorphic chart of M adapted to V such that $\Phi|_U$ is a biholomorphism onto its image, then we call it *adapted to* V *and* Φ and we consider the *special coordinates* $w = z \circ \Phi|_U^{-1} = (w^1, \ldots, w^n)$ over $\Phi(U)$. We call an atlas $\mathfrak{U}$ of local holomorphic charts of M adapted to V and Φ an *atlas adapted to* V *and* Φ.

Let (U, z) be a local holomorphic chart adapted to S' such that $U \cap S' \neq \emptyset$ and let $\pi : TM|_{S'} \to N_{S'}$ be the obvious projection. Then

$$\partial z^1 = \pi \left(\left.\frac{\partial}{\partial z^1}\right|_{S'} \right)$$

is a local holomorphic generator of $N_{S'}$ on $U \cap S'$ and we denote its dual by $\partial^* z^1$, thus

$$\left\{ \partial^* z^1 \otimes \left.\frac{\partial}{\partial z^1}\right|_{S'}, \ldots, \partial^* z^1 \otimes \left.\frac{\partial}{\partial z^n}\right|_{S'} \right\} \tag{2.2.3}$$

is a local holomorphic frame of $N^*_{S'} \otimes TM|_{S'}$ on $U \cap S'$. Let $p \in U \cap S'$ be any point and take any (for the moment) local holomorphic chart (W, w) at $f(p) = g(p)$. Then

$$\left\{ \partial^* z^1 \otimes \left.\left(f^* \frac{\partial}{\partial w^1} \right)\right|_{S'}, \ldots, \partial^* z^1 \otimes \left.\left(f^* \frac{\partial}{\partial w^n} \right)\right|_{S'} \right\} \tag{2.2.4}$$

is a local holomorphic frame of $N^*_{S'} \otimes (f^*TM)|_{S'}$ on $U \cap f^{-1}(W) \cap S'$. We point out that in general $f^* \frac{\partial}{\partial w^j} \neq g^* \frac{\partial}{\partial w^j}$ but clearly $(f^* \frac{\partial}{\partial w^j})|_{S'} = (g^* \frac{\partial}{\partial w^j})|_{S'}$.

In order to express (2.2.2) respect to (2.2.3), we express (2.2.1) respect to (2.2.4) first. Set $f^j = w^j \circ f$ and $g^j = w^j \circ g$ for any $j = 1, \ldots, n$, a trivial computation shows that (2.2.1) is locally

$$\sum_{j=1}^{n} \left.\frac{\partial (f^j - g^j)}{\partial z^1}\right|_{S'} \partial^* z^1 \otimes \left.\left(f^* \frac{\partial}{\partial w^j} \right)\right|_{S'} \tag{2.2.5}$$

since (U, z) is adapted to S'. Assume now (U, z) adapted also to g and take the associated special coordinates $w = z \circ g|_U^{-1}$ on $W = g(U)$. With this choice

$$\left(f^*\frac{\partial}{\partial w^j}\right)\bigg|_q = \left(g^*\frac{\partial}{\partial w^j}\right)\bigg|_q = \frac{\partial}{\partial w^j}\bigg|_{g(q)} = \mathrm{d}g|_q\left(\frac{\partial}{\partial z^j}\bigg|_q\right)$$

for any $q \in U \cap S'$ and $j = 1, \ldots, n$, then by (2.2.5) it follows that (2.2.2) is locally

$$\sum_{j=1}^{n} \frac{\partial(f^j - g^j)}{\partial z^1}\bigg|_{S'} \partial^* z^1 \otimes \frac{\partial}{\partial z^j}\bigg|_{S'} . \tag{2.2.6}$$

Observe now that since (U, z) is adapted to S', by Lemma 2.1.2 it follows that (2.2.6) is identically zero on S' if and only if $\nu_{f,g} > 1$. Thus we would like to define something similar to (2.2.6) but not identically zero on S' if $\nu_{f,g} > 1$, and the idea is to consider higher order derivatives of "$f - g$" respect to z^1 not all vanishing on S'. More precisely, let (U, z) be a local holomorphic chart adapted to S' and g such that $U \cap S' \neq \emptyset$ and consider the associated special coordinates $w = z \circ g|_U^{-1}$ on $W = g(U)$. Set $f^j = w^j \circ f$ and $g^j = w^j \circ g$ for any $j = 1, \ldots, n$ as usual, then by Lemma 2.1.2 there exist local holomorphic functions $h^j \in \mathcal{O}_M$ such that

$$f^j - g^j = h^j (z^1)^{\nu_{f,g}} \tag{2.2.7}$$

for any $j = 1, \ldots, n$. Let us define

$$\mathfrak{D}_{f,g} := \sum_{j=1}^{n} h^j (\mathrm{d}z^1)^{\nu_{f,g}} \otimes \frac{\partial}{\partial z^j}, \tag{2.2.8}$$

where $(\mathrm{d}z^1)^{\nu_{f,g}} = (\mathrm{d}z^1)^{\otimes \nu_{f,g}}$, which is a local holomorphic section of $(TM^{\otimes \nu_{f,g}})^* \otimes TM$ on U. Note that (2.2.8) is not identically zero on S by construction, moreover we have the following fundamental proposition.

Proposition 2.2.2. *Let (U, z) and $(\hat{U}, \hat{z})$ be two local holomorphic chart adapted to S' and g such that $U \cap \hat{U} \cap S' \neq \emptyset$, and let $\mathfrak{D}_{f,g}$ and $\hat{\mathfrak{D}}_{f,g}$ be the corresponding local sections of $(TM^{\otimes \nu_{f,g}})^* \otimes TM$ defined as in (2.2.8). Then*

$$\hat{\mathfrak{D}}_{f,g} = \mathfrak{D}_{f,g} \pmod{\mathcal{I}_S}$$

where they overlap.

Proof. Set $\nu = \nu_{f,g}$ in the following to ease the notation. Since z^1 and $\hat{z}^1$ are both generators of $\mathcal{I}_{S,p}$ for any $p \in U \cap \hat{U} \cap S'$, it follows that $\hat{z}^1 = a z^1$ for a germ $a \in \mathcal{O}_M^*$, then

$$(\mathrm{d}\hat{z}^1)^\nu = a^\nu (\mathrm{d}z^1)^\nu \pmod{\mathcal{I}_S} . \tag{2.2.9}$$

By (2.1.1) and (2.2.7) we have that

$$\hat{h}^j a^\nu (z^1)^\nu = \sum_{k=1}^n h^k (z^1)^\nu \left(\frac{\partial \hat{w}^j}{\partial w^k} \circ g \right) \pmod{\mathcal{I}_S^{2\nu}},$$

hence

$$\hat{h}^j a^\nu = \sum_{k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k} \pmod{\mathcal{I}_S^\nu} \tag{2.2.10}$$

for any $j = 1, \dots, n$, since $\frac{\partial \hat{w}^j}{\partial w^k} \circ g = \frac{\partial \hat{z}^j}{\partial z^k}$ because w are the special coordinates associated to z. Lastly, we need the trivial relation

$$\frac{\partial}{\partial \hat{z}^j} = \sum_{k=1}^n \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial}{\partial z^k} \tag{2.2.11}$$

for any $j = 1, \dots, n$. Then it follows by (2.2.9), (2.2.10) and (2.2.11) that

$$\begin{aligned}
\hat{\mathfrak{D}}_{f,g} &= \sum_{j=1}^n \hat{h}^j (\mathrm{d}\hat{z}^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} = \sum_{j=1}^n \hat{h}^j a^\nu (\mathrm{d}z^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} \pmod{\mathcal{I}_S} \\
&= \sum_{j=1}^n \sum_{k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k} (\mathrm{d}z^1)^\nu \otimes \frac{\partial}{\partial \hat{z}^j} \pmod{\mathcal{I}_S} \\
&= \sum_{k=1}^n \sum_{i=1}^n h^k \left(\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial z^i}{\partial \hat{z}^j} \right) (\mathrm{d}z^1)^\nu \otimes \frac{\partial}{\partial z^i} \pmod{\mathcal{I}_S} \\
&= \sum_{k=1}^n \sum_{i=1}^n h^k (\mathrm{d}z^1)^\nu \otimes \frac{\partial}{\partial z^i} \pmod{\mathcal{I}_S},
\end{aligned}$$

where in the last equality we use that $\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial z^i}{\partial \hat{z}^j} = \frac{\partial z^i}{\partial z^k} = \delta_{ik}$ for any $i, k = 1, \dots, n$. □

Remark 2.2.3. A priori the germs h^j defined in (2.2.7) do not have representatives defined on U, however we can assume it (possibly shrinking U).

We can now introduce the following definition.

Definition 2.2.4. The *canonical section associated to the pair* (f,g) is the global holomorphic section $\mathscr{D}_{f,g}$ of the holomorphic vector bundle $(N_{S'}^{\otimes \nu_{f,g}})^* \otimes TM|_{S'}$ obtained by gluing together *over* S' the local $\mathfrak{D}_{f,g}$ defined as in (2.2.8).

Observe that we can think to $\mathscr{D}_{f,g}$ also as a morphism $\mathscr{D}_{f,g} : N_{S'}^{\otimes \nu_{f,g}} \to TM|_{S'}$ of holomorphic vector bundles. If (U, z) is any local holomorphic chart adapted to S' and g such that $U \cap S' \neq \emptyset$, then $\mathscr{D}_{f,g}$ is given over $U \cap S'$ by

$$\mathscr{D}_{f,g} = \sum_{j=1}^{n} h^j\big|_{S'} (\partial^* z^1)^{\nu_{f,g}} \otimes \frac{\partial}{\partial z^j}\bigg|_{S'},$$

where the h^j are the local holomorphic functions defined as in (2.2.7). We point out that $\mathscr{D}_{f,g}$ is not identically zero on S' by construction.

Definition 2.2.5. A point $p \in S'$ is a *singularity of the pair* (f, g) if the canonical section $\mathscr{D}_{f,g} : N_{S'}^{\otimes \nu_{f,g}} \to TM|_{S'}$ is not injective in p, that is if $\mathscr{D}_{f,g}|_p = 0$. The *singular set of* (f, g) is denoted by $\mathrm{Sing}(f, g)$.

Remark 2.2.6. Let (U, z) be any local holomorphic chart of M adapted to S' and g such that $U \cap S' \neq \emptyset$ and consider the special coordinates $w = z \circ g|_U^{-1}$ on $W = g(U)$. With this choice of coordinates we have that

$$f^j = w^j \circ f = z^j \circ g|_U^{-1} \circ f \qquad \text{and} \qquad g^j = z^j$$

for any $j = 1, \ldots, n$. Then if $\nu_{g|_U^{-1} \circ f}$ is the 'order of contact of $g|_U^{-1} \circ f$ with (a piece of) S' (according with [4, Def.1.1]) we clearly have the equality $\nu_{f,g} = \nu_{g|_U^{-1} \circ f}$, and $\mathscr{D}_{f,g}$ is locally exactly as the canonical section associated to $g|_U^{-1} \circ f$ (see [4, Sec.3]). Consequently if g is a *global* biholomorphism over an open neighborhood of S' then $\nu_{f,g} = \nu_{g^{-1} \circ f}$ and $\mathscr{D}_{f,g}$ is exactly the canonical section associated to $g^{-1} \circ f$.

2.3 Basic definitions about 1-dimensional holomorphic foliations

We stop the discussion about the pair (f, g) for a while in order to make a brief digression about 1-dimensional holomorphic foliations, since our next aim is to construct such geometric objects.

Let S denote only in this section any complex manifold. A 1-*dimensional holomorphic possibly singular foliation* on S is a morphism of holomorphic vector bundles

$$\mathscr{F} : F \longrightarrow TS,$$

where F is a holomorphic line bundle on S called the *tangent bundle of* $\mathscr{F}$. The *singular set of* $\mathscr{F}$ is

$$\mathrm{Sing}(\mathscr{F}) = \{q \in S \text{ s.t. } \mathscr{F}|_q = 0\}\,,$$

and we set $S^0 = S - \mathrm{Sing}(\mathscr{F})$ and $F^0 = F|_{S^0}$. Since $\mathscr{F}|_{S^0}$ is injective we can define the holomorphic vector bundle $N^0_{\mathscr{F}} = TS^0/F^0$ and we call it the *normal bundle of* $\mathscr{F}$ (note that it is defined over S^0). Instead the virtual complex vector bundle $TS - F \in K(S)$ is called the *virtual normal bundle of* $\mathscr{F}$ as $TS^0 - F^0 = N^0_{\mathscr{F}}$ in $K(S^0)$.

If M is another complex manifold and $S \subset M$ then we can even define the holomorphic vector bundle $N^0_{\mathscr{F},M} = TM|_{S^0}/F^0$, which we call the *normal bundle of* $\mathscr{F}$ *in* M. Observe that in this case the double inclusion $F^0 \subset TS^0 \subset TM|_{S^0}$ induces the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow N^0_{\mathscr{F}} \longrightarrow N^0_{\mathscr{F},M} \longrightarrow N_{S^0} \longrightarrow 0.$$

If v is a local generator of the tangent bundle F then the possibly singular holomorphic vector field $\mathscr{F}(v) \in \mathcal{T}_S$ is called a *local generator of* $\mathscr{F}$.

We may equivalently reformulate everything in the language of sheaves. In this case a 1-dimensional holomorphic possibly singular foliation on S is an injective morphism of $\mathcal{O}_S$-modules

$$\mathscr{F} : \mathcal{F} \longrightarrow \mathcal{T}_S,$$

where $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of rank 1 called the *tangent sheaf of* $\mathscr{F}$. Clearly $\mathcal{F}$ is the sheaf of holomorphic sections of the tangent bundle F. The singular set of the foliation is then

$$\mathrm{Sing}(\mathscr{F}) = \{q \in S \text{ s.t. the stalk } (\mathcal{T}_S/\mathcal{F})_q \text{ is not a free } \mathcal{O}_{S,q}\text{-module}\}\,,$$

where the coherent $\mathcal{O}_S$-module $\mathcal{N}_{\mathscr{F}} = \mathcal{T}_S/\mathcal{F}$ is called the *normal sheaf of* $\mathscr{F}$. Clearly $\mathcal{N}^0_{\mathscr{F}} = \mathcal{N}_{\mathscr{F}}|_{S^0}$ is the sheaf of holomorphic sections of the normal bundle $N^0_{\mathscr{F}}$. If M is another complex manifold and $S \subset M$ then $\mathcal{T}_{M,S}$ denotes the sheaf of holomorphic sections of $TM|_S$ and the coherent $\mathcal{O}_S$-module $\mathcal{N}_{\mathscr{F},M} = \mathcal{T}_{M,S}/\mathcal{F}$ is called the *normal sheaf of* $\mathscr{F}$ *in* M. Clearly $\mathcal{N}^0_{\mathscr{F},M} = \mathcal{N}_{\mathscr{F},M}|_{S^0}$ is the sheaf of holomorphic sections of the normal bundle $N^0_{\mathscr{F},M}$. Observe that in this case we have the double inclusion $\mathcal{F} \subset \mathcal{T}_S \subset \mathcal{T}_{M,S}$ which induces the short exact sequence of coherent $\mathcal{O}_S$-modules

$$0 \longrightarrow \mathcal{N}_{\mathscr{F}} \longrightarrow \mathcal{N}_{\mathscr{F},M} \longrightarrow \mathcal{N}_S \longrightarrow 0.$$

2.4 Tangential case

Now let us come back to the discussion about the pair (f, g). Recall that we are assuming that g is a local biholomorphism over an open neighborhood of S' and set in the following $\nu = \nu_{f,g}$ to ease the notation. As showed in Section 2.2 the pair (f, g) induces the canonical section $\mathscr{D}_{f,g} : N_{S'}^{\otimes \nu} \to TM|_{S'}$, which in practice is a 1-dimensional holomorphic distribution over S'. However we would like to have a 1-dimensional holomorphic foliation over S', and in this section we discuss a first condition under which it is possible.

Let $p \in S$ be any point, the pull-back by f and g of germs of holomorphic functions at $f(p) = g(p)$ gives rise to two ring homomorphism

$$f_p^*, g_p^* : \mathcal{O}_{M,f(p)} \longrightarrow \mathcal{O}_{M,p}$$

which are not equal in general. Anyway $(f_p^*)^{-1}(\mathcal{I}_{S,p}) = (g_p^*)^{-1}(\mathcal{I}_{S,p})$ as $f|_S = g|_S$ and we denote by $I_{f(p)}$ this ideal of $\mathcal{O}_{M,f(p)}$.

Remark 2.4.1. Let $p \in S$ be any point. Clearly $I_{f(p)} \supseteq \mathcal{I}_{f(S),f(p)}$ but they may be not equal in general, nevertheless $I_{f(p)}$ may be seen as the stalk at $f(p)$ of a coherent module. In fact, by hypothesis there is an open neighborhood $U \subset M$ of p such that $f(U \cap S) = g(U \cap S)$ is biholomorphic to $U \cap S$ and then it is an analytic subvariety of $g(U) \subset M$. Thus $\mathcal{I}_{f(U\cap S)}$ is a coherent $\mathcal{O}_{g(U)}$-module and $I_{f(p)} = \mathcal{I}_{f(U\cap S),f(p)}$.

Definition 2.4.2. Let $p \in S$ be any point, if

$$\min \left\{ \nu_{f,g}^p(h), \text{ for } h \in I_{f(p)} \right\} > \nu_{f,g}^p$$

we say that the pair (f, g) is *tangential at* $p \in S$.

Lemma 2.4.3. *Let $p \in S$ be any point. The following two statements are true:*

(i) let $\{\rho^1, \ldots, \rho^k\}$ be any set of generators of $I_{f(p)}$, then (f, g) is tangential at p if and only if

$$min \left\{ \nu_{f,g}^p(\rho^1), \ldots, \nu_{f,g}^p(\rho^k) \right\} > \nu_{f,g}^p.$$

(ii) if (f, g) is tangential at p then it is tangential at any point of S.

Proof. (i) Let $h \in I_{f(p)}$ be any germ, then $h = h_1\rho^1 + \cdots + h_k\rho^k$ for some $h_1, \dots, h_k \in \mathcal{O}_{M,f(p)}$ and clearly

$$h \circ f - h \circ g = \sum_{j=1}^{k} (h_j \circ f)\left(\rho^j \circ f - \rho^j \circ g\right) + \sum_{j=1}^{k} (\rho^j \circ g)\,(h_j \circ f - h_j \circ g)\,.$$

The j-th term of the first sum belongs to $\mathcal{I}_{S,p}^{\nu_{f,g}^p(\rho^j)}$, for any $j = 1, \dots, k$, and each term of the second sum belongs to $\mathcal{I}_{S,p}^{\nu_{f,g}^p + 1}$. Thus by definition

$$\nu_{f,g}^p(h) \geq \min\left\{\nu_{f,g}^p(\rho^1), \dots, \nu_{f,g}^p(\rho^k), \nu_{f,g}^p + 1\right\}$$

and since this is true for any $h \in I_{f(p)}$ we have that

$$\min\left\{\nu_{f,g}^p(h),\ \text{for } h \in I_{f(p)}\right\} \geq \min\left\{\nu_{f,g}^p(\rho^1), \dots, \nu_{f,g}^p(\rho^k), \nu_{f,g}^p + 1\right\}.$$

Then recallling Definition 2.4.2 the assertion follows.

(ii) By Remark 2.4.1 if $\{\rho^1, \dots, \rho^k\}$ is any set of generators of $I_{f(p)}$ then the corresponding germs at $f(q)$ are generators of $I_{f(q)}$ for any $q \in S$ close enough to p. Then (ii) of Lemma 2.1.2 and (i) of the current Lemma imply that the set of points of S where (f,g) is tangential is both open and closed. Then the assertion follows because S is connected. □

By (ii) of Lemma 2.4.3 we can give the following definition.

Definition 2.4.4. The pair (f,g) is *tangential along* S if it is tangential at one point $p \in S$.

Let $p \in S'$ be any regular point, let (U,z) be a local holomorphic chart adapted to S' and g such that $p \in U \cap S'$ and take the associated special coordinates $w = z \circ g|_U^{-1}$ over $W = g(U)$. Note that $I_{f(p)}$ is generated by the germ of w^1 at $f(p)$, then by (i) of Lemma 2.4.3

$$(f,g) \text{ tangential at } p \iff h^1 \in \mathcal{I}_{S,p}$$

where h^1 is defined as in (2.2.7). Thus by (ii) of Lemma 2.4.3

$$\begin{aligned}(f,g) \text{ tangential along } S \iff\ & h^1 \in \mathcal{I}_{S,p},\ \text{for any } p \in S' \text{ and any } (U,z) \\ & \text{adapted to } S' \text{ and } g \text{ such that } p \in U \cap S' \\ \iff\ & \mathscr{D}_{f,g} \text{ is tangential to } S' \text{ i.e. } \mathscr{D}_{f,g}(N_{S'}^{\otimes \nu}) \subset TS'\end{aligned}$$

where again the (germs of) holomorphic functions h^1 are defined as in (2.2.7). Therefore *when* (f, g) *is tangential along* S *the canonical section* $\mathscr{D}_{f,g}$ *is in fact a* 1*-dimensional holomorphic possibly singular foliation on* S',

$$\mathscr{D}_{f,g} : N_{S'}^{\otimes\nu} \longrightarrow TS',$$

with tangent bundle $N_{S'}^{\otimes\nu}$ and singular set $\mathrm{Sing}(f, g)$. If (U, z) is any local holomorphic chart adapted to S' and g such that $U \cap S' \neq \emptyset$ then

$$X_{f,g} = \mathscr{D}_{f,g}\left((\partial z^1)^\nu\right) = \sum_{j=2}^{n} h^j\big|_{S'} \left.\frac{\partial}{\partial z^j}\right|_{S'} \tag{2.4.1}$$

is a local generator of $\mathscr{D}_{f,g}$ on $U \cap S'$, where the holomorphic functions h^j are defined as in (2.2.7). We call (2.4.1) the *canonical local generator of* $\mathscr{D}_{f,g}$ *associated to* (U, z).

Example 2.4.5. Let $(\tilde{f}, \tilde{g})$ be a pair of holomorphic self-maps of $\mathbb{C}^n$ as in Example 2.1.4 and let (f, g) be its lifting to the blow-up of $\mathbb{C}^n$ at 0, denoted by $\pi : M \to \mathbb{C}^n$. In this example we want to understand in which case the pair (f, g) is tangential along the exceptional divisor $S = \pi^{-1}(0) \subset M$. Let us refer to the notation already introduced in Example 2.1.4.

Let $p \in U \cap S$ be any point. Since w^1 is a generator of $I_{f(p)}$, by Lemma 2.4.3 the pair (f, g) is tangential along S if and only if there is a $j = 2, \ldots, n$ such that $\nu_{f,g}^p(w^1) > \nu_{f,g}^p(w^j)$. By Example 2.1.4 this is equivalent to say that there is a $j = 2, \ldots, n$ such that

$$\min\left\{d \geq 2 \text{ s.t. } \tilde{P}_d^j(1, z^2, \ldots, z^n) - z^j \tilde{P}_d^1(1, z^2, \ldots, z^n) \neq 0\right\} \leq \tilde{\nu}^1.$$

Suppose for a $j = 2, \ldots, n$ that $\tilde{P}_{\tilde{\nu}}^j(1, z^2, \ldots, z^n) - z^j \tilde{P}_{\tilde{\nu}}^1(1, z^2, \ldots, z^n) \neq 0$, then for such a j

$$\min\left\{d \geq 2 \text{ s.t. } \tilde{P}_d^j(1, z^2, \ldots, z^n) - z^j \tilde{P}_d^1(1, z^2, \ldots, z^n) \neq 0\right\} \leq \tilde{\nu} \leq \tilde{\nu}^1$$

and (f, g) is tangential along S. On the contrary, assume now that $\tilde{P}_{\tilde{\nu}}^j(1, z^2, \ldots, z^n) - z^j \tilde{P}_{\tilde{\nu}}^1(1, z^2, \ldots, z^n) = 0$ for any $j = 2, \ldots, n$. Then necessarily $\tilde{P}_{\tilde{\nu}}^1 \neq 0$ otherwise $\tilde{P}_{\tilde{\nu}}^j = 0$ for any $j = 1, \ldots, n$ and this would be in contradiction with the definition of $\tilde{\nu}$. This implies that $\tilde{P}_{\tilde{\nu}}^j \neq 0$ for any $j = 1, \ldots, n$ and consequently that $\tilde{\nu}^1 = \cdots = \tilde{\nu}^n = \tilde{\nu}$, thus

$$\min\left\{d \geq 2 \text{ s.t. } \tilde{P}_d^j(1, z^2, \ldots, z^n) - z^j \tilde{P}_d^1(1, z^2, \ldots, z^n) \neq 0\right\} > \tilde{\nu} = \tilde{\nu}^1$$

for any $j = 2, \ldots, n$, that is (f, g) is not tangential along S. Summing up, the pair (f, g) is tangential along S if and only if there is a $j = 2, \ldots, n$ such that $\tilde{P}^j_{\tilde{\nu}}(1, z^2, \ldots, z^n) - z^j \tilde{P}^1_{\tilde{\nu}}(1, z^2, \ldots, z^n) \neq 0$, or equivalently such that $\tilde{z}^1 \tilde{P}^j_{\tilde{\nu}} \neq \tilde{z}^j \tilde{P}^1_{\tilde{\nu}}$. As a consequence, if (f, g) is tangential along S then there is a $j = 2, \ldots, n$ such that $\nu^p_{f,g}(w^j) \leq \tilde{\nu} - 1$. However $\nu^p_{f,g}(w^j) \geq \nu_{f,g} \geq \tilde{\nu} - 1$, where the second inequality is proved in Example 2.1.4, then $\nu_{f,g} = \tilde{\nu} - 1$. Conversely, if (f, g) is not tangential along S then $\nu_{f,g} = \nu^p_{f,g}(w^1) = \tilde{\nu}^1 = \tilde{\nu}$, where the second equality is also proved in Example 2.1.4 and the third is proved just above. Concluding, we have found that

$$\tilde{\nu} - 1 \leq \nu_{f,g} \leq \tilde{\nu}$$

and that

$$(f, g) \text{ tangential along } S \Longleftrightarrow \nu_{f,g} = \tilde{\nu} - 1.$$

We end the example with the following remark. As noted in Example 2.1.4, we may consider also the holomorphic map $\tilde{g}^{-1} \circ \tilde{f}$ and its lifting to the blow-up $\pi : M \to \mathbb{C}^n$, which turns out to be $g^{-1} \circ f$. Then, as one shall expect, the pair (f, g) is tangential along S if and only if $g^{-1} \circ f$ is tangential along S (according with [4, Def.1.2]). Compare the present example and Example 2.1.4 to [4, Ex.1.2].

Remark 2.4.6. Let (U, z) be any local holomorphic chart of M adapted to S' and g such that $U \cap S' \neq \emptyset$ and consider the special coordinates $w = z \circ g|_U^{-1}$ on $W = g(U)$. As noted in Remark 2.2.6, with this choice of coordinates we have that $f^j = w^j \circ f = z^j \circ g|_U^{-1} \circ f$ and $g^j = z^j$ for any $j = 1, \ldots, n$. Then clearly (f, g) is tangential along S if and only if '$g|_U^{-1} \circ f$ is tangential along (a piece of) S' (according with [4, Def.1.2]). Consequently if g is a *global* biholomorphism over an open neighborhood of S' then (f, g) is tangential along S if and only if $g^{-1} \circ f$ is tangential along S.

2.5 Non-tangential case

Suppose now that the pair (f, g) is not tangential along S. If S' sits into M in a "favorable way" we can still define a 1-dimensional holomorphic foliation over it. For details about this geometric property of S' see [4, Sec.2] or, for a deeper treatment, [5, Sec.2-3] and [6]. Here we just recall some general definitions and basic facts that we need in the following.

Definition 2.5.1. Let M be a complex manifold of dimension n and let $V \subset M$ be a complex submanifold of codimension $0 < k < n$. We say that V *splits into* M if the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow TV \xrightarrow{\imath} TM|_V \xrightarrow{\pi} N_V \longrightarrow 0$$

splits holomorphically, that is if there exists a morphism of holomorphic vector bundles $\sigma : N_V \to TM|_V$ such that $\pi \circ \sigma = \mathrm{Id}_{N_V}$. We call such morphism σ a *splitting morphism* of V into M.

Let $\mathfrak{U}$ be an atlas of M adapted to V. If for any two local holomorphic charts $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ such that $U \cap \hat{U} \cap V \neq \emptyset$ we have that

$$\frac{\partial \hat{z}^i}{\partial z^j} \in \mathcal{I}_V \tag{2.5.1}$$

for any $i = k+1, \dots, n$ and $j = 1, \dots, k$, we say that $\mathfrak{U}$ is a *splitting atlas adapted to* V.

Proposition 2.5.2. *Let M be a complex manifold of dimension* n *and let $V \subset M$ be a complex submanifold of codimension* $0 < \mathrm{k} < \mathrm{n}$. *The following statements are equivalent:*

(i) *V splits into M.*

(ii) *there exists a morphism of holomorphic vector bundles $\tau : TM|_V \to TV$ such that $\tau \circ \imath = Id_{TV}$.*

(iii) *there exists a splitting atlas adapted to V over M.*

Proof. $(i) \iff (iii)$: see [4, Prop.2.1] or [5, Prop.2.15].
$(i) \iff (ii)$: suppose there exists a splitting morphism $\sigma : N_V \to TM|_V$, then $\pi \circ (\mathrm{Id}_{TM|_V} - \sigma \circ \pi) = 0$ and $\mathrm{im}(\mathrm{Id}_{TM|_V} - \sigma \circ \pi) \subset \ker(\pi) = \mathrm{im}(\imath)$. Thus we can define $\tau = \imath^{-1} \circ (\mathrm{Id}_{TM|_V} - \sigma \circ \pi)$, which is a morphism as in (ii). Conversely, suppose there exists a morphism $\tau : TM|_V \to TV$ as in (ii). Then $\ker(\tau) \cap \mathrm{im}(\imath) = \ker(\tau) \cap \ker(\pi) = \{\text{zero section}\}$ and $\pi|_{\ker(\tau)}$ has the inverse, thus we can define $\sigma = \pi|_{\ker(\tau)}^{-1}$ which is a splitting morphism of V into M. □

Remark 2.5.3. The arrow $(iii) \Longrightarrow (i)$ can be proved by construction in the following way. Let $\mathfrak{U}$ be a splitting atlas adapted to V and let $(U, z) \in \mathfrak{U}$ be any local holomorphic chart such that $U \cap V \neq \emptyset$. Then

$$\left\{ \partial z^1 = \pi \left(\left. \frac{\partial}{\partial z^1} \right|_V \right), \dots, \partial z^k = \pi \left(\left. \frac{\partial}{\partial z^k} \right|_V \right) \right\} \tag{2.5.2}$$

is a local holomorphic frame of N_V on $U \cap V$ and we can define a *local* splitting morphism $\sigma_U : N_{U\cap V} \to TM|_{U\cap V}$ by setting

$$\sigma_U\left(\partial z^j\right) = \left.\frac{\partial}{\partial z^j}\right|_V$$

for any $j = 1, \dots, k$. Let now $(\hat{U}, \hat{z}) \in \mathfrak{U}$ be another local holomorphic chart such that $U \cap \hat{U} \cap V \neq \emptyset$ and let $\sigma_{\hat{U}}$ be the associated local splitting morphism. Since

$$\frac{\partial}{\partial z^j} = \sum_{i=1}^{k} \frac{\partial \hat{z}^i}{\partial z^j} \frac{\partial}{\partial \hat{z}^i} + \sum_{i=k+1}^{n} \frac{\partial \hat{z}^i}{\partial z^j} \frac{\partial}{\partial \hat{z}^i}$$

for any $j = 1, \dots, n$, we have over $U \cap \hat{U} \cap V$ that

$$\begin{aligned}\left(\sigma_U - \sigma_{\hat{U}}\right)\left(\partial z^j\right) &= \sigma_U\left(\partial z^j\right) - \sum_{i=1}^{k} \left.\frac{\partial \hat{z}^i}{\partial z^j}\right|_V \sigma_{\hat{U}}\left(\partial \hat{z}^i\right) \\ &= \left.\frac{\partial}{\partial z^j}\right|_V - \sum_{i=1}^{k} \left.\frac{\partial \hat{z}^i}{\partial z^j}\right|_V \left.\frac{\partial}{\partial \hat{z}^i}\right|_V = \sum_{i=k+1}^{n} \left.\frac{\partial \hat{z}^i}{\partial z^j}\right|_V \left.\frac{\partial}{\partial \hat{z}^i}\right|_V\end{aligned}$$

for any $j = 1, \dots, k$. Then by property (2.5.1) we have that

$$\left(\sigma_U - \sigma_{\hat{U}}\right)\left(\partial z^j\right) = 0$$

for any $j = 1, \dots, k$ and consequently we can glue σ_U and $\sigma_{\hat{U}}$ over $U \cap \hat{U} \cap V$. As this is true for any $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ we get a (global) splitting morphism $\sigma_{\mathfrak{U}} : N_V \to TM|_V$.

Example 2.5.4. Let M be a complex manifold of dimension n and let $\pi : M \to V$ be a holomorphic vector bundle of rank $k \geq 1$. The base V can be embedded into the total space M via the zero section, and in this way it is a complex submanifold of M of codimension k. Let $\mathfrak{U}''$ be an atlas of V such that for any local holomorphic chart $(U'', z'') = (U'', z^{k+1}, \dots, z^n) \in \mathfrak{U}''$ there is a trivialization

$$\varphi : U \xrightarrow{\cong} \mathbb{C}^k \times U'',$$

where $U = \pi^{-1}(U'') \subset M$. We can consider over all these open sets U the local holomorphic coordinates $z = (\mathrm{Id}_{\mathbb{C}^k} \times z'') \circ \varphi = (z^1, \dots, z^n)$, then $\mathfrak{U}''$ induces an atlas $\mathfrak{U} = \{(U, z)\}$ over M. This atlas is a splitting atlas adapted to V, in fact for any $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ we have that $\hat{z}^i = \hat{z}^i(z^{k+1}, \dots, z^n)$ for $i = k+1, \dots, n$, then

$$\frac{\partial \hat{z}^i}{\partial z^j} \equiv 0$$

for any $i = k+1, \dots, n$ and $j = 1, \dots, k$. Thus V splits into M via the zero section.

Let $V \subset M$ be a splitting submanifold of codimension $0 < k < n$ and let $\sigma : N_V \to TM|_V$ be any splitting morphism. We denote by τ^σ the morphism $TM|_V \to TV$ induced by σ as in the proof of Proposition 2.5.2, that is

$$\tau^\sigma = \imath^{-1} \circ (\mathrm{Id}_{TM|_V} - \sigma \circ \pi).$$

Observe that $\tau^\sigma \circ \sigma = \imath^{-1} \circ (\sigma - \sigma) = 0$, hence $\mathrm{im}(\sigma) \subset \ker(\tau^\sigma)$. Moreover $\ker(\tau^\sigma) = \ker(\mathrm{Id}_{TM|_V} - \sigma \circ \pi)$ hence $\mathrm{im}(\sigma) = \ker(\tau^\sigma)$. Let now $\mathfrak{U}$ be a splitting atlas adapted to V over M, which exists by Proposition 2.5.2. For any local holomorphic chart $(U, z) \in \mathfrak{U}$ such that $U \cap V \neq \emptyset$ the set $\{\partial z^1, \dots, \partial z^k\}$ defined as in (2.5.2) is a local holomorphic frame of N_V on $U \cap V$. In general

$$\sigma\left(\partial z^j\right) \neq \left.\frac{\partial}{\partial z^j}\right|_V$$

for any $j = 1, \dots, k$, hence the following definition is justified.

Definition 2.5.5. Let $\sigma : N_V \to TM|_V$ be a splitting morphism of V into M and let $\mathfrak{U}$ be a splitting atlas adapted to V. If for any chart $(U, z) \in \mathfrak{U}$ such that $U \cap V \neq \emptyset$ we have that

$$\sigma\left(\partial z^j\right) = \left.\frac{\partial}{\partial z^j}\right|_V \tag{2.5.3}$$

for any $j = 1, \dots, k$, then we say that $\mathfrak{U}$ is a *splitting atlas adapted to V and σ*. If moreover there are an open neighborhood $W \subset M$ of V and a local biholomorphism $\Phi : W \to M$ and $\mathfrak{U}$ is adapted also to Φ, then we say that $\mathfrak{U}$ is a *splitting atlas adapted to V, Φ and σ*.

Proposition 2.5.6. *Let M be a complex manifold of dimension* n *and let $V \subset M$ be a complex submanifold of codimension* $0 < \mathrm{k} < \mathrm{n}$. *Then*

(i) for any splitting morphism $\sigma : N_V \to TM|_V$ there is always a splitting atlas adapted to V and σ.

(ii) if moreover there are an open neighborhood $W \subset M$ of V and a local biholomorphism $\Phi : W \to M$ then there is always a splitting atlas adapted to V, Φ and σ.

Proof. (i) Let $\sigma : N_V \to TM|_V$ be a splitting morphism, $\mathfrak{U}$ a splitting atlas adapted to V (which exists by Proposition 2.5.2) and $\sigma_{\mathfrak{U}} : N_V \to TM|_V$ the splitting morphism induced by $\mathfrak{U}$ as in Remark 2.5.3. Since $\pi \circ (\sigma - \sigma_{\mathfrak{U}}) = 0$, the image of the morphism $\sigma - \sigma_{\mathfrak{U}}$ lies into TV, consequently if $(U, z) \in \mathfrak{U}$ is any local holomorphic chart such that $U \cap V \neq \emptyset$ and $\{\partial z^1, \ldots, \partial z^k\}$ is the local holomorphic frame of N_V on $U \cap V$ defined as in (2.5.2), then

$$(\sigma - \sigma_{\mathfrak{U}})\left(\partial z^j\right) = \sum_{i=k+1}^{n} \alpha_j^i \left.\frac{\partial}{\partial z^i}\right|_V \tag{2.5.4}$$

for any $j = 1, \ldots, k$, where $\alpha_j^i \in \mathcal{O}_V(U \cap V)$. Actually, we can consider these α_j^i as holomorphic functions defined on U not depending on $z^1, \ldots, z^k$. Thus for any $(U, z) \in \mathfrak{U}$ we can replace the coordinates z with the new ones $z_\sigma = (z_\sigma^1, \ldots, z_\sigma^n)$ defined by $z_\sigma^j = z^j$ for any $j = 1, \ldots, k$ and $z_\sigma^i = z^i - \sum_{j=1}^k \alpha_j^i z^j$ for any $i = k+1, \ldots, n$. Then we get a new atlas $\mathfrak{U}_\sigma = \{(U, z_\sigma)\}$ adapted to V which turns out to be a splitting atlas adapted to V and σ, as showed in the following.

Let $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ be any local holomorphic charts such that $U \cap \hat{U} \cap V \neq \emptyset$, let α_j^i, β_j^i be the holomorphic functions arising as in (2.5.4) respectively for (U, z) and $(\hat{U}, \hat{z})$, and let $(U, z_\sigma), (\hat{U}, \hat{z}_\sigma) \in \mathfrak{U}_\sigma$ be the resulting new local holomorphic charts. By the very definition of the new coordinates

$$\begin{aligned}
\left.\frac{\partial \hat{z}_\sigma^i}{\partial z_\sigma^j}\right|_V &= \left.\frac{\partial \hat{z}_\sigma^i}{\partial z^j}\right|_V + \sum_{r=k+1}^{n} \left.\frac{\partial \hat{z}_\sigma^i}{\partial z^r}\right|_V \left.\frac{\partial z^r}{\partial z_\sigma^j}\right|_V \\
&= \left.\frac{\partial \hat{z}^i}{\partial z^j}\right|_V - \sum_{t=1}^{k} \beta_t^i \left.\frac{\partial \hat{z}^t}{\partial z^j}\right|_V + \sum_{r=k+1}^{n} \alpha_j^r \left.\frac{\partial \hat{z}^i}{\partial z^r}\right|_V
\end{aligned} \tag{2.5.5}$$

for any $i = k+1, \ldots, n$ and $j = 1, \ldots, k$. Since

$$\frac{\partial}{\partial z^j} = \sum_{\ell=1}^{k} \frac{\partial \hat{z}^\ell}{\partial z^j} \frac{\partial}{\partial \hat{z}^\ell} + \sum_{\ell=k+1}^{n} \frac{\partial \hat{z}^\ell}{\partial z^j} \frac{\partial}{\partial \hat{z}^\ell}$$

we have that

$$(\sigma - \sigma_{\mathfrak{U}})\left(\partial z^j\right) = \sum_{r=k+1}^{n} \alpha_j^r \left.\frac{\partial}{\partial z^r}\right|_V = \sum_{i=k+1}^{n} \sum_{r=k+1}^{n} \alpha_j^r \left.\frac{\partial \hat{z}^i}{\partial z^r}\right|_V \left.\frac{\partial}{\partial \hat{z}^i}\right|_V$$

and

$$(\sigma - \sigma_{\mathfrak{U}})\left(\partial z^j\right) = \sum_{t=1}^{k} \left.\frac{\partial \hat{z}^t}{\partial z^j}\right|_V (\sigma - \sigma_{\mathfrak{U}})\left(\partial \hat{z}^t\right) = \sum_{i=k+1}^{n} \sum_{t=1}^{k} \beta_t^i \left.\frac{\partial \hat{z}^t}{\partial z^j}\right|_V \left.\frac{\partial}{\partial \hat{z}^i}\right|_V,$$

for any $j = 1, \dots, k$. Consequently

$$\sum_{r=k+1}^{n} \alpha_j^r \left.\frac{\partial \hat{z}^i}{\partial z^r}\right|_V = \sum_{t=1}^{k} \beta_t^i \left.\frac{\partial \hat{z}^t}{\partial z^j}\right|_V$$

for any $i = k+1, \dots, n$ and $j = 1, \dots, k$, then recalling property (2.5.1) we have by (2.5.5) that

$$\left.\frac{\partial \hat{z}_\sigma^i}{\partial z_\sigma^j}\right|_V \equiv 0$$

for any $i = k+1, \dots, n$ and $j = 1, \dots, k$. Then $\mathfrak{U}_\sigma$ is a splitting atlas adapted to V. Lastly, observe that

$$\frac{\partial}{\partial z_\sigma^j} = \frac{\partial}{\partial z^j} + \sum_{i=k+1}^{n} \frac{\partial z^i}{\partial z_\sigma^j} \frac{\partial}{\partial z^i}$$

for any $j = 1, \dots, k$, hence $\partial z_\sigma^j = \partial z^j$ and

$$\sigma\left(\partial z_\sigma^j\right) = \sigma\left(\partial z^j\right) = \left.\frac{\partial}{\partial z^j}\right|_V + \sum_{i=k+1}^{n} \alpha_j^i \left.\frac{\partial}{\partial z^i}\right|_V = \left.\frac{\partial}{\partial z_\sigma^j}\right|_V \tag{2.5.6}$$

for any $j = 1, \dots, k$. Then $\mathfrak{U}_\sigma$ is adapted also to σ.

(*ii*) Just shrink the local holomorphic charts $(U, z) \in \mathfrak{U}$. $\square$

Now come back to the pair (f, g). It is clear that *if $\sigma : N_{S'} \to TM|_{S'}$ is a splitting morphism of S' into M then we can define the* 1*-dimensional holomorphic possibly singular foliation on S'*

$$\mathscr{D}_{f,g}^\sigma = \tau^\sigma \circ \mathscr{D}_{f,g} : N_{S'}^{\otimes \nu} \longrightarrow TS',$$

whose tangent bundle is again $N_{S'}^{\otimes \nu}$. The singular set may be instead larger than $\mathrm{Sing}(f, g)$ since it depends also on σ (it may be even the whole S', see Example 2.5.7). If $\mathfrak{U}$ is a splitting atlas adapted to S', g and σ (which exists by Proposition 2.5.6) and $(U, z) \in \mathfrak{U}$ is any local holomorphic chart such that $U \cap S' \neq \emptyset$, then

$$\tau^\sigma\left(\left.\frac{\partial}{\partial z^1}\right|_{S'}\right) = \left(\tau^\sigma \circ \sigma\right)\left(\partial z^1\right) = 0$$

and

$$X_{f,g}^\sigma = \mathscr{D}_{f,g}^\sigma\left((\partial z^1)^\nu\right) = \sum_{j=2}^{n} h^j|_{S'} \left.\frac{\partial}{\partial z^j}\right|_{S'} \tag{2.5.7}$$

is a local generator of $\mathscr{D}^\sigma_{f,g}$ on $U \cap S'$, where the holomorphic functions h^j are defined as in (2.2.7). We call (2.5.7) the *canonical local generator of* $\mathscr{D}^\sigma_{f,g}$ *associated to* (U, z).

Note that if S' splits into M and the pair (f, g) is tangential along S at the same time then $\mathscr{D}^\sigma_{f,g} = \mathscr{D}_{f,g}$ and $X^\sigma_{f,g} = X_{f,g}$ for any splitting morphism σ since $\tau^\sigma \circ \imath = \mathrm{Id}_{TS'}$.

Example 2.5.7. In this example we want to furnish a family of pairs of holomorphic self-maps coinciding on a splitting hypersurface which are not tangential along it. The holomorphic foliations induced by these pairs on the hypersurface unfortunately turn out to be identically zero.

Let S be a connected complex manifold of dimension $n-1$ and suppose it is the base of a holomorphic line bundle $\pi : M \to S$. As remarked in Example 2.5.4, S is a splitting hypersurface of M via the zero section. Let $f : M \to M$ be any endomorphism of the line bundle such that $f \neq \mathrm{Id}_M$, then we can consider the pair (f, Id_M) of holomorphic self-maps of M coinciding on S. Consider the atlas $\mathfrak{U}$ of M induced by a trivializing atlas of S as in Example 2.5.4, let $(U, z) \in \mathfrak{U}$ be any local holomorphic charts and set $f^j = z^j \circ f$ for any $j = 1, \ldots, n$. Since f sends any fiber onto itself, it must be

$$f^1 = \Phi(z^2, \ldots, z^n) z^1$$

for a holomorphic function $\Phi(z^2, \ldots, z^n)$ and

$$f^j = z^j$$

for any $j = 2, \ldots, n$. Since z^1 is a local generator of $\mathcal{I}_S$ and $\Phi \neq 1$ (as $f \neq \mathrm{Id}_M$), it follows that $\nu_{f,\mathrm{Id}_M}(z^1) = 1$, while $\nu_{f,\mathrm{Id}_M}(z^j) = +\infty$ for any $j = 2, \ldots, n$. As a consequence $\nu_{f,\mathrm{Id}_M} = 1$ by Lemma 2.1.2 and (f, Id_M) is *not tangential* along S by Lemma 2.4.3.

The canonical section $\mathscr{D}_{f,\mathrm{Id}_M} : N_S \to TM|_S$ is locally given by

$$\mathscr{D}_{f,\mathrm{Id}_M} = (\Phi - 1)\, \partial^* z^1 \otimes \left.\frac{\partial}{\partial z^1}\right|_{U \cap S},$$

then $\mathrm{Sing}(f, \mathrm{Id}_M)$ can be identified via the zero section with the analytic subset $\{q \in S \text{ s.t. } f|_q = \mathrm{Id}\}$. Let now $\sigma : N_S \to TM|_S$ be a splitting morphism of S into M such that $\mathfrak{U}$ is adapted to it. We may define the holomorphic foliation $\mathscr{D}^\sigma_{f,\mathrm{Id}_M} = \tau^\sigma \circ \mathscr{D}_{f,\mathrm{Id}_M}$ as explained above, but note that

$$\tau^\sigma \left(\left.\frac{\partial}{\partial z^1}\right|_S \right) = (\tau^\sigma \circ \sigma)\left(\partial z^1\right) = 0$$

hence $\mathscr{D}^{\sigma}_{f,\mathrm{Id}_M} \equiv 0$ (roughly speaking, $\mathscr{D}_{f,\mathrm{Id}_M}$ is transversal to S so when we project it over S by τ^{σ} we get a foliation singular everywhere).

Remark 2.5.8. Observe that in order to have a foliation on S' when (f, g) is not tangential along S, actually we just need that $S' - \mathrm{Sing}(f, g)$ splits into M. In fact if $\sigma : N_{S'-\mathrm{Sing}(f,g)} \to TM|_{S'-\mathrm{Sing}(f,g)}$ is a splitting morphism we can define the 1-dimensional holomorphic possibly singular foliation

$$\tau^{\sigma} \circ \mathscr{D}_{f,g} : N^{\otimes\nu}_{S'-\mathrm{Sing}(f,g)} \longrightarrow TS'|_{S'-\mathrm{Sing}(f,g)}$$

over $S' - \mathrm{Sing}(f, g)$, and then extend it to $\mathrm{Sing}(f, g)$ by zero.

Remark 2.5.9. In this chapter we have assumed $S \subset M$ of codimension 1 but actually everything we have done up to now can be easily adapted to S of any codimension $0 < k < n$. Instead we need that S is a hypersurface from Chapter 3 onward.

Chapter 3

The first index theorem

Suppose in this chapter that $S \subset M$ is non-singular and let (f, g) be a pair as in Chapter 2. If (f, g) is tangential along S then set $\mathscr{D} = \mathscr{D}_{f,g}$, let $\mathfrak{U}$ denote an atlas adapted to S and g, and for any $(U, z) \in \mathfrak{U}$ set $X = X_{f,g}$. Instead, if S splits into M and σ is a splitting morphism of S into M then set $\mathscr{D} = \mathscr{D}^{\sigma}_{f,g}$, let $\mathfrak{U}$ denote a splitting atlas adapted to S, g and σ, and for any $(U, z) \in \mathfrak{U}$ set $X = X^{\sigma}_{f,g}$. In both cases set $S^0 = S - \mathrm{Sing}(\mathscr{D})$, moreover set $\nu = \nu_{f,g}$ to ease the notation.

We show that if (f, g) is tangential along S or S splits into M then we can define a partial holomorphic connection along $N_{S^0}^{\otimes\nu} \subset TS^0$ on $N^0_{\mathscr{D}}$, and thanks to it we obtain a *Baum-Bott-type index theorem*. Then we find an explicit formula for the computation of the Baum-Bott residues at isolated singular points of $\mathscr{D}$.

3.1 First partial holomorphic connection

Assume that (f, g) is tangential along S or that S splits into M, and fix a splitting morphism $\sigma : N_S \to TM|_S$ in the latter case. We have showed in Chapter 2 that in both cases (f, g) induces a foliation $\mathscr{D} : N_S^{\otimes\nu} \to TS$. Let $\mathcal{N}_{S^0}^{\otimes\nu}$ denote the sheaf of holomorphic sections of $N_{S^0}^{\otimes\nu}$ and recall the notation introduced in Section 2.3. Then we can define a partial holomorphic connection along $N_{S^0}^{\otimes\nu} \subset TS^0$ on $N^0_{\mathscr{D}}$ by setting

$$\begin{aligned} \mathcal{N}^0_{\mathscr{D}} &\xrightarrow{\delta^{bb}} \left(\mathcal{N}_{S^0}^{\otimes\nu}\right)^* \otimes \mathcal{N}^0_{\mathscr{D}} \\ w &\longrightarrow \delta^{bb}(w) \text{ s.t. } \delta^{bb}_v(w) = \pi\left([v, \tilde{w}]\right), \end{aligned} \tag{3.1.1}$$

for any $w \in \mathcal{N}^0_{\mathscr{D}}$ and $v \in \mathcal{N}^{\otimes\nu}_{S^0} \subset \mathcal{T}_{S^0}$, where $\pi : TS^0 \to N^0_{\mathscr{D}}$ is the obvious projection and $\tilde{w} \in \mathcal{T}_{S^0}$ is any holomorphic vector field such that $\pi(\tilde{w}) = w$. Observe that definition (3.1.1) does not depend on the choice of $\tilde{w}$, in fact if $\bar{w} \in \mathcal{T}_{S^0}$ is another holomorphic vector field such that $\pi(\bar{w}) = w$ then $\tilde{w} - \bar{w} \in \mathcal{N}^{\otimes\nu}_{S^0}$, hence $[v, \tilde{w} - \bar{w}] \in \mathcal{N}^{\otimes\nu}_{S^0}$ and consequently $\pi([v, \tilde{w} - \bar{w}]) = 0$. Moreover, δ^{bb} is flat since by the Jacobi identity

$$\begin{aligned}
\delta^{bb}_u \delta^{bb}_v(w) - \delta^{bb}_v \delta^{bb}_u(w) &= \pi\left([u, [v, \tilde{w}]]\right) - \pi\left([v, [u, \tilde{w}]]\right) \\
&= \pi\left([u, [v, \tilde{w}]] - [v, [u, \tilde{w}]]\right) \\
&= \pi\left([[u, v], \tilde{w}]\right) = \delta^{bb}_{[u,v]}(w)
\end{aligned}$$

for any $w \in \mathcal{N}^0_{\mathscr{D}}$ and $u, v \in \mathcal{N}^{\otimes\nu}_{S^0} \subset \mathcal{T}_{S^0}$.

3.2 A Baum-Bott-type index theorem

Consider the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow N^{\otimes\nu}_{S^0} \xrightarrow{\mathscr{D}|_{S^0}} TS^0 \xrightarrow{\pi} N^0_{\mathscr{D}} \longrightarrow 0 \tag{3.2.1}$$

and let ∇^{bb} be any δ^{bb}-connection on $N^0_{\mathscr{D}}$. By (i) of Lemma 1.3.3 we can take C^∞ connections ∇^1_0 and ∇^2_0 respectively on $N^{\otimes\nu}_{S^0}$ and TS^0 such that the triple $(\nabla^1_0, \nabla^2_0, \nabla^{bb})$ is compatible with (3.2.1). Then setting $\nabla^\bullet_0 = (\nabla^1_0, \nabla^2_0)$ we have by (i) of Lemma 1.3.3 and Theorem 1.4.9 that

$$\varphi\left(\nabla^\bullet_0\right) = \varphi(\nabla^{bb}) = 0 \tag{3.2.2}$$

for any homogeneous symmetric polynomial φ of degree $n-1$. Let now $\mathrm{Sing}(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set of $\mathscr{D}$, which is an analytic subvariety of S since it is the zero set of a holomorphic section of a holomorphic vector bundle over S. For any λ let $\tilde{U}_\lambda \subset S$ be an open subset such that $\tilde{U}_\lambda \supset \Sigma_\lambda$ and $\tilde{U}_\lambda \cap \tilde{U}_{\lambda'} = \emptyset$ whenever $\lambda \neq \lambda'$. Set $\tilde{U}_1 = \sqcup_\lambda \tilde{U}_\lambda$ and let ∇^1_1 and ∇^2_1 be any C^∞ connections respectively on $N^{\otimes\nu}_S|_{\tilde{U}_1}$ and $TS|_{\tilde{U}_1}$. Then set $\nabla^\bullet_1 = (\nabla^1_1, \nabla^2_1)$ and consider the open covering $\tilde{\mathcal{U}} = \{S^0, \tilde{U}_1\}$ of S. By (3.2.2) it follows that

$$\varphi\left(TS - N^{\otimes\nu}_S\right) = \left[\left(0, \varphi(\nabla^\bullet_1), \varphi(\nabla^\bullet_0, \nabla^\bullet_1)\right)\right] \in H^{2(n-1)}\left(\tilde{\mathcal{U}}, S^0, \mathbb{C}\right)$$

for any homogeneous symmetric polynomial φ of degree $n-1$, that is for any such polynomial φ we can localize $\varphi(TS - N_S^{\otimes\nu})$ at $\mathrm{Sing}(\mathscr{D})$. Moreover, the localization does not depend on the choice of the C^∞ connections, as stated by the following lemma.

Lemma 3.2.1. *Let $\bar{\nabla}^{bb}$ be another δ^{bb}-connection on $N^0_{\mathscr{D}}$, let $(\bar{\nabla}^1_0, \bar{\nabla}^2_0, \bar{\nabla}^{bb})$ be another triple compatible with (3.2.1) and set $\bar{\nabla}^\bullet_0 = (\bar{\nabla}^1_0, \bar{\nabla}^2_0)$. Moreover, let $\bar{\nabla}^1_1$ and $\bar{\nabla}^2_1$ be other C^∞ connections respectively on $N_S^{\otimes\nu}|_{\tilde{U}_1}$ and $TS|_{\tilde{U}_1}$ and set $\bar{\nabla}^\bullet_1 = (\bar{\nabla}^1_1, \bar{\nabla}^2_1)$. Then for any homogeneous symmetric polynomial φ of degree $n-1$*

$$\Big[\big(0, \varphi(\nabla^\bullet_1), \varphi(\nabla^\bullet_0, \nabla^\bullet_1)\big)\Big] = \Big[\big(0, \varphi(\bar{\nabla}^\bullet_1), \varphi(\bar{\nabla}^\bullet_0, \bar{\nabla}^\bullet_1)\big)\Big]$$

in $H^{2(n-1)}(\tilde{\mathcal{U}}, S^0, \mathbb{C})$.

Proof. By property (1.3.1) we have that

$$\varphi(\nabla^\bullet_1) - \varphi(\bar{\nabla}^\bullet_1) = \mathrm{d}\varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_1), \tag{3.2.3}$$

moreover that

$$\varphi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_1) - \varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_0) - \mathrm{d}\varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_0, \nabla^\bullet_1)$$

and

$$\begin{aligned}\varphi(\bar{\nabla}^\bullet_0, \bar{\nabla}^\bullet_1) &= \varphi(\nabla^\bullet_0, \bar{\nabla}^\bullet_1) + \varphi(\bar{\nabla}^\bullet_0, \nabla^\bullet_0) + \mathrm{d}\varphi(\bar{\nabla}^\bullet_0, \nabla^\bullet_0, \bar{\nabla}^\bullet_1)\\ &= -\varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_0) + \mathrm{d}\varphi(\bar{\nabla}^\bullet_0, \nabla^\bullet_0, \bar{\nabla}^\bullet_1),\end{aligned}$$

where in the last equality we use that $\varphi(\bar{\nabla}^\bullet_0, \nabla^\bullet_0) = \varphi(\bar{\nabla}^{bb}, \nabla^{bb}) = 0$ by (ii) of Lemma 1.3.3 and Theorem 1.4.9. Hence

$$\varphi(\nabla^\bullet_0, \nabla^\bullet_1) - \varphi(\bar{\nabla}^\bullet_0, \bar{\nabla}^\bullet_1) = \varphi(\bar{\nabla}^\bullet_1 \nabla^\bullet_1) - \mathrm{d}\omega, \tag{3.2.4}$$

with $\omega = \varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_0, \nabla^\bullet_1) + \varphi(\bar{\nabla}^\bullet_0, \nabla^\bullet_0, \bar{\nabla}^\bullet_1)$. By (3.2.3) and (3.2.4) and recalling the definition of the Čech-de Rham differential D we then have

$$\big(0, \varphi(\nabla^\bullet_1), \varphi(\nabla^\bullet_0, \nabla^\bullet_1)\big) - \big(0, \varphi(\bar{\nabla}^\bullet_1), \varphi(\bar{\nabla}^\bullet_0, \bar{\nabla}^\bullet_1)\big) = D\big(0, \varphi(\bar{\nabla}^\bullet_1, \nabla^\bullet_1), \omega\big).$$

□

Suppose now S compact, then $\mathrm{Sing}(\mathscr{D}) \subset S$ is also compact and we can assume without loss of generality that $\tilde{U}_1$ is a regular neighborhood of $\mathrm{Sing}(\mathscr{D})$. Then by what we said in this chapter and by the commutative diagram (1.9.1) we gain the following index theorem.

Theorem 3.2.2 (Baum-Bott-type index theorem)**.** *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a non-singular compact connected complex hypersurface. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S, with order of coincidence $\nu = \nu_{f,g}$. Assume that*

(i) (f, g) is tangential along S

or

(ii) S splits into M and σ is a splitting morphism of S into M.

In case (i) set $\mathscr{D} = \mathscr{D}_{f,g}$, while in case (ii) set $\mathscr{D} = \mathscr{D}^{\sigma}_{f,g}$ and suppose $\mathscr{D} \neq 0$. Let $Sing(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set of the foliation $\mathscr{D}$. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $Res_\varphi(\mathscr{D}, TS - N_S^{\otimes \nu}, \Sigma_\lambda)$, one for each connected component Σ_λ, such that

$$\sum_\lambda Res_\varphi \left(\mathscr{D}, TS - N_S^{\otimes \nu}, \Sigma_\lambda\right) = \int_S \varphi \left(TS - N_S^{\otimes \nu}\right).$$

We call Theorem 3.2.2 a *Baum-Bott-type index theorem* because Baum and Bott have introduced this kind of residues and the partial holomorphic connection (3.1.1) (see [9] and [8]). Moreover, it basically follows by the Baum-Bott index theorem (see Theorem 0.1.2).

Remark 3.2.3. Suppose that g is a *global* biholomorphism over an open neighborhood of S and recall Remark 2.2.6 and 2.4.6. In this case Theorem 3.2.2 turns out to be [4, Thm.6.4] for the holomorphic self-map $g^{-1} \circ f$.

Note that we may weaken hypothesis (ii) of Theorem 3.2.2 by asking that only $S - \mathrm{Sing}(f, g)$ splits into M, in place of the whole S (see Remark 2.5.8). In [4, Thm.6.4] the authors assume an embedding property on $S - \mathrm{Sing}(f)$ stronger than (ii), but actually it is not necessary. We are going to talk about this embedding property in the next chapter since it will be necessary in Theorem 5.2.2.

3.3 Computation of the Baum-Bott residues

We end the chapter by deriving an explicit formula for the computation of the residues of Theorem 3.2.2 at isolated singular points of $\mathscr{D}$. A reference for this section is [36, Sec.III.7]. In the following let φ be any homogeneous symmetric polynomial of degree $n-1$.

Let ∇^1_0 and ∇^2_0 be C^∞ connections respectively on $N^{\otimes\nu}_{S^0}$ and TS^0 as in Section 3.2 and set $\nabla^\bullet_0 = (\nabla^1_0, \nabla^2_0)$. Let Σ be any connected component of $\mathrm{Sing}(\mathscr{D})$ and let $\tilde{U} \subset S$ be a regular neighborhood of Σ such that $\tilde{U} \cap \mathrm{Sing}(\mathscr{D}) = \Sigma$. Let ∇^1 and ∇^2 be for the moment any C^∞ connections respectively on $N^{\otimes\nu}_{\tilde{U}} = N^{\otimes\nu}_S|_{\tilde{U}}$ and $T\tilde{U} = TS|_{\tilde{U}}$, and set $\nabla^\bullet = (\nabla^1, \nabla^2)$. Lastly, let $\tilde{R} \subset \tilde{U}$ be any compact real C^∞ submanifold of dimension $2(n-1)$ oriented as S such that $\Sigma \subset \mathrm{Int}\tilde{R}$, and let the boundary $\partial\tilde{R}$ be oriented with the orientation induced by $\tilde{R}$. Then by Section 3.2 and 1.9 the residue of $\varphi(TS - N^{\otimes\nu}_S)$ at Σ is

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TS - N^{\otimes\nu}_S, \Sigma\right) = \int_{\tilde{R}} \varphi\left(\nabla^\bullet\right) - \int_{\partial\tilde{R}} \varphi\left(\nabla^\bullet_0, \nabla^\bullet\right).$$

Suppose that Σ is an isolated point, that is $\Sigma = \{p\}$. In this case up to shrinking $\tilde{U}$ we can assume that $\tilde{U} = U \cap S$ for a local holomorphic chart $(U, z) \in \mathfrak{U}$ centered at p. Thus $(\partial z^1)^\nu$ is a holomorphic generator of $N^{\otimes\nu}_{\tilde{U}}$ and $\{\frac{\partial}{\partial z^2}|_{\tilde{U}}, \ldots, \frac{\partial}{\partial z^n}|_{\tilde{U}}\}$ is a holomorphic frame of $T\tilde{U}$, and consequently we can choose ∇^1 and ∇^2 trivial respect to them, that is

$$\nabla^1\left((\partial z^1)^\nu\right) = 0$$

and

$$\nabla^2\left(\left.\frac{\partial}{\partial z^j}\right|_{\tilde{U}}\right) = 0$$

for any $j = 2, \ldots, n$. With this choice $c(\nabla^\bullet) = c(\nabla^2) \wedge c(\nabla^1)^{-1} = 1 \wedge 1 = 1$, then $\varphi(\nabla^\bullet) = 0$ and the residue of $\varphi(TS - N^{\otimes\nu}_S)$ at p becomes

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TS - N^{\otimes\nu}_S, p\right) = -\int_{\partial\tilde{R}} \varphi\left(\nabla^\bullet_0, \nabla^\bullet\right). \tag{3.3.1}$$

Now in what follows we are going to mimic the strategy adopted in the proof of [36, Thm.III.5.5] (see also [25, Sec.5]). Set $\tilde{U}_0 = \tilde{U} - \{p\}$ to ease the notation. The canonical local generator X of $\mathscr{D}$ associated to (U, z) generates the holomorphic line subbundle $N^{\otimes\nu}_{\tilde{U}_0} \subset T\tilde{U}_0$, and it induces two flat partial holomorphic connections δ_0^1 and δ_0^2 along $N_{\tilde{U}_0}^{\otimes\nu} \subset T\tilde{U}_0$ on $N_{\tilde{U}_0}^{\otimes\nu}$ and $T\tilde{U}_0$. The connection δ_0^1 is defined by setting

$$\left(\delta_0^1\right)_{\alpha X}(\omega) = \alpha \mathscr{D}^{-1}\big([X, \mathscr{D}(\omega)] \big)$$

for any $\omega \in \mathcal{N}_{\tilde{U}_0}^{\otimes\nu}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$, while δ_0^2 is defined by setting

$$\left(\delta_0^2\right)_{\alpha X}(w) = \alpha\,[X, w]$$

for any $w \in \mathcal{T}_{\tilde{U}_0}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$. Clearly the flatness comes from the Jacoby identity, moreover by construction the triple $(\delta_0^1, \delta_0^2, \delta^{bb})$ is compatible with the short exact sequence

$$0 \longrightarrow N_{\tilde{U}_0}^{\otimes\nu} \xrightarrow{\mathscr{D}|_{\tilde{U}_0}} T\tilde{U}_0 \xrightarrow{\pi} N_{\mathscr{D}}^0\big|_{\tilde{U}_0} \longrightarrow 0,$$

which is (3.2.1) restricted to $\tilde{U}_0$. Then by Lemma 1.4.3 we can assume ∇_0^1 and ∇_0^2 respectively a δ_0^1-connection and a δ_0^2-connection (the reason of this choice will be clear in the proof of Lemma 3.3.2). Observe that $\delta_0^1((\partial z^1)^\nu) = 0$ then ∇^1 is a δ_0^1-connection.

Remark 3.3.1. Observe that the C^∞ connections ∇_0^1 and ∇_0^2 just chosen are defined only on $\tilde{U}_0$ and not on the whole S^0. Anyway, this is not a problem since for the computation of the residue we integrate over $\partial\tilde{R} \subset \tilde{U}_0$ (it is a local issue).

Let $L^j \subset T\tilde{U}$ be the holomorphic line subbundle generated by $\frac{\partial}{\partial z^j}|_{\tilde{U}}$, for any $j = 2, \ldots, n$. There are natural flat partial holomorphic connections $\delta^{1,j}$ along L^j on $N_{\tilde{U}}^{\otimes\nu}$ defined by setting

$$\left(\delta^{1,j}\right)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}}}\left(\beta(\partial z^1)^\nu\right) = \alpha \frac{\partial \beta}{\partial z^j}\left(\partial z^1\right)^\nu$$

for any $\alpha, \beta \in \mathcal{O}_{\tilde{U}}$ and $j = 2, \ldots, n$. Moreover, there are natural flat partial holomorphic connections $\delta^{2,j}$ along L^j on $T\tilde{U}$ defined by setting

$$\left(\delta^{2,j}\right)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}}}(w) = \alpha \left[\frac{\partial}{\partial z^j}\bigg|_{\tilde{U}}, w \right]$$

for any $\alpha \in \mathcal{O}_{\tilde{U}}$, $w \in \mathcal{T}_{\tilde{U}}$ and $j = 2, \ldots, n$. Again, the flatness comes from the Jacoby identity. Note that $\delta^{1,j}((\partial z^1)^\nu) = 0$ then ∇^1 is a $\delta^{1,j}$-connection, for any $j = 2, \ldots, n$. Furthermore $\delta^{2,j}(\frac{\partial}{\partial z^k}|_{\tilde{U}}) = 0$ for any $k = 2, \ldots, n$ hence ∇^2 is a $\delta^{2,j}$-connection, for any $j = 2, \ldots, n$.

Let us introduce now the open covering $\tilde{\mathcal{U}}_0 = \{\tilde{U}^2, \ldots, \tilde{U}^n\}$ of $\tilde{U}_0$ defined by setting

$$\tilde{U}^j = \left\{ q \in \tilde{U}_0 \text{ s.t. } h^j(q) \neq 0 \right\}$$

for any $j = 2, \ldots, n$, where the holomorphic functions $h^j \in \mathcal{O}_M$ are the ones defined in (2.2.7). This is a covering of $\tilde{U}_0$ since $X = \sum_{j=2}^n h^j|_{\tilde{U}} \frac{\partial}{\partial z^j}|_{\tilde{U}}$ and then $\{h^2 = \cdots = h^n = 0\} \cap \tilde{U} = \{p\}$ by hypothesis. Observe that

$$\left\{ \left.\frac{\partial}{\partial z^2}\right|_{\tilde{U}^j}, \ldots, \widehat{\left.\frac{\partial}{\partial z^j}\right|_{\tilde{U}^j}}, \ldots, \left.\frac{\partial}{\partial z^n}\right|_{\tilde{U}^j}, X\big|_{\tilde{U}^j} \right\}$$

is a holomorphic frame of $T\tilde{U}^j = TS|_{\tilde{U}^j}$ for any $j = 2, \ldots, n$, thus we can define the C^∞ connections ∇_j^2 on $T\tilde{U}^j$ by asking that are δ_0^2-connections and by setting

$$(\nabla_j^2)_{\alpha \frac{\partial}{\partial z^k}\big|_{\tilde{U}^j}}(w) = \alpha \left[\left.\frac{\partial}{\partial z^k}\right|_{\tilde{U}^j}, w \right]$$

for any $\alpha \in \mathcal{O}_{\tilde{U}^j}$, $w \in \mathcal{T}_{\tilde{U}^j}$ and $k = 2, \ldots, n$ such that $k \neq j$. Moreover, we can define the C^∞ connections ∇_j^1 on $N_{\tilde{U}^j}^{\otimes \nu}$ simply by setting

$$\nabla_j^1 = \nabla^1\big|_{\tilde{U}^j}$$

for any $j = 2, \ldots, n$. Note that in general

$$(\nabla_j^2)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}^j}}(w) \neq \alpha \left[\left.\frac{\partial}{\partial z^j}\right|_{\tilde{U}^j}, w \right]$$

hence ∇_j^2 is not a $\delta^{2,j}$-connection, on the contrary it is a $\delta^{2,k}$-connection for any $k = 2, \ldots, n$ such that $k \neq j$. Moreover, ∇_j^1 is a δ_0^1-connection and a $\delta^{1,k}$-connection for any $k = 2, \ldots, n$ since ∇^1 is.

Set now $\nabla_j^\bullet = (\nabla_j^1, \nabla_j^2)$ for any $j = 2, \ldots, n$ and consider the Čech-de Rham cocycle $\tau \in C_{\tilde{\mathcal{U}}_0}^{2(n-1)-2}$ defined by

$$\tau_{i_1 \cdots i_k} = \varphi\left(\nabla_0^\bullet, \nabla^\bullet, \nabla_{i_1}^\bullet, \ldots, \nabla_{i_k}^\bullet\right) \in \Omega^{2(n-1)-k-1}\left(\tilde{U}^{i_1} \cap \cdots \cap \tilde{U}^{i_k}\right),$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n-1$.

Lemma 3.3.2.

(*i*) $(D\tau)_i = -\varphi(\nabla_0^\bullet, \nabla^\bullet)$ *for any* $i = 2, \ldots, n$.

(*ii*) $(D\tau)_{i_1 \cdots i_k} = 0$ *for any* $2 \leq i_1 < \cdots < i_k \leq n$ *and* $k = 2, \ldots, n-2$.

(*iii*) $(D\tau)_{2 \cdots n} = -\varphi(\nabla^\bullet, \nabla_2^\bullet, \ldots, \nabla_n^\bullet)$.

Proof. Observe preliminarily that by Theorem 1.4.10

$$\varphi\left(\nabla^\bullet_0, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right) = 0 \tag{3.3.2}$$

for any $2 \le i_1 < \cdots < i_k \le n$ and $k = 1, \dots, n-1$, since by what we have remarked the connections $\nabla^1_0, \nabla^1_{i_1}, \dots, \nabla^1_{i_k}$ are all δ^1_0-connections and $\nabla^2_0, \nabla^2_{i_1}, \dots, \nabla^2_{i_k}$ are all δ^2_0-connections. Moreover, again by Theorem 1.4.10

$$\varphi\left(\nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right) = 0 \tag{3.3.3}$$

for any $2 \le i_1 < \cdots < i_k \le n$ and $k = 1, \dots, n-2$, since by what we have remarked the connections $\nabla^1, \nabla^1_{i_1}, \dots, \nabla^1_{i_k}$ are all $\delta^{1,j}$-connections (for any $j = 2, \dots, n$) and there is an $i \in \{2, \dots, n\} - \{i_1, \dots, i_k\}$ such that $\nabla^2, \nabla^2_{i_1}, \dots, \nabla^2_{i_k}$ are all $\delta^{2,i}$-connections.

(i) By definition of D and property (1.3.1)

$$\begin{aligned}(D\tau)_i =& \mathrm{d}\tau_i = \mathrm{d}\varphi\left(\nabla^\bullet_0, \nabla^\bullet, \nabla^\bullet_i\right)\\ =& -\varphi\left(\nabla^\bullet, \nabla^\bullet_i\right) + \varphi\left(\nabla^\bullet_0, \nabla^\bullet_i\right) - \varphi\left(\nabla^\bullet_0, \nabla^\bullet\right)\end{aligned}$$

for any $i = 2, \dots, n$. Then by (3.3.2) and (3.3.3) we are done.

(ii), (iii) By definition of D

$$\begin{aligned}(D\tau)_{i_1\cdots i_k} =& (-1)^{k-1}\mathrm{d}\tau_{i_1\cdots i_k} + \sum_{\ell=1}^{k}(-1)^{\ell-1}\tau_{i_1\cdots\widehat{i_\ell}\cdots i_k}\\ =& (-1)^{k-1}\mathrm{d}\varphi\left(\nabla^\bullet_0, \nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right)\\ &+ \sum_{\ell=1}^{k}(-1)^{\ell-1}\varphi\left(\nabla^\bullet_0, \nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \widehat{\nabla^\bullet_{i_\ell}}, \dots, \nabla^\bullet_{i_k}\right),\end{aligned}$$

for any $2 \le i_1 < \cdots < i_k \le n$ and $k = 2, \dots, n-1$. By property (1.3.1) we have that

$$\begin{aligned}&(-1)^{k-1}\mathrm{d}\varphi\left(\nabla^\bullet_0, \nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right)\\ &\quad = -\varphi\left(\nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right) + \varphi\left(\nabla^\bullet_0, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}\right)\\ &\qquad - \sum_{\ell=1}^{k}(-1)^{\ell-1}\varphi\left(\nabla^\bullet_0, \nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \widehat{\nabla^\bullet_{i_\ell}}, \dots, \nabla^\bullet_{i_k}\right).\end{aligned}$$

Then $(D\tau)_{i_1\cdots i_k} = -\varphi(\nabla^\bullet, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}) + \varphi(\nabla^\bullet_0, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k})$ for any $2 \le i_1 < \cdots < i_k \le n$ and $k = 2, \dots, n-1$, and by (3.3.2) and (3.3.3) we are done. $\square$

Consider the open covering $\{\tilde{U}^2 \cap \partial\tilde{R}, \ldots, \tilde{U}^n \cap \partial\tilde{R}\}$ of $\partial\tilde{R}$ and a system of honey-comb cells $\{\partial\tilde{R}^2, \ldots, \partial\tilde{R}^n\}$ adapted to it. Set $\partial\tilde{R}^{2\cdots n} = \partial\tilde{R}^2 \cap \cdots \cap \partial\tilde{R}^n$, then by Lemma 3.3.2 and recalling the Čech-de Rham integration (1.7.1) we have that

$$0 = \int_{\partial\tilde{R}} D\tau = -\sum_{j=2}^{n} \int_{\partial\tilde{R}^j} \varphi\left(\nabla_0^\bullet, \nabla^\bullet\right) - \int_{\partial\tilde{R}^{2\cdots n}} \varphi\left(\nabla^\bullet, \nabla_2^\bullet, \cdots, \nabla_n^\bullet\right),$$

and consequently by (3.3.1)

$$\operatorname{Res}_\varphi\left(\mathscr{D}, TS - N_S^{\otimes\nu}, p\right) = \int_{\partial\tilde{R}^{2\cdots n}} \varphi\left(\nabla^\bullet, \nabla_2^\bullet, \ldots, \nabla_n^\bullet\right). \tag{3.3.4}$$

Recall by Section 1.3 that $\varphi(\nabla^\bullet, \nabla_2^\bullet, \ldots, \nabla_n^\bullet) = I_*(\varphi(\nabla_{(t)}^\bullet))$, where $\nabla_{(t)}^\bullet = (\nabla_{(t)}^1, \nabla_{(t)}^2)$ is defined by setting

$$\nabla_{(t)}^1 = \left(1 - \sum_{j=2}^{n} t_j\right) \nabla^1 + \sum_{j=2}^{n} t_j \nabla_j^1$$

and

$$\nabla_{(t)}^2 = \left(1 - \sum_{j=2}^{n} t_j\right) \nabla^2 + \sum_{j=2}^{n} t_j \nabla_j^2,$$

and I_* denotes the integration along the fiber. Observe that $\nabla_{(t)}^1 = \nabla^1$, then $c(\nabla_{(t)}^\bullet) = c(\nabla_{(t)}^2) \wedge c(\nabla_{(t)}^1)^{-1} = c(\nabla_{(t)}^2) \wedge c(\nabla^1)^{-1}$. Since we have chosen ∇^1 trivial respect to $(\partial z^1)^\nu$, we have that $c(\nabla^1) = 1$ and then $c(\nabla_{(t)}^\bullet) = c(\nabla_{(t)}^2)$. By the definitions it follows that $\varphi(\nabla_{(t)}^\bullet) = \varphi(\nabla_{(t)}^2)$, thus

$$\varphi\left(\nabla^\bullet, \nabla_2^\bullet, \ldots, \nabla_n^\bullet\right) = \varphi\left(\nabla^2, \nabla_2^2, \ldots, \nabla_n^2\right)$$

and by (3.3.4) the residue of $\varphi(TS - N_S^{\otimes\nu})$ at p is

$$\operatorname{Res}_\varphi\left(\mathscr{D}, TS - N_S^{\otimes\nu}, p\right) = \int_{\partial\tilde{R}^{2\cdots n}} \varphi\left(\nabla^2, \nabla_2^2, \ldots, \nabla_n^2\right). \tag{3.3.5}$$

The last step is now to understand how is done $\varphi(\nabla^2, \nabla_2^2, \ldots, \nabla_n^2)$, which is a $(n-1)$-form over $\tilde{U}^{2\cdots n} = \tilde{U}^2 \cap \cdots \cap \tilde{U}^n$. By definition $\varphi(\nabla^2, \nabla_2^2, \ldots, \nabla_n^2) = I_*(\varphi(\nabla_{(t)}^2))$ where $\nabla_{(t)}^2$ is the C^∞ connection on the complex vector bundle

$$T\tilde{U}^{2\cdots n} \times \mathbb{R}^{n-1} \longrightarrow \tilde{U}^{2\cdots n} \times \mathbb{R}^{n-1} \tag{3.3.6}$$

defined above and I_* is the integration along the fiber. To compute $\varphi(\nabla^2_{(t)})$ we need ultimately to know the curvature matrix $\kappa^2_{(t)}$ of $\nabla^2_{(t)}$ respect to a C^∞ frame of (3.3.6). Recall that if $\theta^2_{(t)}$ is a connection matrix of $\nabla^2_{(t)}$ then we have the relation $\kappa^2_{(t)} = \mathrm{d}\theta^2_{(t)} - \theta^2_{(t)} \wedge \theta^2_{(t)}$, so we just need to compute $\theta^2_{(t)}$. Let $\theta^2, \theta^2_2, \dots, \theta^2_n$ be the connection matrices of $\nabla^2, \nabla^2_2, \dots, \nabla^2_n$ respect to the holomorphic frame

$$\left\{ \frac{\partial}{\partial z^2}\bigg|_{\tilde{U}^{2\cdots n}}, \dots, \frac{\partial}{\partial z^n}\bigg|_{\tilde{U}^{2\cdots n}} \right\}$$

of $T\tilde{U}^{2\cdots n}$. Then $\theta^2_{(t)}$ respect to the induced C^∞ frame of (3.3.6) is

$$\theta^2_{(t)} = \left(1 - \sum_{j=2}^n t_j\right)\theta^2 + \sum_{j=2}^n t_j \theta^2_j = \sum_{j=2}^n t_j \theta^2_j, \tag{3.3.7}$$

where in the second equality we use that $\theta^2 = 0$ because ∇^2 is trivial respect to the fixed frame. By definition

$$(\nabla^2_j)_{\frac{\partial}{\partial z^k}|_{\tilde{U}^j}} \left(\frac{\partial}{\partial z^i}\bigg|_{\tilde{U}^j} \right) = 0$$

for any $j, k, i = 2, \dots, n$ such that $k \neq j$, and

$$\begin{aligned}(\nabla^2_j)_{\frac{\partial}{\partial z^j}|_{\tilde{U}^j}} \left(\frac{\partial}{\partial z^i}\bigg|_{\tilde{U}^j} \right) &= \frac{1}{h^j|_{\tilde{U}^j}} \left[X|_{\tilde{U}^j}, \frac{\partial}{\partial z^i}\bigg|_{\tilde{U}^j} \right] \\ &= -\sum_{\ell=2}^n \frac{1}{h^j|_{\tilde{U}^j}} \frac{\partial h^\ell}{\partial z^i}\bigg|_{\tilde{U}^j} \frac{\partial}{\partial z^\ell}\bigg|_{\tilde{U}^j}\end{aligned}$$

for any $j, i = 2, \dots, n$. This means that

$$\theta^2_j = -\frac{\mathrm{d}z^j}{h^j}\bigg|_{\tilde{U}^j} \cdot H|_{\tilde{U}^j}$$

for any $j = 2, \dots, n$, where H is the $(n-1) \times (n-1)$ matrix of holomorphic functions

$$H = \left(\frac{\partial h^\ell}{\partial z^i} \right)_{i,\ell=2,\dots,n}, \tag{3.3.8}$$

and then by (3.3.7)

$$\theta^2_{(t)} = -\sum_{j=2}^n t_j \frac{\mathrm{d}z^j}{h^j}\bigg|_{\tilde{U}^{2\cdots n}} \cdot H|_{\tilde{U}^{2\cdots n}}.$$

Consequently the curvature matrix of $\nabla^2_{(t)}$ respect to the fixed frame of (3.3.6) is

$$\kappa^2_{(t)} = \mathrm{d}\theta^2_{(t)} - \theta^2_{(t)} \wedge \theta^2_{(t)} = -\sum_{j=2}^{n} \mathrm{d}t_j \wedge \left.\frac{\mathrm{d}z^j}{h^j}\right|_{\tilde{U}^{2\cdots n}} \cdot H\big|_{\tilde{U}^{2\cdots n}}$$
$$+ \text{ terms not containing } \mathrm{d}t_j,$$

then

$$\varphi\left(\nabla^2_{(t)}\right) = (-1)^{\left[\frac{n-1}{2}\right]}(n-1)! \left.\frac{\varphi\left(-H\right)\mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{h^2\cdots h^n}\right|_{\tilde{U}^{2\cdots n}} \wedge \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n$$
$$+ \text{ terms not containing } \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n.$$

By definition of I_* (see [36, Sec.II.4]) and recalling that $\int_{\Delta^{n-1}} \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n = 1/(n-1)!$, we finally have that

$$\varphi\left(\nabla^2, \nabla^2_2, \ldots, \nabla^2_n\right) = (-1)^{\left[\frac{n-1}{2}\right]} \left.\frac{\varphi\left(-H\right)\mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{h^2\cdots h^n}\right|_{\tilde{U}^{2\cdots n}}$$

and by (3.3.5) the residue of $\varphi(TS - N_S^{\otimes\nu})$ at p is

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TS - N_S^{\otimes\nu}, p\right) = \int_{\partial\tilde{R}^{2\cdots n}} \frac{(-1)^{\left[\frac{n-1}{2}\right]}\varphi\left(-H\right)}{h^2\cdots h^n}\, \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n.$$

Let us consider now the specific compact real submanifold $\tilde{R} \subset \tilde{U}$ defined by

$$\tilde{R} = \left\{q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right|^2 + \cdots + \left|h^n(q)\right|^2 \le (n-1)\epsilon^2\right\}$$

for $\epsilon > 0$ small enough, whose boundary is

$$\partial\tilde{R} = \left\{q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right|^2 + \cdots + \left|h^n(q)\right|^2 = (n-1)\epsilon^2\right\}.$$

In this case we can take as a system of honey-comb cells $\{\partial\tilde{R}^2, \ldots, \partial\tilde{R}^n\}$ adapted to $\{\tilde{U}^2 \cap \partial\tilde{R}, \ldots, \tilde{U}^n \cap \partial\tilde{R}\}$ the one defined by setting

$$\partial\tilde{R}^j = \left\{q \in \partial\tilde{R} \text{ s.t. } \left|h^j(q)\right| \ge \left|h^i(q)\right| \text{ for } i = 2, \ldots, n\right\}$$

for any $j = 2, \ldots, n$. Let Γ be defined by

$$\Gamma = \left\{q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right| = \cdots = \left|h^n(q)\right| = \epsilon\right\} \tag{3.3.9}$$

and oriented so that $\mathrm{d}\vartheta^2 \wedge \cdots \wedge \mathrm{d}\vartheta^n > 0$, with $\vartheta^j = \arg(h^j)$, then considering the orientations we have $\Gamma = (-1)^{\left[\frac{n-1}{2}\right]}\partial\tilde{R}^{2\cdots n}$. Summing up, we have proved the following theorem about Baum-Bott residues at isolated singular points of $\mathscr{D}$.

Theorem 3.3.3. *Let M, S and (f,g) be as in Theorem 3.2.2 and assume*

(i) (f,g) is tangential along S

or

(ii) S splits into M and σ is a splitting morphism of S into M.

In case (i) set $\mathscr{D} = \mathscr{D}_{f,g}$ and let $\mathfrak{U}$ be an atlas adapted to S and g, while in case (ii) set $\mathscr{D} = \mathscr{D}^{\sigma}_{f,g}$, let $\mathfrak{U}$ be an atlas adapted to S, g and σ, and suppose $\mathscr{D} \neq 0$. Let $p \in Sing(\mathscr{D})$ be an isolated singular point, let $(U,z) \in \mathfrak{U}$ be a local holomorphic chart centered at p such that $U \cap (Sing(S) \sqcup Sing(\mathscr{D})) = \{p\}$ and let φ be any homogeneous symmetric polynomial of degree $n-1$. Then both in case (i) and (ii) the residue $Res_{\varphi}(\mathscr{D}, TS - N_S^{\otimes\nu}, p)$ of Theorem 3.2.2 is given by the integral formula

$$Res_{\varphi}\left(\mathscr{D}, TS - N_S^{\otimes\nu}, p\right) = \int_{\Gamma} \frac{\varphi\left(-H\right)}{h^2 \cdots h^n}\, dz^2 \wedge \cdots \wedge dz^n, \tag{3.3.10}$$

where the holomorphic functions h^j are defined in (2.2.7), Γ is defined in (3.3.9) and is oriented so that $d\vartheta^2 \wedge \cdots \wedge d\vartheta^n > 0$, with $\vartheta^j = arg(h^j)$, and H is the $(n-1)\times(n-1)$ matrix of holomorphic functions defined in (3.3.8).

Chapter 4

Local canonical extensions of the induced foliation

Now come back to assume S possibly singular and let (f, g) be a pair as in Chapter 2. As usual set $\nu = \nu_{f,g}$ to ease the notation.

We know by Chapter 2 that if (f, g) is tangential along S or S' splits into M then we have a 1-dimensional holomorphic possibly singular foliation on S'. If it had a *global extension* to an open neighborhood of S then we could define certain natural partial holomorphic connections. This is not the case in general, anyway we show in this chapter that it is possible to define "good" *local extensions* of the foliation. To do this in the non-tangential case we need S' to have a geometric property stronger than the splitting property.

4.1 Natural partial holomorphic connections in case of a global extension of the induced foliation

Assume that (f, g) is tangential along S or S' splits into M, let $\mathscr{D} : N_{S'}^{\otimes\nu} \to TS'$ be the foliation induced by (f, g) in both cases and set $S^0 = S' - \mathrm{Sing}(\mathscr{D})$. Suppose there exist an open neighborhood $W \subset M$ of S, a holomorphic line bundle $\tilde{F} \to W$ and a morphism of holomorphic vector bundles

$$\tilde{\mathscr{D}} : \tilde{F} \to TM|_W$$

such that $\tilde{F}|_{S'} = N_{S'}^{\otimes\nu}$ and $\tilde{\mathscr{D}}|_{S'} = \mathscr{D}$ (in other words, suppose that $\mathscr{D}$ has a global extension around S). Under this hypothesis we have two natural partial holomorphic connections along $N_{S^0}^{\otimes\nu} \subset TS^0$, respectively on N_{S^0} and $N^0_{\mathscr{D},M}$ (the normal bundle of $\mathscr{D}$ in M, see Section 2.3). Let $\tilde{\mathcal{F}}$ be the sheaf of holomorphic sections of $\tilde{F}$ and let $\pi : TM|_{S^0} \to N_{S^0}$ and $\rho : TM|_{S^0} \to N^0_{\mathscr{D},M}$ be the obvious projections. Then the partial holomorphic connection on N_{S^0} is defined by setting

$$\begin{aligned} \mathcal{N}_{S^0} &\xrightarrow{\delta^{cs}} \left(\mathcal{N}_{S^0}^{\otimes\nu}\right)^* \otimes \mathcal{N}_{S^0} \\ w &\longrightarrow \delta^{cs}(w) \text{ s.t. } \delta^{cs}_v(w) = \pi\left([\tilde{v},\tilde{w}]|_{S^0}\right) \end{aligned} \tag{4.1.1}$$

for any $w \in \mathcal{N}_{S^0}$ and $v \in \mathcal{N}_{S^0}^{\otimes\nu} \subset \mathcal{T}_{S^0}$, where $\tilde{w} \in \mathcal{T}_M|_{S^0}$ and $\tilde{v} \in \tilde{\mathcal{F}}|_{S^0} \subset \mathcal{T}_M|_{S^0}$ are any holomorphic vector fields such that $\pi(\tilde{w}|_{S^0}) = w$ and $\tilde{v}|_{S^0} = v$. The partial holomorphic connection on $N^0_{\mathscr{D},M}$ is defined in a similar way, that is by setting

$$\begin{aligned} \mathcal{N}^0_{\mathscr{D},M} &\xrightarrow{\delta^{ls}} \left(\mathcal{N}_{S^0}^{\otimes\nu}\right)^* \otimes \mathcal{N}^0_{\mathscr{D},M} \\ w &\longrightarrow \delta^{ls}(w) \text{ s.t. } \delta^{ls}_v(w) = \rho\left([\tilde{v},\tilde{w}]|_{S^0}\right) \end{aligned} \tag{4.1.2}$$

for any $w \in \mathcal{N}^0_{\mathscr{D},M}$ and $v \in \mathcal{N}_{S^0}^{\otimes\nu} \subset \mathcal{T}_{S^0}$, where $\tilde{w} \in \mathcal{T}_M|_{S^0}$ and $\tilde{v} \in \tilde{\mathcal{F}}|_{S^0} \subset \mathcal{T}_M|_{S^0}$ are any holomorphic vector fields such that $\rho(\tilde{w}|_{S^0}) = w$ and $\tilde{v}|_{S^0} = v$. Observe that both (4.1.1) and (4.1.2) does not depend on the choices of $\tilde{w}$ and $\tilde{v}$. For instance, if $\bar{w} \in \mathcal{T}_M|_{S^0}$ is another holomorphic vector field such that $\pi(\bar{w}|_{S^0}) = w$ then $(\tilde{w}-\bar{w})|_{S^0} \in \mathcal{T}_{S^0}$, thus $[\tilde{v},\tilde{w}-\bar{w}]|_{S^0} \in \mathcal{T}_{S^0}$ and consequently $\pi([\tilde{v},\tilde{w}-\bar{w}]|_{S^0}) = 0$. We can argue similarly for (4.1.2), moreover in both the definitions if $\bar{v} \in \tilde{\mathcal{F}}|_{S^0} \subset \mathcal{T}_M|_{S^0}$ is another holomorphic vector field such that $\bar{v}|_{S^0} = v$ then $[\tilde{v}-\bar{v},\tilde{w}]|_{S^0} = 0$. Lastly, as well as (3.1.1) also δ^{cs} and δ^{ls} are flat by the Jacobi identity.

Unfortunately $\mathscr{D} : N_{S'}^{\otimes\nu} \to TS'$ does not admit in general a global extension around S. Anyway we can do something weaker, that is we can extend $\mathscr{D}$ *locally* about points of S' in such a way that we are able to define partial holomorphic connections as in (4.1.1) and (4.1.2). We talk about this sort of 'first order extension' of $\mathscr{D}$ in the following two sections (see [15] to investigate further the topic), while the discussion about the resulting partial holomorphic connections is postponed to Chapter 5 .

4.2 Tangential case

Let (f, g) be tangential along S and let $\mathfrak{U}$ be an atlas adapted to S' and g. Recall that any local holomorphic chart $(U, z) \in \mathfrak{U}$ such that $U \cap S' \neq \emptyset$ induces the canonical local generator $X_{f,g}$ of $\mathscr{D}_{f,g}$ on $U \cap S'$ defined as in (2.4.1). Let us define the local holomorphic vector field

$$\mathcal{X}_{f,g} = \sum_{j=1}^{n} h^j \frac{\partial}{\partial z^j} \tag{4.2.1}$$

and observe that $\mathcal{X}_{f,g}|_{U\cap S'} = X_{f,g}$. We call it the *canonical local extension of* $\mathscr{D}_{f,g}$ *associated to* (U, z), since it generates a 1-dimensional holomorphic foliation on U whose restriction to $U \cap S'$ coincides with $\mathscr{D}_{f,g}|_{U\cap S'}$.

Let now $(\hat{U}, \hat{z}) \in \mathfrak{U}$ be another local holomorphic chart such that $U \cap \hat{U} \cap S' \neq \emptyset$, $\hat{X}_{f,g}$ the corresponding canonical local generator of $\mathscr{D}_{f,g}$ on $\hat{U} \cap S'$ and $\hat{\mathcal{X}}_{f,g}$ the corresponding canonical local extension of $\mathscr{D}_{f,g}$ on $\hat{U}$. As both z^1 and $\hat{z}^1$ are local generators for $\mathcal{I}_S$, there is a (germ of) holomorphic function $a \in \mathcal{O}^*_M$ such that $\hat{z}^1 = az^1$, and

$$\hat{X}_{f,g} = \mathscr{D}_{f,g}\left((\partial \hat{z}^1)^\nu\right) = \mathscr{D}_{f,g}\left(\frac{1}{(a|_{S'})^\nu}(\partial z^1)^\nu\right) = \frac{1}{(a|_{S'})^\nu} X_{f,g} \tag{4.2.2}$$

on the overlap. Instead the relation between $\mathcal{X}_{f,g}$ and $\hat{\mathcal{X}}_{f,g}$ is less obvious than (4.2.2). We describe it in the next Proposition, but first let us fix the following notation: for any $(U, z) \in \mathfrak{U}$ denote

(1) by T_k the holomorphic vector fields of the form $\sum_{j=2}^n b^j \frac{\partial}{\partial z^j}$, with $b^j \in \mathcal{I}_S^k$ (*'tangential' terms vanishing on S with 'order k'*),

(2) by V_k the holomorphic vector fields of the form $c\frac{\partial}{\partial z^1}$, with $c \in \mathcal{I}_S^k$ (*'transversal' terms vanishing on S with 'order k'*).

Note that if $(\hat{U}, \hat{z}) \in \mathfrak{U}$ is another local holomorphic chart then

$$\hat{T}_k = T_k + V_{k+1} \qquad \text{and} \qquad \hat{V}_k = T_k + V_k. \tag{4.2.3}$$

Proposition 4.2.1. *Suppose (f, g) tangential along S and let $\mathfrak{U}$ be an atlas adapted to S' and g. If $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ are any local holomorphic charts such that $U \cap \hat{U} \cap S' \neq \emptyset$ and $\mathcal{X}_{f,g}, \hat{\mathcal{X}}_{f,g}$ are the corresponding canonical local extensions of $\mathscr{D}_{f,g}$ defined as in (4.2.1), then*

$$\hat{\mathcal{X}}_{f,g} = \frac{1}{a^\nu}\mathcal{X}_{f,g} + T_\nu + V_{\nu+1}$$

where they overlap, with $a \in \mathcal{O}^*_M$ such that $\hat{z}^1 = az^1$.

Proof. Let w and $\hat{w}$ be the special coordinates associated to z and $\hat{z}$ respectively and let $b \in \mathcal{O}^*_M$ be the germ such that $\hat{w}^1 = bw^1$. Then by (2.1.1), (2.2.7) and observing that $w^1 \circ f \in \mathcal{I}_S$ we have that

$$\begin{aligned}
\hat{h}^1 a^\nu (z^1)^\nu = \hat{h}^1(\hat{z}^1)^\nu &= \hat{w}^1 \circ f - \hat{w}^1 \circ g \\
&= (b \circ f)(w^1 \circ f) - (b \circ g)(w^1 \circ g) \\
&= (w^1 \circ f)\,(b \circ f - b \circ g) + (b \circ g) h^1 (z^1)^\nu \\
&= (w^1 \circ f)(z^1)^\nu \sum_{k=1}^n h^k \left(\frac{\partial b}{\partial w^k} \circ g \right) + (b \circ g) h^1 (z^1)^\nu \pmod{\mathcal{I}_S^{2\nu+1}}.
\end{aligned}$$

Since (f, g) is tangential along S, we have that $w^1 \circ f = w^1 \circ g \pmod{\mathcal{I}_S^{\nu+1}}$, hence we can rewrite the previous relation as

$$\begin{aligned}
\hat{h}^1 a^\nu &= (w^1 \circ g) \sum_{k=1}^n h^k \left(\frac{\partial b}{\partial w^k} \circ g \right) + (b \circ g) h^1 \pmod{\mathcal{I}_S^{\nu+1}} \\
&= \sum_{k=1}^n h^k \frac{\partial \hat{z}^1}{\partial z^k} \pmod{\mathcal{I}_S^{\nu+1}}, \qquad (4.2.4)
\end{aligned}$$

improving in this way (2.2.10) for $j = 1$. Using (2.2.10), (2.2.11), (4.2.3) and (4.2.4) we have that

$$\begin{aligned}
\hat{\mathcal{X}}_{f,g} &= \sum_{j=1}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j} = \hat{h}^1 \frac{\partial}{\partial \hat{z}^1} + \sum_{j=2}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j} \\
&= \frac{1}{a^\nu} \left(\sum_{k=1}^n h^k \frac{\partial \hat{z}^1}{\partial z^k} \right) \frac{\partial}{\partial \hat{z}^1} + \frac{1}{a^\nu} \sum_{j=2}^n \sum_{k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial}{\partial \hat{z}^j} + \hat{T}_\nu + \hat{V}_{\nu+1} \\
&= \frac{1}{a^\nu} \sum_{j,k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial}{\partial \hat{z}^j} + T_\nu + V_{\nu+1} \\
&= \frac{1}{a^\nu} \sum_{k,i=1}^n h^k \left(\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial z^i}{\partial \hat{z}^j} \right) \frac{\partial}{\partial z^i} + T_\nu + V_{\nu+1}.
\end{aligned}$$

Finally, recalling that $\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k} \frac{\partial z^i}{\partial \hat{z}^j} = \frac{\partial z^i}{\partial z^k} = \delta_{ik}$, for any $i, k = 1, \ldots, n$, we get the theorem. $\square$

4.3 Non-tangential case

Suppose now that S' splits into M and fix a splitting morphism $\sigma : N_{S'} \to TM|_{S'}$. Let $\mathfrak{U}$ be a splitting atlas adapted to S', g and σ. Recall that any local holomorphic chart $(U, z) \in \mathfrak{U}$ such that $U \cap S' \neq \emptyset$ induces the canonical local generator $X^\sigma_{f,g}$ of $\mathscr{D}^\sigma_{f,g}$ on $U \cap S'$ defined as in (2.5.7). Let us define the holomorphic function $h^1_0 = h^1(0, z^2, \ldots, z^n)$ on U, where h^1 is the defined as in (2.2.7), and observe that there exists a germ $k^1 \in \mathcal{O}_M$ such that

$$h^1 - h^1_0 = k^1 z^1 \tag{4.3.1}$$

since z^1 is a local generator for $\mathcal{I}_S$. Then define the local holomorphic vector field

$$\mathcal{X}^\sigma_{f,g} = k^1 z^1 \frac{\partial}{\partial z^1} + \sum_{j=2}^n h^j \frac{\partial}{\partial z^j} \tag{4.3.2}$$

and note that $\mathcal{X}^\sigma_{f,g}|_{U \cap S'} = X^\sigma_{f,g}$. We call it the *canonical local extension of* $\mathscr{D}^\sigma_{f,g}$ *associated* (U, z), since it generates a 1-dimensional holomorphic foliation on U whose restriction to $U \cap S'$ coincides with $\mathscr{D}^\sigma_{f,g}|_{U \cap S'}$.

Let now $(\hat{U}, \hat{z}) \in \mathfrak{U}$ be another local holomorphic chart such that $U \cap \hat{U} \cap S' \neq \emptyset$, $\hat{X}^\sigma_{f,g}$ the corresponding canonical local generator of $\mathscr{D}^\sigma_{f,g}$ on $\hat{U} \cap S'$ and $\hat{\mathcal{X}}^\sigma_{f,g}$ the corresponding canonical local extension of $\mathscr{D}^\sigma_{f,g}$ on $\hat{U}$. If $a \in \mathcal{O}^*_M$ is the (germ of) holomorphic function such that $\hat{z}^1 = a z^1$ then

$$\hat{X}^\sigma_{f,g} = \mathscr{D}^\sigma_{f,g}\left((\partial \hat{z}^1)^\nu\right) = \mathscr{D}^\sigma_{f,g}\left(\frac{1}{(a|_{S'})^\nu}(\partial z^1)^\nu\right) = \frac{1}{(a|_{S'})^\nu} X^\sigma_{f,g} \tag{4.3.3}$$

on the overlap. Again, the relation between $\mathcal{X}^\sigma_{f,g}$ and $\hat{\mathcal{X}}^\sigma_{f,g}$ is less obvious than (4.3.3). We could obtain something similar to Proposition 4.2.1 but the splitting property on S' is not enough now.

Definition 4.3.1. Let M be a complex manifold of dimension n, let $V \subset M$ be a complex submanifold of codimension $0 < k < n$ and let $\mathfrak{U}$ be a splitting atlas adapted to V. If for any two local holomorphic charts $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ such that $U \cap \hat{U} \cap V \neq \emptyset$ we have that

$$\frac{\partial^2 \hat{z}^r}{\partial z^s \partial z^t} \in \mathcal{I}_V \tag{4.3.4}$$

for any $r, s, t = 1, \ldots, k$, we say that $\mathfrak{U}$ is a *comfortable atlas adapted to* V. Then we say that V *is comfortably embedded into* M if there exists a comfortable atlas adapted to V.

If $\sigma : N_V \to TM|_V$ is a splitting morphism of V into M and $\mathfrak{U}$ is also adapted to σ, then we call $\mathfrak{U}$ a *comfortable atlas adapted to V and σ*. Moreover, if there are an open neighborhood $W \subset M$ of V and a local biholomorphism $\Phi : W \to M$ and $\mathfrak{U}$ is adapted also to Φ, then we say that $\mathfrak{U}$ is a *comfortable atlas adapted to V, Φ and σ*.

Roughly speaking, a comfortably embedded submanifold is a sort of first-order approximation of the zero section of a holomorphic vector bundle.

Example 4.3.2. Let M be a complex manifold of dimension n and let $\pi : M \to V$ be a holomorphic vector bundle of rank $k \geq 1$. We have seen in Example 2.5.4 that V splits into M via the zero section, and that an atlas of M induced by a trivializing atlas of V is a splitting atlas adapted to V. Actually V is even comfortably embedded into M. In fact if $\mathfrak{U}$ is an atlas of M as in Example 2.5.4 and $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ are any charts, then the coordinate $\hat{z}^r$ depends linearly on $z^1, \ldots, z^k$ for any $r = 1, \ldots, k$, thus

$$\frac{\partial^2 \hat{z}^r}{\partial z^s \partial z^t} \equiv 0$$

for any $r, s, t = 1, \ldots, k$.

Example 4.3.3. Let $\pi : M \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ at 0 and let $S = \pi^{-1}(0) \subset M$ be the exceptional divisor. In this example we want to show that S is comfortably embedded into M.

Recall that any system of holomorphic coordinates of $\mathbb{C}^n$ centered at 0 induces canonically n local holomorphic charts on M which cover S. As in Example 2.1.4, let us denote by (U_j, z_j) the ones induced by the canonical coordinates of $\mathbb{C}^n$, for $j = 1, \ldots, n$. By their very definition $U_i \cap U_j \cap S \neq \emptyset$ for any $i \neq j$, and the change of coordinates $z_j \circ z_i^{-1}$ is given by

$$z_j^j = z_i^i z_i^j, \qquad z_j^i = 1/z_i^j, \qquad z_j^\ell = z_i^\ell / z_i^j, \;\; \ell \neq i, j, \tag{4.3.5}$$

for any $i \neq j$. Now let us consider for instance (U_1, z_1) and (U_2, z_2). For the sake of simplicity rename $x = z_1$, hence $x^\ell = z_1^\ell$ for any $\ell = 1, \ldots, n$. Moreover, define the local holomorphic coordinates y on U_2 by switching the first two components of z_2, that is

$$y^1 = z_2^2, \qquad y^2 = z_2^1, \qquad y^\ell = z_2^\ell, \;\; \ell = 3, \ldots, n.$$

We do this in order to have a local holomorphic chart (U_2, y) adapted to S according with our Definition 2.2.1. By (4.3.5) the change of coordinates $x \circ y^{-1}$ is given by

$$x^1 = y^1 y^2, \qquad x^2 = y^2, \qquad x^\ell = y^\ell / y^2, \;\; \ell = 3, \ldots, n.$$

Then
$$\frac{\partial x^\ell}{\partial y^1} \equiv 0$$
for any $\ell = 2, \ldots, n$, and
$$\frac{\partial^2 x^1}{(\partial y^1)^2} \equiv 0$$
Making a similar switching of components for the other local holomorphic charts (U_j, z_j) we clearly obtain a comfortable atlas adapted to S.

Now we can state the analogous of Proposition 4.2.1 for the non-tangential case. We still refer to the notation introduced in Section 4.2.

Proposition 4.3.4. *Suppose S' comfortably embedded into M, let $\sigma : N_{S'} \to TM|_{S'}$ be a splitting morphism and let $\mathfrak{U}$ be a comfortable atlas adapted to S', g and σ. If $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ are any local holomorphic charts such that $U \cap \hat{U} \cap S' \neq \emptyset$ and $\mathcal{X}^\sigma_{f,g}, \hat{\mathcal{X}}^\sigma_{f,g}$ are the corresponding canonical local extensions of $\mathscr{D}^\sigma_{f,g}$ defined as in (4.3.2), then for $\nu > 1$*

$$\hat{\mathcal{X}}^\sigma_{f,g} = \frac{1}{a^\nu} \mathcal{X}^\sigma_{f,g} + T_1 + V_2$$

where they overlap, with $a \in \mathcal{O}^*_M$ such that $\hat{z}^1 = a z^1$.

Proof. Let w and $\hat{w}$ be the special coordinates associated to z and $\hat{z}$ respectively and refer to the usual notation. First of all observe that $\frac{\partial z^1}{\partial \hat{z}^1} = \frac{1}{a} (\operatorname{mod} \mathcal{I}_S^2)$ by property (4.3.4) of comfortable atlases (if the atlas is not comfortable we only have that $\frac{\partial z^1}{\partial \hat{z}^1} = \frac{1}{a} (\operatorname{mod} \mathcal{I}_S))$. Thus by (2.2.11) and the other property (2.5.1) of comfortable atlases we have that

$$\frac{\partial}{\partial \hat{z}^1} = \frac{1}{a} \frac{\partial}{\partial z^1} + T_1 + V_2. \tag{4.3.6}$$

Observe moreover that by (2.2.10) and by definition of $\hat{h}^1_0$ and h^1_0 it follows that $\hat{h}^1_0 a^\nu = h^1_0 a$, then

$$\hat{k}^1 \hat{z}^1 a^\nu = -h^1_0 a + \sum_{k=1}^{n} h^k \frac{\partial \hat{z}^1}{\partial z^k} \left(\operatorname{mod} \mathcal{I}_S^\nu\right). \tag{4.3.7}$$

This relation will play here the role of (4.2.4), which is not true if (f, g) is not tangential along S. Then using (2.2.10), (2.2.11), (4.2.3), (4.3.6) and

(4.3.7) we have that

$$\begin{aligned}
\hat{\mathcal{X}}^\sigma_{f,g} &= \hat{k}^1 \hat{z}^1 \frac{\partial}{\partial \hat{z}^1} + \sum_{j=2}^n \hat{h}^j \frac{\partial}{\partial \hat{z}^j} \\
&= \frac{1}{a^\nu}\left(-h^1_0 a\right)\frac{\partial}{\partial \hat{z}^1} + \frac{1}{a^\nu}\sum_{j,k=1}^n h^k \frac{\partial \hat{z}^j}{\partial z^k}\frac{\partial}{\partial \hat{z}^j} + \hat{T}_\nu + \hat{V}_\nu \\
&= \frac{1}{a^\nu}\left(-h^1_0\right)\frac{\partial}{\partial z^1} + \frac{1}{a^\nu}\sum_{k,i=1}^n h^k \left(\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k}\frac{\partial z^i}{\partial \hat{z}^j}\right)\frac{\partial}{\partial z^i} + T_1 + V_{\min(\nu,2)}.
\end{aligned}$$

Recalling that $\sum_{j=1}^n \frac{\partial \hat{z}^j}{\partial z^k}\frac{\partial z^i}{\partial \hat{z}^j} = \frac{\partial z^i}{\partial z^k} = \delta_{ik}$, for $i, k = 1, \ldots, n$, we then have

$$\begin{aligned}
\hat{\mathcal{X}}^\sigma_{f,g} &= \frac{1}{a^\nu}\left(-h^1_0\right)\frac{\partial}{\partial z^1} + \frac{1}{a^\nu}\sum_{k=1}^n h^k \frac{\partial}{\partial z^k} + T_1 + V_{\min(\nu,2)} \\
&= \frac{1}{a^\nu} k^1 z^1 \frac{\partial}{\partial z^1} + \frac{1}{a^\nu}\sum_{k=2}^n h^k \frac{\partial}{\partial z^k} + T_1 + V_{\min(\nu,2)} \\
&= \frac{1}{a^\nu}\mathcal{X}^\sigma_{f,g} + T_1 + V_{\min(\nu,2)}.
\end{aligned}$$

Assuming $\nu > 1$ we get the theorem. □

Observe that Proposition 4.3.4 does not work for $\nu = 1$. In this case we can modify slightly $\mathscr{D}^\sigma_{f,g}$ in order to overcome the problem. By hypothesis there is a well-defined morphism of holomorphic vector bundles

$$d_{f,g} : N_{S'} \longrightarrow N_{S'} \tag{4.3.8}$$

given on the fibers by $[v] \to [(\mathrm{d}g_p^{-1} \circ \mathrm{d}f_p)(v)]$, for any $[v] \in N_{S',p}$ and $p \in S'$. Hence we can define

$$\mathscr{D}^{\sigma,1}_{f,g} = \mathscr{D}^\sigma_{f,g} \circ d_{f,g} : N_{S'} \longrightarrow TS',$$

which is again a 1-dimensional holomorphic possibly singular foliation on S'.

Remark 4.3.5. We may consider $d^{\otimes\nu}_{f,g} : N^{\otimes\nu}_{S'} \to N^{\otimes\nu}_{S'}$ and define

$$\mathscr{D}^{\sigma,\nu}_{f,g} = \mathscr{D}^\sigma_{f,g} \circ d^{\otimes\nu}_{f,g} : N^{\otimes\nu}_{S'} \longrightarrow TS'$$

for any $\nu \geq 1$. Anyway $\mathscr{D}^{\sigma,\nu}_{f,g} = \mathscr{D}^\sigma_{f,g}$ for any $\nu > 1$ since $d_{f,g} \neq \mathrm{Id}_{N_{S'}}$ if and only if (f, g) is not tangential along S and $\nu = 1$.

Clearly the singular set $\mathrm{Sing}(\mathscr{D}^{\sigma,1}_{f,g})$ may be larger than $\mathrm{Sing}(\mathscr{D}^{\sigma}_{f,g})$. If $(U,z)\in\mathfrak{U}$ is any local holomorphic chart such that $U\cap S'\neq\emptyset$ then

$$d_{f,g}\left(\partial z^1\right)=\left.\frac{\partial f^1}{\partial z^1}\right|_{S'}\partial z^1=\left(1+h^1|_{S'}\right)\partial z^1,$$

where $f^1=w^1\circ f$, $w=(w^1,\ldots,w^n)$ are the special coordinates associated to the z and h^1 is the one defined as in (2.2.7). Thus

$$\begin{aligned}X^{\sigma,1}_{f,g}=\mathscr{D}^{\sigma,1}_{f,g}\left(\partial z^1\right)&=\mathscr{D}^{\sigma}_{f,g}\left(\left(1+h^1|_{S'}\right)\partial z^1\right)\\&=\left(1+h^1|_{S'}\right)\sum_{j=2}^{n}h^j|_{S'}\left.\frac{\partial}{\partial z^j}\right|_{S'}\end{aligned}\tag{4.3.9}$$

is a local generator of $\mathscr{D}^{\sigma,1}_{f,g}$ on $U\cap S'$, where the holomorphic functions h^j are the ones appearing in (2.2.7). We call (4.3.9) the *canonical local generator of $\mathscr{D}^{\sigma,1}_{f,g}$ associated to* (U,z). Let h^1_0 and k^1 be the holomorphic functions defined in (4.3.1), then the local holomorphic vector field

$$\mathcal{X}^{\sigma,1}_{f,g}=k^1z^1\frac{\partial}{\partial z^1}+(1+h^1_0)\sum_{j=2}^{n}h^j\frac{\partial}{\partial z^j}\tag{4.3.10}$$

is such that $\mathcal{X}^{\sigma,1}_{f,g}|_{U\cap S'}=X^{\sigma,1}_{f,g}$. We call it the *canonical local extension of $\mathscr{D}^{\sigma,1}_{f,g}$ associated* (U,z) since it generates a 1-dimensional holomorphic foliation on U whose restriction to $U\cap S'$ coincides with $\mathscr{D}^{\sigma,1}_{f,g}|_{U\cap S'}$.

Let now $(\hat{U},\hat{z})\in\mathfrak{U}$ be another local holomorphic chart such that $U\cap\hat{U}\cap S'\neq\emptyset$, $\hat{X}^{\sigma,1}_{f,g}$ the corresponding canonical local generator of $\mathscr{D}^{\sigma,1}_{f,g}$ on $\hat{U}\cap S'$ and $\hat{\mathcal{X}}^{\sigma,1}_{f,g}$ the canonical local extension of $\mathscr{D}^{\sigma,1}_{f,g}$ on $\hat{U}$. Then

$$\hat{X}^{\sigma,1}_{f,g}=\mathscr{D}^{\sigma,1}_{f,g}\left(\partial\hat{z}^1\right)=\mathscr{D}^{\sigma,1}_{f,g}\left(\frac{1}{a|_{S'}}\partial z^1\right)=\frac{1}{a|_{S'}}X^{\sigma,1}_{f,g}\tag{4.3.11}$$

on the overlap, where $a\in\mathcal{O}^*_M$ is the holomorphic function such that $\hat{z}^1=az^1$. The relation between $\mathcal{X}^{\sigma,1}_{f,g}$ and $\hat{\mathcal{X}}^{\sigma,1}_{f,g}$ is instead described by the following proposition.

Proposition 4.3.6. *Suppose S' comfortably embedded into M, let $\sigma:N_{S'}\to TM|_{S'}$ be a splitting morphism and let $\mathfrak{U}$ be a comfortable atlas adapted to S', g and σ. Assume that $\nu=1$. If $(U,z),(\hat{U},\hat{z})\in\mathfrak{U}$ are any local holomorphic charts such that $U\cap\hat{U}\cap S'\neq\emptyset$ and $\mathcal{X}^{\sigma,1}_{f,g},\hat{\mathcal{X}}^{\sigma,1}_{f,g}$ are the corresponding canonical local extensions of $\mathscr{D}^{\sigma,1}_{f,g}$ defined as in (4.3.10), then*

$$\hat{\mathcal{X}}^{\sigma,1}_{f,g}=\frac{1}{a}\mathcal{X}^{\sigma,1}_{f,g}+T_1+V_2$$

where they overlap, with $a \in \mathcal{O}_M^*$ *such that* $\hat{z}^1 = az^1$.

Proof. Let w and $\hat{w}$ be the special coordinates associated to z and $\hat{z}$ respectively and refer to the usual notation. If $\nu = 1$ we need to improve relation (4.3.7). First of all observe that we can improve (2.1.1) by writing

$$
\begin{aligned}
h \circ f - h \circ g = &\sum_{j=1}^{n} \left(f^j - g^j\right) \frac{\partial h}{\partial z^j} \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \left(f^i - g^i\right) \left(f^j - g^j\right) \frac{\partial^2 h}{\partial z^i \partial z^j} \left(\bmod \mathcal{I}_S^{3\nu}\right),
\end{aligned}
$$

for any $h \in \mathcal{O}_M$. Hence if $\nu = 1$ and recallingg (2.2.7) we have that

$$
\hat{h}^1 a = \sum_{j=1}^{n} h^j \frac{\partial \hat{z}^1}{\partial z^j} + \frac{1}{2} z^1 \sum_{i,j=1}^{n} h^i h^j \frac{\partial^2 \hat{z}^1}{\partial z^i \partial z^j} \left(\bmod \mathcal{I}_S^2\right).
$$

Since $\frac{\partial^2 \hat{z}^1}{\partial z^1 \partial z^1} = 2 \frac{\partial a}{\partial z^1} (\bmod \mathcal{I}_S)$, $\frac{\partial^2 \hat{z}^1}{\partial z^i \partial z^1} = \frac{\partial a}{\partial z^i} (\bmod \mathcal{I}_S)$ if $i \geq 2$ and $\frac{\partial^2 \hat{z}^1}{\partial z^i \partial z^j} = 0 (\bmod \mathcal{I}_S)$ if $i, j \geq 2$, the previous relation becomes

$$
\begin{aligned}
\hat{h}^1 a &= \sum_{j=1}^{n} h^j \frac{\partial \hat{z}^1}{\partial z^j} + z^1 h^1 \sum_{j=1}^{n} h^j \frac{\partial a}{\partial z^j} \left(\bmod \mathcal{I}_S^2\right) \\
&= \sum_{j=1}^{n} h^j \frac{\partial \hat{z}^1}{\partial z^j} + h_0^1 \sum_{j=1}^{n} h^j \left(z^1 \frac{\partial a}{\partial z^j}\right) \left(\bmod \mathcal{I}_S^2\right) \\
&= -h^1 h_0^1 a + \left(1 + h_0^1\right) \sum_{j=1}^{n} h^j \frac{\partial \hat{z}^1}{\partial z^j} \left(\bmod \mathcal{I}_S^2\right)
\end{aligned}
$$

and consequently

$$
\hat{k}^1 \hat{z}^1 a = -h_0^1 a - h^1 h_0^1 a + \left(1 + h_0^1\right) \sum_{j=1}^{n} h^j \frac{\partial \hat{z}^1}{\partial z^j} \left(\bmod \mathcal{I}_S^2\right). \tag{4.3.12}
$$

Then using (2.2.10), (2.2.11), (4.2.3), (4.3.6) and (4.3.12) we have that

$$\begin{aligned}
\hat{\mathcal{X}}^{\sigma,1}_{f,g} =& \hat{k}^1\hat{z}^1\frac{\partial}{\partial\hat{z}^1} + \left(1+\hat{h}^1_0\right)\sum_{j=2}^n \hat{h}^j\frac{\partial}{\partial\hat{z}^j}\\
=& \left(-h^1_0 - h^1h^1_0\right)\frac{\partial}{\partial\hat{z}^1} + \frac{1}{a}\left(1+h^1_0\right)\sum_{j,k=1}^n h^k\frac{\partial\hat{z}^j}{\partial z^k}\frac{\partial}{\partial\hat{z}^j} + \hat{T}_\nu + \hat{V}_2\\
=& \frac{1}{a}\left(-h^1_0 - h^1h^1_0\right)\frac{\partial}{\partial z^1} + \frac{1}{a}\left(1+h^1_0\right)\sum_{k,i=1}^n h^k\left(\sum_{j=1}^n\frac{\partial\hat{z}^j}{\partial z^k}\frac{\partial z^i}{\partial\hat{z}^j}\right)\frac{\partial}{\partial z^i}\\
&+ T_1 + V_2\\
=& \frac{1}{a}\left(-h^1_0 - h^1h^1_0\right)\frac{\partial}{\partial z^1} + \frac{1}{a}\left(1+h^1_0\right)\sum_{k=1}^n h^k\frac{\partial}{\partial z^k} + T_1 + V_2\\
=& \frac{1}{a}k^1z^1\frac{\partial}{\partial z^1} + \frac{1}{a}\sum_{k=2}^n h^k\frac{\partial}{\partial z^k} + T_1 + V_2 = \frac{1}{a}\mathcal{X}^{\sigma,1}_{f,g} + T_1 + V_2
\end{aligned}$$

and we get the theorem. □

Remark 4.3.7. Suppose that g is a *global* biholomorphism over an open neighborhood of S'. In this case Proposition 4.2.1 is basically [4, Lem.4.1] for the holomorphic self-map $g^{-1}\circ f$, while Proposition 4.3.4 and 4.3.6 are together [4, Prop.4.2].

Chapter 5

The other index theorems

Let S and (f,g) be as in Chapter 4, set again $\nu=\nu_{f,g}$ and let $[S]$ denote the holomorphic line bundle on M canonically induced by S. Similarly to Chapter 3 we fix the following notation. If (f,g) is tangential along S then set $\mathscr{D}=\mathscr{D}_{f,g}$, let $\mathfrak{U}$ be an atlas adapted to S' and g and for any $(U,z)\in\mathfrak{U}$ set $X=X_{f,g}$ and $\mathcal{X}=\mathcal{X}_{f,g}$. If S' is comfortably embedded into M, σ is a splitting morphism of S' into M and $\nu>1$ then set $\mathscr{D}=\mathscr{D}^{\sigma}_{f,g}$, let $\mathfrak{U}$ be a comfortable atlas adapted to S', g and σ, and for any $(U,z)\in\mathfrak{U}$ set $X=X^{\sigma}_{f,g}$ and $\mathcal{X}=\mathcal{X}^{\sigma}_{f,g}$. Lastly, if S' is comfortably embedded into M, σ is a splitting morphism of S' into M and $\nu=1$ then set $\mathscr{D}=\mathscr{D}^{\sigma,1}_{f,g}$, let $\mathfrak{U}$ be a comfortable atlas adapted to S', g and σ, and for any $(U,z)\in\mathfrak{U}$ set $X=X^{\sigma,1}_{f,g}$ and $\mathcal{X}=\mathcal{X}^{\sigma,1}_{f,g}$. In all the cases $S^0=S'-\mathrm{Sing}(\mathscr{D})$.

We show that if (f,g) is tangential along S or S' is comfortably embedded into M we can define a partial holomorphic connection along $N_{S^0}^{\otimes\nu}\subset TS^0$ on N_{S^0}, and then we get a *Camacho-Sad-type index theorem*. Moreover, if (f,g) is tangential along S and $\nu>1$ we show that we can define a partial holomorphic connection along $N_{S^0}^{\otimes\nu}\subset TS^0$ on $N^0_{\mathscr{D},M}$, and then we get a *Lehmann-Suwa-Khanedani-type index theorem*. In both cases we find explicit formulas for the computation of the residues at isolated singular points of $\mathscr{D}$ and, in a case, of S.

5.1 Second partial holomorphic connection

Suppose (f,g) tangential along S or S' comfortably embedded into M, and fix a splitting morphism $\sigma: N_{S'}\to TM|_{S'}$ in the latter case. We know by

Chapter 2 that in both cases the pair (f, g) induces a foliation $\mathscr{D} : N_{S'}^{\otimes\nu} \to TS'$, and we have showed in Chapter 4 that any local holomorphic chart $(U, z) \in \mathfrak{U}$ induces a canonical local extension $\mathcal{X}$ od $\mathscr{D}$ on U. This means that we are locally in the setting of Section 4.1 and then we can define locally partial holomorphic connections exactly as in (4.1.1). Thus for any $(U, z) \in \mathfrak{U}$ such that $U \cap S' \neq \emptyset$ we can define a partial holomorphic connection along $N_{U\cap S^0}^{\otimes\nu} \subset TS^0|_{U\cap S^0}$ on $N_{U\cap S^0}$ by setting

$$\begin{aligned} \mathcal{N}_{U\cap S^0} &\xrightarrow{\delta^{cs}_{\mathcal{X}}} \left(\mathcal{N}_{U\cap S^0}^{\otimes\nu}\right)^* \otimes \mathcal{N}_{U\cap S^0} \\ w &\longrightarrow \delta^{cs}_{\mathcal{X}}(w) \text{ s.t. } (\delta^{cs}_{\mathcal{X}})_{\alpha X}(w) = \alpha\pi\left([\mathcal{X}, \tilde{w}]|_{U\cap S^0}\right), \end{aligned} \tag{5.1.1}$$

for any $w \in \mathcal{N}_{U\cap S^0}$ and $\alpha \in \mathcal{O}_{U\cap S^0}$, where $\pi : TM|_{S^0} \to N_{S^0}$ is the obvious projection and $\tilde{w} \in \mathcal{T}_M|_{U\cap S^0}$ is any holomorphic vector field such that $\pi(\tilde{w}|_{U\cap S^0}) = w$. Observe that any holomorphic vector field $v \in \mathcal{N}_{U\cap S^0}^{\otimes\nu} \subset \mathcal{T}_{S^0}|_{U\cap S^0}$ is of the form $v = \alpha X$ for a holomorphic function $\alpha \in \mathcal{O}_{U\cap S^0}$ since X is a local generator of $\mathscr{D}$ on $U \cap S'$. Exactly as (4.1.1) the definition of $\delta^{cs}_{\mathcal{X}}$ does not depend on the choice of $\tilde{w}$, moreover it is flat by the Jacobi identity.

The key point is that we can glue together all these local partial holomorphic connections thanks to Proposition 4.2.1, 4.3.4 and 4.3.6, and consequently we have a *global* partial holomorphic connection along $N_{S^0}^{\otimes\nu} \subset TS^0$ on N_{S^0}.

Proposition 5.1.1. *Suppose that (f, g) is tangential along S or S' is comfortably embedded into M, and fix a splitting morphism $\sigma : N_{S'} \to TM|_{S'}$ in the latter case. Let $\mathfrak{U}$ be an atlas according to the case. If $(U, z), (\hat{U}, \hat{z}) \in \mathfrak{U}$ are any local holomorphic charts such that $U \cap \hat{U} \cap S' \neq \emptyset$ and $\delta^{cs}_{\mathcal{X}}, \delta^{cs}_{\hat{\mathcal{X}}}$ are the corresponding local partial holomorphic connections defined as in (5.1.1) then*

$$\delta^{cs}_{\hat{\mathcal{X}}} = \delta^{cs}_{\mathcal{X}}$$

where they overlap. Consequently, there is a well-defined flat partial holomorphic connection

$$\mathcal{N}_{S^0} \xrightarrow{\delta^{cs}} \left(\mathcal{N}_{S^0}^{\otimes\nu}\right)^* \otimes \mathcal{N}_{S^0}$$

along $N_{S^0}^{\otimes\nu} \subset TS^0$ on N_{S^0}.

Proof. Since X is a local generator of $\mathscr{D}$ on $U \cap \hat{U} \cap S'$, we just need to prove that

$$\left(\delta^{cs}_{\hat{\mathcal{X}}}\right)_X(w) = \left(\delta^{cs}_{\mathcal{X}}\right)_X(w)$$

for any $w \in \mathcal{N}_{U\cap\hat{U}\cap S^0}$. This is true because if $\hat{z}^1 = az^1$, with $a \in \mathcal{O}_M^*$, and $\pi : TM|_{S^0} \to N_{S^0}$ is the projection, by (4.2.2), (4.3.3), (4.3.11) and by Proposition 4.2.1, 4.3.4 and 4.3.6 we have that

$$\begin{aligned}\left(\delta^{cs}_{\hat{\mathcal{X}}}\right)_X(w) &= (a|_{S^0})^\nu \left(\delta^{cs}_{\hat{\mathcal{X}}}\right)_{\hat{X}}(w) = (a|_{S^0})^\nu\, \pi\left(\left[\hat{\mathcal{X}}, \tilde{w}\right]\Big|_{S^0}\right)\\ &= (a|_{S^0})^\nu\, \pi\left(\left[\left(\frac{1}{a}\right)^\nu \mathcal{X} + T_1 + V_2, \tilde{w}\right]\Bigg|_{S^0}\right)\\ &= \pi\left([\mathcal{X}, \tilde{w}]|_{S^0}\right) = (\delta^{cs}_{\mathcal{X}})_X(w),\end{aligned}$$

where the last but one equality is due to the fact that $[T_1, \tilde{w}]|_{S^0} \in \mathcal{T}_{S^0}$ and $[V_2, \tilde{w}]|_{S^0} \equiv 0$. $\square$

5.2 A Camacho-Sad-type index theorem

Let ∇^{cs} be any δ^{cs}-connection on N_{S^0} and consider a tubular neighborhood $r : U_0 \to S^0$ of S^0 in M (it always exists, see for example [33, p.465]). By the property of the tubular neighborhood $r^*N_{S^0} \cong [S]|_{U_0}$, hence $r^*\nabla^{cs}$ is a C^∞ connection on $[S]|_{U_0}$ and by Theorem 1.4.9 we have that

$$c_1^{n-1}(r^*\nabla^{cs}) = r^* c_1^{n-1}(\nabla^{cs}) = 0. \tag{5.2.1}$$

Let now $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $\mathrm{Sing}(S)\sqcup\mathrm{Sing}(\mathscr{D})$, which is clearly an analytic subvariety of S. For any λ let $U_\lambda \subset M$ be an open subset such that $U_\lambda \supset \Sigma_\lambda$ and $U_\lambda \cap U_{\lambda'} = \emptyset$ whenever $\lambda \neq \lambda'$. Set $U_1 = \sqcup_\lambda U_\lambda$ and let ∇_1 be any C^∞ connection on $[S]|_{U_1}$, then consider the open neighborhood $U = U_0 \cup U_1$ of S and its covering $\mathcal{U} = \{U_0, U_1\}$. By (5.2.1) it follows that

$$c_1^{n-1}([S]) = \left[\left(0, c_1^{n-1}(\nabla_1), c_1^{n-1}(r^*\nabla^{cs}, \nabla_1)\right)\right] \in H^{2(n-1)}(\mathcal{U}, U_0, \mathbb{C}),$$

that is we can localize $c_1^{n-1}([S])$ at $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$. Moreover, the localization does not depend on the choice of the C^∞ connections, as stated by the following lemma.

Lemma 5.2.1. *Let $\bar{\nabla}^{cs}$ be another δ^{cs}-connection on N_{S^0} and let $\bar{\nabla}_1$ be another C^∞ connection on $[S]|_{U_1}$. Then*

$$\left[\left(0, c_1^{n-1}(\nabla_1), c_1^{n-1}(r^*\nabla^{cs}, \nabla_1)\right)\right] = \left[\left(0, c_1^{n-1}(\bar{\nabla}_1), c_1^{n-1}(r^*\bar{\nabla}^{cs}, \bar{\nabla}_1)\right)\right]$$

in $H^{2(n-1)}(\mathcal{U}, U_0, \mathbb{C})$.

Proof. By property (1.1.2) we have that

$$c_1^{n-1}(\nabla_1) - c_1^{n-1}\left(\bar{\nabla}_1\right) = \mathrm{d}c_1^{n-1}\left(\bar{\nabla}_1, \nabla_1\right), \tag{5.2.2}$$

moreover that

$$\begin{aligned} c_1^{n-1}(r^*\nabla^{cs}, \nabla_1) =& c_1^{n-1}(\bar{\nabla}_1, \nabla_1) + c_1^{n-1}(r^*\nabla^{cs}, \bar{\nabla}_1) \\ & - \mathrm{d}c_1^{n-1}(\bar{\nabla}_1, r^*\nabla^{cs}, \nabla_1) \end{aligned}$$

and

$$\begin{aligned} c_1^{n-1}(r^*\bar{\nabla}^{cs}, \bar{\nabla}_1) =& c_1^{n-1}(r^*\nabla^{cs}, \bar{\nabla}_1) - c_1^{n-1}(r^*\nabla^{cs}, r^*\bar{\nabla}^{cs}) \\ & - \mathrm{d}c_1^{n-1}(r^*\nabla^{cs}, r^*\bar{\nabla}^{cs}, \bar{\nabla}_1) \\ =& c_1^{n-1}(r^*\nabla^{cs}, \bar{\nabla}_1) + dc_1^{n-1}(r^*\nabla^{cs}, \bar{\nabla}_1, r^*\bar{\nabla}^{cs}), \end{aligned}$$

where in last equality we use that $c_1^{n-1}(r^*\nabla^{cs}, r^*\bar{\nabla}^{cs}) = r^*c_1^{n-1}(\nabla^{cs}, \bar{\nabla}^{cs}) = 0$ by Theorem 1.4.9. Hence

$$c_1^{n-1}(r^*\nabla^{cs}, \nabla_1) - c_1^{n-1}(r^*\bar{\nabla}^{cs}, \bar{\nabla}_1) = c_1^{n-1}(\bar{\nabla}_1, \nabla_1) - \mathrm{d}\omega, \tag{5.2.3}$$

with $\omega = c_1^{n-1}(\bar{\nabla}_1, r^*\nabla^{cs}, \nabla_1) + c_1^{n-1}(r^*\nabla^{cs}, \bar{\nabla}_1, r^*\bar{\nabla}^{cs})$. By (5.2.2) and (5.2.3) and recalling the definition of the Čech-de Rham differential D we then have

$$\begin{aligned} & \big(0, c_1^{n-1}(\nabla_1), c_1^{n-1}(r^*\nabla^{cs}, \nabla_1)\big) - \big(0, c_1^{n-1}(\bar{\nabla}_1), c_1^{n-1}(r^*\bar{\nabla}^{cs}, \bar{\nabla}_1)\big) \\ & \qquad = D\big(0, c_1^{n-1}(\bar{\nabla}_1, \nabla_1), \omega\big) \end{aligned}$$

□

Suppose now that S is compact, then $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D}) \subset S$ is also compact and we can assume without loss of generality that U_1 is a regular neighborhood of $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$. Consequently we can assume that U is a regular neighborhood of S. Then by what we said in this chapter and by the commutative diagram (1.9.2) we gain the following index theorem.

Theorem 5.2.2 (Camacho-Sad-type index theorem)**.** *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a globally irreducible compact analytic hypersurface with regular part $S' = S - Sing(S)$. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S', with order of coincidence $\nu = \nu_{f,g}$. Assume that*

(i) (f, g) is tangential along S

or

(ii) *S' is comfortably embedded into M and σ is a splitting morphism of S' into M.*

In case (i) set $\mathscr{D} = \mathscr{D}_{f,g}$, while in case (ii) set $\mathscr{D} = \mathscr{D}^{\sigma}_{f,g}$ if $\nu > 1$ and $\mathscr{D} = \mathscr{D}^{\sigma,1}_{f,g}$ if $\nu = 1$, and suppose $\mathscr{D} \neq 0$. Let $Sing(S) \sqcup Sing(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $Sing(S) \sqcup Sing(\mathscr{D})$. Then there exist complex numbers $Res(\mathscr{D}, S, \Sigma_\lambda)$, one for each connected component Σ_λ, such that

$$\sum_\lambda Res\,(\mathscr{D}, S, \Sigma_\lambda) = \int_S c_1^{n-1}\,([S])\,.$$

Remark 5.2.3. Suppose that g is a *global* biholomorphism over an open neighborhood of S' and recall Remark 2.2.6 and 2.4.6. In this case Theorem 5.2.2 turns out to be [4, Thm.6.2] for the holomorphic self-map $g^{-1} \circ f$.

Observe that we may weaken hypothesis (ii) by asking that only $S' - \mathrm{Sing}(f, g)$ is comfortably embedded into M, instead of the whole S' (see Remark 2.5.8). Similarly, in [4, Thm.6.2] the authors assume that only $S' - \mathrm{Sing}(f)$ is comfortably embedded into M.

5.3 Computation of the Camacho-Sad residues

In this section we derive explicit formulas for the computation of the residues of Theorem 5.2.2 at isolated singular points of $\mathscr{D}$ and, in a case, of S. A reference for this part is [36, Sec.IV.6].

Let ∇^{cs} be any δ^{cs}-connection on N_{S^0}, let $r : U_0 \to S^0$ be a tubular neighborhood of S^0 in M and consider the C^∞ connection $r^*\nabla^{cs}$ on $[S]|_{U_0} \cong r^*N_{S^0}$. Let Σ be any connected component of $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$ and let $U \subset M$ be a regular neighborhood of Σ such that $U \cap (\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})) = \Sigma$. Let ∇ be for the moment any C^∞ connection on $[S]|_U$. Lastly, let $R \subset U$ be any compact real C^∞ submanifold of dimension $2n$ oriented as M such that $\Sigma \subset \mathrm{Int}R$, with the boundary ∂R transverse to S and oriented with the orientation induced by R. Set $\tilde{R} = R \cap S$, then by Section 5.2 and 1.9 the residue of $c_1^{n-1}([S])$ at Σ is

$$\mathrm{Res}\,(\mathscr{D}, S, \Sigma) = \int_{\tilde{R}} c_1^{n-1}\,(\nabla) - \int_{\partial\tilde{R}} c_1^{n-1}\,(r^*\nabla^{cs}, \nabla)\,.$$

From now on set $\tilde{U} = U \cap S$ and $\tilde{U}_0 = \tilde{U} - \{p\}$ to ease the notation. Suppose that Σ is an isolated point, that is $\Sigma = \{p\}$. In this case up to shrinking U we can assume $[S]|_U$ trivial, then we can choose ∇ trivial respect to a holomorphic generator of $[S]|_U$. Consequently, the residue of $c_1^{n-1}([S])$ at p becomes

$$\begin{aligned}\operatorname{Res}(\mathscr{D}, S, p) &= -\int_{\partial\tilde{R}} c_1^{n-1}\left(r^*\nabla^{cs}, \nabla\right) \\ &= -\int_{\partial\tilde{R}} c_1^{n-1}\left(\nabla^{cs}, \nabla|_{\tilde{U}_0}\right), \end{aligned} \tag{5.3.1}$$

where in the second equality we use that $\partial\tilde{R} \subset \tilde{U}_0$ and that by the very definitions $c_1^{n-1}(r^*\nabla^{cs}, \nabla)|_{\tilde{U}_0} = c_1^{n-1}(\nabla^{cs}, \nabla|_{\tilde{U}_0})$.

Now let us consider the case $p \in \operatorname{Sing}(\mathscr{D})$ first. In this case $N_{\tilde{U}} = N_{S'}|_{\tilde{U}}$ does exist since p is a regular point of S, moreover we can assume that U is the open set of a local holomorphic chart $(U, z) \in \mathfrak{U}$ centered at p. Let $\nabla_{\tilde{U}}$ be the C^∞ connection on $N_{\tilde{U}}$ trivial respect to ∂z^1, that is

$$\nabla_{\tilde{U}}\left(\partial z^1\right) = 0,$$

and let $r' : U \to \tilde{U}$ be a tubular neighborhood of $\tilde{U}$ in M. We consider the C^∞ connection $\nabla = (r')^*\nabla_{\tilde{U}}$ over $[S]|_U \cong (r')^*N_{\tilde{U}}$, which is trivial respect to $(r')^*\partial z^1$. Clearly $\nabla|_{\tilde{U}} = \nabla_{\tilde{U}}$, then we can rewrite (5.3.1) as

$$\operatorname{Res}(\mathscr{D}, S, p) = -\int_{\partial\tilde{R}} c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}\right). \tag{5.3.2}$$

As done in Section 3.3, we mimic the strategy adopted in the proof of [36, Thm.III.5.5]. Let $L^j \subset T\tilde{U} = TS'|_{\tilde{U}}$ be the holomorphic line subbundle generated by $\frac{\partial}{\partial z^j}|_{\tilde{U}}$, for any $j = 2, \dots, n$. There are natural flat partial holomorphic connections δ^j along L^j on $N_{\tilde{U}}$ defined by setting

$$\left(\delta^j\right)_{\alpha\frac{\partial}{\partial z^j}\big|_{\tilde{U}}}(\omega) = \alpha\pi\left(\left[\frac{\partial}{\partial z^j}, \tilde{\omega}\right]\bigg|_{\tilde{U}}\right),$$

for any $\omega \in \mathcal{N}_{\tilde{U}}$, $\alpha \in \mathcal{O}_{\tilde{U}}$ and $j = 2, \dots, n$, where $\pi : TM|_{S'} \to N_{S'}$ and $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}}$ is any holomorphic vector field such that $\pi(\tilde{\omega}|_{\tilde{U}}) = \omega$. Clearly the flatness of δ^j comes from the Jacoby identity. Note that $\delta^j(\partial z^1) = 0$ hence $\nabla_{\tilde{U}}$ is a δ^j-connection, for any $j = 2, \dots, n$.

Let $\mathcal{X}$ be the canonical local extension of $\mathscr{D}$ associated to (U, z). It depends on the case we are in, but for the moment we write

$$\mathcal{X} = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial z^j},$$

where $\xi^j \in \mathcal{O}_M$, in order to keep the generality of the argumentation. With this notation the canonical local generator of $\mathscr{D}$ associated to (U, z) is then $X = \mathcal{X}|_{\tilde{U}} = \sum_{j=2}^n \xi^j|_{\tilde{U}} \frac{\partial}{\partial z^j}|_{\tilde{U}}$. Let us introduce the open covering $\tilde{\mathcal{U}}_0 = \{\tilde{U}^2, \ldots, \tilde{U}^n\}$ of $\tilde{U}_0$ defined likewise the one in Section 3.3, that is by setting

$$\tilde{U}^j = \left\{ q \in \tilde{U}_0 \text{ s.t. } \xi^j(q) \neq 0 \right\}$$

for any $j = 2, \ldots, n$. This is a covering of $\tilde{U}_0$ since $\{\xi^2 = \cdots = \xi^n = 0\} \cap \tilde{U} = \{p\}$ by hypothesis. Observe that

$$\left\{ \left.\frac{\partial}{\partial z^2}\right|_{\tilde{U}^j}, \ldots, \widehat{\left.\frac{\partial}{\partial z^j}\right|_{\tilde{U}^j}}, \ldots, \left.\frac{\partial}{\partial z^n}\right|_{\tilde{U}^j}, X|_{\tilde{U}^j} \right\}$$

is a holomorphic frame of $T\tilde{U}^j = TS'|_{\tilde{U}^j}$ for any $j = 2, \ldots, n$, thus we can define the C^∞ connections ∇_j on $N_{\tilde{U}^j}$ by asking that are δ^{cs}-connections and by setting

$$(\nabla_j)_{\alpha \frac{\partial}{\partial z^k}|_{\tilde{U}^j}}(\omega) = \alpha \pi \left(\left.\left[\frac{\partial}{\partial z^k}, \tilde{\omega} \right]\right|_{\tilde{U}^j} \right)$$

for any $\omega \in \mathcal{N}_{\tilde{U}^j}$, $\alpha \in \mathcal{O}_{\tilde{U}^j}$ and $k = 2, \ldots, n$ such that $k \neq j$, where $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}^j}$ is any holomorphic vector field such that $\pi(\tilde{\omega}|_{\tilde{U}^j}) = \omega$. Observe that in general

$$(\nabla_j)_{\alpha \frac{\partial}{\partial z^j}|_{\tilde{U}^j}}(\omega) \neq \alpha \pi \left(\left.\left[\frac{\partial}{\partial z^j}, \tilde{\omega} \right]\right|_{\tilde{U}^j} \right)$$

then ∇_j is not a δ^j-connection, on the contrary it is clearly a δ^k-connection for any $k = 2, \ldots, n$ such that $k \neq j$.

Consider now the Čech-de Rham cocycle $\tau \in C^{2(n-1)-2}_{\tilde{\mathcal{U}}_0}$ defined by

$$\tau_{i_1 \cdots i_k} = c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right) \in \Omega^{2(n-1)-k-1}\left(\tilde{U}^{i_1} \cap \cdots \cap \tilde{U}^{i_k}\right),$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n-1$. Then we have a lemma analogous to Lemma 3.3.2 (the proof is almost the same).

Lemma 5.3.1.

(i) $(D\tau)_i = -c_1^{n-1}(\nabla^{cs}, \nabla_{\tilde{U}})$ *for any* $i = 2, \ldots, n$.

(ii) $(D\tau)_{i_1 \cdots i_k} = 0$ *for any* $2 \leq i_1 < \cdots < i_k \leq n$ *and* $k = 2, \ldots, n-2$.

(iii) $(D\tau)_{2 \cdots n} = -c_1^{n-1}(\nabla_{\tilde{U}}, \nabla_2, \ldots, \nabla_n)$.

Proof. Observe preliminarily that by Theorem 1.4.9

$$c_1^{n-1}\left(\nabla^{cs}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right) = 0 \tag{5.3.3}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n-1$, since by definition $\nabla^{cs}, \nabla_{i_1}, \ldots, \nabla_{i_k}$ are all δ^{cs}-connections. Moreover, again by Theorem 1.4.9

$$c_1^{n-1}\left(\nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right) = 0 \tag{5.3.4}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n-2$, since by what we have remarked there is a $i \in \{2, \ldots, n\} - \{i_1, \ldots, i_k\}$ such that $\nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}$ are all δ^i-connections.

(i) By definition of D and property (1.1.2)

$$\begin{aligned}(D\tau)_i =& \mathrm{d}\tau_i = \mathrm{d}c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_i\right)\\ =& - c_1^{n-1}\left(\nabla_{\tilde{U}}, \nabla_i\right) + c_1^{n-1}\left(\nabla^{cs}, \nabla_i\right) - c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}\right)\end{aligned}$$

for any $i = 2, \ldots, n$. Then by (5.3.3) and (5.3.4) we are done.

$(ii), (iii)$ By definition of D

$$\begin{aligned}(D\tau)_{i_1\cdots i_k} =& (-1)^{k-1}\mathrm{d}\tau_{i_1\cdots i_k} + \sum_{\ell=1}^{k}(-1)^{\ell-1}\tau_{i_1\cdots\widehat{i_\ell}\cdots i_k}\\ =& (-1)^{k-1}\mathrm{d}c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right)\\ &+ \sum_{\ell=1}^{k}(-1)^{\ell-1}c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \widehat{\nabla_{i_\ell}}, \ldots, \nabla_{i_k}\right),\end{aligned}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 2, \ldots, n-1$. By property (1.1.2) we have that

$$\begin{aligned}(-1)^{k-1}&\mathrm{d}c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right)\\ &= - c_1^{n-1}\left(\nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right) + c_1^{n-1}\left(\nabla^{cs}, \nabla_{i_1}, \ldots, \nabla_{i_k}\right)\\ &\quad - \sum_{\ell=1}^{k}(-1)^{\ell-1}c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \widehat{\nabla_{i_\ell}}, \ldots, \nabla_{i_k}\right).\end{aligned}$$

Then $(D\tau)_{i_1\cdots i_k} = -c_1^{n-1}(\nabla_{\tilde{U}}, \nabla_{i_1}, \ldots, \nabla_{i_k}) + c_1^{n-1}(\nabla^{cs}, \nabla_{i_1}, \ldots, \nabla_{i_k})$ for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 2, \ldots, n-1$, and by (5.3.3) and (5.3.4) we are done. □

Consider the open covering $\{\tilde{U}^2 \cap \partial\tilde{R}, \ldots, \tilde{U}^n \cap \partial\tilde{R}\}$ of $\partial\tilde{R}$ and a system of honey-comb cells $\{\partial\tilde{R}^2, \ldots, \partial\tilde{R}^n\}$ adapted to it. Set $\partial\tilde{R}^{2\cdots n} = \partial\tilde{R}^2 \cap$

$\cdots \cap \partial\tilde{R}^n$, then by Lemma 5.3.1 and recalling the Čech-de Rham integration (1.7.1) we have that

$$0 = \int_{\partial\tilde{R}} D\tau = -\sum_{i=2}^{n} \int_{\partial\tilde{R}^i} c_1^{n-1}\left(\nabla^{cs}, \nabla_{\tilde{U}}\right) \\ - \int_{\partial\tilde{R}^{2\cdots n}} c_1^{n-1}\left(\nabla_{\tilde{U}}, \nabla_2, \cdots, \nabla_n\right),$$

and consequently by (5.3.2)

$$\operatorname{Res}(\mathscr{D}, S, p) = \int_{\partial\tilde{R}^{2\cdots n}} c_1^{n-1}\left(\nabla_{\tilde{U}}, \nabla_2, \ldots, \nabla_n\right). \tag{5.3.5}$$

Recall by Section 1.1 that $c_1^{n-1}(\nabla_{\tilde{U}}, \nabla_2, \ldots, \nabla_n)$ is a $(n-1)$-form over $\tilde{U}^{2\cdots n} = \tilde{U}^2 \cap \cdots \cap \tilde{U}^n$ defined by $c_1^{n-1}(\nabla_{\tilde{U}}, \nabla_2, \ldots, \nabla_n) = I_*(c_1^{n-1}(\nabla_{(t)}))$, where $\nabla_{(t)}$ is the C^∞ connection on the complex vector bundle

$$N_{\tilde{U}^{2\cdots n}} \times \mathbb{R}^{n-1} \longrightarrow \tilde{U}^{2\cdots n} \times \mathbb{R}^{n-1} \tag{5.3.6}$$

defined by

$$\nabla_{(t)} = \left(1 - \sum_{j=2}^{n} t_j\right) \nabla_{\tilde{U}} + \sum_{j=2}^{n} t_j \nabla_j$$

and I_* denotes the integration along the fiber. To compute $c_1^{n-1}(\nabla_{(t)})$ we need to know the curvature 2-form $\kappa_{(t)}$ of $\nabla_{(t)}$ respect to a C^∞ generator of (5.3.6). Recall that if $\theta_{(t)}$ is a connection 1-form of $\nabla_{(t)}$ then we have the relation $\kappa_{(t)} = \mathrm{d}\theta_{(t)} - \theta_{(t)} \wedge \theta_{(t)} = \mathrm{d}\theta_{(t)}$, so we just need to compute $\theta_{(t)}$. Let $\theta_{\tilde{U}}, \theta_2, \ldots, \theta_n$ be the connection 1-forms of $\nabla_{\tilde{U}}, \nabla_2, \ldots, \nabla_n$ respect to the holomorphic generator ∂z^1 of $N_{\tilde{U}^{2\cdots n}}$. Then $\theta_{(t)}$ respect to the induced C^∞ generator of (5.3.6) is

$$\theta_{(t)} = \left(1 - \sum_{j=2}^{n} t_j\right) \theta_{\tilde{U}} + \sum_{j=2}^{n} t_j \theta_j = \sum_{j=2}^{n} t_j \theta_j, \tag{5.3.7}$$

where in the second equality we use that $\theta_{\tilde{U}} = 0$ because $\nabla_{\tilde{U}}$ is trivial respect to ∂z^1. Since $\nabla_2, \ldots, \nabla_n$ are all δ^{cs}-connections, the connection 1-forms $\theta_2, \ldots, \theta_n$ depend ultimately on the canonical local extension $\mathcal{X}$ of $\mathscr{D}$. By definition

$$(\nabla_j)_{\left.\frac{\partial}{\partial z^k}\right|_{\tilde{U}^j}} \left(\partial z^1\right) = 0$$

for any $j, k = 2, \dots, n$ such that $k \neq j$, and

$$(\nabla_j)_{\frac{\partial}{\partial z^j}\big|_{\tilde{U}^j}} \left(\partial z^1\right) = \frac{1}{\xi^j|_{\tilde{U}^j}} \pi \left(\left[\mathcal{X}, \frac{\partial}{\partial z^1} \right] \bigg|_{\tilde{U}^j} \right)$$
$$= - \frac{1}{\xi^j|_{\tilde{U}^j}} \left. \frac{\partial \xi^1}{\partial z^1} \right|_{\tilde{U}^j} \partial z^1,$$

for any $j = 2, \dots, n$. This means that

$$\theta_j = - \left. \frac{\mathrm{d}z^j}{\xi^j} \right|_{\tilde{U}^j} \cdot \left. \frac{\partial \xi^1}{\partial z^1} \right|_{\tilde{U}^j}$$

for any $j = 2, \dots, n$, then by (5.3.7)

$$\theta_{(t)} = - \sum_{j=2}^{n} t_j \left. \frac{\mathrm{d}z^j}{\xi^j} \right|_{\tilde{U}^{2\cdots n}} \cdot \left. \frac{\partial \xi^1}{\partial z^1} \right|_{\tilde{U}^{2\cdots n}} .$$

Consequently the curvature 2-form of $\nabla_{(t)}$ respect to the fixed generator of (5.3.6) is

$$k_{(t)} = \mathrm{d}\theta_{(t)} = - \sum_{j=2}^{n} \mathrm{d}t_j \wedge \left. \frac{\mathrm{d}z^j}{\xi^j} \right|_{\tilde{U}^{2\cdots n}} \cdot \left. \frac{\partial \xi^1}{\partial z^1} \right|_{\tilde{U}^{2\cdots n}}$$
$$+ \text{ terms not containing } \mathrm{d}t_j,$$

then

$$c_1^{n-1}\left(\nabla_{(t)}\right) = \left(\frac{\sqrt{-1}}{2\pi} k_{(t)} \right)^{n-1}$$
$$= (-1)^{\left[\frac{n-1}{2}\right]} (n-1)! \left. \frac{\left(\frac{-\sqrt{-1}}{2\pi} \frac{\partial \xi^1}{\partial z^1} \right)^{n-1} \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{\xi^2 \cdots \xi^n} \right|_{\tilde{U}^{2\cdots n}} \wedge \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n$$
$$+ \text{ terms not containing } \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n.$$

By definition of I_* (see [36, Sec.II.4]) and recalling that $\int_{\Delta^{n-1}} \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n = 1/(n-1)!$, we finally have that

$$c_1^{n-1}(\nabla_{\tilde{U}}, \nabla_2, \dots, \nabla_n) = (-1)^{\left[\frac{n-1}{2}\right]} \left. \frac{\left(\frac{-\sqrt{-1}}{2\pi} \frac{\partial \xi^1}{\partial z^1} \right)^{n-1} \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{\xi^2 \cdots \xi^n} \right|_{\tilde{U}^{2\cdots n}}$$

and by (5.3.5)

$$\operatorname{Res}(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_{\partial\tilde{R}^{2\cdots n}} \frac{(-1)^{\left[\frac{n-1}{2}\right]} \left(\frac{\partial \xi^1}{\partial z^1}\right)^{n-1}}{\xi^2 \cdots \xi^n} \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n \tag{5.3.8}$$

Similarly to Section 3.3, let us consider now the specific compact real submanifold $R \subset U$ defined by

$$R = \left\{ q \in U \text{ s.t. } \left|\xi^2(q)\right|^2 + \cdots + |\xi^n(q)|^2 \leq (n-1)\epsilon^2 \right\}$$

for $\epsilon > 0$ small enough. Then

$$\tilde{R} = R \cap S = \left\{ q \in \tilde{U} \text{ s.t. } \left|\xi^2(q)\right|^2 + \cdots + |\xi^n(q)|^2 \leq (n-1)\epsilon^2 \right\}$$

and its boundary is

$$\partial\tilde{R} = \left\{ q \in \tilde{U} \text{ s.t. } \left|\xi^2(q)\right|^2 + \cdots + |\xi^n(q)|^2 = (n-1)\epsilon^2 \right\}.$$

As in Section 3.3, we take the system of honey-comb cells $\{\partial\tilde{R}^2, \ldots, \partial\tilde{R}^n\}$ adapted to $\{\tilde{U}^2 \cap \partial\tilde{R}, \ldots, \tilde{U}^n \cap \partial\tilde{R}\}$ defined by setting

$$\partial\tilde{R}^j = \left\{ q \in \partial\tilde{R} \text{ s.t. } \left|\xi^j(q)\right| \geq \left|\xi^i(q)\right| \text{ for } i = 2, \ldots, n \right\}$$

for any $j = 2, \ldots, n$. Let Γ_ξ be defined by

$$\Gamma_\xi = \left\{ q \in \tilde{U} \text{ s.t. } \left|\xi^2(q)\right| = \cdots = |\xi^n(q)| = \epsilon \right\} \tag{5.3.9}$$

and oriented so that $\mathrm{d}\vartheta^2 \wedge \cdots \wedge \mathrm{d}\vartheta^n > 0$, with $\vartheta^j = \arg(\xi^j)$. Considering the orientations we have $\Gamma_\xi = (-1)^{\left[\frac{n-1}{2}\right]} \partial\tilde{R}^{2\cdots n}$, then we can rewrite (5.3.8) as

$$\operatorname{Res}(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_{\Gamma_\xi} \frac{\left(\frac{\partial \xi^1}{\partial z^1}\right)^{n-1}}{\xi^2 \cdots \xi^n} \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n. \tag{5.3.10}$$

Clearly this formula changes slightly depending on the case. For instance if (f, g) is tangential along S then Γ_ξ is exactly the Γ defined in (3.3.9), and by (4.2.1) we have that $\xi^j = h^j$ for any $j = 1, \ldots, n$, where the h^j are defined in (2.2.7). Moreover, since $h^1 \in \mathcal{I}_S$ there is a holomorphic function $\ell^1 \in \mathcal{O}_M$ such that

$$h^1 = \ell^1 z^1 \tag{5.3.11}$$

and then
$$\left.\frac{\partial\xi^1}{\partial z^1}\right|_{\tilde{U}^{2\cdots n}} = \left.\frac{\partial(\ell^1 z^1)}{\partial z^1}\right|_{\tilde{U}^{2\cdots n}} = \ell^1|_{\tilde{U}^{2\cdots n}}\,.$$
If S' is comfortably embedded into M and $\nu > 1$ we have again that $\Gamma_\xi = \Gamma$ and $\xi^j = h^j$ for $j = 2, \ldots, n$, while by (4.3.2) it follows that $\xi^1 = k^1 z^1$, where k^1 is defined in (4.3.1). In particular in this case
$$\left.\frac{\partial\xi^1}{\partial z^1}\right|_{\tilde{U}^{2\cdots n}} = \left.\frac{\partial(k^1 z^1)}{\partial z^1}\right|_{\tilde{U}^{2\cdots n}} = k^1|_{\tilde{U}^{2\cdots n}}\,.$$
Lastly, if S' is comfortably embedded into M and $\nu = 1$ then Γ_ξ becomes
$$\begin{aligned}\Gamma' = \Big\{q \in \tilde{U} \text{ s.t. } \left|\left(1 + h_0^1(q)\right) h^2(q)\right| = \cdots \\ \cdots = \left|\left(1 + h_0^1(q)\right) h^n(q)\right| = \epsilon\Big\},\end{aligned} \tag{5.3.12}$$
where h_0^1 is defined in (4.3.1). Moreover, by (4.3.10) we have that $\xi^1 = k^1 z^1$ and $\xi^j = (1 + h_0^1)h^j$ for any $j = 2, \ldots, n$, and in this case $\frac{\partial\xi^1}{\partial z^1}|_{\tilde{U}^{2\cdots n}}$ is exactly as for $\nu = 1$. Summing up, by (5.3.10) and taking into account all these remarks we have the following theorem about Camacho-Sad residues at isolated singular points of $\mathscr{D}$.

Theorem 5.3.2. *Let M, S and (f, g) be as in Theorem 5.2.2 and assume*

(i) (f, g) is tangential along S

or

(ii) S' is comfortably embedded into M and σ is a splitting morphism of S' into M.

In case (i) set $\mathscr{D} = \mathscr{D}_{f,g}$ and let $\mathfrak{U}$ be an atlas adapted to S' and g. In case (ii) set $\mathscr{D} = \mathscr{D}^\sigma_{f,g}$ if $\nu > 1$ and $\mathscr{D} = \mathscr{D}^{\sigma,1}_{f,g}$ if $\nu = 1$, let $\mathfrak{U}$ be a comfortable atlas adapted to S', g and σ, and suppose $\mathscr{D} \neq 0$. Let $p \in Sing(\mathscr{D})$ be an isolated singular point and let $(U, z) \in \mathfrak{U}$ be a local holomorphic chart centered at p such that $U \cap (Sing(S) \sqcup Sing(\mathscr{D})) = \{p\}$. If (f, g) is tangential along S then the residue $Res(\mathscr{D}, S, p)$ of Theorem 5.2.2 is given by the integral formula
$$Res\,(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_\Gamma \frac{(\ell^1)^{n-1}}{h^2 \cdots h^n}\, dz^2 \wedge \cdots \wedge dz^n, \tag{5.3.13}$$
where the holomorphic functions $h^2, \ldots, h^n$ are defined in (2.2.7), ℓ^1 is defined in (5.3.11) and Γ is defined in (3.3.9) and is oriented so that

$d\vartheta^2 \wedge \cdots \wedge d\vartheta^n > 0$, with $\vartheta^j = arg(h^j)$. Instead, if S' is comfortably embedded into M and $\nu > 1$ then the residue is given by the integral formula

$$Res\,(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_{\Gamma} \frac{\left(k^1\right)^{n-1}}{h^2 \cdots h^n} dz^2 \wedge \cdots \wedge dz^n, \tag{5.3.14}$$

where the holomorphic function k^1 is defined in (4.3.1). Lastly, if S' is comfortably embedded into M and $\nu = 1$ then the residue is given by the integral formula

$$Res\,(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_{\Gamma'} \frac{\left(k^1\right)^{n-1}}{\left(1 + h_0^1\right)^{n-1} h^2 \cdots h^n} dz^2 \wedge \cdots \wedge dz^n, \tag{5.3.15}$$

where the holomorphic function h_0^1 is defined in (4.3.1) and Γ' is defined in (5.3.12) and is oriented so that $d\vartheta^2 \wedge \cdots \wedge d\vartheta^n > 0$, with $\vartheta^j = arg((1+h_0^1)h^j)$.

Let us conclude with the case $p \in \mathrm{Sing}(S)$. Recall that from the beginning of the section $U \subset M$ is an open neighborhood of p such that $U \cap (\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})) = \{p\}$ and $[S]|_U$ is trivial. The strategy of the non-singular case does not work now since we do not have in general a local holomorphic chart $(U, z) \in \mathfrak{U}$ centered at p. Consequently, we do not have a holomorphic generator γ of $[S]|_U$ such that $\gamma|_{\tilde{U}_0} = \partial z^1$ and we can not rewrite (5.3.1) in (5.3.2). In what follows we are going to see that if $g|_U$ is a biholomorphism onto its image, (f, g) is tangential along S and $\nu > 1$ then we can anyway compute the residue at p in another way.

First of all, observe that up to shrinking U we can assume there is a holomorphic function $x \in \mathcal{I}_S$ which is a local generator of $\mathcal{I}_S$ at any point of U. Now suppose there is a holomorphic vector field $\mathcal{V}$ extending the foliation $\mathscr{D}$ on U, that is $\mathcal{V}|_{\tilde{U}_0}$ generates $\mathscr{D}$ on $\tilde{U}_0$. Such a $\mathcal{V}$ would induce the flat partial holomorphic connection $\delta^{cs}_{\mathcal{V}}$ along $N^{\otimes \nu}_{\tilde{U}_0} \subset T\tilde{U}_0$ on $N_{\tilde{U}_0}$ defined by setting

$$(\delta^{cs}_{\mathcal{V}})_{\alpha \mathcal{V}|_{\tilde{U}_0}}(\omega) = \alpha\pi\left(\left[\mathcal{V}, \tilde{\omega}\right]|_{\tilde{U}_0}\right) \tag{5.3.16}$$

for any $\omega \in \mathcal{N}_{\tilde{U}_0}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$, where $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}_0}$ is any holomorphic vector field such that $\pi(\tilde{\omega}|_{\tilde{U}_0}) = \omega$. Suppose that $\delta^{cs}_{\mathcal{V}}$ coincides with the partial holomorphic connection δ^{cs} of Proposition 5.1.1. Then choosing some holomorphic coordinates $(y^1, \ldots, y^n)$ over U as in [36, Cor.IV.4.5] (see also [27, Thm.2]), namely such that

$$\mathcal{V} = \sum_{j=1}^{n} \upsilon^j \frac{\partial}{\partial y^j}$$

and $\{\upsilon^2 = \cdots = \upsilon^n = 0\} \cap S = \{p\}$, we may compute the residue at p applying directly the integral formula of [36, Thm.IV.6.3] (see also the formula just after [27, Thm.2]). More precisely, we would have that

$$\operatorname{Res}(\mathscr{D}, S, p) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1} \int_{\Gamma_{\mathcal{V}}} \frac{(\mathrm{d}x(\mathcal{V})/x)^{n-1}}{\upsilon^2 \cdots \upsilon^n} \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n \quad (5.3.17)$$

where $\Gamma_{\mathcal{V}}$ is defined by

$$\Gamma_{\mathcal{V}} = \left\{ q \in \tilde{U} \text{ s.t. } \left|\upsilon^2(q)\right| = \cdots = \left|\upsilon^n(q)\right| = \epsilon \right\}, \quad (5.3.18)$$

for $\epsilon > 0$ small enough, and oriented so that $\mathrm{d}\vartheta^2 \wedge \cdots \wedge \mathrm{d}\vartheta^n > 0$, with $\vartheta^j = \arg(\upsilon^j)$. Note that $\mathrm{d}x(\mathcal{V}) \in \mathcal{I}_S$ and that $(\mathrm{d}x(\mathcal{V})/x)|_{\tilde{U}}$ does not depend on the choice of x.

Remark 5.3.3. We could use directly the integral formula (5.3.17) even for the non-singular case, and we would have obtained the same formulas of Theorem 5.3.2. In fact if $p \in \operatorname{Sing}(\mathscr{D})$ and $(U, z) \in \mathfrak{U}$ is such that $p \in U$ we can take as $\mathcal{V}$ the canonical local extension $\mathcal{X}$ associated to (U, z), as $x \in \mathcal{I}_S$ the first coordinate z^1 and as coordinates $(y^1, \ldots, y^n)$ the coordinates $(z^1, \ldots, z^n)$. Then recalling (4.2.1) we have that

$$\frac{\mathrm{d}z^1(\mathcal{X})}{z^1} = \frac{h^1}{z^1} = \ell^1$$

and $\upsilon^j = h^j$ for any $j = 2, \ldots, n$ when (f, g) is tangential alnog S. Moreover, recalling (4.3.2) we have that

$$\frac{\mathrm{d}z^1(\mathcal{X})}{z^1} = \frac{k^1 z^1}{z^1} = k^1$$

and $\upsilon^j = h^j$ for any $j = 2, \ldots, n$ when S' is comfortably embedded into M and $\nu > 1$. Lastly, recalling (4.3.10) we again have that

$$\frac{\mathrm{d}z^1(\mathcal{X})}{z^1} = \frac{k^1 z^1}{z^1} = k^1$$

and $\upsilon^j = (1 + h_0^1)h^j$ for any $j = 2, \ldots, n$ when S' is comfortably embedded into M and $\nu = 1$. Thus by (5.3.17) and (5.3.18) we recover (5.3.13), (5.3.14) and (5.3.15).

Let $(y^1, \ldots, y^n)$ be for the moment any holomorphic coordinates over U, let $(u^1 = y^1 \circ g|_U^{-1}, \ldots, u^n = y^n \circ g|_U^{-1})$ be the corresponding special coordinates on $g(U)$ and set for simplicity $f^j = w^j \circ f$ and $g^j = w^j \circ g$ for any $j = 1, \dots, n$. By Lemma 2.1.2 we have that $f^j - g^j \in \mathcal{I}_S^\nu$ for any $j = 1, \dots, n$, hence we can define the holomorphic vector field over U

$$\mathcal{V}_{f,g} = \sum_{j=1}^n \frac{f^j - g^j}{x^\nu} \frac{\partial}{\partial y^j}. \tag{5.3.19}$$

Moreover, let us define the sets $U^j = \{q \in U \text{ s.t. } \frac{\partial x}{\partial y^j}(q) \neq 0\}$ and $\tilde{U}^j = U^j \cap \tilde{U}_0$, for any $j = 1, \dots, n$. Note that $\{\tilde{U}^1, \dots, \tilde{U}^n\}$ is an open covering of $\tilde{U}_0$ since $\{\frac{\partial x}{\partial y^1} = \cdots = \frac{\partial x}{\partial y^n} = 0\} \cap \tilde{U} = \{p\}$ by hypotheses. We can define over each U^j the holomorphic coordinates $z_j = (z_j^1, \dots, z_j^n)$ by setting

$$\begin{aligned}
z_j^1 &= x, && \text{for any } j = 1, \dots, n,\\
z_j^i &= y^{i-1}, && \text{for } j = 2, \dots, n \text{ and } i = 2, \dots, j,\\
z_j^i &= y^i, && \text{for } j = 1, \dots, n-1 \text{ and } i = j+1, \dots, n.
\end{aligned}$$

Note that (U^j, z_j) is a local holomorphic chart adapted to S' and g for any $j = 1, \dots, n$.

Lemma 5.3.4. *Let $\mathcal{V}_{f,g}$ be the holomorphic vector field over U defined as in (5.3.19). If (f,g) is tangential along S then*

$$\mathcal{X}^j = \mathcal{V}_{f,g}|_{U^j} + V_\nu^j$$

for any $j = 1, \dots, n$, where $\mathcal{X}^j$ is the canonical local extension of $\mathscr{D}$ associated to (U^j, z_j) and V_ν^j is a holomorphic vector field on U^j with coefficients belonging to $\mathcal{I}_S^\nu$.

Proof. We do the proof only for $j = 1$ since it is basically the same for any $j = 1, \dots, n$. For the sake of simplicity set $\mathcal{X} = \mathcal{X}^1$ and $(U, z) = (U^1, z_1)$, and let $w = (w^1, \dots, w^n)$ be the special coordinates associated to the z. Recall by (4.2.1) that

$$\mathcal{X} = \sum_{j=1}^n h^j \frac{\partial}{\partial z^j},$$

where the holomorphic functions $h^j \in \mathcal{O}_M$ are defined in (2.2.7). By definition of the coordinates z, we have for $j = 2, \dots, n$ that

$$\frac{f^j - g^j}{x^\nu} = \frac{w^j \circ f - w^j \circ g}{(z^1)^\nu} = h^j \tag{5.3.20}$$

and

$$0 = \frac{\partial y^1}{\partial y^j} = \sum_{i=1}^{n} \frac{\partial y^1}{\partial z^i}\frac{\partial z^i}{\partial y^j} = \frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j} + \frac{\partial y^1}{\partial z^j},$$

hence for $j = 2, \ldots, n$

$$\frac{\partial y^1}{\partial z^j} = -\frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j}. \tag{5.3.21}$$

Then by (5.3.21) and recalling (2.1.1) we have

$$\begin{aligned} f^1 - g^1 =& \sum_{j=1}^{n} \left(w^j \circ f - w^j \circ g\right)\left(\frac{\partial u^1}{\partial w^j} \circ g\right) \left(\operatorname{mod} \mathcal{I}_S^{2\nu}\right) \\ =& \sum_{j=1}^{n} h^j (z^1)^\nu \frac{\partial y^1}{\partial z^j} \left(\operatorname{mod} \mathcal{I}_S^{2\nu}\right) \\ =& h^1 (z^1)^\nu \frac{\partial y^1}{\partial z^1} - \sum_{j=2}^{n} h^j (z^1)^\nu \frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j} \left(\operatorname{mod} \mathcal{I}_S^{2\nu}\right), \end{aligned}$$

consequently

$$\frac{f^1 - g^1}{x^\nu} = h^1 \frac{\partial y^1}{\partial z^1} - \sum_{j=2}^{n} h^j \frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j} \left(\operatorname{mod} \mathcal{I}_S^{\nu}\right). \tag{5.3.22}$$

Lastly, by the very definition of the coordinates z and by (5.3.21) it follows

$$\frac{\partial}{\partial z^1} = \frac{\partial y^1}{\partial z^1}\frac{\partial}{\partial y^1} \tag{5.3.23}$$

and

$$\frac{\partial}{\partial z^j} = -\frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j}\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^j} \tag{5.3.24}$$

for any $j = 2, \ldots, n$. Then by (5.3.20), (5.3.22), (5.3.23) and (5.3.24) we have that

$$\begin{aligned} \mathcal{X} =& \sum_{j=1}^{n} h^j \frac{\partial}{\partial z^j} = \left(h^1 \frac{\partial y^1}{\partial z^1} - \sum_{j=2}^{n} h^j \frac{\partial y^1}{\partial z^1}\frac{\partial z^1}{\partial y^j} \right) \frac{\partial}{\partial y^1} + \sum_{j=2}^{n} h^j \frac{\partial}{\partial y^j} \\ =& \mathcal{V}_{f,g}|_U + C \frac{\partial}{\partial y^1}, \end{aligned}$$

where $C \in \mathcal{I}_S^{\nu}$. $\square$

Clearly, Lemma 5.3.4 implies that if (f,g) is tangential along S then $\mathcal{V}_{f,g}$ extends $\mathscr{D}$ on U. Secondly, if $\nu > 1$ then the partial holomorphic connection $\delta^{cs}_{\mathcal{V}_{f,g}}$ induced by $\mathcal{V}_{f,g}$ as in (5.3.16) is exactly δ^{cs} restricted to $\tilde{U}_0$. In fact by Lemma 5.3.4 and recalling (5.1.1) we have on each $\tilde{U}^j$ that

$$\begin{aligned}\left(\delta^{cs}_{\mathcal{V}_{f,g}}\right)_{\alpha X^j}(\omega) = \left(\delta^{cs}_{\mathcal{V}_{f,g}}\right)_{\alpha \mathcal{V}_{f,g}|_{\tilde{U}^j}}(\omega)& \\ =\alpha\pi\left(\left[\mathcal{V}_{f,g},\tilde{\omega}\right]\big|_{\tilde{U}^j}\right) &= \alpha\pi\left(\left[\mathcal{X}^j - V^j_\nu,\tilde{\omega}\right]\big|_{\tilde{U}^j}\right)\\ =\alpha\pi\left(\left[\mathcal{X}^j,\tilde{\omega}\right]\big|_{\tilde{U}^j}\right) &= \left(\delta^{cs}\right)_{\alpha X^j}(\omega)\end{aligned}$$

for any $\omega \in \mathcal{N}_{\tilde{U}^j}$, $\alpha \in \mathcal{O}_{\tilde{U}^j}$ and $j = 1,\ldots,n$, where $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}^j}$ is any holomorphic vector field such that $\pi(\tilde{\omega}|_{\tilde{U}^j}) = \omega$. Note that in the third line we use that $[V^j_\nu,\tilde{\omega}]|_{\tilde{U}^j} = 0$ since $\nu > 1$. Consequently, choosing the holomorphic coordinates $(y^1,\ldots,y^n)$ in such a way that

$$\left\{\frac{f^2-g^2}{x^\nu} = \cdots = \frac{f^n-g^n}{x^\nu} = 0\right\} \cap S = \{p\} \tag{5.3.25}$$

we can apply formula (5.3.17). Observe that in this case

$$\frac{\mathrm{d}x\left(\mathcal{V}_{f,g}\right)}{x} = \frac{1}{x^{\nu+1}}\sum_{j=1}^{n}\left(f^j - g^j\right)\frac{\partial x}{\partial y^j}$$

and the $\Gamma_{\mathcal{V}}$ defined in (5.3.18) becomes

$$\Gamma'' = \left\{q \in \tilde{U} \text{ s.t. } \left|\frac{f^j-g^j}{x^\nu}(q)\right| = \epsilon, \text{ for } j = 2,\ldots,n\right\}, \tag{5.3.26}$$

then we have the following theorem about Camacho-Sad residues at isolated singular points of S.

Theorem 5.3.5. *Let M, S and (f,g) be as in Theorem 5.2.2 and assume that (f,g) is tangential along S and $\nu > 1$. Let $p \in Sing(S)$ be an isolated singular point, let $U \subset M$ be an open subset such that $U \cap (Sing(S) \sqcup Sing(\mathscr{D})) = \{p\}$ and suppose that $g|_U$ is a biholomorphism onto its image. Let $x \in \mathcal{I}_S$ be a local generator of $\mathcal{I}_S$ at any point of U, let $(y^1,\ldots,y^n)$ be holomorphic coordinates on U and $(u^1,\ldots,u^n)$ the corresponding special coordinates on $g(U)$. Set $f^j = u^j \circ f$ and $g^j = u^j \circ g$ for any $j = 1,\ldots,n$, and suppose that the coordinates y are such that we have (5.3.25). Then the residue $Res(\mathscr{D},S,p)$ of Theorem 5.2.2 is given by* *the integral formula*

$$Res\,(\mathscr{D},S,p)=\left(\frac{-\sqrt{-1}}{2\pi}\right)^{n-1}\int_{\Gamma''}\frac{\left[\sum_{j=1}^{n}(f^j-g^j)\frac{\partial x}{\partial y^j}\right]^{n-1}}{x^{n-1}\prod_{j=2}^{n}(f^j-g^j)}\;dy^2\wedge\cdots\wedge dy^n, \tag{5.3.27}$$

where Γ'' *is defined as in (5.3.26) and oriented in the usual way.*

5.4 Third partial holomorphic connection

Assume now that (f,g) is tangential along S. As already recalled in Section 5.1 any local holomorphic chart $(U,z)\in\mathfrak{U}$ induces a canonical local extension $\mathcal{X}$ of $\mathscr{D}$ on U, and this implies that we can define locally partial holomorphic connections exactly as in (4.1.2). Thus if $U\cap S'\neq\emptyset$ we can define a partial holomorphic connection along $N^{\otimes\nu}_{U\cap S^0}\subset TS^0|_{U\cap S^0}$ on $N^0_{\mathscr{D},M}|_{U\cap S^0}$ by setting

$$\begin{aligned}\mathcal{N}^0_{\mathscr{D},M}\big|_{U\cap S^0}&\xrightarrow{\delta^{ls}_{\mathcal{X}}}\left(\mathcal{N}^{\otimes\nu}_{U\cap S^0}\right)^*\otimes\mathcal{N}^0_{\mathscr{D},M}\big|_{U\cap S^0}\\ w&\longrightarrow\delta^{ls}_{\mathcal{X}}(w)\text{ s.t. }(\delta^{ls}_{\mathcal{X}})_{\alpha X}(w)=\alpha\rho\left([\mathcal{X},\tilde{w}]|_{U\cap S^0}\right),\end{aligned} \tag{5.4.1}$$

for any $w\in\mathcal{N}^0_{\mathscr{D},M}|_{U\cap S^0}$ and $\alpha\in\mathcal{O}_{U\cap S^0}$, where $\rho:TM|_{S^0}\to N^0_{\mathscr{D},M}$ is the obvious projection and $\tilde{w}\in\mathcal{T}_M|_{U\cap S^0}$ is any holomorphic vector field such that $\rho(\tilde{w}|_{U\cap S^0})=w$. Observe that any holomorphic vector field $v\in\mathcal{N}^{\otimes\nu}_{U\cap S^0}\subset\mathcal{T}_{S^0}|_{U\cap S^0}$ is of the form $v=\alpha X$ for a holomorphic function $\varphi\in\mathcal{O}_{U\cap S^0}$ since X is a local generator of $\mathscr{D}$ on $U\cap S'$. Exactly as (4.1.2) the definition of $\delta^{ls}_{\mathcal{X}}$ does not depend on the choice of $\tilde{w}$, moreover it is flat by the Jacobi identity.

As in Section 5.1, we can glue together all these local partial holomorphic connections thanks to Proposition 4.2.1, but *only when* $\nu>1$. Then only in this case we have a *global* partial holomorphic connection along $N^{\otimes\nu}_{S^0}\subset TS^0$ on $N^0_{\mathscr{D},M}$.

Proposition 5.4.1. *Suppose that* (f,g) *is tangential along* S *and that* $\nu>1$. *Let* $\mathfrak{U}$ *be an atlas adapted to* S' *and* g. *If* $(U,z),(\hat{U},\hat{z})\in\mathfrak{U}$ *are any local holomorphic charts such that* $U\cap\hat{U}\cap S'\neq\emptyset$ *and* $\delta^{ls}_{\mathcal{X}},\delta^{ls}_{\hat{\mathcal{X}}}$ *are the corresponding local partial holomorphic connections defined as in (5.4.1) then*

$$\delta^{ls}_{\hat{\mathcal{X}}}=\delta^{ls}_{\mathcal{X}}$$

where they overlap. Consequently, there is a well-defined flat partial holomorphic connection

$$\mathcal{N}^0_{\mathscr{D},M} \xrightarrow{\delta^{ls}} \left(\mathcal{N}^{\otimes\nu}_{S^0}\right)^* \otimes \mathcal{N}^0_{\mathscr{D},M}$$

along $N^{\otimes\nu}_{S^0} \subset TS^0$ *on* $N^0_{\mathscr{D},M}$.

Proof. Since X is a local generator of $\mathscr{D}$ on $U \cap \hat{U} \cap S^0$, we just need to prove that

$$\left(\delta^{ls}_{\hat{\mathcal{X}}}\right)_X(w) = \left(\delta^{ls}_{\mathcal{X}}\right)_X(w)$$

for any $w \in \mathcal{N}^0_{\mathscr{D},M}|_{U\cap\hat{U}\cap S^0}$. This is true because if $\hat{z}^1 = az^1$, with $a \in \mathcal{O}^*_M$, and $\rho : TM|_{S^0} \to N^0_{\mathscr{D},M}$ is the projection, by (4.2.2), Proposition 4.2.1 and recalling that $\nu > 1$ we have that

$$\begin{aligned}
\left(\delta^{ls}_{\hat{\mathcal{X}}}\right)_X(w) &= (a|_{S^0})^\nu \left(\delta^{ls}_{\hat{\mathcal{X}}}\right)_{\hat{X}}(w) = (a|_{S^0})^\nu \rho\left(\left[\hat{\mathcal{X}}, \tilde{w}\right]\Big|_{S^0}\right)\\
&= (a|_{S^0})^\nu \rho\left(\left[\left(\frac{1}{a}\right)^\nu \mathcal{X} + T_2 + V_3, \tilde{w}\right]\Big|_{S^0}\right)\\
&= \rho\left([\mathcal{X}, \tilde{w}]|_{S^0}\right) = \left(\delta^{ls}_{\mathcal{X}}\right)_X(w),
\end{aligned}$$

where in the last line we use the fact that $[T_2 + V_3, \tilde{w}]|_{S^0} \equiv 0$. □

5.5 A Lehmann-Suwa-Khanedani-type index theorem

Consider the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow N^{\otimes\nu}_{S^0} \xrightarrow{\mathscr{D}|_{S^0}} TM|_{S^0} \xrightarrow{\rho} N^0_{\mathscr{D},M} \longrightarrow 0 \tag{5.5.1}$$

and let ∇^{ls} be any δ^{ls}-connection on $N^0_{\mathscr{D},M}$. By (i) of Lemma 1.3.3 we can take C^∞ connections ∇^1_0 and ∇^2_0 respectively on $N^{\otimes\nu}_{S^0}$ and $TM|_{S^0}$ such that the triple $(\nabla^1_0, \nabla^2_0, \nabla^{bb})$ is compatible with (5.5.1). Let $r : U_0 \to S^0$ be a tubular neighborhood of S^0 in M and consider the C^∞ connections $r^*\nabla^1_0$ and $r^*\nabla^2_0$ respectively on $[S]|^{\otimes\nu}_{U_0} \cong r^*N^{\otimes\nu}_{S^0}$ and $TM|_{U_0} \cong r^*TM|_{S^0}$. Then setting $r^*\nabla^\bullet_0 = (r^*\nabla^1_0, r^*\nabla^2_0)$ we have by (i) of Lemma 1.3.3 and Theorem 1.4.9 that

$$\varphi\left(r^*\nabla^\bullet_0\right) = r^*\varphi\left(\nabla^\bullet_0\right) = r^*\varphi(\nabla^{ls}) = 0 \tag{5.5.2}$$

for any homogeneous symmetric polynomial φ of degree $n-1$. Let now $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$, which is clearly an analytic subvariety of S. For any λ let $U_\lambda \subset M$ be an open subset such that $U_\lambda \supset \Sigma_\lambda$ and $U_\lambda \cap U_{\lambda'} = \emptyset$ whenever $\lambda \neq \lambda'$. Set $U_1 = \sqcup_\lambda U_\lambda$ and let ∇_1^1 and ∇_1^2 be any C^∞ connections respectively on $[S]|_{U_1}^{\otimes \nu}$ and $TM|_{U_1}$. Then set $\nabla_1^\bullet = (\nabla_1^1, \nabla_1^2)$ and consider the open neighborhood $U = U_0 \cup U_1$ of S and its covering $\mathcal{U} = \{U_0, U_1\}$. By (5.5.2) it follows that

$$\varphi\left(TM - [S]^{\otimes \nu}\right) = \Big[\big(0, \varphi(\nabla_1^\bullet), \varphi(r^*\nabla_0^\bullet, \nabla_1^\bullet)\big)\Big] \in H^{2(n-1)}\left(\mathcal{U}, U_0, \mathbb{C}\right)$$

for any homogeneous symmetric polynomial φ of degree $n-1$, that is for a any such polynomial φ we can localize $\varphi(TM - [S]^{\otimes\nu})$ at $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$. Moreover the localization does not depend on the choice of the C^∞ connections, as stated by the following lemma.

Lemma 5.5.1. *Let $\bar\nabla^{ls}$ be another δ^{ls}-connection on $N^0_{\mathscr{D},M}$, let $(\bar\nabla_0^1, \bar\nabla_0^2, \bar\nabla^{bb})$ be another triple compatible with (5.5.1) and set $r^*\bar\nabla_0^\bullet = (r^*\bar\nabla_0^1, r^*\bar\nabla_0^2)$. Moreover, let $\bar\nabla_1^1$ and $\bar\nabla_1^2$ be other C^∞ connections respectively on $[S]|_{U_1}^{\otimes\nu}$ and $TM|_{U_1}$ and set $\bar\nabla_1^\bullet = (\bar\nabla_1^1, \bar\nabla_1^2)$. Then for any homogeneous symmetric polynomial φ of degree $n-1$*

$$\Big[\big(0, \varphi(\nabla_1^\bullet), \varphi(r^*\nabla_0^\bullet, \nabla_1^\bullet)\big)\Big] = \Big[\big(0, \varphi(\bar\nabla_1^\bullet), \varphi(r^*\bar\nabla_0^\bullet, \bar\nabla_1^\bullet)\big)\Big]$$

in $H^{2(n-1)}(\mathcal{U}, U_0, \mathbb{C})$.

Proof. By property (1.3.1) we have that

$$\varphi(\nabla_1^\bullet) - \varphi(\bar\nabla_1^\bullet) = \mathrm{d}\varphi(\bar\nabla_1^\bullet, \nabla_1^\bullet), \tag{5.5.3}$$

moreover that

$$\varphi(r^*\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\bar\nabla_1^\bullet, \nabla_1^\bullet) - \varphi(\bar\nabla_1^\bullet, r^*\nabla_0^\bullet) - \mathrm{d}\varphi(\bar\nabla_1^\bullet, r^*\nabla_0^\bullet, \nabla_1^\bullet)$$

and

$$\begin{aligned}\varphi(r^*\bar\nabla_0^\bullet, \bar\nabla_1^\bullet) &= \varphi(r^*\nabla_0^\bullet, \bar\nabla_1^\bullet) + \varphi(r^*\bar\nabla_0^\bullet, r^*\nabla_0^\bullet) + \mathrm{d}\varphi(r^*\bar\nabla_0^\bullet, r^*\nabla_0^\bullet, \bar\nabla_1^\bullet) \\ &= -\varphi(\bar\nabla_1^\bullet, r^*\nabla_0^\bullet) + \mathrm{d}\varphi(r^*\bar\nabla_0^\bullet, r^*\nabla_0^\bullet, \bar\nabla_1^\bullet),\end{aligned}$$

where in the last equality we use that $\varphi(r^*\bar\nabla_0^\bullet, r^*\nabla_0^\bullet) = r^*\varphi(\bar\nabla^{ls}, \nabla^{ls}) = 0$ by (ii) of Lemma 1.3.3 and Theorem 1.4.9. Hence

$$\varphi(r^*\nabla_0^\bullet, \nabla_1^\bullet) - \varphi(r^*\bar\nabla_0^\bullet, \bar\nabla_1^\bullet) = \varphi(\bar\nabla_1^\bullet, \nabla_1^\bullet) - \mathrm{d}\omega, \tag{5.5.4}$$

with $\omega = \varphi(\bar{\nabla}_1^\bullet, r^*\nabla_0^\bullet, \nabla_1^\bullet) + \varphi(r^*\bar{\nabla}_0^\bullet, r^*\nabla_0^\bullet, \bar{\nabla}_1^\bullet)$. By (5.5.3) and (5.5.4) and recalling the definition of the Čech-de Rham differential D we then have

$$\big(0, \varphi(\nabla_1^\bullet), \varphi(r^*\nabla_0^\bullet, \nabla_1^\bullet)\big) - \big(0, \varphi(\bar{\nabla}_1^\bullet), \varphi(r^*\bar{\nabla}_0^\bullet, \bar{\nabla}_1^\bullet)\big) = D\big(0, \varphi(\bar{\nabla}_1^\bullet, \nabla_1^\bullet), \omega\big).$$

□

Suppose now S compact, then $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D}) \subset S$ is also compact and we can assume without loss of generality that U_1 is a regular neighborhood of $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$. Consequently we can assume that U is a regular neighborhood of S. Then by what we said in this chapter and by the commutative diagram (1.9.2) we gain the following index theorem.

Theorem 5.5.2 (Lehmann-Suwa-Khanedani-type index theorem). *Let M be a complex manifold of dimension* n *and let $S \subset M$ be a globally irreducible compact analytic hypersurface with regular part $S' = S - Sing(S)$. Let (f, g) be a pair of distinct holomorphic self-maps of M such that $f|_S \equiv g|_S$ and g is a local biholomorphism over an open neighborhood of S', with order of coincidence $\nu = \nu_{f,g}$. Assume that (f, g) is tangential along S and that $\nu > 1$, set $\mathscr{D} = \mathscr{D}_{f,g}$ and let $Sing(S) \sqcup Sing(\mathscr{D}) = \sqcup_\lambda \Sigma_\lambda$ be the decomposition in connected components of the singular set $Sing(S) \sqcup Sing(\mathscr{D})$. Then for any homogeneous symmetric polynomial φ of degree $n-1$ there exist complex numbers $Res_\varphi(\mathscr{D}; TM|_S - [S]|_S^{\otimes\nu}; \Sigma_\lambda)$, one for each connected component Σ_λ, such that*

$$\sum_\lambda Res_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, \Sigma_\lambda\right) = \int_S \varphi\left(TM - [S]^{\otimes\nu}\right).$$

We call Theorem 5.5.2 a *Lehmann-Suwa-Khanedani-type index theorem* since Lehmann and Suwa introduced this kind of residues and the partial holomorphic connection (4.1.2), which inspires the one in Proposition 5.4.1 (see [27], [28] and also [23]).

Remark 5.5.3. Suppose that g is a *global* biholomorphism over an open neighborhood of S' and recall Remark 2.2.6 and 2.4.6. In this case Theorem 5.5.2 turns out to be [4, Thm.6.3] for the holomorphic self-map $g^{-1} \circ f$.

In [4, Thm.6.3] the authors assume also that S' is comfortably embedded into M, on the contrary we do not need this hypothesis. The reason is a slightly different definition of the partial holomorphic connection δ^{ls}, which in [4] is called 'Lehmann-Suwa (holomorphic) action' and it is denoted by $\tilde{V}$ (see [4, Thm.5.3]). Both here and in [4] the partial holomorphic connection is defined locally by fixing a local holomorphic chart $(U, z) \in \mathfrak{U}$, and one has to extend $\mathscr{D}(v) \in \mathcal{T}_{S^0}|_{U\cap S^0}$ to a section of $\mathcal{T}_M|_{U\cap S^0}$ for any $v \in \mathcal{N}_{U\cap S^0}^{\otimes\nu}$. The difference between here and [4] is on the way of extending $\mathscr{D}(v)$. In (5.4.1) we observe that $\mathscr{D}(v) = \alpha X$ for a holomorphic function $\alpha \in \mathcal{O}_{U\cap S^0}$, since X is a local generator of $\mathscr{D}$, and then we extend $\mathscr{D}(v)$ to $\alpha\mathcal{X}$. Instead in [4] the authors make a sort of "double extension". They firstly extend v to a section $\tilde{v} \in \mathcal{T}_M|_{U\cap S^0}^{\otimes\nu}$ called the 'local extension of v along the fibers of a splitting morphism σ' (see [4, Def.5.7]), and then extend $\mathscr{D}(v)$ to $\mathfrak{D}(\tilde{v})$ where $\mathfrak{D}$ is a local holomorphic section of $(TM^{\otimes\nu})^* \otimes TM$ on U defined similarly to (2.2.8). In order to have good local extensions $\tilde{v}$, and then good local extensions $\mathfrak{D}(\tilde{v})$, the atlas $\mathfrak{U}$ needs to be comfortable (and adapted to S' and σ).

5.6 Computation of the Lehmann-Suwa-Khanedani residues

We end the chapter and the thesis by deriving explicit formulas for the computation of the residues appearing in Theorem 5.5.2 at isolated singular points of $\mathscr{D}$ and, in a case, of S. To do it, we mix in some sense the strategies of Section 3.3 and 5.3. A reference for this part is [36, Sec.IV.5]. In the following let φ be any homogeneous symmetric polynomial of degree $n-1$.

Let ∇_0^1 and ∇_0^2 be C^∞ connections respectively on $N_{S^0}^{\otimes\nu}$ and $TM|_{S^0}$ as in Section 5.5 and set $\nabla_0^\bullet = (\nabla_0^1, \nabla_0^2)$. Let $r : U_0 \to S^0$ be a tubular neighborhood of S^0 in M, consider the C^∞ connections $r^*\nabla_0^1$ and $r^*\nabla_0^2$ respectively on $[S]|_{U_0}^{\otimes\nu} \cong r^*N_{S^0}^{\otimes\nu}$ and $TM|_{U_0} \cong r^*TM|_{S^0}$, and set $r^*\nabla_0^\bullet = (r^*\nabla_0^1, r^*\nabla_0^2)$. Let Σ be any connected component of $\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})$ and let $U \subset M$ be a regular neighborhood of Σ such that $U \cap (\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})) = \Sigma$. Let ∇^1 and ∇^2 be for the moment any C^∞ connections respectively on $[S]|_U^{\otimes\nu}$ and $TM|_U$, and set $\nabla^\bullet = (\nabla^1, \nabla^2)$. Lastly, let $R \subset U$ be any compact real C^∞ submanifold of dimension $2n$ oriented as M such that $\Sigma \subset \mathrm{Int}R$, with ∂R transverse to S and oriented with the orientation induced by R. Set $\tilde{R} = R \cap S$, then by Section 5.5 and 1.9 the residue of $\varphi(TM - [S]^{\otimes\nu})$ at Σ is

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, \Sigma\right) = \int_{\tilde{R}} \varphi\left(\nabla^\bullet\right) - \int_{\partial\tilde{R}} \varphi\left(r^*\nabla_0^\bullet, \nabla^\bullet\right).$$

From now on set $\tilde{U} = U \cap S$ and $\tilde{U}_0 = \tilde{U} - \{p\}$ to ease the notation. Suppose that Σ is an isolated point, that is $\Sigma = \{p\}$. In this case up to shrinking

U we can assume $[S]|_U^{\otimes\nu}$ and $TM|_U$ trivial, then we can choose ∇^1 and ∇^2 trivial respect to some local holomorphic frames of them. Consequently, the residue of $\varphi(TM - [S]^{\otimes\nu})$ at p becomes

$$\begin{aligned} \mathrm{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) &= -\int_{\partial\tilde{R}} \varphi\left(r^*\nabla_0^\bullet, \nabla^\bullet\right) \\ &= -\int_{\partial\tilde{R}} \varphi\left(\nabla_0^\bullet, \nabla^\bullet|_{\tilde{U}_0}\right), \end{aligned} \tag{5.6.1}$$

where in the second equality we use that $\partial\tilde{R} \subset \tilde{U}_0$ and that by the very definitions $\varphi(r^*\nabla_0^\bullet, \nabla^\bullet)|_{\tilde{U}_0} = \varphi(\nabla_0^\bullet, \nabla^\bullet|_{\tilde{U}_0})$.

Now let us consider the case $p \in \mathrm{Sing}(\mathscr{D})$ first. In this case $N_{\tilde{U}} = N_{S'}|_{\tilde{U}}$ does exist since p is a regular point of S, moreover we can assume that U is the open set of a local holomorphic chart $(U, z) \in \mathfrak{U}$ centered at p. Thus $(\partial z^1)^\nu$ is a holomorphic generator of $N_{\tilde{U}}^{\otimes\nu}$ and $\{\frac{\partial}{\partial z^1}|_{\tilde{U}}, \ldots, \frac{\partial}{\partial z^n}|_{\tilde{U}}\}$ is a holomorphic frame of $TM|_{\tilde{U}}$. Consequently, we can define the C^∞ connection $\nabla_{\tilde{U}}^1$ over $N_{\tilde{U}}^{\otimes\nu}$ trivial respect to $(\partial z^1)^\nu$, that is

$$\nabla_{\tilde{U}}^1\left((\partial z^1)^\nu\right) = 0,$$

and the C^∞ connection $\nabla_{\tilde{U}}^2$ over $TM|_{\tilde{U}}$ trivial respect to the frame, that is

$$\nabla_{\tilde{U}}^2\left(\left.\frac{\partial}{\partial z^j}\right|_{\tilde{U}}\right) = 0$$

for any $j = 1, \ldots, n$. Let $r' : U \to \tilde{U}$ be a tubular neighborhood of $\tilde{U}$ in M and consider the C^∞ connections $\nabla^1 = (r')^*\nabla_{\tilde{U}}^1$ and $\nabla^2 = (r')^*\nabla_{\tilde{U}}^2$ respectively on $[S]|_U^{\otimes\nu} \cong (r')^*N_{\tilde{U}}^{\otimes\nu}$ and $TM|_U \cong (r')^*TM|_{\tilde{U}}$, which are trivial respect to $(r')^*(\partial z^1)^\nu$ and $\{(r')^*\frac{\partial}{\partial z^1}|_{\tilde{U}}, \ldots, (r')^*\frac{\partial}{\partial z^n}|_{\tilde{U}}\}$. Clearly $\nabla^1|_{\tilde{U}} = \nabla_{\tilde{U}}^1$ and $\nabla^2|_{\tilde{U}} = \nabla_{\tilde{U}}^2$, then setting $\nabla_{\tilde{U}}^\bullet = (\nabla_{\tilde{U}}^1, \nabla_{\tilde{U}}^2)$ we can rewrite (5.6.1) as

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = -\int_{\partial\tilde{R}} \varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet\right). \tag{5.6.2}$$

As for the other kinds of residues, we mimic the strategy adopted in the proof of [36, Thm.III.5.5]. What follows is extremely similar to what is done in Section 3.3, anyhow for the sake of completeness and to make the section self-contained we repeat the steps. The canonical local generator X of $\mathscr{D}$ associated to (U, z) generates the holomorphic line subbundle $N_{\tilde{U}_0}^{\otimes\nu} \subset T\tilde{U}_0$, and it induces two flat partial holomorphic connections δ_0^1 and δ_0^2 along $N_{\tilde{U}_0}^{\otimes\nu} \subset T\tilde{U}_0$ on $N_{\tilde{U}_0}^{\otimes\nu}$ and $TM|_{\tilde{U}_0}$. The connection δ_0^1 is defined by setting

$$\left(\delta_0^1\right)_{\alpha X}(\omega) = \alpha \mathscr{D}^{-1}\big(\,[X, \mathscr{D}(\omega)]\,\big)$$

for any $\omega \in \mathcal{N}_{\tilde{U}_0}^{\otimes\nu}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$, while δ_0^2 is defined by setting

$$\left(\delta_0^2\right)_{\alpha X}(w) = \alpha\,[\mathcal{X}, \tilde{w}]|_{\tilde{U}_0}$$

for any $w \in \mathcal{T}_{M,\tilde{U}_0}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$, where $\mathcal{X}$ is the canonical local extension of $\mathscr{D}$ associated to (U, z) and $\tilde{w} \in \mathcal{T}_M|_{\tilde{U}_0}$ is any holomorphic vector field such that $\tilde{w}|_{\tilde{U}_0} = w$. Clearly the flatness comes from the Jacoby identity, moreover by construction the triple $(\delta_0^1, \delta_0^2, \delta^{ls})$ is compatible with the short exact sequence

$$0 \longrightarrow N_{\tilde{U}_0}^{\otimes\nu} \xrightarrow{\mathscr{D}|_{\tilde{U}_0}} TM|_{\tilde{U}_0} \xrightarrow{\rho} N^0_{\mathscr{D},M}|_{\tilde{U}_0} \longrightarrow 0,$$

which is (5.5.1) restricted to $\tilde{U}_0$. Then by Lemma 1.4.3 we can assume ∇_0^1 and ∇_0^2 respectively a δ_0^1-connection and a δ_0^2-connection. Note that $\delta_0^1((\partial z^1)^\nu) = 0$ then $\nabla_{\tilde{U}}^1$ is a δ_0^1-connection.

Let $L^j \subset T\tilde{U}$ be the holomorphic line subbundle generated by $\frac{\partial}{\partial z^j}|_{\tilde{U}}$, for any $j = 2, \ldots, n$. There are natural flat partial holomorphic connections $\delta^{1,j}$ along L^j on $N_{\tilde{U}}^{\otimes\nu}$ defined by setting

$$\left(\delta^{1,j}\right)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}}}\left(\beta(\partial z^1)^\nu\right) = \alpha \frac{\partial \beta}{\partial z^j}\left(\partial z^1\right)^\nu$$

for any $\alpha, \beta \in \mathcal{O}_{\tilde{U}}$ and $j = 2, \ldots, n$. Moreover, there are natural flat partial holomorphic connections $\delta^{2,j}$ along L^j on $TM|_{\tilde{U}}$ defined by setting

$$\left(\delta^{2,j}\right)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}}}(w) = \alpha \left.\left[\frac{\partial}{\partial z^j}, \tilde{w}\right]\right|_{\tilde{U}}$$

for any $\alpha \in \mathcal{O}_{\tilde{U}}$, $w \in \mathcal{T}_{M,\tilde{U}}$ and $j = 2, \ldots, n$, where $\tilde{w} \in \mathcal{T}_M|_{\tilde{U}}$ is any holomorphic vector field such that $\tilde{w}|_{\tilde{U}} = w$. Again, the flatness comes from the Jacoby identity. Note that $\delta^{1,j}((\partial z^1)^\nu) = 0$ then $\nabla_{\tilde{U}}^1$ is a $\delta^{1,j}$-connection, for any $j = 2, \ldots, n$. Furthermore $\delta^{2,j}(\frac{\partial}{\partial z^k}|_{\tilde{U}}) = 0$ for any $k = 1, \ldots, n$ hence $\nabla_{\tilde{U}}^2$ is a $\delta^{2,j}$-connection, for any $j = 2, \ldots, n$.

Let now $\tilde{\mathcal{U}}_0 = \{\tilde{U}^2, \ldots, \tilde{U}^n\}$ be the open covering of $\tilde{U}_0$ defined by setting

$$\tilde{U}^j = \left\{q \in \tilde{U}_0 \text{ s.t. } h^j(q) \neq 0\right\}$$

for any $j = 2, \ldots, n$, where the holomorphic functions $h^j \in \mathcal{O}_M$ are the ones defined in (2.2.7). As already noted in Section 3.3

$$\left\{ \frac{\partial}{\partial z^2}\Big|_{\tilde{U}^j}, \ldots, \widehat{\frac{\partial}{\partial z^j}\Big|_{\tilde{U}^j}}, \ldots, \frac{\partial}{\partial z^n}\Big|_{\tilde{U}^j}, X|_{\tilde{U}^j} \right\}$$

is a holomorphic frame of $T\tilde{U}^j = TS'|_{\tilde{U}^j}$ for any $j = 2, \ldots, n$, thus we can define the C^∞ connections ∇_j^2 on $TM|_{\tilde{U}^j}$ by asking that are δ_0^2-connections and by setting

$$\left(\nabla_j^2\right)_{\alpha \frac{\partial}{\partial z^k}\big|_{\tilde{U}^j}}(w) = \alpha \left[\frac{\partial}{\partial z^k}, \tilde{w}\right]\Big|_{\tilde{U}^j}$$

for any $\alpha \in \mathcal{O}_{\tilde{U}^j}$, $w \in \mathcal{T}_{M,\tilde{U}^j}$ and $k = 2, \ldots, n$ such that $k \neq j$, where $\tilde{w} \in \mathcal{T}_M|_{\tilde{U}^j}$ is any holomorphic vector field such that $\tilde{w}|_{\tilde{U}^j} = w$. Moreover, we can define the C^∞ connections ∇_j^1 on $N_{\tilde{U}^j}^{\otimes\nu}$ simply by setting

$$\nabla_j^1 = \nabla_{\tilde{U}}^1\Big|_{\tilde{U}^j}$$

for any $j = 2, \ldots, n$. Observe that in general

$$\left(\nabla_j^2\right)_{\alpha \frac{\partial}{\partial z^j}\big|_{\tilde{U}^j}}(w) \neq \alpha \left[\frac{\partial}{\partial z^j}, \tilde{w}\right]\Big|_{\tilde{U}^j}$$

hence ∇_j^2 is not a $\delta^{2,j}$-connection, on the contrary it is a $\delta^{2,k}$-connection for any $k = 2, \ldots, n$ such that $k \neq j$. Moreover, ∇_j^1 is a δ_0^1-connection and a $\delta^{1,k}$-connection for any $k = 2, \ldots, n$ since $\nabla_{\tilde{U}}^1$ is.

Set now $\nabla_j^\bullet = (\nabla_j^1, \nabla_j^2)$ for any $j = 2, \ldots, n$ and consider the Čech-de Rham cocycle $\tau \in C_{\tilde{\mathcal{U}}_0}^{2(n-1)-2}$ defined by

$$\tau_{i_1 \cdots i_k} = \varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \ldots, \nabla_{i_k}^\bullet\right) \in \Omega^{2(n-1)-k-1}\left(\tilde{U}^{i_1} \cap \cdots \cap \tilde{U}^{i_k}\right),$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \ldots, n-1$. Then we have a lemma analogous to Lemma 3.3.2 (and the proof is almost the same).

Lemma 5.6.1.

(i) $(D\tau)_i = -\varphi(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet)$ *for any* $i = 2, \ldots, n$.

(ii) $(D\tau)_{i_1 \cdots i_k} = 0$ *for any* $2 \leq i_1 < \cdots < i_k \leq n$ *and* $k = 2, \ldots, n-2$.

(iii) $(D\tau)_{2 \cdots n} = -\varphi(\nabla_{\tilde{U}}^\bullet, \nabla_2^\bullet, \ldots, \nabla_n^\bullet)$.

Proof. Observe preliminarily that by Theorem 1.4.10

$$\varphi\left(\nabla_0^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) = 0 \tag{5.6.3}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \dots, n-1$, since by what we have remarked the connections $\nabla_0^1, \nabla_{i_1}^1, \dots, \nabla_{i_k}^1$ are all δ_0^1-connections and $\nabla_0^2, \nabla_{i_1}^2, \dots, \nabla_{i_k}^2$ are all δ_0^2-connections. Moreover, again by Theorem 1.4.10

$$\varphi\left(\nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) = 0 \tag{5.6.4}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 1, \dots, n-2$, since by what we have remarked the connections $\nabla_{\tilde{U}}^1, \nabla_{i_1}^1, \dots, \nabla_{i_k}^1$ are all $\delta^{1,j}$-connections (for any $j = 2, \dots, n$) and there is an $i \in \{2, \dots, n\} - \{i_1, \dots, i_k\}$ such that $\nabla_{\tilde{U}}^2, \nabla_{i_1}^2, \dots, \nabla_{i_k}^2$ are all $\delta^{2,i}$-connections.

(i) By definition of D and property (1.3.1)

$$\begin{aligned}(D\tau)_i =& \mathrm{d}\tau_i = \mathrm{d}\varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_i^\bullet\right) \\ =& -\varphi\left(\nabla_{\tilde{U}}^\bullet, \nabla_i^\bullet\right) + \varphi\left(\nabla_0^\bullet, \nabla_i^\bullet\right) - \varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet\right)\end{aligned}$$

for any $i = 2, \dots, n$. Then by (5.6.3) and (5.6.4) we are done.

$(ii), (iii)$ By definition of D

$$\begin{aligned}(D\tau)_{i_1 \cdots i_k} =& (-1)^{k-1}\mathrm{d}\tau_{i_1\cdots i_k} + \sum_{\ell=1}^{k}(-1)^{\ell-1}\tau_{i_1\cdots\widehat{i_\ell}\cdots i_k} \\ =& (-1)^{k-1}\mathrm{d}\varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) \\ &+ \sum_{\ell=1}^{k}(-1)^{\ell-1}\varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \widehat{\nabla_{i_\ell}^\bullet}, \dots, \nabla_{i_k}^\bullet\right),\end{aligned}$$

for any $2 \leq i_1 < \cdots < i_k \leq n$ and $k = 2, \dots, n-1$. By property (1.3.1) we have that

$$\begin{aligned}&(-1)^{k-1}\mathrm{d}\varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) \\ &\quad = -\varphi\left(\nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) + \varphi\left(\nabla_0^\bullet, \nabla_{i_1}^\bullet, \dots, \nabla_{i_k}^\bullet\right) \\ &\quad\quad - \sum_{\ell=1}^{k}(-1)^{\ell-1}\varphi\left(\nabla_0^\bullet, \nabla_{\tilde{U}}^\bullet, \nabla_{i_1}^\bullet, \dots, \widehat{\nabla_{i_\ell}^\bullet}, \dots, \nabla_{i_k}^\bullet\right).\end{aligned}$$

Then $(D\tau)_{i_1\cdots i_k} = -\varphi(\nabla^\bullet_{\tilde U}, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k}) + \varphi(\nabla^\bullet_0, \nabla^\bullet_{i_1}, \dots, \nabla^\bullet_{i_k})$ for any $2 \le i_1 < \cdots < i_k \le n$ and $k = 2, \dots, n-1$, and by (5.6.3) and (5.6.4) we are done. $\square$

Consider the open covering $\{\tilde U^2 \cap \partial\tilde R, \dots, \tilde U^n \cap \partial\tilde R\}$ of $\partial\tilde R$ and a system of honey-comb cells $\{\partial\tilde R^2, \dots, \partial\tilde R^n\}$ adapted to it. Set $\partial\tilde R^{2\cdots n} = \partial\tilde R^2 \cap \cdots \cap \partial\tilde R^n$, then by Lemma 5.6.1 and recalling the Čech-de Rham integration (1.7.1) we have that

$$0 = \int_{\partial\tilde R} D\tau = -\sum_{j=2}^n \int_{\partial\tilde R^j} \varphi\left(\nabla^\bullet_0, \nabla^\bullet_{\tilde U}\right) - \int_{\partial\tilde R^{2\cdots n}} \varphi\left(\nabla^\bullet_{\tilde U}, \nabla^\bullet_2, \cdots, \nabla^\bullet_n\right),$$

and consequently by (5.6.2)

$$\operatorname{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = \int_{\partial\tilde R^{2\cdots n}} \varphi\left(\nabla^\bullet_{\tilde U}, \nabla^\bullet_2, \cdots, \nabla^\bullet_n\right) \tag{5.6.5}$$

Recall by Section 1.3 that $\varphi(\nabla^\bullet_{\tilde U}, \nabla^\bullet_2, \dots, \nabla^\bullet_n) = I_*(\varphi(\nabla^\bullet_{(t)}))$, where $\nabla^\bullet_{(t)} = (\nabla^1_{(t)}, \nabla^2_{(t)})$ is defined by setting

$$\nabla^1_{(t)} = \left(1 - \sum_{j=2}^n t_j\right) \nabla^1_{\tilde U} + \sum_{j=2}^n t_j \nabla^1_j$$

and

$$\nabla^2_{(t)} = \left(1 - \sum_{j=2}^n t_j\right) \nabla^2_{\tilde U} + \sum_{j=2}^n t_j \nabla^2_j,$$

and I_* denotes the integration along the fiber. Note that $\nabla^1_{(t)} = \nabla^1_{\tilde U}$, then $c(\nabla^\bullet_{(t)}) = c(\nabla^2_{(t)}) \wedge c(\nabla^1_{(t)})^{-1} = c(\nabla^2_{(t)}) \wedge c(\nabla^1_{\tilde U})^{-1}$. Since we have chosen $\nabla^1_{\tilde U}$ trivial respect to $(\partial z^1)^\nu$, we have that $c(\nabla^1_{\tilde U}) = 1$ and then $c(\nabla^\bullet_{(t)}) = c(\nabla^2_{(t)})$. By the definitions it follows that $\varphi(\nabla^\bullet_{(t)}) = \varphi(\nabla^2_{(t)})$, thus

$$\varphi\left(\nabla^\bullet_{\tilde U}, \nabla^\bullet_2, \dots, \nabla^\bullet_n\right) = \varphi\left(\nabla^2_{\tilde U}, \nabla^2_2, \dots, \nabla^2_n\right)$$

and by (5.6.5) the residue of $\varphi(TM - [S]^{\otimes\nu})$ at p is

$$\operatorname{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = \int_{\partial\tilde R^{2\cdots n}} \varphi\left(\nabla^2_{\tilde U}, \nabla^2_2, \cdots, \nabla^2_n\right) \tag{5.6.6}$$

The last step is now to understand how is done $\varphi(\nabla^2_{\tilde U}, \nabla^2_2, \dots, \nabla^2_n)$, which is a $(n-1)$-form over $\tilde U^{2\cdots n} = \tilde U^2 \cap \cdots \cap \tilde U^n$. By definition

$\varphi(\nabla^2_{\tilde{U}}, \nabla^2_2, \ldots, \nabla^2_n) = I_*(\varphi(\nabla^2_{(t)}))$ where $\nabla^2_{(t)}$ is the C^∞ connection on the complex vector bundle

$$TM|_{\tilde{U}^{2\cdots n}} \times \mathbb{R}^{n-1} \longrightarrow \tilde{U}^{2\cdots n} \times \mathbb{R}^{n-1} \tag{5.6.7}$$

defined above and I_* is the integration along the fiber. To compute $\varphi(\nabla^2_{(t)})$ we need ultimately to know the curvature matrix $\kappa^2_{(t)}$ of $\nabla^2_{(t)}$ respect to a C^∞ frame of (5.6.7). Recall that if $\theta^2_{(t)}$ is a connection matrix of $\nabla^2_{(t)}$ then we have the relation $\kappa^2_{(t)} = \mathrm{d}\theta^2_{(t)} - \theta^2_{(t)} \wedge \theta^2_{(t)}$, so we just need to compute $\theta^2_{(t)}$. Let $\theta^2_{\tilde{U}}, \theta^2_2, \ldots, \theta^2_n$ be the connection matrices of $\nabla^2_{\tilde{U}}, \nabla^2_2, \ldots, \nabla^2_n$ respect to the holomorphic frame

$$\left\{ \left.\frac{\partial}{\partial z^1}\right|_{\tilde{U}^{2\cdots n}}, \ldots, \left.\frac{\partial}{\partial z^n}\right|_{\tilde{U}^{2\cdots n}} \right\}$$

of $TM|_{\tilde{U}^{2\cdots n}}$. Then $\theta^2_{(t)}$ respect to the induced C^∞ frame of (5.6.7) is

$$\theta^2_{(t)} = \left(1 - \sum_{j=2}^n t_j\right) \theta^2_{\tilde{U}} + \sum_{j=2}^n t_j \theta^2_j = \sum_{j=2}^n t_j \theta^2_j, \tag{5.6.8}$$

where in the second equality we use that $\theta^2_{\tilde{U}} = 0$ because $\nabla^2_{\tilde{U}}$ is trivial respect to the fixed frame. By definition

$$(\nabla^2_j)_{\frac{\partial}{\partial z^k}|_{\tilde{U}^j}} \left(\left.\frac{\partial}{\partial z^i}\right|_{\tilde{U}^j} \right) = 0$$

for any $j, k = 2, \ldots, n$ such that $k \neq j$ and $i = 1, \ldots, n$, and

$$\begin{aligned}
(\nabla^2_j)_{\frac{\partial}{\partial z^j}|_{\tilde{U}^j}} \left(\left.\frac{\partial}{\partial z^i}\right|_{\tilde{U}^j} \right) &= \frac{1}{h^j|_{\tilde{U}^j}} \left.\left[\mathcal{X}, \frac{\partial}{\partial z^i} \right]\right|_{\tilde{U}^j} \\
&= -\sum_{\ell=1}^n \frac{1}{h^j|_{\tilde{U}^j}} \left.\frac{\partial h^\ell}{\partial z^i}\right|_{\tilde{U}^j} \left.\frac{\partial}{\partial z^\ell}\right|_{\tilde{U}^j}
\end{aligned}$$

for any $j = 2, \ldots, n$ and $i = 1, \ldots, n$. This means that

$$\theta^2_j = -\left.\frac{\mathrm{d}z^j}{h^j}\right|_{\tilde{U}^j} \cdot H'|_{\tilde{U}^j}$$

for any $j = 2, \ldots, n$, where H' is the $n \times n$ matrix of holomorphic functions

$$H' = \left(\frac{\partial h^\ell}{\partial z^i} \right)_{i,\ell=1,\ldots,n}, \tag{5.6.9}$$

and then by (5.6.8)

$$\theta^2_{(t)} = -\sum_{j=2}^{n} t_j \left.\frac{\mathrm{d}z^j}{h^j}\right|_{\tilde{U}^{2\cdots n}} \cdot H'|_{\tilde{U}^{2\cdots n}}.$$

Consequently the curvature matrix of $\nabla^2_{(t)}$ respect to the fixed frame of (5.6.7) is

$$\begin{aligned}\kappa^2_{(t)} = \mathrm{d}\theta^2_{(t)} - \theta^2_{(t)} \wedge \theta^2_{(t)} = -\sum_{j=2}^{n} \mathrm{d}t_j \wedge \left.\frac{\mathrm{d}z^j}{h^j}\right|_{\tilde{U}^{2\cdots n}} \cdot H'|_{\tilde{U}^{2\cdots n}} \\ + \text{ terms not containing } \mathrm{d}t_j,\end{aligned}$$

then

$$\begin{aligned}\varphi\left(\nabla^2_{(t)}\right) =& (-1)^{\left[\frac{n-1}{2}\right]}(n-1)! \left.\frac{\varphi\left(-H'\right)\mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{h^2\cdots h^n}\right|_{\tilde{U}^{2\cdots n}} \wedge \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n \\ &+ \text{ terms not containing } \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n.\end{aligned}$$

By definition of I_* and recalling that $\int_{\Delta^{n-1}} \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n = 1/(n-1)!$, we finally have that

$$\varphi\left(\nabla^2_{\tilde{U}}, \nabla^2_2, \ldots, \nabla^2_n\right) = (-1)^{\left[\frac{n-1}{2}\right]} \left.\frac{\varphi\left(-H'\right)\mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n}{h^2\cdots h^n}\right|_{\tilde{U}^{2\cdots n}}$$

and by (5.6.6) the residue of $\varphi(TM - [S]^{\otimes\nu})$ at p is

$$\mathrm{Res}_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = \int_{\partial\tilde{R}^{2\cdots n}} \frac{(-1)^{\left[\frac{n-1}{2}\right]}\varphi\left(-H'\right)}{h^2\cdots h^n}\, \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^n.$$

As in Section 3.3 and 5.3, let us consider now the specific compact real submanifold $R \subset U$ defined by

$$R = \left\{q \in U \text{ s.t. } \left|h^2(q)\right|^2 + \cdots + |h^n(q)|^2 \leq (n-1)\epsilon^2\right\}$$

for $\epsilon > 0$ small enough. Then

$$\tilde{R} = R \cap S = \left\{q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right|^2 + \cdots + |h^n(q)|^2 \leq (n-1)\epsilon^2\right\}$$

and its boundary is

$$\partial\tilde{R} = \left\{q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right|^2 + \cdots + |h^n(q)|^2 = (n-1)\epsilon^2\right\}.$$

As usual we take the system of honey-comb cells $\{\partial\tilde{R}^2, \dots, \partial\tilde{R}^n\}$ adapted to $\{\tilde{U}^2 \cap \partial\tilde{R}, \dots, \tilde{U}^n \cap \partial\tilde{R}\}$ defined by setting

$$\partial\tilde{R}^j = \left\{ q \in \partial\tilde{R} \text{ s.t. } \left|h^j(q)\right| \geq \left|h^i(q)\right| \text{ for } i = 2, \dots, n \right\}$$

for any $j = 2, \dots, n$. Let Γ be the one defined in (3.3.9), that is

$$\Gamma = \left\{ q \in \tilde{U} \text{ s.t. } \left|h^2(q)\right| = \cdots = |h^n(q)| = \epsilon \right\},$$

and let it be oriented so that $\mathrm{d}\vartheta^2 \wedge \cdots \wedge \mathrm{d}\vartheta^n > 0$, with $\vartheta^j = \arg(h^j)$. Then considering the orientations we have $\Gamma = (-1)^{\left[\frac{n-1}{2}\right]} \partial\tilde{R}^{2\cdots n}$ and, summing up, we have proved the following theorem about Lehmann-Suwa-Khanedani residues at isolated singular points of $\mathscr{D}$.

Theorem 5.6.2. *Let M, S and (f, g) be as in Theorem 5.5.2, in particular (f, g) is tangential along S and $\nu > 1$. Set $\mathscr{D} = \mathscr{D}_{f,g}$ and let $\mathfrak{U}$ be an atlas adapted to S' and g. Let $p \in Sing(\mathscr{D})$ be an isolated singular point, let $(U, z) \in \mathfrak{U}$ be a local holomorphic chart centered at p such that $U \cap (Sing(S) \sqcup Sing(\mathscr{D})) = \{p\}$ and let φ be any homogeneous symmetric polynomial of degree $n-1$. Then the residue $Res_\varphi(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p)$ of Theorem 5.5.2 is given by the integral formula*

$$Res_\varphi\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = \int_\Gamma \frac{\varphi\left(-H'\right)}{h^2 \cdots h^n}\, dz^2 \wedge \cdots \wedge dz^n, \qquad (5.6.10)$$

where the holomorphic functions h^j are defined in (2.2.7), Γ is defined in (3.3.9) and is oriented so that $d\vartheta^2 \wedge \cdots \wedge d\vartheta^n > 0$, with $\vartheta^j = arg(h^j)$, and H' is the $n \times n$ matrix of holomorphic functions defined in (5.6.9).

We conclude with the case $p \in \mathrm{Sing}(S)$. Recall that from the beginning of the section $U \subset M$ is an open neighborhood of p such that $U \cap (\mathrm{Sing}(S) \sqcup \mathrm{Sing}(\mathscr{D})) = \{p\}$ and $[S]|_U^{\otimes\nu}$ and $TM|_U$ are trivial. As in Section 5.3, the strategy of the non-singular case does not work now since we do not have in general a local holomorphic chart $(U, z) \in \mathfrak{U}$ centered at p, and consequently we can not rewrite (5.6.1) in (5.6.2). Anyhow, if we assume that $g|_U$ is a biholomorphism onto its image we can again compute the residue at p basically in the same way of the Camacho-Sad singular case.

Suppose there is a holomorphic vector field $\mathcal{V}$ extending the foliation $\mathscr{D}$ on U, that is $\mathcal{V}|_{\tilde{U}_0}$ generates $\mathscr{D}$ on $\tilde{U}_0$. Such a $\mathcal{V}$ would induce the flat partial holomorphic connection $\delta_{\mathcal{V}}^{ls}$ along $N_{\tilde{U}_0}^{\otimes\nu} \subset T\tilde{U}_0$ on $N_{\mathscr{D},M}^0|_{\tilde{U}_0}$ defined by setting

$$\left(\delta_{\mathcal{V}}^{ls}\right)_{\alpha\mathcal{V}|_{\tilde{U}_0}}(\omega) = \alpha\rho\left(\left[\mathcal{V}, \tilde{\omega}\right]|_{\tilde{U}_0}\right) \qquad (5.6.11)$$

for any $\omega \in \mathcal{N}^0_{\mathscr{D},M}|_{\tilde{U}_0}$ and $\alpha \in \mathcal{O}_{\tilde{U}_0}$, where $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}_0}$ is any holomorphic vector field such that $\rho(\tilde{\omega}|_{\tilde{U}_0}) = \omega$. Suppose that $\delta^{ls}_{\mathcal{V}}$ coincides with the partial holomorphic connection δ^{ls} of Proposition 5.4.1. Then choosing some holomorphic coordinates $(y^1, \ldots, y^n)$ over U as in [36, Cor.IV.4.5] (see also [27, Thm.2]), namely such that

$$\mathcal{V} = \sum_{j=1}^{n} v^j \frac{\partial}{\partial y^j}$$

and $\{v^2 = \cdots = v^n = 0\} \cap S = \{p\}$, we may compute the residue at p applying directly the integral formula of [36, Thm.IV.5.3]. More precisely, we would have that

$$\operatorname{Res}_{\varphi}\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes \nu}, p\right) = \int_{\Gamma_{\mathcal{V}}} \frac{\varphi(-V)}{v^2 \cdots v^n} \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n, \qquad (5.6.12)$$

where V is the $n \times n$ matrix of holomorphic functions

$$V = \left(\frac{\partial v^{\ell}}{\partial y^i}\right)_{i,\ell=1,\ldots,n} \qquad (5.6.13)$$

and $\Gamma_{\mathcal{V}}$ is defined as in (5.3.18).

Remark 5.6.3. We could use directly the integral formula (5.6.12) even for the non-singular case, and we would have obtained the same formula of Theorem 5.6.2. In fact if $p \in \operatorname{Sing}(\mathscr{D})$ and $(U, z) \in \mathfrak{U}$ is such that $p \in U$ we can take as $\mathcal{V}$ the canonical local extension $\mathcal{X}$ associated to (U, z), as $x \in \mathcal{I}_S$ the first coordinate z^1 and as coordinates $(y^1, \ldots, y^n)$ the coordinates $(z^1, \ldots, z^n)$. Then recalling (4.2.1) we have $V = H'$, where H' is the matrix defined in (5.6.9), and $v^j = h^j$ for any $j = 2, \ldots, n$. Thus by (5.6.12) and (5.3.18) we recover (5.6.10).

As already observed in Section 5.3, up to shrinking U we can assume there is a holomorphic function $x \in \mathcal{I}_S$ which is a local generator of $\mathcal{I}_S$ at any point of U. Let $(y^1, \ldots, y^n)$ be for the moment any holomorphic coordinates over U, let $(u^1 = y^1 \circ g|_U^{-1}, \ldots, u^n = y^n \circ g|_U^{-1})$ be the corresponding special coordinates on $g(U)$ and set for simplicity $f^j = u^j \circ f$ and $g^j = u^j \circ g$ for any $j = 1, \ldots, n$. Moreover, let $\mathcal{V}_{f,g}$ be the holomorphic vector field over U defined in (5.3.19), that is

$$\mathcal{V}_{f,g} = \sum_{j=1}^{n} \frac{f^j - g^j}{x^{\nu}} \frac{\partial}{\partial y^j}.$$

We have already observed in Section 5.3 that if (f, g) is tangential alon S then $\mathcal{V}_{f,g}$ extends $\mathscr{D}$ on U. Furthermore, if $\nu > 1$ the partial holomorphic connection $\delta^{ls}_{\mathcal{V}_{f,g}}$ induced by $\mathcal{V}_{f,g}$ as in (5.6.11) is exactly δ^{ls} restricted to $\tilde{U}_0$. In fact by Lemma 5.3.4 and recalling (5.4.1) we have on each $\tilde{U}^j$ (defined as in Section 5.3) that

$$\begin{aligned}\left(\delta^{ls}_{\mathcal{V}_{f,g}}\right)_{\alpha X^j}(\omega) = &\left(\delta^{ls}_{\mathcal{V}_{f,g}}\right)_{\alpha \mathcal{V}_{f,g}|_{\tilde{U}^j}}(\omega)\\ =&\alpha\rho\left(\left[\mathcal{V}_{f,g}, \tilde{\omega}\right]|_{\tilde{U}^j}\right) = \alpha\rho\left(\left[\mathcal{X}^j - V^j_\nu, \tilde{\omega}\right]|_{\tilde{U}^j}\right)\\ =&\alpha\rho\left(\left[\mathcal{X}^j, \tilde{\omega}\right]|_{\tilde{U}^j}\right) = \left(\delta^{ls}\right)_{\alpha X^j}(\omega)\end{aligned}$$

for any $\omega \in \mathcal{N}^0_{\mathscr{D},M}|_{\tilde{U}^j}$, $\alpha \in \mathcal{O}_{\tilde{U}^j}$ and $j = 1, \ldots, n$, where $\tilde{\omega} \in \mathcal{T}_M|_{\tilde{U}^j}$ is any holomorphic vector field such that $\rho(\tilde{\omega}|_{\tilde{U}^j}) = \omega$. Note that in the third line we use that $[V^j_\nu, \tilde{\omega}]|_{\tilde{U}^j} = 0$ since $\nu > 1$. Consequently choosing the holomorphic coordinates $(y^1, \ldots, y^n)$ such that we have (5.3.25), that is

$$\left\{\frac{f^2 - g^2}{x^\nu} = \cdots = \frac{f^n - g^n}{x^\nu} = 0\right\} \cap S = \{p\},$$

we can apply formula (5.6.12). Observe that in this case the matrix V defined in (5.6.13) is the Jacobian matrix of

$$\left(\frac{f^2 - g^2}{x^\nu}, \ldots, \frac{f^n - g^n}{x^\nu}\right), \tag{5.6.14}$$

which we denote by J for simplicity. Moreover, the $\Gamma_{\mathcal{V}}$ defined in (5.3.18) becomes (5.3.26) as in Section 5.3, that is

$$\Gamma'' = \left\{q \in \tilde{U} \text{ s.t. } \left|\frac{f^j - g^j}{x^\nu}(q)\right| = \epsilon,\ \text{for } j = 2, \ldots, n\right\}.$$

Then finally we have the following theorem about Lehmann-Suwa-Khanedani residues at isolated singular points of S.

Theorem 5.6.4. *Let M, S and (f, g) be as in Theorem 5.5.2, in particular (f, g) is tangential along S and $\nu > 1$. Let $p \in Sing(S)$ be an isolated singular point, let $U \subset M$ be an open subset such that $U \cap (Sing(S) \sqcup Sing(\mathscr{D})) = \{p\}$ and suppose that $g|_U$ is a biholomorphism onto its image. Let $x \in \mathcal{I}_S$ be a local generator of $\mathcal{I}_S$ at any point of U, let $(y^1, \ldots, y^n)$ be holomorphic coordinates on U and $(u^1, \ldots, u^n)$ the corresponding special coordinates on $g(U)$. Set $f^j = u^j \circ f$ and $g^j = u^j \circ g$ for any $j = 1, \ldots, n$, and suppose that* *the coordinates y are such that we have (5.3.25). Let* φ *be any homogeneous symmetric polynomial of degree* $n-1$*, then the residue* $Res_{\varphi}(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p)$ *of Theorem 5.5.2 is given by the integral formula*

$$Res_{\varphi}\left(\mathscr{D}, TM|_S - [S]|_S^{\otimes\nu}, p\right) = \int_{\Gamma''} \frac{x^{\nu(n-1)}\varphi(-J)}{\prod_{j=2}^{n}(f^j - g^j)}\ du^2 \wedge \cdots \wedge du^n, \quad (5.6.15)$$

where J *is the Jacobian matrix of (5.6.14) and* Γ'' *is defined as in (5.3.26) and oriented in the usual way.*

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